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Theme

Generalization of some fixed point results

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Arabic Summary

تتناول هذه المذكرة مبادئ نظرية النقطة الثابتة وتعميمها في الفضاء المتري المستطيل. عبر ثلاثة فصول مفصلة، تم إنشاء المفاهيم الأساسية في الفصل الأول، وتم فحص نظريات باناخ، وكانان، وشاترجيا في الفصل الثاني، وتمت تغطية التعميمات في الفصل الثالث

يساهم هذا العمل بشكل كبير في تقدم الرياضيات وتطبيقاتها

<u>الكلمات المفتاحية</u> : نظرية النقطة الثابتة، الانكماش الرئيسي، الفضاء المتري المستطيل

French summary

Ce mémoire explore les principes de la théorie du point fixe et les généralisation dans l'espace métrique rectangulaire. À travers trois chapitres détaillés, les concepts fondamentaux sont établis dans le premier chapitre, les théorèmes de Banach, Kannan et Chatterjea sont examinées dans le deuxième chapitre, et des généralisations sont traitées dans le troisième chapitre. Ce travail contribue de manière significative à l'avancement des mathématiques et de leurs applications.

Mot clés:Théorème du point fixe, Principale de contraction , Espace métrique complet, Espace métrique rectangulaire.

Abstract

The memory explores the principles of fixed point theory, extending them to rectangular metric space. Through three detailed chapters, the foundational concepts are set in the first chapter, theories by Banach, Kannan, and Chatterjea are reviewed in the second chapter, and generalizations are addressed in the third chapter. This work significantly contributes to the advancement of mathematics and its applications.

keyword: Fixed point theorem, Principle contraction, Complete metric space, Rectangular metric space.

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Dedication

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I dedicate the fruits of this work, hoping that it will be a small brick in the edifice of science and knowledge, and the beginning of further research and development in the field of mathematics.

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INTRODUCTION

The inception of fixed point theory marks a significant milestone in the annals of mathematics, with its roots tracing back to the pioneering work of the eminent mathematician Stefan Banach in 1922 [4]. Banach's groundbreaking insights into the nature of fixed points within mathematical mappings laid the foundation for what would become a cornerstone of mathematical inquiry, propelling the theory into the forefront of academic discourse (see [9], [10], [25], [26]).

Since its inception, fixed point theory has transcended disciplinary boundaries, permeating diverse fields of human endeavor. From its humble origins in pure mathematics, it has blossomed into a versatile tool with applications spanning the breadth of human knowledge. In fields as varied as medicine, engineering, economics, and computer science, the concept of fixed points serves as a guiding light, illuminating pathways to solutions and insights previously unattainable (see [19], [21], [25], [34]).

As we embark on this scholarly journey, each chapter serves as a portal into a distinct realm of inquiry:

Chapter One: Fundamental Concepts of Fixed Point, Within this foundational chapter, we delve into the bedrock upon which fixed point theory rests. Herein lie the essential definitions and concepts, from the fundamental definition of metric spaces to the intricacies of sequences and convergent sequences. Through a meticulous exploration of these foundational elements, we pave the way for a deeper understanding of fixed point theory and its applications (see [11], [29], [26], [7], [25], [32]).

Chapter Two: Key Theorems, In this chapter, we navigate the landscape of key theorems that underpin fixed point theory. Through a detailed examination of Banach's fixed point theorem, Kannan's fixed point theorem, and the Chatterjea fixed point theorem, we uncover the theoretical underpinnings and practical implications of fixed point theory. Moreover, we explore real-world examples illustrating the profound impact of fixed point theory across a spectrum of disciplines ([7], [20], [25], [29], [13], [18]).

Chapter Three: Some Fixed Point Theorems in Rectangular Metric Spaces, in Rectangular Metric Spaces The zenith of our inquiry lies within this pivotal chapter, where we transcend traditional boundaries to explore the intricacies of rectangular metric spaces. Herein, we embark on a journey of generalization, extending previous theories to encompass the complexities of this novel domain. Through rigorous theoretical explorations and illuminating examples, we unveil the broader applicability and nuanced intricacies inherent in extending fixed point theory beyond conventional metric spaces (see [34], [17], [8], [6], [33], [31]).

In essence, this memory is more than a scholarly endeavor; it is a testament to the enduring relevance and adaptability of fixed point theory in the face of ever-evolving challenges. By delving into its theoretical underpinnings, exploring its practical applications, and extending its reach into new frontiers, we aspire not only to expand the horizons of mathematical knowledge but also to inspire future generations of scholars to push the boundaries of human understanding.

CHAPTER 1

Fundamental Concepts of Fixed Point

1.1 Introduction

In this chapter, we lay the groundwork for our exploration of fixed point theory. We begin by defining key concepts such as metric spaces, convergence of sequences, and complete metric spaces. These foundational principles are crucial for understanding the subsequent theorems and their applications. By establishing a solid base, we prepare to delve deeper into the intricacies of fixed point theory, setting the stage for more advanced discussions in the following chapters (see [9], [32]).

This chapter has been devoted to mentioning some of the necessary and basic concepts that were used in this memory.

1.2 Metric Spaces

A metric space is a set X equipped with a function d of two variables which measures the distance between points: $d(x, y)$ is the distance between two points x and y in X .

Definition 1 *[25] Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}$ is said to be a metric or a distance function on X if d satisfies the following properties:*

(i) (Positivity) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$;
(ii) (Symmetry) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.
The pair (X, d) is then called a metric space.

Remark 1 *Usually, only three conditions are used to define a distance function. Indeed, the first of these conditions is a property that follows from the other three, since:*

$$d(x, y) = d(x, y) + d(y, x) \geq d(x, x) = 0. \tag{1.1}$$

Example 1 *Below, we give some examples of metric spaces.*

• $X = (\mathbb{R}, d)$ with

$$d(x,y) = |x-y|\,.$$

•$X = (\mathbb{R}^n, d)$ with the Euclidean distance

$$d(x,y) = \sqrt{\sum_{k=1}^{n}(x_k - y_k)^2}.$$

Where $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$.
•$X = (\mathbb{R}^n, d)$ with

$$d(x,y) = \max_{k=1,..,n} |x_k - y_k|\,.$$

Example 2 *Assume that we want to move from one point $x = (x_1, x_2)$ in the plan to another $y = (y_1, y_2)$, but that we are only allowed to move horizontally and vertically.*

If we first move horizontally from (x_1, x_2) to (y_1, x_2) and then vertically from (y_1, x_2) to (y_1, y_2), the total distance is

$$d(x,y) = |y_1 - x_1| + |y_2 - x_2|.$$

This gives us a metric on $\mathbb{R}^2$ which is different from the usual metric in example (1.1). Also in this case the first two conditions of a metric space are obviously satisfied. To prove the triangle inequality, observe that for any third point $z = (z_1, z_2)$, we have :

$$\begin{aligned} d(x,y) &= |y_1 - x_1| + |y_2 - x_2| \\ &= |(y_1 - z_1) + (z_1 - x_1)| + |(y_2 - z_2) + (z_2 - x_2)| \\ &\leq |y_1 - z_1| + |z_1 - x_1| + |y_2 - z_2| + |z_2 - x_2| \\ &= |z_1 - x_1| + |z_2 - x_2| + |y_1 - z_1| + |y_2 - z_2| \\ &= d(x,z) + d(z,y). \end{aligned}$$

We used the ordinary triangle inequality for real numbers to get from the second to the third line.

Example 3 *There are many ways to measure the distance between functions, and in this example we shall look at some.*

Let X be the set of all continuous functions $f : [a,b] \to \mathbb{R}$. Then :

$$d_1(f,g) = sup\{|f(x) - g(x)| : x \in [a,b]\}$$

is a metric on X. This metric determines the distance between two functions by measuring the distance at the $x - value$ where the graphs are most apart.

This means that the distance between two functions may be large even if the functions in average are quite close.

$$d_2(f,g) = \int_a^b |f(x) - g(x)|\, dx.$$

The metric instead sums up the distance between $f(x)$ or $g(x)$ at all points. A third popular metric is :

$$d_3(f,g) = \left(\int_a^b |f(x) - g(x)|^2\, dx \right)^{\frac{1}{2}}.$$

This metric is a generalization of the usual (euclidean) metric in $\mathbb{R}^n$:

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}.$$

1.2.1 Convergence of sequences

A sequence is a function whose domain is the set of natural numbers $\mathbb{N}$. We usually denote the values of the function by $s(n)$ and call these values the terms of the sequence.

The notion of convergence of a sequence is crucial to understand the behaviour of real valued functions and is helpful in the estimation of sums of series (see [11], [34]).

We will provide some important definitions to bring the concept closer .

Definition 2 *[24] (**Convergent sequences**)*

A sequence is said to be convergent if it has a limit, and divergent otherwise.

When a sequence $(a_n)_{n\in\mathbb{N}}$ has the limit L, we write

$$\lim_{n\to+\infty} a_n = L.$$

The same statement can also be written as "$(a_n)_{n\in\mathbb{N}}$ converges to L ", or as "$a_n \to L$" as "$n \to \infty$" or just as "$a_n \to L$".

Let us look at the following example :

Example 4 *Verify that $\left(\frac{\sin(n)}{n}\right)_{n\in\mathbb{N}}$ converges to 0.*

Let $a_n = \frac{\sin(n)}{(n)}$ and $L = 0$ Take $\varepsilon > 0$ We have

$$|a_n - L| = \left| \frac{\sin(n)}{n} - 0 \right| \leq \frac{1}{n}.$$

Thus

$$|a_n - L| < \varepsilon \text{ if } \frac{1}{n} < \varepsilon \text{ if } (n > \frac{1}{\varepsilon}).$$

Choose

$$n_0 = \frac{1}{\varepsilon}.$$

Now

$$n \geq n_0 \Longrightarrow n \geq \frac{1}{\varepsilon} \Longrightarrow n > \frac{1}{\varepsilon} \Longrightarrow \frac{1}{n} < \varepsilon \Longrightarrow |a_n - L| < \varepsilon.$$

Therefore : $\left(\frac{\sin(n)}{n}\right)_{n\in\mathbb{N}}$ converges to 0.

Theorem 1 *Every convergent sequence is bounded.*

Proof. *Let us consider a convergent sequence $(a_n)_{n\in\mathbb{N}}$ which converges to L.*

Then for a given $\varepsilon > 0$, we have some $n_0 \in \mathbb{N}$ such that for all $n > n_0$:

$$|a_n - L| < \varepsilon.$$

Then for all $n > n_0$

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < |L| + \varepsilon.$$

Let

$$M = \max\{|a_1|, |a_2|, ..., |a_{n_0}|\}$$

and

$$K = \max\{|L| + \varepsilon, M\}.$$

Thus for all $n \geq 1$ we have

$$|a_n| \leq K.$$

This prove that $(a_n)_{n\in\mathbb{N}}$ is bounded. ■

Remark 2 *What is the opposite of the previous theorem statement? "every unbounded sequence is divergent." You know that a statement is true if and only if its contrapositive is true.*

This can help us in proving the divergence of many sequences. See one such example, given below.

Example 5 *Prove that the sequence $(n^n)_{n\in\mathbb{N}}$ is divergent.*

We know that for each $K \in \mathbb{N}$ there exists some $n \in \mathbb{N}$ such that $n > K$.This implies $n^n > K^n > K$.

Thus for each $K \in \mathbb{N}$ some term of $(n^n)_{n\in\mathbb{N}}$ is larger than k. Therefore $(n^n)_{n\in\mathbb{N}}$ is not bounded above.Hence $(n^n)_{n\in\mathbb{N}}$ is unbounded.

Therefore it follows from the previous theory, that $(n^n)_{n\in\mathbb{N}}$ is divergent.

1.2.2 Cauchy sequences

Definition 3 *(**Cauchy sequences**)*

A sequence $(a_n)_{n\in\mathbb{N}}$ is said to be a Cauchy sequence if for every $\varepsilon > 0$, there exists some $n_0 \in \mathbb{N}$ such that :

$$d(a_m, a_n) < \varepsilon \ , \forall m, n \geq n_0$$

- The sequence $(a_n)_{n\in\mathbb{N}} = (\frac{1}{n})_{n\in\mathbb{N}}$ is Cauchy sequance.
To prove : Let $\varepsilon > 0$, Now choose $n_0 \in \mathbb{N}$ such that $\frac{1}{2n_0} < \varepsilon$, i.e $n_0 > \frac{1}{2\varepsilon}$.
Now :

$$\begin{aligned} m &> n \geq n_0 \Longrightarrow \left|\frac{1}{m} - \frac{1}{n}\right| < \frac{1}{2n} \leq \frac{1}{2n_0} < \varepsilon \\ &\Longrightarrow d(a_m, a_n) < \varepsilon. \end{aligned}$$

So, $(a_n)_{n\in\mathbb{N}}$ is Cauchy sequence.

Remark 3 *We have seen above that every convergent sequence is a Cauchy sequence.*

Theorem 2 [34] *Every Cauchy sequence (in $\mathbb{R}$) is convergent.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. We define a set S as follow.

$$S = \{x \in \mathbb{R} : \exists k_0 \in \mathbb{N} \text{ such that } x < a_n, \forall n \geq k_0\}.$$

The proof involves two steps :
Step 1: $S \neq \phi$ and S is bounded above.
Since $(a_n)_{n\in\mathbb{N}}$ is bounded, there exists some $M > 0$ such that :

$$-M < a_n < M \ , \ \forall n \in \mathbb{N}.$$

Hence, $-M \in S$. This shows that $S \neq \phi$.
Let $x \in S$ be arbitrary. There exists some $n_0 \in \mathbb{N}$ such that

$$x < a_n, \forall n \geq n_0.$$

But

$$a_n < M, \forall n \in \mathbb{N}.$$

This shows that $x < M$. Since $x \in S$ is arbitrary, it follows that S is bounded above.The completeness property of $\mathbb{R}$ now implies that the supremum of S exists in $\mathbb{R}$.

Step 2: $\sup S$ is the limit of $(a_n)_{n\in\mathbb{N}}$,

Let $u = \sup S$, $\varepsilon > 0$ be given. Since the sequence is Cauchy, there exists $n_0 \in N$ such that :

$$d(a_m, a_n) < \frac{\varepsilon}{2}, \forall m, n \geq n_0,$$

In particular :

$$\begin{aligned} d(a_m, a_n) &< \frac{\varepsilon}{2}, \forall n \geq n_0 \\ &\Longrightarrow a_{n_0} - \frac{\varepsilon}{2} < a_n < a_{n_0} + \frac{\varepsilon}{2}, \forall n \geq n_0 \\ &\Longrightarrow a_{n_0} - \frac{\varepsilon}{2} \in S \text{ and } a_{n_0} + \frac{\varepsilon}{2} \notin S \\ a_{n_0} - \frac{\varepsilon}{2} &\leq u \leq a_{n_0} + \frac{\varepsilon}{2} \ ; \ \lceil u = \sup S \rceil \\ d(a_{n_0}, u) &\leq \frac{\varepsilon}{2} \ . \end{aligned}$$

Thus, for all $n \geq n_0$ we have :

$$\begin{aligned} d(a_n, u) &= |a_n - a_{n_0} + a_{n_0} - u| \\ &\leq |a_n - a_{n_0}| + |u - a_{n_0}| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Consequently $(a_n)_{n\in\mathbb{N}}$ converges to u . This complete the proof. ■

Example 6 *The sequence* $\left(\frac{(-1)^n}{2^n}\right)_{n\in\mathbb{N}}$ *is a Cauchy sequence .*

The inequality " $2^{m-1} > \frac{1}{\varepsilon}$" is true for all $m \in \mathbb{N}$, if $\varepsilon \geq 1$. For $0 < \varepsilon < 1$, Now we enter the logarithm:

$$\begin{aligned} \ln\left(2^{m-1}\right) &> \ln\left(\frac{1}{\varepsilon}\right) \\ (m-1)\ln(2) &> \ln\left(\frac{1}{\varepsilon}\right) \\ m &> 1 + \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln(2)} . \end{aligned}$$

So let us take : $n_0 = \max\left\{1, 1 + \frac{\ln\left(\frac{1}{\varepsilon}\right)}{\ln(2)}\right\}$,Now we have $n, m \geq n_0$,This implies $d(a_m, a_n) < \varepsilon$.

This proves that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

The sequence in Example has been proved Cauchy, hence convergent. (See Theorem 2)

Corollary 3 • *If the sequences* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *are convergent then* $(a_n + b_n)_{n\in\mathbb{N}}$ *is convergent ,furtheir :*

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

• If for all $n \in \mathbb{N}$, $a_n \neq 0$, and $(a_n)_{n\in\mathbb{N}}$ converges to $L \neq 0$, Then :

$$\left(\frac{1}{a_n}\right)_{n\in\mathbb{N}} \text{ converges to } \frac{1}{L}.$$

Preposition 1 *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence in a metric space* X *and suppose that there exists an* $\varepsilon > 0$ *such that :*

$$d(a_m, a_n) \geq \varepsilon \textit{ for every } m \neq n.$$

Then, the sequence $(a_n)_{n\in\mathbb{N}}$ does not converge.

Proof. Suppose by contradiction that $a_n \to a$,Then, there exists $n_0 \in \mathbb{N}$, such that for every $n > n_0$, we have

$$d(a_n, a) < \frac{\varepsilon}{2}.$$

By the triangle inequality,

$$d(a_{n_0+1}, a_{n_0+2}) \leq d(a_{n_0+1}, a) + d(a, a_{n_0+2}) < \varepsilon,$$

which is a contradiction. ■

1.3 Complete Metric Space

A complete metric space is a fundamental concept in the study of metric spaces in mathematics, particularly in analysis and topology.

Definition 4 *[34] Let (X, d) be a metric space. We say that X is complete metric space if every Cauchy sequence in X converges to a point in X.*

- The spaces $\mathbb{R}^n, \mathbb{C}^n$ with

$$d(x, y) = \sqrt[2]{\sum_{k=1}^{n} |x_k - y_k|^2}.$$

are completes .

- The space l_p with the metric

$$d(x, y) = d(\{x_n\}, \{y_n\}) = \sqrt[p]{\sum_{n=1}^{\infty} |x_n - y_n|^p}.$$

is complete .

- The space l_∞ with the metric

$$d(x, y) = d(\{x_n\}, \{y_n\}) = \sup_n |x_n - y_n| .$$

is complete.

- The space $C([a, b])$ with the metric

$$d_\infty(f, g) = \sup_{x\in[a,b]} |f(x) - g(x)| .$$

is complete .

- The space $\mathbb{Q}$ with $d(x, y) = |x - y|$, is not complete, For example the sequence

$$\begin{aligned} x_1 &= 1 \\ x_{n+1} &= \frac{x_n^2 + 2}{2x_n}. \end{aligned}$$

is in $\mathbb{Q}$, but

$$\lim_{n\to\infty} x_n = \sqrt{2} \notin \mathbb{Q}.$$

- The space $C([-1,1])$,with the metric

$$d_1(x,y) = \int_{-1}^{1} |f(x) - g(x)|\, dx.$$

is not complete.

Take the example

$$f_n(x) = \begin{cases} 1 \ \text{if} \ -1 \leq x \leq 0 \\ 1 - nx \ \text{if} \ 0 \leq x \leq \frac{1}{n} \\ 0 \ \text{if} \ \frac{1}{n} \leq x \leq 1 \end{cases} .$$

Then $\{f_n\}$ is Cauchy but it converges $\chi_{[-1,0]} \notin C([-1,1])$.

1.4 Normed space

A normed space is a vector space equipped with a function called a norm that assigns a non-negative length or size to each vector in the space. Normed spaces are essential in functional analysis and provide a framework for discussing concepts such as distance, convergence, and continuity

Definition 5 [20] *(**Normed Space**)*

A vector space V over field A is called a normed vector space (or normed space) if there is a real-valued function $\|.\|$ on V , called the norm , such that for any $x, y \in V$ and any $\alpha \in A$,

(i) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$;

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$, $\forall \ \alpha \in A$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$, $\forall \ x, y \in V$.

A norm $\|.\|$ of V defines a metric d on V via $d(x,y) = \|x - y\|$, All concepts from metric and topological spaces are applicable to normed spaces.

1.4.1 Banach space

The concept of Banach spaces is crucial in functional analysis and has profound implications in various areas of mathematics and applied sciences.

Definition 6 [30] *(**Banach space**)*

A complete normed linear space is called a Banach space. Equivalently a Banach space is a normed linear space which is complete with respect to a metric induced by a norm.

Theorem 4 *If X be a Banach Space and E be a closed subspace of X then E is a Banach space.*

Proof. If E is a closed subspace of X then E becomes a closed set of a complete metric space X, the metric being induced from the norm on X. And we know that every closed subspace of a complete metric space is a complete metric space (metric subspace) it follows that E is a Banach space (as a subspace of X). ∎

Preposition 2 *finite-dimensional k-vector spaces are Banach spaces .*

Example 7 *Let X a set.$B(X, \mathbb{R})$ is the vector space of bounded functions of X in $\mathbb{R}$, We provide $B(X, \mathbb{R})$ Identified by $\|.\|$:*

$$\forall f \in B(X, \mathbb{R}) \ , \quad \|f\| = \sup_{x \in X} |f(x)| \, .$$

• Prove that $(B(X, \mathbb{R}), \|.\|)$ is a Banach space ?

Let (f_n) be a Cauchy sequence of $B(X, \mathbb{R})$,Let us fix $x \in X$ .For all $p, q \in \mathbb{N}$, we have :

$$|f_p(x) - f_q(x)| \leq \|f_p - f_q\|$$

The sequence (f_n) is therefore Cauchy in $\mathbb{R}$, and as $\mathbb{R}$ is complete,the sequence (f_n) converges.and the limit of this sequence is $f(x)$,

We also know the application f in X is bounded .

(f_n) a Cauchy so it is bounded by a constant $M > 0$. But then, for all $x \in X$,we have :

$$|f_n(x)| \leq \|f_n\| \leq M.$$

By passing to the limit, we have :

$$|f(x)| \leq M.$$

and this proves that $f \in B(X, \mathbb{R})$ with $\|f\| \leq M$, It remains to prove that (f_n) converges to f in $B(X, \mathbb{R})$,

$$\forall \varepsilon > 0, \exists N > 0 \ , \forall \ p, q > N \implies \|f_p - f_q\| \leq \varepsilon.$$

We fix x in X. We have:

$$|f_p(x) - f_q(x)| \leq \varepsilon.$$

Since this inequality is true for all x in X, we get:

$$\|f - f_q\| \leq \varepsilon.$$

This is true for all $q \geq N$,and therefore (f_q) converges to f.

Finally, any sequence of Cauchy (f_n) of $B(X, \mathbb{R})$ converges, therefore $B(X, \mathbb{R})$ is a complete space.

Example 8 *For $1 \leq p < \infty$, the sequence space $l^p(\mathbb{N})$ consists of all infinite sequences $x = (x_n)_{n=1}^{\infty}$ such that :*

$$\sum_{n=1}^{\infty} |x_n| < \infty.$$

with the p-norm :

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n| \right)^{\frac{1}{p}}.$$

For $p = \infty$, the sequence space $l^{\infty}(\mathbb{N})$ consists of all bounded sequences, with :

$$\|x\|_{\infty} = \sup \{|x_n| \ , n = 1, 2, ...\}.$$

Then $l^p(\mathbb{N})$ is Banach space for $1 \leq p \leq \infty$.

In conclusion, Chapter One has provided a thorough examination of the fundamental concepts of fixed point theory. By defining and exploring metric spaces, convergence of sequences, and complete metric spaces, we have laid a robust foundation for understanding the core principles of fixed point theory. These basic yet crucial concepts set the stage for the more advanced theorems and applications discussed in the subsequent chapters. Understanding these foundational elements is essential for grasping the broader implications and applications of fixed point theory in various mathematical and practical contexts.

CHAPTER 2

Key Theorems in metric spaces

2.1 Introduction

Building on the foundational concepts introduced in Chapter One, this chapter focuses on some of the most significant theorems in fixed point theory. We will explore Banach's Fixed Point Theorem, Kannan's Fixed Point Theorem, and Chatterjea's Fixed Point Theorem, among others. Each theorem will be discussed in detail, highlighting both its theoretical framework and practical implications. Through real-world examples

The theory of fixed point dynamically develops in the period of the recent decades. The first important result in the theory of fixed point about contractive mapping is Banach theorem (principle of contractive mapping). Exactly this theorem is very important researching instrument in many different fields of mathematics.The above theorem was presented **1922, in S. Banach** dissertation (see [30]). By applying the stated theorem is proven that an integral equality might be solved.

Further, **R. Kannan 1968** (see [23]) has proven that,if (X, d) is a complete metric space and $S : X \longrightarrow X$,is mapping such that, it exists $\alpha \in \left[0, \frac{1}{2}\right)$ so that for all $x, y \in X$ the inequality :

$$d(Sx, Sy) \leq \alpha(d(x, Sx) + d(y, Sy)).$$

is satisfied, then there is a unique fixed point of S .

Several years later **1972, S. K. Chatterjea** (see [12]) has proven that if (X, d) is a complete metric space and $S : X \longrightarrow X$,is mapping such that, it exists $\alpha \in \left[0, \frac{1}{2}\right)$ so that for all $x, y \in X$ the inequality :

$$d(Sx, Sy) \leq \alpha(d(x, Sy) + d(y, Sx)).$$

is satisfied, then there is a unique fixed point of S .

2.2 Banach's fixed point theorem

Banach's Fixed Point Theorem, also known as The Contraction Theorem, concerns certain mappings (so-called contractions) of a complete metric space into itself. It states conditions sufficient for the existence and uniqueness of a fixed point, which we will see is a point that is mapped to itself. The theorem also gives an iterative process by which we can obtain approximations to the fixed point along with error bounds.

2.2.1 Lipschitz application

A Lipschitz application (or Lipschitz function) is a function between two metric spaces that satisfies a specific condition regarding the distances between points in its domain and their corresponding images in the codomain. This concept is fundamental in analysis and various applied mathematics fields due to its role in ensuring controlled behavior of functions, especially in contexts involving continuity and approximation.

Definition 7 *[30]* ***(Lipschitz application)*** *We say that T is Lipschitzian if there exists a real number $\alpha \geq 0$ such that:*

$$d(Tx, Ty) \leq \alpha d(x, y), \text{ for all } x, y \in X.$$

The smallest constant α which verifies the above inequality is called the Lipschitz constant.

- If $\alpha \leq 1$, the application T is called non-expansive.
- If $\alpha < 1$, the application T is called contraction.

Preposition 3

$$T \textit{ Lipschitzian} \Longrightarrow T \textit{ uniformly continuous} \Longrightarrow T \textit{ continue.}$$

and the converse is false.

2.2.2 Contraction

Definition 8 ***(Contraction)*** *Let (X, d) be a metric space. Then a map $T : X \to X$, is called a contraction on X if there a positive real number $\alpha < 1$,such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq \alpha d(x, y).$$

Example 9 *Let a $T : \mathbb{R} \to \mathbb{R}$ be a mapping defined by :*

$$T(x) = \frac{1}{4}x.$$

for all $x \in \mathbb{R}$ is a contraction for all $x, y \in \mathbb{R}$, we find:

$$|Tx - Ty| \leq \frac{1}{4}|x - y|.$$

Definition 9 ***(Fixed Point)*** *Let X be a non empty set and $T : X \to X$, be a mapping. A point $x \in X$ is said to be a fixed point of T if $x = T(x)$.*

Example 10 *We have the same example*

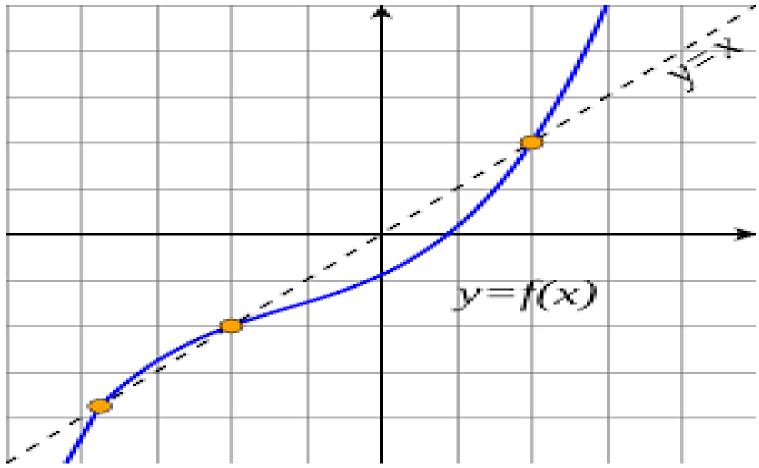
$$T(x) = \frac{1}{6}x.$$

Then $x = 0$ .is the unique fixed point of T.

Example 11 *Let a* $S : \mathbb{R} \to \mathbb{R}$,*be a mapping defined by :*

$$S(x) = x + 12 \text{ ,for every } x \in \mathbb{R} \text{ .}$$

Then T has no fixed point .



Geometric form of fixed point

2.2.3 Banach fixed-point

From above examples we see that in a metric space the fixed points self mapping T may be unique ,finite , infinitely many , or it may have no fixed point.

Basic problems of fixed point theory are :

1)Existence

2)Uniqueness

In 1922, S. Banach gave his cellebrated fixed point theorem which attracted many researches and lots of research paper and books are written in this area .

Theorem 5 *[4] **(Banach fixed-point)** Let (X, d) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T has a unique fixed point.*

Proof. We will prove Banach's theorem in the following steps,

- Construct an iterative sequence :

$$x_{n+1} = Tx_n \text{ fore very } n = 0, 1, 2, ...$$

-

$$d(x_{n+1}, x_n) \leq \alpha^n d(x_n, x_{n-1}).$$

- $\{x_n\}$ is cauchy sequence .
- Fixed point of T.
- Uniquness of fixed point .
  - Let $x \in X$ be fixed,we define a Picard iterative sequence $\{x_n\}$ by

$$\begin{aligned} x_0 = x \ , \ x_1 = T(x_0), \\ x_2 = T(x_1) = T(T(x_0)) = T^2(x_0), \\ x_3 = T(x_2) = T(T^2(x_0)) = T^3(x_0), \\ . \\ . \\ . \\ x_n = T(x_{n-1}) = T(T^{n-1}(x_0)) = T^n(x_0). \end{aligned}$$

- Consider :

$$\begin{aligned} d(x_{m+1}, x_m) &= d(Tx_m, Tx_{m-1}) \\ &\leq \alpha d(x_m, x_{m-1}) \\ &= \alpha d(Tx_{m-1}, Tx_{m-2}) \\ &\leq \alpha^2 d(x_{m-1}, x_{m-2}) \\ &. \\ &. \\ &. \\ &\leq \alpha^m d(x_{m-(m-1)}, x_{m-m}) \\ &= \alpha^m d(x_1, x_0) \\ d(x_{m+1}, x_m) &\Longrightarrow \alpha^m d(x_1, x_0). \end{aligned}$$

- Let $m < n$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + ... + d(x_{n-1}, x_n) \\ &\leq \alpha^m d(x_1, x_0) + \alpha^{m+1} d(x_1, x_0) + ... + \alpha^{n-1} d(x_1, x_0) \\ &= (\alpha^m + \alpha^{m+1} + ... + \alpha^{n-1}) d(x_1, x_0) \end{aligned}$$

is a finite geametric series, we put :

$$\begin{aligned} a_1 = \alpha^m \ , \qquad r = \alpha, \\ a_N = \alpha^{n-1}, \\ a_N = a_1 r^{N-1}. \end{aligned}$$

$$\begin{aligned} \alpha^{n-1} = \alpha^m \alpha^{N-1} = \alpha^{m+N-1}, \\ n - 1 = m + N - 1, \\ N = n - m. \end{aligned}$$

Since $|r| = |\alpha| < 1$, So :

$$S_N = \frac{\alpha^m (1 - \alpha^{n-m})}{(1 - \alpha)}.$$

Therefore

$$\begin{aligned} d(x_m, x_n) &\leq \frac{\alpha^m (1 - \alpha^{n-m})}{(1 - \alpha)} d(x_0, x_1) \\ &\leq \frac{\alpha^m}{(1 - \alpha)} d(x_0, x_1) \ ,\text{(since } (1 - \alpha^{n-m}) \leq 1) \\ \lim_{m \to \infty} d(x_m, x_n) &\leq \lim_{m \to \infty} \frac{\alpha^m}{(1 - \alpha)} d(x_0, x_1) = 0 \ . \\ \text{" because } \lim_{m \to \infty} \alpha^m &= 0 \ , \text{For } 0 < \alpha < 1." \end{aligned}$$

Hence (x_m) is a Cauchy sequence, since X is complete,(x_n) converges, Say $x_n \longrightarrow x$.

- We prove that x is a fixed point of the mapping T.

By continuity of T ,

$$\begin{aligned} x_n &= T(x_{n-1}), \\ \lim_{n\to\infty} x^n &= \lim_{n\to\infty} T(x_{n-1}) = T(\lim_{n\to\infty} x_{n-1}) \\ x &= T(x). \end{aligned}$$

This shows that x is the fixed point of T.

- For uniqueness,Let y be another fixed point of T that is

$$y = Ty.$$

Let $x \neq y$, we have :

$$\begin{aligned} d(x,y) &= d(Tx,Ty), \\ &\leq \alpha d(x,y) < d(x,y) \ . \end{aligned}$$

Which is a contradiction therefore, we have $y = x$ and the fixed point is unique. ■

Corollary 6 *Let (X,d) be a complete metric space and let $T : X \longrightarrow X$ be a map such that one of these iterated T^2 is strictly contracting, i.e. there exists $0 < \beta < 1$,such as*

$$d(T^2(x),T^2(y)) \leq \beta d(x,y) \ . \ \textit{for all} \quad x,y \in X$$

Show that f has a unique fixed point.

So

$$g(y) = T^2(y) = y \ .$$

So y is also a fixed point for g, and $y = x$ .It remains to show that T have a fixed point.

We have :

$$\begin{aligned} T^2(x) &= x \\ &\Longrightarrow T(T^1(x)) = T(x) \\ &\Longrightarrow T^1(T(x)) = T(x) \\ &\Longrightarrow g(T(x)) = T(x). \end{aligned}$$

We have just proven that $T(x)$ is a fixed point of g. As g has a unique fixed point x then $T(x) = x$, So x is indeed a fixed point for T.

2.3 Kannan's fixed point theorem

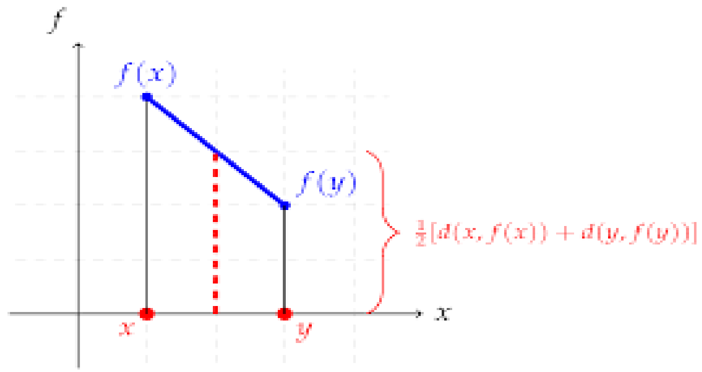
It is almost a century where several mathematicians have improved, extended and enriched the classical Banach contraction principle in different directions along with variety of applications. It is well-known that every Banach contractive mapping is a continuous function. In this sequel, it was a natural question does there exist any contractive map accompanied with fixed point which is not necessarily continuous .

In 1968, R. Kannan was the first mathematician who found the answer and presented the following fixed point result (see [16], [23]).

2.3.1 Kannan-type mapping

Definition 10 *[23] **(Kannan-type mapping)** A mapping $S : X \longrightarrow X$,where (X, d) is a metric space is called a kannan-type mapping if there exists $0 \leq k < \frac{1}{2}$ prove that*

$$d(Sx, Sy) \leq k \{(d(x, Sx) + d(y, Sy)\} \text{, for all } x, y \in X.$$



The following examples confirm that Kannan mappings need not be continuous.

Example 12 *Consider $S : [0, 1] \longrightarrow [0, 1]$ defined by:*

$$S(x) = \begin{cases} 1 - x & x \in \left[0, \frac{1}{3}\right) \\ \frac{x+1}{3} & x \in \left(\frac{1}{3}, 1\right] \end{cases},$$

$$\text{for any } x, y \in [0, 1] \text{ , } d(x, y) = |x - y| .$$

If $x, y \in \left[0, \frac{1}{3}\right)$, Then

$$\begin{aligned} d(Sx, Sy) &= |Sx - Sy| = |x - y| . \\ \text{and} \qquad d(x, Sx) &= |x - Sx| = |2x - 1| . \\ \frac{1}{2} \{(d(x, Sx) + d(y, Sy)\} &= \frac{1}{2} \{|2x - 1| + |2y - 1|\} \\ &= \left|x - \frac{1}{2}\right| + \left|y - \frac{1}{2}\right| \\ &\geq |x - y| = |(x - 1) - (y - 1)| \\ &= d(Sy, Sx). \end{aligned}$$

Similarly, we can prove this inequality for other x, y.

Thus S is a Kannan mapping and this mapping is not non-expansive because it is not continuous at $x = \frac{1}{3}$.

Example 13 *Define* $S : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *by*

$$\begin{aligned} S(x) &= x + 1. \\ \textit{and} \quad d(x,y) &= |x-y| \ , \textit{for any } x, y \in \mathbb{R}^+. \end{aligned}$$

Clearly

$$d(Sx, Sy) = d(x, y) \text{ , for any } x, y \in \mathbb{R}^+$$

i.e. S is non-expansive. But S is not Kannan: Take

$$\begin{aligned} x, y &\in \mathbb{R}^+ \text{ with } |x-y| > 1. \\ d(x, Sx) &= 1, \text{ for any } x \in \mathbb{R}^+ \\ \frac{d(x,Sx) + d(y,Sy)}{2} &= 1 \\ &< |x-y| \\ &= |(1+x) - (1-y)| \\ &= d(Sx, Sy). \end{aligned}$$

Therefore ,

$$d(Sx, Sy) > \frac{d(x,Sx) + d(y,Sy)}{2}, \text{ for any } x, y \in \mathbb{R}^+ \text{ with } |x-y| > 1.$$

Theorem 7 ***(R.Kannan fixed-point)*** *Let* (X, d) *be a complete metric space and* $S : X \longrightarrow X$ *,be a kannan-type mapping .Then* S *has a unique fixed point.*

Proof. We will prove the theorem in the following steps,

- Iteration of a sequence using . ■

- Applying the contraction mapping . (Given)
- To make the iterated sequence to be a cauchy sequence .
- Fixed point of S .
- Uniqueness .
  - Pick an element $x_0 \in X$ arbitrary , Now we construct a sequence $\{x_n\}$ as follows

:

$$\begin{aligned} x_1 &= Sx_0. \\ x_2 &= Sx_1. \\ &. \\ &. \\ &. \\ x_n &= Sx_{n-1}. \\ x_{n+1} &= Sx_n. \end{aligned}$$

is called iterative sequence.

- We have :

$$\begin{aligned}
d(x_1,x_2) &= d(Sx_0,Sx_1)\\
&\leq k\left\{d(x_0,Sx_0)+d(x_1,Sx_1)\right\}\\
&\leq k\left\{d(x_0,x_1)+d(x_1,x_2)\right\}\\
&\leq kd(x_0,x_1)+kd(x_1,x_2)\\
d(x_1,x_2)-kd(x_1,x_2) &\leq kd(x_0,x_1)\\
(1-k)d(x_1,x_2) &\leq kd(x_0,x_1)\\
d(x_1,x_2) &\leq (\frac{k}{1-k})d(x_0,x_1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
d(x_2,x_3) &= d(Sx_1,Sx_2)\\
&\leq k\left\{d(x_1,Sx_1)+d(x_2,Sx_2)\right\}\\
&\leq k\left\{d(x_1,x_2)+d(x_2,x_3)\right\}\\
d(x_2,x_3)-kd(x_2,x_3) &\leq kd(x_1,x_2)\\
(1-k)d(x_2,x_3) &\leq kd(x_1,x_2)\\
d(x_2,x_3) &\leq (\frac{k}{1-k})\left\{d(x_1,x_2)\right\}\\
d(x_2,x_3) &\leq (\frac{k}{1-k})\left\{(\frac{k}{1-k})d(x_0,x_1)\right\}\\
&\leq (\frac{k}{1-k})^2 d(x_0,x_1)\ ,
\end{aligned}$$

Up to so on

$$\begin{aligned}
d(x_n,x_{n+1}) &= d(Sx_{n-1},Sx_n)\\
&\leq k\left\{d(x_{n-1},Sx_{n-1})+d(x_n,Sx_n)\right\}\\
&\quad .\\
&\quad .\\
&\quad .\\
&\leq (\frac{k}{1-k})^n d(x_0,x_1)\ ,\\
&\leq (\beta)^n d(x_0,x_1)\ .
\end{aligned}$$

$$\begin{aligned}
\beta &= \frac{k}{1-k}<1\\
k &< 1-k\ ,\\
k &< \frac{1}{2}\in\left[0,\frac{1}{2}\right[\ .
\end{aligned}$$

- For any positive integer $m > n$

$$\begin{aligned}
d(x_n, x_m) &= d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\
&\leq (\beta)^n d(x_0, x_1) + (\beta)^{n+1} d(x_0, x_1) + ... + (\beta)^{m-1} d(x_0, x_1) \\
&\leq \left((\beta)^n + (\beta)^{n+1} + ... + (\beta)^{m-1}\right) d(x_0, x_1) \quad , \\
&\leq (\beta)^n \left(1 + (\beta)^1 + (\beta)^2 + (\beta)^3 + ...\right) d(x_0, x_1) \quad , \\
&\leq \left(\frac{k^n}{1-k}\right) d(x_0, x_1) \ , \longrightarrow 0 \text{ as, } n \longrightarrow \infty \ .
\end{aligned}$$

becouse $(1 + (\beta)^1 + (\beta)^2 + (\beta)^3 + ...)$ is an infinite geometric series.
This implies that $\{x_n\}$ is a Cauchy sequence.

- Since X is complete ,$\{x_n\}$ converges to say x.

Now we will have to prove that x is a fixed point of the S ,i.e $Sx = x$.
Now

$$\begin{aligned}
d(x, Sx) &= d(x, x_{n+1}) + d(x_{n+1}, Sx) \\
&\leq d(x, x_{n+1}) + d(Sx_n, Sx) \\
&\leq d(x, x_{n+1}) + c\left[d(x_n, Sx_n) + d(x, Sx)\right] \\
&\leq d(x, x_{n+1}) + c\left[d(x_n, x_{n+1}) + d(x, Sx)\right] \\
&\leq d(x, x_{n+1}) + cd(x_n, x_{n+1}) + cd(x, Sx) \\
d(x, Sx) - cd(x, Sx) &\leq d(x, x_{n+1}) + cd(x_n, x_{n+1}) \\
(1-c)d(x, Sx) &\leq d(x, x_{n+1}) + cd(x_n, x_{n+1}) \\
d(x, Sx) &\leq \frac{1}{(1-c)}\left[d(x, x_{n+1}) + cd(x_n, x_{n+1})\right] \\
&\longrightarrow 0 \text{ ,as } n \longrightarrow \infty \ .
\end{aligned}$$

x is a fixed point of S .

- Let $y \neq x$ be an another element of X , prove that $Sy = y$.

Now

$$\begin{aligned}
d(x, Sx) &= d(Sx, Sy) \\
&\leq c\left[d(x, Sx) + d(y, Sy)\right] \\
&\leq c0 + c0.
\end{aligned}$$

by definition of usual metric, Gives us $x = y$. So, the fixed point is unique .

Example 14 *Take $E = [0,1]$ endowed with the usual metric. Define*

$$\begin{aligned}
S &: E \longrightarrow E \ , \textit{ as follow} \\
S(x) &= \left\{ \begin{array}{lll} \frac{x}{3} & \textit{if} & 0 \leq x < 1 \ , \\ \frac{1}{6} & \textit{if} & x = 1 \ , \end{array} \right\}
\end{aligned}$$

The above theorem is applicable and also we have $x = 0$ a unique fixed point ,It is observed that S is not continuous on E .

2.4 Chatterjea's Fixed point theorem

Fixed point theorem due to Chatterjea which was established in the year 1972 and which is actually a sort of dual of the Kannan fixed point theorem .

2.4.1 C-contraction

Definition 11 *([12] [30])**(C-contraction)** A mapping $T : X \longrightarrow X$ where (X, d) is a metric space,is said to bea Chatterjea type contraction and we symbolize it a C-contraction.If there exists $0 \le c < \frac{1}{2}$ such that*

$$d(Tx, Ty) \le c\,\{(d(x, Ty) + d(y, Tx)\} \ \ , \forall \ x, y \in X \ .$$

Theorem 8 *[12] **(Chatterjea Fixed point)** Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a C-contraction. Then T admits a unique fixed point.*

Proof. We will present the proof of the theorem in sequential steps.

- Iteration of a sequence using .
- Applying the contraction mapping . (Given)
- To make the iterated sequence to be a cauchy sequence .
- Fixed point of T .
- Uniqueness .

•Iteration of a sequence using,

Pick an element $x_0 \in X$ arbitrary. Now we construct sequence $\{x_n\}$ as follows :

$$\begin{aligned} x_1 &= Tx_0 \\ x_2 &= Tx_1 \\ &. \\ &. \\ &. \\ x_n &= Tx_{n-1} \\ x_{n+1} &= Tx_n \ . \end{aligned}$$

This sequance called iterative sequence.

$$\begin{aligned} d(x_1, x_2) &= d(Tx_0, Tx_1) \\ &\le c\,\{d(x_0, Tx_1) + d(x_1, Tx_0)\} \\ &\le c\,\{d(x_0, x_2) + d(x_1, x_1)\} \,. \end{aligned}$$

By using triangular inequality, we get

$$\begin{aligned} d(x_1, x_2) &\le c\,\{d(x_0, x_1) + d(x_1, x_2) + 0\} \\ d(x_1, x_2) - cd(x_1, x_2) &\le cd(x_1, x_2) \\ d(x_1, x_2) &\le \frac{c}{1 - c} d(x_0, x_1) \ . \end{aligned}$$

Similarly

$$\begin{aligned} d(x_2,x_3) &= d(Tx_1,Tx_2) \\ &\leq c\{d(x_1,Tx_2)+d(x_2,Tx_1)\} \\ &\leq c\{d(x_1,x_3)+d(x_2,x_2)\}\ . \end{aligned}$$

By using tringular inequality

$$\begin{aligned} d(x_2,x_3) &\leq c\{d(x_1,x_2)+d(x_2,x_3)+0\} \\ d(x_2,x_3)-cd(x_2,x_3) &\leq cd(x_1,x_2) \\ d(x_2,x_3) &\leq \frac{c}{1-c}d(x_1,x_2) \\ d(x_2,x_3) &\leq \frac{c}{1-c}\left\{\frac{c}{1-c}d(x_0,x_1)\right\} \\ d(x_2,x_3) &\leq \left(\frac{c}{1-c}\right)^2 d(x_0,x_1)\ . \end{aligned}$$

Up to so on

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\ &\leq c\{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})\} \\ &\leq c\{d(x_{n-1},x_{n+1})+d(x_n,x_n)\}\ . \end{aligned}$$

By using triangular inequality

$$\begin{aligned} d(x_n,x_{n+1}) &\leq c\{d(x_{n-1},x_n)+d(x_n,x_{n+1})+0\} \\ d(x_n,x_{n+1})-cd(x_n,x_{n+1}) &\leq cd(x_{n-1},x_n) \\ d(x_n,x_{n+1}) &\leq \frac{c}{1-c}d(x_{n-1},x_n) \\ d(x_n,x_{n+1}) &\leq \frac{c}{1-c}\left\{\frac{c}{1-c}d(x_{n-2},x_{n-1})\right\} \\ d(x_n,x_{n+1}) &\leq \left(\frac{c}{1-c}\right)^2\left\{\frac{c}{1-c}d(x_{n-3},x_{n-2})\right\} \\ &\ \ . \\ &\ \ . \\ &\ \ . \\ &\leq \left(\frac{c}{1-c}\right)^n (d(x_0,x_1)) \\ &\leq (\beta)^n (d(x_0,x_1))\ . \end{aligned}$$

So

$$\begin{aligned} \beta &= \frac{c}{1-c} < 1 \\ c &< 1-c \\ 2c &< 1 \\ c &< \frac{1}{2}\ . \end{aligned}$$

And from

$$0 \leq c < \frac{1}{2} .$$

•To make the iteratie sequence to be a Cauchy sequence, for any positive integer $m > n$

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\leq \beta^n d(x_0, x_1) + \beta^{n+1} d(x_0, x_1) + ... + \beta^{m-1} d(x_0, x_1) \\ &\leq \beta^n \left(1 + \beta^1 + \beta^2 + ...\right) d(x_0, x_1) . \end{aligned}$$

By $\{(1 + \beta^1 + \beta^2 + ...)$ is an infinite geometric series, *we* find

$$\leq \frac{\beta^n}{1-\beta} d(x_0, x_1) \quad .$$

$$\longrightarrow 0 \text{ ,as } n \to \infty$$

This implies that$\{x_n\}$ is Cauchy Sequence .

• Fixed point ,Since X is complete $\{x_n\}$converges to say x .

We will have to prove that x is a fixed point of the T ,i-e $\{Tx = x\}$

We use the following inequality

$$\begin{aligned} d(x, Tx) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx) \\ &\leq d(x_n, x_{n+1}) + d(Tx_n, Tx) \\ &\leq d(x_n, x_{n+1}) + c\left(d(x_n, Tx) + d(x, Tx_n)\right) \\ &\leq d(x_n, x_{n+1}) + c\left(d(x_n, Tx) + d(x, x_{n+1})\right) \\ &\leq d(x_n, x_{n+1}) + cd(x_n, Tx) + cd(x, x_{n+1}) \\ &\leq (c+1)d(x_n, x_{n+1}) + cd(x_n, Tx) . \\ &\longrightarrow 0 \text{ ,as } n \to \infty . \end{aligned}$$

Finally, x is a fixed point of T .

• Uniqueness, Let $y \neq x$ be an another element of X, Such that $Ty = y$ by inequality

$$\begin{aligned} d(x, y) &= d(Tx, Ty) \\ &\leq c\left(d(x, Ty) + d(y, Tx)\right) \\ &\leq c\left(d(x, y) + d(y, x)\right) \\ &\leq 2cd(x, y). \end{aligned}$$

This is a contradiction. That is why the fixed point is unique. ■

Example 15 *Take* $X = [0, 1]$ *equipped with usual metric* d *,Define*

$$S : X \longrightarrow X \ ,$$

as

$$S_z = \begin{cases} 0 \text{ if } 0 \leq z < 1 \ , \\ \frac{1}{12} \text{ if } z = 1 \ . \end{cases}$$

The conditions of the above theorem are satisfied and here $z = 0$ is the unique fixed point of S ,It is observed that S is not continuous on X .

Chapter Two has delved into the key theorems that form the backbone of fixed point theory, including Banach's Fixed Point Theorem, Kannan's Fixed Point Theorem, and Chatterjea's Fixed Point Theorem. We have not only explored the theoretical underpinnings of these theorems but also their practical applications across different fields. By examining real-world examples, we have illustrated the profound impact and utility of these theorems. This chapter has highlighted the importance of these fixed point theorems in solving complex problems and provided a comprehensive understanding of their applications, paving the way for their generalization in more complex spaces, which is discussed in the next chapter.

CHAPTER 3

Some Fixed Point Theorems in Rectangular Metric Spaces

3.1 Introduction

In the final chapter, we extend the principles of fixed point theory to rectangular metric spaces. This generalization broadens the scope and applicability of the theory, allowing us to tackle more complex and diverse mathematical problems. We will examine how traditional fixed point theorems can be adapted and applied within this novel framework, demonstrating the versatility and robustness of fixed point theory. This chapter aims to showcase the ongoing evolution of mathematical concepts and their relevance to contemporary challenges

This chapter, we will study some fixed point theorems on different spaces. We will discuss Banach contraction theorem in metric space along with two other fixed point theorems namely Kannan fixed point theorem and Chatterjea fixed point theorem. Moreover, we will observe these theorems in different spaces such as the b-metric space, the rectangular metric space see ([1], [3], [33], [17], [14], [16], [15]).

We explore the generalization of metric spaces, with an emphasis on extending Banach, Kannan, and Chatterjea's theorems to rectangular metric spaces.([22], [8], [2])

In the sequel, Branciari (2000) [[5]] introduced the concept of rectangular metric space (RMS) by replacing the sum on the right side of the trigonometric inequality in the definition of the metric space with a trinomial expression and proved the existence of an isomorphism of the Banach contraction principle in such a space. Metric spaces provide a framework for studying distance and continuity in mathematics. By extending these theories to rectangular scales, providing a more flexible way to measure distances. This chapter examines the theoretical foundations of rectangular measurement, with the aim of deepening our understanding and expanding the application of these basic concepts in mathematics.

3.2 Rectangular metric space

Definition 12 [5] ***(RMS)***

Let X be a nonempty set and the mapping $d : X \times X \longrightarrow [0, \infty)$ satisfies:

- $d(x, y) = 0$ *iff* $x = y$, *for all* $x, y \in X$,
- $d(x, y) = d(y, x)$, *for all* $x, y \in X$,
- $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$, *for all* $x, y \in X$ *and all distinct points* $u, v \in X \backslash \{x, y\}$.

*Then d is called a rectangular metric on X and (X, d) is called a rectangular or generalized metric spaces (in short **RMS**).*

To clarify the definition further, we will give some examples,

Example 16 *Let $A = \{\alpha, \beta, \gamma, \delta\}$ in $\mathbb{R}$,and define $\lambda : A^2 \longrightarrow [0, +\infty)$ as follows:*

$$\begin{aligned}
\lambda(\alpha, \beta) &= \lambda(\beta, \alpha) = 1.3 \ , \ \lambda(\beta, \gamma) = \lambda(\gamma, \beta) = 0.7 \ , \\
\lambda(\alpha, \gamma) &= \lambda(\gamma, \alpha) = 0.2 \ , \ \lambda(\beta, \delta) = \lambda(\delta, \beta) = 1.1 \ , \\
\lambda(\alpha, \delta) &= \lambda(\delta, \alpha) = 0.4 \ , \ \lambda(\gamma, \delta) = \lambda(\delta, \gamma) = 0.8 \ , \\
\lambda(\alpha, \alpha) &= \lambda(\beta, \beta) = \lambda(\gamma, \gamma) = \lambda(\delta, \delta) = 0,
\end{aligned}$$

Then, it is easy to show that (A, λ) is an RMS, but it is not a metric space. Indeed,

$$1.3 = \lambda(\alpha, \beta) > \lambda(\alpha, \gamma) + \lambda(\gamma, \beta) = 0.2 + 0.7 \ ,$$

Example 17 *Let $X = A \cup B$, where*

$$\begin{aligned}
A &= \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}\right\}, \\
\text{and } B &= [1,3] \ .
\end{aligned}$$

$$\begin{aligned}
d(\frac{1}{2}, \frac{2}{3}) &= d(\frac{3}{4}, \frac{4}{5}) = 0.2 \ , \\
d(\frac{1}{2}, \frac{4}{5}) &= d(\frac{2}{3}, \frac{3}{4}) = 0.3 \ , \\
d(\frac{1}{2}, \frac{3}{4}) &= d(\frac{2}{3}, \frac{4}{5}) = 0.6 \ .
\end{aligned}$$

and

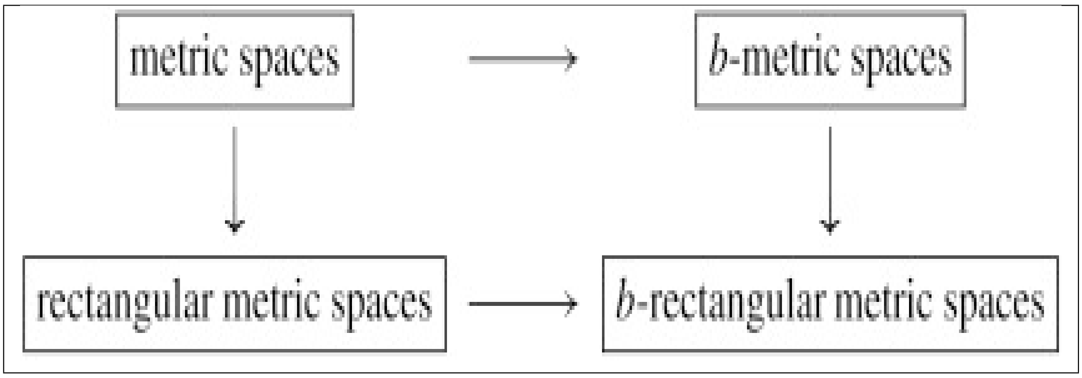
$$d(x, y) = |x - y|$$

It is easy to show that d does not satisfy the triangle inequality on d. Indeed,

$$0.6 = d(\frac{1}{2}, \frac{3}{4}) \geq d(\frac{1}{2}, \frac{2}{3}) + d(\frac{2}{3}, \frac{3}{4}) = 0.2 + 0.3 \ .$$

But the space is a rectangular metric that we need,

$$0.2 = d(\frac{1}{2}, \frac{2}{3}) \leq d(\frac{1}{2}, \frac{3}{4}) + d(\frac{3}{4}, \frac{4}{5}) + d(\frac{4}{5}, \frac{2}{3}) = 0.6 + 0.2 + 0.6 \ .$$



Definition 13 [5] *Let (X,d) be a rectangular metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then*

The sequence $\{x_n\}$ is said to be convergent in (X,d) and converges to x, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and this fact is represented by

$$\lim_{n\to\infty} x_n = x \text{ or } x_n \to x \quad as \quad n \to \infty.$$

The sequence $\{x_n\}$ is said to be Cauchy sequence in (X,d) if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0$, $p > 0$ or equivalently, if

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0, \text{ for all} \quad p > 0.$$

(X,d) is said to be a complete rectangular metric space if every Cauchy sequence in X converges to some $x \in X$.

Example 18 *Let $X = A \cup B$,where*

Si $n > 1$,

$$A = \left\{ \frac{1}{n}, n > 1 \right\},$$

and B is the set of all positive antegers .

Define

$$d(x,y) = \begin{cases} 0 & if \quad x = y \\ 2\alpha & if \quad x, y \in A \\ \frac{\alpha}{2n} & if \quad x \in A \text{ and } y \in \{2,3\} \\ \alpha & if \quad contrarily \end{cases},$$

where $\alpha > 0$ is a constant.

The space (X,d) is a rectangular metric space, as

$$d(\frac{1}{2}, \frac{1}{3}) = 2\alpha < \frac{13}{6}\alpha = d(\frac{1}{2}, 4) + d(4,3) + d(3, \frac{1}{3}).$$

However, we have the following

$$\begin{aligned} d(\frac{1}{2}, \frac{1}{n}) &= 2\alpha < \frac{\alpha(n+1)}{n} = d(\frac{1}{2}, 2) + d(2,3) + d(3, \frac{1}{n}). \\ 2\alpha &< \frac{\alpha(n+1)}{n} \Longrightarrow 2\alpha n < \alpha(n+1) . \end{aligned}$$

This is a contradiction and means that it is not a rectangular metric space.

Note : You cannot judge a metric space to be a rectangle from one example. Al cases must be verified. The example above demonstrates this.

Lemma 1 *[27] Let (X, d) be a metric space, and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to x, y, respectively. Then we have ,*

$$\begin{aligned} d(x, y) &\leq \liminf_{n\to\infty} d(x_n, y_n) \\ &\leq \limsup_{n\to\infty} d(x_n, y_n) \\ &\leq d(x, y). \end{aligned}$$

In particular, if $x = y$, then we have

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Moreover, for each $z \in X$, we have ,

$$\begin{aligned} d(x, z) &\leq \liminf_{n\to\infty} d(x_n, z) \\ &\leq \limsup_{n\to\infty} d(x_n, z) \\ &\leq d(x, z) \ . \end{aligned}$$

Lemma 2 *[27] Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X such that ,*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that for the following four sequences ,

$$\begin{aligned} &\bullet d(x_{m(k)}, y_{n(k)}) \\ &\bullet d(x_{m(k)}, y_{n(k)+1}) \\ &\bullet d(x_{m(k)+1}, y_{n(k)}) \\ &\bullet d(x_{m(k)+1}, y_{n(k)+1}). \end{aligned}$$

It holds:

$$\begin{aligned} \varepsilon &\leq \liminf_{n\to\infty} d(x_{m(k)}, y_{n(k)}) \leq \limsup_{n\to\infty} d(x_{m(k)}, y_{n(k)}) \leq \varepsilon \ , \\ \varepsilon &\leq \liminf_{n\to\infty} d(x_{m(k)}, y_{n(k)+1}) \leq \limsup_{n\to\infty} d(x_{m(k)}, y_{n(k)+1}) \leq \varepsilon \ , \\ \varepsilon &\leq \liminf_{n\to\infty} d(x_{m(k)+1}, y_{n(k)}) \leq \limsup_{n\to\infty} d(x_{m(k)+1}, y_{n(k)}) \leq \varepsilon \ , \\ \varepsilon &\leq \liminf_{n\to\infty} d(x_{m(k)+1}, y_{n(k)+1}) \leq \limsup_{n\to\infty} d(x_{m(k)+1}, y_{n(k)+1}) \leq \varepsilon \ . \end{aligned}$$

Lemma 3 *[27] Let (X, d) be a rectangular metric space and let $\{x_n\}$ be a Cauchy sequence in X such that $x_n \neq x_m$ whenever $n \neq m$. Then $\{x_n\}$ can converge to at most one point.*

Lemma 4 *[27] Let (X,d) be a rectangular metric space*

Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are prove that,

$$\begin{aligned} x_n &\longrightarrow x \text{ and } y_n \longrightarrow y \text{ as } n \to \infty \\ \text{with } x &\neq y \text{ and } x_n \neq x \text{ , } y_n \neq y \text{ for } n \in \mathbb{N} \text{ .} \end{aligned}$$

Then we have

$$\begin{aligned} d(x,y) &\leq \liminf_{n\to\infty} d(x_n,y_n) \\ &\leq \limsup_{n\to\infty} d(x_n,y_n) \\ &\leq d(x,y) \text{ .} \end{aligned}$$

If $y \in X$ and $\{x_n\}$ is a Cauchy sequence in X with $x_n \neq x_m$ for infinitely many $m,n \in \mathbb{N}$, $n \neq m$, converging to $x \neq y$, then

$$\begin{aligned} d(x,y) &\leq \liminf_{n\to\infty} d(x_n,y) \\ &\leq \limsup_{n\to\infty} d(x_n,y) \\ &\leq d(x,y) \text{ for all } x \in X. \end{aligned}$$

3.3 Extending Theorems in RMS

The following theorem corresponds to the Banach Contraction Principle but in the domain of a rectangular metric space.

Theorem 9 *Let (X,d) be a complete rectangular metric space and $T : X \to X$ be a mapping satisfying:*

$$d(Tx,Ty) \leq \alpha d(x,y). \tag{3.1}$$

for all x,y $\in X$, where $\alpha \in [0,1)$,Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$.

We shall show that$\{x_n\}$ is Cauchy sequence.If $x_n = x_{n+1}$ then x_n is fixed point of T.

So, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$.

Put $d(x_n,x_{n+1}) = d_n$, it follows from (3.1)

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \\ &\leq \alpha d(x_{n-1},x_n) \\ d_n &\leq \alpha d_{n-1}. \end{aligned}$$

This process gets,

$$d_n \leq \alpha^n d_0. \tag{3.2}$$

If $x_0 = x_n$, then, by (3.2), for any $n \geq 2$, we find

$$\begin{aligned} d(x_0,Tx_0) &= d(x_n,Tx_n), \\ d(x_0,x_1) &= d(x_n,Tx_{n+1}), \\ d_0 &= d_n, \\ d_0 &\leq \alpha^n d_0. \end{aligned}$$

Contradiction. Therefore, we must have $d_0 = 0$, i.e., $x_0 = x_1$, and so x_0 is a fixed point of T.

Thus, we assume that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$, again

$$d(x_n, x_{n+2}) = d_n^* \ .$$

and using (3.1) for any $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \\ &\leq \alpha d(x_{n-1}, x_{n+1}) \\ d_n^* &\leq \alpha d_{n-1}^* \ . \end{aligned}$$

We conclude by repeating

$$d(x_n, x_{n+2}) \leq \alpha d_0^* \ . \tag{3.3}$$

We have two cases. If p is odd $(p = 2m+1)$ and using (3.3) we obtain

$$\begin{aligned} d(x_n, x_{n+2m+1}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1}) \\ &\leq d_n + d_{n+1} + [d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1})] \\ &\leq d_n + d_{n+1} + d_{n+2} + d_{n+3} + ... + d_{n+2m} \\ &\leq \alpha^n d_0 + \alpha^{n+1} d_0 + \alpha^{n+2} d_0 + ... + \alpha^{n+2m} d_0 \\ &\leq \alpha^n \left(1 + \alpha^2 + \alpha^4 + ...\right) d_0 \\ &= \frac{1+\alpha}{1-\alpha^2} \alpha^n d_0 \ .(\text{note that } \alpha^2 < 1). \end{aligned}$$

Hence

$$d(x_n, x_{n+2m+1}) \leq \frac{1+\alpha}{1-\alpha^2} \alpha^n d_0. \tag{3.4}$$

If p is even $(p = 2m)$, then using (3.2) and (3.3) we get

$$\begin{aligned} d(x_n, x_{n+2m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m}) \ , \\ &\leq d_n + d_{n+1} + [d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m})] \\ &\leq d_n + d_{n+1} + d_{n+2} + d_{n+3} + [d(x_{n+4}, x_{n+5}) + d(x_{n+5}, x_{n+6}) + d(x_{n+6}, x_{n+2m})] \\ &\leq d_n + d_{n+1} + d_{n+2} + d_{n+3} + d_{n+4} + ... + [d_{2m-4} + d_{2m-3} + d(x_{n+2m-2}, x_{n+2m})] \\ &\leq \alpha^n \left[1 + \alpha^2 + \alpha^4 + ... + \alpha^{n+2m-2}\right] d_0^*. \end{aligned}$$

i,e.

$$\begin{aligned} d(x_n, x_{n+2m}) &\leq d(x_n, x_{n+2m}) \leq \frac{1+\alpha}{1-\alpha^2} \alpha^n d_0 + \alpha^{n+2m-2} d_0^* \\ &\leq \frac{1+\alpha}{1-\alpha^2} \alpha^n d_0 + \alpha^{n-2} \alpha^{2m} d_0^* \\ &\leq \frac{1+\alpha}{1-\alpha^2} \alpha^n d_0 + \alpha^{n-2} d_0^* \ . \end{aligned}$$

Then,

$$d(x_n, x_{n+2m}) \leq \frac{1+\alpha}{1-\alpha^2} \alpha^n d_0 + \alpha^{n-2} d_0^*. \tag{3.5}$$

It follows from (3.4) and (3.5)

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 \text{ For all } p > 0. \tag{3.6}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X,d) there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{3.7}$$

We shall show that u is a fixed point of T. Again, for any $n \in \mathbb{N}$ we have,

$$\begin{aligned} d(u,Tu) &\leq d(u,x_n) + d(x_n,x_{n+1}) + d(x_{n+1},Tu) \ , \\ &\leq d(u,x_n) + d_n + d(Tx_n,Tu) \ , \\ &\leq d(u,x_n) + d_n + \alpha d(x_n,u) \ , \end{aligned}$$

Using (3.5) and (3.6) it follows from above inequality that,

$$d(u,Tu) = 0,\text{i.e.},Tu = u.$$

Thus u is a fixed point of T.

For uniqueness, let v be another fixed point of T. Then it follows from (3.1) that

$$d(u,v) = d(Tu,Tv) \leq \alpha d(u,v) < d(u,v),$$

A contradiction. Therefore, we must have

$$d(u,v) = 0,\text{i.e.},u = v.$$

Thus fixed point is unique. ∎

The following theorem is analogous to the Kannan's theorem, but within the framework of a rectangular metric space.

Theorem 10 *Let (X,d) be a complete rectangular metric space,and $T : X \longrightarrow X$ be a mapping satisfying:*

$$d(Tx,Ty) \leq \alpha\left[d(x,Tx) + d(y,Ty)\right]. \tag{3.8}$$

for all $x,y \in X$,where $\alpha \in [0,\frac{1}{2})$,Then T has unique fixed point .

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$.

We shall show that $\{x_n\}$ is Cauchy sequence. If $x_{n+1} = x_n$ then x_n is fixed point of T. So, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$.

Setting $d(x_n,x_{n+1}) = d_n$, it follows from (3.8) that

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \ , \\ &\leq \alpha\left[d(x_{n-1},Tx_{n-1}) + d(x_n,Tx_n)\right] \ , \\ d_n &\leq \alpha\left[d_{n-1},d_n\right] \ , \\ d_n &\leq \frac{\alpha}{\alpha-1}d_{n-1}, \\ &= \delta d_{n-1} \ . \end{aligned}$$

where

$$\delta = \frac{\alpha}{\alpha - 1} < 1 \Rightarrow \alpha < \frac{1}{2} \ .$$

Repeating this process we obtain

$$d_n \leq \delta^n d_0 \ . \tag{3.9}$$

Indeed, if $x_0 = x_n$ then using (3.9), for any $n \geq 2$, we have

a contradiction. Therefore, we must have $d_0 = 0$, i.e. $x_0 = x_1$, and so x_0 is a fixed point of T.

Thus we assume that $x_n \neq x_m$,for all distinct $n, m \in \mathbb{N}$. Again using (3.8) and (3.9) for any $n \in \mathbb{N}$, we get

$$\begin{aligned}
d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}), \\
&\leq \alpha \left[d(x_{n-1}, Tx_{n-1}) + d(x_{n+1}, Tx_{n+1}) \right], \\
&= \alpha \left[d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2}) \right], \\
&= \alpha \left[d_{n-1} + d_{n+1} \right], \\
&\leq \alpha \left(\beta^{n-1} d_0 + \beta^{n+1} d_0 \right), \\
&\leq \alpha \beta^{n-1} \left(1 + \beta^2 \right) d_0, \\
&= \gamma \beta^{n-1} d_0.
\end{aligned}$$

So

$$d(x_n, x_{n+2}) = \gamma \beta^{n-1} d_0. \tag{3.10}$$

where

$$\gamma = \alpha \left(1 + \beta^2 \right) > 0.$$

For the sequence $\{x_n\}$ we consider $d(x_n, x_{n+p})$ in two cases. If p is odd say $2m + 1$ then using (3.9) we obtain

$$\begin{aligned}
d(x_n, x_{n+2m+1}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1}), \\
&\leq d_n + d_{n+1} + \left[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1}) \right], \\
&\leq d_n + d_{n+1} + d_{n+2} + d_{n+3} + \ldots + d_{n+2m} \ , \\
&\leq \beta^n d_0 + \beta^{n+1} d_0 + \beta^{n+2} d_0 + \ldots + \beta^{n+2m} d_0, \\
&\leq \beta^n \left[1 + \beta^2 + \beta^4 + \ldots \right] d_0, \\
&= \frac{1 + \beta}{1 - \beta^2} \beta^n d_0, \quad \text{So } \left(\beta^2 < 1 \right).
\end{aligned}$$

Therefore,

$$d(x_n, x_{n+2m+1}) \leq \frac{1 + \beta}{1 - \beta^2} \beta^n d_0. \tag{3.11}$$

If p is even say $2m$ then using (3.9) and (3.10) we obtain

$$\begin{aligned}
d(x_n, x_{n+2m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m}), \\
&\leq d_n + d_{n+1} + \left[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m}) \right], \\
&\leq d_n + d_{n+1} + d_{n+2} + d_{n+3} + \ldots + d_{2m-4} + d_{2m-3} + d(x_{n+2m-2}, x_{n+2m}), \\
&\leq \beta^n d_0 + \beta^{n+1} d_0 + \beta^{n+2} d_0 + \ldots + \beta^{2m-4} d_0 + \beta^{2m-3} d_0 + \gamma \beta^{n+2m-3} d_0, \\
&\leq \beta^n \left[1 + \beta^2 + \beta^4 + \ldots + \gamma \beta^{n+2m-3} \right] d_0,
\end{aligned}$$

i.e,

$$\begin{aligned} d(x_n, x_{n+2m}) &\leq \frac{1+\beta}{1-\beta^2}\beta^n d_0 + \gamma\beta^{n+2m-3} d_0, \\ &\leq \frac{1+\beta}{1-\beta^2}\beta^n d_0 + \gamma\beta^{2m}\beta^{n-3} d_0, \\ &\leq \frac{1+\beta}{1-\beta^2}\beta^n d_0 + \gamma\beta^{n-3} d_0, \quad (\text{as } \ \beta \leq 1\ ). \end{aligned}$$

So

$$d(x_n, x_{n+2m}) \leq \frac{1+\beta}{1-\beta^2}\beta^n d_0 + \gamma\beta^{n-3} d_0, \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0, \text{ for all } p > 0 \tag{3.13}$$

Thus $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d) there exists $u \in X$ prove that

$$\lim_{n\to\infty} x_n = u. \tag{3.14}$$

We will show that u is a fixed point of T, for any $n \in \mathbb{N}$ we have,

$$\begin{aligned} d(u, Tu) &\leq d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)\ , \\ &\leq d(u, x_n) + d_n + d(x_{n+1}, Tu)\ , \\ &\leq d(u, x_n) + d_n + \alpha\left[d(x_n, Tx_n) + d(u, Tu)\right], \\ &= d(u, x_n) + d_n + \alpha\left[d(x_n, x_{n+1}) + d(u, Tu)\right], \\ (1-\alpha)\, d(u, Tu) &\leq d(u, x_n)\beta^n d_0 + \alpha d(x_n, x_{n+1})\ . \end{aligned}$$

Using (3.13) and (3.14) via condition $\alpha < \frac{1}{2}$, it follows that,

$$d(u, Tu) = 0, \text{i.e.}, Tu = u\ .$$

Thus u is a fixed point of T.

For uniqueness, let v be another fixed point of T. Then it follows from (3.8) that,

$$\begin{aligned} d(u, v) &= d(Tu, Tv), \\ &\leq \alpha\left[d(u, Tu) + d(v, Tv)\right], \\ &\leq \alpha\left[d(u, u) + d(v, v)\right] = 0, \end{aligned}$$

Then, we have,

$$d(u, v) = 0, \text{i.e.}, u = v\ .$$

Finally, fixed point is unique. ■

Theorem 11 ***(Chatterjea)*** *3.2 Let (X, d) be a complete r.m.s, and $T : X \longrightarrow X$ be a C-contraction. Then T admits a unique fixed point .*

Proof. The proof is similar to Kannan's previous theorem . ■

CONCLUSION

This analysis highlights the significant importance and evolution of fixed point theory, which has become a fundamental tool across various fields over time. From its foundational concepts to its generalization into rectangular metric spaces, the theory remains flexible and adaptable to contemporary challenges. This memorandum is more than just an academic endeavor, it serves as a testament to the enduring relevance of fixed point theory and its ability to inspire future generations to continue exploring and advancing the frontiers of human knowledge.

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