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ON SOLVABILITY OF THE INTEGRODIFFERENTIAL HYPERBOLIC EQUATION WITH PURELY NONLOCAL CONDITIONS*

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Abstract In this study, we prove the existence, uniqueness, and continuous dependence upon the data of solution to integro-differential hyperbolic equation with purely nonlocal (integral) conditions. The proofs are based on a priori estimates and Laplace transform method. Finally, we obtain the solution using a numerical technique (Stehfest algorithm) by inverting the Laplace transform.

Key words Integro-differential hyperbolic equation; approximate solution; nonlocal purely conditions

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1 Introduction

During the past decades, the topic of integro-differential equations which are combination of differential and integral has attracted many scientists and researchers due to their applications in many areas; see, for example, [19, 20]. Many mathematical formulation of physical phenomena contain integro-differential equations, and these equations may arise in fluid dynamics, biological models, and chemical kinetics; for more details, see [24, 35]. Integro-differential equations are usually difficult to solve analytically, so it is required to obtain an efficient approximate solution.

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The nonlocal Cauchy problem for abstract evolution differential equations was introduced and studied by Byszewski; see [8–10]. Afterwards, many authors (for example, [11, 12]) discussed the problem for different kinds of nonlinear differential equations as well as integrodifferential equations, which include functional differential equations in Banach spaces. Later, Balachandran and several researchers established the existence of solutions for quasilinear integrodifferential equations with nonlocal conditions; see [13, 14]. N'Guérékata [15], Balachandran and Park made further contributions in [16] regarding the existence of solutions of fractional abstract differential equations with nonlocal initial condition. Ahmad [17] obtained some existence results for boundary value problems of fractional semilinear evolution equations. More recently, Balachandran and Trujillo [18] investigated the nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces.

In this article, we are concerned with the following hyperbolic integrodifferential equation,

$$\frac{\partial^2 v}{\partial t^2}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = \int_0^t a(t-s)\, v(x,s)\, \mathrm{d}s,\ 0 < x < 1,\ 0 < t \le T, \tag{1.1}$$

subject to the initial conditions,

$$\begin{aligned} v(x,0) &= \Phi(x),\ 0 < x < 1, \\ \frac{\partial v(x,0)}{\partial t} &= \Psi(x),\ 0 < x < 1, \end{aligned} \tag{1.2}$$

and the purely nonlocal (integral) conditions,

$$\begin{aligned} \int_0^1 v(x,t)\, \mathrm{d}x &= r(t),\ 0 < t \le T, \\ \int_0^1 x v(x,t)\, \mathrm{d}x &= q(t),\ 0 < t \le T, \end{aligned} \tag{1.3}$$

where v is an unknown function, r, q, $\Phi(x)$, and $\Psi(x)$ are given functions supposed to be sufficiently regular, a is suitably defined function satisfying certain conditions that will be specified later, and T is a positive constant.

Some problems of modern physics and technology can be described in terms of partial differential equations with nonlocal conditions. The nonlocal term of our problem (that is, $\int_0^t a(t-s)\, v(x,s)\, \mathrm{d}s$) appears, for instance, in the modelling of the quasistatic flexure of a thermoelastic rod; see [5] which firstly has been studied, by the second author with more general second-order parabolic equation or a $2m$-parabolic equation in [4] by using of the energy-integrals methods and the Rothe method in [28]. For other models, we refer the reader, for instance, to [3, 6, 29], [26, 33], and references therein. The problem (1.1)–(1.3) is studied by the Rothe method in [7]. Ang [2] considered a one-dimensional heat equation with nonlocal (integral) conditions. The author taken the Laplace transform of the problem and then used numerical technique for the inverse Laplace transform to obtain the numerical solution.

This article is organized as follows. In Section 2, we introduce certain function spaces which are used in the next sections, and we reduce our problem to another equivalent problem with homogeneous integral conditions. In Section 3, we establish the existence of solution by the Laplace transform. In Section 4, we deal with a priori estimates, which give the uniqueness and continuous dependence upon the data.

2 Statement of the Problem and Notation

As integral conditions are inhomogeneous, it is convenient to convert problem (1.1)–(1.3) to an equivalent problem with homogenous integral conditions. For this reason, we introduce a new function $u(x,t)$, which will represent the deviation of the function $v(x,t)$ by

$$u(x,t) = v(x,t) - w(x,t), \ 0 < x < 1, \ 0 < t \leq T, \tag{2.1}$$

where

$$w(x,t) = 6(2q(t) - r(t))x - 2(3q(t) - 2r(t)). \tag{2.2}$$

Problem (1.1)–(1.3) with inhomogeneous integral conditions (1.3) can be equivalently reduced to the problem of finding a function u satisfying

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = \int_0^t a(t-s)\, u(x,s)\,\mathrm{d}s, \ 0 < x < 1, \ 0 < t \leq T, \tag{2.3}$$

$$\begin{aligned} u(x,0) &= \varphi(x), \ 0 < x < 1, \\ \frac{\partial u(x,0)}{\partial t} &= \psi(x), \ 0 < x < 1, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \int_0^1 u(x,t)\,\mathrm{d}x &= 0, \ 0 < t \leq T, \\ \int_0^1 xv(x,t)\,\mathrm{d}x &= 0, \ 0 < t \leq T, \end{aligned} \tag{2.5}$$

where

$$\varphi(x) = \Phi(x) - w(x,0), \quad \psi(x) = \Psi(x) - \frac{\partial w(x,0)}{\partial t}. \tag{2.6}$$

The solution of problem (1.1)–(1.3) will be obtained by the relation (2.1) and (2.2). If H is a Hilbert space with a norm $\|.\|_H$ and $L^2(0,1)$ is the square integrable functions. We introduce the appropriate function spaces that will be used in the rest of the note.

Definition 2.1 (i) We denote by $L^2(0,T;H)$ the set of all measurable abstract functions $u(.,t)$ from $(0,T)$ into H equipped with the norm

$$\|u\|_{L^2(0,T;H)} = \left(\int_0^T \|u(.,t)\|_H^2\,\mathrm{d}t \right)^{1/2} < \infty. \tag{2.7}$$

(ii) The space $C(0,T;H)$ is the set of all continuous functions $u(.,t) : (0,T) \longrightarrow H$ equipped with the norm

$$\|u\|_{C(0,T;H)} = \max_{0 \leq t \leq T} \|u(.,t)\|_H < \infty.$$

We denote by $C_0(0,1)$ the space of continuous functions with compact support in $(0,1)$. As such functions are Lebesgue integrable with respect to x, we can define on $C_0(0,1)$ the bilinear form given by

$$((u,w)) = \int_0^1 J_x^m u . J_x^m w \mathrm{d}x, \ m \geq 1, \tag{2.8}$$

where

$$J_x^m u = \int_0^x \frac{(x-\zeta)^{m-1}}{(m-1)!} u(\zeta,t)\,\mathrm{d}\zeta \quad \text{for } m \geq 1. \tag{2.9}$$

The bilinear form (2.8) is considered as a scalar product on $C_0(0,1)$ but is not a complete space.

Definition 2.2 Denote by $B_2^m(0,1)$ the completion of $C_0(0,1)$ for the scalar product (2.8), which is denoted by $(.,.)_{B_2^m(0,1)}$ and introduced in [4]. By the norm of a function u from $B_2^m(0,1)$, $m \geq 1$, we understand the nonnegative number:

$$\|u\|_{B_2^m(0,1)} = \left(\int_0^1 (J_x^m u)^2 \,\mathrm{d}x\right)^{1/2} = \|J_x^m u\|, \text{ for } m \geq 1. \tag{2.10}$$

Lemma 2.3 For any $m \in \mathbb{N}^*$, the following inequality holds,

$$\|u\|^2_{B_2^m(0,1)} \leq \frac{1}{2}\|u\|^2_{B_2^{m-1}(0,1)}. \tag{2.11}$$

Proof See [4]. □

Corollary 2.4 For any $m \in \mathbb{N}^*$, we have the elementary inequality,

$$\|u\|^2_{B_2^m(0,1)} \leq \left(\frac{1}{2}\right)^m \|u\|^2_{L^2(0,1)}. \tag{2.12}$$

Definition 2.5 We denote by $L^2(0,T;B_2^m(0,1))$ the space of functions which are square integrable in the Bochner sense, with the scalar product

$$(u,w)_{L^2(0,T;B_2^m(0,1))} = \int_0^T (u(.,t), w(.,t))_{B_2^m(0,1)} \,\mathrm{d}t. \tag{2.13}$$

As the space $B_2^m(0,1)$ is a Hilbert space, it can be shown that $L^2(0,T;B_2^m(0,1))$ is also a Hilbert space. The set of all continuous abstract functions in $[0,T]$ equipped with the norm

$$\sup_{0\leq t\leq T} \|u(.,t)\|_{B_2^m(0,1)}$$

is denoted by $C(0,T;B_2^m(0,1))$.

Corollary 2.6 The following imbedding $L^2(0,1) \longrightarrow B_2^m(0,1)$ is continuous for $m \geq 1$.

Lemma 2.7 (Gronwall Lemma) Let $f_1(t), f_2(t) \geq 0$ be two integrable functions on $[0,T]$ and $f_2(t)$ is nondecreasing. If

$$f_1(\tau) \leq f_2(\tau) + c\int_0^\tau f_1(t)\,\mathrm{d}t,\ \forall \tau \in [0,T], \tag{2.14}$$

where $c \in \mathbb{R}^+$, then, we have

$$f_1(t) \leq f_2(t)\exp(ct),\ \forall t \in [0,T]. \tag{2.15}$$

Proof See Lemma 1.3.19 in [25]. □

3 Existence of the Solution

Laplace transform is an efficient method for solving many differential equations and partial differential equations, the only main difficulty with Laplace transform method is in inverting the Laplace domain solution into the real domain. In this section, we shall apply the Laplace transform technique to find solutions of partial differential equations.

Suppose that $v(x,t)$ is defined and is of exponential order for $t \geq 0$, that is, there exists A, $\gamma > 0$, and $t_0 > 0$ such that $|f(t)| \leq A\exp(\gamma t)$ for $t \geq t_0$. Then, the Laplace transform $V(x,s)$ exists and is given by

$$V(x,s) = \int_0^\infty v(x,t)\exp(-st)\,\mathrm{d}t, \tag{3.1}$$

where s is positive real parameter. Applying the Laplace transform on both sides of (1.1) , we have

$$\left(s^2 - A(s)\right) V\left(x, s\right) - \frac{\mathrm{d}^2}{\mathrm{d}x^2} V\left(x, s\right) = s\Phi\left(x\right) + \Psi\left(x\right). \tag{3.2}$$

Similarly, we have

$$\int_0^1 V\left(x, s\right) \mathrm{d}x = R(s), \qquad \int_0^1 xV\left(x, s\right) \mathrm{d}x = Q(s), \tag{3.3}$$

where $R(s)$, $Q(s)$ are the Laplace transforms of $r(t)$ and $q(t)$, respectively. Now, we distinguish the following cases:

- Case 1: If $s^2 - A(s) > 0$.
- Case 2: If $s^2 - A(s) < 0$.
- Case 3: If $s^2 - A(s) = 0$.

We only consider Cases 2 and 3, because Case 1 can be dealt with similarly as in [2]. For $\left(s^2 - A(s)\right) = 0$, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} V\left(x, s\right) = -s\Phi\left(x\right) - \Psi\left(x\right). \tag{3.4}$$

The general solution for Case 3 is given by

$$V\left(x, s\right) = -\int_0^x \int_0^y \left[s\Phi\left(x\right) + \Psi\left(x\right)\right] \mathrm{d}z\mathrm{d}y + C_1\left(s\right) x + C_2(s). \tag{3.5}$$

Putting the integral conditions (3.3) in (3.5), we get

$$\begin{aligned} \frac{1}{2} C_1(s) + C_2(s) &= \int_0^1 \int_0^x \int_0^y \left[s\Phi\left(x\right) + \Psi\left(x\right)\right] \mathrm{d}z\mathrm{d}y + R(s), \\ \frac{1}{3} C_1(s) + \frac{1}{2} C_2(s) &= \int_0^1 \int_0^x \int_0^y x\left[s\Phi\left(x\right) + \Psi\left(x\right)\right] \mathrm{d}z\mathrm{d}y + Q(s), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} C_1(s) &= 12 \int_0^1 \int_0^x \int_0^y x\left[s\Phi\left(x\right) + \Psi\left(x\right)\right] \mathrm{d}z\mathrm{d}y \\ &\quad -6 \int_0^1 \int_0^x \int_0^y \left[s\Phi\left(x\right) + \Psi\left(x\right)\right] \mathrm{d}z\mathrm{d}y + 12Q(s) - 6R(s), \\ C_2(s) &= 4 \int_0^1 \int_0^x \int_0^y \left[s\Phi\left(x\right) + \Psi\left(x\right)\right] \mathrm{d}z\mathrm{d}y \\ &\quad -6 \int_0^1 \int_0^x \int_0^y x\left[s\Phi\left(x\right) + \Psi\left(x\right)\right] \mathrm{d}z\mathrm{d}y - 6Q(s) + 4R(s). \end{aligned} \tag{3.7}$$

For Case 2, that is, $\left(s^2 - A(s)\right) < 0$, using the method of variation of parameter, we have the general solution as

$$\begin{aligned} V\left(x, s\right) = &\frac{1}{\sqrt{A(s) - s^2}} \int_0^x \left(s\Phi\left(x\right) + \Psi\left(x\right)\right) \sin\left(\sqrt{A(s) - s^2}\right)\left(x - \tau\right) \mathrm{d}\tau \\ &+ d_1(s) \cos\sqrt{\left(A(s) - s^2\right)} x + d_2(s) \sin\sqrt{\left(A\left(s\right) - s^2\right)} x. \end{aligned} \tag{3.8}$$

By the integral conditions (3.3), we obtain

$$d_1(s) \int_0^1 \cos\sqrt{\left(A\left(s\right) - s^2\right)} x\mathrm{d}x + d_2(s) \int_0^1 \sin\sqrt{\left(A(s) - s^2\right)} x\mathrm{d}x$$

$$= R(s) - \frac{1}{\sqrt{A(s)-s^2}} \int_0^1 \int_0^x (s\Phi(x) + \Psi(x)) \sin\left(\sqrt{A(s)-s^2}\right)(x-\tau)\,\mathrm{d}\tau\mathrm{d}x,$$

$$d_1(s)\int_0^1 x\cos\sqrt{(A(s)-s^2)}x\mathrm{d}x + d_2(s)\int_0^1 x\sin\sqrt{(A(s)-s^2)}x\mathrm{d}x$$

$$= Q(s) - \frac{1}{\sqrt{A(s)-s^2}} \int_0^1 \int_0^x x(s\Phi(x) + \Psi(x)) \sin\left(\sqrt{A(s)-s^2}\right)(x-\tau)\,\mathrm{d}\tau\mathrm{d}x. \quad (3.9)$$

Thus, d_1 and d_2 are given by

$$\begin{pmatrix} d_1(s) \\ d_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}, \quad (3.10)$$

where

$$\begin{aligned}
a_{11}(s) &= \int_0^1 \cos\sqrt{(A(s)-s^2)}x\mathrm{d}x,\\
a_{12}(s) &= \int_0^1 \sin\sqrt{(A(s)-s^2)}x\mathrm{d}x,\\
a_{21}(s) &= \int_0^1 x\cos\sqrt{(A(s)-s^2)}x\mathrm{d}x,\\
a_{22}(s) &= \int_0^1 x\sin\sqrt{(A(s)-s^2)}x\mathrm{d}x, \quad (3.11)\\
b_1(s) &= R(s) - \frac{1}{\sqrt{A(s)-s^2}} \int_0^1 \int_0^x (s\Phi(x) + \Psi(x)) \sin\left(\sqrt{A(s)-s^2}\right)(x-\tau)\,\mathrm{d}\tau\mathrm{d}x,\\
b_2(s) &= Q(s) - \frac{1}{\sqrt{A(s)-s^2}} \int_0^1 \int_0^x x(s\Phi(x) + \Psi(x)) \sin\left(\sqrt{A(s)-s^2}\right)(x-\tau)\,\mathrm{d}\tau\mathrm{d}x.
\end{aligned}$$

If it is not possible to calculate the integrals directly, then we calculate them numerically. We approximate similarly as done in [2]. If the Laplace inversion is possibly computed directly for (3.5) and (3.8) , we obtain our solution explicitly. Otherwise, we use the suitable approximate method, then we use the numerical inversion of the Laplace transform. Considering $A(s)-s^2 = k(s)$ and using Gauss's formula given in [1], we have the following approximations of the integrals:

$$\begin{aligned}
&\int_0^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \cos\sqrt{k(s)}x\mathrm{d}x \simeq \frac{1}{2}\sum_{i=1}^N w_i \begin{pmatrix} 1 \\ \frac{1}{2}[x_i+1] \end{pmatrix} \cos\left(\sqrt{k(s)}\frac{1}{2}[x_i+1]\right),\\
&\int_0^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \sin\sqrt{k(s)}x\mathrm{d}x \simeq \frac{1}{2}\sum_{i=1}^N w_i \begin{pmatrix} 1 \\ \frac{1}{2}[x_i+1] \end{pmatrix} \sin\left(\sqrt{k(s)}\frac{1}{2}[x_i+1]\right),\\
&\int_0^x (s\Phi(x) + \Psi(x)) \sin\left(\sqrt{k(s)}\right)(x-\tau)\,\mathrm{d}\tau\\
\simeq\ & \frac{x}{2}\sum_{i=1}^N w_i \left[s\Phi\left(\frac{x}{2}[x_i+1]\right) + \Psi\left(\frac{x}{2}[x_i+1]\right)\right] \sin\left(\sqrt{k(s)}\left[x - \frac{x}{2}[x_i+1]\right]\right), \quad (3.12)\\
&\int_0^1 \left[[s\Phi(\tau) + \Psi(\tau)] \int_\tau^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \sin\left(\sqrt{k(s)}\right)(x-\tau)\,\mathrm{d}x\right]\mathrm{d}\tau\\
\simeq\ & \frac{1}{2}\sum_{i=1}^N w_i \left[s\Phi\left(\frac{1}{2}[x_i+1]\right) + \Psi\left(\frac{1}{2}[x_i+1]\right)\right]
\end{aligned}$$

$$\cdot\left(\frac{1-\frac{1}{2}\left[x_i+1\right]}{2}\right)\sum_{i=1}^{N} w_j\left(\frac{1}{\frac{1-\frac{1}{2}[x_i+1]}{2}x_j+\frac{1-\frac{1}{2}[x_i+1]}{2}}\right)$$
$$\cdot\sin\left(\sqrt{k(s)}\left[\frac{1-\frac{1}{2}\left[x_i+1\right]}{2}x_j+\frac{1+\frac{1}{2}\left[x_i+1\right]}{2}-\frac{1}{2}\left(x_i+1\right)\right]\right),$$

where x_i and w_i are the abscissa and weights, defined as

$$x_i : i^{th} \text{ zero of } P_n\left(x\right),\ \omega_i\ = 2/\left(1-x_i^2\right)\left[P_n^{'}\left(x\right)\right]^2.$$

Their tabulated values can be found in [1] for different values of N.

3.1 Numerical Inversion of Laplace Transform

Sometimes, an analytical inversion of the Laplace domain solution is difficult to obtain; therefore, a numerical inversion method must be used. An important comparison of four frequently used numerical Laplace inversion algorithms is given by Hassanzadeh and Pooladi-Darvish [23]. Here, we use Stehfest's algorithm [33] that is easy to implement. This numerical technique was first introduced by Graver [22] and its algorithm then is developed by [33]. Stehfest's algorithm approximates the time domain solution by

$$v\left(x,t\right)\approx\frac{\ln 2}{t}\sum_{n=1}^{2m}\beta_n V\left(x;\frac{n\ln 2}{t}\right), \tag{3.13}$$

where m is the positive integer,

$$\beta_n=\left(-1\right)^{n+m}\sum_{k=\left[\frac{n+1}{2}\right]}^{\min(n,m)}\frac{k^m\left(2k\right)!}{(m-k)!k!\left(k-1\right)!\left(n-k\right)!\left(2k-n\right)!}, \tag{3.14}$$

and $[q]$ denotes the integer part of the real number q.

4 Uniqueness and Continuous Dependence of the Solution

We establish an a priori estimate, and the uniqueness and continuous dependence of the solution with respect to the data are immediate consequences.

Theorem 4.1 If $u\left(x,t\right)$ is a solution of the Problem (2.3)–(2.5) and $f\in C\left(\overline{D}\right)$, then we have

$$\begin{aligned}&\left\|u\left(.,\tau\right)\right\|^2_{L^2(0,1)}\le c_1\left(\left\|\varphi\right\|^2_{L^2(0,1)}+\left\|\psi\right\|^2_{B_2^1(0,1)}\right),\\&\left\|\frac{\partial u\left(.,\tau\right)}{\partial t}\right\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)}\le c_2\left(\left\|\varphi\right\|^2_{L^2(0,1)}+\left\|\psi\right\|^2_{B_2^1(0,1)}\right),\end{aligned} \tag{4.1}$$

where $c_1=\exp\left(a_0T\right),\ c_2=\frac{\exp(a_0T)}{1-a_0}$, $1<a\left(x,t\right)<a_0$, and $0\le\tau\le T$.

Proof Taking the scalar product in $B_2^1\left(0,1\right)$ of equation (2.3) and $\frac{\partial u}{\partial t}$, and integrating over $(0,\tau)$, we have

$$\begin{aligned}&\int_0^\tau\left(\frac{\partial^2u\left(.,t\right)}{\partial t^2},\frac{\partial u\left(.,t\right)}{\partial t}\right)_{B_2^1(0,1)}\mathrm{d}t-\int_0^\tau\left(\frac{\partial^2u\left(.,t\right)}{\partial x^2},\frac{\partial u\left(.,t\right)}{\partial t}\right)_{B_2^1(0,1)}\mathrm{d}t\\=&\int_0^\tau\left(\int_0^t a\left(t-s\right)u\left(x,s\right)\mathrm{d}s,\frac{\partial u\left(.,t\right)}{\partial t}\right)_{B_2^1(0,1)}\mathrm{d}t.\end{aligned} \tag{4.2}$$

Integrating by parts on the left-hand sid of (4.2), we obtain

$$\begin{aligned}&\frac{1}{2}\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{B_2^1(0,1)}-\frac{1}{2}\|\psi\|^2_{B_2^1(0,1)}+\frac{1}{2}\|u(.,\tau)\|^2_{L^2(0,1)}-\frac{1}{2}\|\varphi\|^2_{L^2(0,1)}\\ =&\int_0^\tau\left(\int_0^t a(t-s)\,u(x,s)\,\mathrm{d}s,\frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)}\mathrm{d}t. \qquad (4.3)\end{aligned}$$

By Cauchy inequality, the right-hand side of (4.3) is bounded by

$$\frac{a_0}{2}\int_0^t\|u(x,s)\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)}\mathrm{d}s+\frac{a_0}{2}\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} \qquad (4.4)$$

Substituting (4.4) into (4.3) yields

$$\begin{aligned}&(1-a_0)\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)}+\|u(.,\tau)\|^2_{L^2(0,1)}\\ \le&\left(\|\varphi\|^2_{L^2(0,1)}+\|\psi\|^2_{B_2^1(0,1)}\right)+\frac{a_0}{2}\int_0^t\|u(x,s)\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)}\mathrm{d}s. \qquad (4.5)\end{aligned}$$

By Gronwall Lemma, we have

$$\begin{aligned}&(1-a_0)\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)}+\|u(.,\tau)\|^2_{L^2(0,1)}\\ \le&\exp(a_0T)\left(\|\varphi\|^2_{L^2(0,1)}+\|\psi\|^2_{B_2^1(0,1)}\right). \qquad (4.6)\end{aligned}$$

From (4.6) , we obtain the estimates (4.1). □

Corollary 4.2 If Problem (2.3)–(2.5) has a solution, then this solution is unique and depends continuously on (φ,ψ).

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