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Fractional and non-homogeneous eigenvalue problems

Supervisor : **Abdelhamid GOUASMIA**

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Dr. Kamel Djeddi,	President
Dr. Abdelhamid Gouasmia,	Thesis director
Dr. Billel Gheraibia,	Examinator

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ملخص

الأهداف الرئيسية لهذه المذكرة تتمثل في دراسة وجود وعدم وجود الحلول الضعيف و كذلك إنتظاميتها لمسألة ذات مؤثر كسري غير متجانس باستخدام مبرهنة مرور الجبل.
لقد قسمنا هذا العمل إلى فصلين:
الفصل الأول خصص لجمع بعض النتائج المهمة في التحليل الدالي و التي سنستعملها في هذا العمل ايضافة الى فضاء سوبوليف الكسري $W^{s,p}(\Omega)$ و نتطرق إلى طريقة مبرهنة مرور الجبل.
يدرس الفصل الثاني وجود وعدم وجود والانتظام للحلول الضعيفة بمسألتنا.

الكلمات المفتاحية

مؤثر الكسري p− لابلاس، عدم الوجود، نتائج الانتظام، الوجود، مبرهنة مرور الجبل، حلول غير تافهة، مؤثر غير متجانسة.

Abstract

The main objective of this thesis is to study the existence , non-existence , and regularity of the weak solution of the fractional non-homogenous problem involving fractional and non-homogenous operateurs by using the mountain pass theorem .
We divided this work into two chapters:

In the first chapter, we begin by providing an overview of the functional analysis used in this work and define the fractional sobolev espace $W^{s,p}(\Omega)$,moreover, define method of mountain pass theorem .

the second chapter study the existence,non-existence, and regularity of the weak solutions.

key-words : Fractional p-Laplacian operator, non-existence, regularity results , existence , mountain pass theorem ,nontrivial solutions , non-homogenous operator.

Function spaces

$L^p(\Omega) := \left\{ u : \Omega \to \mathbb{R}^N : u \text{ is measurable and } \int_\Omega |u|^p \, dx < \infty \right\}, 1 \le p < \infty.$

$L^\infty(\Omega) := \{ u : \Omega \to \mathbb{R} : u \text{ is measurable and } |u(x)| \le C \text{ a.e. in } \Omega \text{ for some constant } C \}.$

$C(\Omega)$ space of continuously fanctions on Ω.

$C(\overline{\Omega})$ functions in $C(\Omega)$ where the function $x \mapsto u(x)$ admits a continuous extension to $\overline{\Omega}$.

$C_c^\infty(\Omega) := \left\{ \varphi : \mathbb{R}^N \to \mathbb{R} : \varphi \in C^\infty(\mathbb{R}^N) \quad \text{and} \quad \operatorname{supp}(\varphi) \Subset \Omega \right\}.$

$C^{0,\alpha}(\overline{\Omega}) := \left\{ u \in C(\overline{\Omega}), \sup_{x,y \in \overline{\Omega}, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty \right\}, \text{ with } 0 < \alpha < 1.$

$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy < \infty \right\}, \text{ with } 0 < s < 1 \text{ and } 1 \le p < \infty.$

$W_0^{s,p}(\Omega) := \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.$

Contents

Introduction

In this work, we study some results of the fractional non-homogeneous problem involving fractional and non-homogeneous operators. These kinds of non-local operators have their applications in the real world such as optimization, finance, phase transitions, soft thin films, and image processing. The fractional Laplacian also provides a model to describe certain jump Lévy processes in probability theory and porous media in physical and among others in various fields, see (1; 16; 36; 4; 19) and the references therein.
The non-homogeneous operator in the problem (**P**) (see bellow) is known as fractional $(p,q)-$Laplacian, which is the natural counterpart of the $(p,q)-$Laplace $(-\Delta_p - \Delta_q)$ operator. Here, the latter operator appears in several contexts: biophysics, plasma physics, reaction-diffusion equations, and chemical reactions, we refer to (42; 30; 38; 14) without giving an exhaustive list.

- In the local case ($s_1 = s_2 = 1$), we can rewrite the problem (**P**) as follows:

$$-\Delta_p u - \Delta_q u = g(x,u), \quad u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{on } \partial\Omega. \tag{$\mathbf{P}_{\text{loc}}$}$$

This type of problem has been extensively investigated in the literature with various methods. We start from the work of Marano and Mosconi (38) where they mentioned some existence and multiplicity results about the problem ($\mathbf{P}_{\text{loc}}$) in many papers of research. More precisely, they presented the existence and non-existence of a solution for generalized eigenvalue problems, i.e.,

$$g(x,u) = \alpha |u|^{p-2} u + \beta |u|^{q-2} u,$$

with $(\alpha,\beta) \in \mathbb{R}^2$ (see (43; 40; 37) and the references cited therein for further details).

We point out that Tanaka in (28) obtained the existence and non-existence of a positive weak solution for ($\mathbf{P}_{\text{loc}}$) in the special case:

$$g(x,u) = \lambda\, m_r(x) u^{r-1} \text{ such that } r = p \text{ or } q,$$

where $m_r \in L^\infty(\Omega)$ and the Lebesgue measure of $\{x \in \Omega; \quad m_r(x) > 0\}$ is positive. In addition, the uniqueness of a positive solution for the same problem is given in the article (29), when $r = q$.

Now, we return again to the article (38), where the authors also collect diversified results about the existence and multiplicity of solutions to the problem ($\mathbf{P}_{\text{loc}}$), where g of the type:

$$g(x,u) = \alpha |u|^{p-2} u + \beta |u|^{q-2} u + f(u),$$

where $f \in C^1(\mathbb{R})$ and exhibits a suitable growth rate at $\pm\infty$ and/or at zero. We refer the reader to (31; 26; 39) and (15) for related issues.

In this thesis, we study the following problem:

$$(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \lambda\, a_p(x) u^{p-1}, \quad u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \tag{P}$$

where Ω is bounded in $\mathbb{R}^N$ with boundary of class $C^{1,1}$, $N > s_1 p$, $0 < s_2 \leq s_2 < 1$ and $1 < q \leq p < \infty$, here $(-\Delta)_r^s$ is the fractional r-laplace operator .
defined for $s \in \{s_1, s_2\}$ and $r \in \{p, q\}$ as :

$$(-\Delta)_r^s = 2\text{P.V} \int_{\mathbb{R}^N} \frac{\left|u(x) - u(y)\right|^{r-2} (u(x) - u(y))}{\left|x - y\right|^{N+sr}} \quad dy \tag{1}$$

where P.V denotes the cauchy principal value .

Remark 0.1. *when $s_1 = s_2$ with $p \neq q$ does not hold , for that ,we consider the space $\boldsymbol{W} = W_0^{s_1,p}(\Omega)$,in the case $0 < s_1 < s_2 < 1$,and if $s = s_1 = s_2$, we tak $W = W_0^{s,p}(\Omega) \cap W_0^{s,q}(\Omega)$,equipped with the cartesian norm $\|.\|_{W} = \|.\|_{W_0^{s,p}(\Omega)} + \|.\|_{W_0^{s,q}(\Omega)}$, notice that $W_0^{s,r}$ is a separble reflevixe Banach space with $s \in \{s_1, s_2\}$ and $r \in \{p, q\}$, then $\boldsymbol{W}$ is also a seprable reflexive Banach espace .*

We say that a non-negative function $u \in \mathbf{W}$ is called a weak solution to (**P**) if, for any $\varphi \in \mathbf{W}$ we have:

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x) - u(y)\right|^{p-2} \left(u(x) - u(y)\right)\left(\varphi(x) - \varphi(y)\right)}{|x - y|^{N+s_1 p}} dx dy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x) - u(y)\right|^{q-2} \left(u(x) - u(y)\right)\left(\varphi(x) - \varphi(y)\right)}{|x - y|^{N+s_2 q}} dx dy \\
&\quad = \lambda \int_\Omega a_r(x) u^{r-1} \varphi dx.
\end{aligned} \tag{2}$$

In addition if u satisfies $u > 0$ in Ω, we call u positive weak solution.

We obtain the following results:

Theorem 0.2. *If $\lambda \leq \lambda_{1,s_1,p}(a_p)$ holds, then (**P**) has no nontrivial solutions.*

Theorem 0.3. *If $\lambda > \lambda_{1,s_1,p}(a_p)$. Then (**P**) has at least one positive solution u. In addition, $u \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for any $\sigma \in (0, s_1)$ and $\sigma' > s_1$, there exists a positive constant $c = c(\sigma, \sigma') > 0$, such that:*

$$c^{-1} d^{\sigma'} \leq u \leq c\, d^{\sigma} \quad in \Omega.$$

Theorem 0.4. *We set the following non-local Rayleigh quotient:*

$$\underline{\lambda}_{s_1,s_2,p,q}(a) := \inf_{u \in \boldsymbol{W}} \left\{ \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x) - u(y)\right|^p}{|x - y|^{N+s_1}} dx dy + \frac{p}{q} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x) - u(y)\right|^{s_2}}{|x - y|^{N+s_2 q}} dx dy}{\displaystyle\int_\Omega a_p(x) u^p\, dx} \right\}.$$

Then, $\underline{\lambda}_{s_1,s_2,p,q}(a_p) = \lambda_{1,s,p}(a_p)$. In addition, the infimum is not attained.

CHAPTER 1

Preliminaries and functional setting

This chapter is meant to provide an overview of the functional analysis that will be used afterward. Moreover, we present some basic facts concerning the necessary function of spaces (see (12)).

1 Fractional Sobolev spaces

Considering a measurable function $u:\mathbb{R}^N \to \mathbb{R}$, we adopt. Let $p \in [1,+\infty[$, the norm in the space $L^p(\Omega)$ is denoted by :

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p\,dx\right)^{\frac{1}{p}}.$$

Let Ω be a possibly nonsmooth, open set of the Euclidean space $\mathbb{R}^n$. For any $s>0$, we would define the fractional Sobolev space $W^{s,p}(\Omega)$. In the literature, fractional Sobolev-type spaces are also called Aronszajn, Gagliardo, or Slobodeckij spaces, by the names of the ones who introduced them, almost simultaneously.

If $s>1$ is a positive integer, we denote by $W^{s,p}(\Omega)$ the classical Sobolev space equipped with the standard norm

$$\|u\|_{W^{s,p}(\Omega)} = \sum_{0\le|\alpha|\le s} \left\|D^\alpha u\right\|_{L^p(\Omega)}$$

For every $u \in W^{s,p}(\Omega)$, where here and in what follows $\|.\|_{L^p(\Omega)}$ denotes the usual norm in $L^p(\Omega)$, and D^α stands for the distributional derivative. This section is devoted to the definition of fractional Sobolev spaces; that is, here we are interested in the case where $s \notin \mathbb{N}$.

For a fixed $s\in(0,1)$, we recall that the Sobolev space $W^{s,p}(\Omega)$ is defined as follows:

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x)-u(y)|}{|x-y|^{n/p+s}} \in L^p(\Omega\times\Omega) \right\}.$$

it is endowed with the natural norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\int_{\Omega} |u(x)|^p \, dx + \int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy \right)^{\frac{1}{p}}, \tag{1.1}$$

where the term

$$[u]_{W^{s,p}(\Omega)} := \left(\int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy \right) \tag{1.2}$$

is the Gagliardo seminorm of u.

When $s > 1$ and $s \notin \mathbb{N}$, we can write $s = m + \sigma$, where $m \in \mathbb{N}$ and $\sigma \in (0,1)$. We can define $W^{s,p}(\Omega)$ as follows :

$$W^{s,p}(\Omega) := \left\{ u \in W^{m,p}(\Omega) : D^{\alpha} u \in W^{\sigma,p}(\Omega) \text{ for any } \alpha . s.t. |\alpha| = m \right\}.$$

In this case, $W^{s,p}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} \|D^{\alpha} u\|^p_{W^{\sigma,p}(\Omega)} \right)^{\frac{1}{p}},$$

For every $u \in W^{s,p}(\Omega)$. All in all, the space $W^{s,p}(\Omega)$ is well defined and is a Banach space for every$s > 0$.

As in the classical case ($i, e., s \in \mathbb{N}$) any function in the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ can be approximated by a sequence of smooth functions with compact support. Indeed, for any $s > 0$,

$$\overline{C_0^{\infty}(\mathbb{R}^n)}^{\|.\|_{W^{s,p}(\mathbb{R}^n)}} = W^{s,p}(\mathbb{R}^n);$$

that is, the space $C_0^{\infty}(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$.

In general, if $\Omega \subset \mathbb{R}^n$, the space $C_0^{\infty}(\Omega)$ is not dense in $W^{s,p}(\Omega)$, Hence, we denote by $W_0^{s,p}(\Omega)$ the closure of $C_0^{\infty}(\Omega)$ with respect to the norm $\|.\|_{W^{s,p}(\Omega)}$; that is,

$$W_0^{s,p}(\Omega) := \overline{C_0^{\infty}(\Omega)}^{\|.\|_{W^{s,p}(\Omega)}}.$$

With this definition, we can also construct $W^{s,p}(\Omega)$) when $s < 0$. Indeed, for $s < 0$ and $p \in (1, +\infty)$, we can define

$$W^{s,p}(\Omega) = \left(W_0^{-s,q}(\Omega) \right);$$

that is $W^{s,p}(\Omega)$, is the dual space of $W_0^{-s,q}(\Omega)$, where $1/p + 1/q = 1$ **?** .

2 Fractional Sobolev inequalities and applications :

Theorem 2.1. *Let $s \in (0,1)$ and $p \in [1, +\infty)$ be such that $sp < n$. Then there exists a positive constant $C = C(n, p, s)$ such that, for any measurable and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$, we have*

$$\|f\|^p_{L^{p^*}(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} dxdy \tag{1.3}$$

where $p^* = p^*(n,s)$ is the so-called "fractional critical exponent" and it is equal to $np/(n-sp)$.

Consequently, the space $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p, p^*]$. The above embedding does not generally hold for the space $W^{s,p}(\Omega)$ since it not always possible to extend a function $f \in W^{s,p}(\Omega)$ to a function $\widetilde{f} \in W^{s,p}(\mathbb{R}^n)$. In order to be allowed to do that, we should require further regularity assumptions on Ω.

Theorem 2.2. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ be such that $sp < n$. Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,p}$. Then there exists a positive constant $C = C(n,p,s,\Omega)$ such that, for any $f \in W^{s,p}(\Omega)$, we have*

$$\|f\|_{L^q(\Omega)} \leq C\,\|f\|_{W^{s,p}(\Omega)}, \tag{1.4}$$

for any $q \in [p,p^*]$, i.e., the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [p,p^*]$.

If, in addition, Ω is bounded, then the space $W^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [1,p^*]$.

Proof. Let $f \in W^{s,p}(\Omega)$. Since $\Omega \subseteq \mathbb{R}^n$ is an extension domain for $W^{s,p}$, hen there exists a constant $C_1 = C_1(n,p,s,\Omega) > 0$ such that

$$\|\widetilde{f}\|_{W^{s,p}(\mathbb{R}^n)} \leq C_1\,\|f\|_{W^{s,p}(\Omega)}, \tag{1.5}$$

with $\widetilde{f}$ such that $\widetilde{f}(x) = f(x)$ for x a.e.in Ω.

On the other hand, by (2.1), the space $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p,p^*]$; i.e., there exists a constant $C_2 = C_2(n,p,s) > 0$ such that

$$\|\widetilde{f}\|_{L^q(\mathbb{R}^n)} \leq C_2\,\|\widetilde{f}\|_{W^{s,p}(\Omega)}, \tag{1.6}$$

Combining (1.5) with (1.9), we get

$$\|f\|_{L^q(\Omega)} = \|\widetilde{f}\|_{L^q(\Omega)} \leq \|\widetilde{f}\|_{L^q(\mathbb{R}^n)} \leq C_2\,\|\widetilde{f}\|_{W^{s,p}(\mathbb{R}^n)} \leq C_2C_1\,\|f\|_{W^{s,p}(\Omega)},$$

that gives the inequality in (1.4), by choosing $C = C_2C_1$.

In the case of Ω being bounded, the embedding for $q \in [1,p]$ plainly follows from (1.4), by using the Hölder inequality. □

Remark 2.3. *In the critical case $q = p^*$ the constant C in (2.2) does not depend on Ω: this is a consequence of (1.3) and of the extension property of Ω.*

Corollary 2.4. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ be such that $sp < n$. let $q \in [1,p^*]$ $\Omega \subseteq \mathbb{R}^n$ be a bounded extension domain for $W^{s,p}$ and $\Im$ be a bounded subset of $L^p(\Omega)$. Suppose that*

$$\sup_{f\in\Im}\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy < +\infty.$$

Then $\Im$ is pre-compact in $L^q(\Omega)$.

Proof. first, note that for $1 \leq q \leq p$ the compactness follows. for any $q \in (p,p^*)$, we may take $\theta = \theta(p,p^*,q) \in (0,1)$ such that $1/q = \theta/p + 1 - \theta/p^*$, thus for any $f \in \Im$ and β_j with $j \in \{1,........,N\}$ as in the theorem above, using the Hölder inequality with $p/(\theta q)$ and $p^*/((1-\theta)q)$, we get

$$\|f-\beta_j\|_{L^q(\Omega)} = \left(\int_\Omega |f-\beta_j|^{q\theta}\,|f-\beta_j|^{q(1-\theta)}\,dx\right)^{1/q}$$

$$\leq \left(\int_{\Omega} |f - \beta_j|^p \, dx \right)^{\theta/p} \left(\int_{\Omega} |f - \beta_j|^{p^*} \, dx \right)^{(1-\theta)/p^*}$$

$$= \| f - \beta_j \|_{L^{p^*}(\Omega)}^{1-\theta} \| f - \beta_j \|_{L^p(\Omega)}^{\theta}$$

$$\leq C \| f - \beta_j \|_{W^{s,p}(\Omega)}^{1-\theta} \| f - \beta_j \|_{L^p(\Omega)}^{\theta} \leq \widetilde{C} \varepsilon^{\theta},$$

where the last inequalities come directly and the continuous embedding (see (2.2)). □

Remark 2.5. *As is well known in the classical case $s = 1$ (and, more generally, when s is an integer), also in the fractional case the lack of compactness for the critical embedding $(q = p)$ is not surprising, because of translation and dilation invariance (for various results in this direction, for any $0 < s < n/2$)* **?** .

3 The fractional $p-$laplacian operator :

Given numbers $p \in (1, \infty)$ and $s \in (0, 1)$ we define the Gagliardo functional for smooth integrable functions defined in $\mathbb{R}^N$ as

$$J_{p,s}(u) = \frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy. \tag{1.7}$$

Functional $J_{s,p}$ is a convex, lower semi-continuous and proper functional, and it has an associated Euler–Lagrange operator, given in

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} \left(u(x) - u(y)\right) \left(\Phi(x) - \Phi(y)\right)}{|x - y|^{N+sp}} dx dy, \tag{1.8}$$

where Φ is any smooth variation in $W^{s,p}\left(\mathbb{R}^N\right)$. This defines the operator $J_{p,s}$ as

$$(-\Delta)_p^s \, u(x) = 2p.v. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2} \left(u(x) - u(y)\right)}{|x - y|^{N+sp}} dy, \tag{1.9}$$

where **p.v** stands for the principal value of the integral. Thus defined, it is a positive operator, that corresponds for $p = 2$ to the standard definition of $(-\Delta^s)$ (up to a constant that we do not take into account). Caveat: we can find in the literature the definition with the opposite sign, to look like the usual Laplacian, but we will stick here to this definition as a positive operator, following the tradition in fractional Laplacian operators.

4 Lebesgue's dominated convergence theorem

Let f_n be a sequence of complex-valued measurable functions on a measure space $(S, \sum, \mu)$. Suppose that the sequence converges pointwise to a function f and is dominated by some integrable function g in the sense that

$$|f_n(x)| \leq g(x)$$

for all numbers n in the index set of the sequence and all points $x \in S$. Then f is integrable (in the Lebesgue sense) and

$$\lim_{n\to\infty} \int_S |f_n - f| d\mu = 0$$

which also implies

$$\lim_{n\to\infty} \int_S f_n d\mu = \int_S f d\mu$$

5 Fatou's Lemma

Lemma 5.1. *Giver a mesure space* (Ω, F, μ) *and a set* $X \in F$ *,let* $\{f_n\}$ *be a sequence of* $(F, B_{\mathbb{R}\geq})$ *mesurable non negative functions* $f_n : X \to [0, +\infty]$ *. Define the function* $f : X \to [0, +\infty]$ *by setting* $f(x) = \lim_{n\to\infty} \inf f_n(x)$*, for every* $x \in X$
then f *is* $(F, B_{\mathbb{R}\geq})$ *mesurable , and also* $\int_X f d\mu \leq \liminf_{n\longrightarrow\infty} f_n d\mu$*. where the integrals may be infinite*

Now, we recall the embedding of $W_0^{s_1,p}(\Omega)$ in $W_0^{s_2,q}(\Omega)$ in the following Lemma:

Lemma 5.2. *(see (7, Lemma 2.1)) Let* $1 < q \leq p < \infty$ *and* $0 < s_2 < s_1 < 1$*, then there exists a constant* $C = C(|\Omega|, N, p, q, s_1, s_2) > 0$ *such that*

$$\|u\|_{W_0^{s_2,q}(\Omega)} \leq C \|u\|_{W_0^{s_1,p}(\Omega)}$$

for all $u \in W_0^{s_1,p}(\Omega)$.

For later use, we state here the following Lemma.

Lemma 5.3. *For* $1 < q \leq p$ *and* $d > 0$*, let us set*

$$\boldsymbol{W}_d := \left\{ u \in \boldsymbol{W}:\ \|u\|^p_{W_0^{s_1,p}(\Omega)} + \|u\|^p_{W_0^{s_2,q}(\Omega)} \leq d \|u\|^p_{L^p(\Omega)} \right\}.$$

Then, for each $d > 0$*, there exists* $C = C(d) > 0$ *such that*

$$\|u\|_{\boldsymbol{W}} \leq C \|u\|_{L^q(\Omega)}, \quad \textit{for every}\, u \in \boldsymbol{W}_d.$$

Proof. Our proof uses the method of contradiction. Suppose that there exist some $d > 0$ and a sequence $\{u_n\}_n \subset \mathbf{W}_d$ satisfying:

$$\|u_n\|_{W_0^{s_1,p}(\Omega)} + \|u_n\|_{W_0^{s_2,q}(\Omega)} > n \|u_n\|_{L^q(\Omega)}. \tag{1.10}$$

Now, we set

$$v_n = \frac{u_n}{\|u_n\|_{L^p(\Omega)}}.$$

First, we note that $\{v_n\}_n$ is bounded in $\mathbf{W}$. Indeed, by using the following inequality:

$$a^t + b^t \geq C_t (a+b)^t, \quad \forall a, b \geq 0, \forall t > 1$$

together with $u_n \in \mathbf{W}_d$, we obtain:

$$C_p \|v_n\|^p_{\mathbf{W}} \leq \|v_n\|^p_{W_0^{s_1,p}(\Omega)} + \|v_n\|^p_{W_0^{s_2,q}(\Omega)} \leq d.$$

Then, up to a sub-sequence, we have $v_n \rightharpoonup v$ in $\mathbf{W}$ and $v_n \to v$ in $L^r(\Omega)$ for $1 \leq r < p^*_{s_1}$. Next, we have $\|v_n\|_{L^p(\Omega)} = 1$, then $\|v\|_{L^p(\Omega)} = 1$, we deduce that $v \neq 0$. From (1.10), we have:

$$\|v_n\|_{L^q(\Omega)} < \frac{\|v_n\|_{\mathbf{W}}}{n} \leq \frac{d^{\frac{1}{p}}}{C_p^{\frac{1}{p}}\, n} \to 0 \quad \text{as} \quad n \to \infty.$$

Then $\|v\|_{L^q(\Omega)} = 0$, which is a contradiction. □

6 The Mountain Pass Theoreme :

we begin by the following definition:

Definition 6.1. *(13; 8) let* X *be a a Banach space and* $\Phi : X \longrightarrow \mathbb{R}$ *a* C^1 *functional , we say that* Φ *satisfies the Palais Smale condition , denoted by (PS) if any sequence* (u_n) *in* X *such that .*

$$\text{(PS)}\ \Phi(u_n) \text{ is bounded and } \Phi^{'}(u_n) \longrightarrow 0$$

admits a convergent susbsequence

We have seen in the one dimensional MPT that there is not only a compactness condition needed $[x_1, x_2] \in \mathbb{R}$ is compact a geometric condition , equation $f(x_1) = f(x_2)$ similarly here we require not only Palais smale compactness condition but also a geometric condition which we will introduce now :
but first let us defin the class of all paths joining $u = 0$ and $u = e$;

$$\Gamma = \left\{ y \in ([0,1]; X); y(0) = 0; y(1) = e \right\}$$

where ,
$e \in X; \|e\| > r > 0$
clearly ;$\Gamma \neq \phi$ since $y(t) = te$ is in Γ.Then we assune that the geometric condition :

$$\inf_{u \in S(0,\rho)} \Phi(u) > max\left\{ y(0), y(e) \right\}$$

is satisified for all

$$u \in S(0,\rho) = \left\{ u \in X : \|u - 0\| \leq \rho \right\}$$

notice that Φ might be bounded below on $S(0,\rho)$, for some $\rho > 0$.

Theorem 6.2. *(25)(**The Mountain Pass Theorem**)*
Let X *be a banach space and* $\Phi : X \longrightarrow \mathbb{R}$ *a* C^1 *suppose there existe* $e \in X$ *and two real nombers* $\rho > 0$ *and* $r > 0$ *such that* $\|e\| > r$ *and*
1) $\Phi(u) \geq \rho > 0$ *in* $u \in X | \|u\| \leq r\ 0$.
2) $\Phi(0) = \Phi(e) = 0$.
3) if $(u_n) \subset X$ *with* $0 < \Phi(u_n)$ *,*$\Phi(u_n)$ *bounded above , and* $\Phi'(u_n) \longrightarrow 0$ *then* (u_n) *possesses a convergent subsequence , then*

$$c = \inf_{y \in \Gamma, u \in y([0,1])} max \Phi(u)$$

with S *and* Γ *given by*

$$u \in S(0,\rho) = \left\{ u \in X : \|u - 0\| \leq \rho \right\}$$

$$\Gamma = \left\{ y \in ([0,1]; X); y(0) = 0; y(1) = e \right\}$$

is a critical value of Φ .

Proof. (13) If we view Γ asa normed space for the uniform to pology generated by the norm $\left\|y\right\|_{\Gamma} = max_{t\in[0,1]}\left|y(t)\right|$, for $y \in \Gamma$ and define $\Psi : \Gamma \longrightarrow \mathbb{R}$ by :

$$\Psi(y) = \max_{t\in[0,1]} \Phi(y(t))$$

Then Ψ is lower semicontinous as the upper bounded of a family of lower semicountinous function it is also bounded from below since

$$c = \inf_{\Gamma} \Psi \geq \max\{\Phi(0), \Phi(e)\}$$

hence we ean use Ekeland's variational principle i.e for every $\varepsilon > 0$? there exists a path $y_{\varepsilon} \in \Gamma$ such that :

$$\begin{aligned} \Psi(y_{\varepsilon}) &\leq c + \varepsilon, and \\ \Psi(y) \geq \Psi(y_{\varepsilon}) - \varepsilon \left\|y - y_{\varepsilon}\right\|_{\Gamma} & for \quad all \quad y \in \Gamma \end{aligned}$$

let $\mathrm{M}(\varepsilon) = \left\{t \in [0,1]; \Phi(y_{\varepsilon}(t)) = \max_{s\in[0,1]} \Phi(y_{\varepsilon}(s))\right\}$ then ,$\left\|\Phi^{'}(y_{\varepsilon}(t_{\varepsilon})\right\| \leq \varepsilon$ the (PS) condition with Palais smal sequence $x_n = y_{\frac{1}{n}}(t_{\frac{1}{n}})$ then do the job . □

CHAPTER 2

Non-existence, existence, and Hölder regularity of weak solutions

The purpose of this chapter is to study the existence, non-existence, and regularity of the weak solutions to the following non-linear problem involving fractional operators:

$$(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \lambda\, a_p(x) u^{p-1}, \quad u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \tag{P}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with boundary of class $C^{1,1}$, $N > s_1 p$, $0 < s_2 \le s_1 < 1$ and $1 < q \le p < \infty$.

1 Non-existence of positive weak solutions

We begin by the following definition:

Definition 1.1. *A non-negative function $u \in \mathbf{W}$ is called a weak solution to (**P**) if, for any $\varphi \in \mathbf{W}$ we have:*

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x)-u(y)\right|^{p-2}\left(u(x)-u(y)\right)\left(\varphi(x)-\varphi(y)\right)}{|x-y|^{N+s_1 p}}\,dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x)-u(y)\right|^{q-2}\left(u(x)-u(y)\right)\left(\varphi(x)-\varphi(y)\right)}{|x-y|^{N+s_2 q}}\,dxdy \\
&\quad = \lambda \int_\Omega a_p(x) u^{p-1} \varphi\, dx.
\end{aligned} \tag{2.1}$$

In addition if u satisfies $u > 0$ in Ω, we call u positive weak solution.

We first investigate the non-existence of positive weak solutions for the problem (**P**):

Theorem 1.2. *If $\lambda \le \lambda_{1,s_1,p}(a_p)$ holds, then (**P**) has no nontrivial solutions.*

***Proof of Theorem** (1.2).* Assume by contradiction that $u \in \mathbf{W}$ is a nontrivial solution of (**P**) and $\lambda \le \lambda_{1,s_1,p}(a_p)$. Taking u as a test function in (2.1) and by the definition of $\lambda_{1,s_1,p}(a_p)$, we have that:

$$\lambda_{1,s_1,p}(a_p) \le \frac{\|u\|^p_{W_0^{s_1,p}(\Omega)}}{\int_\Omega a_p(x) u^p\, dx} < \lambda = \frac{\|u\|^p_{W_0^{s_1,p}(\Omega)} + \|u\|^q_{W_0^{s_2,q}(\Omega)}}{\int_\Omega a_p(x) u^p\, dx}.$$

This contradicts our hypothesis. □

2 Existence, uniqueness, and regularity of weak solutions

In the following, we present an important remark about the regularity of weak solutions to fractional and non-homogeneous equations that we will use several times in the sequel:

Remark 2.1. *Let $u_0 \in \mathbf{W} \cap L^\infty(\Omega)$ be a nontrivial weak solution to the problem (P). Then, Theorem 2.3 in (18), Corollary 2.4 and Remark 2.3 in (17) provide the $C^{0,\alpha}(\overline{\Omega})$-regularity of u_0, for some $\alpha \in (0, s_1)$. By (18, Theorem 2.5), we infer that $u_0 > 0$ in Ω. Finally, by the Hopf's Lemma (18, Proposition 2.6) implies that $u_0 \geq k\, d^{s_1+\epsilon_0}(x)$ for some $k = k(\epsilon_0) > 0$ and for any $\epsilon_0 > 0$. Again by using (18, Proposition 3.11), we get that, for all $\sigma \in (0, s_1)$ there exists a constant $K = K(\sigma) > 0$ such that $u_0 \leq K\, d^\sigma(x)$ in Ω.*

Next, we state the results of the existence, uniqueness, and regularity:

Theorem 2.2. *Let $0 < s_2 \leq s_1 < 1$ and $1 < q < p < \infty$. Then, we have:*

If $\lambda > \lambda_{1,s,r}(a_r)$, where $r = p$ (or q), with $s = s_1$ (or s_2, respectively). Then (P) has at least one positive solution u. In addition, $u \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for any $\sigma \in (0, s_1)$ and $\sigma' > s_1$, there exists a positive constant $c = c(\sigma, \sigma') > 0$, such that:

$$c^{-1} d^{\sigma'} \leq u \leq c\, d^\sigma \quad in\,\Omega.$$

Theorem 2.3. *We set the following non-local Rayleigh quotient:*

$$\underline{\lambda}_{s,s*,r,r^*}(a) := \inf_{u \in \mathbf{W}} \left\{ \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^r}{|x-y|^{N+sr}}\,dxdy + \frac{r}{r^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{r^*}}{|x-y|^{N+s^* r^*}}\,dxdy}{\displaystyle\int_\Omega a_r(x) u^r\,dx} \right\}.$$

where $r = p$ (or q), with $s = s_1$ (or s_2) if $r^* = q$ (or p), with $s = s_2$ (or s_1, respectively). Then, $\underline{\lambda}_{s,s^*,r,r^*}(a_r) = \lambda_{1,s,r}(a_r)$. In addition, the infimum is not attained.

Proof. Will use the Mountain Pass Theorem to show the existence of a solution. For this aim, we consider the energy functional $\mathcal{K}$ corresponding to (P), defined on **W** by:

$$\mathcal{K}(u) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1 p}}\,dxdy + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^q}{|x-y|^{N+s_2 q}}\,dxdy - \frac{\lambda}{p}\int_\Omega a_p(x)(u^+)^p\,dx$$

Next, we divided the proof into 2 steps.

Step 1: The functional $\mathcal{K}$ satisfies the Palais-Smale condition at any level $c \in \mathbb{R}$.

We follow the ideas of the proof of Lemma 8 in (28). For the reader's convenience, we have included detailed proof. Let $(u_k)_{k\in\mathbb{N}} \subset \mathbf{W}$ be a Palais–Smale sequence for $\mathcal{K}$ at the level $c \in \mathbb{R}$, that is,

$$\mathcal{K}(u_k) = c + o_k(1) \quad \text{and} \quad \mathcal{K}'(u_k) = o_k(1).$$

So, for any $\varphi \in \mathbf{W}$, we write

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_k(x)-u_k(y)|^{p-2}\big(u_k(x)-u_k(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s_1 p}}\,dxdy \\
&+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_k(x)-u_k(y)|^{q-2}\big(u_k(x)-u_k(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s_2 q}}\,dxdy \\
&- \lambda\int_\Omega a_p(x)(u_k^+)^{p-1}\varphi\,dx = o_k(1). \qquad (2.2)
\end{aligned}$$

Claim 1. *The sequence $(u_k)_{k\in\mathbb{N}}$ is bounded in* $\mathbf{W}$.

To see this, we argue by contradiction, let us suppose:

$$\|u_k\|_{\mathbf{W}} \to \infty, \quad \text{as} \quad k \to \infty.$$

Setting

$$w_k := \frac{u_k}{\|u_k\|_{\mathbf{W}}}.$$

Then, there exists a sub-sequence such that $w_k \rightharpoonup w$ in $\mathbf{W}$ and $w_k \to w$ in $\mathrm{L}^r(\Omega)$ for $1 \le r < p^*_{s_1}$.

• Firstly, we can deduce that $w \ge 0$ in Ω.

Indeed, we choose $(u_k)^-$ as test function in (2.2) and by using the following inequality

$$\left|x^- - y^-\right|^l \le \left|x-y\right|^{l-2}(x^- - y^-)(x-y), \quad \text{for}\, x, y \in \mathbb{R} \quad \text{and} \quad l > 1$$

we obtain

$$\|(u_k)^-\|^p_{\mathrm{W}^{s_1,p}_0(\Omega)} + \|(u_k)^-\|^q_{\mathrm{W}^{s_2,q}_0(\Omega)} = \left\langle \mathscr{K}'(u_k), (u_k)^- \right\rangle \le o_k(1)\,\|(u_k)^-\|_{\mathbf{W}} \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality action. Since $1 < q < p$, we infer that $\|(w_k)^-\|_{\mathbf{W}} \to 0$, as $k \to \infty$. To see that, we need to show $\{(u_k)^-\}_k$ is bounded in $\mathbf{W}$. For this, by contradiction, we have the following three Alternatives:

Alternative 1: $\|(u_k)^-\|_{\mathrm{W}^{s_1,p}_0(\Omega)} \to +\infty$ and $\|(u_k)^-\|_{\mathrm{W}^{s_2,q}_0(\Omega)} \to +\infty$.

Then, for k large enough, we have $\|(u_k)^-\|_{\mathrm{W}^{s_1,p}_0(\Omega)} > 1$. By use the following inequality

$$a^q + b^q \ge \mathrm{C}_q(a+b)^q, \quad \forall\, a, b \ge 0, \forall\, q > 1,$$

and (2.3), we obtain:

$$\mathrm{C}_q\,\|(u_k)^-\|^{q-1}_{\mathbf{W}} \le o_k(1)$$

and this gives a contradiction.

Alternative 2: $\|(u_k)^-\|_{\mathrm{W}^{s_1,p}_0(\Omega)} \to +\infty$ and $\|(u_k)^-\|_{\mathrm{W}^{s_2,q}_0(\Omega)}$ is bounded.

From (2.3) we have

$$1 \le o_k(1)\left(\|(u_k)^-\|^{1-p}_{\mathrm{W}^{s_1,p}_0(\Omega)} + \frac{\|(u_k)^-\|_{\mathrm{W}^{s_2,q}_0(\Omega)}}{\|(u_k)^-\|^p_{\mathrm{W}^{s_1,p}_0(\Omega)}}\right).$$

Since $p > 1$ and passing to the limit as $k \to 0$, we obtain that $1 \le 0$ and this is a contradiction.

Alternative 3: $\|(u_k)^-\|_{\mathrm{W}^{s_2,q}_0(\Omega)} \to +\infty$ and $\|(u_k)^-\|_{\mathrm{W}^{s_1,p}_0(\Omega)}$ is bounded.

Symmetrically to Alternative 2.

Then, we obtain:

$$\|(w_k)^-\|_{\mathbf{W}} = \frac{\|(u_k)^-\|_{\mathbf{W}}}{\|u_k\|_{\mathbf{W}}} \to 0 \quad \text{as} \quad k \to \infty.$$

This implies that $(w_k)^+ \to w$, which yields $w \ge 0$ a.e. in Ω.

• Secondly, we can infer that $w \not\equiv 0$ in Ω.

Indeed, by taking $\phi = \dfrac{w_k - w}{\|u_n\|_{\mathbf{W}}^{p-1}}$ as test function in (2.2), we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|w_k(x)-w_k(y)\right|^{p-2}\left(w_k(x)-w_k(y)\right)\left((w_n-w)(x)-(w_n-w)(y)\right)}{|x-y|^{N+s_1p}}dxdy$$
$$+\frac{1}{\|u_k\|_{\mathbf{W}}^{p-q}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|w_k(x)-w_k(y)\right|^{q-2}\left(w_k(x)-w_k(y)\right)\left((w_k-w)(x)-(w_k-w)(y)\right)}{|x-y|^{N+s_2q}}dxdy$$
$$-\lambda\int_\Omega a_p(x)w_k^{p-1}(w_n-w)dx = o_k(1).$$

On the other hand, by using the Hölder's inequality, we have:

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|w_k(x)-w_k(y)\right|^{q-2}\left(w_k(x)-w_k(y)\right)\left((w_k-w)(x)-(w_k-w)(y)\right)}{|x-y|^{N+s_2q}}dxdy\right|$$
$$\leq \|w_k\|_{\mathbf{W}}^{q-1}\|w_k-w\|_{\mathbf{W}} < c$$

and

$$\left|\int_\Omega a_p(x)w_k^{p-1}(w_n-w)dx\right| \leq c\left\|a_p\right\|_{L^\infty(\Omega)}\|w_k\|_{\mathbf{W}}^{p-1}\|w_k-w\|_{L^p(\Omega)}$$

where $c>0$ is a constant.

Since w_k is bounded in $\mathbf{W}$, $w_k \to w$ in $L^p(\Omega)$ and passing to the limit as $k\to\infty$, we deduce that:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|w_k(x)-w_k(y)\right|^{p-2}\left(w_k(x)-w_k(y)\right)\left((w_k-w)(x)-(w_k-w)(y)\right)}{|x-y|^{N+s_1p}}dxdy = o_k(1).$$

Then $w_k \to w$ in $W_0^{s_1,p}(\Omega)$ (by using **(S)** property of fractional $p-$Laplace operator on $W_0^{s_1,p}(\Omega)$). Since $\|w_k\|_{\mathbf{W}} = 1$, we deduce $w \not\equiv 0$ in Ω.

- Thirdly, we show that w is an eigenfunction of $(-\Delta)_p^{s_1}$ with weight a_p in $W_0^{s_1,p}(\Omega)$.

From (2.2), we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|w_k(x)-w_k(y)\right|^{p-2}\left(w_k(x)-w_k(y)\right)\left(\varphi(x)-\varphi(y)\right)}{|x-y|^{N+s_1p}}dxdy$$
$$+\frac{1}{\|u_k\|_{\mathbf{W}}^{p-q}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|w_k(x)-w_k(y)\right|^{q-2}\left(w_k(x)-w_k(y)\right)\left(\varphi(x)-\varphi(y)\right)}{|x-y|^{N+s_2q}}dxdy$$
$$-\lambda\int_\Omega a_p(x)w_k^{p-1}\varphi dx = \frac{o_n(1)}{\|u_k\|_{\mathbf{W}}^{p-1}}.$$

Passing to the limit as $k\to\infty$, we deduce that w is a non-negative and non-trivial solution of the following problem:

$$(-\Delta)_p^{s_1}w = \lambda\, a_p(x)w^{p-1}, \quad w=0, \quad \text{in } \mathbb{R}^N\setminus\Omega.$$

For regularity and positivity of weak solutions we use the Moser iteration process. Firstly, we claim that all weak solutions to the problem (**P**) belongs to $L^\infty(\Omega)$. To this aim, we follow the approach of (11, Theorem 3.2). Precisely, let $u_0 \in \mathbf{W}$ be a weak solution to (**P**). Setting

$$v_0 = \frac{u_0}{\rho\|u_0\|_{L^q(\Omega)}} \quad \text{where} \quad \rho = \max\left\{1, \|u_0\|_{L^q(\Omega)}^{-1}\right\}.$$

Noting that $v_0 \in \mathbf{W}$ and $\|v_0\|_{L^q(\Omega)} = \rho^{-1}$. Now, we consider the function w_k defined as follows

$$\begin{cases} w_k(x) & := (v_0(x) - (1-2^{-k}))^+ \quad \text{for } k \in \mathbb{N} \\ w_0(x) & = (v_0(x))^+. \end{cases}$$

We first state the following straightforward observations about $w_k(x)$:

$$w_k \in \mathbf{W} \quad \text{and} \quad w_k = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega,$$

and

$$\begin{cases} 0 \le w_{k+1}(x) \le w_k(x) \quad \text{a.e. in } \mathbb{R}^N, \\ v_0(x) < (2^{k+1}+1)\, w_k(x) \quad \text{for } x \in \{w_{k+1} > 0\}. \end{cases} \tag{2.4}$$

Also the inclusion

$$\{w_{k+1} > 0\} \subseteq \{w_k > 2^{-(k+1)}\} \quad \text{holds for all } k \in \mathbb{N}. \tag{2.5}$$

Now, we set $V_k := \|w_k\|^q_{L^q(\Omega)}$.

Claim 2. $V_k \to 0$ *as* $k \to \infty$.

Indeed, since $1 < q < p$, $\rho\,\|u_0\|_{L^q(\Omega)} \ge 1$ and by using the following inequality

$$\left|x^+ - y^+\right|^l \le \left|x-y\right|^{l-2}(x^+ - y^+)(x-y), \quad \text{for } x, y \in \mathbb{R} \quad \text{and} \quad l > 1$$

we obtain

$$\begin{aligned} \|w_{k+1}\|^q_{W_0^{s_2,q}(\Omega)} &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_{k+1}(x)-w_{k+1}(y)|^q}{|x-y|^{N+s_2 q}}\,dx\,dy \\ &\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_{k+1}(x)-w_{k+1}(y)|^p}{|x-y|^{N+s_1 p}}\,dx\,dy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_{k+1}(x)-w_{k+1}(y)|^q}{|x-y|^{N+s_2 q}}\,dx\,dy \\ &\le (\rho\,\|u_0\|_{L^q(\Omega)})^{1-p} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^{p-2}(w_{k+1}(x)-w_{k+1}(y))(u_0(x)-u_0(y))}{|x-y|^{N+s_1 p}}\,dx\,dy \\ &\quad + (\rho\,\|u_0\|_{L^q(\Omega)})^{1-q} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^{q-2}(w_{k+1}(x)-w_{k+1}(y))(u_0(x)-u_0(y))}{|x-y|^{N+s_2 q}}\,dx\,dy \\ &\le (\rho\,\|u_0\|_{L^q(\Omega)})^{1-q}\,\|a_q\|_{L^\infty(\Omega)} \int_{\{w_{k+1}>0\}} u_0^{q-1} w_{k+1}\,dx \\ &= \|a_q\|_{L^\infty(\Omega)} \int_{\{w_{k+1}>0\}} v_0^{q-1} w_{k+1}\,dx. \end{aligned}$$

This fact combined with (2.4), we obtain

$$\|w_{k+1}\|^q_{W_0^{s_2,q}(\Omega)} \le C_1 (2^{k+1}+1)^{q-1} V_k$$

where $C_1 > 0$ is a constant. On the other hand, by the Hölder's inequality and fractional Sobolev imbeddings (9, Theorem 6.5), we obtain

$$V_{k+1} = \int_{\{w_{k+1}>0\}} w_{k+1}^q\,dx \le C_2 \|w_{k+1}\|^q_{W_0^{s_2,q}(\Omega)} \left|\{w_{k+1} > 0\}\right|^{1-\frac{q}{q^*_{s_2}}} \tag{2.6}$$

where $C_2 > 0$ is a constant. Now, from (2.5) we have

$$V_k = \int_\Omega w_k^q\,dx \ge \int_{\{w_{k+1}>0\}} w_k^q\,dx \ge 2^{-(k+1)q} |\{w_{k+1} > 0\}|.$$

Hence, we can write the inequality (2.6) as follows:

$$V_{k+1} \leq C^k V_k^{1+\alpha}, \quad \text{for all } k \in \mathbb{N} \tag{2.7}$$

for a suitable constant $C > 1$ and $\alpha = \dfrac{s_2 q}{N}$. This implies that

$$V_k \leq \frac{\eta^k}{\rho^q}, \quad \text{for all } n \in \mathbb{N} \tag{2.8}$$

where $\eta = C^{-\frac{1}{\alpha}}$ and $\rho = \max\left\{1, \|u_0\|^{-1}_{L^q(\Omega)}, C^{\frac{1}{q\alpha^2}}\right\}$.

Indeed, by induction arguments, we have:

- Clearly $V_0 = \|v_0^+\|^q_{L^q(\Omega)} \leq \|v_0\|^q_{L^q(\Omega)} = \dfrac{1}{\rho^q}$.
- Now we can assume that (2.8) holds for some $k \in \mathbb{N}$, and by use (2.7) we get

$$V_{k+1} \leq C^k V_k^{1+\alpha} \leq \frac{\eta^{k+1}}{\rho^q}.$$

Since $\eta \in (0,1)$, we deduce that

$$\lim_{k\to\infty} V_k = 0 \tag{2.9}$$

Since w_k converges to $(v_0 - 1)^+$ a.e. in $\mathbb{R}^N$, from (2.9) we infer that $w_k \to 0$ a.e. in Ω. Hence $v_0 \leq 1$ a.e. in Ω, which implies $\|u_0\|_{L^\infty(\Omega)} \leq \rho\|u_0\|_{L^q(\Omega)}$. Then, we deduce that $u_0 \in L^\infty(\Omega)$. On the other hand, from Remark (2.1), we infer that $u_0 \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for any $\epsilon_0 > 0$ there exists a constant $K = K(\epsilon_0) > 0$ such that

$$K^{-1} d^{s_1+\epsilon_0} \leq u_0 \leq K d^{s_1-\epsilon_0} \text{ in } \Omega.$$

On the other hand, from Remark (2.1), we infer that $w > 0$ in Ω. Then, from (11, Theorem 4.1) we deduce that $\lambda_{1,s_1,p}(a_p) = \lambda > \lambda_{1,s_1,p}(a_p)$, and this gives a contradiction. Consequently, $(u_k)_{k\in\mathbb{N}}$ is bounded in $\mathbf{W}$. Then there exists a sub-sequence, such that $u_k \rightharpoonup u$ in $\mathbf{W}$ and $u_k \to u$ in $L^p(\Omega)$.

Now, by taking $\varphi = u_k - u$ as test function in (2.2), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_k(x)-u_k(y)|^{p-2}\big(u_k(x)-u_k(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s_1p}}dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_k(x)-u_k(y)|^{q-2}\big(u_k(x)-u_k(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s_2q}}dxdy \\
&\qquad - \lambda\int_\Omega a_p(x)u_k^{p-1}(u_n-u)dx = o_k(1).
\end{aligned}$$

Using the Hölder's inequality, we have

$$\left|\int_\Omega a_p(x)u_k^{p-1}\varphi dx\right| \leq c\left\|a_p\right\|_{L^\infty(\Omega)}\|u_k\|^{p-1}_{\mathbf{W}}\|u_k-u\|_{L^p(\Omega)} \to 0 \quad \text{as} \quad k\to\infty$$

where $c > 0$ is a constant. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_k(x)-u_k(y)|^{p-2}\big(u_k(x)-u_k(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s_1p}}dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_k(x)-u_k(y)|^{q-2}\big(u_k(x)-u_k(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s_2q}}dxdy = o_k(1).
\end{aligned} \tag{2.10}$$

On the other hand, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2q}}dxdy = o_k(1). \tag{2.11}$$

Subtracting (2.10)-(2.11) and using inequalities in (33, Section 10)Lindqvist2017Notes, yields

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))((u_k-u)(x)-(u_k-u)(y))}{|x-y|^{N+s_1p}}dxdy = o_k(1)$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_k(x)-u_k(y)|^{q-2}(u_k(x)-u_k(y))((u_k-u)(x)-(u_k-u)(y))}{|x-y|^{N+s_2q}}dxdy = o_k(1).$$

Then by using again **(S)** property of $(-\Delta)_p^{s_1}$ and $(-\Delta)_q^{s_2}$ on $W_0^{s_1,p}(\Omega)$ and $W_0^{s_2,q}(\Omega)$, respectively, we infer that $u_k \to u$ in $\mathbf{W}$.

Step 2: The functional $\mathcal{K}$ possesses the Mountain Pass geometry.

Now, we show that the functional $\mathcal{K}$ satisfies the following two conditions:

(1) There exist α,ρ such that $\mathcal{K}(u)\geq\alpha$ for $\|u\|_{\mathbf{W}}=\rho$. Indeed, let $u\in\mathbf{W}$ where $\|u\|_{\mathbf{W}}=\rho\in(0,1)$. Then, by taking into account that $\|u\|_{W_0^{s_2,q}(\Omega)}<1$ and using $1<q<p$, we obtain that $\|u\|^p_{W_0^{s_2,q}(\Omega)}\leq \|u\|^q_{W_0^{s_2,q}(\Omega)}$. Now, we distinguish two cases:

Case 1: For $u\notin\mathbf{W}_d$ with $d=2\lambda\|a_p\|_{L^\infty(\Omega)}$ (see Lemma 5.3), by using the following inequality:

$$a^p+b^p\geq C_p(a+b)^p,\quad \forall a,b\geq 0,\ \forall p>1,$$

we obtain:

$$\mathcal{K}(u)\geq\frac{1}{p}\left(\|u\|^p_{W_0^{s_1,p}(\Omega)}+\|u\|^p_{W_0^{s_2,q}(\Omega)}\right)-\frac{\lambda\|a_p\|_{L^\infty(\Omega)}}{p}\|u^+\|^p_{L^p(\Omega)}$$
$$\geq\frac{C_p}{2p}\|u\|^p_{\mathbf{W}}+\frac{d}{2p}\|u\|^p_{L^p(\Omega)}-\frac{\lambda\|a_p\|_{L^\infty(\Omega)}}{p}\|u^+\|^p_{L^p(\Omega)}\geq\frac{C_p}{2p}\|u\|^p_{\mathbf{W}}\qquad\text{(see Lemma 5.3).}$$

Case 2: For $u\in\mathbf{W}_d$, by using definition of $\lambda_{1,s_1,p}(a_p)$ and $\lambda_{1,s_2,q}(1)$ together with $\lambda>\lambda_{1,s_1,p}(a_p)$, we get

$$\mathcal{K}(u)\geq\frac{1}{p}\left(1-\frac{\lambda}{\lambda_{1,s_1,p}(a_p)}\right)\|u\|^p_{W_0^{s_1,p}(\Omega)}+\frac{1}{q}\|u\|^q_{W_0^{s_2,q}(\Omega)}$$
$$\geq\frac{1}{p}\left(1-\frac{\lambda}{\lambda_{1,s_1,p}(a_p)}\right)\|u\|^p_{\mathbf{W}}+\frac{\lambda_{1,s_2,q}(1)}{qC(d)^q}\|u\|^q_{\mathbf{W}}\qquad\text{(see Lemma 5.3).}$$

Since $1<q<p$, we can find $\alpha>0$ such that $\mathcal{K}(u)\geq\alpha$ for $\|u\|_{\mathbf{W}}=\rho$ small enough.

(2) There exists $\phi\in\mathbf{W}$ with $\|\phi\|_{\mathbf{W}}>\rho$ and $\mathcal{K}(\phi)<0$. Indeed, for any $t>0$, we have:

$$\mathcal{K}(t\phi_{1,s_1,p}(a_p))=\frac{t^p}{p}\|\phi_{1,s_1,p}(a_p)\|^p_{W_0^{s_1,p}(\Omega)}+\frac{t^q}{q}\|\phi_{1,s_1,p}(a_p)\|^q_{W_0^{s_2,q}(\Omega)}-\frac{\lambda t^p}{p}\|\phi_{1,s_2,q}(a_p)\|^p_{L^p(\Omega)}$$
$$=t^p\left[\frac{t^{q-p}}{q}\|\phi_{1,s_1,p}(a_p)\|^q_{W_0^{s_2,q}(\Omega)}+\frac{\lambda_{1,s_1,p}(a_p)-\lambda}{p}\|\phi_{1,s_2,q}(a_p)\|^p_{L^p(\Omega)}\right].$$

By using $p > q$ and $\lambda > \lambda_{1,s_1,p}(a_p)$, we obtain

$$\mathcal{K}(t\phi_{1,s_1,p}(a_p)) \to -\infty \quad \text{and} \quad \rho < \left\| t\phi_{1,s_1,p}(a_p) \right\|_{\mathbf{W}} \to +\infty \quad \text{as } t \to +\infty$$

Finally, we can apply the mountain pass Theorem (1) to deduce that, for any $\lambda > \lambda_{1,s_1,p}(a_p)$ there exists $u_0 \in \mathbf{W}$ such that $\mathcal{K}(u_0) = c$ and $\mathcal{K}'(u_0) = 0$, such that

$$c := \inf_{\gamma \in \Sigma} \max_{t \in [0,1]} \mathcal{K}\left(\gamma(t)\right)$$

where

$$\Sigma := \left\{\gamma \in C([0,1], \mathbf{W}); \quad \gamma(0) = 0, \gamma(1) = t\phi_{1,s_1,p}(a_p)\right\}.$$

Since $c \geq \alpha > 0 = \mathcal{K}(0)$, we deduce $u_0 \not\equiv 0$. Moreover, by a similar proof as in **Step 2** of the proof of **Case 1**, we obtain that $u_0 \in L^\infty(\Omega)$. From Remark (2.1), we infer that $u_0 \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for any $\epsilon_0 > 0$ there exists a constant $K = K(\epsilon_0) > 0$ such that

$$K^{-1} d^{s_1+\epsilon_0} \leq u_0 \leq K d^{s_1-\epsilon_0} \quad \text{in} \quad \Omega.$$

□

3 Non-local Rayleigh quotient

Theorem 3.1. *We set the following non-local Rayleigh quotient:*

$$\underline{\lambda}_{s,s*,r,r^*}(a) := \inf_{u \in W} \left\{ \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x)-u(y)\right|^r}{|x-y|^{N+sr}} dxdy + \frac{r}{r^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x)-u(y)\right|^{r*}}{|x-y|^{N+s^*r^*}} dxdy}{\displaystyle\int_\Omega a_r(x)u^r\,dx} \right\}.$$

where $r = p$ (or q), with $s = s_1$ (or s_2) if $r^* = q$ (or p), with $s = s_2$ (or s_1, respectively). Then, $\underline{\lambda}_{s,s^*,r,r^*}(a_r) = \lambda_{1,s,r}(a_r)$. In addition, the infimum is not attained.

Proof. First, we have

$$\begin{aligned}\underline{\lambda}_{s,s*,r,r^*}(a_r) &= \inf_{u \in \mathbf{W}} \left\{ \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x)-u(y)\right|^r}{|x-y|^{N+sr}} dxdy + \frac{r}{r^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x)-u(y)\right|^{r*}}{|x-y|^{N+s^*r^*}} dxdy}{\displaystyle\int_\Omega a_r(x)u^r\,dx} \right\} \\ &\geq \lambda_{1,s,r}(a_r) > -\infty. \end{aligned} \tag{2.12}$$

Hence $\underline{\lambda}_{s,s*,r,r^*}(a_r)$ exists.

Next, we will follow the same idea in (28, Proposition 4). For that, let $t > 0$, $v = t\phi_{1,s,r}(a_r)$ and by the definition of $\underline{\lambda}_{s,s*,r,r^*}(a_r)$ and $\lambda_{1,s,r}(a_r)$, we have that

$$\begin{aligned}\underline{\lambda}_{s,s*,r,r^*}(a_r) &\leq \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|v(x)-v(y)\right|^r}{|x-y|^{N+sr}} dxdy + \frac{r}{r^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|v(x)-v(y)\right|^{r*}}{|x-y|^{N+s^*r^*}} dxdy}{\displaystyle\int_\Omega a_r(x)v^r(x)\,dx} \\ &= \lambda_{1,s,r}(a_r) + \frac{r\,t^{r^*-r}}{r^*} \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|\phi_{1,s,r}(a_r)(x)-\phi_{1,s,r}(a_r)(y)\right|^{r*}}{|x-y|^{N+s^*r^*}} dxdy}{\displaystyle\int_\Omega a_r(x)\phi_{1,s,r}(a_r)^r\,dx}\end{aligned} \tag{2.13}$$

Now, we distinguish two cases:

Case 1: $r = p$ with $s = s_1$ and $r^* = q$ with $s^* = s_2$.

Since $1 < q < p$ and passing to the limit as $t \to +\infty$ in (2.13), we deduce that

$$\underline{\lambda}_{s_1,s_2,p,q}(a_p) \leq \lambda_{1,s_1,p}(a_p).$$

Case 2: $r = q$ with $s = s_2$ and $r^* = p$ with $s^* = s_1$.

Passing to the limit as $t \to 0$ in (2.13), and using again the fact that $1 < q < p$, we get

$$\underline{\lambda}_{s_2,s_1,q,p}(a_q) \leq \lambda_{1,s_2,q}(a_q).$$

Then, from (2.12), we can see that

$$\underline{\lambda}_{s,s^*,r,r^*}(a_r) = \lambda_{1,s,r}(a_r).$$

On the other hand, we suppose by contradiction that there exists a function $u_0 \in \mathbf{W}$, such that

$$\underline{\lambda}_{s,s*,r,r^*}(a_r) = \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u_0(x)-u_0(y)\right|^r}{|x-y|^{N+sr}}dxdy + \frac{r}{r^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u_0(x)-u_0(y)\right|^{r*}}{|x-y|^{N+s^*r^*}}dxdy}{\displaystyle\int_\Omega a_r(x)u_0^r\,dx}$$

$$\geq \lambda_{1,s,r}(a_r) + \frac{\displaystyle\frac{r}{r^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u_0(x)-u_0(y)\right|^{r*}}{|x-y|^{N+s^*r^*}}dxdy}{\displaystyle\int_\Omega a_r(x)u_0^r\,dx} > \lambda_{1,s,r}(a_r) = \underline{\lambda}_{s,s*,r,r^*}(a_r)$$

which is a contradiction. □

Conclusion

He main goals of the present work are to discuss non-existence, existence, uniqueness, and Hölder regularity results, in the problem. More precisely, by using the mountain pass theorem, this is done by proving two conditions represented by the Plais smale condition and Geometry condition, with verification of regularity and positivily of the weak solution.

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