

Recursive prediction error identification of fractional order models

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ABSTRACT

This paper deals with time domain identification of fractional order systems. A new identification technique is developed providing recursive parameters estimation of fractional order models. The identification model is defined by a generalized ARX structure obtained by discretization of a continuous fractional order differential equation. The parameters are then estimated using the recursive least squares and the recursive instrumental variable algorithms extended to fractional order cases. Finally, the quality of the proposed technique is illustrated and compared through the identification of simulated fractional order systems.

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1. Introduction

The fractional calculus based on generalized fractional order integro-differential operators is a very old topic in mathematics. At the end of the 19th century, Liouville and Riemann introduced the first definition of the fractional derivative. However, this field started to be interesting for engineers only in the late 1960s, especially when it was observed that the description of some real systems is more accurate when the fractional derivative is used. For example, modeling the behavior of some materials like polymers and rubber [1], the relaxation phenomena of organic dielectric materials such as semicrystalline polymers [2], the electrochemical processes and flexible robot [3], traffic in information networks [4]. More examples, physical interpretations and areas of applications of fractional calculus are to be found in [5–11].

Due to their long memory behavior, the identification of fractional order models is more difficult compared with those of the integer order models. Several algorithms based frequency domain were proposed to solve this problem [3]. In [12–14,17], time domain identification techniques of fractional order discrete time models based nonrecursive least squares are presented. In several cases, these techniques can have serious computational problems due to the matrix singularity. Also, in some applications, it might be necessary to estimate the model online as in adaptive control, adaptive filtering or adaptive prediction. It might be necessary to investigate possible time variation in the system's properties during the data acquisition. For these purposes, a recursive identification approach is an interesting alternative.

The new technique proposed in this paper extends, to fractional order differentiation, the classical recursive least squares approach (RLS). The identification method is based on equation error using fractional ARX models obtained by discretization of continuous fractional order differential equation (FODE).

The paper is organized as follows: In Section 2, a mathematical background is presented. In Section 3, fractional order ARX model is defined and a new identification technique is described using RLS and recursive instrumental variable (RIV)

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methods. Finally, in Section 4, the quality of the proposed methods is illustrated and compared through the identification of simulated fractional order models.

2. Mathematical background

2.1. Brief review of fractional differentiation

The concept of differentiation to an arbitrary order $\alpha \in \mathbb{R}_+^*$ was defined in the 19th century by Riemann and Liouville as [18]:

$$D^\alpha = \left(\frac{d}{dt} \right)^\alpha \quad (1)$$

The α fractional derivative of $x(t)$ is defined as being an integer derivative of order $m = [\alpha] + 1$ ($[\cdot]$ stands for the floor operator) of a non-integer integral of order $(m - \alpha)$:

$$D^\alpha x(t) = D^m (I^{m-\alpha})x(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt} \right)^m \int_0^t \frac{x(\tau) d\tau}{(t-\tau)^{1-(m-\alpha)}} \quad (2)$$

where $t > 0, \alpha \in \mathbb{R}_+^*$, and the Euler's function Γ is defined as:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \quad (3)$$

A discrete-time definition of fractional derivative was proposed by Grunwald (1867) as:

$$D^\alpha x(t) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor t/h \rfloor} (-1)^k \binom{\alpha}{k} x(t - kh) \quad (4)$$

where h is the sampling time, and the Newton's binomial $\binom{\alpha}{k}$ is generalized to non-integer orders using the Euler's function as:

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha + 1)}{\Gamma(k + 1)\Gamma(\alpha - k + 1)} \quad (5)$$

As Newton's binomial $\binom{\alpha}{k}$ does not converge rapidly to zero with k when α is non integer, fractional operator are known to have a long memory behavior. For real implementation, according to the short memory principle given in [19], Eq. (4) can be approximated using only the recent past values of $x(t)$ as:

$$D^\alpha x(t) = \frac{1}{h^\alpha} \sum_{k=0}^N (-1)^k \binom{\alpha}{k} x(t - kh) \quad (6)$$

where $N = \lfloor T/h \rfloor$ is the number of the approximation addends.

Approximation (6) is known as the simplest tool used in time-domain simulations of fractional systems.

2.2. Fractional order models

Fractional order systems can be represented by the generalized fractional differential equation given by:

$$y(t) + \sum_{i=1}^{n_a} a_i D^{\alpha_i} y(t) = \sum_{j=0}^{n_b} b_j D^{\beta_j} u(t) \quad (7)$$

where $\alpha_1 < \alpha_2 < \dots < \alpha_{n_a}$, $\beta_0 < \beta_1 < \dots < \beta_{n_b}$ are non-integer positive numbers and $(a_i, b_j) \in \mathbb{R}^2$, $i = 1, 2, \dots, n_a$, $j = 0, 1, \dots, n_b$.

Providing the system is relaxed at $t = 0$, the Laplace transform of (7) can be given by [18]

$$G(s) = \frac{\sum_{j=0}^{n_b} b_j s^{\beta_j}}{1 + \sum_{i=1}^{n_a} a_i s^{\alpha_i}} \quad (8)$$

3. Fractional order model identification

Consider the class of stable SISO given by Eq. (7) and consider the estimation of coefficients a_i, b_j where the orders α_i, β_j are supposed known by the user (as is the case for many thermal systems [8]). For parameter estimation using prediction error

method, model (7) should be firstly approximated by a difference equation. Using the Grunwald–Letnikov approximation given by Eq. (6), it can be written in a linear regression form as [17]:

$$y(k+1) = - \sum_{i=1}^{n_a} a'_i Y_i(k) + \sum_{j=0}^{n_b} b'_j U_j(k) \tag{9}$$

where

$$a'_i = \frac{\frac{\alpha_i}{h^{\alpha_i}}}{1 + \sum_{k=1}^{n_a} \frac{\alpha_k}{h^{\alpha_k}}}, \quad b'_j = \frac{\frac{\beta_j}{h^{\beta_j}}}{1 + \sum_{k=1}^{n_b} \frac{\alpha_k}{h^{\alpha_k}}}, \quad 1 \leq i \leq n_a, \quad 0 \leq j \leq n_b, \tag{10}$$

$$Y_i(k) = \sum_{j=1}^N (-1)^j \binom{\alpha_i}{j} y(k+1-j), \quad U_j(k) = \sum_{i=0}^N (-1)^i \binom{\beta_j}{i} u(k+1-i) \tag{11}$$

Consider observed data $u(t)$ and $y^*(t) = y(t) + p(t)$ where $p(t)$ is a perturbation signal, then, regression (9) can be rewritten as:

$$y^*(k+1) = - \sum_{i=1}^{n_a} a'_i Y_i^*(k) + \sum_{j=0}^{n_b} b'_j U_j(k) + e(k+1) \tag{12}$$

where

$$Y_i^*(k) = \sum_{j=1}^N (-1)^j \binom{\alpha_i}{j} y^*(k+1-j) \tag{13}$$

and

$$e(k+1) = p(k+1) + \sum_{i=1}^{n_a} a'_i \sum_{j=1}^N (-1)^j \binom{\alpha_i}{j} p(k+1-j) \tag{14}$$

Model (12) is linear versus coefficients a'_i and b'_j , ($i = 1, \dots, n_a, j = 0, \dots, n_b$), it can be represented by:

$$y(k+1) = \theta^T \phi(k) + e(k+1) \tag{15}$$

where

$$\theta = [a'_1, \dots, a'_{n_a}, b'_0, \dots, b'_{n_b}]^T \tag{16}$$

and

$$\phi(k) = [-Y_1^*(k), \dots, -Y_{n_a}^*(k), U_0(k), \dots, U_{n_b}(k)]^T \tag{17}$$

3.1. Parameter estimation using recursive least squares algorithm (RLS)

Regression (15) represents an accurate discrete-time representation of the considered system. However, in this expression the parameter vector θ is supposed to be unknown, where the fractional orders α_i and β_j , ($i = 1, \dots, n_a, j = 0, \dots, n_b$), are supposed known (fixed) by the user. If the disturbance $e(k)$ is a white noise, we can consider the *a posteriori* output prediction $\hat{y}(k+1)$ given by:

$$\hat{y}(k+1) = \hat{\theta}_{k+1}^T \phi(k) \tag{18}$$

where $\phi(k)$ is given by (17) and

$$\hat{\theta}_{k+1} = [\hat{a}'_1, \dots, \hat{a}'_{n_a}, \hat{b}'_0, \dots, \hat{b}'_{n_b}]^T \tag{19}$$

The estimated vector $\hat{\theta}$ is obtained by minimizing the quadratic least squares criterion:

$$\hat{\theta}_k = \underset{\theta}{\operatorname{argmin}} \frac{1}{k} \sum_{i=1}^k [y(i) - \hat{y}(i, \theta)]^2 \tag{20}$$

Provided the inverse $[\sum_{i=1}^k \phi(i-1)\phi^T(i-1)]^{-1}$ exists, the solution of this problem can be given by the least-squares estimate (as in the classical integer-order models [15]):

$$\hat{\theta}_k = \left[\sum_{i=1}^k \phi(i-1)\phi^T(i-1) \right]^{-1} \sum_{i=1}^k \phi(i-1)y(i) \tag{21}$$

A recursive version of (21) can be given as [20]:

$$\begin{cases} \hat{\theta}_{k+1} = \hat{\theta}_k + F_k \phi(k) \varepsilon(k+1) \\ F_{k+1} = F_k - \frac{F_k \phi(k) \phi^T(k) F_k}{1 + \phi^T(k) F_k \phi(k)} \\ \varepsilon(k+1) = \frac{y(k+1) - \hat{\theta}_k^T \phi(k)}{1 + \phi^T(k) F_k \phi(k)} \end{cases} \quad (22)$$

where the adaptation gain matrix F_k is generally chosen to be started by:

$$F_0 = \frac{1}{\delta} I; \quad 0 < \delta \ll 1 \quad (23)$$

For non stationary systems with slow parameters variation, a forgetting factor λ , ($0 < \lambda < 1$) can be introduced. The recursive least squares algorithm will be [20]:

$$\begin{cases} \hat{\theta}_k = \hat{\theta}_k + F_k \phi(k) \varepsilon(k+1) \\ F_k = \frac{1}{\lambda} \left[F_k - \frac{F_k \phi(k) \phi^T(k) F_k}{\lambda + \phi^T(k) F_k \phi(k)} \right] \\ \varepsilon(k+1) = \frac{y(k+1) - \hat{\theta}_k^T \phi(k)}{1 + \phi^T(k) F_k \phi(k)} \end{cases} \quad (24)$$

3.2. Parameter estimation using recursive instrumental variable algorithm (RIV)

As in the classical case for integer-order models, Cois et al. [14] showed that the least squares estimator (21) of non-integer models is biased in the presence of noisy output. A recursive instrumental variable method can then used as: [21]:

$$\begin{cases} \hat{\theta}_{k+1} = \hat{\theta}_k + F_k \psi(k) \varepsilon(k+1) \\ F_{k+1} = F_k - \frac{F_k \psi(k) \psi^T(k) F_k}{1 + \psi^T(k) F_k \psi(k)} \\ \varepsilon(k+1) = \frac{y(k+1) - \hat{\theta}_k^T \psi(k)}{1 + \psi^T(k) F_k \psi(k)} \end{cases} \quad (25)$$

where ψ is the *instrumental variable regression vector*, of an *auxiliary fractional model* which has a similar dynamics as those of the identified system. A selected model in this context can be given by the following discrete-time regression:

$$y^{IV}(k+1) = - \sum_{i=1}^{n_a} \hat{a}'_i Y_i^{IV}(k) + \sum_{j=0}^{n_b} \hat{b}'_j U_j(k) \quad (26)$$

where \hat{a}'_i and \hat{b}'_j ($i = 1, \dots, n_a$; $j = 0, \dots, n_b$) are the estimated parameters in step k , and

$$Y_i^{IV}(k) = \sum_{j=1}^N (-1)^j \binom{\alpha_i}{j} y^{IV}(k+1-j) \quad (27)$$

The *instrumental variable regression vector* ψ is then given by:

$$\psi(k) = \left[-Y_1^{IV}(k), \dots, -Y_{n_a}^{IV}(k), U_0(k), \dots, U_{n_b}(k) \right]^T \quad (28)$$

Using Eq. (10), parameters a_i ($i = 1, \dots, n$) of the continuous-time fractional order model can be obtained by solving the following equation:

$$\begin{pmatrix} \frac{a'_1 - 1}{h^{\alpha'_1}} & \frac{a'_1}{h^{\alpha'_2}} & \dots & \frac{a'_1}{h^{\alpha'_{n_a}}} \\ \frac{a'_2}{h^{\alpha'_1}} & \frac{a'_2 - 1}{h^{\alpha'_2}} & \dots & \frac{a'_2}{h^{\alpha'_{n_a}}} \\ \vdots & \vdots & \dots & \vdots \\ \frac{a'_{n_a}}{h^{\alpha'_1}} & \frac{a'_{n_a}}{h^{\alpha'_2}} & \dots & \frac{a'_{n_a} - 1}{h^{\alpha'_{n_a}}} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n_a} \end{pmatrix} = \begin{pmatrix} -a'_1 \\ -a'_2 \\ \vdots \\ -a'_{n_a} \end{pmatrix}$$

Thereby, the coefficients b_j ($j = 0, 1, \dots, n_b$) can be obtained using the following expression:

$$b_j = b'_j + b'_j \sum_{i=1}^{n_a} a_i h^{\beta_j - \alpha_i} \quad (29)$$

4. Application examples

In this section, we present numerical examples to illustrate the effectiveness of the proposed methods compared to the least squares (LS) method used in [22]. We illustrate our algorithms using a simple examples involving the following fractional order model:

$$\mathbf{M} : a_2 D^{1.5} y(t) + a_1 D^{0.5} y(t) + y(t) = bu(t) + e(t) \quad (30)$$

where $e(t)$ is a noise signal.

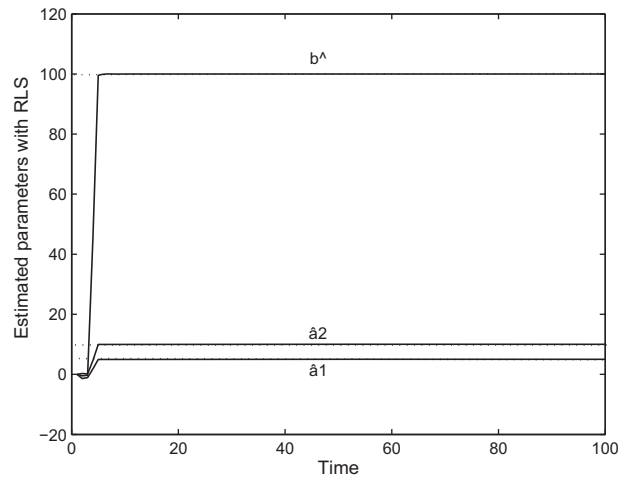


Fig. 1. Evolution of the estimated parameters using RLS without noise.

The sampled model is obtained by discretization of **M** using the Grunwald–Letnikov approximation method with a sampling period $h = 0.5$ s and a number of addends $N = 100$.

Data were generated by simulation of the sampled model with parameter values $a_1 = 5$, $a_2 = 10$ and $b = 100$. The input is a Pseudo Random Binary Signal (PRBS). All the parameters are initialized to zero. The adaptation gain F is initialized as: $F_0 = 1000I$ and the forgetting factor is fixed to $\lambda = 0.95$.

According to different types of the noise signal $e(t)$, three cases can be analyzed:

4.1. Simulation without noise

Fig. 1 presents the evolution of the estimated parameters with RLS method. It can be seen that the estimate $\theta = [a_1, a_2, b]^T$ converged rapidly to the true parameters. Table 1 compare RLS and LS estimation results. it can be seen that the RLS algorithm restor the exact parameters of the simulated model.

4.2. Simulation with equation noise

The noise $e(t)$ is supposed to be white with Noise to Signal Ratio $NSR = \frac{\sqrt{\text{var}(e(t))}}{\sqrt{\text{var}(y(t))}} \times 100 = 45\%$.

Fig. 2 presents the evolution of the parameter estimate with RLS method. It can be seen that this method performs a quite good convergence.

The estimated parameters of the model for one run after 100 samples are given in Table 2 compared to those obtained using the LS method. It can be seen that the RLS method provides the best estimation.

4.3. Simulation with output noise

In this case, the simulated outputs $y(t)$ are corrupted by a white noise such that $y^*(t) = y(t) + p(t)$, where $p(t)$ is a white noise with Noise to Signal Ratio $NSR = \frac{\sqrt{\text{var}(e(t))}}{\sqrt{\text{var}(y(t))}} \times 100 = 45\%$.

Using the RIV method, unbiased parameter estimation can be derived.

Fig. 3 presents the evolution of the parameter estimate. It can be seen that RIV method performs a quite good convergence.

The estimated parameters for one run after 100 samples are given in Table 3. It can be seen that the RIV method provides a satisfactory estimation.

In order to validate the identified models using the proposed algorithms, the whiteness test $RN_{(e, \hat{e})}(i)$ (in the case of identification with RLS) and the uncorrelation test $RN_{(e, \hat{y})}(i)$ (in the case of identification with RIV) [20] are provided:

Table 1
RLS and LS estimations in deterministic environment.

		a_1	a_2	b
Real parameters		5	10	100
Estimated parameters	RLS	5	10	100
	LS	5.022	10.045	99.900

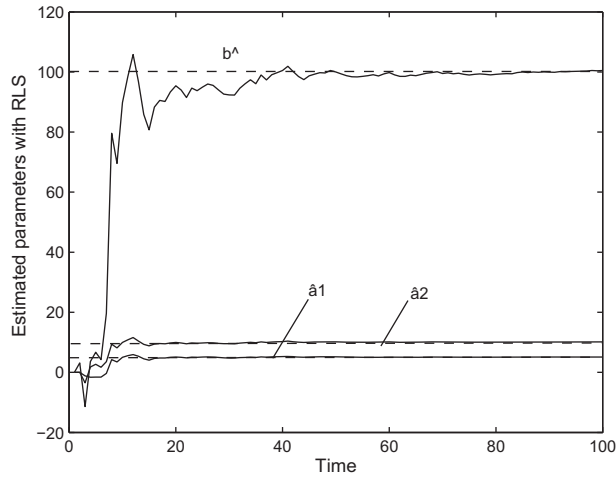


Fig. 2. Evolution of the estimated parameters using RLS (stochastic environment).

Table 2
RLS and LS estimations in stochastic environment.

		a_1	a_2	b
Real parameters	RLS	5.115	10.128	100.480
	LS	4.954	9.937	98.299

$$RN_{(\varepsilon,\varepsilon)}(i) = \frac{\sum_{k=0}^K \varepsilon(k)\varepsilon(k-i)}{\sum_{k=0}^K \varepsilon^2(k)} \tag{31}$$

$$RN_{(\varepsilon,\hat{y}_r)}(i) = \frac{\sum_{k=0}^K \varepsilon(k)\hat{y}_r(k-i)}{\left[\left(\sum_{k=0}^K \varepsilon^2(k) \right) \left(\sum_{k=0}^K \hat{y}_r^2(k) \right) \right]^{1/2}} \tag{32}$$

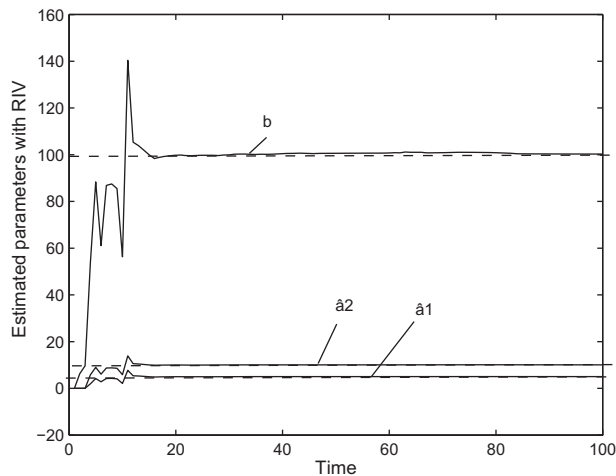


Fig. 3. Evolution of the estimated parameters using RIV algorithm in stochastic environment.

Table 3
RIV estimation in stochastic environment.

	a_1	a_2	b
Real parameters	5	10	100
Estimated parameters	5.076	10.080	100.092

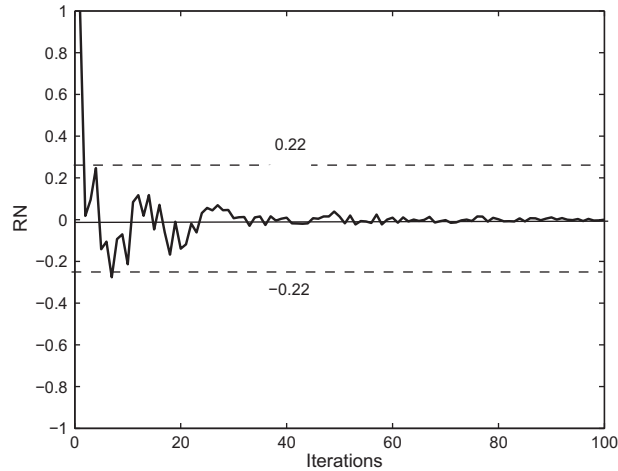


Fig. 4. Whiteness test.

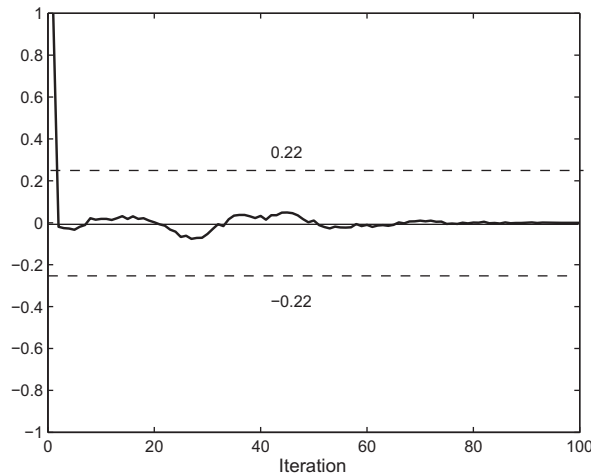


Fig. 5. Uncorrelation test.

$i = 0, 1, \dots, N \dots$

If the residual prediction error sequence is perfectly white, and the number of samples is very large, then $RN_{\varepsilon,x}(0) = 1$; $RN_{\varepsilon,x}(i) = 0, i \geq 1$.

In real situations, one considers as a practical validation criterion:

$$|RN_{(\varepsilon,x)}(i)| \leq \frac{2.17}{\sqrt{K}}, \quad i \geq 1, \quad x = \varepsilon, \hat{y} \tag{33}$$

where K is the number of samples.

Figs. 4 and 5 show model validation results for $K = 100$. As it can be seen, the results obtained are very close. The auto-correlation of the prediction errors in the case of RLS and the correlation between the prediction error and the instrumental predictor in the case of RIV tend toward zero which means unbiased parameter estimation in both cases.

5. Conclusion

We have considered recursive LSs-type identification problems for fractional order systems described by continuous fractional order differential equations. We have shown that the problem can be solved by linear regression obtained by sampling of the continuous time model using the Grnwald approximation. Two parameter estimation methods have been proposed and compared: recursive least square method (RLS) and recursive instrumental variable method (RIV). Simulation examples have been performed to confirm the interest of the proposed approach for fractional order model identification.

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