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Optimal Control and Sentinel applied to the Parabolic Systems

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مُلَخص

الاطروحة مُخَصَصة لِدِرَاسَة النَمَاذِج المُكَافِئة (وَ المُكَافِئة الثَنوِيَة) مِن وِجهَة نَظَر تَحكُم مِثَالِيَة في ظِلِ وُجود اضطِرَابَات وَبَايَنَات غير كَامِلَة. يَتِمُ تَحلِيل وُجود عَنصُر التَحَكُم (التَحَكُم التَقرِيبِي) لِلمُعَادَلات المُكَافِئَة وَالمُرَادِفَة بِعُمق بِاستِخدَام طَرِيقَة العَالِم لِيونَص. تُستَخدَم في فِكرَة التَحَكُم في هَذِهِ المُشكِلات طَريقَة الحَارِس.

مُلَخص هَذِهِ الاعمَال كَالتَالي لَقَد عَمِلنَا عَلَى مُشكِلَة تَحدِيد المُصطَلَحَات الصَاخِبَة (مُصطَلَح التَلَوُث) اَلتِي تُنشَا كَمُشكِلات مُنفَرِدَة، وَالَّتِي تَمَت نَمذَجَتُهَا بِوَاسِطَة مُعَادَلات تَفَاضُلِية خَطِيَة مَع بَيَانَات مَفقُودَة، ثُمَ يَتِمُ تَقدِيم الحَارِس الجَدِيد هُنَا لِتَحدِيد المُصطَلَحَات الصَاخِبَة، مَع العِلمِ انَ المُرَاقَبَة وَ التَحَكُم لَهُمَا دَعَامَتَان في مَجموعَتِينِ مَفتُوحَتِينِ مُختَلِفَتِينِ تَمَامًا. لِذَلِكَ، فَإنَ الغَرَض مِن هَذَا العَمَل هُوَ، تَعدِيل التَعَرِيف القَدِيم لِلحَارِس، حَتَي تَتَمَكَن مِن التَحَكُم وَدَعم المُرَاقَبَة، وَمِن نَاحِيَة اخرَي، تَحدِيد مُصطَلَحَات التَلَوُث اَلتِي تَنشَأُ بِشَكل مُنتَظِم (وَ غير مُنتَظِم) اَلمُشكِلَة اَلتِي يَتِمُ نَمذَجَتُهَا بِوَاسِطَة مُعَادَلَة تَفَاضُلِيَة خَطِيَة (او مُعَادَلَة ثَنوِيَة). مِن مُعَادَلات اصلِيَة مُكَافِئَة لَهَا اَلتِي نَقُومُ بِحَلِهَا. وَ قَد تَمَت دِرَاسَة إمكَانِيَة التَحَكُم التَقرِيبِي لِنِظَام المُعَدلات المُكَافِئة.

أَخيرًا، تَمَ عَرض بَعض نَتَائِج نَظَرِيَة الحَارِس فِمَا يَتَعَلَقُ بِنَظَرِيَة التَحَكُم لِلانظِمَة المُوَزعَة. وَ بِالّتَالي هُنَاكَ مَجمُوعَة مُتَنَوِعَة مِن النَمَاذِج حَيثُ يُمكِن تَطبِقُهَا لِلتَعبِير عَمَا يَدُور في أَذهَانِنَا بِشَكل خَاصِ لِتقرِيب المُعَادلة مِن مُعَادَلَة نَافِير ستوكَس. اَلحُصُول عَلَي مَعلُومَة فَورِيَة عَن مُصطَلَح التَلَوُث في نِظَام نَافير ستوكَس. أَفضَل طَرِيقَة لِحَلِ هَذِهِ المُشكِلَة هيَ طَرِيقَة الحَارِس. اَلَّذي سَيَجعَل مِن المُمكِنِ تَقوِيم مُصطَلَح التَلَوُث اَلذي يَتِمُ البَحث عَنهُ بِشَكل مُستَقِل عَنِ المُصطَلَحِ المَفقُود اَلذِي لَا يَرغَب المَرء في تَحدِيدِهِ. وَ بِالّتَالي نُحَبِذُ وُجود مِثلَ هَذَا الحَارِس الفَورِي مِن خِلَال حَلِ مُشكِلَة امكَانِيَة التَحَكُم بِطَرِيقَة مُثلَى.

Abstract

In this thesis, we were based on the analysis of controllability of parabobilc system and sentinel theory. We have shown that either regional or pointwise sentinel construction depends on the existence of optimal control.

The sentinel theory is given here in its more general and more realistic setting for Stockes problem and parabolic problems: in this case, the observation and the control have their supports in different open sets but not disjoint.

The problem of finding a sentinel is equivalent to a null-controllability problem for the parabolic equation that we solve. Elsewhere, the sentinel theory is an important tool in the estimation of the system pollution terms, independently of its missing terms.

Also we were concerned by the problem of identification of pollution terms which arise, first in Stockes problem and secondly in parabolic problem as for remote sensing problems, and which are modeled by a linear parabolic equation, using the new modification sentinel which change the classic definition of the sentinel method of J.-L. Lions (1992). A new modification of the sentinel is used to identify pollution terms in the general case where the observation and the control supports are disjoint.

Résumé

La thèse est consacrée à étudier des modèles paraboliques (paraboliques singuliers) d'un point de vue de contrôle optimal, en présence de perturbations et de données incomplètes. L'existence de contrôle (contrôle approchée) pour les équations paraboliques et singulières est analysée en profondeur à l'aide du méthode H.U.M. La contrôlabilité de ces problèmes utilise la méthode de sentinelle (J.L.Lions 1992) comme un outil.

Le résumé de ce travail est comme suit: on a travaillé sur le problème d'identification des termes bruités (terme de pollution) qui se posent en problème singulier, et qui est modélisé par une équation parabolique singulière linéaire avec des données manquantes, alors la nouvelle sentinelle est donnée ici pour l'identification des termes bruités, sachant que l'observation et le contrôle ont leur support dans des ensembles ouverts différents. Donc, le but de ce travail est, dans un premier temps, de modifier l'ancienne définition de la sentinelle, afin que l'on puisse séparer le support de contrôle et support d'observation et, d'autre part, d'identifier les termes de pollution qui se posent dans des problèmes réguliers (et singulières) qui sont modélisés par une équation parabolique linéaire (même paraboliques singulières).Le problème de trouver une nouvelle sentinelle de modification revient à trouver l'unique contrôle du système adjoint régulier (ou singulier) de l'équation parabolique que nous résolvons. Ensuite on a étudié aussi la contrôlabilité approchée pour un système d'équations paraboliques de type Navier-Stokes modélisant un écoulement visqueux incompressible.

Finalement, on a présenté quelques résultats de la théorie sentinelle en connexion avec la théorie du contrôle des systèmes distribués. Bien sûr il existe une grande variété de modèles où les résultats à suivre pourraient être appliqués. Ce que nous avons particulièrement en tête, c'est l'ensemble classique des équations de Navier-Stokes. Nous tentons d'obtenir des informations instantanées sur le terme de pollution dans le système Navier-Stokes dans lequel la condition initiale est incomplète. La meilleure méthode qui puisse résoudre ce problème est la méthode sentinelle ; il permet d'estimer le terme de pollution auquel on cherche des informations indépendamment du terme manquant que l'on ne veut pas identifier. Ainsi, nous prouvons l'existence d'une telle sentinelle instantanée en résolvant un problème de contrôlabilité avec une contrainte sur le contrôle.

Contents

0.1 Introduction

First, we are interested in the control of systems governed by the Navier-Stokes equations modeling incompressible viscous flow. Indeed, substantial progress concerning this objective took place in the late 1990s.
Let us say that the control problems and methods which have been discussed and mostly concerned with systems governed by linear diffusion equations of the parabolic type, associated with second order elliptic operators. Indeed, these methods have been applied in, for example, Glowinski, and J.L. Lions [7], to the solution of approximate boundary controllability problems for systems governed by strongly advection dominated linear advection-diffusion equations. These methods can also be applied to systems of linear advection-diffusion equations and to higher-order parabolic equations (or systems of such equations). Motivated by the solution of controllability problems for the Navier-Stokes equations modeling incompressible viscous flow, we will discuss now controllability issues for a system of partial differential equations which is not of the Cauchy-Kowalewska type, namely, the classical Stokes system.

Secondly, we are interested in the identification of the pollution terms which arise in parabolic problems as for remote sensing with active sensors or passive sensors as well. As it is well known, these problems generate heat spread which here are modeled by a linear parabolic equation.

The main tool to identify missing terms is the classical least square method, but this method determines all the missing terms (the pollution terms and the missing initial data terms). We here use the sentinel method of J.-L. Lions (1992), which permits to distinguish between all the missing terms. Here we want to characterize the pollution terms independently of the missing initial data ones. With the sentinel method, there is a gain of time calculations when interested in simulations.

As we will see, the problem of finding a sentinel is equivalent to a null-controllability problem (see the general case in the book by J.-L. Lions [7]) where the control and the observation have their support in the same open set. In our case, the sentinel theory is used in its general setting since we consider two different open sets for the control and the observation. This is a new issue for regular problems in general.

Optimal control of distributed parameter systems governed by a system of parabolic equations is of special importance for propagation processing problems which are generally expressed by the resolution of the heat equation. The use of these equation may however leave a gap between the theoretical solutions and the experimental ones, then the use of optimal control allows to fill the gap, as it permits to optimize the distance between the two solutions (See [15] and the references therein)

As an immediate application, the existence of a discriminating sentinel for a *nonlinear* singular parabolic equation can be discussed, as we will see in the following section. Finally we note that the backward problem appears under this form in the Lions sentinel theory [7]. Let us mention the works by A. Omrane [16] and by Miloudi et al. in [13] and [14] who solved the control with constraints problem for the heat equation using a well adapted Carleman inequality (see also A. Fursikov and O. Yu Imanuvilov [21]).

As noticed above, we will prove that the sentinel problem is equivalent to a null-controllability one. The general null-controllability problem for the heat equation is well understood (see J. L. Lions [7] and C. Bardos *et* al. [19]). Indeed, assuming the geometric control condition (GCC) introduced by [19], one can establish an observation estimate which yields by the HUM method of Lions [7], to the existence of the control v. The (GCC) is a microlocal making a like to bicharacteristic rays of the heat operator. Moreover, it is equivalent to exact controllability of the linear heat equation (with stability with respect to small perturbations). We state a more general null-controllability problem in this article.
The thesis is organized as follows :
In chapter one, we give the notion of distributed Systems Analysis and the different situation of the controllability

In chapter two ,we give the sentinel for the analysis distributed systems with missing terms and their application to the sentinel theory were the control and observation have their supports in two different open sets.

In chapter three, we give the detection of the pollution in the singular parabolic system : In this chapter we are concerned by the problem of identification of noisy terms which arise in singular problem as for remote sensing problems, and which are modeled by a linear singular parabolic equation. Some data are missing, then the sentinel method of J.-L. (1992) is changed and the new sentinel is used ; it is a particular least square-like method which permits to distinguish between the missing terms and the pollution terms. In particular, the new sentinel is given here in its more realistic setting for singular parabolic problems : in this case, the observation and the control have their support in different open sets. The problem of finding a new sentinel is equivalent to finding singular optimality system of the least square control for the parabolic equation that we solve.

In chapter four we give a new modified sentinel for linear parabolic equation : The purpose of this chapter is, firstly, to modify the old definition of the sentinel, introduced by J.L. Lions, so that we can separate the control support to the observation support and, secondly, to identify the pollution terms which arise in regular problems which are modeled by a linear parabolic equation. A new modified of the sentinel is used to identify pollution terms in the general case where the observation and the control supports are disjoint. The problem of finding a new modified sentinel is equivalent to finding the unique control of regular adjoint system of the parabolic equation that we solve.

In chapter five we give the approximate controllability of the the Stokes System : In this chapter we establish some approximate controllability results for system of parabolic equations of the Stokes kind. Motivated by the solution of controllability problems for the Navier-Stokes equations modeling incompressible viscous flow, we will discuss now controllability issues for a system of partial differential equations which is not of the Cauchy-Kowalewska type, namely, the classical Stokes system.

In the last chapter, we give the instantaneous sentinel for the identification of the pollution term in Navier Stokes system : The aim of this chapter is about presenting some results of the Sentinel Theory in Connection with Control Theory of Distributed Systems. There is of course a large variety of models where the results to follow could be applied. What we have particularly in mind is the classical set of Navier-Stokes equations. We shall denote here by the velocity field and the pressure, which is a quite unusual notation in the ŤTurbulenceŤ circle. The reason is simply that in all what is following that we think of as the state of our system, this state depends on control functions, these control functions are being either ŤartificialŤ or ŤnaturalŤ. Also in this work we are(trying) working to get instantaneous information at fixed instant ŤTŤ on pollution term in Navier-Stokes system in which the initial condition is incomplete. The best method which can solve this problem is the sentinel one; it allows the estimation of the pollution term at which we look for information independently of the missing term that we do not want to identify. So, we prove the existence of such instantaneous sentinel by solving a problem of controllability with a constraint on the control.

Chapter 1

Distributed Systems Analysis

1.1 Controllability of Distributed Systems

Let be Ω an open bounded set of R^n with regular boundary Γ and $T > 0$.
We consider the system described by the differential equation:
find $y(t)$ such that :

$$\begin{cases} y'(t) = Ay(t) + Bu(t) \qquad 0 < t < T, \\ y(0) = y_0. \end{cases} \tag{1.1}$$

with

- $A \in L(X)$.
- $B \in L(U, X)$.
- $u(.) \in L^2(0, T; U)$. the function u is said control.
- y : the state of the system with $y(., t) \in X = L^2(\Omega)$.
- y_0 : the initial state.

In the study of the system (1.1), we make the following assumptions:

H$_1$) X, U are separable Hilbert spaces, they represent respectively the state space and the control space.

H$_2$) $u \in L^2(0, T; U), B \in L(U, X)$.

H$_3$) A is auto-adjoint with compact resolvent and generates a strongly continuous semi-group $(S(t))_{t\geq 0}$ on $X = L^2(\Omega)$.

We will note $y_u(.)$ the solution of the system (1.1).
with these assumptions the system (1.1) admits a unique weak solution defined by

$$y_u(t) = S(t)y_0(x) + \int_0^t S(t-s)Bu(x,s)ds. \tag{1.2}$$

- without loss of generality we take $y_0 = 0$.
- Let be H_T a bounded linear operator definded by :

$$\begin{aligned} H_T \quad &: \quad L^2(0,T;U) \to X \ . \\ u \quad &\longmapsto \quad H_T u = \int_0^T S(T-s)Bu(s)ds. \end{aligned} \tag{1.3}$$

H_T will be used, thereafter, for various definitions and properties.

1.2 Exact Controllability and Weak controllability

The notion of controllability is to transfer, in finite time, the initial state to a desired state chosen a priori. Many notions of controllability are presented, for example, the exact controllability, weak controllability,...

1.2.1 Exact Controllability

Definition 1.2.1 *The system (1.1) is said to be exactly controllable on* $[0,T]$ *if :*

$$\forall y_d \in X, \exists u \in L^2(0,T;U) \ \textit{such that} \ : y(T) = y_d$$

Remark 1.2.1 *The last definition is equivalent to* $ImH_T = X$ *(1.3)*

Definition 1.2.2 *Let be* $X_1 \subset X$ *subspace of* X*, the system (1.1) is said to be exactly controllable in* X_1 *if :* $\forall y_d \in X_1$, $\exists u \in L^2(0,T;U)$ *such that:* $y(T) = y_d$.

Remark 1.2.2 *The last definition is equivalent to:* $X_1 \subset ImH_T$.

1.2.2 Characterization of Exact Controllability

Proposition 1.2.1 *The system (1.1) is said to be exactly controllable on* $[0,T]$ *if and only if:*

$$\exists \gamma > 0 \ \textit{tel que} : \|y^*\|_{X'} \leq \gamma \|B^* S^*(.) y^*\|_{L^2(0,T;U')}$$

for all y^* *in* X'.

We use the next lemma to prove the proposition 1.2.1

Lemma 1.2.1 *if* X_1, U_1 *are two reflexive Banach spaces and* $H_1 \in L(U_1, \ X_1)$ *then: there is equivalence between:*
i) $X_1 \subset Im(H_1)$
ii) $\exists \gamma > 0$ *tel que :*$\|y^*\|_{X_1'} \leq \gamma \|H_1^* y^*\|_{U_1'}$, *for any* y^* *in* X_1'

Proof.
we put $U_1 = L^2(0,T;U)$, $X_1 = X$ et $H_1 = H_T$
$U_1' = L^2(0,T;U)$, $(S^*(t))_{t\geq 0}$ is a strongly continuous semi-group on X_1'
we have:

$$\begin{aligned}\langle y^*, H_T u\rangle &= \left\langle y^*, \int_0^T S(T-s)Bu(s)ds \right\rangle \\ &= \int_0^T \langle y^*, S(T-s)Bu(s)\rangle ds \\ &= \int_0^T \langle B^*S^*(T-s)y^*, u(s)\rangle ds\end{aligned}$$

then

$$\left\langle \int_0^T B^*S^*(T-s)y^*ds, u \right\rangle = \langle H_T^* y^*, u\rangle$$

and then

$$H_T^* y^* = B^*S^*(.)y^*$$

which completes the demonstration. ■
The characterization given previously is interesting as far as we brought back the exact controllability to enough simple inequality.

Lemma 1.2.2 *If the system* (1.1) *is exactly controllable on* $[0,T]$ *then*

$$X = ImH_T = \bigcup_{n\in N} H_T(B_n)$$

where B_n *the ball centred at the origin and with radius* n *in* $L^2(0,T;U)$ *(ie* $B_n = B(0,n)$*).*

Proof.
1) $\bigcup_{n\in N} H_T(B_n) \subset ImH_T$ (evident).
2) $ImH_T \subset \bigcup_{n\in N} H_T(B_n)$

$$y_0 \in ImH_T \Rightarrow \exists u_0 L^2(0,T;U), y_0 = H_T(u_0)$$

$u_0 \in L^2(0,T;U) \Rightarrow \exists n_0, (n_0 = E(\|u_0\|)+1), u_0 \in B(0,n_0), y_0 = H_T(u_0) \in H_T(B_{n_0}) \subset \bigcup_{n\in N} H_T(B_n).$

then

$$ImH_T = \bigcup_{n\in N} H_T(B_n).$$

■

Proposition 1.2.2 *If* H_T *is compact then the system* (1.1) *is not exactly controllable.*

Proof.
H_T is compact then H_T transform each bounded part to relativly compact. so $\overline{H_T(B_n)}$ being compact (the image of any bounded set is compact), so

$$\overset{\circ}{\overline{H_T(B_n)}} = \varnothing, \forall n \in N,$$

and according to the theorem of Baire we deduce that

$$\overline{\bigcup_{n\in N} \overset{\circ}{\widehat{H_T(B_n)}}} = \varnothing$$

then

$$\overset{\circ}{\widehat{X}} = \varnothing,$$

so

$$X = \varnothing,$$

it's a contradiction. ■

Corollary 1.2.1 *If $S(t)$ is compact, for any $t > 0$, then the system (1.1) is not exactly controllable.*

Proof. we pose $T_\varepsilon = T - \varepsilon$, it's easy to see that:

$$S(\varepsilon) H_{T_\varepsilon} u + \int_{T-\varepsilon}^{T} S(T-s) Bu(s)ds = H_T u.$$

$S(t)$ is compact for any $t > 0$ and H_{T_ε} is bounded then, $S(\varepsilon)H_{T_\varepsilon}$ is compact. when $\varepsilon \to 0$, $S(\varepsilon)H_{T_\varepsilon}$ converge uniformly to H_T then, H_T is compact. hence the result of corollary. ■

Corollary 1.2.2 *If B is compact then the system (1.1) is not exactly controllable.*

Example 1.2.1

$$\begin{cases} y'(t) = \Delta y(t) + u(t) & sur\ [0,T], \\ y(0) = y_0. \end{cases}$$

this dynamics system is not exactly controllable in $L^2(\Omega)$ on $[0,T]$. this example proves that there are many systems not exactly controllable. For this reason we need to define other kinds of controllability (for example the weak) controllability

1.2.3 Weak Controllability

Definition 1.2.3 *The system (1.1) is said to be weakly controllable on $[0,T]$ if for any y_d in X :*

$$\forall \varepsilon > 0, \exists u \in L^2(0,T;U)\ \text{such that} : \|y(T) - y_d\| \leq \varepsilon$$

1.2.4 Characterization of Weak Controllability

For distributed systems, the notion of weak controllability is more adapted, we can characterize it:

Proposition 1.2.3 *([77])There is equivalence between:*

a) (1.1) *is weakly controllable on*$[0, T]$.

b) $\overline{ImH_T} = X$.

c) $\ker(H_T^*) = \ker(H_T H_T^*) = \{0\}$.

d) $\langle y, S(s)Bv \rangle_X = 0, \forall s \in [0, T]$, *and* $\forall v \in U \Rightarrow y = 0$.

e) if the semi-group $(S(t))_{t \geq 0}$ *is analytic,* $\overline{\bigcup_{n \in N} ImH(A^n S(s)B)} = X$ *for any s in* $]0, T]$.

1.3 Optimal Control

In case the system (1.1) is controllable, there will be infinite controls which solve the problem. The questions that arise are the following:
1- Among these controls , does it exist one control that has a minimum norm?
2- Can we explicitly determine this control according to the various parameters of the problem?
The problem of optimal control serves to minimize the functional

$$J(u) = \int_0^T \|u\|^2 dt.$$

on the space control U. In other terms if $y_d \in L^2(\Omega)$ is a desired state, the problem of optimal control consists to transfer cheaply, the system (1.1) from y_0 to y_d at the time T. so the question then becomes , "is there a control $u \in U$ with minimal energy such that $y(T) = y_d$?. This can be formulated as follows:

$$(P) \begin{cases} \min\limits_{u \in U_{ad}} J(u) \\ U_{ad} = \{u \in U, y(T) = y_d\} \end{cases}$$

Proposition 1.3.1 *If* U_{ad} *is convex closed and not empty then, the problem* (P) *admits a unique solution.*

Proof. The application $u \longrightarrow J(u)$ continious coercive and strictly convex on U. Then, if U_{ad} is not empty, closed and convex; the problem (P) admits unique solution [28]. ■
The goal of this theory is:
1- to study the existence of control $u \in U_{ad}$ solution of problem(p), we call it optimal control
2- to give the conditions of optimality u.
3- to get the properties of optimal control from optimal conditions.

Chapter 2

Sentinel and Distributed Systems with Missing Terms

2.1 Notion of the sentinel

The study of dynamic of spatiotemporal systems has generated wide literature with applications in many fields as such ecology, immunology, desertification, population dynamics, pollution as well as many others. Interesting problem for such systems concerns incomplete data and state measurement on a certain region of its geometric domain. In the case of distributed systems defined on a geometric domain Ω, numerous papers were devoted to the state controllability in the whole domain Ω (see Lions[7, 42] and the references therein). This work caters with regional analysis paradigm developed by Zerrik[6], El Jai [35] and others, by using the weakly sentinel notion introduced by Ayadi[1, 4] for pollution estimation where the measurement region O is either Ω or in the pointwises of Ω. Precisely, we consider a parabolic distributed parameter system defined on the geometric domain Ω and we assume that the following assymptions are given:
- An open regular and bounded set Ω of R^n, $n \geq 1$,with boundary $\Gamma = \partial\Omega$.
- A time interval $[0, T]$we denote Q=$]0, T[\ \times\Omega$ and $\sum =]0, T[\ \times\partial\Omega$
- A second order differential linear operator A with compact resolvent and which generates a strongly continuous semi-group $(S(t))_{t\geq 0}$ on the state space $X = L^2(\Omega)$.
- A^* will denote the adjoint operator of A.

Then the considered system which is described by the following state equations

$$\begin{cases} \frac{\partial y}{\partial t}(t,x) + Ay(t,x) = F(t,x), & \text{in } Q \\ y(0,x) = y_0(x), \ x \in \bar{\Omega} \\ y(t,x) = g(t,x), & \text{in } \sum \end{cases} \tag{2.1}$$

where

$$F \in L^2(Q), g \in L^2(\Sigma) \ \ and \ \ y_0 \in L^2(\Omega)$$

have unique weak solution.

Now we will recall Green's formula:

$$\int_\Omega yAqdx - \int_\Omega qA^*ydx = \int_\Gamma y\frac{\partial}{\partial v_A}qd\Gamma - \int_\Gamma q\frac{\partial}{\partial v_{A*}}yd\Gamma$$

for all y and q in the Sobolev space $H^1(\Omega)$. In systems theory, the sentinel is related to the possibility of finding the state of the adjoint system dynamics independently of the missing and pollution terms, and of the choose of control spaces. The regional (boundary) sentinel explores the notion of sentinel in the particular case where the support of the initial state of adjoint system dynamics is into the subregion (a part of boundary) ω.

2.2 Regional sentinel

In this section, we choose an open O in the interior of Ω and we assume that the considered system is described by the following equations

$$\begin{cases} \frac{\partial y}{\partial t}(t,x;\lambda,\tau) + Ay(t,x;\lambda,\tau) = f_0(t,x) + \lambda f(t,x), & \text{in } Q \\ y(0,x;\lambda,\tau) = y_0(x) + \tau\overline{y}(x), \; x \in \Omega \\ y(t,x;\lambda,\tau) = 0, \;\; \text{in } \sum \end{cases} \tag{2.2}$$

where f_0, y_0 are given ; f, $\overline{y}$ are unknown functions and λ, τ are small unknown parameters. Let h_0 be a function given in $L^2((0,T)\times O)$. One considers a functional defined by the formula

$$S(\lambda,\tau) = \int\int_{(0,T)\times O} (h_0 + \varphi)y(x,t;\lambda,\tau)dxdt \tag{2.3}$$

where $\varphi \in L^2((0,T) \times O)$

Definition 2.2.1 *The functional $S(\lambda,\tau)$ is said to be regional sentinel defined by h_0 if the following properties are satisfied:*
1) there exists $u \in L^2([0,T] \times O)$ such that $\frac{\partial S}{\partial \tau}(\lambda,\tau)_{\lambda=0,\tau=0}=0$, for all
$\overline{y} \in L^2([0,T] \times \omega)$
2) $\|u\| = \inf \|\varphi\|$ for all φ satisfying the property one.

Now we focus on the regional sentinel construction: let $\widehat{y}(t,x)$ be the unique solution of the following equations

$$\begin{cases} \frac{\partial \widehat{y}}{\partial t}(t,x;0,0) + A\widehat{y}(t,x;0,0) = f_0(t,x), & \text{in } Q \\ \widehat{y}(0,x;0,0) = y_0(x) \quad \text{in } \Omega \\ \widehat{y}(t,x;0,0) = 0, \;\text{on } \sum. \end{cases} \tag{2.4}$$

The derivative of the system (2.2) with respect to the parameter λ near $(\lambda = 0, \tau = 0)$ is given by the following equations

$$\begin{cases} \frac{\partial y_\lambda}{\partial t}(t,x) + Ay_\lambda(t,x) = f(t,x), & \text{in } Q \\ y_\lambda(0,x;\lambda,\tau) = 0, \;\text{in } \Omega \\ y_\lambda(t,x) = 0, \;\text{on } \sum \end{cases} \tag{2.5}$$

and also the derivative of the system (2.2) with respect to the parameter τ near $(\lambda = 0, \tau = 0)$ is given by the following equations

$$\begin{cases} \frac{\partial y_\tau}{\partial t}(t,x) + Ay_\tau(t,x) = 0, & \text{in } Q \\ y_\tau(0,x) = \overline{y}, \;\text{in } \Omega \\ y_\tau(t,x) = 0, \;\text{on } \sum. \end{cases} \tag{2.6}$$

The adjoint system associated to (2.6) is defined by the following equations

$$\begin{cases} -\frac{\partial q}{\partial t}(t,x) + A^* q(t,x) = (h_0(t,x) + \varphi(t,x))\chi_{|O}(x), & \text{in } Q \\ q(T) = 0, & \text{in } \Omega \\ q = 0, & \text{on } \sum \end{cases} \tag{2.7}$$

with h_0 and φ in $L^2(]0,T[\times O)$. The system (2.7) is decomposed into two systems, free one and forced one. The free system is given by the following equations

$$\begin{cases} -\frac{\partial q_0}{\partial t}(t,x) + A^* q_0(t,x) = h_0(t,x)\chi_{|O}(x) & \text{in } Q \\ q_0(T) = 0, & x \in \Omega \\ q_0 = 0, & \text{on } \sum \end{cases} \tag{2.8}$$

the forced system is given by the following equations

$$\begin{cases} -\frac{\partial q_1}{\partial t}(t,x) + A^* q_1(t,x) = \varphi(t,x)\chi_{|O}(x) & \text{in } Q \\ q_1(T) = 0, & x \in \Omega \\ q_1 = 0, & \text{on } \sum \end{cases} \tag{2.9}$$

then the solution of (2.7) is written as

$$q = q_0 + q_1$$

Definition 2.2.2 *The dynamic system 2.9 is said to be regionally controllable on the region ω if, for all desired state, there exists a control such that the final state is equal to the considered desired state on ω.*

we consider $q_0(0,.) \in L^2(\Omega)$ as the desired state and we take a region $\omega = \Omega \diagdown O$. Then the regional controllability consists in finding a control u in $L^2(]0,T[\,;L^2(O))$ which permits, in a finite time, to bring the state q_1 of system (2.9) from the initial state $q_1(T,x) = 0$, to the final desired state $-q_0(0,x)$ on this region.

Remark 2.2.1 *If the higher multiplicity of the eigenvalue of A is equal to one, then the system 2.9 is Controllable in $L^2(\omega)$,[6, 35]*

Theorem 2.2.1 *If the system 2.9 is ω regional controllable, then there exists a unique control $u \in L^2(]0,T[\,;L^2(O))$ which satisfies the definition 2.2.1 of the sentinel.*

Proof. If the system (2.9) is regionally controllable on ω then, for $q_0(0)$ is given in $L^2(O)$, there exists a unique control $u \in L^2((0,T)\times O)$ such that $q_1(0)_{|\omega} = -q_0(0)_{|\omega}$, hence we get the first formula of the definition 1. From the equation (2.6) and the equation (2.7) we can deduce

$$-\int_O q(0)\bar{y}dx = \int\int_{(0,T)\times O} (h_0 + u) y_\tau(x,t;\lambda,\tau)dxdt. \tag{2.10}$$

and hence, for any $\bar{y}$ having its support outside O, we have $\int_O q(0)\bar{y}dx = 0$, hence

$$\frac{\partial}{\partial \tau} S(\lambda,\tau)_{\lambda=0,\tau=0} = \int\int_{(0,T)\times O} (h_0 + u) y_\tau(x,t;\lambda,\tau)dxdt = 0. \tag{2.11}$$

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2.3 Estimate of the pollution terms

Now, Let $y_m(t,x)$ be the measured state of the system on the observatory O during the interval $]0,T[$, then the measured regional sentinel is given by formula

$$S_m(\lambda,\tau) = \int\int_{(0,T)\times O} (h_0+u)y_m(x,t;\lambda,\tau)dxdt. \tag{2.12}$$

Theorem 2.3.1 *If the system 2.9 is $\omega-$regionally controllable then we have the following estimation*

$$\lambda \int_{[0,T]\times O} qf dxdt = S_m(\lambda,\tau) - S(0,0).$$

Proof. We know that

$$S(\lambda,\tau) = S(0,0) + \lambda\frac{\partial}{\partial\lambda}S(\lambda,\tau)_{\lambda=0,\tau=0} + \tau\frac{\partial}{\partial\tau}S(\lambda,\tau)_{\lambda=0,\tau=0} \tag{2.13}$$

using the equations (2.11) and (2.12) we have

$$S_m(\lambda,\tau) - S(0,0) = \lambda\frac{\partial}{\partial\lambda}S(\lambda,\tau)_{\lambda=0,\tau=0} \tag{2.14}$$

where

$$\frac{\partial}{\partial\lambda}S(\lambda,\tau)\,|_{\lambda=0,\tau=0} = \int\int_{O\times(0,T)} (h_0+u)y_\lambda(x,t)dxdt, \tag{2.15}$$

and

$$S(0,0) = \int\int_{O\times(0,T)} (h_0+u)\widehat{y}(x,t)dxdt \tag{2.16}$$

using the equations (2.5) and (2.7), we deduce that

$$\int\int_{(0,T)\times O} (h_0+u)y_\lambda(x,t)dxdt = \int_{(0,T)\times\Omega} qf dx$$

hence

$$\lambda\int_{(0,T)\times\Omega} qf dx = S_m(\lambda,\tau) - S(0,0)$$

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2.4 Pointwise sentinel

In this section, we choose $O = \{b\}$ a point in Ω and we assume that the considered system is described by the equation 2.2. Let h_0 be a function given in $L^2(0,T)$,one considers a functional defined by the formula

$$S(\lambda,\tau) = \int_0^T (h_0(t)+\varphi(t))y(b,t;\lambda,\tau)dxdt \tag{2.17}$$

where $\varphi \in L^2(0,T)$.

Definition 2.4.1 *The functional $S(\lambda,\tau)$ is said to be pointwise sentinel defined by h_0 if the following properties are satisfied:*
1) there exists $u \in L^2([0,T])$ such that $\frac{\partial S}{\partial \tau}(\lambda,\tau)_{\lambda=0,\tau=0}=0$ for all $\bar{y} \in L^2([0,T]\times\Omega)$ and $\delta(b-x)\bar{y} \neq 0$ on $[0,T]\times\{b\}$ annd equal zero somewhere else
2) $\|u\| = \inf \|\varphi\|$ for all φ satisfying the property 1

Now we focus on the Pointwise sentinel construction: let $\widehat{y}(t,x)$ be the some solution of 2.4,the derivative solution with respect to the parameter λ is given by 2.5 and also the derivative solution with respect to the parameter τ is given by2.5 The adjoint system associated to (2.5) is defined by the following equations

$$\begin{cases} -\frac{\partial q}{\partial t}(t,x) + A^* q(t,x) = (h_0(t)+\varphi(t))\delta(b-x), & \text{in } Q \\ q(T,x)=0, \ \text{in } \Omega \\ q = 0 \quad \text{on} \sum \end{cases} \tag{2.18}$$

with h_0 and φ in $L^2(0,T)$. The system (2.18) is decomposed into two systems, free one and forced one. The free system is given by the following equations

$$\begin{cases} -\frac{\partial q_0}{\partial t}(t,x) + A^* q_0(t,x) = h_0(t)\delta(b-x) \quad \text{in } Q \\ q_0(T,x)=0 \\ q_0 =, \quad \text{on} \sum \end{cases} \tag{2.19}$$

The forced system is given by the following equations

$$\begin{cases} -\frac{\partial q_1}{\partial t}(t,x) + A^* q_1(t,x) = \varphi(t)\delta(b-x) \quad \text{in } Q \\ q_1(T,x)=0 \quad \text{in} \quad \Omega \\ q_1 = 0 \quad \text{on} \quad \sum \end{cases} \tag{2.20}$$

there is one function φ such that

$$\delta(b-x)q_1(0) = -\delta(b-x)q_0(0), \quad u(t,x) = \varphi(t)\delta(b-x)$$

hence

$$\delta(b-x)q(0) = 0 \text{ and } u(t,x) = \varphi(t)\delta(b-x)$$

$$q = q_0 + q_1.$$

Multiplying the equation (2.6) by q and integrating by parts, we have

$$\int_0^T (h_0+u)\delta(b-x)y_\tau dt = 0$$

Let $y_m(t,x)$ be a measured state of the system on the observatory $\{b\}$ during the interval $]0,T[$, then the measured sentinel is given by

$$S_m(\lambda,\tau) = \int_0^T (h_0+u)y_m(t,b;\lambda,\tau)dt, \tag{2.21}$$

and we write

$$S(\lambda,\tau) = S(0,0) + \lambda\frac{\partial}{\partial\lambda}S(\lambda,\tau)_{\lambda=0,\tau=0} + \tau\frac{\partial}{\partial\tau}S(\lambda,\tau)_{\lambda=0,\tau=0}, \tag{2.22}$$

where

$$S(0,0) = \int_0^T (h_0+u)(t)\widehat{y}(t,b)dt, \tag{2.23}$$

2.5 Estimate of the pollution terms

In this section, the objectif is to estimate the pollution terms independently of the missing terms.

Theorem 2.5.1 *Under the hypothesis of the theorem 2.2.1, the pollution term of the system 2.5 is estimated independently of the missing term by*

$$S_m(\lambda,\tau) - S(0,0) = \int_0^T (h(t)+u(t))(y_m(t,b) - \hat{y}(t,b))dt$$

where $\hat{y}$ *is the solution of 2.4 and* y_m *is the observed state in* $\{b\}$ *during the time interval* $[0,T]$.

Proof. Let $S(\lambda,\tau)$ be the sentinel defined by h_0, from the equation (2.22), we can deduce:

$$\lambda \frac{\partial}{\partial \lambda} S(\lambda,\tau)\,|_{\lambda=0,\tau=0} = S(\lambda,\tau) - S(0,0).$$

as we know that $S(\lambda,\tau) = S_m(\lambda,\tau)$ in the point $\{b\}$, then we deduce from the equations (2.17) and (2.18):

$$\frac{\partial}{\partial \lambda} S(\lambda,\tau)\,|_{\lambda=0,\tau=0} = \int_0^T (h_0+u)y_\lambda(t,b)dt = \int_0^T qf(t,b)dt$$

thus

$$\begin{aligned} \lambda \int_0^T q(t,b)f(t,b)dxdt &= S_m(\lambda,\tau) - S(0,0) \\ &= \int_0^T (h_0+u)(y_m(t,b) - \widehat{y}(t,b))dt. \end{aligned}$$

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2.6 Weak sentinel

2.6.1 Formulation problem

For $n = \{2;3\}$, let Ω be a bounded open subset of $\mathbb{R}^n$ with boundary $\partial\Omega = \Gamma$ of class $\mathcal{C}^2$, $T > 0$, and let $\mathcal{O}$ be an open non empty subset of Ω. Set $\mathcal{Q} = (0,T)\times\Omega$, $\Sigma = (0,T)\times\Gamma$, $\mathcal{U} = (0,T)\times\mathcal{O}$. We consider the parabolic equation:

$$\left\{ \begin{array}{rcccc} y' + \Delta y + f(y) &=& \xi + \lambda\widehat{\xi} & \text{in} & \mathcal{Q} \\ y(0) &=& y_0 + \tau\widehat{y_0} & \text{in} & \Omega \\ y &=& 0 & \text{on} & \Sigma \end{array} \right. \tag{2.24}$$

Where $(.)'$ is the partial derivative with respect to time t.

Remark 2.6.1 *The problem (2.24) admits a unique solution. For the sake of simplicity, we denote* $y(x,t;\lambda,\tau)=y(\lambda,\tau)$.
One supposes that the data ξ *is rather regular, and that the terms of pollution "that one wants to estimate" are rather regular. It will be always supposed that the solution* y *check at least* $y\in L^2(\mathcal{Q})$.

Remark 2.6.2 *One will always indicate by* $\bar{y}$ *the solution* $y(x,t;0,0)$; *thus*

$$\left\{\begin{array}{rcll} \bar{y}'+\Delta\bar{y}+f(\bar{y}) & = & \xi & in \quad \mathcal{Q} \\ \bar{y} & = & 0 & on \quad \Sigma \\ \bar{y}(0) & = & y_0 & on \quad \Omega \end{array}\right. \tag{2.25}$$

The problem considered here consists in trying to estimate $\lambda\widehat{\xi}$ starting from observations, distributed or borders, without seeking to estimate the term lack $\tau\widehat{y_0}$.
One starts with a distributed observation, therefore a distributed sentinel

2.6.2 The weak sentinel method

Definition 2.6.1 *(definition, existence and uniqueness of the sentinel)*
Let $h\in L^2(\mathcal{U})$ *and for any control function* $u\in L^2(\mathcal{U})$, *set*

$$\mathcal{S}(\lambda,\tau)=\int_{\mathcal{Q}}(h+u)\,\chi_{\mathcal{O}}\,y(x,t;\lambda,\tau)\,dxdt \tag{2.26}$$

the functional $\mathcal{S}$ *is said to be weak sentinel if it satisfies the following conditions:*
for all $\epsilon>0$ *there exists* $u\in L^2(\mathcal{U})$ *such as*

$$u\in L^2(\mathcal{U}),\ \ \textit{of minimal norm} \tag{2.27}$$

$$\left|\frac{\partial}{\partial\tau}\mathcal{S}(0,0)\right|\le\epsilon \tag{2.28}$$

Remark 2.6.3 *The function* $u=-h$ *give place to (2.26) so that the problem (2.27, 2.28) admits a single solution, which is defined by* h.
The problem is thus:
(1) to calculate this solution;
(2) to see whether the corresponding sentinel justifies its name, i.e. gives information on pollution $\lambda\widehat{\xi}$.

Adjoint state

The adjoint state q is introduced by

$$\left\{\begin{array}{rcll} -q'+\Delta q+f'(\bar{y})\,q & = & (h+u)\,\chi_{\mathcal{O}} & \text{in } \mathcal{Q} \\ q & = & 0 & \text{on } \Sigma \\ q(T) & = & 0 & \text{in } \Omega \end{array}\right. \tag{2.29}$$

Where $(.)'$ is the partial derivative with respect to time t, $h,u\in L^2(\mathcal{U})$.

Remark 2.6.4 *System (2.29) is the adjoint parabolic problem. It appears under this form in J.L.Lions sentinels theory as the associated adjoint state.*

We multiply (2.29) by y_τ and we integrate by parts, we have

$$\left(q\left(0\right),\widehat{y}_0\right)=\int_{\mathcal{Q}}\left(h+u\right)\chi_{\mathcal{O}}y_\tau\left(x,t;\lambda,\tau\right)dxdt$$

y_τ is defined by

$$\left\{\begin{array}{rclll} y'_\tau+\Delta y_\tau+f'\left(\bar{y}\right)y_\tau & = & 0 & \text{in} & \mathcal{Q}\\ y_\tau\left(0\right) & = & \widehat{y}_0 & \text{in} & \Omega\\ y_\tau & = & 0 & \text{on} & \Sigma \end{array}\right.$$

So we get

$$\frac{\partial}{\partial\tau}\mathcal{S}\left(0,0\right)=\left(q\left(0\right),\widehat{y}_0\right) \tag{2.30}$$

so that (2.28) is equivalent to

$$\left\|q\left(.,0\right)\right\|_{L^2(\Omega)}\leq\epsilon \tag{2.31}$$

There is thus business with a problem of the type "approximate controllability with zero" .

The main result

The main result is the following

Lemma 2.6.1 *Let $v\in L^2\left(\mathcal{O}\right)$.Then there is no $\rho\in L^2\left(\mathcal{Q}\right),\rho\neq0$ such that ρ satisfies*

$$\left\{\begin{array}{rclll} \rho'+\Delta\rho & = & 0 & in & \mathcal{Q}\\ \rho & = & 0 & on & \Sigma\\ \rho\left(0\right)\chi_{\mathcal{O}} & = & v & on & \mathcal{O} \end{array}\right. \tag{2.32}$$

Proof. If the problem (2.32) admits a solution, then it is given by

$$\rho\left(x,t\right)=\sum_{j=1}^{\infty}\alpha_j\left(t\right)u_j\left(x\right) \tag{2.33}$$

Where u_j are eigenfunctions of

$$\left\{\begin{array}{rcll} -\Delta u & = & \lambda u & \text{in }\Omega,\\ u & = & 0 & \text{on }\Gamma. \end{array}\right. \tag{2.34}$$

Differentiate the solution (2.34) once with respect to t and twice with respect to x and substitute these derivatives into the first equation of (2.32). We then obtain

$$\sum_{j=1}^{\infty}\left(\alpha'_j\left(t\right)-\lambda_j\alpha_j\left(t\right)\right)u_j\left(x\right)=0 \tag{2.35}$$

Thus,

$$\alpha'_j\left(t\right)-\lambda_j\alpha_j\left(t\right)=0 \tag{2.36}$$

Because (u_j) form an orthonormal base of $L^2(\otimes)$. Furthermore, the function ρ satisfies the boundary conditions if and only if

$$\sum_{j=1}^{\infty} \alpha_j(0)\, \chi_{\mathcal{O}} u_j(x) = v \tag{2.37}$$

As $v \in L^2(\mathcal{O})$ then

$$v = \sum_{j=1}^{\infty} \langle v, \chi_{\mathcal{O}} u_j \rangle_{L^2(\mathcal{Q})} \chi_{\mathcal{O}} u_j(x) \tag{2.38}$$

Consequently

$$\alpha_j(0) = \langle v, \chi_{\mathcal{O}} u_j \rangle_{L^2(\mathcal{Q})} \tag{2.39}$$

Finally, we have

$$\left\{ \begin{array}{rcl} \alpha_j'(t) - \lambda_j \alpha_j(t) & = & 0 \\ \alpha_j(0) & = & \langle v, \chi_{\mathcal{O}} u_j \rangle_{L^2(\mathcal{Q})} \end{array} \right. \text{ in } (0,T), \tag{2.40}$$

Then the solution of the first order linear is given by

$$\alpha_j(t) = \langle v, \chi_{\mathcal{O}} u_j \rangle_{L^2(\mathcal{Q})} e^{\lambda_j t} \tag{2.41}$$

Consequently, if the problem (2.32) admits a solution, it is necessarily in the form:

$$\rho(x,t) = \sum_{j=1}^{\infty} \langle v, \chi_{\mathcal{O}} u_j \rangle_{L^2(\mathcal{Q})} e^{\lambda_j t} u_j(x) \tag{2.42}$$

We prove now that $\rho \notin L^2(\mathcal{Q})$. Indeed,

$$\begin{aligned} \int_0^T |\alpha_j(t)|^2\, dt &= \left| \langle v, \chi_{\mathcal{O}} u_j \rangle_{L^2(\mathcal{Q})} \right|^2 \int_0^T e^{2\lambda_j t} dt \\ &= \left| \langle v, \chi_{\mathcal{O}} u_j \rangle_{L^2(\mathcal{Q})} \right|^2 \left[\frac{-1}{2\lambda_j} + \frac{1}{2\lambda_j} e^{2\lambda_j T} \right] \end{aligned} \tag{2.43}$$

But, λ_j is the eigenvalue of problem (2.34), then $\lambda_j \underset{j \longmapsto \infty}{\longrightarrow} \infty$. Consequently,

$$\int_0^T |\alpha_j(t)|^2\, dt \underset{j \longmapsto \infty}{\longrightarrow} \infty \tag{2.44}$$

Which means that the series whose general term $\alpha_j(t)$ is not normally convergent. So, problem 2.32 admits no solution. ■

Theorem 2.6.1 *For $\epsilon > 0$, $h \in L^2(\mathcal{U})$, there existes unique control u and some state q such that (2.29) and (2.31) hold.*

Proof. Let q be a solution of the system (2.29 and q_0 a solution of the following system

$$\begin{cases} -q_0' + \Delta q_0 + f'(\bar{y})\, q_0 & = & h\chi_{\mathcal{O}} & \text{in } \mathcal{Q} \\ q_0 & = & 0 & \text{on } \Sigma \\ q_0(T) & = & 0 & \text{in } \Omega \end{cases} \tag{2.45}$$

We put

$$q = q_0 + z \tag{2.46}$$

Then, z is the solution of the following problem

$$\begin{cases} -z' + \Delta z + f'(\bar{y})\, z & = & u\chi_{\mathcal{O}} & \text{in } \mathcal{Q} \\ z & = & 0 & \text{on } \Sigma \\ z(T) & = & 0 & \text{in } \Omega \end{cases} \tag{2.47}$$

We now introduce the set of states reachable at time 0 defined by

$$\mathcal{F}(0) = \left\{ z(u, 0) \text{ such as } u \in L^2(\mathcal{U}) \right\}. \tag{2.48}$$

It is clear that $\mathcal{F}(0)$ is a vector subspace of $L^2(\Omega)$. According to the **HAHN-BANACH** theorem, it will be dense in $L^2(\Omega)$ if and only if its orthogonal in $L^2(\Omega)$ is reduced to zero. As the subset $\{0\} \subset \mathcal{F}^{\perp}(0)$, it remains to show that $\mathcal{F}^{\perp}(0) \subset \{0\}$. Let $\rho^0 \in \mathcal{F}^{\perp}(0)$, then

$$\left\langle \rho^0, z(0) \right\rangle_{L^2(\Omega)} = \int_{\Omega} \rho^0 z(0)\, dx = 0 \tag{2.49}$$

Where z is solution of (2.47). It is therefore natural to define the adjoint ρ of z, this is the solution of the following problem

$$\begin{cases} \rho' + \Delta \rho + f'(\bar{y})\, \rho & = & 0 & \text{in } \mathcal{Q} \\ \rho(0) & = & \rho^0 & \text{in } \Omega \\ \rho & = & 0 & \text{on } \Sigma \end{cases} \tag{2.50}$$

Now we multiply the first equation of system (2.47) by ρ. After integration by parts in $\mathcal{Q}$, it comes

$$\begin{aligned} 0 \;=\; & \int\int_{\Omega\times(0,T)} \rho\left(-z' + \Delta z + f'(\bar{y})z\right) dxdt + \int_{\Omega} \rho(T)\, z(T)\, dx \\ & + \int\int_{\Gamma\times(0,T)} \rho \frac{\partial z}{\partial \nu} \Gamma dt - \int\int_{\Gamma\times(0,T)} \frac{\partial \rho}{\partial \nu} z d\Gamma dt - \int_{\Omega} \rho^0 z(0)\, dx \end{aligned} \tag{2.51}$$

Since z and ρ are solutions of (2.47) and (2.50) respectively, (2.51) becomes

$$\int\int_{\Omega\times(0,T)} \rho u \chi_{\mathcal{O}} dxdt = 0 \tag{2.52}$$

Therefore, ρ satisfies (2.50) and (2.52) and by applying **Lemma 2.6.1**, we deduce that

$$\rho = 0 \quad \text{in} \quad \mathcal{Q}$$

As a consequence, $\rho^0 = 0$ which shows that $\mathcal{F}^{\perp}(0) = \{0\}$. ■

2.7 Characterization of optimal control

In this section, we will characterize the optimal control using a result of **Fenchel-Rockafellar** duality.

The optimality system satisfied by (u,q) is established. Let $\rho^0 \in L^2(\Omega)$ and ρ the associated solution of

$$\begin{cases} \rho' + \Delta\rho + f'(\bar{y})\rho & = & 0 & \text{in } \mathcal{Q} \\ \rho(0) & = & \rho^0 & \text{in } \Omega \\ \rho & = & 0 & \text{on } \Sigma \end{cases} \tag{2.53}$$

We now introduce the functional J_ϵ defined by

$$J_\epsilon(\rho^0) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |\rho|^2 \, dxdt + \epsilon \left\|\rho^0\right\|^2_{L^2(\Omega)} + \int_0^T \int_{\mathcal{O}} \rho h dxdt \tag{2.54}$$

Consider the following unconstrained problem

$$(P_\epsilon) : \begin{cases} \min J_\epsilon(\rho^0) \\ \rho^0 \in L^2(\Omega) \end{cases} \tag{2.55}$$

Then, we have

Proposition 2.7.1 *The functional J_ϵ defined in (2.54) is coercive.*

Proof. To prove that J_ϵ is coercive, it suffices to show the following relation:

$$\lim_{\|\rho^0\|_{L^2(\Omega)} \to \infty} \frac{J_\epsilon(\rho^0)}{\|\rho^0\|_{L^2(\Omega)}} \geq \epsilon \tag{2.56}$$

Let $(\rho_j^0) \subset L^2(\Omega)$ be a sequence of initial data for the adjoint system (2.53) with $\left\|\rho_j^0\right\|_{L^2(\Omega)} \longrightarrow \infty$. We normalize them as follows

$$\widetilde{\rho}_j^0 = \frac{\rho_j^0}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \tag{2.57}$$

So $\left\|\widetilde{\rho}_j^0\right\|_{L^2(\Omega)} \leq 1$. On the other hand, let $\widetilde{\rho}_j$ be the solution of (2.53) with initial data $\widetilde{\rho}_j^0$. Then, we have

$$\begin{aligned} \frac{J_\epsilon(\rho_j^0)}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} &= \frac{1}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \int_0^T \int_{\mathcal{O}} \left(\frac{1}{2}|\rho_j|^2 + \rho_j h\right) dxdt + \epsilon \\ &= \int_0^T \int_{\mathcal{O}} \widetilde{\rho}_j \left(\frac{1}{2}\rho_j + h\right) dxdt + \epsilon \end{aligned} \tag{2.58}$$

We now show that the last integral in equation (2.58) is bounded. Indeed, we know that ρ_j is the solution of the problem

$$\begin{cases} \rho_j' + \Delta\rho_j + f'(\bar{y})\rho_j & = & 0 & \text{in} & \mathcal{Q} \\ \rho_j & = & 0 & \text{on} & \Sigma \\ \rho_j(0) & = & \rho_j^0 & \text{in} & \Omega \end{cases} \tag{2.59}$$

Multiplying the first equation of system (2.59) by ρ_j then **integrating by parts** on $\mathcal{Q}$, yields

$$\begin{aligned} 0 &= \int_0^T \int_\Omega \left(\rho_j' + \Delta\rho_j + f'(\bar{y})\,\rho_j\right)\rho_j dxdt = \frac{1}{2}\left\|\rho_j(T)\right\|^2_{L^2(\Omega)} \\ &-\frac{1}{2}\left\|\rho_j^0\right\|^2_{L^2(\Omega)} + \left\|\nabla\rho_j\right\|^2_{L^2(\mathcal{Q})} \end{aligned} \tag{2.60}$$

By the **Poincaré inequality**, (2.60) becomes,

$$C_0\left\|\rho_j\right\|^2_{L^2(\mathcal{Q})} \leq \left\|\nabla\rho_j\right\|^2_{L^2(\mathcal{Q})} \leq \frac{1}{2}\left\|\rho_j^0\right\|^2_{L^2(\Omega)} \tag{2.61}$$

Now, by **Cauchy Schwartz inequality**, one finds

$$\int_0^T \int_{\mathcal{O}} \frac{h\rho}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} dxdt \leq C_1 \frac{\left\|\rho_j\right\|_{L^2(\mathcal{Q})}}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \tag{2.62}$$

From (2.61), (2.62), we conclude that

$$\int_0^T \int_{\mathcal{O}} \frac{h\rho}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} dxdt \leq C \tag{2.63}$$

Returning to relation (2.58), two cases can occur:
1. $\int_0^T \int_{\mathcal{O}} \widetilde{\rho}_j^2 dxdt > 0$. In this case, we immediately obtain

$$\frac{J_\epsilon\left(\rho_j^0\right)}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \underset{\left\|\rho_j^0\right\|_{L^2(\Omega)} \longmapsto +\infty}{\longrightarrow} +\infty. \tag{2.64}$$

2. $\int_0^T \int_{\mathcal{O}} \widetilde{\rho}_j^2 dxdt = 0$.In this case, since $\left(\widetilde{\rho}_j^0\right)_j$ is bounded in $L^2(\Omega)$, we can extract a subsequence $\left(\widetilde{\rho}_j^0\right)_j$ such that:

$$\begin{cases} \widetilde{\rho}_j^0 \rightharpoonup \psi^0 \text{ weakly in } L^2(\Omega), \\ \widetilde{\rho}_j \rightharpoonup \psi \text{ weakly in } L^2\left(0,T; H_0^1(\Omega)\right). \end{cases} \tag{2.65}$$

Where ψ is solution of system (2.53) with initial data ψ^0. Moreover, by lower semi continuity of the norm, it comes

$$\int_0^T \int_{\mathcal{O}} |\psi|^2 dxdt \leq \liminf \int_0^T \int_{\mathcal{O}} \left|\widetilde{\rho}_j\right|^2 dxdt = 0 \tag{2.66}$$

Therefore,

$$\psi = 0 \quad \text{in} \quad \mathcal{O}\times(0,T) \tag{2.67}$$

And as ψ is solution of (2.53), and in view of (2.67), we have

$$\psi = 0 \quad \text{in} \quad \Omega\times(0,T) \tag{2.68}$$

Thus,

$$\widetilde{\rho}_j \rightharpoonup 0 \text{ weakly in } L^2\left(0,T;H_0^1(\Omega)\right). \tag{2.69}$$

Moreover, from inequality (2.61), we deduce that $\left(\frac{\rho_j}{\|\rho_j^0\|_{L^2(\Omega)}}\right)_j$ is bounded in $L^2\left(0,T;H_0^1(\Omega)\right)$.
Hence

$$\frac{\rho_j}{\|\rho_j^0\|_{L^2(\Omega)}} \rightharpoonup \xi \text{ in } L^2\left(0,T;H_0^1(\Omega)\right) \tag{2.70}$$

But,

$$\widetilde{\rho}_j = \frac{\rho_j}{\|\rho_j^0\|_{L^2(\Omega)}} \rightharpoonup 0 \tag{2.71}$$

From (2.70) and (2.71), we conclude that

$$\xi' + \Delta\xi + f'(\bar{y})\,\xi = 0 \quad \text{in} \quad L^2(\mathcal{Q}) \tag{2.72}$$

So by **Lemma 2.6.1**, it comes

$$\xi = 0 \quad \text{in} \quad \mathcal{Q} \tag{2.73}$$

As a consequence,

$$\widetilde{\rho}_j = \frac{\rho_j}{\|\rho_j^0\|_{L^2(\Omega)}} \longmapsto 0 \tag{2.74}$$

But,

$$\frac{J_\epsilon\left(\rho_j^0\right)}{\|\rho_j^0\|_{L^2(\Omega)}} = \frac{1}{\|\rho_j^0\|_{L^2(\Omega)}} \int_0^T \int_{\mathcal{O}} \left(\frac{1}{2}|\rho_j|^2 + \rho_j h\right) dxdt + \epsilon \tag{2.75}$$

Thus,

$$\liminf_{j \longmapsto +\infty} \frac{J_\epsilon\left(\rho_j^0\right)}{\|\rho_j^0\|_{L^2(\Omega)}} \geq \epsilon \tag{2.76}$$

Hence relation (2.56) is satisfied. ■

Theorem 2.7.1 *Problem (2.55) has a unique solution $\widehat{\rho}^0 \in L^2(\Omega)$. Furthermore, if $\widehat{\rho}$ is the solution of (2.53) associated to $\widehat{\rho}^0$, then $(\widehat{u} = \widehat{\rho})$ is solution such that (2.31) hold.*

Proof. As J_ϵ attains its minimum value at $\widehat{\rho}^0 \in L^2(\Omega)$, then, for any $\psi^0 \in L^2(\Omega)$ and any $r \in \mathbb{R}$ we have

$$J_\epsilon\left(\widehat{\rho}^0\right) \leq J_\epsilon\left(\widehat{\rho}^0 + r\psi^0\right) \Longrightarrow J_\epsilon\left(\widehat{\rho}^0 + r\psi^0\right) - J_\epsilon\left(\widehat{\rho}^0\right) \geq 0 \tag{2.77}$$

On the other hand,

$$J_\epsilon\left(\widehat{\rho}^0\right) = \int_0^T \int_{\mathcal{O}} \left(\frac{1}{2}|\widehat{\rho}|^2 + \widehat{\rho}h\right) dxdt + \epsilon \left\|\widehat{\rho}^0\right\|_{L^2(\Omega)}$$

$$J_\epsilon\left(\widehat{\rho}^0+r\psi^0\right) = \frac{1}{2}\int_0^T\int_{\mathcal{O}}|\widehat{\rho}|^2\,dxdt+\frac{r^2}{2}\int_0^T\int_{\mathcal{O}}|\psi|^2\,dxdt + r\int_0^T\int_{\mathcal{O}}\widehat{\rho}\psi dxdt+\sqrt{\varepsilon}\left\|\widehat{\rho}^0+r\psi^0\right\|_{L^2(\Omega)} + \int_0^T\int_{\mathcal{O}}h\left(\widehat{\rho}+r\psi\right)dxdt \tag{2.78}$$

Substituting (2.78) in (2.77) and after simplifications, we find

$$\begin{aligned} 0 &\leq J_\epsilon\left(\widehat{\rho}^0+r\psi^0\right)-J_\varepsilon\left(\widehat{\rho}^0\right) \\ 0 &\leq \frac{r^2}{2}\int_0^T\int_{\mathcal{O}}|\psi|^2\,dxdt+\epsilon\left[\left\|\widehat{\rho}^0+r\psi^0\right\|_{L^2(\Omega)}-\left\|\widehat{\rho}^0\right\|_{L^2(\Omega)}\right] \\ &+r\int_0^T\int_{\mathcal{O}}\psi\left(\widehat{\rho}+h\right)dxdt \end{aligned} \tag{2.79}$$

On the other hand,

$$\left\|\widehat{\rho}^0+r\psi^0\right\|_{L^2(\Omega)}-\left\|\widehat{\rho}^0\right\|_{L^2(\Omega)}\leq |r|\,.\left\|\psi^0\right\|_{L^2(\Omega)} \tag{2.80}$$

From (2.79) and (2.80), we obtain for any $\psi^0\in L^2(\Omega)$ and $r\in\mathbb{R}$,

$$0\leq\frac{r^2}{2}\int_0^T\int_{\mathcal{O}}|\psi|^2\,dxdt+\epsilon\,|r|\,.\left\|\psi^0\right\|_{L^2(\Omega)}+r\int_0^T\int_{\mathcal{O}}\psi\left(\widehat{\rho}+h\right)dxdt$$

Dividing by $r>0$ and by passing to the limit $r\to 0$, we obtain

$$\epsilon.\left\|\psi^0\right\|_{L^2(\Omega)}+\int_0^T\int_{\mathcal{O}}\psi\left(\widehat{\rho}+h\right)dxdt\geq 0$$

The same calculations with $r<0$ give

$$\left|\int_0^T\int_{\mathcal{O}}\psi\left(\widehat{\rho}+h\right)dxdt\right|\leq\epsilon\left\|\psi^0\right\|_{L^2(\Omega)};\forall\psi^0\in L^2\left(\Omega\right).$$

so if we take $\widehat{u}=\widehat{\rho}\chi_{\mathcal{O}}$ in (2.58) and we multiply the first equation of the system (2.58) by ψ solution of (2.53) and we get after integration by parts over $\mathcal{Q}$,

$$\int_\Omega q(0)\psi^0dx=\int_0^T\int_{\mathcal{O}}\left(h+\widehat{\rho}\right)\psi dxdt \tag{2.81}$$

It comes from the last two relations:

$$\left|\int_\Omega q(0)\psi^0dx\right|\leq\epsilon\left\|\psi^0\right\|_{L^2(\Omega)};\forall\psi^0\in L^2\left(\Omega\right).$$

Consequently,

$$\left\|q\left(.,0\right)\right\|_{L^2(\Omega)}\leq\epsilon. \tag{2.82}$$

■

2.8 use of the concept of sentinel: Detection of pollution AndFurtivity

It is noted that

$$\mathcal{S}(\lambda, \tau) \simeq \mathcal{S}(0,0) + \lambda \frac{\partial \mathcal{S}}{\partial \lambda}(0,0) + \tau \frac{\partial \mathcal{S}}{\partial \tau}(0,0) \tag{2.83}$$

And

$$\text{observation of } y = y\chi_{\mathcal{O}} = \text{function } m_0(x,t) \text{ of } L^2(\mathcal{O} \times (0,T)) \tag{3.2}$$

With the notation (2.62) for the observation of y, and while using (2.55), one thus has

$$\lambda \frac{\partial \mathcal{S}}{\partial \lambda}(0,0) \simeq \int\int_{\mathcal{O}\times(0,T)} (h+u)\, m_0 dxdt - \mathcal{S}(0,0) - \tau \frac{\partial \mathcal{S}}{\partial \tau}(0,0) \tag{2.84}$$

Such as

$$\mathcal{S}(0,0) = \int\int_{\mathcal{O}\times(0,T)} (h+u)\, \bar{y} dxdt$$

But

$$\frac{\partial \mathcal{S}}{\partial \lambda}(0,0) = \int\int_{\mathcal{O}\times(0,T)} (h+u)\, y_\lambda dxdt \tag{2.85}$$

In (2.85), y_λ is defined by

$$\begin{cases} y'_\lambda + \Delta y_\lambda + f'(\bar{y})\, y_\lambda & = & \widehat{\xi} & \text{in} & \mathcal{Q} \\ y_\lambda(0) & = & 0 & \text{in} & \Omega \\ y_\lambda & = & 0 & \text{on} & \Sigma \end{cases} \tag{2.86}$$

By multiplying the corresponding equation (2.29) by y_λ, one finds, after integration by parts, that

$$\frac{\partial \mathcal{S}}{\partial \lambda}(0,0) = \int\int_{\mathcal{O}\times(0,T)} q\widehat{\xi} dxdt \tag{2.87}$$

Consequently

$$\int\int_{\mathcal{O}\times(0,T)} () \left\{\lambda\widehat{\xi}\right\} dxdt \simeq \int\int_{\mathcal{O}\times(0,T)} (h+u)(m_0 - \bar{y})\, dxdt - \tau \frac{\partial \mathcal{S}}{\partial \tau}(0,0) \tag{2.88}$$

So

$$\int\int_{\mathcal{O}\times(0,T)} q \left\{\lambda\widehat{\xi}\right\} dxdt \simeq \int\int_{\mathcal{O}\times(0,T)} (h+u)(m_0 - \bar{y})\, dxdt + \tau\epsilon \tag{2.89}$$

Remark 2.8.1 *Pollution $\lambda\widehat{\xi}$ is furtive for the sentinel defined by h if*

$$\int\int_{\mathcal{O}\times(0,T)} q \left\{\lambda\widehat{\xi}\right\} dxdt = 0 \tag{2.90}$$

There are thus always furtive pollution for a sentinel.

Chapter 3

Detecting Pollution In the Singular Linear Parabolic Systems

3.1 Introduction

We are interested in the detection of the noisy terms which arise in singular parabolic problems as for remote sensing with active sensors or passive sensors as well. As it is well known, these problems generate heat spread which here are modeled by a linear parabolic equation.

The electromagnetic (EM) radiation that is reflected back from different patterns of the Earth surface is measured by remote sensing tools. The measurements which consist of the evaluation of different wavelengths allow to distinguish the type of ocean or land covering: the water, the vegetation and the soil in general.
The noisy terms for which we refer to pollution terms in this article are unknown and deterministic. They are found in the boundary of the domain for high wave numbers. The initial data are supposed unknown too, and we do not want to find them.

The main tool to detect missing terms is the classical least square method, but this method determines all the missing terms (the noisy terms and the missing initial data terms). We here use the sentinel method of J.-L. Lions (1992), which permits to distinguish between all the missing terms. Here we want to characterize the noisy terms independently of the missing initial data ones. With the sentinel method, there is a gain of time calculations when interested in simulations.

As we will see, the problem of finding a sentinel is equivalent to a null-controllability problem (see the general case in the book by J.-L. Lions [7]) where the control and the observation have their support in the same open set. In our case, the sentinel theory is used in its general setting since we consider two different open sets for the control and the observation. This is a new issue for singular problems in general.

Optimal control of distributed parameter systems governed by a system of parabolic equations is of special importance for propagation processing problems which are generally expressed by the resolution of the heat equation. The use of these equation may however leave a gap between the theoretical solutions and the experimental ones, then the use of optimal control allows to fill the gap, as it permits to optimize the distance between the two solutions (See [15] and the references therein)

As an immediate application, the existence of a discriminating sentinel for a *nonlinear* singular parabolic equation can be discussed, as we will see in the following section. Finally we note that the backward problem appears under this form in the Lions sentinel theory [7]. Let us mention the works by A. Omrane [16] and by Miloudi et al. in [13] and [14] who solved the control with constraints problem for the heat equation using a well adapted Carleman inequality (see also A. Fursikov and O. Yu Imanuvilov [21]).

As noticed above, we will prove that the sentinel problem is equivalent to a null-controllability one. The general null-controllability problem for the heat equation is well understood (see J. L. Lions [7] and C. Bardos *et* al. [19]). Indeed, assuming the geometric control condition (GCC) introduced by [19], one can establish an observation estimate which yields by the HUM method of Lions [7], to the existence of the control v. The (GCC) is a microlocal making a like to bicharacteristic rays of the heat operator. Moreover, it is equivalent to exact controllability of the linear heat equation (with stability with respect to small perturbations). We state a more general singular null-controllability problem in this article.
The paper is organized as follows :
In section 3.2, we give an application to the Sentinel theory of Lions for nonlinear heat problems with incomplete data, and show the existence of a non trivial sentinel for the heat equation, where the control and observation have their supports in two different open sets.
In section 3.3, we state and prove the Singular null-controllability under the constraints which corresponds to the search of discriminating sentinels. We use a well adapted version of the classical HUM method of Lions [7]. In particular, the observability inequality is established.

3.2 Towards a sentinel problem

. we set $Q = (0,T)\times\Omega$, $\Sigma = (0,T)\times\Gamma$, and we consider the following nonlinear heat problem in several dimensions :

$$\frac{\partial y}{\partial t} - \Delta y + f(y) \;=\; F \qquad \text{in } Q \tag{3.1}$$

f being a $\mathcal{C}^1$ function, with the following initial data of incomplete information :

$$y(T) \;=\; y^0 + \tau\hat{y}^0 \qquad \text{in } \Omega, \tag{3.2}$$

where the function $y^0 \in L^2(\Omega)$ is known, and where $\tau\hat{y}^0$ is unknown (see Lions [7] page 156). Without loss of generality, we may assume that

$$\text{for all } \hat{y}^0 \text{ such that } \left\|\hat{y}^0\right\|_{L^2} \leq 1 \qquad \text{and } \tau \text{ small .}$$

Noisy terms appear in a part of the domain as follows:

$$F \;=\; \xi_0 + \lambda\hat{\xi}_0 \qquad \text{in } Q \tag{3.3}$$

where $\xi_0 \in L^2(Q)$ is given, and where $\lambda\hat{\xi}_0$ is not known too.

The goal is to find a method to estimate the noise (missing) term $\lambda\hat{\xi}_0$. Several methods can be used. The famous one is the least squares method. However with this method, the pollution and initial unknown terms $\tau\hat{y}^0$, $\lambda\hat{\xi}_0$ are computed together and we cannot really separate them (see Lions [7], and Ainseba [8] or [13] for the parabolic case).

Here, we use the sentinel method of Lions which is a method of detecting one parameter, independently of the others. To have a chance to detect pollution, we observe the system in some open subset $\mathcal{O} \subset \Omega$ called observatory, during time T. We denote by y_{obs} this observation, which is formulated by

$$y_{obs} = m_0 + \sum_{i=1}^{N} \beta_i m_i. \tag{3.4}$$

The functions $m_0, m_1, \cdots, m_N$ are known in $L^2(\mathcal{O} \times (0,T))$ but the β_i which are real numbers are unknown. We assume that the β_i are "small". The β_i terms are called the interference (or noisy) terms as well. Without loss of generality, we can assume that for $1 \leq i \leq N$: the functions m_i are linearly independent.

Remark 3.2.1 *In the case of the heat equation, the observatory $\mathcal{O}$ can be chosen arbitrarily small, as well as the final time T.*

Sentinel We now introduce the notion of *sentinel* following the definition in the book chapter by A. Omrane [16] (see also [14]). In this definition, the observation and the control may have different support sets but are not disjoined. Indeed, one can observe somewhere in the domain, and control in another part of the domain Ω. This *natural* definition leads to nontrivial controllability problems.
Let h_0 be a given function on $(0,T) \times \mathcal{O}$ such that

$$h_0 \geq 0, \quad \int_0^T \int_{\mathcal{O}} h_0 \, dxdt = 1. \tag{3.5}$$

Let besides ω be an open and non empty subset of Ω. For a control function $u \in L^2((0,T)\times\omega)$, we introduce the functional

$$\mathcal{S}(\lambda,\tau) = \int_0^T \int_{\mathcal{O}} h_0 \, y(t,x;\lambda,\tau) \, dxdt + \int_0^T \int_{\omega} u \, y(t,x;\lambda,\tau) \, dxdt. \tag{3.6}$$

We shall say that S defines a sentinel (for the system (3.1)-(3.3) and (3.5)) if there exists u such that the pair (u, S) satisfies to the following two conditions:
- S is insensitive at first order with respect to the missing terms $\tau\hat{y}^0$, which means

$$\frac{\partial \mathcal{S}}{\partial \tau}(0,0) = 0, \tag{3.7}$$

- and u is of minimal norm in $L^2((0,T)\times\omega)$, in the sense

$$\|u\|_{L^2((0,T)\times\omega)} \quad = \inf_{v\in L^2((0,T)\times\omega)} \|v\|. \tag{3.8}$$

Remark 3.2.2 *The classical point of view of Lions lies on h_o and u, having their support in the same open set of observation $\mathcal{O}=\omega$. In this case, the question of the existence of a sentinel such that (3.7) holds is evident. Indeed, $h_o=-u$ is a solution, and the only question is the calculus of the optimal control (3.8).*

The point of view considered here is a sentinel notion defined by the function h_o, an observation y_{obs} and a control u, but with h_o having its support in $\mathcal{O}$ and u of support in ω with $\omega\neq\mathcal{O}$. In this case, the existence of a sentinel is not guaranted.

Remark 3.2.3 *The case of noisy observation (3.4) is discussed in subsection 3.4 below.*

3.2.1 The adjoint state. A controllability problem

Denote by $y_\tau=\frac{\partial y}{\partial\tau}(0,0)$ for $\lambda=\tau=0$. Then y_τ satisfies to the following system :

$$\left|\begin{array}{rcll} \Box_{a_0} y_\tau &=& 0 & \text{in} \quad Q, \\ y_\tau &=& 0 & \text{on} \quad \Sigma, \\ y_\tau(T) &=& \hat{y}^0 & \text{in} \quad \Omega, \end{array}\right. \tag{3.9}$$

where $\Box_{a_0}$ given by:

$$\Box_{a_0} = \frac{\partial}{\partial t} - \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} + a_0\, I, \tag{3.10}$$

and $\Box_{a_0}^*$ given by:

$$\Box_{a_0}^* = -\frac{\partial}{\partial t} - \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} + a_0\, I, \tag{3.11}$$

is the d'Alembertian with potential

$$a_0 := f'(y_o) \in L^\infty(Q) \tag{3.12}$$

where a_0 is real valued, and $y_o=y(t,x;0,0)$. It is well known that these linear problems admits each one a unique solution in the space $\mathcal{C}([0,T];H_0^1(\Omega))\cap\mathcal{C}^1([0,T];\,L^2(\Omega))$.

We immediately deduce that (3.7) is equivalent to

$$\int_0^T\int_{\mathcal{O}} h_0\, y_\tau(t,x;\lambda,\tau)\,dxdt + \int_0^T\int_\omega w\, y_\tau(t,x;\lambda,\tau)\,dxdt = 0, \tag{3.13}$$

We then prove the following result :

Proposition 3.2.1 *Let be q the solution to the ill-posed backward problem:*

$$\begin{cases} \Box^*_{a_0} q &= h_o\chi_{\mathcal{O}} + w\chi_\omega & in \quad Q, \\ q &= 0 & on \quad \Sigma, \\ q(0) &= 0 & in \quad \Omega. \end{cases} \tag{3.14}$$

Then, the existence of a sentinel for (3.1)-(3.3), insensitive to the missing data (i.e. such that (3.13) hold), is equivalent to the null-controllability problem (3.14) together with

$$q(T) = 0 \quad in \ \ \Omega. \tag{3.15}$$

Proof. Multiplying (3.14) by y_τ and integrating by parts we find :

$$\int_Q \Box^*_{a_0} q \, y_\tau \, dxdt = \int_Q q\Box_{a_0} y_\tau \, dxdt + \int_\Omega q(T)y_\tau(T)\,dx - \int_\Omega q(0)y_\tau(0)\,dx + \int_\Sigma \frac{\partial y_\tau}{\partial \nu} q \, d\sigma - \int_\Sigma \frac{\partial q}{\partial \nu} y_\tau \, d\sigma$$

Then

$$\int_\Omega q(T)y_\tau(T)\,dx \ = \ \int_Q (h_o\chi_{\mathcal{O}} + w\chi_\omega y_\tau)\,dxdt$$

But, y_τ is solution to (3.9).
Thus, $q(T) = 0$ in Ω; that is finally (3.15). The converse is obvious. ■

3.3 Null-Controllability

In this section, we consider the singular parabolic system

$$\begin{cases} \Box^*_{a_0} q &= v & \text{in} \quad Q, \\ q &= 0 & \text{on} \quad \Sigma, \\ q(0) &= q_0 & \text{in} \quad \Omega, \end{cases} \tag{3.16}$$

where the d'Alembertian $\Box^*_{a_0}$ with potential a_0, is given by (3.11) such that (3.12). It is well known that given $v \in$ sub space dense $\in L^1([0,T];\,L^2(\Omega))$ and $q_0 \in H^1_0(\Omega)$, the problem (3.16) admits a unique solution:

$$q \in \mathcal{C}([0,T]; H^1_0(\Omega)) \cap \mathcal{C}^1([0,T];\, L^2(\Omega)).$$

Now, we state the problem of exact controllability for solutions of system (3.16). Let ω be an open subset of Ω; denote by $(0,T)\times\omega$ the interior cylinder and χ_ω its characteristic function. Given $q_0 \in H^1_0(\Omega)$, the goal is to find a source v in $L^2((0,T)\times\omega)$ such that the unique solution q of (3.16) satisfies

$$q(T) = 0 \qquad \text{in} \quad \Omega. \tag{3.17}$$

The inverse problem is to determine the conductivity distribution in Ω from boundary measurements.

3.4 Case of a Discriminating sentinel

It is worth noticing that an application to the above result on controllability with constraints on the control, the sentinel theory of Lions for parabolic problems with missing data can be analyzed.

Definition 3.4.1 *We say that S is a discriminating sentinel for the system (3.1)-(3.4) and (3.5), if there exists w such that the pair (w, S) satisfies to (3.7)-(3.8), and if*
- S is insensitive to interference terms $\beta_i m_i$, i.e.:

$$\int_0^T \int_{\mathcal{O}} h_0 \, m_i \, dxdt + \int_0^T \int_\omega w \, m_i \, dxdt \quad = \quad 0, \quad 1 \le i \le N. \tag{3.18}$$

Let be $\mathcal{K}$ the vector subspace generated in $L^2((0,T) \times \omega)$ by the N independent functions $m_i \chi_\omega$, $1 \le i \le N$. It is easy to see that there exists a unique $k_0 \in \mathcal{K}$ such that

$$\int_0^T \int_{\mathcal{O}} h_0 \, m_i \, dxdt + \int_0^T \int_\omega k_0 \, m_i \, dxdt = 0, \quad 1 \le i \le N.$$

Remark 3.4.1 *The vector space $\mathcal{K}$ plays the same role as in the previous section. Moreover, condition (3.18) is equivalent to*

$$w - k_0 = v \in \mathcal{K}^\perp. \tag{3.19}$$

Finally and to sum up, the problem consisting in obtaining the control w such that the pair (w, S) satisfies (3.7) with (3.18) (the same as (4.23) with (3.19)) is equivalent to finding the control v such that the pair (v, q) satisfies the following system

$$\left\{ \begin{array}{lclll} & v \in \mathcal{K}^\perp, & & & \\ \Box_{a_0}^* q & = & h + v\chi_\omega & \text{in} & Q, \\ q & = & 0 & \text{on} & \Sigma, \\ q(0) & = & 0 & \text{in} & \Omega, \end{array} \right. \tag{3.20}$$

and

$$q(T) = 0 \ \text{ in } \Omega. \tag{3.21}$$

where $h = h_0 \chi_{\mathcal{O}} + k_0 \chi_\omega$. Hence, we are considering the original problem with the controllability under constraints.

3.4.1 Null-controllability under constraints

We present here a fundamental theorem by B. Dehman and A. Omrane [16], that we should use later. Here, we need the following hypothesis *A1* and *A2* (we recall that $\mathcal{K}$ is the finite dimension linear subspace of $L^2((0,T) \times \omega)$ defining the constraints).
A1. The couple (ω, T) satisfies the geometric control condition (GCC).
A2. The only element $k \in \mathcal{K}$ satisfying $\Box_{a_0}^* k = 0$ in Q is the trivial element $k \equiv 0$.

Then we have the theorem :

Theorem 3.4.1 *Under assumptions A1 and A2, for every $q_0 \in H_0^1(\Omega)$, there exists a constrained control function $v \in L^2((0,T) \times \omega)$ such that the state solution q of problem*

$$\left\{\begin{array}{rcll} \Box_{a_0}^* q &=& \chi_\omega v & in \quad Q, \\ q &=& 0 & on \quad \Sigma, \\ q(0) &=& q_0 & in \quad \Omega, \end{array}\right. \tag{3.22}$$

satisfies (3.17).

Proof. For the proof of Theorem 3.4.1, a suitable version of the HUM method is used (see B. Dehman and A. Omrane [16]). ■

3.4.2 Discriminating sentinel.

We finish this work by proving the existence of a discriminating sentinel:

Proposition 3.4.1 *Under assumptions A1 and A2 of theorem 3.4.1, there exists a unique and non trivial discriminating sentinel for the problem (3.1)-(3.4), insensitive to the missing data and to the noise terms $\beta_i m_i$, $1 \le i \le N$.*

Proof. From the above theorem, the existence of a sentinel, insensitive to the missing data (i.e. here such that (3.7) and (3.19) hold) is guaranteed, if we prove that there exists $v \in \mathcal{K}^\perp$ such that we have (3.15). The above system (3.20) is equivalent to the system (3.16) under constraints. Indeed, if we solve

$$\left\{\begin{array}{rcll} \Box_{a_0}^* z &=& h & \text{in} \quad Q, \\ z &=& 0 & \text{on} \quad \Sigma, \\ z(0) &=& 0 & \text{in} \quad \Omega \end{array}\right.$$

and we put $\bar{q} = q - z$, then $\bar{q}(0) = q(0)$. And so, controlling q or $\bar{q}$ solution of

$$\left\{\begin{array}{rcll} & v \in \mathcal{K}^\perp, & & \\ \Box_{a_0}^* \bar{q} &=& v\chi_\omega & \text{in} \quad Q, \\ \bar{q} &=& 0 & \text{on} \quad \Sigma, \\ \bar{q}(0) &=& 0 & \text{in} \quad \Omega \end{array}\right. \tag{3.23}$$

is the same.
Moreover, the system (3.23) with constraints on the control is null-controllable thanks to Theorem 3.4.1. ■

3.5 Conclusion

In this chapter we were concerned by the problem of identification of noisy terms which arise, first in Stockes problem and secondly in singular problem as for remote sensing problems, and which are modeled by a linear singular parabolic equation. Some data are missing, then the sentinel method of J.-L. Lions (1992) is used; it is a particular least square-like method

which permits to distinguish between the missing terms and the pollution terms.
In particular, the sentinel theory is given here in its more general and more realistic setting for Stockes problem and parabolic problems: in this case, the observation and the control have their supports in different open sets but not disjoint .
The problem of finding a sentinel is equivalent to a singular null-controllability problem for the parabolic equation that we solve.

Chapter 4

New modification Sentinel for Linear Parabolic Equation

4.1 Introduction

We here use a new modified sentinel which changes the classic definition of the sentinel method of J.-L. Lions (1992). The sentinel permits to distinguish between the pollution terms and the missing terms.
We are interested in the identification of the pollution terms which arise in regular parabolic problems. As it is well known, these problems generate distributed parameters which here are modeled by a heat linear parabolic equation. Here we want to estimate the pollution terms independently of the missing initial data ones. In general, the sentinel method is an important tool when one is interested in numerical simulations.

As we will see, the problem of finding the classical sentinel is equivalent to finding the optimal control of adjoint system of heat equation (see Ainseba B.-E. [8] and the general case in the book by J.-L. Lions [7]) where the control and the observation have their supports in the same open set. In our case, the new modificatied sentinel is used in its general setting since we consider two different open sets for the control and the observation.

Optimal control of distributed parameter systems governed by a system of parabolic equations is of special importance for propagation processing problems which are generally expressed by the resolution of the heat equation (see Fursikov [21],Brezis [3]). The use of these equations may however leave a gap between the theoretical solutions and the experimental ones, then the use of optimal control allows to fill the gap, as it permits to optimize the distance between the two solutions.
As an immediate application, the existence of a new modificatied sentinel for a *linear* parabolic equation can be discussed, as we will see in the following section. We note that the backward problem appears under this form in the Lions sentinel theory [7].
As noticed above, we will prove that the sentinel problem is equivalent to finding the unique optimal control of the adjoint system. We will follow the same techniques of the null-controllability problem for the heat equation which is well understood (see J. L. Lions [7],A. Omrane [16]). We state a more general null-controllability problem in this work.

4.2 Towards a sentinel problem

Let Ω be a bounded open subset of $\mathbb{R}^d$, $d \in \mathbb{N}^*$ with boundary Γ of class $\mathcal{C}^2$. For $T > 0$, we set $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \Gamma$, and we consider the following linear heat problem in several dimensions:

$$\frac{\partial y}{\partial t} - \Delta y \; = \; F \qquad \text{in } \; Q \tag{4.1}$$

with the following initial data of incomplete information :

$$y(0) \; = \; \tau \hat{y}^0 \qquad \text{in } \; \Omega \tag{4.2}$$

and with boundary condition

$$y \; = \; 0 \qquad \text{on } \; \Sigma \tag{4.3}$$

where $\tau \hat{y}^0 \in L^2(\Omega)$ is unknown. (see Lions [7] page 156). Without loss of generality, we may assume that

$$\|\hat{y}_0\|_{L^2} \leq 1 \qquad \text{and } \tau \; \text{ small .}$$

Pollution terms appear in a part of the domain as follows:

$$F \; = \; \lambda \hat{\xi}_0 \qquad \text{in } \; Q \tag{4.4}$$

where $\lambda \hat{\xi}_0 \; \in \; L^2(Q)$ is not known too.

Now we write the equation (4.1) in two equations:

$$\left| \begin{array}{rcll} Ly_\lambda & = & \hat{\xi}_0 & \text{in} \quad Q \\ y_\lambda & = & 0 & \text{on} \quad \Sigma \\ y_\lambda(0) & = & 0 & \end{array} \right. \tag{4.5}$$

$$\left| \begin{array}{rcll} Ly_\tau & = & 0 & \text{in} \quad Q \\ y_\tau & = & 0 & \text{on} \quad \Sigma \\ y_\tau(0) & = & \hat{y}_0 & \end{array} \right. \tag{4.6}$$

where L and L^* are given respectively by:

$$L = \frac{\partial}{\partial t} - \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \tag{4.7}$$

$$L^* = -\frac{\partial}{\partial t} - \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} \tag{4.8}$$

Then

$$y = \tau y_\tau + \lambda y_\lambda$$

And we have the following informations :

$$y_\tau, \;\; y_\lambda \;\text{ in } L^2(0,T;H_0^1(\Omega)) \tag{4.9}$$

we have too

$$t \;\to y_\tau, \;\; y_\lambda(t) \textit{ are continuous from } [0,T] \to L^2(\Omega) \tag{4.10}$$

Remark 4.2.1 *Let us recall that*

$$\Big| \; \textit{each of two systems } (4.5) \textit{ and}(4.6) \textit{ admits a unique solution} \tag{4.11}$$

see Lions ([7])
The goal is to find a method to estimate the pollution term $\lambda\hat{\xi}_0$. Several methods can be used. The famous one is the least square method. However with this method, the pollution and initial unknown terms $\tau\hat{y}^0$, $\lambda\hat{\xi}_0$ are computed together and we cannot really separate them (see Lions [7], and Ainseba [8] or [13] for the parabolic case).

Here, we use the new sentinel method which is a method of identifying one parameter, independently of the others. To have a chance to identify pollution, we observe the system in some open subset $\mathcal{O} \subset \Omega$ called observatory, during time T. We denote by y_{obs} this observation, which is known in $L^2(\mathcal{O} \times (0,T))$.

Remark 4.2.2 *In the case of the heat equation, the observatory $\mathcal{O}$ can be chosen arbitrarily small.*

We now will call back the notion of *sentinel* following the definition in the book chapter by Lions[7] (see also [14]). In this definition, the observation and the control may have different support sets but must be not disjoint.

Definition 4.2.1 *Let h_0 be a given function on $(0,T) \times \mathcal{O}$ such that*

$$h_0 \geq 0, \qquad \int_0^T \int_{\mathcal{O}} h_0 \, dxdt = 1. \tag{4.12}$$

Let besides ω be an open and non empty subset of Ω. For a control function $u \in L^2((0,T)\times\omega)$, we introduce the functional

$$\mathcal{S}(\lambda,\tau) = \int_0^T \int_{\mathcal{O}} h_0 \, y(t,x;\lambda,\tau) \, dxdt + \int_0^T \int_{\omega} u \, y(t,x;\lambda,\tau) \, dxdt. \tag{4.13}$$

We shall say that S defines a sentinel (for the system (4.1)-(4.3) and (4.12)) if there exists u such that the pair (u,S) satisfies to the following two conditions:
- S is insensitive at first order with respect to the missing terms $\tau\hat{y}^0$, which means

$$\frac{\partial \mathcal{S}}{\partial \tau}(0,0) = 0, \tag{4.14}$$

- and u is of minimal norm in $L^2((0,T)\times\omega)$, in the sense

$$\|u\|_{L^2((0,T)\times\omega)} = \inf_{v\in L^2((0,T)\times\omega)} \|v\|. \tag{4.15}$$

In the above sentinel, one can not observe the state of the system somewhere in the domain, and control in another part of the domain Ω. Our goal is to study this situation.

4.3 Controllability

We give the function G from $L^2(\omega)$ to $L^2(\mathcal{O})$ which verifies:
There is a positive constant c such that $\|G^*z\| \geq c\|z\|$, for all z in $L^2(\mathcal{O})$ And, let be the regular parabolic system:

$$\left| \begin{array}{rcll} L^*p &=& \chi_{\mathcal{O}}(h_0+Gv) & \text{in} \quad Q \\ p &=& 0 & \text{on} \quad \Sigma \\ p(T) &=& 0 & \end{array} \right. \tag{4.16}$$

We divide this equation into two equations:

$$\left| \begin{array}{rcll} L^*p^0 &=& \chi_{\mathcal{O}}Gv & \text{in} \quad Q \\ p^0 &=& 0 & \text{on} \quad \Sigma \\ p^0(T) &=& 0 & \end{array} \right. \tag{4.17}$$

and

$$\left| \begin{array}{rcll} L^*p^1 &=& \chi_{\mathcal{O}}h_0 & \text{in} \quad Q \\ p^1 &=& 0 & \text{on} \quad \Sigma \\ p^1(T) &=& 0 & \end{array} \right. \tag{4.18}$$

where χ_ω such that

$$\chi_\omega u = \begin{cases} u & \textit{on} \quad (0,T)\times\omega \\ 0 & \textit{else} \end{cases}$$

Remark 4.3.1 *For every given v in $L^2(\omega)$, the system (4.16) admits a unique solution $p = p(v)$ in $L^2(0,T;H_0^1(\Omega)$ for the detail see [3].*

The null controllability problem for (4.16) consists in finding v in $L^2(0,T;L^2(\omega))$ such that if p is solution of (4.16) then

$$P(0) = 0. \tag{4.19}$$

We can obtain this unique solution with the use of the following: First, we introduce the equation:

$$\left| \begin{array}{rcll} L\varphi &=& 0 & \text{in} \quad Q \\ \varphi &=& 0 & \text{on} \quad \Sigma \\ \varphi(0) &=& \rho & \end{array} \right. \tag{4.20}$$

And we pose $v = -B^*\varphi$ and we define the operator Λ by $\Lambda\rho = p^0(0)$ then we have

$$\prec \Lambda\rho, \rho \succ \;=\; |B^*\varphi|^2_{L^2(\Omega)} \tag{4.21}$$

Let F be the completeness of $L^2(\Omega)$ with the norm $\|\rho\|^2_F = |B^*\varphi|^2_{L^2(\Omega)}$ and F^* is the dual space of F, then the operator Λ is ismorphism from F to F^* so for $-p^1(0) \in F^*$ there existe a $\rho \in F$ such that $p^0(0) + p^1(0) = 0 = p(0)$.

4.4 Existence of a non trivial new modification sentinel

Now , we can give the new definition of sentinel which one can observe somewhere in the domain, and control in another part of the domain Ω

Definition 4.4.1 *we introduce the functional*

$$\mathcal{S}(\lambda, \tau) = \int_0^T \int_{\mathcal{O}} (h_0 + G(u))\, y(t, x; \lambda, \tau)\, dxdt. \tag{4.22}$$

Where y is the solution of the system 4.1.
We shall say that S defines a sentinel (for the system (4.1)-(4.3) and (4.12)) if there exists u such that the system 4.3 have unique solution and the pair (u, S) satisfies to the following two conditions:
- S is insensitive at first order with respect to the missing terms $\tau\hat{y}^0$, which means

$$\frac{\partial \mathcal{S}}{\partial \tau}(0, 0) = 0, \tag{4.23}$$

- and u is of minimal norm in $L^2((0, T) \times \omega)$, in the sense

$$\|u\|_{L^2((0,T)\times\omega)} \;=\; \inf_{v\in L^2((0,T)\times\omega)} \|G(v)\|. \tag{4.24}$$

Remark 4.4.1 *The classical point of view of Lions lies on h_o and u, having their supports in the same open set of observation $\mathcal{O} = \omega$. In this case, the question of the existence of a sentinel such that (4.23) holds is evident. Indeed, $h_o = -G(u)$ is a solution, and the only question is the calculus of the optimal control (4.24).*
The point of view considered here is a sentinel notion defined by the function h_o, an observation y_{obs} and a control u, but with h_o having its support in $\mathcal{O}$ and u of support in ω with $\omega \neq \mathcal{O}$. In this case, the existence of a sentinel is not guaranted.

4.4.1 The identification of the pollution term

We deduce that (4.23) is equivalent to

$$\frac{\partial \mathcal{S}}{\partial \tau}(0, 0) = \int_0^T \int_{\mathcal{O}} (h_0 + G(u))\, y_\tau(t, x; \lambda, \tau)\, dxdt = 0, \tag{4.25}$$

Now multiplying (4.16) by y_τ and integrating by parts we find:

$$\int_Q y_\tau L^* p \;\; dxdt = \int_Q pLy_\tau \; dxdt + \int_\Omega p(T)y_\tau(T)\; dx - \int_\Omega p(0)y_\tau(0)\; dx \qquad + \int_\Sigma \frac{\partial y_\tau}{\partial \nu} p\; d\sigma - \int_\Sigma \frac{\partial p}{\partial \nu} y_\tau \; d\sigma$$

because that

$$\int_\Omega p(T)y_\tau(T)\; dx - \int_\Omega p(0)y_\tau(0)\; dx = 0$$

and

$$\int_\Sigma \frac{\partial y_\tau}{\partial \nu} p\, d\sigma - \int_\Sigma \frac{\partial p}{\partial \nu} y_\tau\, d\sigma = 0$$

So, we deduce:

$$\int_0^T \int_{\mathcal{O}} y_\tau(h_0 + G(u)\; dxdt \;\; = \;\; \frac{\partial S}{\partial \tau}(0,0) \;\; = \;\; 0$$

due to the fact that, y_τ is solution of (4.6)
and then

$$S(\lambda,\tau) = S(0,0) + \tau\frac{\partial S}{\partial \tau}(0,0) + \lambda\frac{\partial S}{\partial \lambda}(0,0) \;\; then \;\; S_{obs} = S(0,0) + \lambda\frac{\partial S}{\partial \lambda}(0,0)$$

where

$$S_{obs} = \int_0^T \int_{\mathcal{O}} (h_0 + G(u))y_{obs}\; dxdt \quad \text{and} \; \frac{\partial S}{\partial \lambda}(0,0) = \int_0^T \int_\Omega y_\lambda(h_0 + G(u))\, dxdt = \int_Q \hat{\xi}_0 p dxdt$$

Then

$$\textstyle\int_Q (\lambda\hat{\xi}_0)pdxdt = S_{obs} - S(0,0) = S_{obs},$$

since $S(0,0) = 0$

Remark 4.4.2 *If G is equal to identity then we have the classical definition of the sentinel 4.13*

Chapter 5

Approximate Controlability of the Stokes system

5.1 Generalities. Synopsis

The content of this work was considered as a preliminary step to a more ambitious goal, namely, the control of systems governed by the Navier-Stokes equations modeling incompressible viscous flow. Indeed, substantial progress concerning this objective took place in the late 1990s.
Let us say that the control problems and methods which have been discussed so far in liturature have been mostly concerned with systems governed by linear diffusion equations of the parabolic type, associated with second order elliptic operators. Indeed, these methods have been applied in, for example, Berggren [65]. Glowinski, and J.L. Lions [44], to the solution of approximate boundary controllability problems for systems governed by strongly advection dominated linear advection-diffusion equations. These methods can also be applied to systems of linear advection-diffusion equations and to higher-order parabolic equations (or systems of such equations). Motivated by the solution of controllability problems for the Navier-Stokes equations modeling incompressible viscous flow, we will discuss now controllability issues for a system of partial differential equations which is not of the Cauchy-Kowalewska type, namely, the classical Stokes system.

5.2 Formulation of the Stokes system. A fundamental controllability result

Let Ω be a bounded connected open set (that is, a bounded domain) in $\mathbb{R}^n$.
We shall also assume that $\Gamma = \partial\Omega$ is "sufficiently smooth," which is also not mandatory. Let $\mathcal{O}$ be an open subset of Ω. We emphasize here, at the very beginning, that $\mathcal{O}$ can be arbitrary "small". The control function u will be with support in $\mathcal{O}$; it is a distributed control. The state equation is given by

$$\left\{ \begin{array}{rcll} \dfrac{\partial y}{\partial t} - \Delta y &=& u\chi_{\mathcal{O}} - \nabla\pi & \text{in} \quad \mathcal{Q}, \\ \nabla . y &=& 0 & \text{in} \quad \mathcal{Q}, \end{array} \right. \tag{5.1}$$

subject to the following initial and boundary conditions:

$$y(0) = 0 \quad \text{and} \quad y = 0 \quad \text{on} \quad \Sigma (= (0,T)) \times \Gamma. \tag{5.2}$$

where $\chi_{\mathcal{O}}$ is the characteristic function of the set $\mathcal{O}$. From now on, we shall denote by $\mathcal{Q}$ the space-time domain $(0,T) \times \Omega$.
In (5.1), we shall assume that

$$u \in \mathcal{U} = \quad \text{a closed subspace of} \quad \left(L^2(0,T) \times (\mathcal{O})\right)^n \tag{5.3}$$

To fix ideas, we shall take $n = 3$, and consider the following cases for $\mathcal{U}$:

$$\mathcal{U} = \left(L^2(0,T) \times (\mathcal{O})\right)^3, \tag{5.4}$$

$$\mathcal{U} = \left\{u, u = \{u_1, u_2, 0\}, \{u_1, u_2\} \in \left(L^2(0,T) \times (\mathcal{O})\right)^2\right\}, \tag{5.5}$$

$$\mathcal{U} = \left\{u, u = \{u_1, 0, 0\}, u_1 \in L^2(0,T) \times (\mathcal{O})\right\}. \tag{5.6}$$

Problem (5.1), (5.2) has a unique solution, such that (in particular)

$$\left| \begin{array}{l} y(u) \in L^2\left(0,T; (H_0^1(\Omega))^3\right), \\ \nabla . y(u) = 0, \\ \dfrac{\partial y(u)}{\partial t} \in L^2(0,T;V'), \end{array} \right. \tag{5.7}$$

where V' is the dual space of

$$V = \left\{\varphi, \varphi \in \left(H_0^1(\Omega)\right)^3, \quad \nabla . \varphi = 0\right\}. \tag{5.8}$$

Above, $H^1(\Omega)$ and $H_0^1(\Omega)$ are the functional spaces defined as follows:

$$H^1(\Omega) = \left\{\phi, \phi \in L^2(\Omega), \frac{\partial \phi}{\partial x_i} \in L^2(\Omega), \quad \forall i = 1, ..., n\right\},$$

and

$$H_0^1(\Omega) = \left\{\phi, \phi \in H^1(\Omega), \quad \phi = 0 \quad \text{on} \quad \Gamma\right\}.$$

It follows from (5.7) that

$$t \longrightarrow y(t;u) \quad \text{belongs to} \quad \mathcal{C}^0([0,T];H), \tag{5.9}$$

where

$$\begin{aligned} H = & \quad \text{closure of} \quad V \quad \text{in} \quad \left(L^2(\Omega)\right)^3 \\ = & \left\{\varphi, \varphi \in \left(L^2(\Omega)\right)^3, \quad \nabla . \varphi = 0, \quad \varphi . \upsilon = 0 \quad \text{on} \quad \Gamma\right\} \end{aligned} \tag{5.10}$$

(υ denotes the unit outward normal vector at Γ).
We are now going to prove the following:

Proposition 5.2.1 *If $\mathcal{U}$ is defined by either (5.4) or (5.5), then the space spanned by $y(T;u)$ is dense in H.*

Proof. It suffices to prove the above results for the case where $\mathcal{U}$ is defined by (5.5). Let us therefore consider $f \in H$ such that,

$$\int_{\Omega} y(T;u).f dx = 0, \quad \forall u \in \mathcal{U}. \tag{5.11}$$

With f we associate the solution q of the following backward Stokes problem:

$$\left\{ \begin{array}{rcll} -\dfrac{\partial q}{\partial t} - \Delta q & = & -\nabla \sigma & \text{in} \quad \mathcal{Q}, \\ \nabla .q & = & 0 & \text{in} \quad \mathcal{Q}, \end{array} \right. \tag{5.12}$$

$$q\left(T\right) = f, \quad q = 0 \quad \text{on} \quad \Sigma. \tag{5.13}$$

Multiplying the first equation in (5.12) by $y = y(u)$ and integrating by parts we obtain

$$\int_{\mathcal{Q}} q.udxdt = 0, \quad \forall u \in \mathcal{U}. \tag{5.14}$$

Suppose that $q = \{q_1, q_2, q_3\}$; it follows then from (5.14) that

$$q_1 = q_2 = 0 \quad \text{in} \quad \mathcal{O} \times (0,T)\,. \tag{5.15}$$

Since q is (among other things) continuous in t and real analytic in x in $\mathcal{Q}(= \Omega \times (0,T))$, it follows from (5.15) that

$$q_1 = q_2 = 0 \quad \text{in} \quad \mathcal{Q}. \tag{5.16}$$

Since (from (5.12)) $\nabla .q = 0$, it follows from (5.16) that $\frac{\partial q_3}{\partial x_3} = 0$ in $\mathcal{Q}$, which combined with the boundary condition $q_3 = 0$ on Σ implies that $q_3 = 0$ on $\mathcal{Q}$; the t-continuity of q implies that $q(T) = 0$, that is, $f = 0$ (from (5.13)), which completes the proof. ■

Remark 5.2.1 *The above density result does not always hold if $\mathcal{U}$ is defined by (5.6), as shown in Diaz and Fursikov [21].*

Remark 5.2.2 *Proposition 5.2.1 was proved in the lectures of the second author (J.L. Lions) at College de France in 1990-91. Other results along these lines are due to Fursikov [21].*

The density result in Proposition 5.2.1 implies (at least) approximate controllability. Thus, we shall formulate and discuss, in the following sections, two approximate controllability problems.

5.3 Two approximate controllability problems

The first problem is defined by

$$\min_{u\in\mathcal{V}_f}\frac{1}{2}\int_{\mathcal{Q}}|u|^2\,dxdt, \tag{5.17}$$

where

$$\mathcal{V}_f=\{u,u\in\mathcal{U},(u,y)\ \text{verifies (5.1), (5.2), and}\ y\,(T)\in\{y_T+\beta B_H\}\}\,; \tag{5.18}$$

in (5.18), y_T is given in H, β is a positive number arbitrarily small, B_H is the unit ball of H and "to fix ideas" the control space $\mathcal{U}$ is defined by (5.5).
The second problem is obtained by penalization of the final condition $y(T)=y_T$; we have then

$$\min_{u\in\mathcal{U}}\left\{\frac{1}{2}\int_{\mathcal{Q}}|u|^2\,dxdt+\frac{1}{2}k\int_{\Omega}|y\,(T)-y_T|^2\,dx\right\}, \tag{5.19}$$

where, in (5.19), k is an arbitrarily large positive number, y is obtained from u via (5.1), (5.2), and $\mathcal{U}$ is as above.
It follows from Proposition 5.2.1 that both control problems (5.17) and (5.19) have a unique solution.

5.4 Optimality conditions and dual problems

We start with problem (5.19), since it is (by far) simpler than problem (5.17). If we denote by J_k the cost functional in (5.19), we have

$$\begin{aligned}(J_k'\,(u)\,,w)&=\lim_{\substack{\theta\to 0\\ \theta\neq 0}}\frac{J_k\,(u+\theta w)-J_k\,(u)}{\theta} \\ &=\int_{\mathcal{Q}}(u-\psi)\,.wdxdt,\quad \forall u,w\in\mathcal{U},\end{aligned} \tag{5.20}$$

where, in (5.20), the adjoint velocity field ψ is solution of the following backward Stokes problem:

$$\begin{cases}-\dfrac{\partial\psi}{\partial t}-\Delta\psi+\nabla\sigma&=&0\ \text{ in }\ \mathcal{Q},\\ \nabla.\psi&=&0\ \text{ in }\ \mathcal{Q},\end{cases} \tag{5.21}$$

$$\psi=0\ \text{ on }\ \Sigma,\quad \psi(T)=k\,(y\,(T)-y_T)\,. \tag{5.22}$$

Suppose now that v is the unique solution of problem (5.19); it is characterized by

$$\left|\begin{array}{l}v\in\mathcal{U},\\ (J_k'\,(v)\,,w)=0,\quad\forall w\in\mathcal{U},\end{array}\right. \tag{5.23}$$

which implies in turn that the optimal triple $\{v,y,\psi\}$ is characterized by

$$\left|\begin{array}{l}v_1=\psi_1|_{\mathcal{O}}\,,\\ v_2=\psi_2|_{\mathcal{O}}\,,\\ v_3=0,\end{array}\right. \tag{5.24}$$

$$\left\{ \begin{array}{rcll} \dfrac{\partial y}{\partial t} - \Delta y + \nabla \pi & = & v\chi_{\mathcal{O}} & \text{in} \quad \mathcal{Q}, \\ \nabla . y & = & 0 & \text{in} \quad \mathcal{Q}, \end{array} \right. \tag{5.25}$$

$$y\left(0\right) = 0, \quad y = 0 \quad \text{on} \quad \Sigma, \tag{5.26}$$

to be completed by (5.21), (5.22).
To obtain the dual problem of (5.19) from the above optimality conditions, we proceed by introducing an operator $\Lambda \in \mathcal{L}\left(H, H\right)$ defined as follows:

$$\Lambda g = \Phi_g\left(T\right), \quad \forall g \in H, \tag{5.27}$$

where to obtain $\Phi_g\left(T\right)$ we solve first

$$\left\{ \begin{array}{rcll} -\dfrac{\partial F_g}{\partial t} - \Delta F_g + \nabla \sigma_g & = & 0 & \text{in} \quad \mathcal{Q}, \\ \nabla . F_g & = & 0 & \text{in} \quad \mathcal{Q}, \end{array} \right. \tag{5.28}$$

$$F_g\left(T\right) = g, \quad F_g = 0 \quad \text{on} \quad \Sigma, \tag{5.29}$$

and then (with obvious notation)

$$\left\{ \begin{array}{rcll} -\dfrac{\partial \phi_g}{\partial t} - \Delta \phi_g + \nabla \pi_g & = & \left\{F_{1g}, F_{2g}, 0\right\}\chi_{\mathcal{O}} & \text{in} \quad \mathcal{Q}, \\ \nabla . \Phi_g & = & 0 & \text{in} \quad \mathcal{Q}, \end{array} \right. \tag{5.30}$$

$$\Phi_g\left(0\right) = 0, \quad \Phi_g = 0 \quad \text{on} \quad \Sigma, \tag{5.31}$$

(the two above Stokes problems are well posed).
Integrating by parts in time and using Green's formula, we can show (again with obvious notation) that

$$\int_{\Omega} \left(\Lambda g\right) . g' dx = \int_{\mathcal{Q}} \left(F_1 F_1' + F_2 F_2'\right) dx dt, \quad \forall g, g' \in H. \tag{5.32}$$

It follows from relation (5.32) that the operator Λ is symmetric and positive semi definite over H; indeed, using the approach taken in Section 2 to prove Proposition 5.2.1, we can show that the operator Λ is positive definite over H. Back to the optimality conditions, let us denote by f the function $\psi(T)$; it follows then from (5.22) and from the definition of Λ that f satisfies

$$k^{-1} f + \Lambda f = y_T \tag{5.33}$$

which is precisely the dual problem of (5.19). From the symmetry and positivity of Λ, the dual problem (5.33) can be solved by a conjugate gradient algorithm operating in the space H.
We consider now the control problem (5.17); applying, as done previously, the Fenchel-Rockafellar duality theory it can be shown that the unique solution v of problem (5.17) can be obtained via

$$\left| \begin{array}{l} v_1 = \psi_1 \chi_{\mathcal{O}}, \\ v_2 = \psi_2 \chi_{\mathcal{O}}, \\ v_3 = 0, \end{array} \right. \tag{5.34}$$

where, in (5.34), ψ is the solution of the following backward Stokes problem:

$$\begin{cases} -\dfrac{\partial \psi}{\partial t} - \Delta\psi + \nabla\sigma &= 0 \quad \text{in} \quad \mathcal{Q}, \\ \nabla.\psi &= 0 \quad \text{in} \quad \mathcal{Q}, \end{cases} \tag{5.35}$$

$$\psi(T) = f, \quad \psi = 0 \quad \text{on} \quad \Sigma, \tag{5.36}$$

where, in (5.36), f is the solution of the following variational inequality:

$$\left| \begin{array}{l} f \in H, \quad \forall g \in H, \\ \int_\Omega (\Lambda f).(g-f)\,dx + \beta \|g\|_H - \beta \|f\|_H \geq \int_\Omega y_T.(g-f)\,dx, \end{array} \right. \tag{5.37}$$

with $\|g\|_H = \left(\int_\Omega |g|^2\,dx\right)^{\frac{1}{2}}$. Problem (5.37) can be viewed as the dual of problem (5.17).

5.5 Application of the sentinel

The notion of sentinel was introduced by J.L.Lions to study systems of incomplete data [7]. The notion permits to distinguish and to analyze two types of incomplete data: the so called pollution terms on which we look for information's, independently of the other type of incomplete data which is the missing terms, and that we do not want to identify.
Typically, the Lions' sentinel is a functional defined from an open set $\mathcal{O}$ on which we consider three functions: the "observation" y_{obs} corresponding to measurements, a given "mean" function h_0, and a control function u to be determined.
In this chapter, we propose a notion of sentinel which revisits the one of Lions.
Let us remind that Lions' sentinel theory [7] relies on the following three features: the state equation y which is governed by a system of PDE, the observation system and some particular evaluation function: the sentinel itself.
We consider in the first step the Navier-Stokes system

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + \nabla\pi &= \xi + \lambda\widehat{\xi} \quad \text{in} \quad \mathcal{Q} \\ \operatorname{div} y &= 0 \quad \text{in} \quad \mathcal{Q} \\ y(0) &= y_0 + \tau\widehat{y}_0 \quad \text{in} \quad \Omega \\ y &= 0 \quad \text{on} \quad \Sigma \end{cases} \tag{5.38}$$

We are interested in systems with data that are not completely known. The functions ξ and y_0 are known with ξ in $L^2(\mathcal{Q})$ and y_0 in $L^2(\Omega)$. However, the terms $\lambda\widehat{\xi}$ and $\tau\widehat{y}_0$ are unknown, but are such that

$$\left| \begin{array}{l} \left\|\widehat{\xi}\right\|_{L^2(\mathcal{Q})} \leq 1, \quad \|\widehat{y}_0\|_{L^2(\Omega)} \leq 1, \\ \text{and that the reals } \lambda \text{ and } \tau \text{ are small enough.} \end{array} \right. \tag{5.39}$$

This growth condition is classical. Under this growth condition, it is proved in **[75]**, p. 63 that there exists $\alpha > 0$ such that when

$$\left\|\xi + \lambda\widehat{\xi}\right\|_{L^2(\mathcal{Q})} + \|y_0 + \tau\widehat{y}_0\|_{L^2(\Omega)} \leq \alpha$$

the problem (5.38) admits a unique solution. For the sake of simplicity, we denote

$$y\left(x,t;\lambda,\tau\right)=y\left(\lambda,\tau\right) \tag{5.40}$$

The general question we want to address is

$$\left| \begin{array}{l} \text{given some observation of the state of system, can one obtain} \\ \lambda\widehat{\xi} \text{ without any attempt at computing } \tau\widehat{y}_0? \end{array} \right. \tag{5.41}$$

In this context, we refer to $\lambda\widehat{\xi}$ as pollution term, the one we are trying to identify, and the term $\tau\widehat{y}_0$ as the missing one that on which we do not want to identify.
To make things more specific, we consider in the second step the observation process. The observation is the knowledge, along some time period, of some function y_{obs} which is defined on the strip $\mathcal{O}\times(0,T)$ over some nonempty open subset $\mathcal{O}\subset\Omega$, called observatory. The function y_{obs} is assumed to be of the form

$$y_{obs}=m_0+\sum_{i=1}^{M}\eta_i m_i. \tag{5.42}$$

where the functions $m_0, m_1, ..., m_M$ are given measurements of y in $L^2\left(\mathcal{O}\times(0,T)\right)$, but where the real coefficients η_i are unknown. We assume that η_i are small. We refer to the terms $\eta_i m_i$ as the interference terms. We can assume without loss of generality that

$$\text{the functions } m_i \text{ are linearly independent on } \mathcal{O}\times(0,T)\,. \tag{5.43}$$

Finally, we introduce now the notion of sentinel. Let h_0 be a given function on $\mathcal{O}\times(0,T)$ such that

$$h_0\geq 0,\quad \int_0^T\int_{\mathcal{O}} h_0\,dxdt=1. \tag{5.44}$$

Moreover let ω be an open and non-empty subset of Ω. For any control function $u\in L^2\left(\omega\times(0,T)\right)$, set

$$\mathcal{S}(\lambda,\tau)=\int_0^T\int_\omega uy(\lambda,\tau)dxdt+\int_0^T\int_{\mathcal{O}} h_0y(\lambda,\tau)dxdt. \tag{5.45}$$

The role of the function u appears in the following definition. We shall say that $\mathcal{S}$ defines a discriminating sentinel (for the system (5.38), (5.42) and (5.44)) if there exists u such that the functional $\mathcal{S}$ satisfies the following conditions:
(i) $\mathcal{S}$ is stationary at first order with respect to the missing terms $\tau\widehat{y}_0$, for all $\epsilon\geq 0$ we have

$$\left|\frac{\partial\mathcal{S}\left(0,0\right)}{\partial\tau}\right|\leq\epsilon,\quad\forall\widehat{y}_0 \tag{5.46}$$

(ii) $\mathcal{S}$ is stationary with respect to the interference terms $\eta_i m_i$, that is

$$\int_0^T\int_\omega um_i\,dxdt+\int_0^T\int_{\mathcal{O}} h_0m_i\,dxdt=0,\quad 1\leq i\leq M. \tag{5.47}$$

(iii) u is of minimal norm in $L^2\left(\omega\times(0,T)\right)$ among control functions in $L^2\left(\omega\times(0,T)\right)$ which satisfy the above conditions. That is

$$\|u\|_{L^2(\omega\times(0,T))}=\text{ minimum}. \tag{5.48}$$

Remark 5.5.1 *At this point, some comments must be made*
1. According to (5.48), the function $\mathcal{S}$ *if it exists, is unique. We refer to* $\mathcal{S}$ *as the sentinel.*
2. If the functions m_i, $1 \leq i \leq M$, *are null functions, the sentinel* $\mathcal{S}$ *is defined only by (5.46) and (5.48). If* $m_i \neq 0$, *the sentinel* $\mathcal{S}$ *is defined only by (5.46), (5.47) and (5.48) and it is called a discriminating sentinel.*
3. The original Lions' sentinel $\mathcal{S}$ *corresponds to the case* $\omega = \mathcal{O}$. *In this case; if we choose* $u = -h_0$, *then (5.46) and (5.47) hold true, so that problem (5.46)-(5.47) admits a unique solution. Of course this solution may have an interest only if* $u \neq -h_0$. *Now,* $\omega \neq \mathcal{O}$ *and if the support supp(h) of h does not in* ω, *we cannot have* $u = -h_0$ *except when* $u = -h_0 = 0$. *Therefore, the previous definition introduces a generalization of Lions's discriminating sentinel to the case where the observation and the control have their supports in two different open subsets.*
4. The support supp(m_i) *of functions* m_i *is assumed to be included in* $\mathcal{O}$. *Suppose* $\omega \cap \mathcal{O} = \emptyset$ *then* $\int_0^T \int_\omega u m_i dx dt = 0$. *Therefore, it suffices to choose* h_0 *such that* h_0 *is orthogonal to each* m_i *and then (5.47) would be readily verified. Therefore, for all* ω *can neglect the part of* ω *which is out of* $\mathcal{O}$. *So, without loss of generality, it may be assumed that*

$$\omega \subset \mathcal{O} \tag{5.49}$$

5.5.1 Orientation

A detailed study of the notion of sentinel is given in [76], in particular sufficient conditions on h_0 to ensure the existence of a sentinel. We can also find in [76] other examples of sentinels in the case $\omega \subset \mathcal{O}$. They lead to new controllability problems with constraints on the control.

Chapter 6

Instantaneous Sentinel for the Identification of the pollution term in Navier-Stokes system

6.1 Introduction

The aim of this chapter is to present some findings and remarks to some extent on turbulence theory in connection with control theory of distributed systems as well as how to apply the sentinel method to assess the amount of pollution term.
The first remark, address the question of using control theory to derive possibly "optimal", numerical algorithms. One has of course to make precise what is meant by "optimal", i.e. to define a "cost function" in the terminology of control theory. In general terms, one wants to find algorithms which give the information we are looking for and which "suppress" the difficulties introduced by turbulence or chaos.
The second remarki , address the question of controlling turbulence. This situation arises if we can act on the system through control functions. Several approaches are, at least theoretically, conceivable. They also depend on the "cost function". We consider two families of "cost functions". The first one to be considered expresses the "mixing" of scales which is one of the properties of turbulence. The second family of "cost functions" is connected with exact controllability.
There is of course a large variety of models where the remarks to follow would apply.
What we have particularly in mind is the classical set of Navier-Stokes equations.
We shall denote here by y the velocity field and by P the pressure, a quite unusual notation in the "turbulence" circle. From now on y_0 and f are sufficiently small (in the neighborhood of zero), so the problem has a unique solution (see [71],[79], [72]])
The reason is simply that in all what follows we think of y as the state of our system, this state depending on control functions, these control functions being either "artificial" or "natural".
We shall write therefore for the state equation :

$$\left\{ \begin{array}{rcll} \dfrac{\partial y}{\partial t}+(y\nabla)\,y-\Delta y+f\,(y) & = & -\nabla P & \text{in} \quad \mathcal{Q}=\Omega\times(0,T)\,, \\ \operatorname{div} y & = & 0 & \text{in} \quad \mathcal{Q}, \end{array} \right. \tag{6.1}$$

we assume here that $f:\mathbb{R}^n \longrightarrow \mathbb{R}^n, n=2,3$ is of class $\mathcal{C}^{1.}$
Subject to initial conditions and to boundary conditions.
In (6.1) Ω is a bounded open set of $\mathbb{R}^n, n=2,3$.
The control functions, when there are of the "natural" type, will be either in the "source terms", in the right hand side of (6.1), or on the boundary ; more precisely

$$\left\{\begin{array}{rcll} \dfrac{\partial y}{\partial t}+(y\nabla)\,y-\Delta y+f\,(y) & = & -\nabla P+u\chi_{\mathcal{O}} & \text{in} \quad \mathcal{Q}=\Omega\times(0,T)\,, \\ \text{div}y & = & 0 & \text{in} \quad \mathcal{Q}, \\ y & = & 0 & \text{on} \quad \Sigma=\partial\Omega\times(0,T)\,, \end{array}\right. \tag{6.2}$$

where

$$\left|\begin{array}{rcl} \mathcal{O} & = & \text{open set contained in } \Omega, \\ \chi_{\mathcal{O}} & = & \text{characteristic function of } \mathcal{O}, \\ u=u\,(x,t) & = & \text{control function.} \end{array}\right.$$

Equations (6.2) express the fact that we can act on the system in the region $\mathcal{O}$ (not necessarily connected) and which can be "very small".
According to Lesieur [86] it means that the system has two properties

$$\text{"the" state } y \text{ sudden "unpredictable"} \tag{6.3}$$

$$\left|\begin{array}{l} \text{the system satisfies the "increased mixing property"} \\ \text{(the flow is able to mix transported quantities much more rapidly than} \\ \text{if only molecular diffusion processes were involved).} \end{array}\right. \tag{6.4}$$

In order to make (6.3) more precise, we introduce initial conditions in a particular form. We shall assume that

$$y\,(x,0)=y_0\,(x)+\tau\widehat{y_0}\,(x) \quad \text{in} \quad \Omega, \tag{6.5}$$

the function y_0 is known with $y_0 \in H$ "H = Hilbert space". But, the term : $\tau\widehat{y_0}$ (so-called missing term) is unknown, with norm $\leq$1, and where $\tau \in \mathbb{R}$ is "small".
We shall assume that "the" state y is "very sensitive" to τ. So the equations (6.2) and (6.5) uniquely define the solution

$$y=y(x,t;\tau\widehat{y_0},u). \tag{6.6}$$

The "contradiction" is apparent since the strategy that we shall (try to) follow is to derive equations based on (6.6) and on control theory. But these equations are meaningful independently of (6.6) and we will (try to) study directly these equations.
It remains to make precise hypothesis (6.3), (6.4) using "the" state as in (6.6).

6.2 Approximation Methods

The initial condition is given by

$$y\,(x,0)=y_0\,(x) \quad \text{in} \quad \Omega. \tag{6.7}$$

One way (of course not the only one !) to look at this is to rewrite (6.1) under the form

$$\begin{cases} \dfrac{\partial y}{\partial t} + (y\nabla)\, y - \Delta y + w & = & -\nabla P & \text{in} & \mathcal{Q} = \Omega \times (0,T)\,, \\ \operatorname{div} y & = & 0 & \text{in} & \mathcal{Q}, \end{cases} \tag{6.8}$$

with the constraint

$$w = f\,(y)\,. \tag{6.9}$$

The next step is to relax (6.9). Let us consider a functional $\phi\,(y)$ where we think of y, for the time being, as "the" solution of (6.8), (6.7), with Direchelet condition, and that we would like to keep "small".

Example 6.2.1 *If we do not want to consider turbulence on* $\mathcal{O} \subset \Omega$ *and during the time interval* $(0,T)$*, we could take*

$$\phi\,(y) = \int\int_{\mathcal{O}\times(0,T)} (curl\ y)^2 dxdt, \tag{6.10}$$

for example curl contours and curl in the surface stress.

The problem of optimal control we introduce is now the following.
Let $y(w)$ be the solution of (6.8), (6.7) and the conditions of (6.2). We introduce

$$J_k\,(w) = \phi\,(y\,(w)) + k \int_{\mathcal{O}\times(0,T)} (w - f\,(y\,(w)))^2\, dxdt, \tag{6.11}$$

where $k > 0$ plays the role of a penalty term.
We consider then the problem

$$\inf_{w\in L^2(\mathcal{O}\times(0,T))} J_k\,(w)\,. \tag{6.12}$$

This problem admites one unique solution w^0 associed to the solution y^0 of the system (6.8).

Remark 6.2.1 *One can write the optimality system corresponding to problem (6.12). One can add constraints of other types on* w*. And very many variants are of course possible. One can in particular decompose the solution* y *in two parts, say* $y = z + \zeta$ *and apply the above ideas to one part, say* ζ*.*

6.3 Control of Turbulence

Let us consider now (6.2), (6.5) and let us assume for the time being that (6.6) holds true.
We assume that we are in the situation (6.3), (6.4).
Let us consider a functional $\phi_1\,(y)$ which is, in principle, "unpredictable" if there are small variations in the initial condition (6.5).
We shall say that the control u is insensitive with respect to the functional ϕ_1 if

$$\frac{d}{d\tau}\phi_1\,(y\,(\tau,u))\,|_{\tau=0} = 0, \quad \forall \widehat{y}_0. \tag{6.13}$$

In (6.13), we have written $y(\tau,u)$ for $y(x,t;\tau\widehat{y}_0,u)$ and we assume that the τ derivative does exist near the origin.
If f and ϕ_1 are smooth, then (6.13) is equivalent to

$$\left(\phi_1'\left(y\left(u\right)\right),y_\tau\right)=0,\quad \forall\widehat{y}_0. \tag{6.14}$$

Where

$$\begin{aligned}\phi_1' = &\quad \text{Frechet's derivative of} \quad \phi_1,\\ y\left(u\right) = &\quad \text{solution of (6.2) and (6.5) where} \quad \tau=0,\\ y_\tau = &\frac{d}{d\tau}y\left(x,t;\tau\widehat{y}_0,u\right)|_{\tau=0}\ .\end{aligned}$$

If f is smooth, y_τ is given by

$$\left\{\begin{array}{rcll}\dfrac{\partial}{\partial t}y_\tau+\left(y^0\nabla\right)y_\tau+y_\tau\left(\nabla y^0\right)-\Delta y_\tau+f'\left(y^0\right)y_\tau & = & 0 & \text{in} \quad \mathcal{Q},\\ y_\tau\left(0\right) & = & \widehat{y}_0 & \text{in} \quad \Omega,\\ y_\tau & = & 0 & \text{on} \quad \Sigma.\end{array}\right. \tag{6.15}$$

Where $f'(y^0)$ denotes the derivative of f at point y^0 and where y^0 is the solution of the problem of 6.1 with initial condition 6.7 and bouandary condition of 6.2
We restrict the choice of u to those controls - provided they exist - which satisfy (6.14), the insensitive controls.
We consider next a second functional $\phi_2\left(u\right)$ which, in principle, is related to the "increased mixing property" (6.4).
For instance, in the case of Navier-Stokes equations, we could consider

$$\phi_2\left(u\right)=\phi\left(y\left(u\right)\right)\quad \text{as given by} \quad (6.10).$$

Then the problem of controlling turbulence can be expressed as

$$\inf_{u\in L^2(\mathcal{O}\times(0,T))}\phi_2\left(u\right), \tag{6.16}$$

where

$$\left\{\begin{array}{rcll}\dfrac{\partial y\left(u\right)}{\partial t}+\left(y\left(u\right)\nabla\right)y\left(u\right)-\Delta y\left(u\right)+\chi_\omega f\left(y\left(u\right)\right) & = & u\chi_{\mathcal{O}} & \text{in} \quad \mathcal{Q},\\ y\left(u\right)|_{t=0} & = & y_0 & \text{in} \quad \Omega,\\ y\left(u\right) & = & 0 & \text{on} \quad \Sigma.\end{array}\right. \tag{6.17}$$

And where u is subject to (6.14) (insensitivity).
We proceed by transforming (6.14). To this effect, we introduce as it is classical in sensitivity studies, the adjoint state $q\left(u\right)$ defined by

$$\left\{\begin{array}{rcll}-\dfrac{\partial q\left(u\right)}{\partial t}+\left(y^0\nabla\right)q\left(u\right)+q\left(u\right)\left(\nabla y^0\right)-\Delta q\left(u\right)+f'\left(y^0\right)^*q\left(u\right) & = & \phi_1'\left(y\left(u\right)\right) & \text{in} \quad \mathcal{Q},\\ q\left(x,T;u\right) & = & 0 & \text{in} \quad \Omega,\\ q\left(u\right) & = & 0 & \text{on} \quad \Sigma.\end{array}\right. \tag{6.18}$$

Then (6.14) becomes, using (6.15) and integration by parts

$$\int_{\Omega} q(x,0;u)\,\widehat{y}_0 dx = 0, \quad \forall \widehat{y}_0, \tag{6.19}$$

we take $\widehat{y}_0(x) = q(x,0;u)$,i.e.

$$q(x,0;u) = 0 \quad \text{in} \quad \Omega. \tag{6.20}$$

The insensitive controls with respect to ϕ_1 are those which are such· that (6.17), (6.18), (6.20) hold true.

Remark 6.3.1 *The existence of such insensitive controls seems to be in general a non trivial problem (see [82]).*

The problem can now be formulated, find

$$\inf_{u\in L^2(\mathcal{O}\times(0,T))} \phi_2(u), \quad u \quad \text{subject to (6.17), (6.18), (6.20).} \tag{6.21}$$

A quite realistic way to avoid the difficulty indicated in Remark 6.3.1 (but of course the question raised in this Remark remains of interest, at least we think !) is to relax (6.20) and to consider

$$\inf_{u\in L^2(\mathcal{O}\times(0,T))} \left[\phi_2(u) + k\int_{\Omega} q(x,0;u)^2\,dx\right], \quad k>0. \tag{6.22}$$

In this formulation one can add other constraints on u.

6.4 Exact controllability

Let us now consider the equations

$$\frac{\partial y}{\partial t} + (y\nabla)\,y - \Delta y + f(y) = u\chi_{\mathcal{O}}, \tag{6.23}$$

subject to

$$y = 0 \quad \text{on} \quad \Gamma\times(0,T), \tag{6.24}$$

and where

$$y(0) = y_0 \quad \text{is given.} \tag{6.25}$$

We formulate the problem of exact controllability in a way which makes sense even if existence and uniqueness of a solution of (6.23), (6.24) and (6.25) is not known.
The first formulation is as follows: let T be given and let z_0 be given in a suitable function space. We say that the system is exactly controllable at time T if for every couple y_0, z_0, there exists a couple (u,y) satisfying (6.23), (6.24) and (6.25) such that

$$y(T) = z_0 \quad , \quad (y(T) = y(x,T))\,. \tag{6.26}$$

Moreover, if there is exact controllability, one wants to find u such that (6.26) holds true and which minimizes (for instance)

$$\int\int_{\mathcal{O}\times(0,T)} u^2 dxdt. \tag{6.27}$$

After J.L. Lions, this formulation may be too restrictive and it can be replaced by the following
Minimize

$$\int\int_{\mathcal{O}\times(0,T)} u^2 dxdt + k\int_{\Omega} |y(T) - z_0|^2 dx, \quad k > 0, \tag{6.28}$$

among all couples (u, y) subject to (6.23), (6.24) and (6.25).
Still another possibility is to consider

$$\inf_{u\in L^2(\mathcal{O}\times(0,T))} \int_{\Omega} |y(T) - z_0|^2 dx, \tag{6.29}$$

among all couples (u, y) subject to (6.23), (6.24), (6.25).
Conjecture: 1. We conjecture that for Navier-Stokes equations the inf in (6.29) equals 0, i.e. that the attainable set (the set spanned in

$$H = \left\{\varphi \mid \varphi \in \left(L^2(\Omega)\right)^n, \operatorname{div}\varphi = 0, \varphi\nu = 0 \text{ on } \Gamma\right\}$$

by all possible states $(y(T))$ is dense in H.
2. It seems possible that the in (6.28) decreases as the viscosity u decreases in Navier-Stokes equations.
3. A vague and fuzzy conjecture is that the more "chaotic" is asystem ,closest it is from exact controllability.

6.5 Instantanneous Sentinel

Let $y(x, t, \lambda, \tau) = y(\lambda, \tau)$ with $\tau = (\tau_1, \tau_2, ..., \tau_N)$, be the unique solution of the problem.

$$\begin{cases} \dfrac{\partial y}{\partial t} + (y\nabla) y - \Delta y + f(y) & = & -\nabla P + \lambda\widehat{\xi} & \text{in} & \mathcal{Q}, \\ \operatorname{div} y & = & 0 & \text{in} & \mathcal{Q}, \\ y(0) & = & \sum_{i=1}^{i=N} \tau_i \widehat{y_0^i} & \text{in} & \Omega, \\ y & = & 0 & \text{on} & \Sigma. \end{cases} \tag{6.30}$$

The function ξ is known. But, the terms : $\lambda\widehat{\xi}$ (so-called pollution term) is unknown, $\widehat{\xi}$ is renormalized and represent the size of pollution.
We denote by

$$y(x, T; \lambda, \tau) = y_{obs}, \forall x \in \mathcal{O}, \tag{6.31}$$

an observation which is a measure of the concentration of the pollution taken at the fixed time T and on a non empty open subset $\mathcal{O} \subset \Omega$ called observatory.
Let h be some function in $L^2(\mathcal{O})$, for any control function $u \in L^2(\mathcal{O})$, we introduce the functional $\mathcal{S}(\lambda, \tau)$ as follows

$$\mathcal{S}(\lambda, \tau) = \int_{\Omega} (h + u)\, \chi_{\mathcal{O}} y(x, T; \lambda, \tau) dx. \tag{6.32}$$

Definition 6.5.1 *Let $\mathcal{S}$ be a real function (6.32) depending only on the parameters λ and τ. S is said a sentinel defined by h if there existe a control $u \in L^2(\mathcal{O})$ such that the following conditions are satisfied*

$$\frac{\partial \mathcal{S}}{\partial \tau}(\lambda, \tau)\Big|_{\lambda=0, \tau=0} = 0, \tag{6.33}$$

i.e.

$$\frac{\partial \mathcal{S}}{\partial \tau_i}(\lambda, \tau)\Big|_{\lambda=0, \tau_i=0} = 0, \; 1 \leq i \leq N.$$

and

$$\|u\|_{L^2(\mathcal{O})} = \min_{\alpha \in U} \|\alpha\|, \tag{6.34}$$

where $U = \left\{ \alpha \in L^2(\mathcal{O}), \text{ such that } \frac{\partial \mathcal{S}}{\partial \tau}(\lambda, \tau)\Big|_{\lambda=0, \tau=0} = 0 \right\}.$

• Condition (6.33) express insensitivity of $\mathcal{S}$ with respect to small variations of τ and assume the existence of the derivate.
• According to (6.34) which consists in an optimal criterion of selection for (6.33).
We consider the function y_0 and P_0 which solve the problem (6.30) for $\lambda = 0$ and $\tau_i = 0$

$$\left\{ \begin{array}{rcll} \frac{\partial y0}{\partial t} + (y0\nabla)\, y0 - \Delta y0 + f\left(y^0\right) & = & -\nabla P_0 + \xi & \text{in} \quad \mathcal{Q}, \\ \operatorname{div} y0 & = & 0 & \text{in} \quad \mathcal{Q}, \\ y0(0) & = & y_0 & \text{in} \quad \Omega, \\ y0 & = & 0 & \text{on} \quad \Sigma. \end{array} \right. \tag{6.35}$$

We consider the function y_{τ_i} defined by $y_{\tau_i} = \frac{\partial y}{\partial \tau_i}(0,0)$, which is the unique solution of the problem

$$\left\{ \begin{array}{rcll} \frac{\partial y_{\tau_i}}{\partial t} + (y0\nabla)\, y_{\tau_i} + y_{\tau_i}(\nabla y0) - \Delta y_{\tau_i} + f'(y0)\, y_{\tau_i} & = & -\nabla P_{\tau_i} & \text{in} \quad \mathcal{Q}, \\ \operatorname{div} y_{\tau_i} & = & 0 & \text{in} \quad \mathcal{Q}, \\ y_{\tau_i}(0) & = & \widehat{y}_0^i & \text{in} \quad \Omega, \\ y_{\tau_i} & = & 0 & \text{on} \quad \Sigma. \end{array} \right. \tag{6.36}$$

The condition (6.33) holds if and only if

$$\int_\Omega (h+u)\, \chi_{\mathcal{O}} y_{\tau_i}(T) dx = 0, \quad 1 \leq i \leq N. \tag{6.37}$$

In order to transform this equation, we introduce the classical adjoint state

$$\left\{ \begin{array}{rcll} -\frac{\partial q}{\partial t} + (y0\nabla)\, q + q(\nabla y0) - \Delta q + f'(y0)^* q & = & -\nabla \Psi & \text{in} \quad \mathcal{Q}, \\ \operatorname{div} q & = & 0 & \text{in} \quad \mathcal{Q}, \\ q(T) & = & (h+u)\, \chi_{\mathcal{O}} & \text{in} \quad \Omega, \\ q & = & 0 & \text{on} \quad \Sigma. \end{array} \right. \tag{6.38}$$

Theorem 6.5.1 *Let $q = (q_1, q_2, q_3)$ be the solution to the backward problem (6.38), then the existence of an instantaneous sentinel insensitive to the missing data is equivalent to the null-controllability problem*

$$\int_\Omega q(0)\widehat{y}_0^i dx = 0, \quad 1 \leq i \leq N, \tag{6.39}$$

i.e

$$q(0) \in G^\perp,$$

where $G^\perp$ is the orthogonal of G in $L^2(\Omega)$.

where G the subspace generated by $\widehat{y}_0^i$
Proof. Multiplying the first equation in (6.38) by y_{τ_i}, and integrating by parts over Ω, we find

$$\int_\Omega q(0)y_{\tau_i}(0)dx - \int_\Omega q(T)y_{\tau_i}(T)dx = 0, \qquad 1 \leq i \leq N,$$

then

$$\int_\Omega q(0)\widehat{y}_0^i dx - \int_\Omega (h+u)\,\chi_{\mathcal{O}} y_{\tau_i}(T)dx = 0, \quad 1 \leq i \leq N,$$

thanks to (6.37), we have

$$\int_\Omega q(0)\widehat{y}_0^i dx = 0, \quad 1 \leq i \leq N.$$

So that

$$q(0) \perp \widehat{y}_0^i, \quad 1 \leq i \leq N.$$

■

6.5.1 Optimal control problem

In this section we are interested to solve the problem (6.34), so we consider the optimization problem

$$\min_{u \in M} \|u\|^2_{L^2(\mathcal{O})}, \tag{6.40}$$

with $M = \{u \in L^2(\mathcal{O})$ such hat, we have (6.33) and $\int_\Omega q(0)\widehat{y}_0^i dx = 0, \quad 1 \leq i \leq N$ where q is the solution of (6.38)$\}$.

Lemma 6.5.1 *The problem (6.40) admits an unique solution.*

Proof. The set M is a non empty, colsed and convex set. The mapping

$$v \to \|v\|^2_{L^2(\mathcal{O})},$$

is continuous, coercive and strictly convex, therefore, the problem (6.40) admits an unique solution denoted by $\widehat{v} \in M$ which satisfies

$$\|\widehat{v}\|_{L^2(\mathcal{O})} \leq \|v\|_{L^2(\mathcal{O})}, \ \forall v \in M.$$

■

6.5.2 Characterization of optimal control

To characterize the optimal control, let us introduce $q0$ by

$$\begin{cases} -\dfrac{\partial q0}{\partial t} + (y0\nabla)\, q0 + q0\,(\nabla y0) - \Delta q0 + f'\,(y0)^*\, q0 &= -\nabla\Psi_1 \quad \text{in} \quad \mathcal{Q}, \\ \operatorname{div} q0 &= 0 \quad \text{in} \quad \mathcal{Q}, \\ q0(T) &= h\chi_{\mathcal{O}} \quad \text{in} \quad \Omega, \\ q0 &= 0 \quad \text{on} \quad \Sigma. \end{cases} \tag{6.41}$$

and let us define $z = z(u)$ as the solution of

$$\begin{cases} -\dfrac{\partial z}{\partial t} + (y0\nabla)\, z + z\,(\nabla y0) - \Delta z + f'\,(y0)^*\, z &= -\nabla\Psi_2 \quad \text{in} \quad \mathcal{Q}, \\ \operatorname{div} z &= 0 \quad \text{in} \quad \mathcal{Q}, \\ z(T) &= u\chi_{\mathcal{O}} \quad \text{in} \quad \Omega, \\ z &= 0 \quad \text{on} \quad \Sigma. \end{cases} \tag{6.42}$$

Then

$$q = q0 + z = q0 + z(u), \quad \Psi = \Psi_1 + \Psi_2,$$

we want to find u such that

$$\int_\Omega z(0;u)\widehat{y}_0^i dx = -\int_\Omega q0(0)\widehat{y}_0^i dx, \quad 1 \le i \le N. \tag{6.43}$$

We define ρ as the solution of

$$\begin{cases} -\dfrac{\partial \rho}{\partial t} + (y0\nabla)\, \rho + \rho\,(\nabla y0) - \Delta \rho &= -\nabla\Xi \quad \text{in} \quad \mathcal{Q}, \\ \operatorname{div} \rho &= 0 \quad \text{in} \quad \mathcal{Q}, \\ \rho(0) &= \sum_{i=1}^N \alpha_i \widehat{y}_0^i \quad \text{in} \quad \Omega, \\ \rho &= 0 \quad \text{on} \quad \Sigma. \end{cases} \tag{6.44}$$

where α_i is not determined. Let ξ is the solution of the system

$$\begin{cases} -\dfrac{\partial \xi}{\partial t} + (y0\nabla)\, \xi + \xi\,(\nabla y0) - \Delta \xi + f'\,(y0)^*\, \xi &= -\nabla\Upsilon \quad \text{in} \quad \mathcal{Q}, \\ \operatorname{div} \xi &= 0 \quad \text{in} \quad \mathcal{Q}, \\ \xi(T) &= \rho\,(T)\,\chi_{\mathcal{O}} \quad \text{in} \quad \Omega, \\ \xi &= 0 \quad \text{on} \quad \Sigma. \end{cases} \tag{6.45}$$

and we want to determine $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_N\} \in \mathbb{R}^N$ such that

$$\int_\Omega \xi(0)\widehat{y}_0^i dx = -\int_\Omega q0(0)\widehat{y}_0^i dx, \quad 1 \le i \le N.$$

We introduce the linear operator Λ by

$$\Lambda\alpha = \left\{ \int_\Omega \xi(0)\widehat{y}_0^1 dx, \int_\Omega \xi(0)\widehat{y}_0^2 dx, ..., \int_\Omega \xi(0)\widehat{y}_0^N dx \right\}. \tag{6.46}$$

Then

$$\Lambda \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N),$$

Theorem 6.5.2 *According to the unique continuation theorem of Mizohata [77], we have at least one sentinel given by*

$$\mathcal{S}(\lambda,\tau)=\int_{\Omega}(h+\rho(T))\chi_{\mathcal{O}}y(x,T;\lambda,\tau)dx,$$

where ρ is the solution of (6.44), so

$$u=\rho(T)\chi_{\mathcal{O}},$$

is the solution of (6.33)-(6.34).

Proof. We multiply (6.44) by $\widetilde{\rho}$ corresponding to $\widetilde{\alpha}$, and we integrate by parts. We obtain

$$\langle\Lambda\alpha,\widetilde{\alpha}\rangle=\int_{\mathcal{O}}\rho(T)\widetilde{\rho}(T)dx. \tag{6.47}$$

Therefore, Λ is a symmetric and positive matrix. Let us now set

$$\|\alpha\|_F=\left(\int_{\mathcal{O}}\rho(T)^2dx\right)^{1/2}, \tag{6.48}$$

and let $y_i(x,t)$ the solution of

$$\begin{cases}\dfrac{\partial y_i}{\partial t}+(y_i\nabla)\,y_i-\Delta y_i+f\,(y_i) &= -\nabla P_i \quad \text{in} \quad \mathcal{Q},\\ \operatorname{div} y_i &= 0 \quad \text{in} \quad \mathcal{Q},\\ y_i(0) &= \widehat{y}_0^i \quad \text{in} \quad \Omega,\\ y_i &= 0 \quad \text{on} \quad \Sigma.\end{cases}$$

We define in this way a norm on the space F of the real vector α, where the Hilbert space F is the completion for the norm (6.48) (indeed if $\|\alpha\|_F=0$ then $\rho=0$ on $\mathcal{O}$ and according to the unique continuation theorem of Mizohata $\rho=0$ on $\mathcal{Q}$ so that $\alpha=0$). Then if F' denotes the dual of F, we have

$$\Lambda:F\longrightarrow F' \text{ is an isomorphism.}$$

Therefore, the equation

$$\Lambda\alpha=-\left\{\int_{\Omega}q(0)\widehat{y}_0^1dx,\int_{\Omega}q(0)\widehat{y}_0^2dx,...,\int_{\Omega}q(0)\widehat{y}_0^N dx\right\}, \tag{6.49}$$

admits a unique solution if

$$-\int_{\Omega}q(0)\widehat{y}_0^idx\in F',\qquad 1\le i\le N. \tag{6.50}$$

We set

$$\beta=\left\{\int_{\Omega}q(0)\widehat{y}_0^1dx,\int_{\Omega}q(0)\widehat{y}_0^2dx,...,\int_{\Omega}q(0)\widehat{y}_0^N dx\right\},$$

then the solution of (6.49) is given by

$$\alpha = -\Lambda^{-1}\beta.$$

If we multiplying (6.41) by ρ, and integrating over $\mathcal{Q}$ we obtain

$$u = \rho(T)\chi_{\mathcal{O}}, \tag{6.51}$$

is the solution of (6.34), (6.39). ■

Remark 6.5.1 *The space F is identical to $\mathbb{R}^N$, and its norm is equivalent to the Euclidian norm. Then, Λ is a isomorphism from $\mathbb{R}^N$ to $\mathbb{R}^N$.*

Identification of the pollution term

Let $\mathcal{S}_{obs}$ be the measured sentinel corresponding to the state of the system on the observatory $\mathcal{O}$ at the time T

$$\mathcal{S}_{obs}(\lambda,\tau) = \int_{\Omega}(h+u)\chi_{\mathcal{O}}y_{obs}(x,T;\lambda,\tau)dx. \tag{6.52}$$

Theorem 6.5.3 *The pollution term is estimated as follows*

$$\int_0^T\int_{\Omega} q(h)\lambda\widehat{\xi}dxdt = \mathcal{S}_{obs}(\lambda,\tau) - \mathcal{S}(0,0), \tag{6.53}$$

where $\mathcal{S}(0,0)$ is the sentinel corresponding to the state $y(x,T;0,0)$.

Proof. We have

$$\mathcal{S}_{obs}(\lambda,\tau) = \lambda\frac{\partial\mathcal{S}}{\partial\lambda}(\lambda,\tau_i)\Big|_{\lambda=0,\tau_i=0} + \circ(\lambda,\tau_i),\ \text{for } \lambda,\ \tau \text{ small}, \tag{6.54}$$

and

$$\frac{\partial\mathcal{S}}{\partial\lambda}(\lambda,\tau) = \int_{\mathcal{O}}(h+u)y_\lambda dx, \tag{6.55}$$

where y_λ defined by $y_\lambda = \dfrac{\partial y}{\partial\lambda}(0,0)$ (which depends only on $\widehat{\xi}$ and the other known data) is the unique solution of

$$\left\{\begin{array}{rcll} \dfrac{\partial y_\lambda}{\partial t} + (y0\nabla)\,y_\lambda + y_\lambda\,(\nabla y0) - \Delta y_\lambda + f'\,(y0)\,y_\lambda & = & -\nabla P_\lambda + \widehat{\xi} & \text{in} \quad \mathcal{Q}, \\ \operatorname{div} y_\lambda & = & 0 & \text{in} \quad \mathcal{Q}, \\ y_\lambda(0) & = & 0 & \text{in} \quad \Omega, \\ y_\lambda & = & 0 & \text{on} \quad \Sigma, \end{array}\right. \tag{6.56}$$

and

$$\lambda\frac{\partial\mathcal{S}}{\partial\lambda}(\lambda,\tau)\Big|_{\lambda=0,\tau=0} = \mathcal{S}_{obs}(\lambda,\tau) - \mathcal{S}(0,0). \tag{6.57}$$

We designate by $q(h)$ the unique solution of (6.38) depending on h.

Multiply (6.38) by y and integrate by part, we obtain

$$\int_{\Omega}(h+u)\chi_{\mathcal{O}}y_{\lambda}dx=\int_{0}^{T}\int_{\Omega}q(h)\lambda\widehat{\xi}dxdt. \tag{6.58}$$

Therefore, the pollution term can be characterized

$$\int_{0}^{T}\int_{\Omega}q(h)\lambda\widehat{\xi}dxdt=\mathcal{S}_{obs}\left(\lambda,\tau\right)-\mathcal{S}\left(0,0\right).$$

■

6.5.3 Conclusion

In this chapter we are working to get instantaneous information at fixed instant T on pollution term in Navier-Stokes system in which the initial condition is incomplete. The best method which can solve this problem is the sentinel method; it allows estimating the pollution term at which we look for information independently of the missing term that we do not want to identify. So, we prove the existence of such instantaneous sentinel by solving a problem of controllability with constraint on the control. From this we conclude that there is a harmony between the theory of turbulence and sentinel with the theory of control.

General Conclusion

I carried out my thesis under the supervision of Dr. Rezzoug Imad and Professor Ayadi Abdelhamid it is entitled (Optimal Control and Sentinel Applied to the Parabolic System). we have studied approximate and singular parabolic models from an optimal control point of view, in the presence of perturbation and incomplete data. The existence of control for approximate and singular parabolic equations is analyzed in depth using the H.U.M. The controllability of these problems uses the sentinel method. It is reported that the work carried out by Professor Ayadi, on the spot, indispensable the feasibility of much of this work. The summary of the work is as follows:

In the first work entitled by: Pollution detection for the singular linear parabolic equation, accompanied by Professor Omrane and Ayadi: In this work we are concerned by the problem of identification of noisy terms which arise in singular problem as for remote sensing problems, and which are modeled by a linear singular parabolic equation. A new sentinel is given here in its more realistic setting for singular parabolic problems: in this case, the observation and the control have their support in different open sets.

In the second work entitled: New modification sentinel of linear parabolic equation, accompanied by professor ayadi : The purpose of this work is, firstly, to modify the old definition of the sentinel, introduced by J.L. Lions, so that we can separate the control support to the observation support and, secondly, to identify the pollution terms which arise in regular problems which are modeled by a linear parabolic equation. A new modification of the sentinel is used to identify pollution terms in the general case where the observation and the control supports are disjoint. The problem of finding a new modification sentinel is equivalent to finding the unique control of regular adjoint system of the parabolic equation that we solve.

In the third work entitled: Approximate Controllability of the Stokes system, accompanied by doctor rezzoug and berhail : In this paper we establish some approximate controllability results for system of parabolic equations of the Stokes kind. Motivated by the solution of controllability problems for the Navier-Stokes equations modeling incompressible viscous. Now, we will discuss now controllability issues for a system of partial differential equations which is not of the Cauchy-Kowalewska type, namely, the classical Stokes system.

In the last work (published in 2021 in Proyecciones Journal of Mathematics entitled: Instantaneous sentinel for the identification of the pollution term in Navier-Stokes system, accompanied by doctor rezzoug and oussaeif: The aim of this work is about presenting some results of the Sentinel Theory in Connection with Control Theory of Distributed Systems. There is of course a large variety of models where the results to follow could be applied. What we have particularly in mind is the classical set of Navier-Stokes equations. In this work we are(trying) working to get instantaneous information in which the initial condition is incomplete. The best method which can solve this problem is the sentinel one; it allows the estimation of the pollution term at which we look for information independently of the missing term that we do not want to identify. So, we prove the existence of such instantaneous sentinel by solving a problem of controllability with a constraint on the control.

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