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Entitled

INVERSE PROBLEMS WITH MISSING DATA

Before the jury, composed of

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To the soul of my dear sister,
Amina
. To my beloved Aunt
Warda's spirit
To my Professor's soul
AIROUD Nacer

Dedication

In the name of Allah the merciful

This thesis is dedicated to:
The sake of ALLAH, my Creator and, my Master. My great teacher and messenger, Mohammed (May Allah bless and grant him), who taught us the purpose of life. My great parents, who never stop giving of themselves in countless ways, and my beloved husband, My beloved sister, who has always been by my side. who have supported me all the way since the beginning of my studies, My darling Chahd, Meriem, Nada, and Selsabil. My dear nephews, Moncif, Anis, Mouad, Ahmed, Ammar, and Yakoub, To everyone in my family and circle of friends My friends who motivate and encourage me. To those who have been deprived of their right to study and to all those who believe in the richness of learning.
♡

Chafia. L

Mathematics

Mathematics rightly viewed possesses not only truth but supreme beauty.

[Bertrand Russell]

Mathematics is as much an aspect of culture as it is a collection of algorithms.

[Carl Benjamin Boyer]

Mathematics is the art of giving the same name to different things.

[Henri Poincare]

If I feel unhappy, I do mathematics to become happy. If I am happy, I do mathematics to keep happy."

[Alfred Renyi]

Mathematics is the queen of the sciences.

[Carl Friedrich Gauss]

If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

[John von Neumann]

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Contents

Abstract

In this thesis, we are interested in solving optimal control problems associated with inverse problems. We have a special interest in the optimal control of partially known coupled systems. We are concentrating on a number of key concepts, which are documented in two research papers.

In the first work, we are interested in the environmental pollution problem. That's exactly the water pollution problem. The main objectives are to control the concentration of dissolved oxygen because it is of prime importance in considering the water quality, give an assessment of the biochemical oxygen demand, and study its physiochemical characteristics.

The posed coupled systems considered here are given with unknown initial conditions that present some barriers. For this reason, we try to find the optimal control independent of the missing data variation. The main tool used here is to use the concept of "no regret control," adapted by Lions, to control distributed systems with missing data. The numerical resolution of the obtained relations will allow examining the level of dissolved oxygen and studying the physicochemical characteristics of the water.

The research reported in the last work deals with the sentinel of the fractional problem of coupled thermo-elasticity systems implicating the Riemann-Liouville fractional derivatives. We are interested in examining the deformation of composite materials. Generally, this type of deformation is not a strictly mechanical phenomenon. The main purpose is to apply the sentinel method to an inverse fractional coupled thermoelastic system for studying the interaction between thermal and mechanical effects in elastic bodies. For this reason, we monitor the elastic displacements with the effect of the temperature concentration measured at a few points. The main tool used to solve the sentinel problem is to study a null controllability problem. The right Caputo fractional derivative is more suitable to introduce the fractional coupled adjoint state systems. The identification problem with the Riemann Liouville and Caputo fractional derivative senses suggested in this work is the generalization of classical identification problems in the no-fractional case.

The main idea of this thesis should be of interest to readers in the areas of biosystems, thermo-elasticity systems, and inverse problems, as well as in aspects of the field of control and automation, control theory, and PDEs.

keywords: Inverse problem; Optimal control; Problem with missing data; No regret control; Low regret controls; Riemann-Liouville fractional derivative; Caputo fractional derivative; Sentinel method.

Résumé

Dans cette thèse, nous nous intéressons à la résolution de problèmes de contrôle optimal associés à des problèmes inverses. Nous sommes particulièrement intéressés au contrôle optimal des systèmes couplés partiellement connus. Nous nous concentrons sur un certain nombre de concepts clés, qui sont documentés dans deux documents de recherche.

Dans le premier travail, nous nous intéressons au problème de la pollution de l'environnement. C'est exactement le problème de la pollution de l'eau. Les principaux objectifs sont de contrôler la concentration en oxygène dissous car elle est primordiale dans la prise en compte de la qualité de l'eau, d'évaluer la demande biochimique en oxygène et d'étudier ses caractéristiques physico-chimiques.

Le système couplé posé considéré ici est donné avec des conditions initiales inconnues qui présentent des barrières. Pour cette raison, nous essayons de trouver le contrôle optimal indépendant de la variation des données manquantes. L'outil principal utilisé ici c'est le concept de "contrôle sans regret", adapté par Lions, pour contrôler les systèmes distribués avec des données manquantes. La résolution numérique des relations permettra d'examiner le niveau d'oxygène dissous et d'étudier les caractéristiques physico-chimiques de l'eau.

Les recherches rapportées dans le dernier ouvrage portent sur la sentinelle du problème fractionnaire des systèmes couplés de thermo-élasticité impliquant les dérivées fractionnaires de Riemann-Liouville. L'objectif principal est d'appliquer la méthode de sentinelle sur un système thermoélastique couplé fractionnaire inverse pour étudier l'interaction entre les effets thermiques et mécaniques dans les corps élastiques. Pour cette raison, nous surveillons les déplacements élastiques sous l'effet de la concentration de température mesurée en quelques points. L'outil principal utilisé pour résoudre le problème sentinelle est d'étudier un problème de contrôlabilité nulle. La dérivée fractionnaire de Caputo droite est plus appropriée pour introduire les systèmes d'états adjoints couplés fractionnaires. Le problème d'identification avec les sens des dérivées fractionnaires de Riemann Liouville et Caputo suggéré dans ce travail est la généralisation des problèmes d'identification classiques dans le cas non fractionnaire.

L'idée principale de cette thèse devrait intéresser les lecteurs dans les domaines des biosystèmes, des systèmes de thermo-élasticité et des problèmes inverses, ainsi que dans les aspects du domaine du contrôle et de l'automatisation, de la théorie du contrôle et des EDP.

Mots clés: Problème inverse; Contrôle optimal; Problème avec des données manquantes; Contrôle de sans regret; Dérivées fractionnaires de Riemann-Liouville; Dérivée fractionnaire de Caputo; Méthode de la sentinelle.

ملخص

في هذه الأطروحة نهتم بالتحليل الرياضي والتحكم الأمثل للأنظمة المقترنة المعرفة جزئيا سيركز على ثلاث أفكار رئيسية تم تحقيقها في منشوريين بحثيين

في العمل الأول ، اهتممنا بمشكلة التلوث البيئي ، بالضبط في مشكلة تلوث المياه هدفنا الرئيسي كان محاولة التحكم في تركيز الأكسجين المذاب لأنه ذو أهمية قصوى في الحكم على جودة المياه وإعطاء تقييم للطلب الكيميائي الحيوي للأكسجين ودراسة خصائصه الفيزيائية والكيميائية أيضًا .

النظام المقترن المطروح هنا معطى بشروط أولية غير معروفة وهذا ما سيخلق لنا بعض العوائق. لهذا السبب نحاول إيجاد تحكم امثل لهذا النظام المقترن بشكل مستقل عن تلك البيانات المفقودة. الأداة الرئيسية المستخدمة هنا هي مفهوم التحكم دون فقدا ن و دون خسارة المستخدم للتحكم في الأنظمة الموزعة ذات البيانات المفقودة والمبتكر من قبل جاك لويس ليونس. سيسمح القرار العددي للعلاقات التي تم الحصول عليها بفحص مستوى الأكسجين المذاب ودراسة الخصائص الفيزيوكيميائية لمياهه.

البحث الثاني نستخدم طريقة الحارس للمشكلة الجزئية لأنظمة المرونة الحرارية المقترنة التي تنطوي على مشتقات ريمان ليوفيل الكسرية. نحن مهتمون بفحص تشوه المواد المركبة. بشكل عام ، هذا النوع من التشوه ليس ظاهرة ميكانيكية بحتة. الغرض الرئيسي هو تطبيق طريقة الحارس على نظام مرن حراري مترابط كسري عكسي لدراسة التفاعل بين التأثيرات الحرارية والميكانيكية في الأجسام المرنة. لهذا السبب ، نقوم بمراقبة عمليات الإزاحة المرنة بتأثير تركيز درجة الحرارة المقاس عند نقاط قليلة. الأداة الرئيسية المستخدمة لحل مشكلة الحارس هي دراسة مشكلة قابلية التحكم المنعدمة. مشتقات كابيتو الكسرية هي أكثر ملاءمة لإدخال أنظمة الحالة المتقاربة الكسرية المقترنة. إن مشكلة التحديد مع حواس مشتقات كسور ريمان ليوفيل وكابوتو المقترحة في هذا العمل هي تعميم مشاكل تحديد الهوية الفكرة الرئيسية لهذه الأطروحة ذات أهمية للقراء في مجالات النظم الحيوية .الكلاسيكية في حالة عدم وجود كسور PDE وأنظمة المرونة الحرارية والمشاكل العكسية وفي جوانب مجال التحكم والأتمتة ونظرية التحكم وأجهزة

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The List Of Works

1. C. Laouar, A. Ayadi, A. Hafdallah, Identification problem of a fractional thermoelastic deformation system with incomplete data : A sentinel method. Nonlinear studies journal, www.nonlinearstudies.com Vol. 29, No. 2, pp. 1-13, **(2022)**.
2. C. Laouar, A. Ayadi, A. Hafdallah, Optimal control of a partially known coupled system of BOD and DO. International Journal of Analysis and Applications, vol 19(6), 984-996. **(2021)**.
3. A. Hafdallah, C. Laouar, A. Ayadi, No-regret optimal control characterization for an Ill-posed wave equation. International Journal of Mathematics Trends and Technology (IJMTT).Vol 41(3), 283-288. **(2017)**.

Unpublished papers

1. C. Laouar, Stability of a fractional deformation coupled thermoplastic problem with missing data.
2. C. Laouar, Discriminating distributed sentinel for a partially known coupled system of BOD and DO
3. C. Laouar, A.Chettouh, K.Saoudi. Identification of an unknown time-dependent source parameter in a time-fractional Sobolev-type problem from over determination condition.

Notations

Ω	An open set in $\mathbb{R}^n$ with boundary $\partial\Omega$.
$\mathcal{U}_{ad}$	Sef of admissible controls.
$\mathcal{J}(v)$	The cost function.
$L^2(\Omega)$	espace of functions square integrable on Ω.
$\mathcal{H}^m(\Omega)$	$= \{\varphi \in L^2(\Omega), \cdots, D^\alpha\varphi \in L^2(\Omega), \lvert\alpha\rvert \leq m,\ \alpha = (\alpha_1, \cdots \alpha_n), \lvert\alpha\rvert = \alpha_1 + \cdots \alpha_n\}$.
$\mathcal{H}_0^m(\Omega)$	$= \{\varphi \in H^m(\Omega),\ D^\alpha\varphi = 0 \text{ on } \partial\Omega,\ \lvert\alpha\rvert \leq m-1\}$.
$\mathcal{H}_0^s(\Omega)$	= Fractional Sobolev space of prder s on Ω.
$\mathcal{L}(f(t))$	The Laplace transform of function $f : \mathbb{R}^+ \to \mathbb{R}$.
$\mathcal{A}^*$	The adjoint operator of $\mathcal{A}$.
I^α	Riemann-Liouville fractional integral of order α.
D_{RL}^α	Riemann-Liouville fractional derivative of order α.
$\mathcal{D}_C^\alpha$	Caputo fractional derivative of order α.
$S(\lambda, \tau)$	Sentinel function.
$\mathcal{O}$	Observatory.
$\chi_{\mathcal{W}}$	Characteristic function of the set $\mathcal{W}$.
$\mathcal{L}(A, B)$	The space of linear bounded operators from A to B.
$\Gamma(z)$	Gamma function of z.
$E_\alpha(z)$	The Mittag-Leffler function of z.
$\rightharpoonup$	Weak convergence
$\to$	Strong convergence
$\lVert.\rVert_{\mathcal{H}}$	A norm in Banach space $\mathcal{H}$.
$\langle\ .\ \rangle_{\mathcal{H}}$	A scalar product in Hilbert space $\mathcal{H}$.
$\langle\ .\ \rangle_{\mathcal{H},\mathcal{H}'}$	Duality product between $\mathcal{H}$ and $\mathcal{H}'$.
$\lvert.\rvert$	A semi-norm in $\mathcal{H}$.
$\mathcal{C}^2$	The class of functions with continuous first and second derivative.
$\dfrac{\partial y}{\partial \nu} = \nabla y.\nu$	Conormal derivative.
$\Delta = \sum_{i=1}^n \dfrac{\partial^2}{\partial x_i^2}$	Laplacien operator.
$\nabla = \Big(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \cdots \dfrac{\partial}{\partial x_n}\Big)$	Gradient operator.
$\mathrm{d}iv\vec{A} = \vec{\nabla}.\vec{A} = \sum_{i=1}^m \frac{\partial A_i}{\partial x_i}$	Divergence operator.
OS	Optimality system.
SOS	Singular optimality system.
θ_ϵ	Correctors functions.
u^γ	Low regret control.
BOD	Biochemical oxygen demand.
DO	Dissolved oxygen.

Introduction

In science, an inverse problem is a situation in which one tries to determine the causes of a phenomenon based on experimental observations of its effects. Inverse problems are so named because they begin with the results and then calculate the causes. The inverse of a forward problem is one that begins with the causes and then calculates the results. Partially known systems are one of the most important inverse problems. More information can be found at [31, 43, 46, 51].

Let Ω be a bounded open subset of $\mathbb{R}^n$. For a time $t \in [0,T],\ T > 0$, the state in the general form of a partial differential is given by the solution to

$$\left| \begin{array}{l} \dfrac{\partial y}{\partial t} + \mathcal{A}(y) = \text{the source} \quad \text{in} \quad \Omega \times [0,T], \\ \quad \text{Initial conditions,} \\ \text{Boundary conditions,} \end{array} \right. \tag{1}$$

where $\mathcal{A}$ denotes a partial differential operator. To determine the state of the system, we must know the coefficients of the operator $\mathcal{A}$, as well as the possibility of non-linearity, the source terms, the domain Ω, and the limit conditions. If at least one of the above details is unknown, the last system is said to be partially known. Visit for more details [6, 7, 8, 10, 23, 59].

The majority of inverse problem formulations start with the setting of an optimization problem. For example, we are referring to a system that is partially well-known.

Many tools, such as mathematical models with missing data, are often seen as effective tools that could provide information and improve economic, environmental, and pollution control. Several pollution problems contain incomplete data, among others, in the initial conditions or in boundary conditions. For example, in pollution cases, it may be difficult to determine where and how much pollution is produced, or the pollution may be voluntarily dissimilitude by the polluter. More information is available at [2, 9, 23, 31, 33, 41, 43, 45, 46, 48].

One of the most significant ideas in mathematical control theory is controllability. In both deterministic and stochastic control systems, controllability is critical. Control theory is an area of mathematics that studies how far the state of a system can be changed based on the system's fundamental qualities and how we can act on them. For example, one might question if a solid's temperature can be brought to a constant in a finite amount of time by heating and cooling only a portion of the solid. Since 1995, this problem, known as the null-controllability of the heat equation, has been solved. Controllability roughly translates to the ability to direct a dynamical control system from an arbitrary initial state to an arbitrary final state using the admissible controls.

Many mathematical concepts and methods from differential geometry, functional analysis, topology, matrix analysis, theory of ordinary and partial differential equations, and theory of difference equations are used to solve controllability problems for various types of dynamical systems. The controllability of diverse classes of systems can be studied using state-space models of dynamical systems, which gives a reliable and general method. On the other hand, optimal control is concerned with the problem of determining a control law for a given system that meets a predetermined optimality condition.

A cost functional is a function of state and control variables in a control problem. The optimal control theory's goals are as follows: Obtaining necessary (or possibly necessary and sufficient) conditions for the control to be an extreme (or minimum), studying the structure and properties of the equations expressing these conditions, and obtaining constructive algorithms amenable to numerical computations of the admissible controls that determine the inf (such a control is referred to as an "optimal control"). Optimal control can be used in a variety of sectors, including biology, economics, ecology, engineering, finance, management, and medicine. See also [52]-[59] and the references therein.

Controllability theory for various systems with fractional derivatives and fractional has advanced significantly since the publication of research publications such as [27]-[30] and the monograph [?]. This theory has formed the basis of a very active research topic since it provides a natural framework for mathematical modelling of many physical phenomena and validation of existing ones. Fractional differential equations have recently proved to be strong tools in the modeling of many phenomena in various fields of engineering, physics, and economics. As a consequence, there was an intensive development of the theory of fractional differential equations. Due to this fact, the fractional order models are capable of describing more realistic situations than the integer order models. Many articles have been devoted to the existence of solutions for fractional differential equations, for example [22]. Existence, uniqueness, stability, controllability, and other quantitative and qualitative features of evolutionary equation solutions have recently gotten a lot of attention. For more information, see [7, 17, 18, 22, 24, 29, 30, 39, 49].

Fractional differential equations are applied in a variety of fields, including fractals, chaos, electrical engineering, and medicine. In recent years, there has been a lot of progress in the field of fractional differential equations. For instance, we refer to the monographs of Abbas and al. [15], Kilbas et al. [30], Miller and Ross [49], Podlubny [39], and other documents.

The sentinel of a partially known fractional coupled system and the study of optimal control of distributed coupled systems with incomplete data (inverse problem) have been the subjects of extensive research, and this is what we will focus on in this thesis.

This thesis is divided into three chapters, conclusion, and an appendix.

In the first one, we introduce preliminaries and basic properties of Pareto control, no-regret control, and a sequence of low regret controls in the stationary case, and we will briefly recall some basic properties of no-regret control and low regret control in the evolution case. Then, we discuss the elliptic and parabolic regularization procedures, singular distributed problems, and correctors. We'll go through the basics of fractional calculus. We'll also discuss the existence and uniqueness of fractional partial differential equation solutions. We'll cover the fundamentals of using the sentinel method at the end of this chapter.

On the second one, we will present the relationship between biochemical oxygen demand and dissolved oxygen by introducing and applying the ideas of no-regret control and sequence of low regret controls to a coupled system with missing initial data. We aim to control the level of dissolved oxygen by assessing the biological oxygen demand and analyzing its physiochemical features because it is very important when considering the quality of water. Other times, the coupled systems are given unknown initial conditions, which creates certain difficulties. The major goal of our research is to find the best control. We dispose of the missing data by introducing the ideas of no-regret control and the sequence of least-regret controls to obtain the characterization of this optimal control. No regret control optimality coupled systems are built by pushing to the limit.

The third chapter is devoted to the controllability of fractional differential coupled systems. Our goal is to use the sentinel approach to an inverse fractional coupled thermoelastic system to investigate the interaction between thermal and mechanical forces in elastic bodies. First, we introduce the model under consideration and define our problem. In the next part, we'll develop an observatory domain and a sentinel function based on the mean of the observations provided. The sentinel problem will be reduced to a null controllability problem with a constraint, and the release of control and presentation of optimality coupled systems will be covered. Finally, the pollution term is identified, and the stealthiness connection is derived from it. Finally, we reached an agreement on the paper's conclusion.

We conclude the thesis with a conclusion and views section that summarizes the main findings and offers suggestions for future research studies on the subject.

Chapter 1

Preliminary Background

In this chapter, we summarize basic definitions and facts about the concepts of Pareto control, no-regret control, sequence of low regret controls, and the Sentinel method. We also give some definitions, lemmas, and preliminary fractional calculus.

1.1 Preliminaries and fundamental properties of Pareto Control, No-regret control, and sequence of low regret controls in the stationary case

The no-regret concept was introduced by Savage [58] and J. Louis in [52]. This concept is motivated by a number of applications in economics and ecology. The Pareto concept which, is equivalent to this one, is also used by Lions in [55]. As far as we know, he was the first to use these two principles to control distributed systems with incomplete data in several fields of applied mathematics. In [28, 35, 37, 44, 52, 55], decision criteria are added to the uncertainties in closed subspace, and they improve the results obtained so far by extending the notion of low-regret control to many agents in economics.

In this section, we mention notations, definitions, lemmas, and preliminary thoughts about Pareto control, no-regret control, and the sequence of llow regret control facts needed to establish our main results. See [7, 28, 35, 37, 44, 52, 55] for more details.

1.1.1 Pareto control and no-regret control for a partially known distributed system

Let $\mathcal{V}$ be a real Hilbert space of dual $\mathcal{V}'$, $A \in \mathcal{L}(\mathcal{V};\mathcal{V}')$ an elliptic differential operator modelling a distributed system, $\mathcal{U}$ the Hilbert space of controls, and $B \in \mathcal{L}(\mathcal{U};\mathcal{V}')$. Let G be a non-empty closed vector subspace of the Hilbert space of uncertainties F, and $\beta \in \mathcal{L}(F;\mathcal{V}')$.
For $f \in \mathcal{V}'$, the state equation related to the control $v \in \mathcal{U}$ and to the uncertainty $g \in G$ is given by

$$Ay(v,g) = f + Bv + \beta g. \tag{1.1}$$

Denote the unique solution to the well-posed problem (1.1) in $\mathcal{V}$ by $y = y(v,g)$. [1]
Let us define the cost function as the least square functional:

$$J(v,g) = \|Cy(v,g) - z_d\|^2_{\mathcal{H}} + N\|v\|^2_{\mathcal{U}}, \quad \forall g \in G, \tag{1.2}$$

where $C \in \mathcal{L}(\mathcal{V};\mathcal{H})$, $\mathcal{H}$ is a Hilbert space, $z_d \in \mathcal{H}$ fixed, and $N > 0$.

[1] It satisfies the properties of well-posedness according to Hadamard.

Finally, we minimize the functional $J(v,g)$ on $\mathcal{U}$. More precisely, we seek to

$$\inf_{v\in\mathcal{U}} J(v,g), \quad \forall g \in G. \tag{1.3}$$

This makes no sense when $G \neq \{0\}$.[2] The main idea is to solve the minimization problem described below:

$$\inf_{v\in\mathcal{U}} \Big(\sup_{g\in G} J(v,g) \Big).$$

However, unlike $\sup_{g\in G} J(v,g) = +\infty$, J is not upper bounded.

Definition 1.1.1 (*See [38]*)

We say that $u \in \mathcal{U}$ is a Pareto control for (1.1)-(1.3) if

$$J(u,g) \leq J(v,g), \quad \forall v \in \mathcal{U},\ \forall g \in G.$$

Moreover, if there exists at least $g_0 \in G$ such that

$$J(u,g_0) \leq J(v,g_0), \quad \forall v \in \mathcal{U}.$$

Definition 1.1.2 (*See [38]*)

Let $u \in \mathcal{U}$ is a Pareto control. We say that u is related to a control $u_0 \in \mathcal{U}$ if

$$J(u,g) \leq J(u_0,g) \quad \forall g \in G.$$

Definition 1.1.3 (*See [38]*)

If u is a solution to the following problem, we say it is a no-regret control for (1.1)-(1.2) related to a control $u_0 \in \mathcal{U}$.

$$\inf_{v\in\mathcal{U}} \Big(\sup_{g\in G} (J(v,g) - J(u_0,g)) \Big). \tag{1.4}$$

Remark 1.1.1

When $u_0 = 0$, Definition 1.1.3 reduces to the definition of the no-regret control of Lions.

Lemma 1.1.1 (*See [38]*)

For any $u_0 \in \mathcal{U}$ and any $v \in \mathcal{U}$ we have

$$J(v,g) - J(u_0,g) = J(v,0) - J(u_0,0) + 2\langle \beta^*\zeta(v-u_0), g\rangle_{G',G} \quad \forall g \in G,$$

where $\zeta(v) \in \mathcal{V}$ is defined for all $v \in \mathcal{U}$ by

$$A^*\zeta(v) = C^*C\big(y(v,0) - y(0,0)\big),$$

where A^* (resp., β^*) being the adjoint of A (resp., β).

Proof. We refer the reader to [28]. □

Remark 1.1.2

- *For the sake of simplicity, we denote by $S(v) = \beta^*\zeta(v)$ the linear function for $v \in \mathcal{U}$. Then we have*

$$J(v,g) - J(u_0,g) = J(v,0) - J(u_0,0) + 2\langle S(v-u_0), g\rangle_{G',G} \quad \forall g \in G. \tag{1.5}$$

[2] G being an infinite space.

- *This is realized for the no-regret control v if $v \in K + u_0$, as shown in (1.5), where*
$$K = \{w \in \mathcal{U}, \langle S(w), g\rangle = 0, \quad \forall g \in G\}.$$
- *Of course, the problem (1.4) is defined only for the controls $v \in \mathcal{U}$ where*
$$\sup_{g \in G} \Big(J(v,g) - J(u_0,g)\Big) < \infty.$$

Proposition 1.1.1 (*See [55]*)

Let $u_0 \in \mathcal{U}$ be a given control. Then there exists a unique Pareto control related to u_0. Furthermore, it is the unique element of the set $K + u_0$, which minimizes the functional $J(v,0)$ on $K + u_0$.

Now we can present the following theorem.

Theorem 1.1.1

Let $u_0 \in \mathcal{U}$ be a given control. The control $u \in \mathcal{U}$ is the Pareto control related to u_0 if u is the no-regret control related to u_0.

Proof. We refer the reader to [38]. □

From the Theorem 1.1.1 and the Proposition 1.1.1, the existence of a unique no-regret control related to u_0 and that the Pareto control and no-regret control for the problem (1.1)-(1.2) are actually the same.

1.1.2 Low-regret control, or sequence of least regret controls

In this part, we are interested in the existence and characterization of the no-regret (or Pareto) control related to u_0. In [52], J.L. Lions applied the Pareto control and associated it with a sequence of low-regret controls defined by a quadratic perturbation for deterministic distributed systems with incomplete data. As in [38], we define the low-regret control by relaxing the problem (1.4) as follows:

$$\inf_{v \in \mathcal{U}} \Big(\sup_{g \in G}(J(v,g) - J(u_0,g) - \gamma \|g\|_G^2 \Big), \tag{1.6}$$

where $u_0 \in \mathcal{U}$ is a given control, and where γ is a strictly positive parameter. The solution to the problem (1.6), if it exists, will be the low-regret control related to $u_0 \in \mathcal{U}$, of the problem (1.1)-(1.2). The best possible choice of v is then given by (1.6).

Lemma 1.1.2

The Problem (1.6) is equivalent to

$$\inf_{v \in \mathcal{U}} \Big[J(v,0) - J(u_0,0) + \sup_{g \in G} \Big(2\langle S(v-u_0), g\rangle - \gamma \|g\|_G^2 \Big)\Big]. \tag{1.7}$$

Proof. For all $v \in \mathcal{U}$, and for all $g \in G$, we have

$$J(v,g) - J(u_0,g) = J(v,0) - J(u_0,0) + 2\langle S(v-u_0), g\rangle_{G',G}, \quad \forall g \in G. \tag{1.8}$$

From the equation (1.8), the problem (1.6) is written as (1.7). □

Lemma 1.1.3

The problem (1.7) is equivalent to

$$\inf_{v \in \mathcal{U}} \mathcal{J}^\gamma(v), \tag{1.9}$$

where

$$\mathcal{J}^{\gamma}(v) = J(v,0) - J(u_0,0) + \frac{1}{\gamma} \|S(v-u_0)\|^2_G \,. \tag{1.10}$$

We recognize this as a standard optimization problem with a quadratic cost function.

1.1.3 Optimality system

In this part, we present the optimality system for the low-regret control u^{γ}.

Proposition 1.1.2

The problem (1.9)-(1.10) has a unique sequence of solution u^{γ} is referred to as the sequence of low regret controls related to u_0.

Proof. We refer the reader to [38]. □

We now present the optimality system for low-regret control.

Proposition 1.1.3

The unique sequence of low regret controls u^{γ} is characterized by the unique solutions of the optimality system (OS):

$$\begin{cases} Ay^{\gamma} = f + Bu^{\gamma}, & A^*\zeta^{\gamma} = C^*C(y^{\gamma} - y(0,0)), \\ A\rho^{\gamma} = \frac{1}{\gamma}\beta\beta^*\zeta^{\gamma}, & A^*P^{\gamma} = C^*(Cy^{\gamma} - z_d) + C^*C\rho^{\gamma}, \\ P^{\gamma} + Nu^{\gamma} = 0 & \forall u \in \mathcal{U}. \end{cases}$$

1.1.4 Singular optimality system

This section includes the singular optimality system for the no-regret control of u. Let $\tilde{\mathcal{G}}$ be the completion of $\mathcal{G}$ in $\mathcal{F}$[3]. We'll start with solving

$$A\rho = \beta g, \qquad g \in \mathcal{G}, \quad \rho \in \mathcal{V}.$$

Then we come up with a solution:

$$A^*\sigma = C^*C\rho, \quad \sigma \in \mathcal{V}.$$

Let $\mathcal{R}$ be an operator defined as follows:

$$\mathcal{R}g = B^*\sigma.$$

We're assuming[4]

$$\exists c > 0, \quad \forall g \in \mathcal{G} \quad \|\mathcal{R}g\|_{\tilde{\mathcal{G}}} \geq C \|g\|_{\mathcal{G}} \,. \tag{1.11}$$

Lemma 1.1.4

Suppose that (1.11) holds true. The unique no regret control u related to u_0 is then characterized by the unique solutions of the singular optimality system (SOS):

$$\begin{cases} Ay = f + u, & A^*P = C^*(Cy - z_d) + C^*C\rho, \\ A\rho = \lambda, & P + Nu = 0, \end{cases}$$

with $\lambda \in \tilde{\mathcal{G}}$.

Proof. We refer the reader to [38]. □

[3] The space $\tilde{\mathcal{G}}$ is in fact the completion of $\mathcal{G}$ for a subspace $(\mathcal{H}, \|.\|)$ of $\mathcal{F}$ which can be bigger than $\mathcal{G}$.

[4] The hypothesis (1.11) is very useful theoretically but is not necessary in practice. We need only to make sure that the adjoint state P^{γ} of Proposition 1.1.3 is bounded in a suitable Hilbert space.

1.2 No-regret control and low regret control in the evolution case

1.2.1 No-regret control and low regret control for parabolic systems

In this section, we define an elliptic differential operator $A \in \mathcal{L}(\mathcal{V}; \mathcal{V}')$ as well as

$$\langle Av, v\rangle_{\mathcal{V}',\mathcal{V}} \geq \alpha \|v\|_{\mathcal{V}}^2, \quad \alpha > 0.$$

Let's introduce an operator $B \in \mathcal{L}(\mathcal{U}; L^2(0,T;\mathcal{V}'))$, and a real Hilbert space of uncertainties $\mathcal{F}$ such that $\mathcal{V} \subset \mathcal{F} \subset \mathcal{V}'$. Consider $\mathcal{G}$ to be the closed vector subspace of $\mathcal{F}$. For $f \in L^2(0,T;\mathcal{V}')$, the state equation that we consider is

$$\begin{cases} \dfrac{\partial y}{\partial t} + Ay = f + Bv, \\ y(0,v,g) = y_0 + g, \end{cases} \tag{1.12}$$

where y_0 is a given data in $\mathcal{F}$ and where $g \in \mathcal{G}$.
The problem (1.12) admits a unique solution $y(v,g) \in L^2(0,T;\mathcal{V})$ for chosen v and g. For a fixed $t \in [0,T]$, and for any $g \in \mathcal{G}$, we add a cost function that comes from

$$J(v,g) = \int_0^T \|Cy(v,g) - z_d\|_{\mathcal{H}}^2\, dt + N \int_0^T \|v\|_{\mathcal{U}}^2\, dt, \tag{1.13}$$

where $\mathcal{H}$ is a Hilbert space, and

$$C \in \mathcal{L}(L^2(0,T;\mathcal{V};\mathcal{H}), \quad z_d \text{ fixed in } \mathcal{H}, \quad N > 0.$$

When $\mathcal{G} \neq \{0\}$, We'll keep doing things the same way we did in the evolution case. [5]

Low regret control and the optimality system

Referring to [55], for all $v \in \mathcal{U}, g \in G$, we have

$$J(v,g) - J(u_0,g) = J(v,0) - J(u_0,0) + 2\langle S(v-u_0), g\rangle_{G',G}, \quad \forall g \in G, \tag{1.14}$$

where

$$S(v) = \zeta(t=0,v), \tag{1.15}$$

and where ζ is the solution to the backwards problem,

$$\begin{cases} -\zeta' + A^*\zeta = C^*C(y(v,0) - y(0,0)), \\ \zeta(T,v) = 0, \end{cases} \tag{1.16}$$

with $\zeta' = \frac{\partial \zeta}{\partial t}$. Then, the low-regret control associated with the problem (1.12)-(1.13) is defined by

$$\inf_{v\in\mathcal{U}} \mathcal{J}^\gamma(v), \tag{1.17}$$

where

$$\mathcal{J}^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}\|\zeta(0,v-u_0)\|_{\mathcal{G}'}^2, \tag{1.18}$$

where $\mathcal{G}'$ is the dual of $\mathcal{G}$ which is identified as $\mathcal{G}$.
The problem (1.17)-(1.18) has a unique solution u^γ called low-regret control.

Proposition 1.2.1

The unique sequence of low regret controls u^γ solution to (1.17)-(1.18) is characterized by the

[5] When $\mathcal{G} = \{0\}$, a standard control problem is to find

$$\inf_{v\in\mathcal{U}} J(v,0).$$

unique solutions $y^\gamma, \zeta^\gamma, \rho^\gamma, P^\gamma$ of the optimality system (OS):

$$\begin{cases} y^{\gamma'} + Ay^\gamma = f + Bu^\gamma, & y^\gamma(0) = y_0, \\ -\zeta^{\gamma'} + A^*\zeta^\gamma = C^*(Cy^\gamma - y(0,0)), & \zeta^\gamma(T) = 0, \\ \rho^{\gamma'} + A\rho^\gamma = 0, & \rho^\gamma(0) = \frac{1}{\gamma}\zeta^\gamma, \\ -P^{\gamma'} + A^*P^\gamma = C^*(Cy^\gamma - z_d) + C^*C\rho^\gamma, & P^\gamma(T) = 0, \\ B^*P^\gamma + Nu^\gamma = 0 & \text{in } \mathcal{U}. \end{cases}$$

Proof. We refer the reader to [38]. □

We now give the optimality system for the no-regret control. We need a supplementary hypothesis. Let $\rho \in L^2(0,T;\mathcal{V})$, $\sigma \in L^2(0,T;\mathcal{V})$ be defined by

$$\begin{cases} \rho' + A\rho = 0, & \rho(0) = g, \ g \in \mathcal{G}, \\ -\sigma' + A^*\sigma = \rho, & \sigma(T) = 0. \end{cases}$$

We define a $\mathcal{R}$ operator such that $\mathcal{R}g = B^*\sigma$. The continuous operator $g \to \mathcal{R}g$ from F to $\mathcal{U}$ is then defined. We will put this hypothesis

$$\exists C > 0, \forall g \in \mathcal{G} : \|\mathcal{R}g\|_{\tilde{\mathcal{G}}} > C \|g\|_{\mathcal{G}} . \tag{1.19}$$

Lemma 1.2.1

Suppose that (1.19) holds true. Then, the unique no regret control u related to u_0, for the system (1.12)-(1.13) is characterized by the unique solution $\{y, \zeta, \rho, P\}$ to the singular optimality system (SOS)

$$\begin{cases} y' + Ay = f + Bu, & y(0) = y_0, \\ -\zeta' + A^*\zeta = C^*C(y - y(0,0)), & \zeta(T) = 0, \\ \rho' + A\rho = 0, & \rho(0) = \lambda, \\ -P' + A^*P = C^*(Cy - z_d) + C^*C\rho, & P(T) = 0, \\ B^*P^\gamma + Nu^\gamma = 0 & \text{in } \mathcal{U}. \end{cases}$$

with $\lambda \in \tilde{\mathcal{G}}$.

Proof. We refer the reader to [38]. □

1.2.2 No-regret control for well-posed systems of the Petrowsky type

We now consider an elliptic differential operator A such as $A^* = A$. We consider the state equation

$$\begin{cases} y'' + Ay = v, \\ y(0) = y_0 + g_0, \quad y'(0) = y_1 + g_1, \end{cases} \tag{1.20}$$

where $\{y_0, y_1\}$ is bounded in $\mathcal{V} \times \mathcal{F}$ and where

$$\begin{cases} g_0 \in G_0, \ G_0 \text{ closed vector subspace of } \mathcal{V}, \\ g_1 \in G_1, \ G_1 \text{ closed vector subspace of } \mathcal{F}. \end{cases}$$

The problem (1.20) has a unique solution $y(v,g)$ such that

$$y \in L^\infty(0,T;\mathcal{V}), \qquad y' \in L^\infty(0,T;\mathcal{F}).$$

The quadratic cost function $J(v,g)$ associated with (1.20) is defined by (1.13). We define $y = y(v,0)$ and $\zeta(t,v)$, respectively, by

$$\begin{cases} y'' + Ay = v, & \zeta'' + A\zeta = C^*Cy(v,0), \\ y(0) = y_0, \ y'(0) = y_1, & \zeta(T) = 0, \quad \zeta'(T) = 0. \end{cases}$$

Setting $z = y(0,g) - y(0,0)$. Then z is the solution of

$$\begin{cases} z'' + Az = 0, \\ z(0) = g_0,\ z'(0) = g_1. \end{cases}$$

Using the Green formula, we get

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2\langle \zeta(0), g_1 \rangle_{G_0 \times G_1} + 2\langle \zeta(0), g_1 \rangle_{G_1 \times G_0}.$$

The control solution with the low of regret is described as the

$$\inf_{v \in \mathcal{U}} \Big(\sup_{g \in G_1 \times G_2} (J(v,g) - J(0,g) - \gamma \|g_0\|^2_{G_0} - \gamma \|g_1\|^2_{G_1}) \Big).$$

The method of low-regret control is as follows:

$$\inf_{v \in \mathcal{U}} \mathcal{J}^\gamma(v),$$

where

$$\mathcal{J}^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma} \|\zeta(0)\|^2_{G_1} + \frac{1}{\gamma} \|\zeta'(0)\|^2_{G_0}.$$

We get the following result for Petrowsky systems.

Lemma 1.2.2

The unique no regret control u related to $u_0 = 0$, is characterized by the unique solutions y, ζ, ρ, P to the singular optimality system (SOS):

$$\begin{cases} y'' + Ay = u, & y(0) = y_0, & y'(0) = y_1, \\ \zeta'' + A\zeta = y - y(0,0), & \zeta^(T) = 0, & \zeta'(T) = 0, \\ \rho'' + A\rho = 0, & \rho(0) = \lambda_0, & \rho'(0) = \lambda_1, \\ P'' + AP = y - z_d + \rho, & P(T) = 0 & P'(T) = 0, \\ P + Nu = 0 & \text{in} & \mathcal{U}, \end{cases}$$

with

$$\lambda_0 = \lim_{\gamma \longrightarrow 0} \frac{1}{\gamma} \zeta(0), \quad \lambda_1 = \lim_{\gamma \longrightarrow 0} \frac{1}{\gamma} \zeta'(0), \quad \lambda_0 \in \tilde{G}_0, \quad \lambda_1 \in \tilde{G}_1.$$

Proof. We refer the reader to [38]. □

1.3 Procedures for elliptic and parabolic regularization

The process of elliptic and parabolic regularization has been employed by J. L. Lions in [57, 59]. Using this procedure, the system of equations determining the optimal control may be deduced by passing to the limit. By passing to the limit, the system of equations determining the optimal control can be determined using this technique.

Let V and H be two Hilbert spaces over $\mathbb{R}$, $V \subset H$, V is dense in H. Identifying H with its dual, we have the embeddings

$$V \hookrightarrow H \hookrightarrow V'.$$

We consider the operator $A \in \mathcal{L}(V; V')$, and we set

$$\Big(A\varphi, \psi \Big) = a(\varphi, \psi), \qquad \forall \varphi, \psi \in V.$$

We make the assumption that

a. The bilinear form a is continuous, symmetric on V.

b. there exists $\alpha > 0$ such that $a(\varphi, \varphi) \geq \alpha \|\varphi\|^2,\ \forall \varphi \in V$.

1.3.1 Elliptic regularization

Assume that $\mathcal{U}$ is a real Hilbert space and that the state y is given by

$$\begin{cases} y' + Ay = f + Bv, \\ y(0,.) = 0, \end{cases} \tag{1.21}$$

where $f \in L^2(0,T;V')$, $B \in \mathcal{L}(\mathcal{U}; L^2(0,T;V'))$, $v \in \mathcal{U}$, $y_0 \in H$ are given. It is shown in [59, Chapitre 3] that there exists a unique function $y = y(v)$ which is a solution to (1.21), satisfying $y \in L^2(0,T;V)$. We can approximate the problem (1.21) by a problem of elliptic type. Let $\epsilon > 0$ be given, we have the following results (see [59, Remark 1.6, page 389] or [57, page 407]):

Theorem 1.3.1

There is a unique function y_ϵ that satisfies the equation (1.22) for each $\epsilon > 0$.

$$\begin{cases} -\epsilon y_\epsilon'' + y_\epsilon' + Ay_\epsilon = f + Bv, \\ y_\epsilon(0,.) = 0, \ y_\epsilon(T,.) = 0, \end{cases} \tag{1.22}$$

where

$$y_\epsilon \in L^2(0,T;V).$$

Furthermore, when $\epsilon \to 0$, we have

$$y_\epsilon \to y \text{ in } L^2(0,T;V),$$

where y is the solution to (1.20).

Proof. We refer the reader to [59]. □

We can consider more generally instead of (1.22) the equation (see [57, Remark 1.5, page. 412]):

$$(-1)^m \epsilon \frac{\partial^{2m} y_\epsilon}{\partial t^{2m}} + \frac{\partial y_\epsilon}{\partial t} + Ay_\epsilon = f + Bv,$$

where $y_\epsilon(0,.) = 0$, $(m-1)$ are other boundary conditions for $t = 0$, and m are boundary conditions for $t = T$. This passage is called the procedure of *elliptic regularization*.

1.3.2 Parabolic regularization

We consider the evolution problem (without any control[6])

$$\begin{cases} y'' + Ay = f, \\ y(0,.) = y_0, \ y'(0,.) = y_1, \end{cases} \tag{1.23}$$

where $f \in L^2(0,T;H)$, $y_0 \in V$, $y_1 \in H$ are given. There is a unique function y that is a solution to (1.23). This solution is satisfying

$$y \in L^2(0,T;V), \quad y' \in L^2(0,T;H).$$

We'll use a parabolic problem to approximate problem (1.23). Let $\epsilon > 0$ be given. The following theorems are proposed. For more details see [59].

Theorem 1.3.2

There is a unique function y_ϵ that satisfies the equation (1.24) for each $\epsilon > 0$.

$$\begin{cases} y_\epsilon'' + Ay_\epsilon + \epsilon Ay_\epsilon' = f, \\ y_\epsilon(0,.) = 0, \ y_\epsilon'(0,.) = y_1, \end{cases} \tag{1.24}$$

[6] To simplify the exposition

where

$$y_\epsilon \in L^2(0,T;V),\ y'_\epsilon \in L^2(0,T;V).$$

Proof. We refer the reader to [59]. □

Theorem 1.3.3

As $\epsilon \to 0$, we have

$$y_\epsilon \to y \text{ in } L^2(0,T;V), \quad y'_\epsilon \to y' \text{ in } L^2(0,T;H),$$

where y is the solution to (1.23).

Proof. We refer the reader to [59]. □

1.4 Singular distributed problems and correctors

Singular distributed problems are those in which the state has instabilities, explosion phenomena, multiple solutions, or bifurcation phenomena. In these types of problems, one needs to formally describe the optimal control-state pair by solving optimization problems subject to constraints on the control and additional constraints on the state. The theories of controllability, stability, and optimal control have become an accelerated field of research thanks to the highly influential, ground-breaking research and work of many people. The notion of a corrector has been studied by J.L. Lions [57, Chapter 5, page 66, Section 3.2] and [51]. Its main role is to correct the behavior of the solution. Assume y is the solution to the well-posed problem for $\epsilon > 0$. We're interested in how y behaves when $\epsilon \to 0$. This behavior is singularin one or more of the following ways:

Situation (i) The state y_ϵ has no limit under any topology introduced, but if we delete a singular part θ_ϵ from y_ϵ, then:

$$y_\epsilon - \theta_\epsilon \to y, \quad \text{in a suitable topology.}$$

Situation (ii) The state y_ϵ lives in some functional space F, but it does not converge in . In a a space $\mathcal{F}$ greater than F, we can find a limit y to y_ϵ. Then we can attempt to construct correctors for θ_ϵ, which we can calculate or estimate, so that $\theta_\epsilon \in \mathcal{F}$ and

$$y_\epsilon - (y + \theta_\epsilon) \to 0 \text{ in } F.$$

Example 1.4.1

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, with smooth boundary Γ. For all $T > 0$, we denote by $Q = \Omega \times (0,T)$. The following boundary problem is considered. For all $\epsilon > 0$, We're working on a solution to the Dirichlet problem:

$$\begin{cases} -\epsilon\Delta y_\epsilon + y_\epsilon = f & \text{in} \quad Q, \\ y_\epsilon(0) = 0 & \text{on} \quad \Gamma. \end{cases} \tag{1.25}$$

If it is assumed that $f \in L^2(\Omega)$, then

$$y_\epsilon \in H^2(\Omega) \cap H_0^1(\Omega).$$

We pose

$$a(y,\varphi) = \int_\Omega \nabla y.\nabla\varphi \,\mathrm{d}x, \quad y,\varphi \in H_0^1(\Omega),$$
$$b(y,\varphi) = \int_\Omega y\varphi \mathrm{d}x.$$

The problem (1.25) *therefore becomes equivalent to*

$$\epsilon a(y_\epsilon,\varphi) + b(y_\epsilon,\varphi) = \langle f,\varphi\rangle,\ \forall\varphi, y_\epsilon \in H_0^1(\Omega). \tag{1.26}$$

Putting $\varphi = y_\epsilon$ *gives us*

$$\epsilon a(y_\epsilon, y_\epsilon) + b(y_\epsilon, y_\epsilon) = \langle f, y_\epsilon \rangle.$$

After that, there's

$$\|y_\epsilon\|_{L^2(\Omega)} \leq C, \quad \sqrt{\epsilon}\|y_\epsilon\|_{H^1(\Omega)} \leq C.$$

As a result, we can extract a subsequence, which will be labeled as y_ϵ *such that* $y_\epsilon \to y$ *weakly in* $L^2(\Omega)$. *In (1.26), we reach the limit.*

$$b(y, \varphi) = \langle f, \varphi \rangle, \quad \forall \varphi \in H_0^1(\Omega) \quad \Leftrightarrow \quad y = f.$$

So we can say when $\epsilon \to 0$ *the solution* y_ϵ *of (1.25) converges in* $L^2(\Omega)$ *weakly (strongly) to the solution* $y = f$. *In this case, the aim was to find out which sense* y_ϵ *converges to* y.

(a). *If we assume that* $f \in H^1(\Omega)$ *(without being in* $H_0^1(\Omega)$*). If this is the case,* y_ϵ *does not converge in* $H^1(\Omega)$. *Indeed, as* $H_0^1(\Omega)$ *is closed in* $H^1(\Omega)$*, and if*

$$y_\epsilon \to y,$$

we have : $y \in H_0^1(\Omega)$. *Because* $f = y$*, this results in a contradiction.*

(b). *If we assume the following:*

$$f \in H^1(\Theta), \ \forall \Theta \ \text{ open with } \bar{\Theta} \subset \Omega,$$

then

$$y_\epsilon \to y \in H^1(\Omega).$$

"The estimation of $\|y_\epsilon - y\|_{L^2(\Omega)}$ *in* ϵ *using the attributes of* f*" is another natural question.*

(c). *In case (a), we see that the boundary* Γ *plays a particular role: there is some loss of boundary conditions in the passage to the limit, so the convergence is less regular in the neighborhood of* Γ *than in the interior. Hence, the problem is:* By what should we correct the difference $(y_\epsilon - y)$?.

As a result, we conclude that the convergence for the new difference $(y_\epsilon - y - \theta_\epsilon)$ *to* 0 *is better. For example, it takes place in* $H^1(\Omega)$.

This problem must be solved precisely in order to provide a correction. We are attempting to define some θ_ϵ *correctors, which are*

- *Concentrating in the neighborhood of* Γ.
- *It's simple to compute.*

1.5 Fractional Calculus

Fractional calculus is a theory of integrals and derivatives of arbitrary real or even complex orders. It is a generalization of the classical calculus and therefore preserves many of its basic properties. Fractional calculus was first mentioned in a letter from Hospital to Leibniz in 1695. In this letter, Hospital inquires about Leibniz's work from 1646, in which he defines the derivative of order n of a function f with $n \in \mathbb{N}$. WhenHospital asks what happens if $n = \frac{1}{2}$, Leibniz says, *"This leads to a conundrum from which we shall one day extract valuable conclusions"*. Many mathematicians have studied the issue since its discovery, with the goal of generalizing the findings established for integer-order derivatives to the case of arbitrary-order derivatives.

Fractional calculus is the name given to the theory of arbitrary order integrals and derivatives, which unifies and generalizes integer-order differentiation and n-fond integration. In other words, fractional derivatives and integrals can be considered as an *interpolation* of the infinite sequence, [39].

$$\cdots, \int_a^t \int_a^{\tau_1} f(\tau_2)\mathrm{d}\tau_2\mathrm{d}\tau_1,\ \int_a^t f(\tau_1)\mathrm{d}\tau_1,\ f(t),\ \frac{\mathrm{d}f(t)}{\mathrm{d}t},\ \frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}, \cdots$$

of the classical n fold integrals and n fold derivatives.

Let's review some fundamental fractional calculus definitions and results. We'll go through the definitions and desired outcomes that will help us introduce integral and fractional derivatives, as well as solve our diffusion and fractional wave equations. For more information, look up the references [30, 39, 49].

1.5.1 Special functions

This part is about the collection of functions, we'll use in fractional theory. To begin, the Gamma function will be defined as follows:

Gamma function

Definition 1.5.1 (*[49, 39]*)

The Gamma function, denoted by $\Gamma(z)$ *is a generalization of the factorial function* $n!$, *i.e.,*

$$\Gamma(n) = (n-1)!\quad \forall n \in \mathbb{N}.$$

For complex arguments with positive real part it is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \qquad \mathcal{R}e(z) > 0.$$

This function has the following essential results:

Proposition 1.5.1 (*[49, 39]*)

For a complex argument z *with positive real part* $\mathcal{R}e(z) > 0$. *So we have the following result:*

$$\Gamma(z+1) = z\Gamma(z).$$

Some of the most important values are

$$\begin{aligned}
\Gamma(1) = \Gamma(2) &= 1,\\
\Gamma(\tfrac{1}{2}) &= \sqrt{\pi},\\
\Gamma(n+\tfrac{1}{2}) &= \frac{\sqrt{\pi}(2n-1)!}{2^n},\ \forall n \in \mathbb{N}.
\end{aligned}$$

Beta function

Definition 1.5.2 (*[49, 39]*)

The Beta function is defined by the integral

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\mathrm{d}t, \quad \mathcal{R}e(z) > 0, \quad \mathcal{R}e(w) > 0.$$

The Beta function is used sometimes for convenience to replace a combination of Gamma function. This relation between the Gamma function and Beta function is given by (see [26])

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

It should also be mentioned that the Beta function is symmetric, i.e.,

$$B(z,w) = B(w,z).$$

The Mittag-Leffler function

While the Gamma function is a generalization of the factorial function, the Mittag-Leffler function is a generalization of the exponential function (see [49, 39])

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)}.$$

First introduced as a one parameter function by the series [39]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad z,\alpha \in \mathbb{C},\ \mathcal{R}e(\alpha) > 0.$$

Later, the two parameter generalization is introduced by Agarwal

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z,\alpha,\beta \in \mathbb{C},\ \mathcal{R}e(\alpha) > 0,\ \mathcal{R}e(\beta) > 0,$$

which is of great importance for the fractional calculus. It is called two parameter function of Mittag-Leffler type. Some of its interesting values are [39]

$$\begin{aligned} E_{1,1}(z) &= e^z, \\ E_{2,1}(z^2) &= \cosh(z), \\ E_{2,2}(z^2) &= \frac{\sinh(z)}{z}, \\ E_{\alpha,2}(z) &= E_\alpha(z), \\ E_{\frac{1}{2},1}(z) &= e^{z^2}\mathrm{erfc}(-z). \end{aligned}$$

This function has the following essential results:

Proposition 1.5.2 (*[49, 39]*)

For a complex argument z with $\mathcal{R}e(z) > 0$, we have the following result:

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)},$$

$$\frac{\mathrm{d}}{\mathrm{d}z}E_{\alpha,\beta}(z) = \frac{1}{\alpha z}\Big[E_{\alpha,\beta-1}(z) + (\beta-1)E_{\alpha,\beta}(z)\Big].$$

We'll need to build estimates in order to illustrate the uniqueness of each solution in the next section. We'll use the following two outcomes to do so:

Lemma 1.5.1 (*[7]*)

For positive integers m, λ and α, we have

As well as

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}E_{\alpha,1}(-\lambda z^\alpha) = -\lambda z^{\alpha-n}E_{\alpha,\alpha-n+1}(-\lambda z^\alpha), \quad z > 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}z}(zE_{\alpha,2}(-\lambda z^\alpha)) = E_{\alpha,1}(-\lambda z^\alpha), \quad z > 0.$$

Theorem 1.5.1 ([7])

Let $0 < \alpha < 2$, β is an arbitrary real, and we assume that μ is such that

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}.$$

Then there exists a constant $C = C(\alpha, \beta, \mu) > 0$ such that

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1 + |z|}, \quad \mu \leq |\arg(z)| \leq \pi.$$

The definition of the generalized Mittag-Leffler function is now given.

Definition 1.5.3 ([7])

Let $\alpha, \beta, \rho \in \mathbb{C}$ such as $\mathcal{R}e(\alpha) > 0$ and $\mathcal{R}e(\beta) > 0$. The generalized Mittag-Leffler function is thus defined as follows:

$$\varsigma^{\rho}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\rho)_n z^n}{\Gamma(\alpha n + \beta) n!}, \quad \forall z \in \mathbb{C},$$

where

$$(\rho)_n = \rho(\rho + 1)...(\rho + n - 1).$$

Remark 1.5.1

Note that when $\rho = 1$ we have

$$\varsigma^{\rho}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z).$$

We'll need the following Lemma in the sequel:

Lemma 1.5.2

Let $\alpha, \beta, \rho \in \mathbb{C}$ such as $\mathcal{R}e(\alpha) > 0$ and $\mathcal{R}e(\beta) > 0$. Then, we have

$$\frac{d^n}{dz^n} \varsigma^{\rho}_{\alpha,\beta}(z) = (\rho)_n \varsigma^{\rho+n}_{\alpha,\beta+\alpha n}(z), \quad z \in \mathbb{C}, \; n \in \mathbb{N},$$
$$\alpha\rho \varsigma^{\rho+1}_{\alpha,\beta}(z) = (1 + \alpha\rho - \beta) \varsigma^{\rho}_{\alpha,\beta}(z) + \varsigma^{\rho}_{\alpha,\beta-1}(z), \quad z \in \mathbb{C}.$$

We utilize the Laplace transform to solve our fractional differential equations, just as we did with integer differential equations. As a result, we provide the following definition:

Definition 1.5.4

Let $f : \mathbb{R}^+ \to \mathbb{R}$. The Laplace transform of function is defined by:

$$(\mathcal{L}f)(s) = \mathcal{L}[f(t)](s) = \hat{f}(s) := \int_0^{\infty} \exp(-st) f(t) dt, \quad s > 0.$$

On occasion, we will run across transforms of the form,

$$H(s) = F(s)G(s),$$

that can't be dealt with easily using partial fractions. We would like a way to take the inverse transform of such a transform. We can use a convolution integral to do this.

Definition 1.5.5

If $f(t)$ and $g(t)$ are piecewise continuous function on $[0, +\infty]$ then the convolution integral of $f(t)$

and $g(t)$ is,

$$(f \star g)(t) = \int_0^t f(t-s)g(s)\mathrm{d}s.$$

A nice property of convolution integrals is

$$(f \star g)(t) = (g \star f)(t).$$

Or,

$$\int_0^t f(t-s)g(s)\mathrm{d}s = \int_0^t f(s)g(t-s)\mathrm{d}s.$$

The following fact will allow us to take the inverse transforms of a product of transforms.

$$\mathcal{L}\{f \star g\}(t) = F(s)G(s), \quad \mathcal{L}^{-1}F(s)G(s) = \{f \star g\}(t).$$

Lemma 1.5.3

Let $\alpha, \beta, \rho \in \mathbb{C}$ such as $\mathcal{R}e(\alpha) > 0$, $\mathcal{R}e(\rho) > 0$ and $\mathcal{R}e(\beta) > 0$. Then, we have

$$\mathcal{L}^{-1}\Big[\frac{s^{\rho-1}}{s^{\alpha} + as^{\beta} + b}; z\Big] = t^{\alpha-\rho}\sum_{k=0}^{\infty}(-a)^k z^{k(\alpha-\beta)}\varsigma^{k+1}_{\alpha,\alpha+(\alpha-\beta)k-\rho+1}(-bz^{\alpha}),$$

where $|\dfrac{as^{\beta}}{s^{\alpha}+b}| < 1$. We also assume that the preceding equality's series is convergent.

1.5.2 Riemann-Liouville fractional integral

Calculations of integrals and derivatives of arbitrary real or complex order are referred to as "fractional calculations." In this thesis, we are only concerned with Riemann-Liouville and Caputo derivatives.

Definition 1.5.6 (*See [32]*)

Cauchy's formula for repeated integration is given by

$$I^n f(t) := \int_a^t \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} f(\tau)\mathrm{d}\tau \cdots \mathrm{d}\tau_2 \mathrm{d}\tau_1$$
$$= \frac{1}{(n-1)!}\int_a^t f(\tau)(t-\tau)^{n-1}\mathrm{d}\tau, \ \forall n \in \mathbb{N}, a, t \in \mathbb{R}, t > 0.$$

If n is substituted by a positive real number α and $(n-1)!$ by its generalization $\Gamma(\alpha)$ a formula for fractional integration is obtained.

Definition 1.5.7

The Riemann-Liouville integral is defined by

$$I^{\alpha}f(t) := \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{1-\alpha}f(s)\mathrm{d}s, \quad t > a,$$

where Γ is the gamma function and a is an arbitrary but fixed base point. The integral is well defined provided f is a locally integrable function, and α is a complex number in the half-plane $\Re(\alpha) > 0$ is referred to as Riemann-Liouville fractional integral of order α.

Proposition 1.5.3

- *By convention*
$$I^0 f(t) := f(t),\ \text{i.e., } I^0 := I \text{ is the identity operator.}$$
- *The linearity*
$$I^\alpha(\lambda f(t) + g(t)) = \lambda I^\alpha f(t) + I^\alpha g(t),\ \alpha \in \mathbb{R}_+, \lambda \in \mathbb{C}.$$
- *If $f(t)$ is continuous for $t \geq 0$ the following equalities hold*
$$lim_{\alpha\to 0} I^\alpha f(t) = f(t),$$
$$I^\alpha(I^\beta f(t)) = I^\beta(I^\alpha f(t)) = I^{\alpha+\beta} f(t) \quad \alpha, \beta \in \mathbb{R}_+, \lambda \in \mathbb{C}.$$

Definition 1.5.8

The Laplace transform of Riemann-Liouville fractional integral is defined by:

$$\begin{aligned}\mathcal{L}[I^\alpha f(x)] &= \frac{1}{\Gamma(\alpha)}\mathcal{L}(x^{\alpha-1} \star f(x)) \\ &= \frac{1}{s^\alpha}\mathcal{L}[f(x)].\end{aligned}$$

1.5.3 Riemann-Liouville fractional derivative operator

Definition 1.5.9

The Riemann-Liouville fractional derivative or the Riemann-Liouville fractional differential operator of order α is defined by

$$\begin{aligned}D^\alpha_{RL} f(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}(I^{n-\alpha} f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t (t-s)^{n-\alpha-1} f(s)\ \mathrm{d}s, \quad t > 0, \quad \alpha \in (n-1, n),\ n \in \mathbb{N}.\end{aligned}$$

In the following Lemma, we give some relations between the Riemann-Liouville fractional derivative and the Riemann-Liouville fractional integral.

Lemma 1.5.4

Let $u \in \mathbb{C}^n([0,T])$, $\alpha \in (n-1, n)$, $n \in \mathbb{N}$ and $v \in \mathbb{C}^1([0,T])$.

- *The Riemann-Liouville fractional differential operator D^α is the left inverse operator of the fractional integral I^α, i.e.,*
$$D^\alpha_{RL} I^\alpha = I,$$
By convention it is defined
$$D^0_{RL} v(t) := v(t),\ \text{i.e., } D^0_{RL} := I \text{ is the identity operator.}$$
- *We have*
$$D^\alpha_{RL} v(t) = \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha} v(t), \quad n = 1,$$
$$D^\alpha_{RL} v(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} I^{2-\alpha} v(t), \quad n = 2,$$
$$I^\alpha D^\alpha_{RL} u(t) = u(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}(I^{\alpha-1} u)(0).$$

Remark 1.5.2

As we can see from the previous definition, the Riemann-Liouville fractional derivative of a constant is non-zero, unlike the integer order derivative of a constant C. To be more specific, the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ of a constant C is given by

$$D^{\alpha}_{RL} I^{\alpha} C = \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)}.$$

Definition 1.5.10

The Laplace transform of the Riemann-Liouville fractional derivative is defined by:

$$\begin{aligned} \mathcal{L}[D^{\alpha}_{RL} f(t)] \quad &= \mathcal{L}\Big[\frac{\mathrm{d}^n}{\mathrm{d}t^n}(I^{n-\alpha} f(t))\Big] \\ &= s^{\alpha}\mathcal{L}(f(t)) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1} f(0) \\ &= s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1} f(0), \end{aligned}$$

where the Laplace transform of $f^{(n)}$ is defined as follows:

$$\mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - \sum_{k=0}^{n-1} s^k f^{n-k-1}(0),$$

and

$$D^{\alpha}_{RL} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}(I^{n-\alpha} f(t)) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}(D^{(\alpha-n)} f(t)).$$

As a result, we've arrived at the following two theorems:

Theorem 1.5.2 (*See [7, 8]*)

Let $0 < \alpha < 1$. The Riemann-Liouville fractional derivative of order α is then transformed by the Laplace transform:

$$\mathcal{L}[D^{\alpha}_{RL} f(t)] = s^{\alpha} F(s) - \lim_{t\to 0} I^{1-\alpha} f(t).$$

Theorem 1.5.3 (*See [7, 8]*)

Assume that $1 < \alpha < 2$. The Riemann-Liouville fractional derivative of order α is then transformed by the Laplace transform:

$$\mathcal{L}[D^{\alpha}_{RL} f(t)] = s^{\alpha} F(s) - s \lim_{t\to 0} I^{2-\alpha} f(t) - \lim_{t\to 0} \frac{\mathrm{d}}{\mathrm{d}t} I^{2-\alpha} f(t).$$

In the formulation of the Laplace transforms, we can see the terms $\lim_{t\to 0} I^{1-\alpha} f(t)$, $\lim_{t\to 0} I^{2-\alpha} f(t)$ and $\lim_{t\to 0} \frac{\mathrm{d}}{\mathrm{d}t} I^{2-\alpha} f(t)$. Contrary, in integer order derivatives, where we can see the initial values of the functions f and f'.

1.5.4 The left and right Caputo fractional derivatives

The concepts of left and right Caputo fractional derivatives will be discussed here.

Definition 1.5.11

If $f(t)$ is defined in $\mathcal{C}^n[a, \infty)$, then the left Caputo fractional derivative or left Caputo fractional *differential operator of order α is defined as*

$$\begin{aligned}\mathcal{D}_C^{\alpha} f(t) &= I^{n-\alpha}\left(\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\right) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t f^{(n)}(s)(t-s)^{n-\alpha-1} ds, \quad t>0, \quad \alpha \in (n-1,n), \ n \in \mathbb{N}.\end{aligned}$$

The right Caputo fractional derivative or the right Caputo fractional differential operator of order α is defined by

$$\mathcal{D}_C^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^T f^{(n)}(s)(s-t)^{n-\alpha-1} ds, \quad 0<t<T, \quad \alpha \in (n-1,n), \ n \in \mathbb{N}.$$

A constant's Caputo derivative is equal to zero.

The adjoint operator of the right fractional derivative is represented by the left fractional derivative. In the following lemma, we give some relations between the Riemann-Liouville fractional integral and the Caputo fractional derivative:

Lemma 1.5.5

Let $u \in \mathbb{C}^n([0,T])$, $\alpha \in (n-1,n)$, $n \in \mathbb{N}$ and $v \in \mathbb{C}^1([0,T])$.

$$\mathcal{D}_C^{\alpha} I^{\alpha} v(t) = v(t);$$

$$I^{\alpha}\mathcal{D}_C^{\alpha} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0);$$

$$I^{\alpha}\mathcal{D}_C^{\alpha} u(t) = u(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}(I^{1-\alpha-}u)(0), \quad n=1;$$

$$I^{\alpha}\mathcal{D}_C^{\alpha} u(t) = u(t) - u(0), \quad n=1.$$

Lemma 1.5.6

Let $(n-1) < \alpha < n$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $f(t)$ be such that $\mathcal{D}_C^{\alpha} f(t)$ exists. Then

$$\mathcal{D}_C^{\alpha} f(t) = I^{n-\alpha} D^n f(t) = I^{n-\alpha} \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t).$$

This implies that the Caputo fractional differential operator is equivalent to an $(n-\alpha)$-fold integration following an n-th order differentiation.

Proposition 1.5.4

In general, the two operators, Riemann-Liouville and Caputo, do not coincide, i.e.,

$$D_{RL}^{\alpha} f(t) \neq \mathcal{D}_C^{\alpha} f(t).$$

Lemma 1.5.7

Let $(n-1) < \alpha < n$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $f(t)$ be such that $\mathcal{D}^{\alpha} f(t)$ exists. Then the following properties for the Caputo operator hold:

$$\lim_{\alpha \longrightarrow n} \mathcal{D}_C^{\alpha} f(t) = f^{(n)}(t),$$

$$\lim_{\alpha \longrightarrow n-1} \mathcal{D}_C^{\alpha} f(t) = f^{(n-1)}(t) - f^{(n-1)}(0).$$

Proof. We refer the reader to[39]. □

For the Riemann-Liouville fractional differential operator, the corresponding interpolation property reads

$$\lim_{\alpha\longrightarrow n} D^{\alpha}_{RL} f(t) = f^{(n)}(t),$$
$$\lim_{\alpha\longrightarrow n-1} D^{\alpha}_{RL} f(t) = f^{(n-1)}(t).$$

Lemma 1.5.8

- *Let $(n-1) < \alpha < n,\ n, m \in \mathbb{N},\ \alpha \in \mathbb{R}$ and the functions $f(t)$ and $g(t)$ be such that both $\mathcal{D}^{\alpha}_{C} f(t)$ and $\mathcal{D}^{\alpha}_{C} g(t)$ exist. Then the Caputo fractional derivative is a linear operator, i.e.,*

$$\mathcal{D}^{\alpha}_{C}(\lambda f(t) + g(t)) = \lambda \mathcal{D}^{\alpha}_{C} f(t) + \mathcal{D}^{\alpha}_{C} g(t),\ \ \alpha \in \mathbb{R}_{+}, \lambda \in \mathbb{C}.$$

- *The Riemann-Liouville fractional differential operator satisfies*

$$D^{\alpha}_{RL}(\lambda f(t) + g(t)) = \lambda D^{\alpha}_{RL} f(t) + D^{\alpha}_{RL} g(t),\ \ \alpha \in \mathbb{R}_{+}, \lambda \in \mathbb{C}.$$

- *Let $(n-1) < \alpha < n,\ n, m \in \mathbb{N},\ \alpha \in \mathbb{R}$ and the functions $f(t)$ is such that $\mathcal{D}^{\alpha}_{C} f(t)$ exists. Then in general*

$$\mathcal{D}^{\alpha}_{C} D^{m} f(t) = \mathcal{D}^{\alpha+m}_{C} f(t) \neq D^{m} \mathcal{D}^{\alpha}_{C} f(t).$$

- *Suppose that $(n-1) < \alpha < n,\ 0 < \beta = \alpha-(n-1) < 1,\ n \in \mathbb{N},\ \alpha, \beta \in \mathbb{R}$ and the function$f(t)$ is such that both $\mathcal{D}^{\alpha}_{C} f(t)$ exists. Then*

$$\mathcal{D}^{\alpha}_{C} f(t) = \mathcal{D}^{\beta}_{C} D^{n-1} f(t).$$

Proof. We refer the reader to[32]. □

Definition 1.5.12

The Laplace transform of Caputo's fractional derivative is defined by:

$$\begin{aligned}
\mathcal{L}\big[\mathcal{D}^{\alpha}_{C} f(t)\big] \quad &= \mathcal{L}\big[I^{n-\alpha}(\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t))\big] \\
&= s^{\alpha-n}\mathcal{L}\big[\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\big] \\
&= s^{\alpha}\mathcal{L}(f(t)) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \\
&= s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0).
\end{aligned}$$

1.5.5 Fractional Green's formula

Let Ω be a bounded open subset of $\mathbb{R}^n$, with a smooth boundary Γ of class $\mathcal{C}^2$. For all $T>0$, we denote by $Q=\Omega\times(0,T), \Sigma=\partial\Omega\times(0,T)$. Let $y,\phi\in\mathcal{C}^\infty([0,T]\times\bar{\Omega}),\ T>0$. We have the two following results:

Lemma 1.5.9

Let be $0<\alpha<1$. For any $y,\phi\in\mathcal{C}^\infty([0,T]\times\bar{\Omega})$, we have

$$\begin{aligned}&\int_0^T\int_\Omega\Big[D^\alpha_{RL}y(x,t)-\Delta y(x,t)\Big]\phi(x,t)\mathrm{d}x\mathrm{d}t\\ &=\int_\Omega\phi(x,T)I^{1-\alpha}y(x,T)\mathrm{d}x-\int_\Omega\phi(x,0)I^{1-\alpha}y(x,0)\mathrm{d}x+\int_0^T\int_{\partial\Omega}y(\sigma,t)\frac{\partial\phi}{\partial\nu}(\sigma,t)\mathrm{d}\sigma\mathrm{d}t\\ &\quad-\int_0^T\int_{\partial\Omega}\phi(\sigma,t)\frac{\partial y}{\partial\nu}(\sigma,t)\mathrm{d}\sigma\mathrm{d}t+\int_0^T\int_\Omega\Big[-\mathcal{D}^\alpha_C\phi(x,t)-\Delta\phi(x,t)\Big]y(x,t)\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where $\mathcal{D}^\alpha_C$ is the right fractional Caputo derivative of order $0<\alpha<1$.

Lemma 1.5.10

Let be $1<\alpha<2$. Then, for any $y,\phi\in\mathcal{C}^\infty([0,T]\times\bar{\Omega})$, we have

$$\begin{aligned}&\int_0^T\int_\Omega\Big[D^\alpha_{RL}y(x,t)-\Delta y(x,t)\Big]\phi(x,t)\mathrm{d}x\mathrm{d}t\\ &=\int_\Omega\phi(x,T)\frac{\partial}{\partial t}I^{2-\alpha}y(x,T)\mathrm{d}x-\int_\Omega\phi(x,0)\frac{\partial}{\partial t}I^{2-\alpha}y(x,0^+)\mathrm{d}x-\int_\Omega I^{2-\alpha}y(x,T)\frac{\partial\phi}{\partial t}(x,T)\mathrm{d}x\\ &\quad+\int_\Omega I^{2-\alpha}y(x,0)\frac{\partial\phi}{\partial t}(x,0)\mathrm{d}x+\int_0^T\int_{\partial\Omega}y(\sigma,t)\frac{\partial\phi}{\partial\nu}(\sigma,t)\mathrm{d}\sigma\mathrm{d}t-\int_0^T\int_{\partial\Omega}\phi(\sigma,t)\frac{\partial y}{\partial\nu}(\sigma,t)\mathrm{d}\sigma\mathrm{d}t\\ &\quad+\int_0^T\int_\Omega\Big[\mathcal{D}^\alpha_C\phi(x,t)-\Delta\phi(x,t)\Big]y(x,t)\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where $\mathcal{D}^\alpha_C$ is the right fractional Caputo derivative of order $1<\alpha<2$.

1.5.6 Existence and Uniqueness of solutions to Fractional Partial Differential Equations

Spectral methods are a family of techniques used in applied mathematics and scientific computing to numerically solve certain differential equations, using the fast Fourier transform as a possible component. The concept is to describe the differential equation solution as a sum of certain "basic functions" (for example, a Fourier series, which is a sum of sinusoids) and then choose the coefficients of the sum to satisfy the differential equation as well as possible.
Let Ω be a bounded open subset of $\mathbb{R}^n$, with a smooth boundary Γ of class $\mathcal{C}^2$. For all $T>0$, we denote by $Q=\Omega\times(0,T), \Sigma=\partial\Omega\times(0,T)$. Let $z,\phi\in\mathcal{C}^\infty([0,T]\times\bar{\Omega}),\ T>0$.

Fractional diffusion equation with the Riemann-Liouville fractional derivative

The fractional diffusion equation is as follows:

$$\begin{cases}D^\beta_{RL}z(x,t)-\Delta z(x,t)=f(x,t) & \text{in}\quad Q,\\ I^{1-\beta}z(x,0^+)=z^0 & \text{in}\quad \Omega,\\ z(\sigma,t)=0 & \text{on}\quad \Sigma,\end{cases}\tag{1.27}$$

where $0<\beta<1,\ f\in L^2(Q),\ \ z^0\in H^1_0(\Omega)$ and $I^{1-\beta}z(x,0^+)=\lim_{t\to0}I^{1-\beta}z(x,t)$.

We suppose that the solution of problem (1.27) yields $z \in \mathcal{C}^\infty(\bar{Q})$. We multiply the first equation of (1.27) by a function $v \in H_0^1(\Omega)$ and we integrate by parts over Ω. We get the following equation using Green's formula:

$$\int_\Omega D_{RL}^\beta z(x,t)v(x)\mathrm{d}x - \int_\Omega \nabla z(x,t)\nabla v(x)\mathrm{d}x = \int_\Omega f(x,t)v(x)\mathrm{d}x. \tag{1.28}$$

Now let's put,

$$\forall \varphi, \psi \in L^2(\Omega) \quad (\varphi,\psi)_{L^2(\Omega)} = \int_\Omega \varphi(x)\psi(x)\mathrm{d}x. \tag{1.29}$$

We define the scalar product on $L^2(\Omega)$ and we denote the associated norm $\|.\|_{L^2(\Omega)}$. We set also,

$$\forall \varphi, \psi \in H_0^1(\Omega) \;\; a(\varphi,\psi) = \int_\Omega \nabla\varphi(x)\nabla\psi(x)\mathrm{d}x, \quad \forall \varphi, \psi \in H_0^1(\Omega). \tag{1.30}$$

Then the form $a(.,.)$ defined as such, is the scalar product on $H_0^1(\Omega)$ and we will denote the associated norm

$$\|\varphi\|^2_{H_0^1(\Omega)} = a(\varphi,\varphi).$$

On the other hand, the negative of the Laplacian is given by

$$-\Delta\varphi = -\sum_{i=1}^d \partial_i^2\varphi,$$

is a uniformly elliptic operator. Then, there is a sequence of real eigenvalues $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ with $\lambda_k \to \infty$ when $k \to \infty$. Furthermore, there is a Hilbertian basis $\{\omega_k\}_{k=1}^\infty$ of $L^2(\Omega)$ that is orthonormal, where $\omega_k \in H_0^1(\Omega)$ is the eigenvector associated with λ_k such that

$$-\Delta\omega_k = \lambda_k\omega_k.$$

Then, we have

$$a(\omega_k, p) = \lambda_k(\omega_k,p)_{L^2(\Omega)}, \quad \forall p \in H_0^1(\Omega). \tag{1.31}$$

We also have $\{\frac{\omega_k}{\sqrt{\lambda_k}}\}_{k=1}^\infty$ which is an orthonormal Hilbert basis of $H_0^1(\Omega)$ for the scalar product $a(.,.)$, where we've come from

$$\|\phi\|^2_{H_0^1(\Omega)} = \sum_{i=1}^\infty \lambda_i(\phi,\omega_i)_{L^2(\Omega)}, \quad \forall \phi \in H_0^1(\Omega).$$

As a result of problem (1.27),

$$\begin{cases} D_{RL}^\beta(z(t),v)_{L^2(\Omega)} + a(z(t),v) = (f(t),v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega), \\ I^{1-\beta}z(0) = z^0, \\ z(t) = 0. \end{cases} \tag{1.32}$$

Take the following problem: For a given $0 < \beta < 1, z^0 \in H_0^1(\Omega), f \in L^2(Q)$, we look for

$$z \in L^2((0,T);H_0^1(\Omega)), \tag{1.33}$$

$$I^{1-\beta}z \in \mathcal{C}((0,T);H_0^1(\Omega)), \tag{1.34}$$

such as

$$D_{RL}^\beta(z(t),v)_{L^2(\Omega)} + a(z(t),v) = (f(t),v)_{L^2(\Omega)} \quad \forall t \in (0,T), \tag{1.35}$$

$$I^{1-\beta}z(0) = z^0. \tag{1.36}$$

Theorem 1.5.4

For $\frac{1}{2} < \beta < 1$. Also, consider the bilinear form defined by $a(.,.)$ in (1.30). Then the problem (1.33)-(1.36) admits a unique weak solution $z \in L^2((0,T); H_0^1(\Omega))$, such that $I^{1-\beta}z \in \mathcal{C}((0,T); H_0^1(\Omega))$. This weak solution is given by

$$z(t) = \sum_{i=1}^{\infty}\Big[t^{\beta-1}E_{\beta,\beta}(-\lambda_i t^{\beta})z_i^0 + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}\big(-\lambda_i(t-s)^{\beta}\big)f_i(s)\mathrm{d}s\Big]\omega_i, \tag{1.37}$$

where λ_i is the eigenvalue associated with the operator $(-\Delta)$ corresponding to the eigenvector ω_i. The values

$$z_i^0 = (z^0, \omega_i)_{L^2(\Omega)}, \quad f_i(t) = (f(t), \omega_i)_{L^2(Q)},$$

are respectively the i-th component of z^0 and f in the orthonormal basis $\{\omega_i\}_{i=1}^{\infty}$. Moreover, there is a constant $C > 0$ such that

$$\|z\|_{L^2(0,T;H_0^1(\Omega))} \le \Lambda\Big(\|z^0\|_{H_0^1(\Omega)} + \|f\|_{L^2(Q)}\Big); \tag{1.38}$$

$$\|I^{1-\beta}z\|_{\mathcal{C}(0,T;H_0^1(\Omega))} \le \hat{\Lambda}\Big(\|z^0\|_{H_0^1(\Omega)} + \|f\|_{L^2(Q)}\Big), \tag{1.39}$$

where

$$\Lambda = \max\Big(C\sqrt{\frac{2T^{2\beta-1}}{2\beta-1}}, \frac{C}{\beta}\sqrt{\frac{2T}{\lambda_i}}\Big), \quad \hat{\Lambda} = \sup\Big(C\sqrt{2}, C\sqrt{\frac{2T^{1-\beta}}{1-\beta}}\Big).$$

Proof. We refer the reader to[7, 8].

Fractional wave equation with the Riemann-Liouville derivative

Consider the following equation for a fractional wave:

$$\begin{cases} D_{RL}^{\alpha}y(x,t) - \Delta y(x,t) = f(x,t) & \text{in} \quad Q, \\ I^{2-\alpha}y(x,0^+) = y^0, \quad \frac{\partial}{\partial t}I^{2-\alpha}y(x,0^+) = y^1 & \text{in} \quad \Omega, \\ y(\sigma,t) = 0 & \text{on} \quad \Sigma, \end{cases} \tag{1.40}$$

where $1 < \alpha < 2$, $f \in L^2(Q)$, $y^0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $y^1 \in L^2(Q)$, $I^{2-\alpha}y(x,0^+) = \lim_{t\to 0} I^{2-\alpha}y(x,t)$ and $\frac{\partial}{\partial t}I^{2-\alpha}y(x,0^+) = \lim_{t\to 0}\frac{\partial}{\partial t}I^{2-\alpha}y(x,t)$.
For all $t \in (0,T)$, The problem (1.40) is equivalent to the following:

$$\begin{cases} D_{RL}^{\alpha}(y(t),v)_{L^2(\Omega)} + a(y(t),v) = (f(t),v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega), \\ I^{1-\alpha}y(0) = y^0, \quad \frac{\partial}{\partial t}I^{2-\alpha}y(0) = y^1, \\ y(t) = 0. \end{cases} \tag{1.41}$$

Take the following problem: For a given $1 < \alpha < 2$, $f \in L^2(Q)$, $y^0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $y^1 \in L^2(Q)$, we are looking for

$$y \in L^2((0,T); H_0^1(\Omega)), \tag{1.42}$$

$$I^{2-\alpha}y \in \mathcal{C}((0,T); H_0^1(\Omega)), \tag{1.43}$$

$$\frac{\partial}{\partial t}(I^{2-\alpha}y) \in \mathcal{C}((0,T); L^2(\Omega)). \tag{1.44}$$

such as

$$\forall v \in H_0^1(\Omega), \quad D_{RL}^{\alpha}(y(t),v)_{L^2(\Omega)} + a(y(t),v) = (f(t),v)_{L^2(\Omega)} \quad \forall t \in (0,T), \tag{1.45}$$

$$I^{1-\alpha}y(0) = y^0, \quad \frac{\partial}{\partial t}I^{2-\alpha}y(0) = y^1. \tag{1.46}$$

As a result, we've arrived at this Theorem :

Theorem 1.5.5

For $\frac{3}{2} < \alpha < 2$. Also, consider the bilinear form defined by $a(.,.)$ in (1.30). Then the problem (1.42)-(1.46) admits a unique weak solution $y \in L^2((0,T);H_0^1(\Omega))$, such that $I^{2-\alpha}y \in \mathcal{C}((0,T);H_0^1(\Omega))$, $\frac{\partial}{\partial t}(I^{2-\alpha}y) \in \mathcal{C}((0,T);L^2(\Omega))$. This weak solution is given by

$$y(t) = \sum_{i=1}^{\infty}[t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_i t^{\alpha})y_i^0 + t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^{\alpha})y_i^1 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(t-s)^{\alpha})f_i(s)\mathrm{d}s]\omega_i, \tag{1.47}$$

where λ_i is the eigenvalue corresponding to the eigenvector ω_i. The values

$$y_i^0 = (y^0,\omega_i)_{L^2(\Omega)}, \quad y_i^1 = (y^1,\omega_i)_{L^2(\Omega)} \quad f_i(t) = (f(t),\omega_i)_{L^2(Q)},$$

are respectively the i-th component of y^0, y^1 and f in the orthonormal basis $\{\omega_i\}_{i=1}^{\infty}$. Moreover, there is a constant $C > 0$ such that

$$\|y\|_{L^2(0,T;H_0^1(\Omega))} \le \Theta_1\Big(\|y^0\|_{H_0^1(\Omega)} + \|y^1\|_{L^2(\Omega)} + \|f\|_{L^2(Q)}\Big); \tag{1.48}$$

$$\|I^{2-\alpha}y\|_{\mathcal{C}(0,T;H_0^1(\Omega))} \le \Theta_2\Big(\|y^0\|_{H_0^1(\Omega)} + \|y^1\|_{L^2(\Omega)} + \|f\|_{L^2(Q)}\Big), \tag{1.49}$$

$$\|\frac{\partial}{\partial t}I^{2-\alpha}y\|_{\mathcal{C}(0,T;L^2(\Omega))} \le \Theta_3\Big(\|y^0\|_{H_0^1(\Omega)} + \|y^1\|_{L^2(\Omega)} + \|f\|_{L^2(Q)}\Big), \tag{1.50}$$

where

$$\Theta_1 = \max\Big(C\sqrt{\frac{2T^{2\alpha-3}}{2\alpha-3}}, C\sqrt{\frac{2T^{\alpha-1}}{\alpha-1}}, C\sqrt{\frac{2T^{\alpha}}{\alpha(\alpha-1)}}\Big),$$

$$\Theta_2 = \max\Big(C\sqrt{2},\ C\sqrt{2T^{2-\alpha}}, C\sqrt{\frac{2T^{3-\alpha}}{3-\alpha)}}\Big),$$

$$\Theta_3 = \max\Big(C\sqrt{2}T^{\alpha-1},\ C\sqrt{2}\Big).$$

Proof. We refer the reader to [7, 8].

We'll concentrate on the fractional diffusion equation and the fractional onde equation in this part, with the derivatives taken into account in the Caputo sense. As a result, we will see in Chapter 3 that the adjoining equations are fractional rétrograde equations with the right fractional Caputo derivatives, and it is for this reason that we provide the following results.

Fractional diffusion equation with the right Caputo fractional derivative

In the following, we present the fractional diffusion equation with the right Caputo fractional derivative

$$\begin{cases} \mathcal{D}_{C.L}^{\alpha}y - \Delta y = f & \text{in} \quad Q, \\ y(0) = 0 & \text{in} \quad \Omega, \\ y = 0 & \text{on} \quad \Sigma, \end{cases} \tag{1.51}$$

where $0 < \alpha < 1$, and $\mathcal{D}_C^{\alpha}$ is the left Caputo fractional derivative. So, here's what we've come up with:

Theorem 1.5.6

Let $f \in L^2(Q)$. Then, the problem (1.51) admits a unique solution $y \in L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))$ such that $\dfrac{\partial y}{\partial t} \in L^2(Q)$. Furthermore, there is a constant $C > 0$ such that

$$\|y\|_{L^2(0,T;H^2(\Omega))} + \|\frac{\partial y}{\partial t}\|_{L^2(Q)} \le C\|f\|_{L^2(Q)}. \tag{1.52}$$

We also have a corollaire that follows.

Corollaire 1.5.1

Let $0 < \alpha < 1$, $f \in L^2(Q)$. $\mathcal{D}^{\alpha}_{C.R}$ is the right Caputo fractional derivative. So, the problem:

$$\begin{cases} -\mathcal{D}^{\beta}_{C.R}\psi - \Delta\psi = f & in \quad Q, \\ \varphi(T) = 0 & in \quad \Omega, \\ \varphi = 0 & on \quad \Sigma, \end{cases} \tag{1.53}$$

has a unique solution $\psi \in L^2(0,T; H^2(\Omega) \cap H^1_0(\Omega))$ such that $\dfrac{\partial \psi}{\partial t} \in L^2(Q)$. Furthermore, there is a constant $C > 0$ such that

$$\|\psi\|_{L^2(0,T;H^2(\Omega))} + \|\frac{\partial \psi}{\partial t}\|_{L^2(Q)} \leq C\|f\|_{L^2(Q)}. \tag{1.54}$$

Proof. We refer the reader to [7, 8]. □

Fractional wave equation with the left Caputo fractional derivative.

The following, we present the fractional wave equation with the Caputo derivative

$$\begin{cases} \mathcal{D}^{\alpha}_{C.L}p(x,t) - \Delta p(x,t) = 0 & \text{in} \quad Q, \\ p(x,0) = 0 \quad \dfrac{\partial p}{\partial t}(x,0) = p^1 & \text{in} \quad \Omega, \\ q(\sigma,t) = 0 & \text{on} \quad \Sigma, \end{cases} \tag{1.55}$$

where $1 < \alpha < 2$, $p^1 \in L^2(\Omega)$, and $\mathcal{D}^{\alpha}_{C.L}$ is the left Caputo fractional derivative. So, here's what we've come up with:

Theorem 1.5.7

Let $p^1 \in L^2(\Omega)$. Then, the problem (1.55) admits a unique solution $p \in \mathcal{C}(0,T; L^2(\Omega))$ such that $\dfrac{\partial p}{\partial t} \in \mathcal{C}(0,T; L^2(\Omega))$. Furthermore, there is a $C > 0$ such that

$$\|p\|_{\mathcal{C}(0,T;L^2(\Omega))} + \|\frac{\partial p}{\partial t}\|_{\mathcal{C}(0,T;L^2(\Omega))} \leq C\|p^1\|_{L^2(\Omega)}. \tag{1.56}$$

The following corollary follows from the previous result:

Corollaire 1.5.2

Let $1 < \alpha < 2$, $p^1 \in L^2(\Omega)$. Consider the following equation for a fractional wave where $\mathcal{D}^{\alpha}_{C.R}$ is the right Caputo fractional derivative

$$\begin{cases} \mathcal{D}^{\alpha}_{C.R}\psi - \Delta\psi = 0 & in \quad Q, \\ \psi(T) = 0 \quad \dfrac{\partial \psi}{\partial t}(T) = p^1 & in \quad \Omega, \\ \psi = 0 & on \quad \Sigma, \end{cases} \tag{1.57}$$

has a unique solution $\psi \in \mathcal{C}(0,T; L^2(\Omega))$ such that $\dfrac{\partial \varphi}{\partial t} \in \mathcal{C}(0,T; L^2(\Omega))$. Furthermore, there is a $C > 0$ such that

$$\|\psi\|_{\mathcal{C}(0,T;L^2(\Omega))} + \|\frac{\partial \psi}{\partial t}\|_{\mathcal{C}(0,T;L^2(\Omega))} \leq C\|p^1\|_{L^2(\Omega)}. \tag{1.58}$$

Proof. We refer the reader to [7, 8]. □

Now have a look at the fractional wave equation.

$$\begin{cases} \mathcal{D}^{\alpha}_{C.L} y - \Delta y = f & \text{in} \quad Q, \\ y(x,0) = 0 \ \dfrac{\partial y}{\partial t}(x,0) = q^1 & \text{in} \quad \Omega, \\ q = 0 & \text{on} \quad \Sigma, \end{cases} \tag{1.59}$$

where $\frac{3}{2} < \alpha < 2$, $f \in L^2(Q)$, and $\mathcal{D}^{\alpha}_{L.C}$ is the left Caputo fractional derivative. So, here's what we've come up with:

Theorem 1.5.8

Let $f \in L^2(Q)$. Then, the problem (1.59) admits a unique week solution $y \in \mathcal{C}(0,T;H^1_0(\Omega))$ such that $\dfrac{\partial y}{\partial t} \in \mathcal{C}(0,T;L^2(\Omega))$. Furthermore, This solution comes from

$$y(t) = \sum_{i=1}^{+\infty} \Big(\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_i (t-s)^{\alpha}) f_i(s) \mathrm{d}s \Big) \omega_i.$$

Moreover, there is a constant $C > 0$ such that,

$$\|y\|_{\mathcal{C}(0,T;H^1_0(\Omega))} \leq C \sqrt{\frac{T^{\alpha-1}}{\alpha-1}} \|f\|_{L^2(Q)},$$
$$\|\frac{\partial y}{\partial t}\|_{\mathcal{C}(0,T;L^2(\Omega))} \leq C \sqrt{\frac{T^{2\alpha-3}}{2\alpha-3}} \|f\|_{L^2(Q)}.$$

Proof. We refer the reader to [7, 8]. □

As a result, we have the following:

Corollaire 1.5.3

Let $\frac{3}{2} < \alpha < 2$, $f \in L^2(Q)$. Consider the following equation for a fractional wave where $\mathcal{D}^{\alpha}_{R.C}$ is the right Caputo fractional derivative

$$\begin{cases} \mathcal{D}^{\alpha}_{R.C} \psi - \Delta \psi = f & \text{in} \quad Q, \\ \psi(x,T) = 0 \ \dfrac{\partial \psi}{\partial t}(x,T) = 0 & \text{in} \quad \Omega, \\ \psi = 0 & \text{on} \quad \Sigma, \end{cases}$$

has a unique solution $\psi \in \mathcal{C}(0,T;H^1_0(\Omega))$ such that $\dfrac{\partial \varphi}{\partial t} \in \mathcal{C}(0,T;L^2(\Omega)$. Furthermore, there is a $C > 0$ such that

$$\|\psi\|_{\mathcal{C}(0,T;H^1_0(\Omega))} \leq C \sqrt{\frac{T^{\alpha-1}}{\alpha-1}} \|f\|_{L^2(Q)},$$
$$\|\frac{\partial \psi}{\partial t}\|_{\mathcal{C}(0,T;L^2(\Omega))} \leq C \sqrt{\frac{T^{2\alpha-3}}{2\alpha-3}} \|f\|_{L^2(Q)}.$$

1.6 Sentinel Method

The initial conditions are not fully known in practically all pollution, meteorological, and oceanographic situations. Boundary circumstances, such as on a section of the boundary, may be inaccessible to measurements. All of the aforementioned issues are referred to as "identification problems." The sentinel approach" is one of the methods used to analyze this type of problem.

1.6.1 Perturbation terms, Pollution (or Noisy) terms

We suppose that the state y of the system (1) (see Introduction) is represented by a scalar and $\mathcal{A}$ is a differential elliptic operator of order 2 and its coefficients are known. We also assume that equation (1) is written in this format.

$$\frac{\partial y}{\partial t} + \mathcal{A}(y) = \xi + \lambda\hat{\xi}, \tag{1.60}$$

where ξ is a well-known term in the appropriate space $\mathcal{Y}$. The unknown term $\lambda\hat{\xi}$ (sometimes known as "pollution" or "noise") is satisfying

$$\|\hat{\xi}\|_{\mathcal{Y}} \leq 1, \quad \lambda \text{ is a small enough.} \tag{1.61}$$

The initial data is defined as follows:

$$y(0) = y^0 + \tau\hat{y}^0, \tag{1.62}$$

where y^0 is a well-known term in a suitable space $\mathcal{K}$. The missing term $\tau\hat{y}^0$ (also known as the perturbation term) is satisfied:

$$\|\hat{y}^0\|_{\mathcal{K}} \leq 1, \quad \tau \text{ is a small enough.} \tag{1.63}$$

For example, the boundary condition is provided by

$$y = 0 \qquad \forall x \in \partial\Omega, \quad \forall t \in [0, T]. \tag{1.64}$$

The following hypotheses iare proposed:

(H_1): There is only one solution to problem (1.60)-(1.64), which is differentiable according to λ and τ.

(H_2): The solution of (1.60)-(1.64) is available on Ω at $t = T$, i.e., $y(x, T; \lambda, \tau)$ is known.

The question here is how to determine the pollution term $\lambda\hat{\xi}$ in the state equation without taking into account the variance $\tau\hat{y}^0$ in the initial data.

1.6.2 The sentinel method

The above question is natural and led to some developments; some answers are given by the least squares method. The method entails considering the unknowns $\{\lambda\hat{\xi}, \tau\hat{y}^0\} = \{v, w\}$ as control variables, and then driving the state $y(x, t; v, w)$ as close to m_0 as possible. This leads to a problem of optimal control. As a result, we search for the pair (v, w); yet, there is no real chance of finding v or w independently.

Lions J.-L. sentinel's approach is a special least squares method suited for parameter identification in ecosystems with insufficient information; various models may be found in the literature. The sentinel notion is based on three objects that must be determined : a state equation, an observation function, and a control function.

Observation of the system

For a time $(0, T)$, we monitor the state on an observatory $\mathcal{O}$. The observatory $\mathcal{O}$ can be distributed in one of two ways.

$$\mathcal{O} \subset \Omega,$$

or a border observation station

$$\mathcal{O} \subset \Gamma = \partial\Omega.$$

A time-dependent observatory can also be considered.

$$\mathcal{O} = \mathcal{O}(t), \quad t \in [0, T].$$

Assuming that the state $y(x,t;\lambda,\tau)$ is observed in $L^2(0,T;\mathcal{O})$, where $\mathcal{O}$ is a non-empty open subset of Ω. This observation was noted by

$$y\chi_{\mathcal{O}} = m_0(x,t) \quad \text{where } m_0 \text{ is given.}$$

We can assume that $y\chi_{\mathcal{O}_T}$ is either not directly measurable or contains a small amount of noise. As a result, we can write it in the following format:

$$y\chi_{\mathcal{O}_T} = m_0 + \sum_{i=1}^{N} \beta_i m_i, \quad \beta_i \neq 0, \quad 1 \leq i \leq N,$$

where $m_i \in L^2(0,T;\mathcal{O})$ are known and linearly independent interference functions. The noise terms β_i are small enough and unknown.

Creating a distributed sentinel function

The observation and the control share the same support. Sentinels can be constructed using the specified positive function h_0 in $L^2(0,T;\mathcal{O})$ in the following way:

$$h_0 \geq 0, \quad \iint_{\mathcal{O}\times(0,T)} h_0(x,t)\mathrm{d}x\mathrm{d}t = 1,$$

and a defined control function $\omega \in L^2(0,T;\mathcal{O})$.

Definition 1.6.1 (*See J. L Lions in [52]*)

Consider the real function $\mathcal{S}$, which is defined by (assumed different to zero)

$$\mathcal{S}(\lambda,\tau) = \iint_{\mathcal{O}\times(0,T)} (h_0+\omega)y(x,t;\tau,\lambda)\mathrm{d}x\mathrm{d}t. \tag{1.65}$$

If there exists a control function $\hat{\omega} \in L^2(0,T;\mathcal{O})$ that satisfies these two constraints, the function $\mathcal{S}$ is termed a distributed sentinel described by h_0.

1. *Insensitivity condition for $S(\lambda,\tau)$ with respect to the missing initial data $\tau_0\hat{y}^0$:*

$$\frac{\partial S}{\partial \tau}(0,0) = 0, \quad \forall y^0. \tag{1.66}$$

2. *The control function $\hat{\omega}$ should be kept to a minimum.*

$$\|\hat{\omega}\|_{L^2(0,T;\mathcal{O})} = \min \|\omega\|_{L^2(0,T;\mathcal{O})}. \tag{1.67}$$

There is always a sentinel function in this instance (at least in the case where $\omega = h_0$). The only thing left is to discover the ω solution to (1.67). Finally, we must ensure that $\omega \neq h_0$ is valid. Then the problem is only to find ω solution of (1.67). At last, we have to be sure that $\omega \neq h_0$.

The observation and the control have different supports.

In this case, The sentinel function can be formed from the given positive function $h_0 \in L^2(0,T;\mathcal{O})$, and a determined control function $\omega \in L^2(0,T;\mathcal{W})$, where $\mathcal{W}$ is a non empty open subset of $\mathcal{O} \subset \Omega$.

Definition 1.6.2 (*See [23, 25]*)

Let $\mathcal{S}$ be the real function whose definition is as follows (assumed different to zero):

$$\mathcal{S}(\lambda,\tau) = \int_0^T \int_{\mathcal{O}} h_0 y(x,t;\tau,\lambda)\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathcal{W}} \omega y(x,t;\tau,\lambda)\mathrm{d}x\mathrm{d}t. \tag{1.68}$$

If there is a control function $\omega \in L^2(0,T;\mathcal{W})$ that satisfies the insensitivity condition (1.66) and the inf *condition (1.69), the function $\mathcal{S}$ is termed a distributed sentinel defined by h_0.*

- *The control function $\hat{\omega}$ must be at a minimum,*

$$\|\hat{\omega}\|_{L^2(0,T;\mathcal{W})} = \min_{\omega\in\mathcal{U}} \|\omega\|_{L^2(0,T;\mathcal{W})}, \tag{1.69}$$

where

$$\mathcal{U} = \{\omega \in L^2(0,T;\mathcal{W});\ \omega \ \text{ verifies (1.66) and (1.68)}\}.$$

This point of view $\mathcal{W} \neq \mathcal{O}$ was initially discussed in [23, 25]. We suppose that $\mathcal{W} \subset \Omega \subset \mathcal{O}$. The presence of the sentinel function is not assured in this instance.

The information provided by the sentinel function

If the function $S(\lambda,\tau)$ is differentiable at $(0,0)$, it has a linear approximation close to this point. As a result of (1.66), the function S can be expressed

$$S(\lambda,\tau) \simeq S(0,0) + \lambda\frac{\partial S}{\partial\lambda}(0,0), \tag{1.70}$$

The impacts of condition (1.67) (or (1.69)) are taken into account. A calculable solution to (1.71) is denoted by $y_0 = y(x,t;0,0)$.

$$\begin{cases} \dfrac{\partial y_0}{\partial t} + \mathcal{A}(y_0) = \xi, \\ y_0(0) = y^0, \\ y_0 = 0. \end{cases} \tag{1.71}$$

We can conclude the following from (1.65):

$$S(0,0) = \iint_Q (h_0+\omega)y_0 \mathrm{d}x\mathrm{d}t \ \text{ is given.} \tag{1.72}$$

The following data is derived from the relationships (1.70)-(1.72),

$$\lambda\frac{\partial S}{\partial\lambda}(0,0) \simeq \iint_{\mathcal{O}\times(0,T)} (h_0+\omega)(m_0-y_0)\mathrm{d}x\mathrm{d}t. \tag{1.73}$$

If the function S can be differentiated at λ, the partial derivative of S at λ is

$$\frac{\partial S}{\partial\lambda}(0,0) = \iint_{\mathcal{O}\times(0,T)} (h_0+\omega)y_\lambda \mathrm{d}x\mathrm{d}t, \tag{1.74}$$

where y_λ represents the partial derivatives of λ.

$$\begin{cases} \dfrac{\partial y_\lambda}{\partial t} + \mathcal{A}(y_\lambda) = \hat{\xi}, \\ y_\lambda(0) = 0, \\ y_\lambda = 0. \end{cases} \tag{1.75}$$

Sentinel's data is provided by

$$\lambda\frac{\partial S}{\partial\lambda}(0,0) = \iint_{\mathcal{O}\times(0,T)} (h_0+\omega)(m_0-y_0)\mathrm{d}x\mathrm{d}t = \iint_{\mathcal{O}\times(0,T)} (h_0+\omega)\lambda y_\lambda \mathrm{d}x\mathrm{d}t. \tag{1.76}$$

Remark 1.6.1

In the case of $\mathcal{O} \subset \Omega$, we have

$$\mathcal{S}(0,0) = \int_0^T\int_{\mathcal{O}} h_0 y_0 \mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathcal{W}} \omega y_0 \mathrm{d}x\mathrm{d}t. \tag{1.77}$$

After that, there's

$$\lambda\frac{\partial S}{\partial\lambda}(0,0)\simeq\int_0^T\int_{\mathcal{O}}h_0(m_0-y_0)\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathcal{W}}\omega(m_0-y_0)\mathrm{d}x\mathrm{d}t. \tag{1.78}$$

Sentinel's data is provided by

$$\begin{aligned}\lambda\frac{\partial S}{\partial\lambda}(0,0)&=\int_0^T\int_{\mathcal{O}}h_0(m_0-y_0)\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathcal{W}}\omega(m_0-y_0)\mathrm{d}x\mathrm{d}t\\&=\int_0^T\int_{\mathcal{O}}\lambda h_0y_\lambda\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathcal{W}}\omega\lambda y_\lambda\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{1.79}$$

We'll hunt for a function control ω in the following part. This means that in a null controllability problem with a constraint on the control function restriction, the sentinel problem will be decreased.

In the next part , we are looking for a function control ω. This fact meaning that the problem of sentinel will be reduced in a null controllability problem with a constraint on the control function.

1.6.3 A null-controllability problem

Transforming the sensitivity condition to account for the missing initial data

We'll suppose that the partial derivative $\dfrac{\partial y}{\partial\tau}=y^\tau$ is computed for $\lambda=0,\tau=0$

$$\begin{cases}\dfrac{\partial y_\tau}{\partial t}+\mathcal{A}(y_\tau)=0,\\ y_\tau(0)=\hat{y}^0,\\ y_\tau=0.\end{cases} \tag{1.80}$$

It is obvious from (1.66) that

$$\frac{\partial S}{\partial\tau}(0,0)=\int_0^T\int_{\mathcal{O}}(h_0+\omega)y^\tau\mathrm{d}x\mathrm{d}t=0,\quad\forall y^0, \tag{1.81}$$

which must be considered (1.67) (or (1.69)).

Remark 1.6.2

In the case of $\mathcal{O}\subset\Omega$, we have

$$\frac{\partial S}{\partial\tau}(0,0)=\int_0^T\int_{\mathcal{O}}h_0y_\tau\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathcal{W}}\omega y_\tau\mathrm{d}x\mathrm{d}t=0,\quad\forall y^0. \tag{1.82}$$

The adjoint state

By introducing the classical adjoint state, we may now modify the relation (1.81). The adjoint state p problem is therefore defined as follows:

$$\begin{cases}-\dfrac{\partial p}{\partial t}+\mathcal{A}^*(p)=(h_0+\omega)\chi_{\mathcal{O}},\\ p(T)=0,\\ p=0,\end{cases} \tag{1.83}$$

where $\chi_{\mathcal{O}}$, is the characteristic function of $\mathcal{O}$.

Proposition 1.6.1

Let $p=p(\omega)$ be the unique solution to (1.83), where ω is the control function to be found. The *following relation is similar to the condition of insensitivity for $S(\lambda, \tau)$ with respect to missing initial data (1.81) (or (1.82) in the case of $\mathcal{O} \subset \Omega$).*

$$p(0) = 0 \quad \text{in} \quad \Omega. \tag{1.84}$$

Proof. We refer the reader to [52]. □

The functions p_0 and z are introduced as solutions to this problem:

$$\begin{cases} -\dfrac{\partial p_0}{\partial t} + \mathcal{A}^*(p_0) = h_0 \chi_{\mathcal{O}}, \\ p_0(T) = 0, \\ p_0 = 0, \end{cases} \tag{1.85}$$

and

$$\begin{cases} -\dfrac{\partial z}{\partial t} + \mathcal{A}^*(z) = \omega \chi_{\mathcal{O}}, \\ z(T) = 0, \\ z = 0. \end{cases} \tag{1.86}$$

Finally, we must solve a new control problem in which the new state $z(\omega)$ must satisfy the following conditions:

$$\begin{cases} z(0;\omega) = -p_0(0), \\ \|\omega\|_{L^2(0,T;\mathcal{O})} = \text{is of minimal norm.} \end{cases} \tag{1.87}$$

Optimization problem with a penalty

Further, set $\epsilon > 0$, the penalized cost functional $\mathcal{J}_\epsilon$ is defined as follows:

$$\mathcal{J}_\epsilon(\omega, z) = \frac{1}{2}\|\omega\|^2_{L^2(0,T;\mathcal{O})} + \frac{1}{2\epsilon}\| - \frac{\partial z}{\partial t} + \mathcal{A}^*(z) - \omega\chi_{\mathcal{O}}\|^2_{L^2(0,T;\Omega)},$$

where the state z satisfies the following requirements:

$$\begin{cases} -\dfrac{\partial z}{\partial t} + \mathcal{A}^*(z) \in L^2(0,T;\Omega), \\ z(T) = 0, \;\; z(0) = -p_0(0), \\ z = 0. \end{cases} \tag{1.88}$$

Let's have a look at the next optimization problem.

$$\inf \mathcal{J}_\epsilon(\omega, z). \tag{1.89}$$

The existence and characterization of the solution to the penalized issue are established by the following Theorem.

Theorem 1.6.1

The penalized optimization problems (1.89) admit a single optimal pair control-state $\{\omega^\epsilon, z^\epsilon\}$, which is defined as follows:

$$\begin{cases} -\dfrac{\partial z_\epsilon}{\partial t} + \mathcal{A}^*(z_\epsilon) = \rho_\epsilon \chi_{\mathcal{O}}, \quad \dfrac{\partial \rho_\epsilon}{\partial t} + \mathcal{A}(\rho_\epsilon) = 0, \\ z_\epsilon(T) = 0, \quad \rho_\epsilon(0) = \rho^0, \\ z_\epsilon = 0, \quad \rho_\epsilon = 0, \end{cases} \tag{1.90}$$

and

$$\omega_\epsilon = \rho_\epsilon \chi_{\mathcal{O}}, \tag{1.91}$$

without any knowledge of ρ^0.

Proof. We refer the reader to [52]. □

Passage to the limit and Calculation of ρ^0

We suppose that $\rho_\epsilon \to \rho$ have an appropriate topology. We get to the situation where we get to the limit $\epsilon \to 0$.

$$\begin{cases} -\dfrac{\partial z}{\partial t} + \mathcal{A}^*(z) = \rho\chi_{\mathcal{O}}, \quad \dfrac{\partial \rho}{\partial t} + \mathcal{A}(\rho) = 0, \\ z(T) = 0, \ \rho(0) = \rho^0, \\ z = 0, \ \rho = 0, \end{cases} \tag{1.92}$$

$$\omega = \rho\chi_{\mathcal{O}}, \tag{1.93}$$

where ρ^0 is currently unknown. It is required to have the condition (1.87) and check that the function $S \neq 0$. Assuming ρ^0 is a suitably regular quantity. The following linear operator is defined.

$$\Lambda\rho^0 = z(0).$$

As a result of (1.87) we have

$$\Lambda\rho^0 = -p_0(0). \tag{1.94}$$

We obtain by multiplying the second equality in (1.92) by ρ, applying Green's formula, and finally combining the last two equalities.

$$\langle \Lambda\rho^0, \rho\rangle = \int_0^T \int_{\mathcal{O}} \rho^2 \mathrm{d}x\mathrm{d}t.$$

After that, we introduce the norm:

$$\|\rho^0\|_{\mathcal{F}} = \Big(\int_0^T \int_{\mathcal{O}} \rho^2 \mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{2}}. \tag{1.95}$$

We also define the dual space of $\mathcal{F}'$ as $\mathcal{F}$. As a result, the operator Λ is an isomorphism between $\mathcal{F}$ and $\mathcal{F}'$. As a consequence,

$$\rho^0 = -\Lambda^{-1} p_0(0). \tag{1.96}$$

Last but not least, the needed sentinel is provided by

$$S(\lambda, \tau) = \int_0^T \int_{\mathcal{O}} (h_0 + \rho) m_0 \mathrm{d}x\mathrm{d}t. \tag{1.97}$$

Identification of the pollution term $\lambda\hat{\xi}$

As a consequence of (1.76) and (1.97) we get

$$\lambda \frac{\partial S}{\partial \lambda}(0,0) = \int_0^T \int_{\mathcal{O}} p\hat{\xi} \mathrm{d}x\mathrm{d}t.$$

This crucial information is provided by the desired sentinel, which depends on h_0.

$$\int_0^T \int_{\mathcal{O}} (p_0 + z)\lambda\hat{\xi} \mathrm{d}x\mathrm{d}t \simeq \int_0^T \int_{\mathcal{O}} (h_0 + \rho)(m_0 - y_0) \mathrm{d}x\mathrm{d}t.$$

The pollution term $\lambda\hat{\xi}$, which affects the sentinel S to be unable to observe, is then referred to as stealthy if

$$\int_0^T \int_{\mathcal{O}} p\lambda\hat{\xi} \mathrm{d}x\mathrm{d}t = 0.$$

Chapter 2

Optimal control of a partially known coupled system of BOD and DO

2.1 Introduction

The environmental pollution problem is one of the most serious problems facing the world today. It is always linked to some terrible problems for which we are unable to find a solution and which cause irreparable natural damage. The presence of a sufficient concentration of dissolved oxygen DO is all-important and necessary to preserve aquatic life. If more oxygen is consumed than is produced, dissolved oxygen levels decline, and some sensitive animals may move away, weaken, or die. Oxygen is gained from the atmosphere and plants as a result of photosynthesis. Running water, because of its churning, dissolves more oxygen than still water. Respiration by aquatic animals, decomposition, and various chemical reactions consume oxygen. The required quantity of dissolved oxygen by aerobic biological organisms, which is used for decomposing organic material under aerobic conditions at a specified temperature, is called biochemical oxygen demand BOD. The decrease of BOD is the way of judging the effectiveness of water purification [21, 33, 36]. Many studies have been published in the context of improving the quality of the methods and procedures that can be used to reduce the BOD level. We are referring to works by D.M. Reynolds, S.R. Ahmad in [42], Salguero, Jazmin and Valverde, Jhonny in [9], Magdalena Zajda and Urszula Aleksander-Kwaterczak in [4], etc.

This work provides the main insights into the debate on optimal control choice of an evolution coupled system that presents the relation between biochemical oxygen demand and dissolved oxygen. Because the concentration of dissolved oxygen is of prime importance in considering the quality of water, we try to control its level by giving an assessment of the biochemical oxygen demand and studying its physiochemical characteristics. Elsewhere, the posed coupled systems are given with unknown initial conditions that present some barriers. The main aim of our work is to determine the optimal control. To find the characterization of this optimal control, we dispose of the incomplete data by introducing the concepts of no-regret control and the sequence of low regret controls. The optimality coupled systems of the no regret control are formed by going to the limit.

This work is based on a paper that was published in International Journal of Analysis and Applications, vol 19(6), 984-996. **(2021)**, by C. Laouar, A. Ayadi, A. Hafdallah, (see [2]).

2.2 Setting the problem

In this section, we present a mathematical model that is used for studying the pollution problem. This examined model is not standard because it contains some missing initial conditions. We consider a fixed final time $T > 0$, and Ω which is a bounded open subset of $\mathbb{R}^N,\ N \in \mathbb{N}$ of smooth boundary Γ. We denote the space-time cylinder by $Q = \Omega\times]0, T[$, its boundary by $\Sigma = \Gamma\times]0, T[$. We are interested in an evolutionary organic pollution problem in surface waters, for example, lakes or estuaries, which is reduced to this reaction-dispersion/diffusion problem with uncertainly.

$$\begin{cases} \dfrac{\partial y}{\partial t} - \mathrm{d}iv(d(x)\nabla y) + r(x)y = 0 & \\ \dfrac{\partial z}{\partial t} - \mathrm{d}iv(d(x)\nabla z) + \tilde{r}(x)z + r(x)y = \omega\chi_{\mathcal{O}} & \text{in} \quad Q, \\ y(x,0) = g_1, \quad z(x,0) = g_2 & \text{in} \quad \Omega, \\ z = 0, \quad \dfrac{\partial z}{\partial \nu} = 0 & \text{on} \quad \Sigma, \end{cases} \tag{2.1}$$

where y, z are BOD and DO in a given water sample at a certain temperature over a specific time period.
In the control region of $\mathcal{O}$, the control function ω represents the sources of dissolved oxygen from the atmosphere and photosynthesis of plants, and $\chi_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$. We suppose that for all $\omega \in \mathcal{U}_{ad}$, we have

$$\mathcal{U}_{ad} = \{\omega \in L^2(Q) : \omega_{min} \leq \omega \leq \omega_{max}\} \text{ is non-empty closed, convex.} \tag{2.2}$$

In the absence of other factors, ω_{min} and ω_{max} represent the minimum and maximum concentrations of dissolved oxygen that would be present in water at a specific temperature.
The initial conditions $(g_1, g_2) \in \mathcal{G} \subset H^{-\frac{1}{2}}(\Omega) \times H_0^1(\Omega)$ assumed to be unknown. The boundary conditions $(z, \dfrac{\partial z}{\partial \nu}) \in H^{-\frac{1}{2}}(\Sigma) \times H^{-\frac{1}{2}}(\Sigma)$. The functions r, $\tilde{r}$ and d are reaction coefficients.
The coupled systems (2.1) have a unique pair solution $(y, g) = (y(\omega, g), z(\omega, g))$ in which case

$$(y, z) \in L^2(Q) \cap \mathcal{C}^\infty(0, T; H^{-\frac{1}{2}}(\Omega)) \times L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \cap \mathcal{C}^\infty(0, T; H_0^1(\Omega)).$$

There are many factors that can reduce the level of dissolved oxygen in the air. The respiration of plant life and animal life, the decomposition of organic matter, the reduction due to other gases, the temperature increase, and others. For these reasons, the main goal of this work is to control the evolutionary organic pollution problem. Exactly, we control the level of dissolved oxygen. We point out here that we did not insert any control function in the biochemical oxygen demand equations.

For fixed pair $(y_d, z_d) \in (L^2(\Omega))^2$ and for $N > 0, g = (g_1, g_2) \in \mathcal{G}$, we define the quadratic cost function associated with (2.1)

$$J(\omega, g) = \|y(\omega, g) - y_d\|^2_{L^2(Q)} + \|z(\omega, g) - z_d\|^2_{L^2(Q)} + \int_0^T \int_{\mathcal{O}} N\omega^2 \mathrm{d}t\mathrm{d}x. \tag{2.3}$$

The problem of optimal control is

$$\inf_{\omega \in \mathcal{U}_{ad}} J(\omega, g) \quad \forall g \in \mathcal{G}. \tag{2.4}$$

In the case where the space $\mathcal{G}$ is infinite, the problem (2.4) makes no sense.Then, if $\mathcal{G}$ is finite, we try to solve the inf ˘ sup problem

$$\inf_{\omega \in \mathcal{U}_{ad}} \sup_{g \in \mathcal{G}} J(\omega, g). \tag{2.5}$$

However, in this situation, it is very difficult to ensure that $\sup_{g \in \mathcal{G}} J(\omega, g)$ is bounded.

In (1992), J.L.Lions had a good idea by adding an additional concept which is called "no regret control". The concept of *no-regret control* (or, equivalently, *Pareto control*) of distributed systems with missing data is used by J.L. Lions in [55, Pareto control of distributed systems, page 90].

In [44, 52, 55, 56], J.L. Lions applied the Pareto control and he associated it with a *sequence of low-regret controls* defined by a quadratic perturbation for deterministic distributed systems with incomplete data. In citation [37], O. Nakolima, R. Dorville, and A. Omrane demonstrate how to apply the no regret control to the hyperbolic case. They also generalized these concepts in the case of ill-posed deterministic problems, without assuming Slater's condition [28, 34]. In [3], Hafdallah

A, and Ayadi A applied the no regret and low regret concepts to control a thermoelastic body with missing initial conditions. A. Hafdallah, A. Ayadi, and C. Laouar applied the no-regret control notion to control an ill-posed wave equation, see [5].

The principle of this idea is based on looking for controls such that

$$J(\omega, g) \leq J(0, g) \quad \forall g \in \mathcal{G}. \tag{2.6}$$

The condition (2.6) implies

$$\sup_{g\in\mathcal{G}} \big[J(\omega, g) - J(0, g)\big] \text{ is bounded.}$$

In this case, we solve the following problem:

$$\inf_{\omega\in\mathcal{U}_{ad}} \sup_{g\in\mathcal{G}} \big[J(\omega, g) - J(0, g)\big]. \tag{2.7}$$

In the following, we define the no regret control for the partially known problem (2.1).

2.3 Finding the no-regrets control

We say that $\hat{\omega} \in \mathcal{U}_{ad}$ defines a no-regret control for (2.1) if it is the optimal solution of (2.7).

Lemma 2.3.1

For every $\omega \in \mathcal{U}_{ad}$ the problem (2.7) is equivalent to

$$\inf_{\omega\in\mathcal{U}_{ad}} \Big(J(\omega, 0) - J(0, 0) + 2\sup_{g\in\mathcal{G}} \int_\Omega [g_1.\zeta(\omega)(x, 0) + g_2.\xi(\omega)(x, 0)]\mathrm{d}x, \qquad g = (g_1, g_2) \in \mathcal{G}, \tag{2.8}$$

where $(\zeta, \xi) = (\zeta(\omega, 0)(x, t), \xi(\omega, 0)(x, t))$ satisfies the following backward coupled equations

$$\begin{cases} -\dfrac{\partial\zeta}{\partial t} - \mathrm{d}iv(d(x)\nabla\zeta) + r(x)\zeta + r(x)\xi = y(\omega, 0) & \\ \quad -\dfrac{\partial\xi}{\partial t} - \mathrm{d}iv(d(x)\nabla\xi) + \tilde{r}(x)\xi = z(\omega, 0) & \text{in} \quad Q, \\ \qquad \zeta(x, T) = 0, \;\; \xi(x, T) = 0 & \text{in} \quad \Omega, \\ \qquad \zeta = 0, \;\; \dfrac{\partial\zeta}{\partial\nu} = 0 & \text{on} \quad \Sigma. \end{cases} \tag{2.9}$$

Proof. By linearity, we can write the solution to (2.7) in the form

$$y(\omega, g) = y(\omega, 0) + y(0, g), \quad z(\omega, g) = z(\omega, 0) + z(0, g).$$

Then, the functional $J(\omega, g)$ can be written

$$J(\omega, g) = J(\omega, 0) - J(0, 0) + 2\iint_Q \Big[y(\omega, 0)y(0, g) + z(\omega, 0)z(0, g)\Big]\mathrm{d}t\mathrm{d}x.$$

We introduce $(\zeta(\omega), \xi(\omega))$ the solution of (2.9). Then, we use integration by parts to obtain

$$\iint_Q -(\frac{\partial\zeta}{\partial t} + \mathrm{d}iv(d(x)\nabla\zeta) + r(x)\zeta + r(x)\xi)y(0, g)\mathrm{d}t\mathrm{d}x = \int_\Omega g_1\zeta(\omega)(x, 0)\mathrm{d}x + \iint_Q \xi r(x)y(0, g)\mathrm{d}t\mathrm{d}x, \tag{2.10}$$

and

$$\begin{aligned} &\iint_Q (-\frac{\partial\xi}{\partial t} - \mathrm{d}iv(d(x)\nabla\xi) + \tilde{r}(x)\xi)z(0, g)\mathrm{d}t\mathrm{d}x \\ &= \int_\Omega g_2.\xi(\omega)(x, 0)\mathrm{d}x + \iint_Q \xi(\frac{\partial z}{\partial t}(0, g) - \mathrm{d}iv(d(x)\nabla z(0, g)) + \tilde{r}(x)z(0, g))\mathrm{d}t\mathrm{d}x. \end{aligned} \tag{2.11}$$

When we add equation (2.10) to equation (2.11), we get

$$\iint_Q [y(\omega,0)y(0,g)+z(\omega,0)z(0,g)]\mathrm{d}t\mathrm{d}x = \int_\Omega [g_1.\zeta(\omega)(x,0)+g_2.\xi(\omega)(x,0)]\mathrm{d}x.$$

□

Remark 2.3.1

The no regret control exist only if g_1 and $\zeta(\omega)(x,0)$ (respectively g_2 and $\zeta(\omega)(x,0)$) are perpendicular to each other in $H^{-\frac{1}{2}}(\Omega)$ (respectively in $H_0^1(\Omega)$). For this reason, we consider the following set of admissible controls

$$\hat{\mathcal{U}}_{ad} = \{\omega \in \mathcal{U}_{ad}:\ \langle g_1, \zeta(\omega)(x,0)\rangle_{H^{-\frac{1}{2}}(\Omega)} = 0,\ \langle g_2, \xi(\omega)(x,0)\rangle_{H_0^1(\Omega)} = 0\}.$$

In [44, 52, 55, 56], J.L. Lions applied the no control and associated it with a sequence of low-regret controls defined by a quadratic perturbation for deterministic distributed systems with incomplete data. The sequence of low-regret controls is expected to converge to the no-regret control.

2.4 Defining the sequence of low-regret controls

For every $\gamma > 0$, we relax the problem (2.8) by introducing a quadratic perturbation such that

$$J(\omega,g) - J(0,g) \le \gamma \|g\|^2_{\mathcal{G}}, \quad \forall g \in \mathcal{G}.$$

We say that $\hat{\omega}^\gamma \in \mathcal{U}_{ad}$ is the sequence of low-regret controls for (2.1) if $\hat{\omega}^\gamma$ is the solution to

$$\inf_{\omega\in\mathcal{U}_{ad}} \sup_{g\in\mathcal{G}} \Big[J(\omega,g) - J(0,g) - \gamma(\|g_1\|^2_{H^{-\frac{1}{2}}(\Omega)} + \|g_2\|^2_{H_0^1(\Omega)})\Big]. \tag{2.12}$$

Lemma 2.4.1

The problem (2.12) can be written as

$$\inf_{\omega\in\mathcal{U}_{ad}} \mathcal{J}^\gamma(\omega), \tag{2.13}$$

where

$$\mathcal{J}^\gamma(\omega) = J(\omega,0) - J(0,0) + \frac{1}{\gamma}\|\zeta(\omega)(x,0)\|^2_{H^{-\frac{1}{2}}(\Omega)} + \frac{1}{\gamma}\|\xi(\omega)(x,0)\|^2_{H_0^1(\Omega)}. \tag{2.14}$$

Proof. From (2.8) and (2.9), the problem (2.12) is written as

$$\inf_{\omega\in\mathcal{U}_{ad}} \Big(J(\omega,0) - J(0,0) + \sup_{g\in\mathcal{G}} \int_\Omega \Big[(2g_1\zeta(\omega)(x,0) - \gamma g_1^2) + (2g_2\xi(\omega)(x,0) - \gamma g_2^2)\Big]\mathrm{d}x\Big).$$

The functions $f: g_1 \mapsto (2g_1\zeta(\omega)(x,0) - \gamma g_1^2)$ and $\tilde{f}: g_2 \mapsto (2g_2\xi(\omega)(x,0) - \gamma g_2^2)$ are concave. Then, it's absolutely clear that

$$\sup_{g_1\in H^{-\frac{1}{2}}(\Omega)} f(g_1) = \frac{1}{\gamma}\|\zeta(\omega)(x,0)\|^2_{H^{-\frac{1}{2}}(\Omega)}, \quad \sup_{g_2\in H_0^1(\Omega)} \tilde{f}(g_2) = \frac{1}{\gamma}\|\xi(\omega)(x,0)\|^2_{H_0^1(\Omega)}.$$

□

Lemma 2.4.2

The problem (2.13)-(2.14) has a unique solution $\hat{\omega}^\gamma$, which is called the sequence of low regret controls. *Furthermore, when $\gamma \to 0$, the control $\hat{\omega}^\gamma$ converges weakly to the unique no regret control $\hat{\omega}$.*

Proof. Since the set of admissible controls $\mathcal{U}_{ad}$ is non-empty, closed and bounded, we have

$$\mathcal{J}^\gamma(\omega) \geq -J(0,0) = -\|y_d\|^2_{L^2(\Omega)} - \|z_d\|^2_{L^2(\Omega)}, \quad \forall \omega \in \mathcal{U}_{ad}.$$

Following that, there is

$$d^\gamma := \inf_{\omega \in \mathcal{U}_{ad}} \mathcal{J}^\gamma(\omega) \geq 0.$$

Let $(\omega_n^\gamma) \in \mathcal{U}_{ad}$ be a minimizing sequence such that

$$d^\gamma = \lim_{n\to\infty} \mathcal{J}^\gamma(\omega_n^\gamma) = \mathcal{J}^\gamma(\omega^\gamma).$$

Then again, we get

$$d^\gamma \leq \mathcal{J}^\gamma(\omega_n^\gamma) < d^\gamma + \frac{1}{n} < d^\gamma + 1.$$

So, we deduce the bounds

$$\begin{aligned} &\|\omega_n^\gamma\|_{L^2(0,T;\mathcal{O})} \leq C^\gamma, \ \|y(\omega_n^\gamma, 0)\|_{L^2(Q)} \leq C^\gamma, \ \ \|z(\omega_n^\gamma, 0)\|_{L^2(Q)} \leq C^\gamma, \\ &\frac{1}{\sqrt{\gamma}}\|\zeta(\omega_n^\gamma)(x,0)\|_{H^{-\frac{1}{2}}(\Omega)} \leq C^\gamma, \ \frac{1}{\sqrt{\gamma}}\|\xi(\omega_n^\gamma)(x,0)\|_{H_0^1(\Omega)} \leq C^\gamma. \end{aligned} \tag{2.15}$$

where C^γ is a positive constant and $(y_n^\gamma, z_n^\gamma) = (y(\omega_n^\gamma, 0), z(\omega_n^\gamma, 0))$ solves the the following coupled systems

$$\begin{cases} \dfrac{\partial y_n^\gamma}{\partial t} - \operatorname{div}(d(x)\nabla y_n^\gamma) + r(x) y_n^\gamma = 0 & \\ \dfrac{\partial z_n^\gamma}{\partial t} - \operatorname{div}(d(x)\nabla z_n^\gamma) + \tilde{r}(x) z_n^\gamma + r(x) y_n^\gamma = \omega_n^\gamma \chi_{\mathcal{O}} & \text{in} \quad Q, \\ y_n^\gamma(x,0) = 0, \quad z_n^\gamma(x,0) = 0, & \text{in} \quad \Omega, \\ z_n^\gamma = 0, \quad \dfrac{\partial z_n^\gamma}{\partial \nu} = 0 & \text{on} \quad \Sigma. \end{cases} \tag{2.16}$$

Multiplying the first equality of (2.16) by y_n^γ and the second equality by z_n^γ. We integrate over Ω, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |y_n^\gamma(t)|^2 \mathrm{d}x + \int_\Omega r(x)|y_n^\gamma(t)|^2 \mathrm{d}x - \int_\Omega \operatorname{div}\ (d(x) y_n^\gamma(t)) y_n^\gamma(t) \mathrm{d}x = 0,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |z_n^\gamma(t)|^2 \mathrm{d}x + \int_\Omega \tilde{r}(x)|z_n^\gamma(t)|^2 - \operatorname{div}(\ d(x).z_n^\gamma(t)) z_n^\gamma(t) + r(x) y_n^\gamma(t) z_n^\gamma(t) \mathrm{d}x = \int_{\mathcal{O}} \omega_n^\gamma(t) z_n^\gamma(t) \mathrm{d}x.$$

By integrating over $[0,T]$ and by applying the Gronwall lemma, we obtain

$$\|y_n^\gamma\|_{L^\infty(0,T;\mathcal{Y})} \leq C^\gamma, \quad \|z_n^\gamma\|_{L^\infty(0,T;H_0^1(\Omega))} \leq C^\gamma,$$

where C^γ denotes a positive constant. We can deduce from (2.15)

$$\|\frac{\partial z_n^\gamma}{\partial t} - \operatorname{div}(d(x)\nabla z_n^\gamma) + \tilde{r}(x) z_n^\gamma + r(x) y_n^\gamma\|_{L^2(0,T;\mathcal{O})} \leq C^\gamma.$$

Then there is a subsequence of (ω_n^γ), which we denote with the same indices, such that when n goes to $+\infty$,

$$\begin{aligned} &\omega_n^\gamma \rightharpoonup \hat{\omega}^\gamma \ \text{ weakly in } \ L^2(0,T;\mathcal{O}), \ y_n^\gamma \rightharpoonup \hat{y}^\gamma \ \text{ weakly in } L^\infty(0,T;\mathcal{Y}), \\ &z_n^\gamma \rightharpoonup \hat{z}^\gamma \ \text{ weakly in } \ L^\infty(0,T;H_0^1(\Omega)), \\ &\frac{\partial z_n^\gamma}{\partial t} - \operatorname{div}(d(x)\nabla z_n^\gamma) + \tilde{r}(x) z_n^\gamma + r(x) y_n^\gamma \rightharpoonup f \ \text{ weakly in } \ L^2(0,T;\mathcal{O}). \end{aligned}$$

The space $L^\infty(Q)$ (respectively $L^\infty(0,T;H_0^1(\Omega))$) is continuously embedded in $L^2(Q)$ (respectively $L^2(0,T;H_0^1(\Omega))$). Clearly, we have

$$y_n^\gamma \rightharpoonup \hat{y}^\gamma \ \text{ weakly in } L^2(Q), \quad z_n^\gamma \rightharpoonup \hat{z}^\gamma \ \text{ weakly in } \ L^2(0,T;H_0^1(\Omega)). \tag{2.17}$$

Multiplying two equalities in (2.16) by two test functions $\varphi, \psi \in D(Q)$, we discover

$$\langle y_n^\gamma, -\frac{\partial \varphi}{\partial t} - \mathrm{div}(d(x)\nabla\varphi) + r(x)\varphi\rangle_{L^2(Q)} = 0,$$
$$\langle z_n^\gamma, -\frac{\partial \psi}{\partial t} - \mathrm{div}(d(x)\nabla\psi) + \tilde{r}(x)\psi\rangle_{L^2(Q)} + \langle y_n^\gamma, r(x)\psi\rangle_{L^2(Q)} = \langle \omega_n^\gamma, \psi\rangle_{L^2(Q)}.$$

We get by combining the last two equalities and passing to the limit.

$$\begin{cases} \dfrac{\partial \hat{y}^\gamma}{\partial t} - \mathrm{div}(d(x)\nabla\hat{y}^\gamma) + r(x)\hat{y}^\gamma = 0 \\ \dfrac{\partial \hat{z}^\gamma}{\partial t} - \mathrm{div}(d(x)\nabla\hat{z}^\gamma) + \tilde{r}(x)\hat{z}^\gamma + r(x)\hat{y}^\gamma = \hat{\omega}^\gamma\chi_{\mathcal{O}} \end{cases} \quad \text{in} \quad L^2(0,T,\mathcal{O}). \tag{2.18}$$

From (2.17) and (2.18), we get

$$\hat{y}^\gamma(x,0) = 0, \quad \hat{z}^\gamma(x,0) = 0.$$

Now we must demonstrate that $(\zeta_n^\gamma, \xi_n^\gamma)$ converges to $(\hat{\zeta}^\gamma, \hat{\xi}^\gamma)$. Let $\zeta_n^\gamma = \zeta(\omega_n^\gamma)$ and $\xi_n^\gamma = \xi(\omega_n^\gamma)$. Reverse time variable by taking $\tilde{\zeta}_n^\gamma(x,t) = \zeta_n^\gamma(x,T-t)$, $\tilde{\xi}_n^\gamma(x,t) = \xi_n^\gamma(x,T-t)$, $\tilde{y}_n^\gamma(x,t) = y_n^\gamma(x,T-t)$ and $\tilde{z}_n^\gamma(x,t) = z_n^\gamma(x,T-t)$.

Then, we have

$$\begin{cases} -\dfrac{\partial \tilde{\zeta}_n^\gamma}{\partial t} - \mathrm{div}(d(x)\nabla\tilde{\zeta}_n^\gamma) + r(x)\tilde{\zeta}_n^\gamma + r(x)\tilde{\xi}_n^\gamma = \tilde{y}_n^\gamma & \\ -\dfrac{\partial \tilde{\xi}_n^\gamma}{\partial t} - \mathrm{div}(d(x)\nabla\tilde{\xi}_n^\gamma) + \tilde{r}(x)\tilde{\xi}_n^\gamma = \tilde{z}_n^\gamma & \text{in} \quad Q, \\ \tilde{\zeta}_n^\gamma(x,0) = 0, \ \tilde{\xi}_n^\gamma(x,0) = 0 & \text{in} \quad \Omega, \\ \tilde{\zeta}_n^\gamma = 0, \ \dfrac{\partial \tilde{\zeta}_n^\gamma}{\partial \nu} = 0 & \text{on} \quad \Sigma. \end{cases}$$

Then we come to the conclusion that

$$\tilde{\zeta}_n^\gamma \rightharpoonup \hat{\zeta}^\gamma \ \text{ weakly in } L^2(Q), \quad \tilde{\xi}_n^\gamma \rightharpoonup \hat{\xi}^\gamma \ \text{ weakly in } \ L^2(0,T;H_0^1(\Omega)).$$

Hence, the

$$\tilde{\zeta}_n^\gamma(x,0) \rightharpoonup \hat{\zeta}^\gamma(x,0) \ \text{ weakly in } H^{-\frac{1}{2}}(\Omega), \ \tilde{\xi}_n^\gamma(x,0) \rightharpoonup \hat{\xi}^\gamma(x,0) \ \text{ weakly in } H_0^1(\Omega).$$

Finally, there is

$$\lim_{n\to\infty} \mathcal{J}^\gamma(\omega_n^\gamma) = \mathcal{J}^\gamma(\omega^\gamma) = \inf_{\omega\in\mathcal{U}_{ad}} \mathcal{J}^\gamma(\omega).$$

The functional $\mathcal{J}^\gamma$ quadratic coercive, thus $\hat{\omega}^\gamma$ is unique. □

The characterization of the sequence of low regret controls is given in the following proposition:

Proposition 2.4.1

The unique sequence of low regret controls, $\hat{\omega}^\gamma$ is characterized by the following coupled system

$$\begin{cases} \dfrac{\partial \hat{y}^\gamma}{\partial t} - \mathrm{div}(d(x)\nabla\hat{y}^\gamma) + r(x)\hat{y}^\gamma = 0 & \\ \dfrac{\partial \hat{z}^\gamma}{\partial t} - \mathrm{div}(d(x)\nabla\hat{z}^\gamma) + \tilde{r}(x)\hat{z}^\gamma + r(x)\hat{y}^\gamma = \hat{\omega}^\gamma\chi_{\mathcal{O}} & \text{in} \quad Q, \\ \hat{y}^\gamma(x,0) = 0, \quad \hat{z}^\gamma(x,0) = 0 & \text{in} \quad \Omega, \\ \hat{z}^\gamma = 0, \quad \dfrac{\partial \hat{z}^\gamma}{\partial \nu} = 0 & \text{on} \quad \Sigma, \end{cases} \tag{2.19}$$

$$\begin{cases} -\dfrac{\partial \hat{\zeta}^\gamma}{\partial t} - \mathrm{div}(d(x)\nabla\hat{\zeta}^\gamma) + r(x)\hat{\zeta}^\gamma + r(x)\hat{\xi}^\gamma = y(\omega - \hat{\omega}^\gamma) & \\ -\dfrac{\partial \hat{\xi}^\gamma}{\partial t} - \mathrm{div}(d(x)\nabla\hat{\xi}^\gamma) + \tilde{r}(x)\hat{\xi}^\gamma = z(\omega - \hat{\omega}^\gamma) & \text{in} \quad Q, \\ \hat{\zeta}^\gamma(x,T) = 0, \ \hat{\xi}^\gamma(x,T) = 0 & \text{in} \quad \Omega, \\ \hat{\zeta}^\gamma = 0, \ \dfrac{\partial \hat{\zeta}^\gamma}{\partial \nu} = 0 & \text{on} \quad \Sigma, \end{cases} \tag{2.20}$$

$$\begin{cases} \dfrac{\partial \hat{\rho}^\gamma}{\partial t} - \text{div}(d(x)\nabla\hat{\rho}^\gamma) + r(x)\hat{\rho}^\gamma = 0 & \\ \dfrac{\partial \hat{\sigma}^\gamma}{\partial t} - \text{div}(d(x)\nabla\hat{\sigma}^\gamma) + \tilde{r}(x)\hat{\sigma}^\gamma + r(x)\hat{\rho}^\gamma = 0 & \text{in} \quad Q, \\ \hat{\rho}^\gamma(x,0) = -\dfrac{1}{\gamma}\zeta(\omega - \hat{\omega}^\gamma)(x,0), \quad \hat{\sigma}^\gamma(x,0) = -\dfrac{1}{\gamma}\xi(\omega - \hat{\omega}^\gamma)(x,0) & \text{in} \quad \Omega, \\ \hat{z}^\gamma = 0, \quad \dfrac{\partial \hat{z}^\gamma}{\partial \nu} = 0 & \text{on} \quad \Sigma, \end{cases} \tag{2.21}$$

and

$$\begin{cases} -\dfrac{\partial \hat{p}^\gamma}{\partial t} - \text{div}(d(x)\nabla\hat{p}^\gamma) + r(x)\hat{p}^\gamma + r(x)\hat{q}^\gamma = \hat{y}^\gamma - y_d + \hat{\rho}^\gamma & \\ -\dfrac{\partial \hat{q}^\gamma}{\partial t} - \text{div}(d(x)\nabla\hat{q}^\gamma) + \tilde{r}(x)\hat{q}^\gamma = \hat{z}^\gamma - z_d + \hat{\sigma}^\gamma & \text{in} \quad Q, \\ \hat{p}^\gamma(x,T) = 0, \ \hat{q}^\gamma(x,T) = 0 & \text{in} \quad \Omega, \\ \hat{p}^\gamma = 0, \ \dfrac{\partial \hat{q}^\gamma}{\partial \nu} = 0 & \text{on} \quad \Sigma. \end{cases} \tag{2.22}$$

Furthermore, for all $\omega \in \mathcal{U}_{ad}$, we have

$$\int_0^T \int_{\mathcal{O}} (\hat{q}^\gamma + N\hat{\omega}^\gamma)(\omega - \hat{\omega}^\gamma)\mathrm{d}x\mathrm{d}t \geq 0. \tag{2.23}$$

Proof. The functional $\mathcal{J}^\gamma$ is quadratic coercive, thus it possesses a unique minimum $\hat{\omega}^\gamma$. This minimum is a solution to the Euler equation, thus for all $\omega \in \mathcal{U}_{ad}$, we have

$$\lim_{h\to 0} \frac{J^\gamma(\hat{\omega}^\gamma + h(\omega - \hat{\omega}^\gamma)) - J^\gamma(\hat{\omega}^\gamma)}{h} \geq 0.$$

So, we have

$$\begin{aligned} &\iint_Q [y(\omega - \hat{\omega}^\gamma)(\hat{y}^\gamma - y_d) + z(\omega - \hat{\omega}^\gamma)(\hat{z}^\gamma - z_d)]\mathrm{d}t\mathrm{d}x + \int_0^T \int_{\mathcal{O}} N\hat{\omega}^\gamma(\omega - \hat{\omega}^\gamma)\mathrm{d}t\mathrm{d}x \\ &+ \frac{1}{\gamma}\int_\Omega \Big[\hat{\zeta}^\gamma(x,0)\zeta(\omega - \hat{\omega}^\gamma)(x,0) + \hat{\xi}^\gamma(x,0)\xi(\omega - \hat{\omega}^\gamma)(x,0)\Big]\mathrm{d}x \geq 0. \end{aligned} \tag{2.24}$$

We introduce $(\hat{\rho}^\gamma, \hat{\sigma}^\gamma) = (\rho(\hat{\omega}^\gamma, 0)(x,t), \sigma(\hat{\omega}^\gamma, 0)(x,t))$ solution to (2.21). By integration of parts, we get

$$\begin{aligned} \iint_Q \hat{\zeta}^\gamma(\frac{\partial \hat{\rho}^\gamma}{\partial t} - \text{div}(d(x)\nabla\hat{\rho}^\gamma) + r(x)\hat{\rho}^\gamma)\mathrm{d}t\mathrm{d}x &= \iint_Q \hat{\rho}^\gamma(-\frac{\partial \hat{\zeta}^\gamma}{\partial t} - \text{div}(d(x)\nabla\hat{\zeta}^\gamma) + r(x)\hat{\zeta}^\gamma)\mathrm{d}t\mathrm{d}x \\ &+ \frac{1}{\gamma}\int_\Omega \hat{\zeta}^\gamma(x,0)\zeta(\omega - \hat{\omega}^\gamma)(x,0)\mathrm{d}x \\ &= 0, \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} \iint_Q \hat{\xi}^\gamma(\frac{\partial \hat{\sigma}^\gamma}{\partial t} - \text{div}(d(x)\nabla\hat{\sigma}^\gamma) + \tilde{r}(x)\hat{\sigma}^\gamma + r(x)\hat{\rho}^\gamma)\mathrm{d}t\mathrm{d}x \ &= \iint_Q \hat{\sigma}^\gamma z(\omega - \hat{\omega}^\gamma)\mathrm{d}t\mathrm{d}x + \iint_Q \hat{\rho}^\gamma \tilde{r}(x)\hat{\xi}^\gamma\mathrm{d}t\mathrm{d}x \\ &+ \frac{1}{\gamma}\int_\Omega \hat{\xi}^\gamma(x,0)\xi(\omega - \hat{\omega}^\gamma)(x,0)\mathrm{d}x \\ &= 0. \end{aligned} \tag{2.26}$$

The result of adding (2.25) to (2.26) is

$$\frac{1}{\gamma}\int_\Omega [\hat{\zeta}^\gamma(x,0)\zeta(\omega - \hat{\omega}^\gamma)(x,0) + \hat{\xi}^\gamma(x,0)\xi(\omega - \hat{\omega}^\gamma)(x,0)]\mathrm{d}x = \iint_Q [\hat{\rho}^\gamma y(\omega - \hat{\omega}^\gamma) + \hat{\sigma}^\gamma z(\omega - \hat{\omega}^\gamma)]\mathrm{d}t\mathrm{d}x. \tag{2.27}$$

By substituting (2.27)for (2.24), we get

$$\iint_Q [y(\omega-\hat{\omega}^\gamma)(\hat{y}^\gamma - y_d + \hat{\rho}^\gamma) + z(\omega-\hat{\omega}^\gamma)(\hat{z}^\gamma - z_d + \hat{\sigma}^\gamma)]\mathrm{d}t\mathrm{d}x + \int_0^T \int_{\mathcal{O}} N\hat{\omega}^\gamma(\omega-\hat{\omega}^\gamma)\mathrm{d}t\mathrm{d}x \geq 0. \quad (2.28)$$

We introduce the coupled adjoint state $(\hat{p}^\gamma, \hat{q}^\gamma) = (p(\hat{\omega}^\gamma, 0)(x,t), q(\hat{\omega}^\gamma, 0)(x,t))$ solution to (2.22). Finally, we get (2.23) by substituting (2.22)for (2.28) and integration by parts. □

We need some a priori estimations, which we make in the following lemma.

Lemma 2.4.3

There exist some positive constants $\mathcal{C}$ independent of γ that satisfy the following estimations:

$$\begin{aligned} &\|\hat{\omega}^\gamma\|_{L^2(0,T,\mathcal{O})} \leq \mathcal{C}, \ \|\hat{y}^\gamma\|_{L^2(Q)} \leq \mathcal{C}, \ \|\hat{z}^\gamma\|_{L^2(Q)} \leq \mathcal{C}, \\ &\frac{1}{\sqrt{\gamma}}\|\hat{\zeta}^\gamma(x,0)\|_{H^{-\frac{1}{2}}(\Omega)} \leq \mathcal{C}, \ \frac{1}{\sqrt{\gamma}}\|\hat{\xi}^\gamma(x,0)\|_{H^1_0(\Omega)} \leq \mathcal{C}, \end{aligned} \quad (2.29)$$

and,

$$\|\hat{y}^\gamma\|_{L^2(Q)} \leq \mathcal{C}, \quad \|\frac{\partial \hat{y}^\gamma}{\partial t}\|_{L^2(Q)} \leq \mathcal{C}, \quad \|\hat{z}^\gamma\|_{L^2(0,T;H^1_0(\Omega))} \leq \mathcal{C}, \quad \|\frac{\partial \hat{z}^\gamma}{\partial t}\|_{L^2(Q)} \leq \mathcal{C}, \quad (2.30)$$

$$\|\hat{\zeta}^\gamma\|_{L^2(Q)} \leq \mathcal{C}, \quad \|\frac{\partial \hat{\zeta}^\gamma}{\partial t}\|_{L^2(Q)} \leq \mathcal{C}, \quad \|\hat{\xi}^\gamma\|_{L^2(0,T;H^1_0(\Omega))} \leq \mathcal{C}, \quad \|\frac{\partial \hat{\xi}^\gamma}{\partial t}\|_{L^2(Q)} \leq \mathcal{C}, \quad (2.31)$$

$$\|\hat{\rho}^\gamma\|_{L^2(Q)} \leq \mathcal{C}, \quad \|\frac{\partial \hat{\rho}^\gamma}{\partial t}\|_{L^2(Q)} \leq \mathcal{C}, \quad \|\hat{\sigma}^\gamma\|_{L^2(0,T;H^1_0(\Omega))} \leq \mathcal{C}, \quad \|\frac{\partial \hat{\sigma}^\gamma}{\partial t}\|_{L^2(Q)} \leq \mathcal{C}, \quad (2.32)$$

$$\|\hat{p}^\gamma\|_{L^2(Q)} \leq \mathcal{C}, \quad \|\frac{\partial \hat{p}^\gamma}{\partial t}\|_{L^2(Q)} \leq \mathcal{C}, \quad \|\hat{q}^\gamma\|_{L^2(0,T;H^1_0(\Omega))} \leq \mathcal{C}, \quad \|\frac{\partial \hat{q}^\gamma}{\partial t}\|_{L^2(Q)} \leq \mathcal{C}. \quad (2.33)$$

Proof. Since $\hat{\omega}^\gamma$ is the sequence of low regret controls, we have

$$\mathcal{J}^\gamma(\hat{\omega}^\gamma) \leq \mathcal{J}^\gamma(\omega) \qquad \forall \omega \in \mathcal{U}_{ad}.$$

Specifically, when $\omega = 0$, we get

$$J(\hat{\omega}^\gamma, 0) - J(0,0) + \frac{1}{\gamma}\|\hat{\zeta}^\gamma(x,0)\|^2_{H^{-\frac{1}{2}}(\Omega)} + \frac{1}{\gamma}\|\hat{\xi}^\gamma(x,0)\|^2_{H^1_0(\Omega)} \leq 0.$$

Therefore, we have

$$\begin{aligned} \|\hat{y}^\gamma - y_d\|^2_{L^2(Q)} + \|\hat{z}^\gamma - z_d\|^2_{L^2(Q)} + N\|\|\hat{\omega}^\gamma\|^2_{L^2(Q)} + \frac{1}{\gamma}\|\hat{\zeta}^\gamma(x,0)\|^2_{H^{-\frac{1}{2}}(\Omega)} + \frac{1}{\gamma}\|\hat{\xi}^\gamma(x,0)\|^2_{H^1_0(\Omega)} \\ \leq \|y_d\|^2_{L^2(\Omega)} + \|z_d\|^2_{L^2(\Omega)} = Constant. \end{aligned} \quad (2.34)$$

So (2.29) holds. Multiplying the first equality of (2.19) by $\hat{y}^\gamma$ and the second equality by $\hat{z}^\gamma$ and we integrate over Ω, we find

$$\begin{aligned} &\int_\Omega \hat{y}^\gamma(t)\Big(\frac{\partial \hat{y}^\gamma(t)}{\partial t} - \mathrm{div}(d(x)\nabla \hat{y}^\gamma(t)) + r(x)\hat{y}^\gamma(t)\Big)\mathrm{d}x \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\hat{y}^\gamma(t)|^2\mathrm{d}x + \int_\Omega r(x)|\hat{y}^\gamma(t)|^2\mathrm{d}x - \int_\Omega \mathrm{div}\ (d(x)\hat{y}^\gamma(t))\hat{y}^\gamma(t)\mathrm{d}x \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega \hat{z}^\gamma(t)\Big(\frac{\partial \hat{z}^\gamma(t)}{\partial t} - \mathrm{div}(d(x)\nabla \hat{z}^\gamma(t)) + \tilde{r}(x)\hat{z}^\gamma(t) + r(x)\hat{y}^\gamma(t)\Big)\mathrm{d}x \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\hat{z}^\gamma(t)|^2\mathrm{d}x + \int_\Omega \tilde{r}(x)|\hat{z}^\gamma(t)|^2\mathrm{d}x - \int_\Omega \mathrm{div}(\ d(x).\hat{z}^\gamma(t))\hat{z}^\gamma(t) + \int_\Omega r(x)\hat{y}^\gamma(t)\hat{z}^\gamma(t)\mathrm{d}x \\ &= \int_{\mathcal{O}} \hat{\omega}^\gamma(t)\hat{z}^\gamma(t)\mathrm{d}x. \end{aligned}$$

By integrating over $[0, T]$ and by applying the Gronwall lemma, we obtain

$$\|\hat{y}^\gamma\|_{L^2(Q)} \leq \mathcal{C}_\gamma,\ \|\frac{\partial \hat{y}^\gamma}{\partial t}\|_{L^2(Q)} \leq \mathcal{C}_\gamma,\ \|\hat{z}^\gamma\|_{L^2(0,T;H_0^1(\Omega))} \leq \mathcal{C}_\gamma,\ \ \|\frac{\partial \hat{z}^\gamma}{\partial t}\|_{L^2(Q)} \leq \mathcal{C}_\gamma,$$

where $\mathcal{C}_\gamma$ is a positive constant. From the last estimations we got (2.30). We follow a similar method to demonstrate (2.30) for finding (2.31). □

When we pass to the limit when $\gamma \to 0$, the sequence of low regret controls converges $\hat{\omega}^\gamma$ to the no regret control $\hat{\omega}$.

Theorem 2.4.1

The no regret control $\hat{\omega} = \lim_{\gamma\to 0} \hat{\omega}^\gamma$ is characterized by the unique set $\{(\hat{y}, \hat{z}), (\hat{\zeta}, \hat{\xi}), (\hat{\rho}, \hat{\sigma}), (\hat{p}, \hat{q})\}$ solution to the following coupled optimality system

$$\begin{cases} \dfrac{\partial \hat{y}}{\partial t} - \mathrm{div}(d(x)\nabla \hat{y}) + r(x)\hat{y} = 0 & \\ \dfrac{\partial \hat{z}}{\partial t} - \mathrm{div}(d(x)\nabla \hat{z}) + \tilde{r}(x)\hat{z} + r(x)\hat{y} = \hat{\omega}\chi_{\mathcal{O}} & \text{in} \quad Q, \\ \hat{y}(x,0) = 0, \quad \hat{z}(x,0) = 0, & \text{in} \quad \Omega, \\ \hat{z} = 0, \quad \dfrac{\partial \hat{z}}{\partial \nu} = 0 & \text{on} \quad \Sigma, \end{cases} \tag{2.35}$$

$$\begin{cases} -\dfrac{\partial \hat{\zeta}}{\partial t} - \mathrm{div}(d(x)\nabla \hat{\zeta}) + r(x)\hat{\zeta} + r(x)\hat{\xi} = y(\omega - \hat{\omega}) & \\ -\dfrac{\partial \hat{\xi}}{\partial t} - \mathrm{div}(d(x)\nabla \hat{\xi}) + \tilde{r}(x)\hat{\xi} = z(\omega - \hat{\omega}) & \text{in} \quad Q, \\ \hat{\zeta}(x,T) = 0, \ \hat{\xi}(x,T) = 0 & \text{in} \quad \Omega, \\ \hat{\zeta} = 0, \ \dfrac{\partial \hat{\zeta}}{\partial \nu} = 0 & \text{on} \quad \Sigma, \end{cases} \tag{2.36}$$

$$\begin{cases} \dfrac{\partial \hat{\rho}}{\partial t} - \mathrm{div}(d(x)\nabla \hat{\rho}) + r(x)\hat{\rho} = 0 & \\ \dfrac{\partial \hat{\sigma}}{\partial t} - \mathrm{div}(d(x)\nabla \hat{\sigma}) + \tilde{r}(x)\hat{\sigma} + r(x)\hat{\rho} = 0 & \text{in} \quad Q, \\ \hat{\rho}(x,0) = \rho_0, \quad \hat{\sigma}(x,0) = \sigma_0 & \text{in} \quad \Omega, \\ \hat{\sigma} = 0, \quad \dfrac{\partial \hat{\sigma}}{\partial \nu} = 0 & \text{on} \quad \Sigma, \end{cases} \tag{2.37}$$

and

$$\begin{cases} -\dfrac{\partial \hat{p}}{\partial t} - \mathrm{div}(d(x)\nabla \hat{p}) + r(x)\hat{p} + r(x)\hat{q} = \hat{y} - y_d + \hat{\rho} & \\ -\dfrac{\partial \hat{q}}{\partial t} - \mathrm{div}(d(x)\nabla \hat{q}) + \tilde{r}(x)\hat{q} = \hat{z} - z_d + \hat{\sigma} & \text{in} \quad Q, \\ \hat{p}(x,T) = 0, \ \hat{q}(x,T) = 0 & \text{in} \quad \Omega, \\ \hat{p} = 0, \ \dfrac{\partial \hat{p}}{\partial \nu} = 0 & \text{on} \quad \Sigma, \end{cases} \tag{2.38}$$

with regard to the variational inequality

$$\int_0^T \int_{\mathcal{O}} (\hat{q} + N\hat{\omega})(\omega - \hat{\omega})\mathrm{d}x\mathrm{d}t \geq 0, \tag{2.39}$$

with the following limits:

$$\rho_0 = -\lim_{\gamma\to 0} \frac{1}{\gamma}\zeta(\omega - \hat{\omega}^\gamma)(x,0), \quad \sigma_0 = -\lim_{\gamma\to 0} \frac{1}{\gamma}\xi(\omega - \hat{\omega}^\gamma)(x,0).$$

Proof. From inequality (2.34), we may extract some subsequences of of $(\hat{\omega}^\gamma, \hat{y}^\gamma, \hat{z}^\gamma)_\gamma$ that we denote with the same indices such that, when γ goes to 0, we have

$$\hat{\omega}^\gamma \rightharpoonup \hat{\omega} \text{ weakly in } L^2(0,T;\mathcal{O}),\ (\hat{y}^\gamma,\hat{z}^\gamma) \rightharpoonup (\hat{y},\hat{z}) \text{ weakly in } L^2(Q)\times L^2(0,T;H_0^1(\Omega)).$$

On the other hand, from the optimality coupled systems in the Proposition 2.4.1, the sequences $(\frac{\partial \hat{y}^\gamma}{\partial t} - \mathrm{d}iv(d(x)\nabla \hat{y}^\gamma) + r(x)\hat{y}^\gamma)_\gamma$ and $(\frac{\partial \hat{z}^\gamma}{\partial t} - \mathrm{d}iv(d(x)\nabla \hat{z}^\gamma) + \tilde{r}(x)\hat{z}^\gamma + r(x)\hat{y}^\gamma)_\gamma$ are bounded in $L^2(Q)$. So, we have

$$\frac{\partial \hat{y}^\gamma}{\partial t} - \mathrm{d}iv(d(x)\nabla \hat{y}^\gamma) + r(x)\hat{y}^\gamma \rightharpoonup \frac{\partial \hat{y}}{\partial t} - \mathrm{d}iv(d(x)\nabla \hat{y}) + r(x)\hat{y} \text{ weakly in } L^2(Q),$$
$$\frac{\partial \hat{z}^\gamma}{\partial t} - \mathrm{d}iv(d(x)\nabla \hat{z}^\gamma) + \tilde{r}(x)\hat{z}^\gamma + r(x)\hat{y}^\gamma \rightharpoonup \frac{\partial \hat{z}}{\partial t} - \mathrm{d}iv(d(x)\nabla \hat{z}) + \tilde{r}(x)\hat{z} + r(x)\hat{y} \text{ in } L^2(Q).$$

Taking to the limit $\gamma \to 0$, we get (2.35). Also, from to a priori estimates of Lemma 2.4.3, and by using the same method we found (2.36)-(2.38). From (2.29), we have

$$-\frac{1}{\gamma}\Big(\zeta(\omega-\hat{\omega}^\gamma)(x,0), \xi(\omega-\hat{\omega}^\gamma)(x,0)\Big) \rightharpoonup (\rho_0,\sigma_0) \text{ weakly in } H^{-\frac{1}{2}}(\Omega)\times H_0^1(\Omega).$$

In conclusion, the inequality (2.39) can be deduced using the weak convergence of $\hat{p}^\gamma, \hat{q}^\gamma$ and $\hat{\omega}^\gamma$. □

2.5 Conclusion

Chapter 2 examined an evolution coupled system with missing initial conditions that presented the relationship between biochemical oxygen demand BOD and dissolved oxygen DO. Since the decrease in BOD is a good way to judge the effectiveness of water purification, our main objective was to control the level of dissolved oxygen to give more information about it. We characterized optimal control using the concept of "no regret control," in which we solved an optimal control problem with uncertainty. The obtained method optimization problem is to be transformed into classical optimal control via the notion of low-regret control. Finally, the coupled optimality system for the least regret control converges weakly to the coupled optimality systems for no regret control or optimal control.

Chapter 3

Identification problem of a fractional thermoelastic deformation system with incomplete data: A sentinel method

3.1 Introduction

In the inverse problem, the values of the system's parameters are concluded on the basis of actual observations [31]. An inverse problem, according to J.P. Kernevez, is one in which one must determine the cause from its effect, whereas a direct problem is one in which one must determine how to derive the effect from the cause [43]. J.L. Lions invented and constructed the sentinel strategy in [51, 53], which is today one of the most well-known strategies for investigating the behavior of dynamic systems using limited data. Sentinels are weighted integrals with selectivity-sensitive values for one unknown and insensitivity to the others [41, 46, 50]. As a result of the sentinel problem, a control problem is being investigated [41, 46, 48, 50, 51, 53]. Many researchers have used the sentinel strategy in a variety of numerical applications, particularly O.Bodart, T. Mannikko, and J.P. Kernévez in[48, 50]. Control theory based on fractional-order calculus has been used in a range of fields, including electrical engineering, physics, economics, fractals, chaos, and medical science. In recent years, many publications have appeared on control for fractional distributed concerns, with fractional derivatives and integrals being used to define the controlled dynamical fractional systems [17, 18]. As indicated by [6, 17, 18], the majority of research findings are frequently linked to Riemann-Liouville and Caputo derivatives.

The main objective of this study is to explore the difficulties of identifying a fractional with insufficient data. We're keen to know more about composite material deformation. This type of deformation isn't always strictly mechanical. It is linked to thermal effects, which is why composite thermoelastic behavior must be examined. A. Hafdallah and A. Abdelhamid employed the no-regret control concept to regulate the deformation and temperature of a thermoelastic body by acting on it with an external force delivered to a segment of the body in [3]. Our purpose is to analyze the interplay between thermal and mechanical forces in elastic bodies using the sentinel approach to an inverse fractional coupled thermoelastic system. According to [13], these kinds of issues have recently benefited aeronautics, nuclear reactors, and cryogenic applications.

The following is how the paper is structured: We'll go over some basic fractional calculus definitions in the next section. For more details on fractional derivatives and integrals, see [17, 18, 19, 49, 39]. In Section 3.3, we present the model under consideration and define our problem. In the fourth section, we construct an observatory domain and build the sentinel function using the mean of the supplied observations. In Section 3.5, we will reduce the sentinel problem to a null controllability problem with a constraint. The release of control and presentation of optimality coupled systems will be covered in Section 3.6. We identify the pollution term in Section 3.7 and derive the stealthiness relationship from it. Finally, we come to a conclusion for the paper. This work is based on an article that was published in Nonlinear studies journal, www.nonlinearstudies.com Vol. 29, No. 2, pp. 1-13, **(2022)**, by C. Laouar, A. Ayadi, A. Hafdallah (see [1]).

3.2 Basic definitions of fractional calculus

In all the sequel of this section, $f : [0,T] \to \mathbb{R}$ be a function, $\alpha > 0$, and integer $n \in \mathbb{N}$ satisfying $n-1 < \alpha < n$. Let $y, z, \phi \in \mathcal{C}^\infty([0,T] \times \bar{\Omega})$, $T > 0$, we have the two following Lemmas:

Lemma 3.2.1

For all $0 < \beta < 1$, *and, for any* $y, \phi \in \mathcal{C}^\infty([0,T] \times \bar{\Omega})$, *we have*

$$\begin{aligned} &-\iint_Q \mathcal{D}_C^\beta z(x,t)\phi(x,t)\mathrm{d}x\mathrm{d}t \\ &\quad = -\int_\Omega z(x,T)I^{1-\beta}\phi(x,T)\mathrm{d}x + \int_\Omega z(x,0)I^{1-\beta}\phi(x,0)\mathrm{d}x + \iint_Q D_{RL}^\beta\phi(x,t)z(x,t)\mathrm{d}x\mathrm{d}t, \end{aligned}$$

where $\mathcal{D}_C^\beta z$ *is the right fractional Caputo derivative for* z *which is defined by*

$$\mathcal{D}_C^\beta z(x,t) = \frac{1}{\Gamma(1-\beta)}\int_t^T (s-t)^{-\beta}\frac{\partial}{\partial s}z(x,s)\mathrm{d}s, \tag{3.1}$$

and $D_{RL}^\beta\phi$ *is the left Riemann-Liouville fractional derivative for* ϕ *which is given by*

$$D_{RL}^\beta\phi(x,t) = \frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial t}\int_0^t (t-s)^{-\beta}\phi(x,s)\mathrm{d}s. \tag{3.2}$$

Furthermore, the left Riemann-Liouville fractional integral $I^{1-\beta}$ *for* ϕ *is defined by*

$$I^{1-\beta}\phi(x,t) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\phi(x,s)\mathrm{d}s. \tag{3.3}$$

Proof. When we use the integration by parts method, we discover

$$\begin{aligned} -\iint_Q \mathcal{D}_C^\beta z(x,t)\phi(x,t)\mathrm{d}x\mathrm{d}t &= -\int_\Omega \Big[\int_0^T \phi(x,t)\Big(\frac{1}{\Gamma(1-\beta)}\int_t^T (s-t)^{-\beta}\frac{\partial}{\partial s}z(x,s)\mathrm{d}s\Big)\mathrm{d}t\Big]\mathrm{d}x \\ &= -\int_\Omega \Big[\int_0^T \frac{\partial}{\partial s}z(x,s)\Big(\frac{1}{\Gamma(1-\beta)}\int_0^t (s-t)^{-\beta}\phi(x,t)\mathrm{d}t\Big)\mathrm{d}s\Big]\mathrm{d}x \\ &= -\iint_Q \frac{\partial}{\partial t}z(x,t)I^{1-\beta}\phi(x,t)\mathrm{d}t\mathrm{d}x \\ &= -\int_\Omega z(x,T)I^{1-\beta}\phi(x,T)\mathrm{d}x + \int_\Omega z(x,0)I^{1-\beta}\phi(x,0)\mathrm{d}x \\ &\quad + \iint_Q D_{RL}^\beta\phi(x,t)z(x,t)\mathrm{d}x\mathrm{d}t. \end{aligned}$$

□

Lemma 3.2.2

For all $\frac{3}{2} < \alpha < 2$, and, for any $z, \phi \in \mathcal{C}^\infty([0,T] \times \bar{\Omega})$, we have

$$\begin{aligned}\iint_Q \mathcal{D}_C^\alpha y(x,t)\phi(x,t)\mathrm{d}x\mathrm{d}t = &\int_\Omega \frac{\partial}{\partial t} y(x,T) I^{2-\alpha}\phi(x,T)\mathrm{d}x - \int_\Omega \frac{\partial}{\partial t} y(x,0) I^{2-\alpha}\phi(x,0)\mathrm{d}x \\ &- \int_\Omega y(x,T)\frac{\partial}{\partial t} I^{2-\alpha}\phi(x,T)\mathrm{d}x + \int_\Omega y(x,0)\frac{\partial}{\partial t} I^{2-\alpha}\phi(x,0)\mathrm{d}x \\ &+ \iint_Q D_{RL}^\alpha \phi(x,t) y(x,t)\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{3.4}$$

where $\mathcal{D}_C^\alpha y$ is the right fractional Caputo derivative for y, which is defined by

$$\mathcal{D}_C^\alpha y(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_t^T (s-t)^{1-\alpha} \frac{\partial^2}{\partial s^2} y(x,s)\mathrm{d}s, \tag{3.5}$$

and $D_{RL}^\alpha \phi$ is the left Riemann-Liouville fractional derivative for ϕ which is given by

$$D_{RL}^\alpha \phi(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial t^2} \int_0^t (t-s)^{1-\alpha}\phi(x,s)\mathrm{d}s. \tag{3.6}$$

The left Riemann-Liouville fractional integral $I^{2-\alpha}$ for ϕ is represented by

$$I^{2-\alpha}\phi(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha}\phi(x,s)\mathrm{d}s. \tag{3.7}$$

Proof. Using integration by part, we arrive at

$$\begin{aligned}\iint_Q \mathcal{D}_C^\alpha y(x,t)\phi(x,t)\mathrm{d}x\mathrm{d}t &= \int_\Omega \Big[\int_0^T \phi(x,t)\Big(\frac{1}{\Gamma(2-\alpha)}\int_t^T (s-t)^{1-\alpha}\frac{\partial^2}{\partial s^2} y(x,s)\mathrm{d}s\Big)\mathrm{d}t\Big]\mathrm{d}x \\ &= \int_\Omega \Big[\int_0^T \frac{\partial^2}{\partial s^2} y(x,s)\Big(\frac{1}{\Gamma(2-\alpha)}\int_0^t (s-t)^{1-\alpha}\phi(x,t)\mathrm{d}t\Big)\mathrm{d}s\Big]\mathrm{d}x \\ &= \iint_Q \frac{\partial^2}{\partial t^2} y(x,t) I^{2-\alpha}\phi(x,t)\mathrm{d}t\mathrm{d}x,\end{aligned}$$

which yields (3.4). □

The identification problem for fractional coupled thermoelastic deformation systems with insufficient information will be discussed in the following section.

3.3 Setting the problem

Let be Ω a bounded open subset of $\mathbb{R}^3$, with a smooth boundary Γ of class $\mathcal{C}^2$, Γ_0 a non-empty subset of Γ. For all $T > 0$, we denote by $Q = \Omega \times (0,T), \Sigma = \Gamma \times (0,T)$ and $\Sigma_0 = \Gamma_0 \times (0,T)$. For all $1 < \alpha < 2$, $0 < \beta < 1$, we examine perturbed fractional coupled thermoelastic systems, which are defined as follows:

$$\left\{\begin{array}{lll} D_{RL}^\alpha y - \Delta y + \gamma \nabla z + y + z = 0 & \text{in} & Q, \\ D_{RL}^\beta z - \Delta z + \gamma(\mathrm{div} D_{RL}^\beta y) + y + z = 0 & & \\ I^{2-\alpha}y(0) = y^0 + \tau_0 \hat{y}^0,\ \frac{\partial}{\partial t} I^{2-\alpha} y(0) = y^1 + \tau_1 \hat{y}^1,\ I^{1-\beta}z(0) = z^0 + \tau_2 \hat{z}^0 & \text{in} & \Omega, \\ z = 0 & & \Sigma, \\ y = \xi + \lambda\hat{\xi} & \text{on} & \Sigma_0, \\ y = 0 & & \Sigma|\Sigma_0. \end{array}\right. \tag{3.8}$$

The relationship between thermal and mechanical factors in elastic body behavior is described using perturbed fractional coupled systems (3.8).

The state of the system (3.8) is represented by a vector $y = y(x,t) = (y_1, y_2, y_3)$, where y_1, y_2 and y_3 are the elastic displacements at the moment t. The scalar function $z = z(x,t)$ represents the temperature. The coefficient γ is the coupling positive parameter.
Furthermore, in the two equalities (3.2) and (3.6) the left Riemann-Liouville fractional derivatives expressions D^{β}_{RL} and D^{α}_{RL} are given. The left Riemann-Liouville fractional integrals for $I^{1-\beta}$ and $I^{2-\alpha}$ and are defined in (3.3) and (3.7).

The problem (3.1) is polluted in the following way:

$$y^0 \in (H^2(\Omega) \cap H^1_0(\Omega))^3,\ y^1 \in (L^2(\Omega))^3,\ z^0 \in L^2(\Omega),\ \xi_0 \in H^{\frac{1}{2}}(\Sigma_0) \text{ are known.} \tag{3.9}$$

Furthermore, the terms $\tau_0 \hat{y}^0, \tau_1 \hat{y}^1, \tau_2 \hat{z}^0$ and $\lambda \hat{\xi}$ are unknown.

We only have that

$$\begin{gathered} \|\hat{y}^0\|_{\left(H^1_0(\Omega)\right)^3} \le 1,\quad \|\hat{y}^1\|_{\left(L^2(\Omega)\right)^3} \le 1,\quad \|\hat{z}^0\|_{L^2(\Omega)} \le 1,\quad \|\hat{\xi}\|_{H^{\frac{1}{2}}(\Sigma_0)} \le 1, \\ \tau_0, \tau_1, \tau_2, \lambda \text{ are small enough in } \mathbb{R}. \end{gathered} \tag{3.10}$$

For the given scalars $\lambda, \tau_i, i \in \{0,1,2\}$, and from (3.9)-(3.10), the problem (3.8) has a unique solution denoted by $(y(x,t;\lambda,\tau), z(x,t;\lambda,\tau)) \in L^2(0,T; H^1_0(\Omega) \cap H^2(\Omega)) \times L^2(Q)$ where $\tau = (\tau_0, \tau_1, \tau_2)$. For more information see [51].

3.4 Application of the sentinel method

Consider a non-empty small domain assigned $\mathcal{O}$ such that

$$\mathcal{O} \subset \Omega \qquad \text{is an observatory domain.}$$

In this observatory domain, we measure the temperature z over the time interval $(0,T)$. Then we put $\mathcal{Q} = \mathcal{O} \times (0,T)$, and the observation of z in $\mathcal{Q}$ is denoted by

$$z_{obs} = m_0(x,t) \quad \text{a given function.}$$

The problem then becomes one of determining which pollution terms $\lambda \hat{\xi}$ are influenced by the initial data movements around y^0, y^1 and z^0.

This problem can be approached in two ways. The first is to use the least square classical method, [45]. The second method is the Sentinel Method, which was introduced by J. L. Lions in [51, 53], and is better suited to solving our problem.

The sentinel approach requires building a functional sentinel.

$$S(\lambda,\tau) = \iint_{\mathcal{Q}} (h_0 + v) z_{obs} \mathrm{d}x\ \mathrm{d}t, \tag{3.11}$$

where the given function $h_0 \in L^2(\mathcal{Q})$ and $v \in L^2(\mathcal{Q})$ is a control function. Then we look for $v \in L^2(\mathcal{Q})$, which must meet the following requirements.

$$\frac{\partial S}{\partial \tau_i}(0,0) = 0, \quad i = \{0,1,2\},\ \forall (y^0, y^1, z^0) \in (H^2(\Omega) \cap H^1_0(\Omega))^3 \times L^2(\Omega)^3 \times L^2(\Omega), \tag{3.12}$$

$$\|v\|_{L^2(\mathcal{Q})} = \text{minimum.} \tag{3.13}$$

If the function $S(\lambda,\tau)$ is differentiable at $(0,0)$, it has a linear approximation close to this point. As a result, the S function can be expressed as

$$S(\lambda,\tau) \simeq S(0,0) + \lambda \frac{\partial S}{\partial \lambda}(0,0). \tag{3.14}$$

The implications of condition (3.14) are taken into account (3.13). We denote by $(y_0, z_0) = (y(x,t;0,0), z(x,t;0,0))$ a calculable solution to

$$\begin{cases} D^{\alpha}_{RL} y_0 - \Delta y_0 + \gamma \nabla z_0 + y_0 + z_0 = 0 & \text{in} \quad Q, \\ D^{\beta}_{RL} z_0 - \Delta z_0 + \gamma (\mathrm{div} D^{\beta}_{RL} y_0) + y_0 + z_0 = 0 & \\ I^{2-\alpha} y_0(0) = y^0, \ \dfrac{\partial}{\partial t} I^{2-\alpha} y_0(0) = y^1, \ I^{1-\beta} z_0(0) = z^0 & \text{in} \quad \Omega, \\ z_0 = 0 & \Sigma, \\ y_0 = \xi & \text{on} \quad \Sigma_0, \\ y_0 = 0 & \Sigma | \Sigma_0. \end{cases} \tag{3.15}$$

Then, we deduce

$$S(0,0) = \iint_Q (h_0 + v) z_0 \mathrm{d}x\mathrm{d}t \ \text{ is given.} \tag{3.16}$$

The following information is sourced from the relationships (3.14),(3.15) and (3.16).

$$\lambda \frac{\partial S}{\partial \lambda}(0,0) \simeq \iint_Q (h_0 + v)(m_0 - z_0) \mathrm{d}x\mathrm{d}t. \tag{3.17}$$

The function S is differentiable at λ. The partial derivative of S at λ is given by

$$\frac{\partial S}{\partial \lambda}(0,0) = \iint_Q (h_0 + v) z_\lambda \mathrm{d}x\mathrm{d}t, \tag{3.18}$$

The partial derivatives with respect to z_λ and y_λ are defined as follows:

$$\begin{cases} D^{\alpha}_{RL} y_\lambda - \Delta y_\lambda + \gamma \nabla z_\lambda + y_\lambda + z_\lambda = 0 & \text{in} \quad Q, \\ D^{\beta}_{RL} z_\lambda - \Delta z_\lambda + \gamma (\mathrm{div} D^{\beta}_{RL} y_\lambda) + y_\lambda + z_\lambda = 0 & \\ I^{2-\alpha} y_\lambda(0) = 0, \ \dfrac{\partial}{\partial t} I^{2-\alpha} y_\lambda(0) = 0, \ I^{1-\beta} z_\lambda(0) = 0 & \text{in} \quad \Omega, \\ y_\lambda = \hat{\xi} & \Sigma_0, \\ y_\lambda = 0 & \text{on} \quad \Sigma | \Sigma_0, \\ z_\lambda = 0 & \Sigma. \end{cases} \tag{3.19}$$

The information provided by the sentinel function's is given by

$$\begin{aligned} \lambda \frac{\partial S}{\partial \lambda}(0,0) &= \iint_Q (h_0 + v)(m_0 - z_0) \mathrm{d}x\mathrm{d}t \\ &= \iint_Q (h_0 + v) \lambda z_\lambda \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.20}$$

We'll search for a function control $v \in L^2(Q)$ in the next section. This means that the sentinel problem can be simplified to a null controllability problem with a constraint on the control function.

3.5 Equivalence to a controllability problem

Finding a control function v comes back to transforming the condition of insensitivity (3.12) into a null controllability problem with a constraint on the control function. Right fractional Caputo derivatives are an excellent technique to introduce coupled adjoint state systems.

Proposition 3.5.1

The following null controllability problem is similar to the condition of insensitivity (3.12).

$$p(0;v) = 0, \quad \frac{\partial p}{\partial t}(0;v) = 0, \quad q(0;v) = 0 \quad \textit{in} \quad \Omega, \tag{3.21}$$

where $(p,q) = (p(x,t;v), q(x,t;v) \in (L^2(Q))^2$ is the solution of the following coupled systems of the adjoint state,

$$\begin{cases} \mathcal{D}_C^{\alpha} p - \Delta p + \gamma \nabla \mathcal{D}_C^{\beta} q + p + q = 0 & \\ -\mathcal{D}_C^{\beta} q - \Delta q - \gamma \text{ div } p + p + q = (h_0 + v)\chi_{\mathcal{O}} & \text{in} \quad Q, \\ p(T) = 0, \quad \dfrac{\partial p}{\partial t}(T) = 0, \quad q(T) = 0 & \text{in} \quad \Omega, \\ p = 0, \quad q = 0 & \text{on} \quad \Sigma, \end{cases} \tag{3.22}$$

where the right fractional Caputo derivatives $\mathcal{D}_C^{\beta}$ and $\mathcal{D}_C^{\alpha}$ are defined in equalities (3.1) and (3.5) in the Section3.2.

Proof. Suppose that the partial derivatives y_τ and z_τ have been computed for $\lambda = \tau = 0$. Then there's

$$\begin{cases} D_{RL}^{\alpha} y_\tau - \Delta y_\tau + \gamma \nabla z_\tau + y_\tau + z_\tau = 0 & \\ D_{RL}^{\beta} z_\tau - \Delta z_\tau + \gamma (\text{div} D_{RL}^{\beta} y_\tau) + y_\tau + z_\tau = 0 & \text{in} \quad Q, \\ I^{2-\alpha} y_\tau(0) = \hat{y}^0, \ \dfrac{\partial}{\partial t} I^{2-\alpha} y_\tau(0) = \hat{y}^1, \ I^{1-\beta} z_\tau(0) = \hat{z}^0 & \text{in} \quad \Omega, \\ y_\tau = 0, \quad z_\tau = 0 & \text{on} \quad \Sigma. \end{cases} \tag{3.23}$$

We can see from (3.12) that

$$\iint_Q (h_0 + v) z_\tau \mathrm{d}x\mathrm{d}t = 0, \tag{3.24}$$

which must be taken into account (3.13). We multiply the first equality in (3.22) by y_τ, and we multiply the second equation in (3.22) by z_τ. We get the following result using Green's formula (see Section(3.2)).

$$-\int_\Omega \frac{\partial}{\partial t} p(x,0) \hat{y}^0 \mathrm{d}x + \int_\Omega p(x,0) \hat{y}^1 \mathrm{d}x + \iint_Q (D_{RL}^{\alpha} y_\tau - \Delta y_\tau + y_\tau) p + (\gamma (\text{div } D_{RL}^{\beta}) y_\tau + y_\tau) q \mathrm{d}x\mathrm{d}t = 0,$$

$$\int_\Omega q(x,0) \hat{z}^0 \mathrm{d}x + \iint_Q (D_{RL}^{\beta} z_\tau - \Delta z_\tau + z_\tau) q + (\gamma \text{div} z_\tau + z_\tau) p \mathrm{d}x\mathrm{d}t = 0.$$

Putting the two last equations together, we get (3.21). □

We introduce the functions $\bar{p}, \bar{q}$ (respectively. φ, ψ) solutions of the coupled systems

$$\begin{cases} \mathcal{D}_C^{\alpha} \bar{p} - \Delta \bar{p} + \gamma \nabla \mathcal{D}_C^{\beta} \bar{q} + \bar{p} + \bar{q} = 0 & \\ -\mathcal{D}_C^{\beta} \bar{q} - \Delta \bar{q} - \gamma \ div \ \bar{p} + \bar{p} + \bar{q} = h_0 \chi_{\mathcal{O}} & \text{in} \quad Q, \\ \bar{p}(T) = 0, \ \dfrac{\partial \bar{p}}{\partial t}(T) = 0, \ \bar{q}(T) = 0 & \text{in} \quad \Omega, \\ \bar{p} = 0, \ \bar{q} = 0 & \text{on} \quad \Sigma, \end{cases} \tag{3.25}$$

respectively,

$$\begin{cases} \mathcal{D}_C^{\alpha} \varphi - \Delta \varphi + \gamma \nabla \mathcal{D}_C^{\beta} \psi + \varphi + \psi = 0 & \\ -\mathcal{D}_C^{\beta} \psi - \Delta \psi - \gamma \text{ div} \varphi + \varphi + \psi = v \chi_{\mathcal{O}} & \text{in} \quad Q, \\ \varphi(T) = 0, \ \dfrac{\partial \varphi}{\partial t}(T) = 0, \ \psi(T) = 0 & \text{in} \quad \Omega, \\ \varphi = 0, \ \psi = 0 & \text{on} \quad \Sigma. \end{cases} \tag{3.26}$$

Finally, we must solve a new control problem in which the new coupled states satisfy the requirements:

$$\begin{cases} \varphi(0) = -\bar{p}(0), \quad \dfrac{\partial \varphi}{\partial t}(0) = -\dfrac{\partial \bar{p}}{\partial t}(0), \quad \psi(0) = -\bar{q}(0), \\ \|v\|_{L^2(Q)} = \text{minimum}. \end{cases} \tag{3.27}$$

3.6 Optimal control problem and optimality coupled systems

It is possible to convert the optimization problem (3.27) with a constraint into an optimization problem without a constraint using the penalization technique. Further, set $\epsilon > 0$, we define the penalized cost functional $\mathcal{J}_\epsilon$ as follows:

$$\mathcal{J}_\epsilon(v,\varphi,\psi) = \frac{1}{2}\|v\|^2_{L^2(\mathcal{Q})} + \frac{1}{2\epsilon}\| - \mathcal{D}^\beta_C\psi - \Delta\psi - \gamma\ div\ \varphi + \varphi + \psi - v\chi_{\mathcal{O}}\|^2_{L^2(Q)}$$
$$+ \frac{1}{2\epsilon}\|\mathcal{D}^\alpha_C\varphi - \Delta\varphi + \gamma\nabla\mathcal{D}^\beta_C\psi + \varphi + \psi\|^2_{L^2(Q)},$$

where the coupled states (φ,ψ) satisfies

$$\begin{cases} \begin{matrix} -\mathcal{D}^\beta_C\psi - \Delta\psi - \gamma\ div\ \varphi + \varphi + \psi \in L^2(Q) \\ \mathcal{D}^\alpha_C\varphi - \Delta\varphi + \gamma\nabla\mathcal{D}^\beta_C\psi + \varphi + \psi = 0 \end{matrix}, & \text{in} \quad Q, \\ \varphi(T) = 0,\ \dfrac{\partial\varphi}{\partial t}(T) = 0, \varphi(0) = -\bar{p}(0),\ \psi(T) = 0\ , \psi(0) = -\bar{q}(0) & \text{in} \quad \Omega, \\ \varphi = 0, \quad \psi = 0 & \text{on} \quad \Sigma. \end{cases}$$

Let us now consider the following optimization problems:

$$\inf \mathcal{J}_\epsilon(v;\varphi,\psi), \quad (v;\varphi,\psi) \in L^2(\mathcal{Q}) \times L^2(Q) \times L^2(Q). \tag{3.28}$$

Theorem 3.6.1

The penalized optimization problems (3.28) admit a unique optimal pair control-state $\{v^\epsilon,(\varphi^\epsilon,\psi^\epsilon)\}$, which is characterized by

$$\begin{cases} \mathcal{D}^\alpha_C\varphi^\epsilon - \Delta\varphi^\epsilon + \gamma\nabla\mathcal{D}^\beta_C\psi^\epsilon + \varphi^\epsilon + \psi^\epsilon = 0 & in \quad Q, \\ -\mathcal{D}^\beta_C\psi^\epsilon - \Delta\psi^\epsilon - \gamma\ div\varphi^\epsilon + \varphi^\epsilon + \psi^\epsilon = \sigma^\epsilon\chi_{\mathcal{O}} & \\ \varphi^\epsilon(T) = 0,\ \dfrac{\partial\varphi^\epsilon}{\partial t}(T) = 0,\ \psi^\epsilon(T) = 0 & in \quad \Omega, \\ \varphi^\epsilon = 0,\ \psi^\epsilon = 0 & on \quad \Sigma, \end{cases} \tag{3.29}$$

where σ^ϵ is given by

$$\begin{cases} D^\alpha_{RL}\rho^\epsilon - \Delta\rho^\epsilon + \gamma\nabla\sigma^\epsilon + \rho^\epsilon + \sigma^\epsilon = 0 & in \quad Q, \\ D^\beta_{RL}\sigma^\epsilon - \Delta\sigma^\epsilon + \gamma(\text{div}\ D^\beta_{RL}\rho^\epsilon) + \rho^\epsilon + \sigma^\epsilon = 0 & \\ \rho^\epsilon = 0,\ \sigma^\epsilon = 0 & on \quad \Sigma, \end{cases} \tag{3.30}$$

and

$$v^\epsilon = \sigma^\epsilon \quad in \quad \mathcal{Q}, \tag{3.31}$$

without any information on $\rho^\epsilon(0)$, $\dfrac{\partial\rho^\epsilon}{\partial t}(0)$ and $\sigma^\epsilon(0)$.

Proof. The function $\mathcal{J}_\epsilon$ being strictly convex, differentiable and coercive, it has a unique minimum. The Euler-Lagrange optimality conditions for $\{v^\epsilon,(\varphi^\epsilon,\psi^\epsilon)\}$ are:

For all $\hat{v} \in L^2(\mathcal{Q})$

$$\lim_{h\to 0} \frac{J_\epsilon\Big(v^\epsilon + h(\hat{v} - v^\epsilon), \varphi^\epsilon, \psi^\epsilon\Big) - J_\epsilon\Big(v^\epsilon, \varphi^\epsilon, \psi^\epsilon\Big)}{h} = 0,$$

for all $\hat{\varphi} \in L^2(Q)$,

$$\lim_{h\to 0} \frac{J_\epsilon\Big(v^\epsilon, \varphi^\epsilon + h(\hat{\varphi} - \varphi^\epsilon), \psi^\epsilon\Big)\Big) - J_\epsilon\Big(v^\epsilon, \varphi^\epsilon, \psi^\epsilon\Big)}{h} = 0,$$

for all $\hat{\psi} \in L^2(Q)$,

$$\lim_{h\to 0} \frac{J_\epsilon\Big(v^\epsilon, \varphi^\epsilon, \psi^\epsilon + h(\hat{\psi} - \psi^\epsilon)\Big) - J_\epsilon\Big(v^\epsilon, \varphi^\epsilon, \psi^\epsilon\Big)}{h} = 0.$$

The adjoint state coupled systems are introduced by

$$\left| \begin{array}{l} \sigma^\epsilon = \dfrac{1}{\epsilon}\Big(-\mathcal{D}_C^\beta \psi^\epsilon - \Delta\psi^\epsilon - \gamma \ \mathrm{div}\ \varphi^\epsilon + \varphi^\epsilon + \psi^\epsilon - v^\epsilon \chi_{\mathcal{O}}\Big), \\ \rho^\epsilon = \dfrac{1}{\epsilon}\Big(\mathcal{D}_C^\alpha \varphi^\epsilon - \Delta\varphi^\epsilon + \gamma\nabla\mathcal{D}_C^\beta\psi^\epsilon + \varphi^\epsilon + \psi^\epsilon\Big), \end{array} \right.$$

where $(\varphi^\epsilon, \psi^\epsilon) = (\varphi^\epsilon(x,t), \psi^\epsilon(x,t))$. Then, $\forall \hat{v} \in L^2(\mathcal{Q})$ the unique solution $\{v^\epsilon, (\varphi^\epsilon, \psi^\epsilon)\}$ is characterized by

$$\langle \hat{v}, v^\epsilon\rangle_{L^2(\mathcal{Q})} + \langle \sigma^\epsilon, -\mathcal{D}_C^\beta \hat{\psi}^\epsilon - \Delta\hat{\psi}^\epsilon - \gamma \ \mathrm{div}\ \hat{\varphi}^\epsilon + \hat{\varphi}^\epsilon + \hat{\psi}^\epsilon - \hat{v}^\epsilon \chi_{\mathcal{O}}\rangle_{L^2(Q)} \\ + \langle \rho^\epsilon, \mathcal{D}_C^\alpha \hat{\varphi}^\epsilon - \Delta\hat{\varphi}^\epsilon + \gamma\nabla\mathcal{D}_C^\beta \hat{\psi}^\epsilon + \hat{\varphi}^\epsilon + \hat{\psi}^\epsilon\rangle_{L^2(Q)} = 0.$$

For all $\hat{v} \in L^2(\mathcal{Q})$ and for all $(\hat{\varphi}, \hat{\psi})$ verifies

$$\left\{ \begin{array}{lll} -\mathcal{D}_C^\beta \hat{\psi} - \Delta\hat{\psi} - \gamma \ \mathrm{div}\hat{\varphi} + \hat{\varphi} + \hat{\psi} - \hat{v}\chi_{\mathcal{O}} \in L^2(Q)), & & \\ \mathcal{D}_C^\alpha \hat{\varphi} - \Delta\hat{\varphi} + \gamma\nabla\mathcal{D}_C^\beta\hat{\psi} + \hat{\varphi} + \hat{\psi} = 0 & \text{in} & Q \\ \hat{\varphi}(T) = 0,\ \dfrac{\partial\hat{\varphi}}{\partial t}(T) = 0,\ \hat{\psi}^(T) = 0,\ \hat{\varphi}(0) = 0,\ \hat{\psi}(0) = 0 & \text{in} & \Omega, \\ \hat{\varphi} = 0,\ \hat{\psi} = 0 & \text{on} & \Sigma, \end{array} \right.$$

which yields (3.22), (3.30) and (3.31). □

We suppose that $\sigma^\epsilon \to \sigma$ in a suitable topology. We pass to the limit $\epsilon \to 0$ we obtain

$$\left\{ \begin{array}{lll} \mathcal{D}_C^\alpha \varphi - \Delta\varphi + \gamma\nabla\mathcal{D}_C^\beta\psi + \varphi + \psi = 0 & \text{in} & Q, \\ -\mathcal{D}_C^\beta\psi - \Delta\psi - \gamma \ \mathrm{div}\varphi + \varphi + \psi = \sigma\chi_{\mathcal{O}} & & \\ \varphi(T) = 0,\ \dfrac{\partial\varphi}{\partial t}(T) = 0,\ \psi(T) = 0 & \text{in} & \Omega, \\ \varphi = 0,\ \psi = 0 & \text{on} & \Sigma, \end{array} \right. \tag{3.32}$$

and

$$\left\{ \begin{array}{lll} D_{RL}^\alpha \rho - \Delta\rho + \gamma\nabla\sigma + \rho + \sigma = 0 & \text{in} & Q, \\ D_{RL}^\beta \sigma - \Delta\sigma + \gamma(\mathrm{div}D_{RL}^\beta)\rho) + \rho + \sigma = 0 & & \\ I^{2-\alpha}\rho(0) = \rho^0,\ \dfrac{\partial}{\partial t}I^{2-\alpha}\rho(0) = \rho^1,\ I^{1-\beta}\sigma(0) = \sigma^0 & \text{in} & \mathcal{O}, \\ \rho = 0,\ \sigma = 0 & \text{on} & \Sigma, \end{array} \right. \tag{3.33}$$

$$v = \sigma \quad \text{in} \quad \mathcal{Q}, \tag{3.34}$$

where $\rho^0,\ \rho^1,\ \sigma^0$ are unknown at present.

3.6.1 Calculation of $\rho^0,\ \rho^1,\ \sigma^0$

Assuming that $\rho^0,\ \rho^1,\ \sigma^0$ are sufficiently regular. For technical reasons, we define the following linear operator as

$$\Lambda\{\rho^0, \rho^1, \sigma^0\} = \{\varphi(0), \frac{\partial\varphi}{\partial t}(0), \psi(0)\}.$$

As a result of (3.27) we have

$$\Lambda\{\rho^0, \rho^1, \sigma^0\} = -\{\bar{p}(0), \frac{\partial\bar{p}}{\partial t}(0), \bar{q}(0)\}. \tag{3.35}$$

We multiply the first equality in (3.32) by ρ, then, the second equality in (3.32) by σ, and we apply Green's formula,

$$-\int_\Omega \frac{\partial}{\partial t}\varphi(x,0)\rho^0 \mathrm{d}x + \int_\Omega \varphi(x,0)\rho^1 \mathrm{d}x + \iint_Q (D^\alpha_{RL}\rho - \Delta\rho + \rho)\varphi + (\gamma(\mathrm{div}D^\beta_{RL}\rho) + \rho)\psi \mathrm{d}x\mathrm{d}t = 0, \tag{3.36}$$

$$\int_\Omega \psi(x,0)\sigma^0 \mathrm{d}x + \iint_Q (D^\beta_{RL}\sigma - \Delta\sigma + \sigma)\psi + (\gamma \mathrm{div}\sigma + \sigma)\varphi \mathrm{d}x dt = \iint_Q \sigma^2 \mathrm{d}x\mathrm{d}t. \tag{3.37}$$

When we add the last two equalities together, we get

$$-\int_\Omega \frac{\partial}{\partial t}\bar{p}(x,0)\rho^1 \mathrm{d}x + \int_\Omega \bar{p}(x,0)\rho^0 \mathrm{d}x + \int_\Omega \bar{q}(x,0)\sigma^0 \mathrm{d}x = \iint_Q h_0\sigma \mathrm{d}x\mathrm{d}t.$$

Then $-\int_\Omega \frac{\partial}{\partial t}\bar{p}(x,0)\rho^1 \mathrm{d}x + \int_\Omega \bar{p}(x,0)\rho^0 \mathrm{d}x + \int_\Omega \bar{q}(x,0)\sigma^0 \mathrm{d}x$ is a function of h_0.
Setting

$$Mh_0 = \{\bar{p}(0), \frac{\partial \bar{p}}{\partial t}(0), \bar{q}(0)\}, \tag{3.38}$$

where $M \in \mathcal{L}\Big(L^2(\mathcal{Q}), (L^2(\Omega))^3 \times L^2(\Omega))^3 \times L^2(\Omega)\Big)$. We can deduce from relations (3.35) and (3.38):

$$\Lambda\{\rho^0, \rho^1, \sigma^0\} = -Mh_0.$$

The adjoint operator of M is denoted by M^*, where

$$M^*\{\rho^0, \rho^1, \sigma^0\} = \sigma\chi_{\mathcal{O}}.$$

We now propose a Hilbert space F, that defines a semi norm on $L^2(\mathcal{Q})$. We multiply the first equality in (3.32) by ρ, then, the second equality in (3.32) by σ and we apply Green's formula,

$$-\int_\Omega \frac{\partial}{\partial t}\varphi(x,0)\rho^1 \mathrm{d}x + \int_\Omega \varphi(x,0)\rho^0 \mathrm{d}x + \iint_Q (D^\alpha\rho - \Delta\rho + \rho)\varphi + (\gamma(\mathrm{div}D^\beta\rho) + \rho)\psi \mathrm{d}x\mathrm{d}t = 0, \tag{3.39}$$

$$\int_\Omega \psi(x,0)\sigma^0 \mathrm{d}x + \iint_Q (D^\beta\sigma - \Delta\sigma + \sigma)\psi + (\gamma \mathrm{div}\sigma + \sigma)\varphi \mathrm{d}x\mathrm{d}t = \iint_Q \sigma^2 \mathrm{d}x\mathrm{d}t. \tag{3.40}$$

Combining (3.39) and (3.40) we find

$$\langle \Lambda\{\rho^0, \rho^1, \sigma^0\}, \{\rho^0, \rho^1, \sigma^0\}\rangle = \iint_Q \sigma^2 \mathrm{d}x\mathrm{d}t.$$

Then we introduce the norm

$$\|\{\rho^0, \rho^1, \sigma^0\}\|_{\mathcal{F}} = \Big(\iint_Q \sigma^2 \mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{2}}. \tag{3.41}$$

We define too $\mathcal{F}'$ the dual space of $\mathcal{F}$. So, the operator Λ is an isomorphism of $\mathcal{F}$ into $\mathcal{F}'$. As a consequence,

$$\sigma\chi_{\mathcal{O}} = -M^*\Lambda^{-1}Mh_0. \tag{3.42}$$

Finally, the wished-for sentinel function is provided by

$$S(\lambda, \tau) = \iint_Q (h_0 - M^*\Lambda^{-1}Mh_0)m_0 \mathrm{d}x\mathrm{d}t.$$

3.7 Identification of the pollution term $\lambda\hat{\xi}$

We multiply the first equality in (3.22) by y_λ and the second equality by z_λ. We apply Green's formula and we obtain

$$\iint_\Sigma \frac{\partial p}{\partial \nu}\lambda\hat{\xi}\mathrm{d}\Sigma_0 + \lambda \iint_Q (D^\alpha_{RL} y_\lambda - \Delta y_\lambda + y_\lambda)p + (\gamma \ div \ D^\beta_{RL} y_\lambda + y_\lambda)q\mathrm{d}x\mathrm{d}t = 0, \tag{3.43}$$

and

$$\iint_Q (D^\beta_{RL} z_\lambda - \Delta z_\lambda + z_\lambda)q + (\gamma \mathrm{d}iv z_\lambda + z_\lambda)p\mathrm{d}x\mathrm{d}t = \iint_Q (h_0 + v)\lambda z_\lambda \mathrm{d}x\mathrm{d}t. \tag{3.44}$$

As a result of equalities (3.43)-(3.44):

$$\iint_\Sigma \frac{\partial p}{\partial \nu}\lambda\hat{\xi}\mathrm{d}\Sigma_0 = \iint_Q (h_0 + v)\lambda z_\lambda \mathrm{d}x\mathrm{d}t.$$

The crucial information is provided by the desired sentinel, which depends on h_0.

$$\iint_\Sigma \frac{\partial p}{\partial \nu}\lambda\hat{\xi}\mathrm{d}\Sigma_0 \simeq \iint_Q (h_0 - M^*\Lambda^{-1}Mh_0)(m_0 - z_0)\mathrm{d}x\mathrm{d}t. \tag{3.45}$$

The pollution term $\lambda\hat{\xi}$ is then called stealthy if it makes the sentinel S to be unable to observe.

$$\iint_\Sigma \frac{\partial p}{\partial \nu}\lambda\hat{\xi}\mathrm{d}\Sigma_0 = 0.$$

The relation's numerical resolution (3.45) allows the transverse displacement of y to be valued in terms of the temperature difference z.

3.8 Conclusion

The study reported in this paper was about using the sentinel method to resolve a fractional thermoelastic deformation system with incomplete data. The problem considered was formulated by the Riemann-Liouville fractional derivatives. In the left Riemann-Liouville fractional integral meaning, the partial known initial conditions are defined. The goal was to use the sentinel approach to detect the pollution terms based on a particular temperature data z. The initial data moves had no effect on the information obtained. The null controllability problem is the principal tool used to resolve the sentinel problem. The right fractional Caputo derivatives of fractional order were the best way to introduce the coupled of adjoint states. The numerical value of the transverse displacement of y in relation to the temperature difference can be calculated using the assessment obtained in relation (3.45). The sentinel method given in this paper is a generalization of the sentinel method in the context of fractionally distributed problems.

Conclusion and Perspectives

The major purpose of this thesis was to look at the subject of optimal control for some distributed coupled systems with incomplete data as well as the detection of a fractional coupled system with partial knowledge. We've been interested in the subject of environmental pollution, specifically water pollution. We tried to select the best control that was independent of the missing data variation. The main technique was the use of Lions' concept of "no regret control" to regulate distributed systems with missing data. In the same approach, the deformation of composite materials has been discussed. This type of deformation isn't always simply mechanical. We looked into several results involving the Riemann-Liouville fractional derivatives as a sentinel of the fractional problem of coupled thermo-elasticity systems. For analyzing the interaction between thermal and mechanical effects in elastic bodies, we applied the sentinel approach to an inverse fractional coupled thermoelastic system. Furthermore, when introducing fractional coupled adjoint state systems, we preferred to use the right Caputo fractional derivative. The identification issue proposed in this paper with the Riemann-Liouville and Caputo fractional derivative senses might be considered a generalization of classical identification problems in the non-fractional situation.

Some of the previous results will be quantitatively verified in the future, and our studies on the optimal control for some distributed coupled systems with imperfect information and on the sentinel of a partially known fractional coupled system in a stochastic situation will be expanded. We'll also try to express these results numerically.

We are considering employing other fractional time derivatives, such as those of Caputo-Fabrizio or probably Atangana-Baleanu, in the near future. Finally, the stability of a fractional coupled system, the Bissel equation, and its applications in controlability theory are of interest to me.

Appendix

3.9 Demonstration of the Theorem 1.5.4

Let's prove the existence. We assume that the series defined in (1.37) is convergent in beforehand. By replacing v by ω_i in (1.35) and using the fact that (For more information see [7, 8])

$$a(z(t), \omega_i) = \lambda_i (z(t), \omega_i,)_{L^2(\Omega)} = \lambda_i z_i. \tag{3.46}$$

From (1.35)-(1.36), we may deduce that y_i is a solution of the ordinary differential equation :

$$\begin{cases} D^{\beta}_{RL} z_i(t) + \lambda_i z_i(t) = f_i(t), \quad \forall t \in (0,T), \\ I^{1-\beta} z_i(0) = z_i^0. \end{cases} \tag{3.47}$$

Let's solve the last (3.47) problem using the Laplace transform. We can get the following equation

$$\mathcal{L}\{D^{\beta}_{RL} z_i(t)\} + \lambda_i \mathcal{L}\{z_i(t)\} = \mathcal{L}\{f_i(t)\}. \tag{3.48}$$

We have

$$\mathcal{L}[D^{\beta}_{RL} z_i(t)] = s^{\beta}\mathcal{L}(z_i(t)) - \lim_{t\to 0} I^{1-\beta} z_i(t).$$

As a result, when (3.48) is added, we get

$$s^{\beta}\mathcal{L}(z_i(t)) - z_i^0 + \lambda_i \mathcal{L}\{z_i(t)\} = \mathcal{L}\{f_i(t)\}.$$

It denotes

$$\mathcal{L}(z_i(t)) = \frac{z_i^0}{s^{\beta} + \lambda_i} + \frac{\mathcal{L}\{f_i(t)\}}{s^{\beta} + \lambda_i}.$$

We know this because of Lemma 1.5.3:

$$\mathcal{L}[\frac{1}{s^{\beta} + \lambda_i}] = t^{\beta-1} E_{\beta,\beta}(-\lambda_i t^{\beta}),$$

hence

$$z(t) = t^{\beta-1} E_{\beta,\beta}(-\lambda_i t^{\beta}) z_i^0 = \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(-\lambda_i (t-s)^{\beta}\big) f_i(s) \mathrm{d}s.$$

Otherwise; we have $z(t) = \sum_{i=1}^{\infty} z_i(t)\omega_i$.

Uniqueness of solution

- *To begin, we show that there is a solution to the approximate problem of (1.33)- (1.34).* Let V_m be a subspace of $H_0^1(\Omega)$ generated by $\{\omega_k\}_{k=1}^m$. Consider the following approximate problem associated with (1.33)- (1.34):

  We are looking for $z_m : t \in [0,T] \to z_m(t) \in V_m$ solution of

$$D^{\beta}_{RL} (z_m(t), v)_{L^2(\Omega)} + a(z_m(t), v) = (f(t), v)_{L^2(\Omega)} \quad \forall v \in V_m, \tag{3.49}$$

$$I^{1-\beta} z_m(0) = z_m^0 = \sum_{i=1}^{m} z_i^0 \omega_i. \tag{3.50}$$

Since $z_m(t) \in V_m$, we have

$$z_m(t) = \sum_{i=1}^{m} (z(t), \omega_i)_{L^2(\Omega)} \omega_i = \sum_{i=1}^{m} z_i(t)\omega_i.$$

We check that the function z_m is the solution to (3.49)-(3.50) and that it is given by,

$$z_m(t) = \sum_{i=1}^{m} \Big[t^{\beta-1} E_{\beta,\beta}(-\lambda_i t^\beta) z_i^0 = \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(-\lambda_i (t-s)^\beta\big) f_i(s) \mathrm{d}s \Big] \omega_i. \tag{3.51}$$

- *We show that the sequences $(z_m) \in L^2(0,T; H_0^1(\Omega))$ and $(I^{1-\beta} z_m) \in \mathcal{C}(0,T; H_0^1(\Omega))$ are Cauchy.* Let m and p be two integers such that $p > m \geq 1$. We then have

$$z_p(t) - z_m(t) = \sum_{i=m+1}^{p} z_i(t)\omega_i.$$

Thus,

$$\begin{aligned} a(z_p(t) - z_m(t), z_p(t) - z_m(t)) &= \sum_{i=m+1}^{p} \lambda_i (z_i)^2. \\ &\leq 2 \sum_{i=m+1}^{p} \lambda_i t^{2\beta-2} E_{\beta,\beta}^2(-\lambda_i t^\beta) |z_i^0|^2 \\ &\quad + 2 \sum_{i=m+1}^{p} \lambda_i \Big(\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(-\lambda_i (t-s)^\beta\big) f_i(s) \mathrm{d}s \Big)^2. \end{aligned}$$

whence

$$\begin{aligned} \|z_p(t) - z_m(t)\|^2_{L^2(0,T;H_0^1(\Omega))} &= \int_0^T a(z_p(t) - z_m(t), z_p(t) - z_m(t)) \mathrm{d}t \\ &\leq 2 \sum_{i=m+1}^{p} \lambda_i |z_i^0|^2 \int_0^T t^{2\beta-2} E_{\beta,\beta}^2(-\lambda_i t^\beta) \mathrm{d}t \\ &\quad + 2 \sum_{i=m+1}^{p} \int_0^T \lambda_i \Big(\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(-\lambda_i (t-s)^\beta\big) f_i(s) \mathrm{d}s \Big)^2 \mathrm{d}t \\ &\leq A_p + B_p \\ &\leq \frac{2C^2 T^{2\beta-1}}{2\beta - 1} \sum_{i=m+1}^{p} \lambda_i |z_i^0|^2 + \frac{2C^2 T}{\lambda_i \beta^2} \sum_{i=m+1}^{p} \Big(\int_0^t |f_i(s)|^2 \mathrm{d}s \Big). \end{aligned} \tag{3.52}$$

So, we get

$$\|z_p(t) - z_m(t)\|_{L^2(0,T;H_0^1(\Omega))} \leq C \sqrt{\frac{2T^{2\beta-1}}{2\beta-1}} \Big(\sum_{i=m+1}^{p} \lambda_i |z_i^0|^2 \Big)^{\frac{1}{2}} + \frac{C}{\beta} \sqrt{\frac{2T}{\lambda_i}} \Big[\sum_{i=m+1}^{p} \Big(\int_0^t |f_i(s)|^2 \mathrm{d}s \Big) \Big]^{\frac{1}{2}}. \tag{3.53}$$

Remark 3.9.1

- *We have $(-\lambda_i t^\beta) > 0$ then $\arg(-\lambda_i t^\beta) = \pi$.*

– *From Theorem (1.5.1), for all* $\frac{1}{2} < \beta < 1$ *we have*

$$\begin{aligned} A_p &= 2 \sum_{i=m+1}^{p} \lambda_i |z_i^0|^2 \int_0^T t^{2\beta-2} E^2_{\beta,\beta}(-\lambda_i t^\beta) \mathrm{d}t \\ &\leq 2C^2 \sum_{i=m+1}^{p} \lambda_i |z_i^0|^2 \int_0^T t^{2\beta-2} \mathrm{d}t \\ &= \frac{2C^2 T^{2\beta-1}}{2\beta-1} \sum_{i=m+1}^{p} \lambda_i |z_i^0|^2. \end{aligned}$$

– *By the Cauchy-Schwarz inequity, we have*

$$\begin{aligned} B_p &= 2 \sum_{i=m+1}^{p} \int_0^T \lambda_i \Big(\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\big(-\lambda_i (t-s)^\beta\big) f_i(s) \mathrm{d}s \Big)^2 \mathrm{d}t \\ &\leq 2 \sum_{i=m+1}^{p} \int_0^T \{\lambda_i \Big(\int_0^t \big[(t-s)^{\beta-1} E_{\beta,\beta}(-\lambda_i (t-s)^\beta)\big]^2 \mathrm{d}s \Big) \Big(\int_0^t |f_i(s)|^2 \mathrm{d}s \Big)\} \mathrm{d}t. \end{aligned} \tag{3.54}$$

Now, if we set $Z = t - s$*, we get*

$$\begin{aligned} \int_0^t \Big[(t-s)^{\beta-1} E_{\beta,\beta}(-\lambda_i (t-s)^\beta)\Big]^2 \mathrm{d}s &= \int_0^t \Big[Z^{\beta-1} E_{\beta,\beta}(-\lambda_i Z^\beta)\Big]^2 \mathrm{d}Z \\ &= \int_0^t \Big[-\frac{1}{\lambda_i} \frac{\mathrm{d}}{\mathrm{d}Z} E_{\beta,\beta}(-\lambda_i Z^\beta)\Big]^2 \mathrm{d}Z \\ &= \frac{1}{\lambda_i^2} \int_0^t \Big[\varsigma^2_{\beta,\beta+1}(-\lambda_i Z^\beta)\Big]^2 \mathrm{d}Z \\ &= \frac{1}{\lambda_i^2} \int_0^t \Big[\frac{1}{\beta} E_{\beta,\beta}(-\lambda_i Z^\beta)\Big]^2 \mathrm{d}Z \\ &= \frac{1}{\lambda_i^2} \frac{1}{\beta^2} \int_0^t E^2_{\beta,\beta}(-\lambda_i Z^\beta) \mathrm{d}Z. \end{aligned}$$

As a result of (3.54), we have

$$\begin{aligned} B_p &\leq 2 \sum_{i=m+1}^{p} \frac{\lambda_i}{\lambda_i^2 \beta^2} \int_0^T \Big\{ \Big(\int_0^t E^2_{\beta,\beta}(-\lambda_i Z^\beta) \mathrm{d}Z]^2 \mathrm{d}s \Big) \Big(\int_0^t |f_i(s)|^2 \mathrm{d}s \Big) \Big\} \mathrm{d}t \\ &\leq \frac{2C^2 T}{\lambda_i \beta^2} \sum_{i=m+1}^{p} \Big(\int_0^t |f_i(s)|^2 \mathrm{d}s \Big). \end{aligned}$$

On the other hand, we have

$$I^{1-\beta}(z_p(t) - z_m(t)) = C_p(t) + D_p(t),$$

where

$$
\begin{aligned}
C_p(t) &= \frac{1}{\Gamma(1-\beta)} \sum_{i=m+1}^{p} \Big\{ z_i^0 \int_0^t (t-s)^{-\beta} s^{\beta-1} E_{\beta,\beta}(-\lambda_i s^\beta) \mathrm{d}s \Big\} \omega_i \\
&= \frac{1}{\Gamma(1-\beta)} \sum_{i=m+1}^{p} \Big\{ z_i^0 \sum_{k=0}^{\infty} \frac{(-\lambda_i)^k}{\Gamma(\beta k+\beta)} \int_0^t (t-s)^{-\beta} s^{\beta k+\beta-1} \mathrm{d}s \Big\} \omega_i \\
&= \frac{1}{\Gamma(1-\beta)} \sum_{i=m+1}^{p} \Big\{ z_i^0 \sum_{k=0}^{\infty} \frac{(-\lambda_i)^k}{\Gamma(\beta k+\beta)} t^{\beta k} \frac{\Gamma(1-\beta)\Gamma(\beta k+\beta)}{\Gamma(1+k\beta)} \Big\} \omega_i \qquad (3.55) \\
&= \sum_{i=m+1}^{p} \Big\{ z_i^0 \sum_{k=0}^{\infty} \frac{(-\lambda_i)^k t^{\beta k}}{\Gamma(\beta k+1)} \Big\} \omega_i \\
&= \sum_{i=m+1}^{p} \Big\{ z_i^0 E_{\beta,1}(-\lambda_i t^\beta) \Big\} \omega_i .
\end{aligned}
$$

Remark 3.9.2

Using the Beta function's definition, we get

$$
\begin{aligned}
\int_0^t (t-s)^{-\beta} s^{\beta k+\beta-1} \mathrm{d}s &= \int_0^t t^{-\beta} (1-\frac{s}{t})^{-\beta} (\frac{s}{t})^{\beta k+\beta-1} t^{\beta k+\beta-1} \mathrm{d}s \\
&= t^{\beta k-1} \int_0^t (1-\frac{s}{t})^{-\beta} (\frac{s}{t})^{\beta k+\beta-1} \mathrm{d}s \\
&= t^{\beta k-1} \int_0^1 (1-u)^{-\beta} (u)^{\beta k+\beta-1} t \mathrm{d}u \\
&= t^{\beta k} \int_0^1 (1-u)^{-\beta} (u)^{\beta k+\beta-1} \mathrm{d}u \\
&= t^{\beta k} B(1-\beta, \beta k+\beta) \\
&= t^{\beta k} \frac{\Gamma(1-\beta)\Gamma(\beta k+\beta)}{\Gamma(1+k\beta)} .
\end{aligned}
$$

We have too

$$
\begin{aligned}
Z_p(t) &= \frac{1}{\Gamma(1-\beta)} \sum_{i=m+1}^{p} \Big\{ \int_0^t (t-s)^{-\beta} \Big[\int_0^s (s-\tau)^{\beta-1} E_{\beta,\beta}(-\lambda_i (s-\tau)^\beta) f_i(\tau) \mathrm{d}\tau \Big] \mathrm{d}s \Big\} \omega_i \\
&= \frac{1}{\Gamma(1-\beta)} \sum_{i=m+1}^{p} \Big\{ \int_0^t f_i(\tau) \Big[\int_\tau^t (t-s)^{-\beta} (s-\tau)^{\beta-1} E_{\beta,\beta}(-\lambda_i (s-\tau)^\beta) \mathrm{d}s \Big] \mathrm{d}\tau \Big\} \omega_i \\
&= \frac{1}{\Gamma(1-\beta)} \sum_{i=m+1}^{p} \Big\{ \int_0^t f_i(\tau) \Big[\sum_{k=0}^{\infty} \frac{(-\lambda_i)^k}{\Gamma(\beta k+\beta)} \int_\tau^t (t-s)^{-\beta} (s-\tau)^{\beta k+\beta-1} \mathrm{d}s \Big] \mathrm{d}\tau \Big\} \omega_i \qquad (3.56) \\
&= \frac{1}{\Gamma(1-\beta)} \sum_{i=m+1}^{p} \Big\{ \int_0^t f_i(\tau) \sum_{k=0}^{\infty} \frac{(-\lambda_i)^k}{\Gamma(\beta k+\beta)} \int_\tau^t (t-\tau)^{\beta k} \frac{\Gamma(1-\beta)\Gamma(\beta k+\beta)}{\Gamma(1+k\beta)} \mathrm{d}\tau \Big\} \omega_i \\
&= \sum_{i=m+1}^{p} \Big\{ \int_0^t f_i(\tau) E_{\beta,1}\big(-\lambda_i (t-\tau)^\beta\big) \mathrm{d}\tau \Big\} \omega_i .
\end{aligned}
$$

Using the Beta function's definition, we get

$$\begin{aligned}\int_\tau^t (t-s)^{-\beta}(s-\tau)^{\beta k+\beta-1}\mathrm{d}s &= \int_\tau^t [(t-s)-(s-\tau)]^{-\beta}\Big[\frac{s-\tau}{t-\tau}\Big]^{\beta k+\beta-1}(t-\tau)^{\beta k+\beta-1}\mathrm{d}s\\ &= \int_\tau^t (t-\tau)^{-\beta}\Big[1-\frac{s-\tau}{t-\tau}\Big]^{-\beta}\Big[\frac{s-\tau}{t-\tau}\Big]^{\beta k+\beta-1}(t-\tau)^{\beta k+\beta-1}\mathrm{d}s\\ &= \int_\tau^t (t-\tau)^{\beta k-1}\Big[1-\frac{s-\tau}{t-\tau}\Big]^{-\beta}\Big[\frac{s-\tau}{t-\tau}\Big]^{\beta k+\beta-1}\mathrm{d}s\\ &= (t-\tau)^{\beta k-1}\int_0^1 (1-u)^{-\beta}\, u^{\beta k+\beta-1}(t-\tau)\mathrm{d}u\\ &= (t-\tau)^{\beta k}B(1-\beta,\beta k+\beta)\\ &= (t-\tau)^{\beta k}\frac{\Gamma(1-\beta)\Gamma(\beta k+\beta)}{\Gamma(1+k\beta)}.\end{aligned}$$

Combining (3.55) and (3.56) , we get

$$I^{1-\beta}(z_p(t)-z_m(t)) = \sum_{i=m+1}^{p}\Big\{z_i^0 E_{\beta,1}(-\lambda_i t^\beta)\Big\}\omega_i + \sum_{i=m+1}^{p}\Big\{\int_0^t f_i(\tau)E_{\beta,1}\big(-\lambda_i(t-\tau)^\beta\big)\mathrm{d}\tau\Big\}\omega_i.$$

We can write the following relation using Theorem 1.5.1, and the Cauchy-Schwartz inequality.

$$\begin{aligned}\|I^{1-\beta}(z_p(t)-z_m(t))\|^2_{H_0^1(\Omega)} &= a(I^{1-\beta}(z_p(t)-z_m(t)), I^{1-\beta}(z_p(t)-z_m(t)))\\ &\le 2\sum_{i=m+1}^{p}\lambda_i|z_i^0|^2E^2_{\beta,1}(-\lambda_i t^\beta) + 2\sum_{i=m+1}^{p}\lambda_i\Big(\int_0^t f_i(\tau)E_{\beta,1}(-\lambda_i(t-\tau)^\beta)\mathrm{d}\tau\Big)^2\\ &\le 2C^2\sum_{i=m+1}^{p}\lambda_i|z_i^0|^2 + 2C^2\sum_{i=m+1}^{p}\Big[\int_0^t |f_i(\tau)|^2\mathrm{d}\Big]\Big[\int_0^t (t-\tau)^{-\beta}\mathrm{d}\tau\tau\Big]\\ &= 2C^2\sum_{i=m+1}^{p}\lambda_i|z_i^0|^2 + \frac{2C^2t^{1-\beta}}{1-\beta}\sum_{i=m+1}^{p}\Big[\int_0^t |f_i(\tau)|^2\mathrm{d}\tau\Big].\end{aligned}$$

As a result,

$$\sup_{t\in[0,T]}\|I^{1-\beta}(z_p(t)-z_m(t))\|^2_{H_0^1(\Omega)} \le \sqrt{2}C\Big(\sum_{i=m+1}^{p}\lambda_i|z_i^0|^2\Big)^{\frac12} + C\sqrt{\frac{2t^{1-\beta}}{1-\beta}}\Big(\sum_{i=m+1}^{p}\int_0^T |f_i(\tau)|^2\mathrm{d}\tau\Big)^{\frac12}.$$

Since $z^0 \in H_0^1(\Omega)$ and $f \in L^2(Q)$, we deduce that

$$\lim_{m,p\to+\infty}\Big(\sum_{i=m+1}^{p}\int_0^T |f_i(\tau)|^2\mathrm{d}\tau\Big)^{\frac12} = 0, \quad \lim_{m,p\to+\infty}\Big(\sum_{i=m+1}^{p}\lambda_i|z_i^0|^2\Big)^{\frac12} = 0. \qquad (3.57)$$

So from (3.53) and (3.57), we have,

$$\lim_{m,p\to+\infty}\int_0^T \|z_p(t)-z_m(t)\|^2_{H_0^1(\Omega)}\mathrm{d}t = 0, \quad \sup_{t\in[0,T]}\|I^{1-\beta}(z_p(t)-z_m(t))\|^2_{H_0^1(\Omega)} = 0.$$

This indicates that the sequences (z_m) and $(I^{1-\beta}z_m)$ are both Cauchy respectively in $L^2(0,T;H_0^1(\Omega))$ and $\mathcal{C}(0,T;H_0^1(\Omega))$. Then, we have

$$z_m \to y \;\text{ in }\; L^2(0,T;H_0^1(\Omega)), \quad I^{1-\beta}z_m \to I^{1-\beta}z \;\text{ in }\; \mathcal{C}(0,T;H_0^1(\Omega)) \qquad (3.58)$$

- *We demonstrate that z is the solution to the problem (1.31)-(1.36).* Let $\mathbb{D}(0,T)$ be the space of functions ∞ in $(0,T)$ with compact support, and let $\varphi \in \mathbb{D}(0,T)$. Let also $\mu \geq 1$ be an integer. Then by (3.49), we have for all $m \geq \mu$,

$$\int_0^T (f(t),v)_{L^2(\Omega)}\varphi(t)\mathrm{d}t = \int_0^T D^{\beta}_{RL}(z_m(t),v)_{L^2(\Omega)}\varphi(t)\mathrm{d}t + \int_0^T a(z_m(t),v)\varphi(t)\mathrm{d}t \quad \forall v \in V_\mu,$$
$$= -\int_0^T (z_m(t),v)_{L^2(\Omega)}\mathcal{C}^{\beta}_{C}\varphi(t)\mathrm{d}t + \int_0^T a(z_m(t),v)\varphi(t)\mathrm{d}t \quad \forall v \in V_\mu,$$
$$= -\int_0^T (z(t),v)_{L^2(\Omega)}\mathcal{C}^{\beta}_{C}\varphi(t)\mathrm{d}t + \int_0^T a(z(t),v)\varphi(t)\mathrm{d}t \quad \forall v \in V_\mu.$$

As $U_{\mu\geq 1} \subset V_\mu$ is dense in $H^1_0(\Omega)$ because (ω_i) is a basis of $H^1_0(\Omega)$. We obtain for all $v \in H^1_0(\Omega)$

$$\int_0^T (f(t),v)_{L^2(\Omega)}\varphi(t)\mathrm{d}t = -\int_0^T (z(t),v)_{L^2(\Omega)}\mathcal{C}^{\beta}_{C}\varphi(t)\mathrm{d}t + \int_0^T a(z(t),v)\varphi(t)\mathrm{d}t \quad \forall v \in H^1_0(\Omega),$$
$$= \int_0^T D^{\beta}_{RL}(z(t),v)_{L^2(\Omega)}\varphi(t)\mathrm{d}t + \int_0^T a(z(t),v)\varphi(t)\mathrm{d}t \quad \forall v \in H^1_0(\Omega).$$

Which means that for all $v \in H^1_0(\Omega)$, we have

$$(f(t),v)_{L^2(\Omega)}\varphi(t) = D^{\beta}_{RL}(z(t),v)_{L^2(\Omega)}\varphi(t) + a(z(t),v)\varphi(t) \quad \forall t \in (0,T).$$

From (3.58), we deduce that

$$I^{1-\beta}z_m(0) \to I^{1-\beta}z(0) \ \text{ in } \ H^1_0(\Omega), \quad I^{1-\beta}z_m(0) = \sum_{i=1}^{m} z_i^0\omega_i \to \sum_{i=1}^{\infty} z_i^0\omega_i = z^0.$$

So, we get

$$I^{1-\beta}z(0) = z^0.$$

- *Let us demonstrate the relations (1.38) and (1.39).* If z is the solution to problem (1.33)-(1.36) , then z is ginen by (1.37). Then, using the results of the previous calculations, we demonstrate that we have

$$\|z(t)\|_{L^2(0,T;H^1_0(\Omega))} \leq C\sqrt{\frac{2T^{2\beta-1}}{2\beta-1}}\Big(\sum_{i=m+1}^{p}\lambda_i|z_i^0|^2\Big)^{\frac{1}{2}} + \frac{C}{\beta}\sqrt{\frac{2T}{\lambda_i}}\Big[\sum_{i=m+1}^{p}\Big(\int_0^t |f_i(s)|^2\mathrm{d}s\Big)\Big]^{\frac{1}{2}}.$$

We also have,

$$\sup_{t\in[0,T]}\|I^{1-\beta}z(t))\|_{H^1_0(\Omega)} \leq \sqrt{2}C\Big(\sum_{i=m+1}^{p}\lambda_i|z_i^0|^2\Big)^{\frac{1}{2}} + C\sqrt{\frac{2t^{1-\beta}}{1-\beta}}\Big(\sum_{i=m+1}^{p}\int_0^T |f_i(\tau)|^2\mathrm{d}\tau\Big)\Big)^{\frac{1}{2}}.$$

3.10 Demonstration of the Theorem 1.5.5

Existence of solution Let's prove the existence. We assume that the series defined in (1.47) is convergent in beforehand. By replacing v by ω_i in (1.45) and using the fact that (For more information see [7, 8])

$$a(y(t),\omega_i) = \lambda_i(y(t),\omega_i,)_{L^2(\Omega)} = \lambda_i y_i.$$

From (1.45)-(1.46), we may deduce that y_i is a solution of the ordinary differential equation :

$$\begin{cases} D^{\alpha}_{RL}y_i(t) + \lambda_i y_i(t) = f_i(t), \quad \forall t \in (0,T), \\ I^{2-\alpha}y_i(0) = y_i^0, \\ \frac{\partial}{\partial t}I^{2-\alpha}y_i(0) = y_i^1. \end{cases} \tag{3.59}$$

Let's solve the last (3.59) problem using the Laplace transform. We can get the following equation

$$\mathcal{L}\{D^{\alpha}_{RL}y_i(t)\} + \lambda_i \mathcal{L}\{y_i(t)\} = \mathcal{L}\{f_i(t)\}. \tag{3.60}$$

We have

$$\mathcal{L}[D^{\alpha}_{RL}f(t)] = s^{\alpha}F(s) - s\lim_{t\to 0} I^{2-\alpha}f(t) - \lim_{t\to 0}\frac{\mathrm{d}}{\mathrm{d}t}I^{2-\alpha}f(t).$$

As a result, when (3.60) is added, we get

$$\mathcal{L}(y_i(t)) = \frac{sy_i^0}{s^{\alpha}+\lambda_i} + \frac{sy_i^1}{s^{\alpha}+\lambda_i} + \frac{\mathcal{L}\{f_i(t)\}}{s^{\alpha}+\lambda_i},$$

hence,

$$y_i(t) = \sum_{i=1}^{\infty}\Big[t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_i t^{\alpha})y_i^0 + t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^{\alpha})y_i^1 + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_i(t-s)^{\alpha}\big)f_i(s)\mathrm{d}s.$$

Otherwise; we have $y(t) = \sum_{i=1}^{\infty} y_i(t)\omega_i$, we deduce (1.47).

Uniqueness of solution

- *To begin, we show that there is a solution to the approximate problem of (1.33)- (1.34).* Let V_m be a subspace of $H_0^1(\Omega)$ generated by $\{\omega_k\}_{k=1}^m$. Consider the following approximate problem associated with (1.42)- (1.46):

  We are looking for $y_m : t \in [0,T] \to y_m(t) \in V_m$ solution of

$$D^{\alpha}_{RL}\big(y_m(t), v\big)_{L^2(\Omega)} + a(y_m(t), v) = \big(f(t), v\big)_{L^2(\Omega)} \quad \forall v \in V_m, \tag{3.61}$$

$$I^{2-\alpha}y_m(0) = y_m^0 = \sum_{i=1}^m y_i^0\omega_i, \quad \frac{\partial}{\partial t}I^{2-\alpha}y_m(0) = y_m^1 = \sum_{i=1}^m y_i^1\omega_i. \tag{3.62}$$

  Since $y_m(t) \in V_m$, we have

$$y_m(t) = \sum_{i=1}^m (y(t),\omega_i)_{L^2(\Omega)}\omega_i = \sum_{i=1}^m y_i(t)\omega_i.$$

  We check that the function y_m is the solution to (1.42)- (1.46) and that it is given by (1.47).

- *We show that the sequences* $(y_m) \in L^2(0,T; H_0^1(\Omega))$, $(I^{2-\alpha}y_m) \in \mathcal{C}(0,T;H_0^1(\Omega))$ *and* $(\frac{\partial}{\partial t}I^{2-\alpha}y_m) \in \mathcal{C}(0,T;L^2(\Omega))$ *are Cauchy.* Let m and p be two integers such that $p > m \geq 1$. We then have

$$y_p(t) - y_m(t) = \sum_{i=m+1}^{p} y_i(t)\omega_i.$$

  Thus,

$$\begin{aligned}
a(y_p(t)-y_m(t), y_p(t)-y_m(t)) &= \sum_{i=m+1}^{p}\lambda_i(y_i(t))^2. \\
&\leq 2\sum_{i=m+1}^{p}\lambda_i t^{2\alpha-4}E^2_{\alpha,\alpha-1}(-\lambda_i t^{\alpha})|y_i^0|^2 \\
&\quad + 2\sum_{i=m+1}^{p}\lambda_i t^{2\alpha-2}E^2_{\alpha,\alpha}(-\lambda_i t^{\alpha})|y_i^1|^2 \\
&\quad + 2\sum_{i=m+1}^{p}\lambda_i\Big(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_i(t-s)^{\alpha}\big)f_i(s)\mathrm{d}s\Big)^2.
\end{aligned}$$

whence

$$\begin{aligned}\|y_p(t)-y_m(t)\|^2_{L^2(0,T;H^1_0(\Omega))} &= \int_0^T a(y_p(t)-y_m(t),y_p(t)-y_m(t))\mathrm{d}t\\ &\leq 2\sum_{i=m+1}^{p}\lambda_i|y_i^0|^2\int_0^T t^{2\alpha-4}E^2_{\alpha,\alpha-1}(-\lambda_i t^\alpha)\mathrm{d}t\\ &\quad+2\sum_{i=m+1}^{p}\lambda_i|y_i^1|^2\int_0^T t^{2\alpha-2}E^2_{\alpha,\alpha}(-\lambda_i t^\alpha)\mathrm{d}t\\ &\quad+2\sum_{i=m+1}^{p}\int_0^T\lambda_i\Big(\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_i(t-s)^\alpha\big)f_i(s)\mathrm{d}s\Big)^2\mathrm{d}t\\ &\leq A_p+B_p+C_p.\end{aligned}\tag{3.63}$$

So, from Remark 3.10.1 we get

$$\begin{aligned}\|y_p(t)-y_m(t)\|_{L^2(0,T;H^1_0(\Omega))} \leq \frac{2C^2T^{2\alpha-3}}{2\alpha-3}\sum_{i=m+1}^{p}\lambda_i|y_i^0|^2+\frac{2C^2T^{\alpha-1}}{\alpha-1}\sum_{i=m+1}^{p}\lambda_i|y_i^1|^2\\ 2\frac{C^2T^\alpha}{\alpha(\alpha-1)}\sum_{i=m+1}^{p}\big(\int_0^t|f_i(s)|^2\mathrm{d}s\big).\end{aligned}\tag{3.64}$$

Remark 3.10.1

From Theorem (1.5.1), for all $\frac{3}{2} < \alpha < 2$*, we have*

$$\begin{aligned} A_p &= 2 \sum_{i=m+1}^{p} \lambda_i |y_i^0|^2 \int_0^T t^{2\alpha-4} E^2_{\alpha,\alpha-1}(-\lambda_i t^\alpha) \mathrm{d}t \\ &\leq 2C^2 \sum_{i=m+1}^{p} \lambda_i |y_i^0|^2 \int_0^T t^{2\alpha-4} \mathrm{d}t \\ &= \frac{2C^2 T^{2\alpha-3}}{2\alpha-3} \sum_{i=m+1}^{p} \lambda_i |y_i^0|^2. \end{aligned}$$

We have too

$$\begin{aligned} B_p &= 2 \sum_{i=m+1}^{p} \lambda_i |y_i^1|^2 \int_0^T t^{2\alpha-2} E^2_{\alpha,\alpha-1}(-\lambda_i t^\alpha) \mathrm{d}t \\ &\leq 2C^2 \sum_{i=m+1}^{p} \lambda_i |y_i^1|^2 \int_0^T t^{2\alpha-2} \mathrm{d}t \\ &= \frac{2C^2 T^{\alpha-1}}{\alpha-1} \sum_{i=m+1}^{p} \lambda_i |y_i^1|^2. \end{aligned}$$

By the Cauchy-Schwarz inequity, we have

$$\begin{aligned} C_p &\leq 2 \sum_{i=m+1}^{p} \int_0^T \Big\{ \lambda_i \Big(\int_0^t \Big[(t-s)^{2\alpha-2} E_{\alpha,\alpha}(-\lambda_i (t-s)^\alpha) \Big]^2 \mathrm{d}s \Big) \Big(\int_0^t |f_i(s)|^2 \mathrm{d}s \Big) \Big\} \mathrm{d}t \\ &\leq 2 \sum_{i=m+1}^{p} \int_0^T \Big\{ \frac{C^2}{\lambda_i} \Big(\int_0^t |f_i(s)|^2 \mathrm{d}s \Big) \Big(\int_0^t (t-s)^{\alpha-2}) \mathrm{d}s \Big) \mathrm{d}t \\ &\leq 2 \int_0^T \frac{C^2 t^{\alpha-1}}{\alpha-1} \mathrm{d}t \sum_{i=m+1}^{p} \Big(\int_0^t |f_i(s)|^2 \mathrm{d}s \Big) \\ &\leq 2 \frac{C^2 T^\alpha}{\alpha(\alpha-1)} \sum_{i=m+1}^{p} \Big(\int_0^t |f_i(s)|^2 \mathrm{d}s \Big). \end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
&I^{2-\alpha}(y_p(t)-y_m(t))\\
&=\frac{1}{\Gamma(2-\alpha)}\sum_{i=m+1}^{p}\Big\{y_i^0\int_0^t(t-s)^{1-\alpha}s^{\alpha-1}E_{\alpha,\alpha-1}(-\lambda_i s^\alpha)\mathrm{d}s\Big\}\omega_i\\
&\quad+\frac{1}{\Gamma(2-\alpha)}\sum_{i=m+1}^{p}\Big\{y_i^1\int_0^t(t-s)^{1-\alpha}s^{\alpha-1}E_{\alpha,\alpha-1}(-\lambda_i s^\alpha)\mathrm{d}s\Big\}\omega_i\\
&\quad+\frac{1}{\Gamma(2-\alpha)}\sum_{i=m+1}^{p}\Big\{\int_0^t(t-s)^{1-\alpha}\Big[\int_0^s(s-\tau)^{\alpha-1}E_{\alpha,\alpha-1}(-\lambda_i(s^\alpha-\tau)f_i(\tau)\mathrm{d}\tau\Big]\mathrm{d}s\Big\}\omega_i\\
&=\frac{1}{\Gamma(2-\alpha)}\sum_{i=m+1}^{p}\Big\{y_i^0\sum_{k=0}^{\infty}\frac{(-\lambda_i)^k}{\Gamma(\alpha k+\alpha-1)}\int_0^t(t-s)^{1-\alpha}s^{\alpha k+\alpha-2}\mathrm{d}s\Big\}\omega_i\\
&\quad+\frac{1}{\Gamma(2-\alpha)}\sum_{i=m+1}^{p}\Big\{y_i^1\sum_{k=0}^{\infty}\frac{(-\lambda_i)^k}{\Gamma(\alpha k+\alpha)}\int_0^t(t-s)^{1-\alpha}s^{\alpha k+\alpha-1}\mathrm{d}s\Big\}\omega_i\\
&\quad+\frac{1}{\Gamma(2-\alpha)}\sum_{i=m+1}^{p}\Big\{\int_0^t f_i(\tau)\Big[\int_0^s(t-s)^{1-\alpha}(s-\tau)^{\alpha-1}E_{\alpha,\alpha-1}(-\lambda_i(s^\alpha-\tau)\mathrm{d}s\Big]\mathrm{d}\tau\Big\}\omega_i.
\end{aligned}
$$

Using the Beta function's definition, we get

$$
\begin{aligned}
\int_0^t(t-s)^{1-\alpha}s^{\alpha k+\alpha-2}\mathrm{d}s&=\int_0^t t^{1-\alpha}(1-\frac{s}{t})^{1-\alpha}(\frac{s}{t})^{\alpha k+\alpha-2}t^{\alpha k+\alpha-2}\mathrm{d}s\\
&=t^{\alpha k}\int_0^1(1-\tau)^{1-\alpha}(\tau)^{\alpha k+\alpha-2}\mathrm{d}\tau\\
&=t^{\alpha k}B(2-\alpha,\alpha k+\alpha-1)\\
&=t^{\alpha k}\frac{\Gamma(2-\alpha)\Gamma(\alpha k+\alpha-1)}{\Gamma(1+k\alpha)}.
\end{aligned}
$$

In addition, we have

$$
\begin{aligned}
\int_0^t(t-s)^{1-\alpha}s^{\alpha k+\alpha-1}\mathrm{d}s&=\int_0^t t^{1-\alpha}(1-\frac{s}{t})^{1-\alpha}(\frac{s}{t})^{\alpha k+\alpha-1}t^{\alpha k+\alpha-1}\mathrm{d}s\\
&=t^{\alpha k+1}\int_0^1(1-\tau)^{1-\alpha}(\tau)^{\alpha k+\alpha-1}\mathrm{d}\tau\\
&=t^{\alpha k+1}B(2-\alpha,\alpha k+\alpha)\\
&=t^{\alpha k}\frac{\Gamma(2-\alpha)\Gamma(\alpha k+\alpha)}{\Gamma(2+k\alpha)}.
\end{aligned}
$$

Moreover, Using the Beta function's definition, we get

$$\begin{aligned}
&\int_\tau^t (t-s)^{1-\alpha}(s-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(s-\tau)^\alpha)\mathrm{d}s \\
&= \int_\tau^t [(t-s)-(s-\tau)]^{1-\alpha}(s-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i(s-\tau)^\alpha)\mathrm{d}s \\
&= \int_\tau^t (t-\tau)^{1-\alpha}\Big[1-\frac{s-\tau}{t-\tau}\Big]^{1-\alpha}\Big[\frac{s-\tau}{t-\tau}\Big]^{\alpha-1}(t-\tau)^{\alpha-1}\sum_{k=0}^{\infty}\frac{(-\lambda_i)^k(s-\tau)^{\alpha k}}{\Gamma(\alpha k+\alpha)}\mathrm{d}s \\
&= \sum_{k=0}^{\infty}\frac{(-\lambda_i)^k}{\Gamma(\alpha k+\alpha)}\int_\tau^t (t-\tau)^{1-\alpha}\Big[1-\frac{s-\tau}{t-\tau}\Big]^{1-\alpha}\Big[\frac{s-\tau}{t-\tau}\Big]^{\alpha-1}(t-\tau)^{\alpha k+\alpha-1}\mathrm{d}s \\
&= \sum_{k=0}^{\infty}\frac{(-\lambda_i)^k}{\Gamma(\alpha k+\alpha)}\int_\tau^t (t-\tau)^{\alpha k}\Big[1-\frac{s-\tau}{t-\tau}\Big]^{1-\alpha}\Big[\frac{s-\tau}{t-\tau}\Big]^{\alpha-1}\mathrm{d}s \\
&= (t-\tau)\sum_{k=0}^{\infty}\frac{(-\lambda_i)^k(t-\tau)^{\alpha k}}{\Gamma(\alpha k+\alpha)}\int_0^1 (1-u)^{1-\alpha}\,u^{\alpha k+\alpha-1}(t-\tau)\mathrm{d}u \\
&= (t-\tau)\sum_{k=0}^{\infty}\frac{(-\lambda_i)^k(t-\tau)^{\alpha k}}{\Gamma(\alpha k+\alpha)}B(2-\alpha,\alpha k+\alpha) \\
&= (t-\tau)\sum_{k=0}^{\infty}\frac{(-\lambda_i)^k(t-\tau)^{\alpha k}}{\Gamma(\alpha k+\alpha)}\frac{\Gamma(2-\alpha)\Gamma(\alpha k+\alpha)}{\Gamma(2+k\alpha)} \\
&= (t-\tau)\Gamma(2-\alpha)E_{\alpha,2}(-\lambda_i(s-\tau)^\alpha).
\end{aligned}$$

We therefore deduce that

$$\begin{aligned}
I^{2-\alpha}(y_p(t)-y_m(t)) = &\sum_{i=m+1}^{p} y_i^0 E_{\alpha,1}(-\lambda_i t^\alpha)\omega_i + \sum_{i=m+1}^{p} y_i^1 t E_{\alpha,1}(-\lambda_i t^\alpha)\omega_i \\
&\sum_{i=m+1}^{p}\Big\{\int_0^t f_i(\tau)(t-\tau)E_{\alpha,2}\big(-\lambda_i(t-\tau)^\alpha\big)\mathrm{d}\tau\Big\}\omega_i.
\end{aligned} \tag{3.65}$$

We can write the following relation using Theorem 1.5.1, and the Cauchy-Schwartz inequality.

$$\begin{aligned}
\|I^{2-\alpha}(y_p(t)-y_m(t))\|^2_{H_0^1(\Omega)} &= a(I^{2-\alpha}(y_p(t)-y_m(t)), I^{2-\alpha}(y_p(t)-y_m(t))) \\
&\le 2\sum_{i=m+1}^{p}\lambda_i|y_i^0|^2E^2_{\alpha,1}(-\lambda_i t^\alpha) + 2\sum_{i=m+1}^{p}\lambda_i|y_i^1|^2t^2E^2_{\alpha,2}(-\lambda_i t^\alpha) \\
&+ 2\sum_{i=m+1}^{p}\lambda_i\Big(\int_0^t f_i(\tau)(t-\tau)E_{\alpha,2}\big(-\lambda_i(t-\tau)^\alpha\big)\mathrm{d}\tau\Big)^2 \\
&\le 2C^2\sum_{i=m+1}^{p}\lambda_i|y_i^0|^2 + 2C^2T^{2-\alpha}\sum_{i=m+1}^{p}|y_i^1|^2 \\
&+ 2C^2\sum_{i=m+1}^{p}\Big[\int_0^t|f_i(\tau)|^2\mathrm{d}\tau\Big]\Big[\int_0^t(t-\tau)^{2-\alpha}\mathrm{d}\tau\Big] \\
&\le 2C^2\sum_{i=m+1}^{p}\lambda_i|y_i^0|^2 + 2C^2T^{2-\alpha}\sum_{i=m+1}^{p}\lambda_i|y_i^1|^2 \\
&+ \frac{2C^2T^{1-\alpha}}{3-\alpha}\sum_{i=m+1}^{p}\Big[\int_0^t|f_i(\tau)|^2\mathrm{d}\tau\Big].
\end{aligned}$$

As a result,

$$\sup_{t\in[0,T]} \|I^{2-\alpha}(y_p(t)-y_m(t))\|^2_{H_0^1(\Omega)} \leq \sqrt{2}C\Big(\sum_{i=m+1}^{p}\lambda_i|y_i^0|^2\Big)^{\frac{1}{2}} + \sqrt{2T^{2-\alpha}}C\Big(\sum_{i=m+1}^{p}|y_i^1|^2\Big)^{\frac{1}{2}} + C\sqrt{\frac{2T^{3-\alpha}}{3-\alpha}}\Big(\sum_{i=m+1}^{p}\int_0^T|f_i(\tau)|^2\mathrm{d}\tau\Big)^{\frac{1}{2}}. \tag{3.66}$$

We have,

$$\begin{aligned}\frac{\partial}{\partial t}I^{2-\alpha}(y_p(t)-y_m(t)) &= \sum_{i=m+1}^{p} y_i^0\frac{\partial}{\partial t}(E_{\alpha,1}(-\lambda_i t^\alpha))\omega_i + \sum_{i=m+1}^{p} y_i^1\frac{\partial}{\partial t}(tE_{\alpha,1}(-\lambda_i t^\alpha))\omega_i \\ &+ \sum_{i=m+1}^{p}\frac{\partial}{\partial t}\Big\{\int_0^t f_i(\tau)(t-\tau)E_{\alpha,2}\big(-\lambda_i(t-\tau)^\alpha\big)\mathrm{d}\tau\Big\}\omega_i \\ &= \sum_{i=m+1}^{p} y_i^0\big(-\lambda_i t^{\alpha-1}E_{\alpha,1}(-\lambda_i t^\alpha)\big)\omega_i + \sum_{i=m+1}^{p} y_i^1 E_{\alpha,1}(-\lambda_i t^\alpha)\omega_i \\ &+ \sum_{i=m+1}^{p}\int_0^t f_i(\tau)\frac{\partial}{\partial t}\Big\{(t-\tau)E_{\alpha,2}\big(-\lambda_i(t-\tau)^\alpha\big)\Big\}\mathrm{d}\tau \\ &= \sum_{i=m+1}^{p} y_i^0\big(-\lambda_i t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha)\big)\omega_i + \sum_{i=m+1}^{p} y_i^1 E_{\alpha,1}(-\lambda_i t^\alpha)\omega_i \\ &+ \sum_{i=m+1}^{p}\int_0^t f_i(s)E_{\alpha,1}\big(-\lambda_i(t-s)^\alpha\big)\mathrm{d}s.\end{aligned}$$

Where did it come from

$$\begin{aligned}\|\frac{\partial}{\partial t}I^{2-\alpha}(y_p(t)-y_m(t))\|^2_{L^2(\Omega)} &\leq 2\sum_{i=m+1}^{p}\lambda_i^2|y_i^0|^2(t^{2\alpha-2}E^2_{\alpha,\alpha}(-\lambda_i t^\alpha)) + 2\sum_{i=m+1}^{p}|y_i^1|^2E^2_{\alpha,1}(-\lambda_i t^\alpha) \\ &+ 2\sum_{i=m+1}^{p}\Big|\int_0^t f_i(s)E_{\alpha,1}\big(-\lambda_i(t-s)^\alpha\big)\mathrm{d}s\Big|^2 \\ &\leq 2C^2T^{2\alpha-2}\sum_{i=m+1}^{p}\lambda_i^2|y_i^0|^2 + 2C^2\sum_{i=m+1}^{p}|y_i^1|^2 \\ &+ 2C^2\sum_{i=m+1}^{p}\int_0^T|f_i(s)|^2\mathrm{d}s.\end{aligned}$$

That implies

$$\sup_{t\in[0,T]}\|\frac{\partial}{\partial t}I^{2-\alpha}(y_p(t)-y_m(t))\|^2_{L^2(\Omega)} \leq \sqrt{2}CT^{\alpha-1}\Big(\sum_{i=m+1}^{p}\lambda_i^2|y_i^0|^2\Big)^{\frac{1}{2}} + \sqrt{2}C\Big(\sum_{i=m+1}^{p}|y_i^1|^2\Big)^{\frac{1}{2}} + \sqrt{2}C\Big(\sum_{i=m+1}^{p}\int_0^T|f_i(s)|^2\mathrm{d}s\Big)^{\frac{1}{2}}.$$

Since $y^0\in H^2(\Omega)\cap H_0^1(\Omega)$, $y^1(\Omega)$ and $f\in L^2(Q)$, we deduce that

$$\lim_{m,p\to+\infty}\Big(\sum_{i=m+1}^{p}\int_0^T|f_i(s)|^2\mathrm{d}s\Big)^{\frac{1}{2}} = 0, \quad \lim_{m,p\to+\infty}\Big(\sum_{i=m+1}^{p}\lambda_i^2|y_i^0|^2\Big)^{\frac{1}{2}} = 0,$$

$$\lim_{m,p\to+\infty}\Big(\sum_{i=m+1}^{p}|y_i^1|^2\Big)^{\frac{1}{2}} = 0.$$

As a result

$$\lim_{m,p\to+\infty}\int_0^T \|y_p(t)-y_m(t)\|^2_{H_0^1(\Omega)}\mathrm{d}t=0,$$

$$\sup_{t\in[0,T]}\|I^{2-\alpha}(y_p(t)-y_m(t))\|^2_{H_0^1(\Omega)}=0,\quad \sup_{t\in[0,T]}\|\frac{\partial}{\partial t}I^{2-\alpha}(y_p(t)-y_m(t))\|^2_{L^2(\Omega)}=0.$$

This indicates that the sequences (y_m), $(I^{2-\alpha}y_m)$ and $(\frac{\partial}{\partial t}I^{2-\alpha}y_m)$ are Cauchy respectively in $L^2(0,T;H_0^1(\Omega))$, $\mathcal{C}(0,T;H_0^1(\Omega))$ and $\mathcal{C}(0,T;L^2(\Omega))$. Then, we have

$$y_m\to y \text{ in } \quad L^2(0,T;H_0^1(\Omega)),$$

$$I^{1-\alpha}y_m\to I^{1-\alpha}y \quad\text{in}\quad \mathcal{C}(0,T;H_0^1(\Omega)),\quad \frac{\partial}{\partial t}I^{2-\alpha}y_m\to\frac{\partial}{\partial t}I^{2-\alpha}y \quad\text{in}\quad \mathcal{C}(0,T;L^2(\Omega)).$$

- *We demonstrate that y is the solution to the problem (1.42)-(1.46).*

  Let $\mathbb{D}(0,T)$ be the space of functions ∞ in $(0,T)$ with compact support, and let $\varphi\in\mathbb{D}(0,T)$. Let also $\mu\geq 1$ be an integer. Then by (3.61), we have for all $m\geq\mu$,

$$\begin{aligned}\int_0^T (f(t),v)_{L^2(\Omega)}\varphi(t)\mathrm{d}t &= \int_0^T D^\alpha_{RL}(y_m(t),v)_{L^2(\Omega)}\varphi(t)\mathrm{d}t+\int_0^T a(y_m(t),v)\varphi(t)\mathrm{d}t \quad \forall v\in V_\mu,\\ &= \int_0^T (y_m(t),v)_{L^2(\Omega)}\mathcal{C}^\alpha_C\varphi(t)\mathrm{d}t+\int_0^T a(y_m(t),v)\varphi(t)\mathrm{d}t \quad \forall v\in V_\mu,\\ &= \int_0^T (y(t),v)_{L^2(\Omega)}\mathcal{C}^\alpha_C\varphi(t)\mathrm{d}t+\int_0^T a(y(t),v)\varphi(t)\mathrm{d}t \quad \forall v\in V_\mu.\end{aligned}$$

  As $U_{\mu\geq 1}\subset V_\mu$ is dense in $H_0^1(\Omega)$ because (ω_i) is a basis of $H_0^1(\Omega)$. We obtain for all $v\in H_0^1(\Omega)$

$$\int_0^T (f(t),v)_{L^2(\Omega)}\varphi(t)\mathrm{d}t=\int_0^T D^\alpha_{RL}(y(t),v)_{L^2(\Omega)}\varphi(t)\mathrm{d}t+\int_0^T a(y(t),v)\varphi(t)\mathrm{d}t \quad \forall v\in H_0^1(\Omega).$$

  Which means that for all $v\in H_0^1(\Omega)$, we have

$$(f(t),v)_{L^2(\Omega)}\varphi(t)=D^\alpha_{RL}(y(t),v)_{L^2(\Omega)}\varphi(t)+a(y(t),v)\varphi(t)\quad \forall t\in(0,T).$$

  Finally, we deduce that

$$I^{2-\alpha}y(0)=I^{2-\alpha}y^0\quad,\frac{\partial}{\partial t}I^{2-\alpha}y(0)=\frac{\partial}{\partial t}I^{2-\alpha}y^1.$$

- *Let us demonstrate the relations (1.48)-(1.50).* If y is the solution to problem (1.42)-(1.46), then y is ginen by (1.47). Then, using the results of the previous calculations, we demonstrate that we have

$$\|y\|^2_{L^2(0,T;H_0^1(\Omega))}\leq\frac{2C^2T^{2\alpha-3}}{2\alpha-3}\|y^0\|^2_{H_0^1(\Omega)}+\frac{2C^2T^{\alpha-1}}{\alpha-1}\|y^1\|^2_{L^2(\Omega)}+2\frac{C^2T^\alpha}{\alpha(\alpha-1)}\|f\|^2_{L^2(Q)}.$$

  we have too

$$\|I^{2-\alpha}y(t)\|^2_{H_0^1(\Omega)}\leq 2C^2\sum_{i=1}^\infty\lambda_i|y_i^0|^2+2C^2T^{2-\alpha}\sum_{i=1}^\infty\lambda_i|y_i^1|^2+2C^2\frac{T^{3-\alpha}}{3-\alpha}\sum_{i=1}^\infty\int_0^T|f_i(s)|^2\mathrm{d}s.$$

  Let be

$$\sup_{t\in[0,T]}\|I^{2-\alpha}y(t))\|_{H_0^1(\Omega)}\ \leq\sqrt{2}C\|y^0\|_{H_0^1(\Omega)}+C\sqrt{2T^{2-\alpha}}\|y^1\|_{L^2(\Omega)}+C\sqrt{\frac{2T^{3-\alpha}}{3-\alpha}}\|f\|_{L^2(Q)}.$$

Finally, we've got

$$\|\frac{\partial}{\partial t}I^{2-\alpha}y(t)\|^2_{L^2(\Omega)} \leq 2C^2T^{2\alpha-2}\sum_{i=1}^{\infty}\lambda_i^2|y_i^0|^2 + 2C^2\sum_{i=1}^{\infty}|y_i^1|^2 + 2C^2\sum_{i=1}^{\infty}\int_0^T |f_i(s)|^2\mathrm{d}s.$$

Then

$$\sup_{t\in[0,T]}\|\frac{\partial}{\partial t}I^{2-\alpha}y(t)\|^2_{L^2(\Omega)} \leq \sqrt{2}C\sqrt{T^{2\alpha-2}}(\sum_{i=1}^{\infty}\lambda_i^2|y_i^0|^2)^{\frac{1}{2}} + \sqrt{2}C(\sum_{i=1}^{\infty}|y_i^1|^2)^{\frac{1}{2}} + \sqrt{2}C(\sum_{i=1}^{\infty}\int_0^T |f_i(s)|^2\mathrm{d}s)^{\frac{1}{2}}.$$

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