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Dedication

In the name of Allah, The Most Beneficent, The Most Merciful

I dedicate this humble work:

To my cherished **Dad**, *the man I always look at as a hero, you have always been there when I needed you the most. Seeing the pride in your eyes is a delight that I will forever be grateful for.*

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To me!

Abstract

The aim of this dissertation is to investigate a class of nonlinear parabolic problems with different boundary conditions (local, non-local and nonlinear conditions), where we began with a reminder of some basic preliminary concepts and tools required in this work as a first chapter.

The second chapter concerning a nonlinear parabolic problem with classical Neumann boundary conditions; where we show the existence and the uniqueness of weak solution by using the energy inequality method and an iterative process based on a priori estimate. After that, we moved to dynamic issue, exactly we study the blow-up solution by the energy function method.

The third chapter devoted to study a nonlinear problem with nonlocal conditions of the second type; we present firstly the solvability of the associated linear problem where we separate it into two linear problems and showing their existence using the variable separation method and the energy inequality method. Then by using the Linearization method we prove the existence and the uniqueness of the weak solution of the nonlinear problem. We study also the finite time blow-up of the solutions.

Finally, in the last chapter, we examined the existence of weak solution of initial boundary problem for anonlinear parabolic equation with nonlinear boundary conditions by using Faedo-Galerkin method.

Key words: Nonlinear parabolic equations, nonlocal conditions, existence, uniqueness, energy inequality, blow-up solution, Faedo-Galerkin method.

ملخص

الهدف من هذا العمل هو دراسة بعض المسائل غير الخطية لمعادلات ذات قطع مكافئ مع شروط حدية مختلفة (محلية وغير محلية وغير خطية)، حيث بدأنا بالتذكير ببعض المفاهيم والأدوات الأولية الأساسية المطلوبة في هذا العمل كفصل أول.

الفصل الثاني يتناول مسألة غير خطية لمعادلة قطع مكافئ غير خطية مع شروط حدية من نوع نيومان؛ حيث أثبتنا وجود و وحدانية الحل الضعيف باستخدام طريقة متراجحات الطاقة وطريقة تعتمد على عمليات تكرارية تستند على نتائج المرحلة الأولى. بعد ذلك انتقلنا إلى دراسة ديناميكية الحل، أين درسنا بالضبط الوقت المنتهي لانفجار الحل باستخدام طريقة دوال الطاقة.

الفصل الثالث مخصص لدراسة مسالة ذات قطع مكافئ غير خطية ذات شروط غير محلية من نوع تكامل من النوع الثاني، ندرس أولاً وجود ووحدانية الحل للمسألة الخطية حيث قمنا بفصلها إلى مسألتين خطيتين لإثبات وجود الحل لكل منهما باستخدام طريقة فصل المتغيرات وطريقة متراجحات الطاقة على التوالي. ثم باستخدام طريقة الخطية التي تعتمد على تقنية النقطة الثابتة نثبت وجود ووحدانية الحل الضعيف للمسألة غير الخطية. ندرس كذلك وقت الانفجار المنتهي للحلول.

أخيرًا،في الفصل الرابع، درسنا وجود حل ضعيف لمسألة حدية لمعادلة غير خطية ذات قطع مكاف غير خطية بشروط حدية غير خطية باستخدام طريقة فايدو غالاركين.

كلمات مفتاحية: مسألة ذات قطع مكافئ غير خطية، شرط حدي من نوع تكامل، الوجود، الوحدانية، طريقة متراجحات الطاقة، انفجار الحلول، طريقة فايدو غالاركين.

Résumé

Le but de ce travail est d'étudier une classe de problèmes paraboliques non linéaires avec différentes conditions aux limites (conditions locales, non locales et non linéaires), où nous avons commencé par un rappel de quelques concepts et outils préliminaires de base nécessaires à ce travail dans un premier chapitre.

Le deuxième chapitre concerne un problème parabolique non linéaire avec des conditions aux limites de Neumann classiques ; où nous montrons l'existence et l'unicité de la solution faible en utilisant la méthode des inégalités d'énergie et un processus itératif basé sur une estimation a priori. Après cela, nous sommes passés au problème dynamique, nous étudions exactement l'explosion de la solution par la méthode de la fonction énergétique.

Le troisième chapitre est consacré à l'étude d'un problème non linéaire avec des conditions non locales dedeuxième type ; nous présentons d'abord la solvabilité du problème linéaire associé où nous le séparons en deux problèmes linéaires et montrons l'existence de chaque problème en utilisant la méthode de séparation des variables et la méthode des inégalités d'énergie. Ensuite, en utilisant la méthode de linéarisation, nous prouvons l'existence et l'unicité de la solution faible du problème non linéaire. Nous étudions aussi l'existence entemps fini de la solution.

Enfin, dans le dernier chapitre, nous avons examiné l'existence d'une solution faible du problème pour une équation parabolique non linéaire avec des conditions aux limites non linéaires en utilisant la méthode de Faedo-Galerkin.

Mots clés : Équations paraboliques non linéaires, conditions non locales, existence, unicité, inégalité d'énergie, explosion de la solution, méthode de Faedo-Galerkin.

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Introduction

Our understanding of the fundamental processes of the natural world is based to a large extent on partial differential equations [9]. they arise in every field of science such as the vibrations of solids, the flow of fluids, the diffusion of chemicals, the spread of heat, the structure of molecules, the interactions of photons and electrons and the radiation of electromagnetic waves. In order to obtain a well-defined solution, the equations are supplemented with suitable additional conditions (usually, initial and boundary conditions). The standard differential theories involve linear operators and for them an extensive theory has been developed. Nonlinear partial differential equations and systems exhibit a number of properties which are absent from the linear theories; these nonlinear properties are often related to important features of the real world phenomena which the mathematical model is supposed to describe; at the same time these new properties are closely connected with essential new difficulties of the mathematical treatment[6].

The study of partial differential equations (PDE's) started in the 18th century in the work of Euler, d.Alembert, Lagrange and Laplace as a central tool in the description of mechanics of continua and more generally, as the principal mode of analytical study of models in the physical science. The analysis of physical models has remained to the present day one of the fundamental concerns of the development of PDEs. Beginning in the middle of the 20th century, particularly with the work of Riemann, PDE's also became an essential tool in other branches of mathematics[4]. Many modern physical, mechanical, biological and tachnological phenomena and problems have been shaped by partial differential equations (PDE's), parabolic or hyperbolic equations with nonlocal boundary conditions in the last forty years. Thus the boundary conditions of the integral type can be used when it is impossible to directly measure the quantity sought on the boundary where its total or average value is known.(12) We are intersting to phenomena whose modeled by parabolic equations.One of the most remarkable properties that distinguish nonlinear evolution problems from the linear ones is the possibility of eventual occurrence of singularities starting from perfectly smooth data, more specifically, from classes of data for which a theory of existence, uniqueness and continuous dependence can be established for small time intervals, so-called well-posedness in the small[6]. There is many usiful methods and theories have been developed to solve parabolic equations-especialy since the 1960s- such us the semigroup method, compactness (Faedo Galerkin) method, the monotone operator method[11] and the fixed point method, there is also a method which called the method of functional analysis or the method of a priori estimates, such estimates are the keys to analysing PDE's; they can be used to prove existence, uniqueness and continuous dependence results as well as to provide other qualitative informations[5].

In this context, the question is: in the case of nonlinear parabolic equation with different boundary conditions, is the solution $u = u(x, t)$ exists? if yes, is it unique for any x in bounded domain Ω and any $t \in (0, T)$?

In the other hand, we are interested in the situation where, starting from a smooth initial configuration, and after a first period of classical evolution, the solution (or in some cases its derivatives) becomes infinite in finite time due to the cumulative effect of the nonlinearities. This is what we call a blow-up phenomenon or explosion in Latin languages[6].

Let us review some historical facts with special attention to the early developments by the Russian school. The subject of blow-up was posed in the 1940's and 50's in the context of Semenov's chain reaction theory, adiabatic explosion and combustion theory. A strong influence was also due to blow-up singularities in gas dynamics, the intense explosion (focusing) problem with second kind self-similar solutions considered by Bechert, Guderley and Sedov in the 1940's. First analysis of the most striking effect of space localization of blow-up boundary regime (S-regime of blow-up generated by a blow-up standing wave) in quasilinear diffusion equations was performed by Samarskii and Sobolev in 1963. An essential increase of attention to the blow-up research in gas dynamics, laser fusion and combustion in the 70's was initiated by the numerical results (announced by E. Teller in 1972) on the possibility of the laser blow-up-like compression of deuterium-tritium (DT) drop to super-high densities without shock waves. The problem of localization of blow-up solutions in reaction-diffusion equations was first proposed by Kurdyumov in 1974[6].

Motivation by this in this work we are interested to study the solvability and finite time blow-up for nonlinear parabolic problems with different types of boundary conditions, we divided this work into four chapters

The first chapter is devoted to a reminder of some fundamental preliminary concepts and tools needed in this work. In particular, we present some fundamental results on the properties of unbounded linear operators, functional spaces and methods like the energy inequality method and the compactness method (Faedo-Galerkin method).

The second chapter studies Neumann's boundary-value problem for a class of non-linear parabolic equations. We will display the existence and the uniqueness of the strong solution to the linear problem by the energy inequality method. Then, by applying an iterative process based on the results obtained for the linear problem, to prove the existence and the uniqueness of the weak solution of the nonlinear problem. Finally, we determine the finite-time blow-up of the solutions.

Also, in the third chapter, on a the solvability and finite time blow-up of weak solutions of nonlinear problems with nonlocal boundary conditions of the second type. We study the solvability of the associated linear problem by concentrating on linear problems and showing their existence using the variable separation method and the energy inequality method. Following that, we demonstrate the problem's uniqueness. Then, we use the Linearization method to demonstrate that the weak solution to the main nonlinear problem appears and is unique.

Finally, in the last chapter, we study the existence of weak solutions to a nonlinear parabolic problem with nonlinear conditions defined by the power of the solution and generalized integral condition of the second type.

Chapter 1

Preliminary

In this introductory chapter, we present well known facts of the analysis which will be used below in our treatement.

1.1 measure spaces:

1.1.1 $L^2(Q)$ Space

For the study of some problems, we need to recall some functional spaces. Is $L^2(0,d)$, $d \in \mathbb{R}_+^*$, the usual Hilbert space provided with a scalar product noted $(.,.)_{L^2(0,d)}$and an associated norm $\|.\|_{L^2(0,d)}$ Hilbert's space $L^2(Q) = L^2((0,T), L^2(0,d))$ $(Q = (0,d) \times (0,T))$ consists of (classes of) defined and square functions that can be integrated into Q. Scalar product in $L^2(Q)$ is noted $(.,.)_{L^2(\Omega)}$ defined by:

$$(u,v)_{L^2(Q)} = (u,v)_{L^2(Q)} = \int_0^d (u(x,.), v(x,.))_{L^2(0,T)}\, dx$$

and an associated norm denoted $\|.\|_{L^2(Q)}$ defined by:

$$\|u\|_{L^2(Q)} = \|u\|_{L^2(Q)} = \left(\int_0^d \|u(x,.)\|_{L^2(0,T)}\, dx \right)^{1/2}.$$

1.1.2 $L^2_\rho(\Omega)$ space

We denote by $L^2_\rho(\Omega)$ the Hilbert space of functions of integrable squares with weight endowed with a scalar product denoted $(\bullet,\bullet)_{L^2_\rho(\Omega)}$, defined by :

$$(u,v)_{L^2_\rho(\Omega)} = (\sqrt{\rho}u, \sqrt{\rho}v)_{L^2(\Omega)}$$

and an associated norm denoted $\|u\|_{L^2_\rho(\Omega)}$ defined by :

$$\|u\|_{L^2_\rho(\Omega)} = \|\sqrt{\rho}u\|_{L^2(\Omega)} = \left(\int_\Omega \rho(x)\,|u(x,.)|^2\,dx\right)^{1/2}.$$

If $\rho(x) = 1$, $L^2_\rho(\Omega)$ are identified with standard spaces $L^2(\Omega)$.

1.1.3 Sobelev space

Let Ω be a bounded or unbounded domain of $\mathbb{R}^n$ with smooth boundary Γ. For $m \in \mathbb{N}$, $1 \leq p \leq \infty$, $W^{m,\,p}(\Omega)$ is defined to be the space of functions u in $L^p(\Omega)$ whose distribution derivatives of order up to m are also in $L^p(\Omega)$. Then, it is known that $W^{m,\,p}(\Omega)$ is a Banach space for the norm

$$\|u\|_{W^{m,\,p}(\Omega)} = (\sum_{|\alpha|\leq m} \|\partial^\alpha u\|^p_{L^p(\Omega)})^{\frac{1}{p}},$$

where $\alpha = \{\alpha_1, ..., \alpha_n\} \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + ... + \alpha_n$, and

$$\partial^\alpha u = \frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}.$$

is the distributional derivative of u.

When $p = 2$, we usually denote $W^{m,\,p}(\Omega)$ by $H^m(\Omega)$ and this is a Hilbert space for the induced inner product.[11]

The n-tuple of the first-order distributional derivatives$\left(\frac{\partial^k}{\partial x_1}u, ..., \frac{\partial^k}{\partial x_n}u\right)$is denoted by ∇u and called a gradient of u. We will now consider $\Omega \subset \mathbb{R}^n$ open with $mes_n(\Omega) < +\infty$, such a set will be called a domain. For $p < +\infty$, we define a Sobolev space

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega); \nabla u \in L^p(\Omega, \mathbb{R}^n)\},$$

As, by Rademacher's theorem, Lipschitz functions are a.e. differentiable, it holds that

$$W^{1,\infty}(\Omega) = C^{0,1}(\Omega).$$

Analogously, for $k > 1$ integer, we define

$$W^{k,p}(\Omega) := \left\{ u \in L^p(\Omega, \mathbb{R}^m); \nabla^k u \in L^p(\Omega, R^{n^k}) \right\}$$

where $\nabla^k u$ denotes the set of all k-the order partial derivatives in the distributional sense.

1.2 Unbounded linear operator

Definition 1.1 *An operator T from E to F is said to be linear if and only if :*

$$\forall u_1, u_2 \in E, \ \forall \mu, \lambda \in \mathbb{C} \text{ on a : } T(\lambda u_1 + \mu u_2) = \lambda T(u_1) + \mu T(u_2),$$

where $\mathbb{C}$ is the scalar field of E and F.

Definition 1.2 *Let E and F be two Banach spaces. An unbounded linear operator from E into F is a linear map $A : D(A) \subset E \to F$ defined on a linear subspace $D(A) \subset E$ with values in F. The set $D(A)$ is called the domain of A*

Any linear operator T is completely defined by its graph $G(T)$ which is a vector subspace of $E \times F$ defined by :

$$G(T) = \{(u, Tu)\,,\ u \in D(T)\}\,,$$

where $D(T)$ is the domain of definition of the operator T.

Definition 1.3 *We say that S is an extension of T if $D(T) \subset D(S)$ and $Tu = Su$ for all $u \in D(T)$. Thus, $G(T) \subset G(S)$.*

Remark 1.1 *It is not true that every subspace of $E \times F$ is the graph of an linear operator.*

Definition 1.4 *We say that T is closed if its graph $G(T)$ is closed by $E \times F$.*

Proposition 1.1 *A subspace $G \subset E \times F$ is the graph of a linear operator if and only if :*

$$(0, y) \in G \Rightarrow y = 0. \tag{1.1}$$

Definition 1.5 *We say that a linear operator T is closable in E if it admits a closed extension. In other way T is closable if and only if for any sequence $(u_n)_{n\in\mathbb{N}} \subset D(T)$ such that $u_n \to 0$ and $Tu_n \to v$, then $v = 0$.*

We say that a linear operator T is closable in E if it admits a closed prolongation.

We immediately check that T is closable in E if and only if the adhesion $\overline{G(T)}$ of its graph is a graph (Because, we have $T \subset \overline{T}$ implies $G(T) \subset G(\overline{T})$, as the continuation $\overline{T}$ is closed, then $G(\overline{T})$ is closed, so $G(T) \subset G(\overline{T})$ implies $\overline{G(T)} \subset \overline{G(\overline{T})} = G(\overline{T})$). In other words T is closable if and only if for any sequence $(u_n) \subset D(T)$ such that $u_n \to 0$ and $Tu_n \to v$, then $v = 0$.

The closed operator $\overline{T}$ whose graph $G(\overline{T}) = \overline{G(T)}$ is called closure of T. ($G(\overline{T}) = \overline{G(T)}$ implies that $\overline{G(T)}$ is a graph :

$\forall (u_n, Tu_n) \in G(T) \Rightarrow (\lim_{n\to+\infty} u_n, \lim_{n\to+\infty} Tu_n) \in \overline{G(T)}$ and $(0, v) \in \overline{G(T)}$, which requires $v = 0$ to ensure that $\overline{G(T)}$ is a graph).

Theorem 1.1 *(Isomorphism theorem).*

- Let E and F be two Banach spaces and let T be a continuous and bijective linear operator from E to F. Then T^{-1} is continuous from F to E.

Theorem 1.2 *(Closed graph theorem).*

- Let E and F be two Banach spaces. Let T be a linear operator from E to F. Assume that the graph of T, $G(T)$, is closed in $E \times F$. Then T be continuous.

1.3 Relation between orthogonality and density in Hilbert spaces

Definition 1.6 *Let M be a vector subspace of the Hilbert space F, we define $M^{\perp}$ the orthogonal of M, by*

$$M^{\perp} = \{f \in F, \ \langle f, \ g\rangle_F = 0, \ \forall g \in M\}.$$

Proposition 1.2 *Let M a vector subspace of the Hilbert space F. We say that M is dense in F if and only if $M^{\perp} = \{0\}$.*

1.4 The energy inequality method

The method of energy inequalities is an effective technique for studying the existence and the uniqueness of the solution of partial differential equations, it is also called the method of functional analysis or the method of a priori estimates.this method has a higher character which one can draw the existence theorem from the solution of the problem posed, starting from the uniqueness theorem. The difficult points of this method lie in the choice of the functional spaces E and F and in the choice of the multiplier Mu. The scheme of the method can be summarized as follows :

1. First we write the problem posed in the form of an operational equation :

$$Lu = \mathcal{F}, \quad u \in D(L);$$

where the operator L is consideblack from a Banach space E in a Hilbert space F suitably chosen.

2. Then we establish the a priori estimate for the operator L.

3. Then we demonstrate the density of the set of values of this operator in space F.

More precisely we follow in this work the following schema:

We demonstrate the energy inequality of the type

$$\|u\|_B \leq c\,\|Lu\|_F\,. \tag{1.2}$$

This type of a priori estimates is obtained by multiplying the equation consideblack by an integrro-differential operator Mu (containing the function u or its derivatives) defined on the domain Q_T.

The choice of the operator Mu is fundamental, it is dictated by the equation and the boundary conditions.

Then, we show that the operator L of B in F admits a closure L, therefore the solution of the operational equation:

$$\overline{L}u = \mathcal{F}, \quad u \in D\left(\overline{L}\right), \tag{1.3}$$

is called generalized strong solution of the problem consideblack.

By passing to the limit, the estimate (1.2) will be extended to L, that is to say:

$$\|u\|_B \leq c\left\|\overline{L}u\right\|_F.$$

Thus, we deduce the uniqueness of the solution from equation (1.3).

As the image of the operator $\overline{L}$ is closed in F and sence $R\left(\overline{L}\right) = \overline{R(L)}$, establishing the density of the set $R\left(\overline{L}\right)$ in F guarantees the existence of the strong solution of problem (1.3).

1.5 The compactness method (Faedo-Galerkin method)

The Faedo–Galerkin method is one of the most efficient techniques to determine approximate solutions for a given differential equation in an abstract space. A sequence of approximate solutions constructed in this method that converges uniformly to the exact solution of the given problem. In a way, the existence of an exact solution is additionally suggested. Moreover, the Faedo–Galerkin procedure creates a system of ordinary differential equations (ODEs) that can unravel numerically.The schema of the method can be summarized in the following steps:

1- Use the Faedo-Galerkin method, i.e, choose certain base functions in an appropriate Sobolev space to search for approximate solutions.

2- Obtain the a priori estimates for the solution of the approximate problem.

3- Pass to the limit according to the compactness properties.

1.6 Some definitions, lemmas and theorems

Definition 1.7 *Let E be a Banach space*

(i) Strong convergence : Let $(u_n)_n \subset E$, $u \in E$, we say that $(u_n)_n$ converges strongly towards u in E when $n \longrightarrow +\infty$, if

$$\|u_n - u\|_E \longrightarrow 0.$$

(ii) Weak convergence : Let $(u_n)_n \subset E$, $u \in E$, we say that $(u_n)_n$ weak convergence towards u in E when $n \longrightarrow +\infty$, if

$$\text{for all } f \in E^*, \quad f(u_n) \longrightarrow f(u).$$

where E^* dual espace of E.

Lemma 1.1 *[11]In a reflexive Banach space, any bounded set is weakly compact, i.e., any sequence in a bounded set has a weakly converging subsequence.*

Lemma 1.2 *[11]Let $\Omega \in \mathbb{R}^n$, $n \geq 1$, and let $u_n(x)$, $u(x)$ be two real functions in $L^p(\Omega)$, $(1 \leq p \leq +\infty)$ such that u_n converges strongly towards u in $L^p(\Omega)$. Then if $1 \leq p < +\infty$, $(u_n)_n$ admits a subsequence converges almost everywhere towards u, and if $p = +\infty$, then $(u_n)_n$ converges almost everywhere to u.*

Lemma 1.3 *[11]Suppose that Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, Let $u_n(x)$ a bounded sequence in $L^p(\Omega)$, $(1 \leq p < +\infty)$ such that u_n converges almost everywhere to u.Then $u \in L^p(\Omega)$, and u_n converges weakly towards u in $L^p(\Omega)$.*

Lemma 1.4 *[11]Let B be a Banach space such $B = (B^*)'$ where B^* is a reflexive Banach space. Then any bounded set in B is weakly star compact, i.e., any sequence in a bounded set in B has a weakly star converging subsequence.*

Lemma 1.5 *Let* $v : [0,T] \longrightarrow H$ *be an integrable function of Bochner and* $A \subset [0,T]$ *any measurable subset. So*

(i) the function $\|v(.)\|_H : [0,T] \longrightarrow \mathbb{R}$ *is Lebesgue integrable and*

$$\left\| \int_A v(t)dt \right\|_H \leq \int_A \|v(t)\|_H \, dt \ ,$$

(ii) for all $\varphi \in H$, *the function* $(v(.), \varphi)$: $[0,T] \longrightarrow \mathbb{R}$ *is Lebesgue integrable and*

$$\left(\int_A v(t)dt, \varphi \right)_H = \int_A (v(t), \varphi)_H \, dt \ ,$$

(iii) for all $t_0 \in [0,T]$, *the function* $t \mapsto u(t) = \int_{t_0}^{t} v(s)ds$ *continuous on* $[0,T]$ *,and differentiable, means*

$$t \in [0,T] \ \ with \ \frac{du(t)}{dt} = v(t).$$

Theorem 1.3 *(Rellich's theorem) Let* $\Omega \subset \mathbb{R}^n, n \geq 1$. *a regular bounded open. Then any bounded part of* $H^1(\Omega)$ *is relatively compact in* $L^2(\Omega)$.

Consequently, as any weakly convergent sequence is bounded, Rellich's theorem implies that any weakly convergent sequence in $H^1(\Omega)$ *has a subsequence which converges strongly in* $L^2(\Omega)$ *(in other words, which converges for the topology induced by* L^2 *standard on* Ω*).*

Theorem 1.4 *[11] Let* B_0, B, B_1 *be three Banach spaces where* B_0 *and* B_1 *are reflexive. Supposing that* B_0 *is continuously imbedded into* B, *which is also continuously imbedded into* B_1, *and imbedding from* B_0 *into* B *is compact. For any given* p_0, p_1 *with* $1 < p_0, p_1 < \infty$, *let*

$$W = \{v \mid v \in L^{p_0}([0,T], B_0), v_t \in L^{p_1}([0,T], B_1)\}$$

Then the imbedding from W *into* $L^{p_0}([0,T], B)$ *is compact.*

Remark 1.2 *It can be seen from the proof that if the assumption of reflexivity of* B_0, B_1 *is replaced by the assumption that* B_0 *and* B_1 *are the dual spaces of reflexive Banach spaces* B_0^* *and* B_0^* *respectively, then the conclusion of theorem (1.3) holds still. Accordingly, in the proof, weak convergence should be replaced by weakly star convergence.*

Nemytskii mappings

Definition 1.8 *Considering integers $j, m_0, m_1, ..., m_j$, we say that a mapping*

$$a : \Omega \times R^{m_1} \times \cdots \times R^{m_j} \to R^{m_0}$$

is a Carathéodory mapping if

$$a(\cdot, r_1, ..., r_j) : \Omega \to R^{m_0}$$

is measurable for all

$$(r_1, ..., r_j) \in R^{m_1} \times ... \times R^{m_j} \text{ and } a(x, \cdot) : R^{m_1} \times ... \times R^{m_j} \to R^{m_0}$$

is continuous for a.a. $x \in \Omega$. Then the so-called Nemytskii mappings N_a map functions

$$u_i : \Omega \to R^{m_i}, i = 1, ..., j,$$

to a function

$$N_a(u_1, ..., u_j) : \Omega \to R^{m_0}$$

defined by

$$[N_a(u_1, ..., u_j)](x) = a(x, u_1(x), ..., u_j(x)). \tag{1.4}$$

Theorem 1.5 *(Nemytskii mappings in Lebesgue spaces)*

If

$$a : \Omega \times R^{m_1} \times \cdots \times R^{m_j} \to R^{m_0}$$

is a Carathéodory mapping and the functions

$$u_i : \Omega \to R^{m_i}, i = 1, ..., j$$

are measurable, then $N_a(u_1, ..., u_j)$ is measurable. Moreover, if a satisfies the growth condition

$$a(x, r_1, ..., r_j) \leq \gamma(x) + C \sum_{i=1}^{j} |r_i|^{\frac{p_i}{p_0}} \text{ for some } \gamma \in L^{p_0}(\Omega),$$

with $1 \leq p_i < +\infty, 1 \leq p_0 < +\infty$, then N_a is a bounded continuous mapping

$$L^{p_1}(\Omega, R^{m_1}) \times \cdots \times L^{p_j}(\Omega, R^{m_j}) \to L^{p_0}(\Omega, R^{m_0}).$$

If some $p_i = +\infty, i = 1, ..., j$, the same holds if the respective term $|.|^{\frac{p_i}{p_0}}$ is replaced by any continuous function.

Bessel's function

Definition 1.9 *For* $m \in \mathbb{R}, m \geq 0$. *Let the following Bessel's equation of the order* m *on* $]0, +\infty[$:

$$x^2y'' + xy' + (x^2 - m^2)y = 0, \tag{1.5}$$

there are 2 linearly independent solutions to this equation:

Bessel function of the first kind of the order m :

$$J_m(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!(m+k)!} (\frac{x}{2})^{2k+m},$$

Bessel function of the second kind of the order m :

$$Y_m(x) = \lim_{\alpha \longrightarrow m} \frac{J_m(x)\cos(\alpha\pi) - J_{-m}(x)}{\sin(\alpha\pi)},$$

such that m *is integer;* $m = 1, 2, ...$

Theorem 1.6 *If* m *is integer, all solutions of the differential equation* (1.5) *are of the following form*

$$y(x) = c_1 J_m(x) + c_2 Y_m(x).$$

If m *is noninteger then all solutions of the equation* (1.5) *are of the form*

$$y(x) = c_1 J_m(x) + c_2 J_{-m}(x).$$

1.7 Some useful inequalities

The following elementary inequalities are very useful and will be frequently referblack to in the remainder of this work. (songmu)

Let $\Omega \subset \mathbb{R}^n$:

1-Cauchy inequality :

$$\forall u, v \in L^2(\Omega), \left| \int_\Omega uvdx \right| \leq \left(\int_\Omega |u|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |v|^2 \, dx \right)^{\frac{1}{2}}.$$

2-Cauchy inequality with ε :

(Also called ε −inequality)

$$|xy| \leq \frac{\varepsilon}{2} |x|^2 + \frac{1}{2\varepsilon} |y|^2, \text{ for } \varepsilon > 0 \text{ and } x, y \text{ arbitrary(real)}.$$

3-Cauchy-Schwartz inequality :

Let V be a Hilbert space

$$\forall u, v \in V, |\langle u, v\rangle_V| \leq \|u\|_V \|v\|_V .$$

4-Jensen inequality :

Let

$$\varphi(u) : u \in [\alpha, \beta] \longrightarrow \mathbb{R}$$

be a convex function. Suppose that

$$f : t \in [a, b] \longrightarrow [\alpha, \beta]$$

and $P(t)$ are continuous functions with

$$P(t) \geq 0, P(t) \neq 0$$

Then the following inequality holds

$$\varphi\left(\frac{\int_a^b f(t)P(t)dt}{\int_a^b P(t)dt}\right) \leq \frac{\int_a^b \varphi(f(t))\, P(t)dt}{\int_a^b P(t)dt}.$$

5-Gronwall Lemma :

If a, b are non-negative and integrable functions on $(0, T)$, as function b non-decreasing on $(0, T)$, and $\lambda \in L^1(0, T)$, $\lambda > 0$, we have :

$$a(t) \leq b(t) + \int_0^t \lambda(s) \cdot a(s)\, ds, \tag{1.6}$$

then

$$a(t) \leq b(t) \cdot \exp(\Lambda(t)),$$

$$\Lambda(t) = \int_0^t \lambda(s)\, ds.$$

Chapter 2

Existence, uniqueness and finite time blow-up for Neumann nonlinear parabolic problem

In this chapter we will present the study of Neumann boundary-value problem for a class of nonlinear parabolic equations. We will display the existence and the uniqueness of the strong solution of the linear problem by the energy inequality method. Then, by applying an iterative process based on the results obtained for the linear problem, to prove the existence and the uniqueness of the weak solution of the nonlinear problem. Finally, we determinate the finite time blow-up of the solutions.

2.1 Formulation of the nonlinear problem

Let $T > 0$, $\Omega = (0,1)$ and $Q = \Omega \times (0,T) = \{(x,t) \in \mathbb{R}^2,\ x \in \Omega \text{ and } 0 < t < T\}$. we mainly consider nonlinear parabolic problems of the form:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}\left(a(x,t)\dfrac{\partial u}{\partial x}\right) = f\left(x,t,u,\dfrac{\partial u}{\partial x}\right), & \forall\,(x,t) \in Q \\ u\,(x,0) = \varphi\,(x)\,, & \forall x \in \Omega \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0. & \forall t \in (0,T) \end{cases} \tag{P1}$$

Assuming that $f \in L^2(Q)$, $\varphi \in L^2\,(\Omega)$ are known functions, with the following conditions :

<u>Condition</u> 2.1 *The function f is lipschitzian, i.e : there exists a positive constant k such that*

:

$$\|f(x,t,u_1,v_1) - f(x,t,u_2,v_2)\|_{L^2(Q)} \leq k\left(\|u_1-u_2\|_{L^2(Q)} + \|v_1-v_2\|_{L^2(Q)}\right), \qquad (2.1)$$
$$\forall (u_1,v_1),(u_2,v_2) \in \left(L^2(Q)\right)^2.$$

<u>**Condition**</u> **2.2** *For each $(x,t) \in Q$, assuming that:*

$$a_0 \leq a(x,t) \leq a_1 \text{ and } a_2 \leq \frac{\partial a(x,t)}{\partial t} \leq a_3. \qquad (2.2)$$

such that : $a_i, i = \overline{0,1}$ are positive canstants. $a_i, i = \overline{2,3}$ are negative canstants

2.2 Study of the associated linear problem

For simplicity the study of the main problem, we formulate most of our assertions for the model case where reaction term is linear, we based the solving of the problem by the energy inequality method. Since this well-posedness of the linear problem will play a crucial role in many subsequent sections.

2.2.1 Position of the associated linear problem

In bounded rectangular area $Q = \Omega \times (0,T)$, with $T < \infty$, we consider the following linear problem :

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) = f(x,t), & \forall (x,t) \in Q \\ u(x,0) = \varphi(x), & \forall x \in (0,1) \\ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0. & \forall t \in (0,T) \end{cases}, \qquad (2.3)$$

Whose parabolic equation is given as follows :

$$\mathcal{L}u = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) = f(x,t), \qquad (2.4)$$

with the initial condition :

$$\ell u = u(x,0) = \varphi(x), \quad x \in (0,1), \qquad (2.5)$$

and the boundry conditions of Neumann type :

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0. \qquad (2.6)$$

Where f and φ are known functions . We note that the function φ satisfies the compatibility condition (2.6):

$$\frac{\partial \varphi}{\partial x}(1) = 0, \ \frac{\partial \varphi}{\partial x}(0) = 0. \tag{2.7}$$

2.2.2 A priori estimate (the uniqueness of the solution)

The problem (2.3) can be written in the following operational form :

$$Lu = \mathcal{F}, \tag{2.8}$$

where $L = (\mathcal{L}, \ell)$, with a domain of definition $D(L)$ consists of functions $u \in L^2(Q)$, such that $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x} \in L^2(Q)$ and u satisfies the boundary conditions in (2.6); the operator L is considered to be from E to F, where E is the Banach space of functions $u \in L^2(Q)$ and whose norm :

$$\|u\|_E^2 = \|u\|^2_{L^\infty(0,T,\ L^2(\Omega))} + \|\partial_x u\|^2_{L^2(0,T,\ L^2(\Omega))},$$

is finite, and F is the Hilbert space made up of all the elements $\mathcal{F} = (f, \varphi)$ whose norm :

$$\|\mathcal{F}\|_F^2 = \|f\|^2_{L^2(Q)} + \|\varphi\|^2_{L^2(\Omega)},$$

is finite.

Theorem 2.1 *For any function $u \in D(L)$, we have the estimate :*

$$\|u\|_E \leq k\, \|Lu\|_F \tag{2.9}$$

where k is a positive constant independent of u, such that :

$$k = \sqrt{\frac{e^T}{\min_{\{1,2a_0\}}}}$$

Proof Assuming a solution to the problem exists, multiply the equation (2.4) by the following multiplier Mu :

$$Mu = u\ ,$$

and by integrating on the domain $Q_\tau = (0,1)\times(0,\tau)$,where $\tau \in [0,T]$,we obtain :

$$\begin{aligned}
&\int\limits_{Q_\tau} \mathcal{L}u \cdot Mudxdt \\
&= \int\limits_{Q_\tau} \left[\partial_t u - \partial_x \left(a(x,t)\partial_x u\right)\right] \cdot u(x,t)dxdt \\
&= \int\limits_{Q_\tau} \partial_t u \cdot udxdt - \int\limits_{Q_\tau} \partial_x(a(x,t)\partial_x u) \cdot u(x,t)dxdt \\
&= \int\limits_{Q_\tau} f \cdot udxdt.
\end{aligned}$$

Where $\partial_x u, \partial_t u$ indicate the partial derivative to x, t respectively. We note:

$$\begin{aligned}
I_1 &= \int\limits_{Q_\tau} \partial_t u \cdot udxdt \\
I_2 &= -\int\limits_{Q_\tau} \partial_x(a(x,t)\partial_x u) \cdot u(x,t)dxdt \\
I_3 &= \int\limits_{Q_\tau} f \cdot udxdt.
\end{aligned}$$

Calculation of I_2 : based on integration by parts and using the Neumann conditions, we get:

$$\begin{aligned}
&-\int\limits_{Q_\tau} \partial_x(a(x,t)\partial_x u) \cdot udxdt \\
&= -\int\limits_0^\tau \int\limits_0^1 \partial_x(a(x,t)\partial_x u) \cdot udxdt \\
&= -\int\limits_0^\tau \left. ua(x,t)\partial_x u\right|_0^\tau dt + \int\limits_0^\tau \int\limits_0^1 (\partial_x u)^2 a(x,t)dxdt
\end{aligned}$$

We now use the boundary conditions to eliminate the boundary terms in the equation above, so we have:

$$I_2 = \int\limits_{Q\tau} (\partial_x u)^2 a(x,t)dxdt$$

Calculation of I_3 : by using the Cauchy inequality, we find :

$$\begin{aligned}&\int_{Q_\tau} f \cdot u dx dt \\ &\leq \frac{1}{2}\int_{Q\tau} f^{2} dx dt + \frac{1}{2}\int_{Q\tau} u^2 dx dt \\ &\leq \frac{1}{2}\int_{Q} f^{2} dx dt + \frac{1}{2}\int_{Q} u^2 dx dt\end{aligned}$$

so, it comes

$$\begin{aligned}&\int_{Q_\tau}\left[\partial_t u - \frac{\partial}{\partial x}\left(a(x,t)\partial_x u\right)\right]\cdot u dx dt \\ &= \frac{1}{2}\int_0^1 (u(x,\tau))^2 dx - \frac{1}{2}\int_0^1 (\varphi\,(x))^2 dx + \int_{Q\tau} (\partial_x u)^2 a(x,t) dx dt \\ &\leq \frac{1}{2}\int_Q f^2 dx dt + \frac{1}{2}\int_Q u^2 dx dt,\end{aligned}$$

thus

$$\begin{aligned}&\int_0^1 u(x,\tau)^2 dx + 2\int_{Q_\tau} (\partial_x u)^2 a(x,t) dx dt \\ &\leq \int_Q (f)^2 dx dt + \int_0^1 (\varphi\,(x))^2 dx + \int_Q (u)^2 dx dt\end{aligned}$$

Applying Gronwall's lemma, we find :

$$\begin{aligned}&\int_0^1 u(x,\tau)^2 dx + 2\int_{Q_\tau} (\partial_x u)^2 a(x,t) dx dt \\ &\leq e^{\int_0^T dt}\Big(\int_Q f^2 dx dt + \int_0^1 (\varphi\,(x))^2 dx\Big)\end{aligned}$$

By using the condition (2.2) , we get :

$$\begin{aligned}
&\int_0^1 u(x,\tau)^2dx + 2a_0\int_{Q_\tau}(\partial_x u)^2dxdt \\
&\leq \int_0^1 u(x,\tau)^2dx + 2\int_{Q_\tau}(\partial_x u)^2a(x,t)dxdt \\
&\leq e^T(\int_Q f^2dxdt + \int_0^1(\varphi\,(x))^2dx)
\end{aligned}$$

So we have now

$$\int_0^1 u(x,\tau)^2dx + 2a_0 \parallel \partial_x u \parallel^2_{L^2(Q_T)} \leq e^T(\parallel f \parallel^2_{L^2(Q)} + \parallel \varphi \parallel^2_{L^2(\Omega)})$$

Since the right part of the last inequality does not depend on t, we can pass to the maximum on the left part, so we find :

$$\begin{aligned}
&\max_{0<t<T}\int_0^1 u(x,\tau)^2dx + \parallel \partial_x u \parallel^2_{L^2(Q_T)} \\
&\leq \frac{e^T}{\min_{\{1,2a_0\}}}(\parallel f \parallel^2_{L^2(Q)} + \parallel \varphi \parallel^2_{L^2(\Omega)})
\end{aligned}$$

with

$$C = \frac{e^T}{\min_{\{1,2a_0\}}}.$$

Thus, we write

$$\|u\|^2_{L^\infty(0,T,\ L^2(0,1))} + \|\partial_x u\|^2_{L^2(0,T,L^2(0,1))} \leq C\left(\|f\ \|^2_{L^2(Q)} + \|\varphi\|^2_{L^2(0,1)}\right).$$

Which is equivalent to

$$\|u\|_E \leq k\,\|\mathcal{F}\|_F\,,\ \text{Where } k=\sqrt{C}.$$

Corollary 2.1 *If for any function $u \in D\,(L)$, we have the following estimate :*

$$\|u\|_E \leq k\,\|\mathcal{F}\|_F\,, \tag{2.10}$$

then the solution of the problem (2.3) *if it exists, it is unique.*

■

Proof Let u_1 and u_2 two solutions of the problem (2.3) :

$$\begin{cases} Lu_1 = \mathcal{F} \\ Lu_2 = \mathcal{F} \end{cases} \Longrightarrow Lu_1 - Lu_2 = 0,$$

and since L is linear we then obtain:

$$L(u_1 - u_2) = 0,$$

according to(2.10)

$$\|u_1 - u_2\|_E^2 \leq k \|0\|_F^2 = 0,$$

which gives

$$u_1 = u_2.$$

This completes the proof. ■

2.2.3 Study of the existence of the solution

To demonstrate that the solution is exist we prove the following items:

1. The operator
$$L : E \longrightarrow F$$
is closable.

2. $R(L)$ is dense in F for any $u \in E$ and for any arbitrary $\mathcal{F} = (f, \varphi) \in F$.

Proposition 2.1 *The operator L of E in F is closable.*

Proof let $\{u_n\} \in D(L)$ be a sequence such that :

$$u_n \longrightarrow 0 \text{ in } E, \tag{2.11}$$

and

$$Lu_n \longrightarrow (f; \varphi) \text{ in } F, \tag{2.12}$$

we must prove that

$$f \equiv 0 \text{ and } \varphi = 0$$

The convergence of u_n towards 0 in E implies :

$$u_n \longrightarrow 0 \ \text{ in } D^{'}(Q). \tag{2.13}$$

From the continuity of the derivation of $D^{'}(Q)$ in $D^{'}(Q)$. the relation (2.13) implies:

$$\mathcal{L}u_n \longrightarrow 0 \ \text{ in } D^{'}(Q), \tag{2.14}$$

Moreover, the convergence of $\mathcal{L}u_n$ towards f in $L^2(Q)$ generates:

$$\mathcal{L}u_n \longrightarrow f \ \text{ in } D^{'}(Q). \tag{2.15}$$

By vertue of the uniqueness of the limit in $D^{'}(Q)$, we conclude from (2.14) and (2.15) that

$$f = 0.$$

Then, it is generated from (2.12) that :

$$\ell u_n \longrightarrow \varphi \ \text{ in } L^2(\Omega).$$

On the other hand :

$$\begin{aligned}
&\|u_n\|_E^2 = \|u_n\|^2_{L^2(0,T,\ L^2(\Omega))} + \|\partial_x u_n\|^2_{L^2(Q)} \\
&\|u_n\|_E^2 \geq \|u_n\|^2_{L^2(0,T,\ L^2(\Omega))} \\
&\|u_n\|_E^2 \geq \|u_n(x,0)\|^2_{L^2(\Omega)}.
\end{aligned}$$

By crossing the limit, we find :

$$\lim_{n\to+\infty} \|u_n\|_E^2 \geq \|\varphi(x)\|^2_{L^2(\Omega)},$$

Since $u_n \longrightarrow 0$ in E thus $\|u_n\|_E^2 \longrightarrow 0$ in E, we find :

$$\|\varphi(x)\|^2_{L^2(0,1)} \leq 0,$$

from where $\varphi = 0$. Which is our result. ■

Let $\overline{L}$ be the closure of L,and $D(\overline{L})$ the domaine of definition of $\overline{L}$:

Definition 2.1 *The solution of the equation*

$$\overline{L}u = \mathcal{F},$$

is called the generalized strong solution of the problem (2.3).

The theorem (2.1) is valid for a strong generalized solution,that is to say; we have the inequality:

$$\|u\|_E \leq m \left\|\overline{L}u\right\|_F \ \ \forall u \in D(\overline{L}). \tag{2.16}$$

Consequently,this last inequality leads to the following corollaries :

Corollary 2.2 *The solution of the problem* (2.3) *if it exists,it is unique and depends continuously on $\mathcal{F} \in F$.*

Corollary 2.3 *The set of value $R(\overline{L})$ of the operator $\overline{L}$ is equal to the closure $\overline{R(L)}$ of $R(L)$.*

Proof Let $z \in \overline{R(L)}$, so there is a cauchy sequence $(z_n)_{n\in\mathbb{N}}$in F constituted of the elements of the set $R(L)$ such as

$$\lim_{n\longrightarrow+\infty} z_n = z.$$

There is then a corresponding sequence $u_n \in D(L)$ such as

$$z_n = Lu_n.$$

The estimate (2.9) , we get

$$\|u_p - u_q\|_E \leq k \|Lu_p - Lu_q\|_F \to 0,$$

Where p, q tend towards infinity. We can deduce that $(u_n)_{n\in\mathbb{N}}$is a cauchy sequence in E, so like E is a Banach space, it exists $u \in E$ such as

$$\lim_{n\longrightarrow+\infty} u_n = u \ \text{ in } E.$$

By virtue of the definition of $\bar{L}$ ($\lim_{n\longrightarrow+\infty} u_n = u$ in E ; If $\lim_{n\longrightarrow+\infty} Lu_n = \lim_{n\longrightarrow+\infty} z_n = z$, then $\lim_{n\longrightarrow+\infty} \bar{L}u_n = z$ as like $\bar{L}$ and is closed, so $\bar{L}u = z$), the function u check :

$$u \in D\left(\bar{L}\right), \ \bar{L}u = z.$$

Then $z \in R(\bar{L})$, so

$$\overline{R(L)} \subset R(\bar{L})$$

Also we conclude here that $R(\bar{L})$ is closed becauce it is banach (any complete subspace of a metric space (not necessarily complete) is closed. It remains to show the reverse inclusion

Either $z \in R(\bar{L})$ then it exists a cauchy sequence $(z_n)_{n\in\mathbb{N}}$in F constituted of the elements of the set $R(\bar{L})$ such that

$$\lim_{n\longrightarrow+\infty} z_n = z$$

or $z \in R(\bar{L})$, because $R(\bar{L})$ is a closed subset a completed F, So $R(\bar{L})$ is complet. There is then a corresponding sequence $u_n \in D(\bar{L})$ such that

$$\bar{L}u_n = z_n.$$

We get from (2.9):

$$\|u_p - u_q\|_E \leq C \left\|\bar{L}u_p - \bar{L}u_q\right\|_F \to 0,$$

Where p, q tend towards infinity. We can deduce that $(u_n)_{n\in\mathbb{N}}$is a cauchy sequence in E, so like E is a Banach space, it exists $u \in E$ such as

$$\lim_{n\longrightarrow+\infty} u_n = u \ \text{ in } E.$$

Once again, there is a corresponding sequel $(Lu_n)_{n\in\mathbb{N}} \subset R(L)$ such as

$$\bar{L}u_n = Lu_n \text{ on } R\left(L\right), \forall n \in \mathbb{N}.$$

So

$$\lim_{n\longrightarrow+\infty} Lu_n = z,$$

Consequently $z \in \overline{R\left(L\right)}$, then we conclude that

$$R\left(\bar{L}\right) \subset \overline{R\left(L\right)}.$$

■

Theorem 2.2 *For each $\mathcal{F} = (f, \varphi) \in F$, the problem* (2.3) *admets a unique strong solution* $u = \overline{L}^{-1}\mathcal{F}$.

Proposition 2.2 *If for $\omega \in L^2\left(Q\right)$ and for any $u \in D_0\left(L\right)$, we have*

$$\int_Q \mathcal{L}u \cdot \omega dxdt = 0, \tag{2.17}$$

then ω vanishes almost everywhere in Q.

Proof The scalar product of F is defined by :

$$(Lu, W)_F = \int_Q \mathcal{L}u \cdot \omega dxdt + \int_0^1 \left(\ell u\right) \cdot \left(\omega_0\right) dx, \tag{2.18}$$

where $W = (\omega, \omega_0)$. The equality (2.17) can be written as follows :

$$\int_Q \frac{\partial u}{\partial t} \cdot \omega dxdt - \int_Q \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) \cdot \omega dxdt = 0, \tag{2.19}$$

which implies
$$\int_Q \frac{\partial u}{\partial t} \cdot \omega dxdt = \int_Q \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) \cdot \omega dxdt, \tag{2.20}$$
where u, $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x} \in L^2(Q)$, with u satisfies the boundary conditions of (2.3). We put
$$u(x,t) = \int_0^t z(x,\tau)d\tau = \Im_t z \tag{2.21}$$
by replacing (2.21) in (2.20) we get
$$\int_Q z \cdot \omega dxdt = \int_Q \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial \Im_t z}{\partial x}\right) \omega dxdt. \tag{2.22}$$
During the establishement of the function ω,and from this last equality, we give the function ω in terms of the function z as follows:
$$\omega = x\Im_t z$$
since z satisfies the same conditions as the function u in (2.3), then z, $\frac{\partial z}{\partial x} \in L^2(Q)$, so $\omega \in L^2(Q)$. Now replacing ω in (2.22)we obtain:
$$\int_Q xz\Im_t z dxdt = \int_Q \Im_t z \cdot \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial \Im_t z}{\partial x}\right) dxdt.$$
According to an integration by part and using the boundary conditionof Neuman, we get :
$$= \int_0^1 \frac{x}{2}(\Im_t z)^2\Big|_{\tau=0}^{\tau=T} dx = \int_Q a(x,t)\left(\frac{\partial \Im_t z}{\partial x}\right)^2 dxdt \leqslant 0;$$
which gives
$$\int_Q a(x,t)\left(\Im_t z\right)^2 dxdt = 0.$$
So
$$(\Im_t z) = 0.$$
Therefore, it becomes $u = 0$ in Q, which gives $\omega = 0$ in Q. This was to be demonstrated. Let as now return to the proof of the theorem (2.2). We consider a function $W = (\omega, \omega_0) \in R(L)^{\perp}$ and for any $u \in D(L)$, then W verifies the following equality:
$$(Lu, W)_F = \int_Q \mathcal{L}u \cdot \omega dxdt + \int_0^1 (\ell u) \cdot (\omega_0) dx = 0.$$

Then, we must prove that $W = 0$. We suppose $u \in D_0(L)$ in (2.18), we obtain:

$$\int_Q \mathcal{L}u \cdot \omega \, dxdt = 0, \ u \in D_0(L).$$

From the proposition (2.2), we deduce that $\omega = 0$. Thus (2.18) take the form:

$$\int_0^1 (\ell u) \cdot (\omega_0) \, dx = 0, \ u \in D(L). \tag{2.23}$$

Since the set of value of the operator ℓ is dense everywhere in the Hilbert space F with the norm

$$\left(\int_0^1 \left[(\ell u)^2 \right] \right)^{\frac{1}{2}},$$

the equality (2.23) implies that $\omega_0 = 0$. Therefore $W = 0$ implies $\overline{R(L)} = F$. This which completes the proof of the theorem (2.2). ■

2.3 Existence and uniqueness of a weak solution of the non-linear problem (Linearization method)

This section is devoted to the proof of the existence, the uniqueness of the solution of the nonlinear problem $(P1)$. We consider the following auxiliary problem with the homogeneous equation:

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\partial}{\partial x}\left(a(x,t) \dfrac{\partial w}{\partial x} \right) = 0 \\ w\left(x,0\right) = \varphi\left(x\right) \\ \dfrac{\partial w}{\partial x}(0,t) = \dfrac{\partial w}{\partial x}(1,t) = 0 \end{cases}, \tag{P2}$$

If u is a solution of the problem $(P1)$ and w is a solution of the problem $(P2)$, then $y = u - w$ satisfies

$$\frac{\partial y}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t) \frac{\partial y}{\partial x} \right) = G\left(x, t, y, \frac{\partial y}{\partial x} \right), \tag{2.24}$$

$$y(x,0) = 0, \tag{2.25}$$

$$\frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(1,,t) = 0 \tag{2.26}$$

Where $G\left(x,t,y,\dfrac{\partial y}{\partial x}\right)=f\left(x,t,y+w,\dfrac{\partial y}{\partial x}+\dfrac{\partial w}{\partial x}\right)$. As the function f is lipschitzian the G function is also lipschitzian, i.e there exists a positive constant k such that

$$\left\|G\left(x,t,u_1,v_1\right)-G\left(x,t,u_2,v_2\right)\right\|_{L^2(Q)}\leq k\left(\left\|u_1-u_2\right\|_{L^2(Q)}-\left\|v_1-v_2\right\|_{L^2(Q)}\right),\quad \forall\left(u_1,v_1\right),\left(u_2,v_2\right)\in\left(L^2\left(Q\right)\right)^2. \tag{2.27}$$

From the result of the previous section, we deduce that the problem $(P2)$ has a unique solution which depends continuously on the data. So it remains to prove that the problem $(2.24)-(2.26)$ admits a unique weak solution.

First, we propose the concept of stadied solution. Let $v=v\left(x,t\right)$ any function of $L^2\left(0,T;\ H^1\left(0,1\right)\right)$. Then, multiplying (2.24) by v then we integrate it on Q

$$\int_Q\frac{\partial y}{\partial t}vdxdt-\int_Q\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial y}{\partial x}\right)vdxdt=\int_Q G\left(x,t,y,\frac{\partial y}{\partial x}\right)vdxdt.$$

Next, using integration by parts and the conditions on y, it comes:

$$\int_Q\frac{\partial y}{\partial t}vdxdt+\int_Q a(x,t)\frac{\partial y}{\partial x}\frac{\partial v}{\partial x}dxdt=\int_{Q_\tau}vG\left(x,t,y,\frac{\partial y}{\partial x}\right)dxdt, \tag{2.28}$$

It follows then from (2.28) that

$$A\left(y,v\right)=\int_{Q_\tau}vG\left(x,t,y,\frac{\partial y}{\partial x}\right)dxdt, \tag{2.29}$$

Where

$$A\left(y,v\right)=\int_Q\frac{\partial y}{\partial t}vdxdt+\int_Q a(x,t)\frac{\partial y}{\partial x}\frac{\partial v}{\partial x}dxdt.$$

Definition 2.2 *A function $y\in L^2\left(0,T;\ H^1\left(0,1\right)\right)$ is said to be a weak solution of the problem $(2.24)-(2.26)$ if (2.29),(2.25) and (2.26) are fulfilled.*

Constructing a recurrent sequence starting with $y^{(0)}=0$,the sequence $\left(y^{(n)}\right)_{n\in\mathbb{N}}$ is defined as follows : given the element $y^{(n-1)}$, then for $n=1,2,3,....$ We will solve the following problem :

$$\begin{cases}\dfrac{\partial y^{(n)}}{\partial t}-\dfrac{\partial}{\partial x}\left(a(x,t)\dfrac{\partial y^{(n)}}{\partial x}\right)=G\left(x,t,y^{(n-1)},\dfrac{\partial y^{(n-1)}}{\partial x}\right)\\ y^{(n)}\left(x,0\right)=0\\ \dfrac{\partial y^{(n)}}{\partial x}(0,t)=\dfrac{\partial y^{(n)}}{\partial x}(1,t)=0\end{cases}, \tag{P3}$$

according to the theorem (2.2) ,and each time we fix the n,the problem $(P3)$ admits a unique solution $y^{(n)}(x,t)$.

Suppose now $z^{(n)}(x,t) = y^{(n+1)}(x,t) - y^{(n)}(x,t)$, so we get a new problem :

$$\begin{cases} \dfrac{\partial z^{(n)}}{\partial t} - \dfrac{\partial}{\partial x}\left(a(x,t)\dfrac{\partial z^{(n)}}{\partial x}\right) = p^{(n-1)}(x,t) \\ z^{(n)}(x,0) = 0 \\ \dfrac{\partial z^{(n)}}{\partial x}(0,t) = \dfrac{\partial z^{(n)}}{\partial x}(1,t) = 0 \end{cases}, \tag{P$_Z$}$$

where

$$p^{(n-1)}(x,t) = G\left(x,t,y^{(n)},\frac{\partial y^{(n)}}{\partial x}\right) - G\left(x,t,y^{(n-1)},\frac{\partial y^{(n-1)}}{\partial x}\right).$$

Lemma 2.1 *Suppose the condition (2.27) is satisfied. So for the problem (P_Z), we have a priori estimate :*

$$\|z^n\|_{L^2\left(0,T;\ H^1(0,1)\right)} \leq c\|z^{n-1}\|_{L^2\left(0,T;\ H^1(0,1)\right)}$$

Where

$$c = \sqrt{\frac{2k^2\exp(T)}{\min\{1,2a_0\}}}$$

Proof By multiplying

$$\frac{\partial z^{(n)}}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial z^{(n)}}{\partial x}\right) = p^{(n-1)}(x,t), \tag{2.30}$$

by $z^{(n)}(x,t)$ and we integrate it on Q_τ,we obtain:

$$\int\limits_{Q_\tau} \frac{\partial z^{(n)}}{\partial t}(x,t)\cdot z^{(n)}(x,t)\,dxdt - \int\limits_{Q_\tau} \frac{\partial}{\partial x}(a(x,t)\frac{\partial z^{(n)}}{\partial x}(x,t))\cdot z^{(n)}(x,t)\,dxdt$$
$$= \int\limits_{Q_\tau} p^{(n-1)}(x,t)\cdot z^{(n)}(x,t)\,dxdt.$$

putting

$$I_1 = \int\limits_{Q_\tau} \frac{\partial z^{(n)}}{\partial t}\cdot z^{(n)}dxdt$$

$$I_{2=} - \int\limits_{Q_\tau} \frac{\partial}{\partial x}(a(x,t)\frac{\partial z^{(n)}}{\partial x})\cdot z^{(n)}dxdt$$

$$I_3 = \int\limits_{Q_\tau} p^{(n-1)}\cdot z^{(n)}dxdt.$$

Starting from the calculation of I_1, we use an integration by parts considering initial and boundary conditions, we find:

$$I_1 = \int\limits_{Q_\tau} \frac{\partial z^{(n)}}{\partial t} \cdot z^{(n)} dxdt$$
$$= \frac{1}{2}\int\limits_0^1 (z^{(n)}(x,\tau))^2 dx.$$

For I_2, using integration by parts, the boundary conditions and the condition $a_0 \leq a(x,t) \leq a_1$, we get:

$$I_2 = -\int\limits_{Q_\tau} \frac{\partial}{\partial x}(a(x,t)\frac{\partial z^{(n)}}{\partial x}) \cdot z^{(n)} dxdt.$$
$$= \int\limits_{Q_T} (\frac{\partial z^{(n)}}{\partial x})^2 a(x,t) dxdt$$
$$\geq a_0 \int\limits_{Q_T} (\frac{\partial z^{(n)}}{\partial x})^2 dxdt$$

To calculate I_3, by using cauchy inequality, we find that:

$$I_3 = \int\limits_{Q_\tau} p^{(n-1)} \cdot z^{(n)} dxdt$$
$$\leq \frac{1}{2} \parallel p^{(n-1)} \parallel^2_{L^2(Q_T)} + \frac{1}{2} \parallel z^{(n)} \parallel^2_{L^2(Q_T)},$$
$$\leq \frac{1}{2} \parallel p^{(n-1)} \parallel^2_{L^2(Q)} + \frac{1}{2} \parallel z^{(n)} \parallel^2_{L^2(Q)} .$$

On other hand we have:

$$\left|p^{n-1}\right|^2 = \left|G\left(x,t,y^{(n)},\frac{\partial y^{(n)}}{\partial x}\right) - G\left(x,t,y^{(n-1)},\frac{\partial y^{(n-1)}}{\partial x}\right)\right|^2,$$

as G is lipshitzian and that $(a+b)^2 \leq 2\left(a^2+b^2\right)$, we get that:

$$\left|p^{n-1}\right|^2 \leq k^2\left(\left|y^{(n)} - y^{(n-1)}\right| + \left|y_x^{(n)} - y_x^{(n-1)}\right|\right)^2$$
$$= k^2\left(\left|z^{(n-1)}\right| + \left|z_x^{(n-1)}\right|\right)^2$$
$$\leq 2k^2\left(\left|z^{(n-1)}\right|^2 + \left|z_x^{(n-1)}\right|^2\right),$$

By integrating on Q, we get:

$$\int\limits_Q \left|p^{n-1}\right|^2 dxdt \leq 2k^2 \int\limits_Q \left(\left(\left|z^{(n-1)}\right|\right)^2 + \left(\left|z_x^{(n-1)}\right|\right)^2\right) dxdt,$$

from where

$$\|p^{n-1}\|^2_{L^2(Q)} \leq 2k^2 \|z^{(n-1)}\|^2_{L^2\left(0,T;\ H^1(0,1)\right)}. \tag{2.31}$$

So, we have

$$I_3 \leq \frac{1}{2} \parallel z^{(n-1)} \parallel^2_{L^2(0,T;H^1(0,1))} + \frac{1}{2} \parallel z^{(n)} \parallel^2_{L^2(Q)}$$

Finally, we obtain

$$\frac{1}{2}\int\limits_0^1 (z^{(n)}(x,\tau))^2 dx + a_0 \int\limits_{Q\tau} (\frac{\partial z^{(n)}}{\partial x})^2 dxdt$$
$$\leq k^2 \parallel z^{(n-1)} \parallel^2_{L^2(0,T;H^1(0,1))} + \frac{1}{2} \parallel z^{(n)} \parallel^2_{L^2(Q)}$$

So

$$\int\limits_0^1 (z^{(n)}(x,\tau))^2 dx + 2a_0 \int\limits_{Q\tau} (\frac{\partial z^{(n)}}{\partial x})^2 dxdt$$
$$\leq 2k^2 \parallel z^{(n-1)} \parallel^2_{L^2(0,T;H^1(0,1))} + \parallel z^{(n)} \parallel^2_{L^2(Q)}$$

we apply Gronwall's lemma then pass to the maximum on $(0,T)$, we find:

$$\|z^n\|^2_{L^\infty\left(0,T;\ L^2(0,1)\right)} + \|z^n\|^2_{L^2\left(0,T;\ H^1(0,1)\right)}$$
$$\leq \frac{2k^2 \exp(T)}{\min\{1, 2a_0\}} \|z^{(n-1)}\|^2_{L^2\left(0,T;\ H^1(0,1)\right)},$$

So, we get

$$\|z^n\|_{L^2\left(0,T;\ H^1(0,1)\right)} \leq c\|z^{n-1}\|_{L^2\left(0,T;\ H^1(0,1)\right)},$$

where

$$c = \sqrt{\frac{2k^2 \exp(T)}{\min\{1, 2a_0\}}},$$

this acheived the demenstration. ■

Let be the series $\sum\limits_{n=1}^{\infty} z^{(n)}$, according to the criterion of the convergence of the series,it is said to be convergent if

$$\sqrt{\frac{2k^2 \exp(T)}{\min\{1, 2a_0\}}} < 1,$$

$$k^2 < \frac{\min\{1, 2a_0\}}{2\exp(T)}$$

Which implies that

$$k < \sqrt{\frac{\min\{1, 2a_0\}}{2\exp(T)}}.$$

As $z^{(n)}(x,t) = y^{(n+1)}(x,t) - y^{(n)}(x,t)$ and $y^{(0)}(x,t) = 0$, we have

$$\begin{aligned}\sum_{i=0}^{n-1} z^{(i)} &= \sum_{i=0}^{n-1} \left(y^{(i+1)} - y^{(i)}\right) \\ &= y^{(1)} - y^{(0)} + y^{(2)} - y^{(1)} + ... + y^{(n)} - y^{(n-1)} \\ &= y^{(n)},\end{aligned}$$

then we have the sequence $\left(y^{(n)}\right)_{n\in\mathbb{N}}$ defined by

$$y^{(n)} = \sum_{i=0}^{n-1} z^{(i)},$$

is convergente to an element $y \in L^2\left(0, T;\ H^1(0,1)\right)$.

Now,we will prove that $\lim_{n\longrightarrow\infty} y^{(n)}(x,t) = y(x,t)$ is a solution of the problem $(P3)$ by showing that y satisfies

$$A(y, v) = \int_{Q_\tau} v(x,t)G(x,t)dxdt.$$

Therefore we consider the weak formulation of the following problem $(P3)$:

$$A\left(y^{(n)}, v\right) = \int_Q \frac{\partial y^{(n)}}{\partial t} v dxdt + \int_Q a(x,t)\frac{\partial y^{(n)}}{\partial x}\frac{\partial v}{\partial x}dxdt.$$

From the linearity of A we have:

$$\begin{aligned}A\left(y^{(n)}, v\right) &= A\left(y^{(n)} - y, v\right) + A(y, v) \\ &= \int_Q \frac{\partial(y^{(n)} - y)}{\partial t} v dxdt + \int_Q a(x,t)\frac{\partial(y^{(n)} - y)}{\partial x}\frac{\partial v}{\partial x}dxdt + A(y, v).\end{aligned} \tag{2.32}$$

We apply the Cauchy Schwartz inequality on $A\left(y^{(n)} - y, v\right)$ and the condition $a_0 \leq a(x,t) \leq a_1$, we get

$$\begin{aligned}&\int_Q \frac{\partial(y^{(n)} - y)}{\partial t} v dxdt + \int_Q a(x,t)\frac{\partial(y^{(n)} - y)}{\partial x}\frac{\partial v}{\partial x}dxdt \\ &\leq \int_Q \frac{\partial(y^{(n)} - y)}{\partial t} v dxdt + a_0\int_Q \frac{\partial(y^{(n)} - y)}{\partial x}\frac{\partial v}{\partial x}dxdt, \\ &\leq \max\{1, a_0\} \cdot \|v_x\|_{L^2(Q)}\left[\|\left(y^{(n)} - y\right)_t\|_{L^2\left(0,T;\ H^1(0,1)\right)} + \|\left(y^{(n)} - y\right)\|_{L^2\left(0,T;\ H^1(0,1)\right)}\right].\end{aligned}$$

On the other hand as

$$y^{(n)} \longrightarrow y \qquad \text{in } L^2\left(0,T;\ H^1(0,1)\right),$$

So

$$\begin{aligned} y^{(n)} &\longrightarrow y && \text{in } L^2\left(0,T;\ L^2(0,1)\right), \\ y_t^{(n)} &\longrightarrow y_t && \text{in } L^2\left(0,T;\ L^2(0,1)\right), \\ y_x^{(n)} &\longrightarrow y_x && \text{in } L^2\left(0,T;\ L^2(0,1)\right). \end{aligned}$$

Let us pass to the limit when $n \longrightarrow +\infty$, we find

$$\lim_{n\longrightarrow+\infty} A\left(y^{(n)} - y, v\right) = 0. \tag{2.33}$$

According to (2.33) and by passing to the limite in (2.32) we obtain

$$\lim_{n\longrightarrow+\infty} A\left(y^{(n)}, v\right) = A(y,v).$$

Thus, we have proved the following result.

Theorem 2.3 *If the condition* (2.27) *is satisfied. And*

$$k < \sqrt{\frac{\min\{1, 2a_0\}}{2\exp(T)}}.$$

Then the problem (2.24) – (2.26) *admits a weak solution belonging to* $L^2(0,T;\ L^2(0,1))$.

Now, we will show that the solution of the problem (2.24) – (2.26) is unique.

Theorem 2.4 *If the condition* (2.27) *is verified, then the solution is unique.*

Proof Let y_1, y_2 in $L^2(0,T\ ;H^1(0,1))$ be two solution of (2.24) – (2.26), then

$$y = y_1 - y_2,$$

is also a solution for the problem. So $\forall (x,t) \in Q$, we get the following problem

$$\begin{cases} \dfrac{\partial y}{\partial t} - \dfrac{\partial}{\partial x}\left(a(x,t)\dfrac{\partial y}{\partial x}(x,t)\right) = G\left(x,t,y_1,\dfrac{\partial y_1}{\partial x}\right) - G\left(x,t,y_2,\dfrac{\partial y_2}{\partial x}\right) & \forall (x,t) \in Q \\ y(x,0) = 0 & \forall x \in \Omega \\ \dfrac{\partial y}{\partial x}(0,t) = \dfrac{\partial y}{\partial x}(1,t) = 0 & \forall t \in [0,T\ ] \end{cases},$$

By putting

$$\Psi(x,t) = G\left(x,t,y_1,\frac{\partial y_1}{\partial x}\right) - G\left(x,t,y_2,\frac{\partial y_2}{\partial x}\right).$$

We obtain

$$\begin{cases} \dfrac{\partial y}{\partial t} - \dfrac{\partial}{\partial x}\left(a(x,t)\dfrac{\partial y}{\partial x}\right) = \Psi(x,t), & \forall\,(x,t)\in Q \\ y\,(x,0) = 0, & \forall x\in\Omega \\ \dfrac{\partial y}{\partial x}(0,t) = \dfrac{\partial y}{\partial x}(1,t) = 0 & \forall t\in[0,T\,]. \end{cases}$$

Using the lemma (2.1) , we get

$$\|y\|_{L^2(0,T,H^1(0,1))} \leq c\|y\|_{L^2(0,T,H^1(0,1))},$$

from where

$$(1-c)\,\|y\|_{L^2(0,T,H^1(0,1))} \leq 0,$$

and as $c \leq 1$, we get then

$$\|y\|_{L^2(0,T,H^1(0,1))} = 0,$$

from where

$$y_1 = y_2,$$

which gives the uniqueness of the solution. ■

2.4 The blowing up solutions in a finite time

In this section, we are interested in studying the finite-time explosion of solutions of a class of nonlinear parabolic equations in special cases where $f\left(x,t,u,\dfrac{\partial u}{\partial x}\right) = u^p(x,t)$.

2.4.1 Statement of problem

Let $T>0$, $\Omega=(0,1)$ and $Q=\Omega\times(0,T)=\{(x,t)\in\mathbb{R}^2,\ x\in\Omega \text{ and } 0<t<T\}$. Consider the following nonlinear problem :

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial}{\partial x}\left(a(x,t)\dfrac{\partial u}{\partial x}(x,t)\right) = u^p(x,t), & \forall\,(x,t)\in Q \\ u\,(x,0) = \varphi\,(x)\,, & \forall x\in\Omega \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(1,t) = 0. & \forall t\in(0,T) \end{cases} \tag{Pb}$$

Assuming that $\varphi \in L^2(\Omega)$ is known function and $p > 1$.

Whose parabolic equation is given as follows:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) = u^p, \quad \forall (x,t) \in Q \tag{2.34}$$

With the initial condition:

$$u(x,0) = \varphi(x), \quad \forall x \in \Omega$$

And the boundry condition of Neuman type :

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0, \quad \forall t \in (0,T)$$

<u>Notation</u> 2.1 *To simplify writing we put* $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial t}(x,t)$ *and* $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}(x,t)$; $x \in \Omega$, $t \in (0,t)$. *which denote to the partial derivative with respect to* t, x *respectively and* $u = u(x,t); \forall x \in \Omega, t \in (0,t)$.

2.4.2 Finite time blow up main results

<u>Theorem</u> 2.5 *For* $p > 1$, $\varphi \in L^2(\Omega)$ *is known function and under the assumption that* $\frac{\partial a}{\partial t}$ *is a negative function* $\forall (x,t) \in Q$, *the problem* (Pb) *blows up in a finite time* T^**such that:*

$$T^* = \frac{p+1}{(1-p)^2}\Pi(0)^{\frac{1-p}{2}}, \text{ where } \Pi(0) = \int_\Omega \varphi^2 dx.$$

for a negative intial energy function and initial data are sufficiently large .

<u>Proof</u> the proof is devised to two main steps; the first one we definite the energy function, then we determine the time T^* of the explosion phenomena as a second step.

Step1: Definition of the energy function

To determine the energy function, we multiplying the equation (2.34) by $\frac{\partial u}{\partial t}$ and integrating over Ω, we obtain the following result :

$$\int_\Omega \frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial t} dx - \int_\Omega \left(\frac{\partial}{\partial x}(a(x,t)\frac{\partial u}{\partial x})\right) \cdot \frac{\partial u}{\partial t} dx = \int_\Omega u^p \cdot \frac{\partial u}{\partial t} dx,$$

An integration by parts of the previous equation over Ω gives :

$$\int_\Omega \frac{\partial u}{\partial t}^2 dx - \left[\frac{\partial u}{\partial t} a(x,t) \frac{\partial u}{\partial x}\Big|_0^1 - \int_\Omega \frac{\partial u}{\partial x \partial t} a(x,t) \frac{\partial u}{\partial x} dx \right]$$
$$= \frac{1}{p+1} \frac{d}{dt} \int_\Omega u^{p+1} dx,$$

Then, we find :

$$\int_\Omega \frac{\partial u}{\partial t}^2 dx + \int_\Omega a(x,t) \frac{1}{2} \frac{d}{dt} \frac{\partial u}{\partial x}^2 dx = \frac{1}{p+1} \frac{d}{dt} \int_\Omega u^{p+1} dx, \tag{2.35}$$

By using the derivation low, we obtain :

$$\frac{d}{dt}\left(a(x,t) \frac{\partial u}{\partial x}^2 \right) = \frac{d}{dt}(a(x,t))(\frac{\partial u}{\partial x})^2 + \frac{d}{dt}\left(\frac{\partial u}{\partial x}^2 \right) a(x,t),$$

Which means that :

$$\frac{d}{dt}\left(\frac{\partial u}{\partial x}^2 (x,t) \right) a(x,t) = \frac{d}{dt}\left(a(x,t) \frac{\partial u}{\partial x}^2 (x,t) \right) - \frac{d}{dt}(a(x,t)) \frac{\partial u}{\partial x}^2 (x,t), \tag{2.36}$$

Combining (2.36) with (2.35) yeilds :

$$\int_\Omega \frac{\partial u}{\partial t}^2 dx + \frac{1}{2} \int_\Omega \frac{d}{dt}\left(a(x,t) \frac{\partial u}{\partial x}^2 \right) dx - \frac{1}{2} \int_\Omega \frac{d}{dt}(a(x,t)) \frac{\partial u}{\partial x}^2 dx,$$
$$= \frac{1}{p+1} \frac{d}{dt} \int_\Omega u^{p+1} dx,$$

Similary, we get

$$\frac{1}{2} \int_\Omega \frac{d}{dt}\left(a(x,t) \frac{\partial u}{\partial x}^2 \right) dx - \frac{1}{2} \int_\Omega \frac{d}{dt}(a(x,t)) \frac{\partial u}{\partial x}^2 dx - \frac{1}{p+1} \frac{d}{dt} \int_\Omega u^{p+1} dx$$
$$= -\int_\Omega \frac{\partial u}{\partial t}^2 dx,$$

Under the assumption (2.2), we get

$$\frac{d}{dt} \int_\Omega \left(a(x,t) \frac{\partial u}{\partial x}^2 \right) dx - a_3 \int_\Omega \frac{\partial u}{\partial x}^2 dx - \frac{2}{p+1} \frac{d}{dt} \int_\Omega u^{p+1} dx$$
$$\leq 0,$$

And for $-a_3 > 0$, we get:

$$\frac{d}{dt}\int_{\Omega}\left(a(x,t)\frac{\partial u}{\partial x}^2\right)dx - \frac{2}{p+1}\frac{d}{dt}\int_{\Omega}u^{p+1}dx$$
$$\leq -2\int_{\Omega}\frac{\partial u}{\partial t}^2 dx,$$

Then

$$\frac{d}{dt}\left(\int_{\Omega}\left(a(x,t)\frac{\partial u}{\partial x}^2\right)dx - \frac{2}{p+1}\int_{\Omega}u^{p+1}dx\right)$$
$$< 0.$$

Finally, we obtain the energy function, which defined in the following form:

$$E(t) = \int_{\Omega}\left(a(x,t)\frac{\partial u}{\partial x}^2\right)dx - \frac{2}{p+1}\int_{\Omega}u^{p+1}dx.$$

Step2: Determination of the finite time blow-up:

After finding the expression of the energy function E and from the pervious step we conclude that E is a decreasing function which means that

$$\forall t \in (0,T),\ E(t) < E(0),$$

Under the assumption $E(0) < 0$ we obtain that

$$\forall t \in (0,T),\ E(t) < 0,$$

On the other hand, by multiplying (2.34) by u and integrating over Ω, we find

$$\int_{\Omega}\frac{\partial u}{\partial t}\cdot u - \int_{\Omega}\left(\frac{\partial}{\partial x}(a(x,t)\frac{\partial u}{\partial x})\right)\cdot u dx = \int_{\Omega}u^p \cdot u dx,$$

by using the Green's formula, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}u^2 dx - \left[a(x,t)u\frac{\partial u}{\partial x}\Big|_0^1 - \int_{\Omega}a(x,t)\frac{\partial u}{\partial x}^2 dx\right] - \int_{\Omega}u^{p+1}dx = 0,$$

also, by using the limit conditions we find

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}u^2 dx + \int_{\Omega}a(x,t)\frac{\partial u}{\partial x}^2 dx - \int_{\Omega}u^{p+1}dx = 0,$$

So it comes

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} u^2 + E(t) = \frac{p-1}{p+1}\int_{\Omega} u^{p+1}dx,$$

Then

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} u^2 dx = \frac{p-1}{p+1}\int_{\Omega} u^{p+1}dx - E(t),$$

Since $-E(t) \geq 0$,we result that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} u^2 dx \geq \frac{p-1}{p+1}\int_{\Omega} u^{p+1}dx. \tag{2.37}$$

On the other hand, by using the Holder inequality we get

$$\begin{aligned}\int_{\Omega} u^2 dx &\leq \left(\int_{\Omega} 1^q dx\right)^{\frac{1}{q}} \cdot \left(\int_{\Omega} \left(u^2\right)^{\frac{p+1}{2}}\right)^{\frac{2}{p+1}} \\ &\leq \left(\int_{\Omega} u^{p+1} dx\right)^{\frac{2}{p+1}}.\end{aligned}$$

So we get

$$\int_{\Omega} u^{p+1}dx \geq \left(\int_{\Omega} u^2 dx\right)^{\frac{p+1}{2}}. \tag{2.38}$$

Collecting (2.37) with (2.38), we attain the following inequality

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} u^2 dx \geq \frac{p-1}{p+1}\left(\int_{\Omega} u^2 dx\right)^{\frac{p+1}{2}}.$$

By putting

$$\Pi(t) = \int_{\Omega} u^2 dx,$$

We get

$$\frac{1}{2}\frac{d}{dt}\Pi(t) \geq \frac{p-1}{p+1}\Pi(t)^{\frac{p+1}{2}},$$

Then

$$\frac{d\Pi(t)}{\Pi(t)^{\frac{p+1}{2}}} \geq \frac{2(p-1)}{p+1}dt,$$

By integrating over $(0,t)$, we obtain

$$\int_0^t \frac{d\Pi(\tau)}{\Pi(\tau)^{\frac{p+1}{2}}} \geq \int_0^t \frac{2(p-1)}{p+1} d\tau,$$

So

$$\frac{2}{1-p}\Pi(t)^{\frac{1-p}{2}} - \frac{2}{1-p}\Pi(0)^{\frac{1-p}{2}} \geq \frac{2(p-1)}{p+1}t,$$

attention to

$$p > 1 \Rightarrow \frac{1-p}{2} < 0.$$

we get

$$\frac{2}{1-p}\Pi(t)^{\frac{1-p}{2}} \geq \frac{2(p-1)}{p+1}t + \frac{2}{1-p}\Pi(0)^{\frac{1-p}{2}},$$

Then

$$\Pi(t)^{\frac{1-p}{2}} \leq \frac{-(1-p)^2}{p+1}t + \Pi(0)^{\frac{1-p}{2}},$$

after that

$$\frac{1}{\Pi(t)^{\frac{p-1}{2}}} \leq \frac{-(1-p)^2}{p+1}t + \Pi(0)^{\frac{1-p}{2}},$$

finally, we get

$$\left(\frac{1}{\frac{-(1-p)^2}{p+1}t + \Pi(0)^{\frac{1-p}{2}}}\right)^{\frac{2}{P-1}} \leq \Pi(t),$$

for $\Pi(t) \to \infty$ where $t \to T^*$, must be

$$\frac{-(1-p)^2}{p+1}t + \Pi(0)^{\frac{1-p}{2}} \to 0 \Rightarrow t = \frac{p+1}{(1-p)^2}\Pi(0)^{\frac{1-p}{2}}$$

So, we conclude that the blow up phenomena occur in a finite time T^* where

$$T^* = \frac{p+1}{(1-p)^2}\Pi(0)^{\frac{1-p}{2}}.$$

■

Chapter 3

Solvability and blow-up of the weak solution for a semi-linear Bessel problem with Neumann integral condition

This chapter focuses on the solvability and finite time blow-up of weak solutions of nonlinear problems with nonlocal boundary conditions of the second type.

First, we study into the solvability of the associated linear problem by concentrating on linear problems and showing their existence using the variable separation method and the energy inequality method. Following that, we demonstrate the problem's uniqueness. Then, we use the Linearization method to demonstrate that the weak solution to the main nonlinear problem appears and is unique.

Finally, we use the eigen functions method to measure the finite time of explosion when ever the solution blows up.

3.1 Formulation of the nonlinear problem

Let $Q = \{(x,t) \in \mathbb{R}^2,\ x \in \Omega =]0,1[\text{ and } 0 < t < T\}$. This work devoted to the study of a solution $u(x,t)$ satisfying the following parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{a}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial u}{\partial x}\right) = f\left(x,t,u\right), & \forall\left(x,t\right) \in Q \\ u\left(x,0\right) = \varphi\left(x\right) & \forall x \in \left(0,1\right) \\ \dfrac{\partial u}{\partial x}(0,t) = \displaystyle\int_0^1 xu(x,t)dx & \forall t \in \left(0,T\right) \\ \dfrac{\partial u}{\partial x}(1,t) = \displaystyle\int_0^1 k_2(x)u(x,t)dx & \forall t \in \left(0,T\right). \end{cases} \tag{P}$$

For bounded domain Ω of $\mathbb{R}$ with smooth boundary $\partial\Omega$. Also, f, φ and the weight function k_2 are known functions such that k_2 is continuous on $[0,1]$. And we have the following conditions:

Condition 3.1 *The function f is Lipschitzian, which means that there exists a positive constant k such that :*

$$\|f\left(x,t,u_1\right) - f\left(x,t,u_2\right)\|_{L^2(Q)} \leq k(\|u_1 - u_2\|_{L^2(Q)}), \qquad \forall u_1, u_2 \in L^2\left(Q\right). \tag{3.1}$$

Condition 3.2 *The function k_2 is bounded, such that*

$$k_2(x) \leq \beta\sqrt{x}.$$

Notation 3.1 *We will denote u_t and u_x to the partial derivative with respect to t, x respectively. And $u = u(x,t); \forall(x,t) \in (0,1) \times (0,T)$.*

3.2 Study of the linear problem

3.2.1 Position of the problem

In the domaine $Q = \{(x,t) \in \mathbb{R}^2,\ 0 < x < 1 \text{ and } 0 < t < T\}$. Consider the following linear problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{a}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial u}{\partial x}\right) = f(x,t) & \forall (x,t) \in Q \\ u(x,0) = \varphi(x) & \forall x \in (0,1) \\ u_x(0,t) = \int\limits_0^1 xu(x,t)dx, & \forall t \in (0,T) \\ u_x(1,t) = \int\limits_0^1 k_2(x)u(x,t)dx, & \forall t \in (0,T). \end{cases} \qquad (P_1)$$

Where the functions f ,φ and k_2 are known functions.

Whose parabolic equation is given as follows:

$$\mathcal{L}u = \frac{\partial u}{\partial t} - \frac{a}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) = f(x,t)$$

With the initial condition:

$$u(x,0) = \varphi(x), \qquad \forall x \in (0,1),$$

And the integral conditions of the second type

$$u_x(0,t) = \int\limits_0^1 xu(x,t)dx, \quad t \in (0,T), \tag{3.2}$$

$$u_x(1,t) = \int\limits_0^1 k_2(x)u(x,t)dx, \quad t \in (0,T). \tag{3.3}$$

We divise the main linear problem (P_1) to two other linear problems which are:

$$\begin{cases} v_t - \dfrac{a}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial v}{\partial x}\right) = 0 & \forall (x,t) \in Q \\ v(x,0) = \varphi(x) & \forall x \in (0,1) \\ v_x(0,t) = \int\limits_0^1 xv(x,t)dx & \forall t \in (0,T) \\ v_x(1,t) = \int\limits_0^1 k_2(x)v(x,t)dx & \forall t \in (0,T). \end{cases} \qquad (P_2)$$

and

$$\begin{cases} w_t - \dfrac{a}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial w}{\partial x}\right) = f(x,t) & \forall (x,t) \in Q \\ w(x,0) = 0 & \forall x \in (0,1) \\ w_x(0,t) = w_x(1,t) = 0 & \forall t \in (0,T) \end{cases} \qquad (P_3)$$

3.2.2 Solving the problem (P_2)

To solving the homogenous problem (P_2), we use the variable separation method:

We pose

$$v(x,t) = X(x)T(t). \qquad (3.4)$$

Replacing (3.4) in (P_2), $we\ gettheproblem$:

$$\begin{cases} T'X - \dfrac{a}{x}TX' - aTX'' = 0 \\ X(x)T(0) = \varphi(x) \\ X'(0)T(t) = \displaystyle\int_0^1 xX(x)T(t)dx \\ X'(1)T(t) = \displaystyle\int_0^1 xX(x)T(t)dx \end{cases}.$$

For $\lambda = \omega^2 > 0$, we get:

$$\frac{a}{x}\frac{X'}{X} + a\frac{X''}{X} = \frac{T'}{T} = -\omega^2, \qquad (3.5)$$

▶**Find X(x)** : The equality (3.5) gives the following Sturm-Liouville problem :

$$\begin{cases} ax^2X''(x) + xX(x)' + \omega^2 X = 0 \\ X'(0) = \displaystyle\int_0^1 xX(x)dx \\ X'(1) = \displaystyle\int_0^1 xX(x)dx \end{cases}$$

So the solution of this problem is given by

$$X_n(x) = AJ + By,$$

▶**Find T(t):** According to the principle superposition, we put:

$$v(x,t) = \sum_{n\geq 0} X_n(x) \cdot T_n(t).$$

which implies

$$\begin{aligned} v(x,0) &= \sum_{n\geq 0} X_n(x)T_n(0) \\ &= \varphi(x) \\ &= \sum_{n\geq 0} \varphi_n \cdot X_n(x)dx, \end{aligned}$$

then

$$\varphi_n = \int_0^1 \varphi(x) \cdot X_n(x)dx,$$

so

$$T_n(0) = \varphi_n.$$

$$T_n(t) = \varphi_n e^{-w_n^2 t}$$

3.2.3 Solvability of the problem (P_3) by the energy inequality method

$$\begin{cases} w_t - \dfrac{a}{x}(xw_x)_x = f(x,t) & \forall (x,t) \in Q \\ w(x,0) = 0 & \forall x \in (0,1) \\ w_x(0,t) = w_x(1,t) = 0 & \forall t \in (0,T) \end{cases} \qquad (P_3)$$

Where f is known function and $\forall a \gneqq 0$.

Whose parabolic equation is given as follows :

$$\mathcal{L}w = \frac{\partial w}{\partial t} - \frac{a}{x}\frac{\partial}{\partial x}\left(x\frac{\partial w}{\partial x}\right) = f(x,t) \qquad (3.6)$$

With the initial condition

$$w(x,0) = 0, \qquad \forall x \in (0,1),$$

And the integral conditions of the second type

$$\begin{aligned} w_x(0,t) &= 0, \quad t \in (0,T). \\ w_x(1,t) &= 0, \quad t \in (0,T). \end{aligned} \qquad (3.7)$$

Uniqueness of the solution of the problem (P_3) :

Theorem 3.1 *For any function* $w \in C(0,T,\ L^2_{\sqrt{x}}(0,1))$, *we obtain the estimate :*

$$\|w\|_E \leq k\,\|Lw\|_F \tag{3.8}$$

Where k *is a positive constant independent of* w, *such that :*

$$k = \sqrt{\frac{e^T}{\min\{1, 2a\}}}$$

Where E *is the Banach space with finite norm*

$$\|w\|_E^2 = \|w\|^2_{C(0,T,\ L^2_{\sqrt{x}}(0,1))} + \|w_x\|^2_{L^2(0,T,\ L^2_{\sqrt{x}}(0,1))}$$

And F *is a Hilbert space with the finite norm*

$$\|w\|_F^2 = \|f\ \|^2_{L^2(Q)}$$

Proof We first multiplying the equation (3.6) by the following multiplier Mw:

$$Mw = xw,$$

We get

$$\left[w_t - \frac{a}{x}(xw_x)_x\right] \cdot Mw = f \cdot Mw.$$

Integrating both sides of this identity over $Q_\tau = (0,1)\times(0,\tau)$, where $\tau \in [0,T]$, gives us:

$$\begin{aligned}
&\int\limits_{Q_\tau} \left[w_t - \frac{a}{x}(xw_x)_x\right] \cdot Mwdxdt \\
&= \int\limits_{Q_\tau} \left[w_t - \frac{a}{x}(xw_x)_x\right] \cdot xwdxdt \\
&= \int\limits_{Q_\tau} w_t \cdot xwdxdt - \int\limits_{Q_\tau} \left(\frac{a}{x}(xw_x)_x\right) \cdot xwdxdt \\
&= \int\limits_{Q_\tau} f \cdot xwdxdt.
\end{aligned}$$

Where w_x, w_t indicate the partial derivative to x, t respectively, such that $w_x = w_x(x,t)$ and $w_t = w_t(x,t)$.

Let us use an integration by parts for each term by taking account of the initial condition and the boundary conditions, we find :

$$\frac{1}{2}\int_\Omega xw^2(x,\tau)dx + a\int\limits_{Q_\tau} xw_x^2dxdt = \int\limits_{Q_\tau} f \cdot xwdxdt.$$

Thus: we apply the Cauchy inequality, it comes:

$$\frac{1}{2}\int_{\Omega} xw^2(x,\tau)dx + a\int_{Q_\tau} xw_x^2 dxdt \leq \frac{1}{2}\int_{Q} f^2 dxdt + \frac{1}{2}\int_{Q} (xw)^2 \, dxdt$$

Applying Gronwall's lemma, we find :

$$\int_{\Omega} xw^2(x,\tau)dx + 2a\int_{Q_\tau} xw^2 dxdt \leq (\int_{Q} f(x,t)^2 dxdt) e^{\int_0^T dt}$$

which implies that

$$\max_{0<t<T}\int_{\Omega} xw^2(x,\tau)dxdt + \int_{Q_\tau} xw_x^2 dxdt,$$

$$\leq \frac{e^T}{\min\{1,2a\}}(\int_{Q} f^2 dxdt),$$

Therefore, we obtain

$$\|w\|^2_{C(0,T,\, L^2_{\sqrt{x}}(0,1))} + \|w_x\|^2_{L^2(0,T,\, L^2_{\sqrt{x}}(0,1))} \leq C \, \|f\,\|^2_{L^2(Q)},$$

Where

$$C = \frac{e^T}{\min\{1,2a\}}$$

finally, it follows that

$$\|w\|_E \leq k\,\|\mathcal{F}\|_F, \; where \;\; k = \sqrt{C}.$$

This completes the proof. ■

Corollary 3.1 *If for any function $w \in D(L)$, we have the following estimate :*

$$\|w\|_E \leq k\,\|\mathcal{F}\|_F,$$

then the solution of the problem (P_3) if it exists, it is unique.

Existence of the solution of the problem (P_3) :

In this part, we shall establish existence of solutions for the second linear problem. Specifically, we shall prove the following items:

1. The operator

$$L = (\mathcal{L}, \ell) : E \longrightarrow F$$

is closable.

2. $R(L)$ is dense in F for any $w \in E$ and for any arbitrary $\mathcal{F} = (f, \varphi) \in F$.

Proposition 3.1 *The operator L of E in F is closable.*

Proof let $\{w_n\} \in D(L)$ be a sequence such that :

$$w_n \longrightarrow 0 \ \text{ in } E,$$

and

$$Lw_n \longrightarrow (f; 0) \ \text{ in } F, \tag{3.9}$$

we must prove that

$$f \equiv 0 \ .$$

The convergence of w_n towards 0 in E implies:

$$w_n \longrightarrow 0 \ \text{ in } D'(Q) \, . \tag{3.10}$$

From the continuity of the derivation of $D'(Q)$ in $D'(Q)$. the relation (3.10) implies:

$$\mathcal{L}w_n \longrightarrow 0 \ \text{ in } D'(Q) \, , \tag{3.11}$$

Moreover, the convergence of $\mathcal{L}_n$ towards f in $L^2(Q)$ generates:

$$\mathcal{L}w_n \longrightarrow f \ \text{ in } D'(Q) \, . \tag{3.12}$$

By vertue of the uniqueness of the limit in $D'(Q)$, we conclude from (3.11) and (3.12) that

$$f = 0.$$

Which is the result. ■

Let $\overline{L}$ be the closure of L,and $D(\overline{L})$ the domaine of definition of $\overline{L}$.

Theorem 3.2 *If for $\vartheta \in L^2(Q)$ and for any $w \in C(0, T, \, L^2_{\sqrt{x}}(0, 1))$, we have*

$$\int_Q \mathcal{L}w \cdot \vartheta dxdt = 0, \tag{3.13}$$

then ϑ vanishes almost everywhere in Q.

Proof The scalar product of F is defined by:

$$(Lw, W)_F = \int_Q \mathcal{L}w \cdot \omega dxdt,$$

Where $W = (\omega, \omega_0)$. The equality (3.13) can be written as follows:

$$\int_Q \frac{\partial w}{\partial t} \cdot \omega dxdt - \int_Q \frac{a}{x}\frac{\partial}{\partial x}\left(x\frac{\partial w}{\partial x}\right) \cdot \omega dxdt = 0, \tag{3.14}$$

Which implies

$$\int_Q \frac{\partial w}{\partial t} \cdot \omega dxdt = \int_Q \frac{a}{x}\frac{\partial}{\partial x}\left(x\frac{\partial w}{\partial x}\right) \cdot \omega dxdt, \tag{3.15}$$

Where w, $\frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial x} \in L^2(Q)$, with w satisfies the boundary conditions (3.7). We put

$$w(x,t) = \int_0^t z(x,\tau)d\tau = \Im_t z \tag{3.16}$$

By replacing (3.16) in (3.15) we get

$$\int_Q z \cdot \omega dxdt = a\int_Q \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial \Im_t z}{\partial x}\right)\omega dxdt. \tag{3.17}$$

During the etablishement of the function ω,and from this last equality, we give the function ω in terms of the function z as follows:

$$\omega = x\Im_t z$$

Since z satisfies the same conditions as the function w in (3.7), then z, $\frac{\partial z}{\partial x} \in L^2(Q)$, so $\omega \in L^2(Q)$.

Now replacing ω in (3.17), we obtain :

$$\int_Q xz\Im_t z dxdt = a\int_Q \Im_t z \cdot \frac{\partial}{\partial x}\left(x\frac{\partial \Im_t z}{\partial x}\right) dxdt.$$

According to an integration by parts and using the boundary conditions of Neuman, we get :

$$= \int_0^1 \frac{x}{2}(\Im_t z)^2\Big|_{\tau=0}^{\tau=T} dx = -a\int_Q x\left(\frac{\partial \Im_t z}{\partial x}\right)^2 dxdt \leqslant 0;$$

Which gives

$$\int_Q a(x,t)\left(\Im_t z\right)^2 dxdt = 0.$$

So

$$(\Im_t z) = 0.$$

Therefore, it becomes $w = 0$ in Q, which gives $\omega = 0$ in Q. Finally, we have

$$\overline{R(L)} = F.$$

This was to be demonstrated. ■

3.3 The uniqueness of the linear problem

In this section we will study the uniqueness of the linear problem (P_1) .

Theorem 3.3 *For any function $u \in D(L)$, we have the estimate :*

$$\|u\|_E \leq R\,\|Lu\|_F$$

where k is a positive constant independent of u, such that :

$$R = \frac{\max\left\{1, \frac{1}{\varepsilon}\right\}}{\min\left\{1, 2a(1-\delta)\right\}} e^{2a\delta T + \frac{a\beta^2}{\delta}T + \varepsilon T}$$

Proof Assuming that a solution of the problem exists, multiplying the equation of the problem (P_1) by the following multiplicator Mu:

$$Mu = xu\ ,$$

and by integrating on the domain $Q_\tau = (0,1) \times (0,\tau)$,where $\tau \in [0,T]$,we obtain :

$$\begin{aligned}
&\int\limits_{Q_\tau} \mathcal{L}u \cdot Mudxdt \\
&= \int\limits_{Q_\tau} \left[\partial_t u - \frac{a}{x}\frac{\partial}{\partial x}(x\frac{\partial u}{\partial x})\right] \cdot xudxdt \\
&= \int\limits_{Q_\tau} \partial_t u \cdot xudxdt - a\int\limits_{Q_\tau} \frac{1}{x}\frac{\partial}{\partial x}(x\frac{\partial u}{\partial x}) \cdot xudxdt \\
&= \int\limits_{Q_\tau} f \cdot xudxdt.
\end{aligned}$$

After an integration by parts and using the boundary conditions, we find

$$\frac{1}{2}\int_0^1 xu^2(x,\tau)dx - \frac{1}{2}\int_0^1 x\varphi^2(x)dx - a\int_0^\tau u(1,t)u_x(1,t)dt + a\int_{Q_\tau} xu_x^2(x,t)dxdt$$
$$= \int_{Q_\tau} \sqrt{x}f(x,t)\cdot\sqrt{x}u(x,t)dxdt,$$

Using the Cauchy with ε inequality, we obtain

$$\frac{1}{2}\int_0^1 xu^2(x,\tau)dx + a\int_{Q_\tau} xu_x^2 dxdt - a\int_0^\tau u(1,t)u_x(1,t)dt, \tag{3.18}$$
$$\leq \frac{1}{2\varepsilon}\int_{Q_\tau} xf^2dxdt + \frac{\varepsilon}{2}\int_{Q_\tau} xu^2dxdt + \frac{1}{2}\int_0^1 x\varphi^2(x)dx,$$

To find our estimate, we must give an estimate for the third part of the left hand side in the inequality (3.18), by using integral conditions(3.3), (3.3) and the cauchy with δ inequality, so we obtain:

$$\int_0^\tau u(1,t)u_x(1,t)dt,$$
$$\leq \frac{\delta}{2}\int_0^\tau u^2(1,t)dt + \frac{1}{2\delta}\int_0^\tau u_x^2(1,t)dt,$$
$$= \frac{\delta}{2}\int_0^\tau u^2(1,t)dt + \frac{1}{2\delta}\int_0^\tau (\int_0^1 k_2(x)udx)^2dt,$$
$$\leq \frac{\delta}{2}\int_0^\tau u^2(1,t)dt + \frac{1}{2\delta}\int_0^\tau (\int_0^1 \beta\sqrt{x}udx)^2dt,$$
$$\leq \frac{\delta}{2}\int_0^\tau u^2(1,t)dt + \frac{1}{2\delta}\int_0^\tau\int_0^1 (\beta\sqrt{x}u)^2dxdt,$$
$$\leq \frac{\delta}{2}\int_0^\tau u^2(1,t)dt + \frac{\beta^2}{2\delta}\int_{Q^\tau} xu^2dxdt, \tag{3.19}$$

Now, we must find the estimate of first part of the right hand side of (3.19). Let's put:

$$u(1,t) = \int_x^1 \frac{\partial}{\partial \xi}(\sqrt{x}u(\xi,t))d\xi + \sqrt{x}u(x,t),$$

Then, we use the inequality $|a+b|^2 \leq 2a^2 + 2b^2$, we find:

$$u^2(1,t) = (\int_x^1 \frac{\partial}{\partial \xi}(\sqrt{x}u(\xi,t))d\xi + \sqrt{x}u)^2,$$

$$\leq 2(\int_x^1 \frac{\partial}{\partial \xi}(\sqrt{x}u(\xi,t))d\xi)^2 + 2xu^2,$$

By applying Holder inequality, we get

$$u^2(1,t) \leq 2\int_x^1 1^2 d\xi \cdot \int_x^1 (\frac{\partial}{\partial \xi}(\sqrt{x}u(\xi,t)))^2 d\xi + 2xu^2,$$

Integrating over $(0,\tau)$, we find

$$\int_0^\tau u^2(1,t) \leq 2\int_0^\tau\int_0^1 (\frac{\partial}{\partial x}(\sqrt{x}u))^2 dxdt + 2\int_0^\tau xu^2 dt,$$

So

$$\frac{\delta}{2}\int_0^\tau u^2(1,t) \leq \delta\int_0^\tau\int_0^1 (x(\frac{\partial}{\partial x}u(x,t))^2 dxdt + \delta\int_0^\tau xu^2(x,t)dt, \tag{3.20}$$

$$\leq \delta\int_{Q^\tau} x(\frac{\partial u}{\partial x}(x,t))^2 dxdt + \delta\int_0^\tau xu^2(x,t)dt,$$

Under the previous inequalities (3.19) and (3.20), the inequality (3.18) becomes:

$$\frac{1}{2}\int_0^1 xu^2(x,\tau)dx + a\int_{Q^\tau} x(\frac{\partial u}{\partial x})^2 dxdt,$$

$$\leq a\delta\int_{Q^\tau} x(\frac{\partial u}{\partial x})^2 dxdt + a\delta\int_0^\tau xu^2 dt + \frac{a\beta^2}{2\delta}\int_{Q^\tau} xu^2 dxdt,$$

$$+ \frac{1}{2\varepsilon}\int_{Q_\tau} xf^2 dxdt + \frac{\varepsilon}{2}\int_{Q_\tau} xu^2 dxdt + \frac{1}{2}\int_0^1 x\varphi^2(x)dx.$$

Integrating one more time over $(0,1)$, we get:

$$\frac{1}{2}\int_0^1 xu^2(x,\tau)dx + a\int_{Q^\tau} x(\frac{\partial u}{\partial x})^2 dxdt,$$
$$\leq a\delta\int_{Q^\tau} x(\frac{\partial u}{\partial x})^2 dxdt + a\delta\int_0^1\int_0^\tau xu^2 dtdx + \frac{a\beta^2}{2\delta}\int_{Q^\tau} xu^2 dxdt,$$
$$+\frac{1}{2\varepsilon}\int_{Q_\tau} xf^2 dxdt + \frac{\varepsilon}{2}\int_{Q_\tau} xu^2 dxdt + \frac{1}{2}\int_0^1 x\varphi^2(x)dx.$$

Then, we have

$$\frac{1}{2}\int_0^1 xu^2(x,\tau)dx + a(1-\delta)\int_{Q^\tau} x(\frac{\partial u}{\partial x})^2 dxdt,$$
$$\leq (a\delta + \frac{a\beta^2}{2\delta} + \frac{\varepsilon}{2})\int_{Q_\tau} xu^2 dxdt + \frac{1}{2\varepsilon}\int_{Q_\tau} xf^2 dxdt + \frac{1}{2}\int_0^1 x\varphi^2(x)dx,$$

By applying Gronwall's lemma, we get

$$\int_0^1 xu^2(x,\tau)dx + \int_{Q^\tau} x(\frac{\partial u}{\partial x})^2 dxdt,$$
$$\leq \frac{\max\left\{1,\frac{1}{\varepsilon}\right\}}{\min\left\{1,2a(1-\delta)\right\}} e^{2a\delta T + \frac{a\beta^2}{\delta}T + \varepsilon T}(\int_{Q_\tau} xf^2 dxdt + \int_0^1 x\varphi^2(x)dx).$$

Finally we put

$$\|u\|^2_{L^\infty(0,T;L^2_{\sqrt{x}}(\Omega))} + \left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(0,T;L^2_{\sqrt{x}}(\Omega))}$$
$$\leq R(\|f\|^2_{L^2(0,T;L^2_{\sqrt{x}}(\Omega))} + \|\varphi\|^2_{L^2(0,T;L^2_{\sqrt{x}}(\Omega))}$$

Where

$$R = \frac{\max\left\{1,\frac{1}{\varepsilon}\right\}}{\min\left\{1,2a(1-\delta)\right\}} e^{2a\delta T + \frac{a\beta^2}{\delta}T + \varepsilon T}$$

Which complete the proof. ■

3.4 Solvability of the weak solution of the nonlinear problem

Based to the last section, this section devoted to the mainly consider nonlinear parabolic problem to proof the existence and the uniqueness by using the the linearization method :

$$\begin{cases} u_t - \dfrac{a}{x}(xu_x)_x = f(x,t,u) & \forall (x,t) \in Q \\ u(x,0) = \varphi(x) & \forall x \in (0,1) \\ u_x(0,t) = \displaystyle\int_0^1 xu(x,t)dx & \forall t \in (0,T) \\ u_x(1,t) = \displaystyle\int_0^1 k_2(x)u(x,t)dx & \forall t \in (0,T). \end{cases} \tag{P}$$

Putting

$$u = y + \theta,$$

Such that w is a solution to the following problem:

$$\begin{cases} \theta_t - \dfrac{a}{x}(x\theta_x)_x = 0 & \forall (x,t) \in Q \\ \theta(x,0) = \varphi(x) & \forall x \in (0,1) \\ \theta_x(0,t) = \displaystyle\int_0^1 x\theta(x,t)dx & \forall t \in (0,T) \\ \theta_x(1,t) = \displaystyle\int_0^1 k_2(x)\theta(x,t)dx & \forall t \in (0,T) \end{cases}, \tag{P_4}$$

And the solution

$$y = u - w$$

satisfied the following problem

$$\mathcal{L}y = y_t - \frac{a}{x}(xy_x)_x = G(x,t,y), \tag{3.21}$$

$$y(x,0) = 0, \qquad \forall x \in (0,1), \tag{3.22}$$

$$y_x(0,t) = y_x(1,t) = 0 \qquad \forall t \in (0,t). \tag{3.23}$$

Where

$$G(x,t,y) = f(x,t,y+\theta).$$

As the function f, the function G is also Lipschitzian, so there is a positive constant k such that:

$$\| G(x,t,u_1) - G(x,t,u_2) \|_{L^2(Q)} \leq k \left(\|u_1 - u_2\|_{L^2(0,T,H^1(0,1))} \right). \tag{3.24}$$

From the result of the previous section, we deduce that the problem (P_4) has a unique solution which depends continuously on the data. So it remains to prove that the problem $(3.21)-(3.23)$ admits a unique weak solution. First, we propose the concept of studied solution.

Let $v = v(x;t)$ any function of $L^2(0;T;H^1(0,1))$. Then, multiplying (3.21) by xv then integrating both sides over $Q = (0,1)\times(0,T)$, where gives us:

$$\int_Q \frac{\partial y}{\partial t} \cdot xvdxdt - a\int_Q \frac{\partial}{\partial x}(x\frac{\partial y}{\partial x}) \cdot vdxdt$$
$$= \int_Q G(x,t,u) \cdot xvdxdt.$$

Then by using an integration by parts and the conditions (3.22) and (3.23) we find:

$$\int_Q \frac{\partial y}{\partial t} \cdot vdxdt + a\int_Q x\frac{\partial y}{\partial x}\frac{\partial v}{\partial x} = \int_Q G(x,t,u) \cdot xvdxdt \tag{3.25}$$

It follows from(3.25) that:

$$A(y,v) = \int_Q G(x,t,u) \cdot xvdxdt \tag{3.26}$$

Where

$$A(y,v) = \int_Q \frac{\partial y}{\partial t} \cdot vdxdt + a\int_Q x\frac{\partial y}{\partial x}\frac{\partial v}{\partial x}dxdt$$

Definition 3.1 *A function $y \in L^2\left(0,T;\ H^1(0,1)\right)$ is said to be a weak solution of the problem $(3.21)-(3.23)$ if (3.26), and (3.23) are fulfilled.*

Building a recurring sequence starting with $y^{(0)} = 0$. The sequence $\left(y^{(n)}\right)_{n\in\mathbb{N}}$ is defined as follows : given the element $y^{(n-1)}$, then for $n = 1,2,3,...$, we will solve the following problem:

$$\begin{cases} \dfrac{\partial y^{(n)}}{\partial t} - \dfrac{a}{x}(xy_x^{(n)})_x = G\left(x,t,y^{(n-1)}\right) \\ y^{(n)}(x,0) = 0 \\ y_x^{(n)}(0,t) = y_x^{(n)}(1,t) = 0. \end{cases}, \tag{P_5}$$

According to the study of the previous linear problem each time we fix the n, the problem (P_5) admits a unique solution $y^{(n)}(x,t)$. Now we suppose

$$z^{(n)} = y^{(n+1)} - y^{(n)},$$

So we get a new problem which is:

$$\begin{cases} \dfrac{\partial z^{(n)}}{\partial t} - \dfrac{a}{x}(xz_x^{(n)})_x = p^{(n-1)} \\ z^{(n)}(x,0) = 0 \\ z_x^{(n)}(0,t) = z_x^{(n)}(1,t) = 0. \end{cases}, \qquad (P_6)$$

Where

$$p^{(n-1)} = G\left(x,t,y^{(n)}\right) - G\left(x,t,y^{(n-1)}\right).$$

Lemma 3.1 *Assuming that the condition* (3.24) *is satisfied. So we have the following estimate*

$$\| z^{(n)} \|^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} \leq C \| z^{(n-1)} \|^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))}$$

Where

$$C = \frac{k^2 e^T}{\min\{1, 2a\}}$$

Proof Multiplying

$$\frac{\partial z^{(n)}}{\partial t} - \frac{a}{x}(xz_x^{(n)})_x = p^{(n-1)}$$

by $xz^{(n)}$, and integrate it on Q_τ, we get:

$$\int\limits_{Q_\tau} \frac{\partial z^{(n)}}{\partial t} \cdot xz^{(n)} dxdt - a\int\limits_{Q_\tau} (xz_x^{(n)})_x \cdot z^{(n)} dxdt$$
$$= \int\limits_{Q_\tau} p^{(n-1)} \cdot xz^{(n)} dxdt.$$

If we use an integration by parts technique for each term, taking into consideration the initial and boundary conditions, we get:

$$\frac{1}{2}\int\limits_0^1 x(z^{(n)}(x,\tau))^2 dx + a\int\limits_{Q_\tau} x\left(\frac{\partial z^{(n)}}{\partial x}\right)^2 dxdt$$
$$= \int\limits_{Q_\tau} p^{(n-1)} \cdot xz^{(n)} dxdt.$$

When the Cauchy inequality is applied to the second part of the equation, the following result is obtained:

$$\begin{aligned}&\int\limits_{Q_\tau} p^{(n-1)} \cdot xz^{(n)} dxdt\\ &= \int\limits_{Q_\tau} \sqrt{x}p^{(n-1)} \cdot \sqrt{x}z^{(n)} dxdt\\ &\leqslant \frac{1}{2}\int_{Q^\tau} x\left(p^{(n-1)}\right)^2 dxdt + \frac{1}{2}\int_{Q^\tau} x\left(z^{(n)}\right)^2 dxdt,\end{aligned}$$

On the other hand, we have

$$\left|p^{(n-1)}\right|^2 = \left|G\left(x,t,y^{(n)}\right) - G\left(x,t,y^{(n-1)}\right)\right|^2$$

Like G function is Lipschitzian, we find :

$$\begin{aligned}\left|p^{(n-1)}\right|^2 &\leq k^2\left|y^{(n)} - y^{(n-1)}\right|^2\\ &= k^2\left|z^{(n-1)}\right|^2\end{aligned}$$

Multiplying by x and integrating over Q, we find:

$$\int\limits_{Q} x\left|p^{(n-1)}\right|^2 dxdt \leq k^2 \int\limits_{Q} x\left|z^{(n-1)}\right|^2 dxdt,$$

Then

$$\left\|p^{(n-1)}\right\|^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} \leq k^2\left\|z^{(n-1)}\right\|^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} \tag{3.27}$$

This result give us the following inequality

$$\begin{aligned}&\frac{1}{2}\int\limits_0^1 x(z^{(n)}(x,\tau))^2 dx + a\int\limits_{Q_\tau} x\left(\frac{\partial z^{(n)}}{\partial x}\right)^2 dxdt,\\ &\leq \frac{1}{2}\int_{Q^\tau} x\left(p^{(n-1)}\right)^2 dxdt + \frac{1}{2}\int_{Q^\tau} x\left(z^{(n)}\right)^2 dxdt,\\ &\leq \frac{k^2}{2} \parallel z^{(n-1)} \parallel^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} + \frac{1}{2} \parallel z^{(n)} \parallel^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))}.\end{aligned}$$

Thus, it is easy to get that

$$\| z^{(n)} \|^2_{L^2_{\sqrt{x}}(0,1)} + 2a \| \partial_x z^{(n)} \|^2_{L^2_{\sqrt{x}}(Q^\tau)},$$
$$\leq k^2 \| z^{(n-1)} \|^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} + \| z^{(n)} \|^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} .$$

Now we shall apply Gronwall's lemma, we get:

$$\| z^{(n)} \|^2_{L^2_{\sqrt{x}}(0,1)} + 2a \| \partial_x z^{(n)} \|^2_{L^2_{\sqrt{x}}(Q^\tau)},$$
$$\leqslant k^2 e^{\int_0^T dt} \| z^{(n-1)} \|^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))},$$

Result enables us to pass to the maximum in the right part of the last inequality, and we obtain:

$$\| z^{(n)} \|^2_{L^\infty(0,T;L^2_{\sqrt{x}}(0,1))} + \| \partial_x z^{(n)} \|^2_{L^2_{\sqrt{x}}(Q^\tau)},$$
$$\leqslant \frac{k^2 e^T}{\min\{1, 2a\}} \| z^{(n-1)} \|^2_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} .$$

finally, we get :

$$\| z^{(n)} \|_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} \leq C \| z^{(n-1)} \|_{L^2(0,T;H^1_{\sqrt{x}}(0,1))} .$$

Where

$$C = \sqrt{\frac{k^2 e^T}{\min\{1, 2a\}}}.$$

■

According to the convergence criterion of the series, gives that the serie $\sum_{n=1}^{\infty} z^{(n)}$ converges if $|C| < 1$, which implies:

$$\sqrt{\frac{k^2 e^T}{\min\{1, 2a\}}} < 1$$
$$k^2 < \frac{\min\{1, 2a\}}{e^T}$$
$$k < \sqrt{\frac{\min\{1, 2a\}}{e^T}}$$

As

$$z^{(n)} = y^{(n+1)} - y^{(n)}$$

So

$$y^{(n)} = \sum_{i=1}^{n-1} z^{(i)}$$

Then $y^{(n)}$ converges to an element $y \in L^2(0,T;H^1_{\sqrt{x}}(0,1))$. Now, we will prove that $\lim_{n\longrightarrow\infty} y^{(n)}(x,t) = y(x,t)$ is a solution of the problem (P$_5$) *by* showing that y *satisfies*

$$A(y,v) = \int\limits_Q G(x,t,u)\cdot xvdxdt. \tag{3.28}$$

Therefore we consider the weak formulation of the problem (P$_5$)$A(y^{(n)},v) = \int\limits_Q \frac{\partial y^{(n)}}{\partial t}\cdot xvdxdt +$ $a\int\limits_Q x\frac{\partial y^{(n)}}{\partial x}\frac{\partial v}{\partial x}dxdt$,From the linearity of A we have

$$\begin{aligned} A\left(y^{(n)},v\right) &= A\left(y^{(n)}-y,v\right)+A\left(y,v\right), \\ &= \int\limits_Q xvdxdt + a\int\limits_Q x\frac{\partial (y^{(n)}-y)}{\partial x}\frac{\partial v}{\partial x}dxdt + A\left(y,v\right). \end{aligned} \tag{3.29}$$

When the Cauchy-Schwartz inequality is applied to $A(y^{(n)}-y,v)$, we get

$$A\left(y^{(n)},v\right) \leq \|v\|_{L^2(0,T;H^1_{\sqrt{x}}(0,1))}\left\|(y^{(n)}-y)_t\right\|$$

On the other hand as

$$y^{(n)} \longrightarrow y \qquad \text{in } L^2\left(0,T;\ H^1_{\sqrt{x}}(0,1)\right),$$

so

$$\begin{aligned} y^{(n)} &\longrightarrow y && \text{in } L^2\left(0,T;\ L^2_{\sqrt{x}}(0,1)\right), \\ y_t^{(n)} &\longrightarrow y_t && \text{in } L^2\left(0,T;\ L^2_{\sqrt{x}}(0,1)\right), \\ y_x^{(n)} &\longrightarrow y_x && \text{in } L^2\left(0,T;\ L^2_{\sqrt{x}}(0,1)\right). \end{aligned}$$

Let us pass to the limit when $n \longrightarrow +\infty$, we find

$$\lim_{n\longrightarrow+\infty} A\left(y^{(n)}-y,v\right) = 0. \tag{3.30}$$

According to (3.30) and by passing to the limite in (3.29) we obtain

$$\lim_{n\longrightarrow+\infty} A\left(y^{(n)},v\right) = A\left(y,v\right).$$

Thus, we have proved the following result :

Theorem 3.4 *If the condition* (3.24) *is satisfied. And*

$$k < \sqrt{\frac{\min\{1, 2a\}}{e^T}}.$$

Then the problem $(3.21) - (3.23)$ *admits a weak solution belonging to* $L^2\left(0, T;\ L^2_{\sqrt{x}}(0,1)\right)$.

Now,we will show that the solution of the problem $(3.21) - (3.23)$ is unique.

Theorem 3.5 *If the condition* (3.24) *is verified, then the solution is unique.*

Proof Let y_1, y_2 be two solutions of in $L^2\left(0, T\ ; H^1_{\sqrt{x}}(0,1)\right)$ $(3.21) - (3.23)$,then

$$y = y_1 - y_2,$$

is also a solution in $L^2\left(0, T\ ; H^1_{\sqrt{x}}(0,1)\right)$ and check

$$\frac{\partial y}{\partial t} - a\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right) = G(x, t, y), \tag{3.31}$$

$$y(x, 0) = 0, \tag{3.32}$$

$$\frac{\partial y}{\partial x}(0, t) = \frac{\partial y}{\partial x}(1, t) = 0, \tag{3.33}$$

$$\begin{cases} \dfrac{\partial y}{\partial t} - \dfrac{a}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial y}{\partial x}\right) = \Psi(x, t), \qquad \forall (x, t) \in Q \\ y(x, 0) = 0, \\ \dfrac{\partial y}{\partial x}(0, t) = \dfrac{\partial y}{\partial x}(1, t) = 0 \end{cases}$$

and

$$\Psi(x, t) = G(x, t, y_1) - G(x, t, y_2).$$

Using the lemma (3.1) , we can conclude that

$$\|y\|_{L^2\left(0,T,H^1_{\sqrt{x}}(0,1)\right)} \leq C\|y\|_{L^2\left(0,T,H^1_{\sqrt{x}}(0,1)\right)},$$

from where

$$(1 - C)\|y\|_{L^2\left(0,T,H^1_{\sqrt{x}}(0,1)\right)} \leq 0,$$

and as $C \leq 1$, then we get to

$$\|y\|_{L^2(0,T,H^1(0,1))} = 0,$$

from where

$$y_1 = y_2,$$

This contributes to the solutions uniqueness. ■

3.5 Finite Time Blow-Up for nonlinear problem by using Kaplan's First Eigenvalue Method

In this section we are interesting to the study the blow up phenomena for the nonlinear problem for the diffusion term is given in the following manner $f(x,t,u) = u^p$

3.5.1 Statement of the problem

Let $T > 0$, $\Omega = (0,1)$ and $Q = \Omega \times (0,T) = \{(x,t) \in \mathbb{R}^2,\ x \in \Omega \text{ and } 0 < t < T\}$.

Consider the following nonlinear problem :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{a}{x}\dfrac{\partial}{\partial x}\left(x\dfrac{\partial u}{\partial x}\right) = u^p, & \forall\,(x,t) \in Q \\ u\,(x,0) = \varphi\,(x) & \forall x \in \Omega \\ \dfrac{\partial u}{\partial x}(0,t) = \displaystyle\int_0^1 xu(x,t)dx & \forall t \in \Omega \\ \dfrac{\partial u}{\partial x}(1,t) = \displaystyle\int_0^1 k_2(x)u(x,t)dx & \forall t \in (0,T)\,. \end{cases} \tag{P_7}$$

Assuming that $f \in L^2\,(Q)$, k_1, k_2 are known functions such that $k_2(x) > \omega > 0$ and $p > 1$. Whose parabolic equation is given as follows

$$\frac{\partial u}{\partial t} - \frac{a}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) = u^p,\ x \in \Omega \text{ and } 0 < t \leq T, \tag{3.34}$$

with the initial condition

$$u(x,0) = \varphi(x),\ 0 < x < 1 \tag{3.35}$$

and the nonlocal boundary conditions

$$u(0,t) = \int_0^1 xu(x,t)dx, \qquad 0 < t \leq T, \tag{3.36}$$

$$u(1,t) = \int_0^1 k_2(x)u(x,t)dx, \qquad 0 < t \leq T, \tag{3.37}$$

3.5.2 Determining the finite-time blow up solution by using Kaplan's first eigenvalue method

Let $\psi(x)$ be the normalized eigenfunction corresponding to the eigenvalue λ of the following sturmliouville problem:

$$\begin{cases} -\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial \psi}{\partial x}\right) = \lambda\psi \\ \psi(0) = 0 \\ \psi_x(1) = 0 \end{cases} \tag{P$_8$}$$

Let's find the solution $\psi(x)$:

By multiplying the equation in the problem (P_8) byx,$weget$ $-\frac{\partial \psi}{\partial x} - x\frac{\partial^2 \psi}{\partial x^2} - \lambda x\psi = 0,$
Which is the Bissel's function, with the following general solution

$$\psi(x) = c_1 J_0(\sqrt{\lambda}x) + c_2 Y_0(\sqrt{\lambda}x),$$

Where c_1,c_2 are constants and we have

$$J_0(\sqrt{\lambda}x) = \sum_{k=0}^{+\infty}\frac{(-1)^k}{(k!)^2}(\frac{\sqrt{\lambda}x}{2})^{2k},$$

And

$$Y_0(\sqrt{\lambda}x) = \lim_{\alpha\to 0}\frac{\cos(\alpha\pi)J_0(\sqrt{\lambda}x) - J_0(\sqrt{\lambda}x)}{\sin(\alpha\pi)}$$

Since Y_0 is not bounded as $x \to 0^+$, then we must have $c_2 = 0$ where the solution becomes

$$\psi(x) = c_1 J_0(\sqrt{\lambda}x),$$

Let λ_1 be the first eigenvalue of the problem (P_8) $where \mathrm{J}_0'(\sqrt{\lambda_1}) = 0.$

Theorem 3.6 *For $p > 1$ and $\forall\,(x,t) \in Q$, the solution of the problem (P_7) blows up in a finite time T^*such that:*

$$T^* = \frac{1}{K}\ln\left(\frac{\frac{r(1-p)}{K}}{\left((\Pi(0))^{1-p} - \frac{r(1-p)}{K}\right)}\right), where\ r = (\int_0^1 x\psi dx)^{1-p}$$

$$\Pi(0) = \left(B + \frac{r(1-p)}{K}\right)^{\frac{1}{1-p}} \quad and\ B = (\Pi(0))^{1-p} - \frac{r(1-p)}{K}$$

a

Proof We based this proof on a sufficiently large initial data, for the study of one of the most profiles importantant of explosion phenomenon for the solutions of problem (P_7)

To estimate the finite time blow up of the main problem we use the Kaplans method so by multiplying the equation in (3.34) by $(x\psi)$, then integrating by parts over the domain $\Omega = (0,1)$, we get

$$\int_0^1 x\psi \cdot \frac{\partial u}{\partial t}dx - a\int_0^1 \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)\cdot \psi dx = \int_0^1 x\psi \cdot u^p dx,$$

Then

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot u(x,t)dx - a\left[x\psi(x)u_x|_0^1 - \int_0^1 x\psi_x \cdot u_x\right] = \int_0^1 x\psi \cdot u^p dx,$$

After using the boundary conditions and integration by parts one more time the last inequality becomes

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot udx - a\left[\psi(1)u_x(1,t) + \int_0^1 u\frac{\partial}{\partial x}(x\psi_x)dx\right] = \int_0^1 x\psi \cdot u^p dx,$$

Then

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot udx - ac_1 J_0(\sqrt{\lambda})\int_0^1 k_2(x)udx - a\int_0^1 u\frac{\partial}{\partial x}(x\psi_x)dx = \int_0^1 x\psi \cdot u^p dx, \tag{3.38}$$

In the other hand, we have

$$\int_0^1 u\frac{\partial}{\partial x}(x\psi_x)dx = -\lambda\int_0^1 xu\psi dx,$$

So, the equality (3.38) becomes

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot udx - ac_1 J_0(\sqrt{\lambda})\int_0^1 k_2(x)udx + a\lambda\int_0^1 x\psi \cdot u(x,t)dx = \int_0^1 x\psi \cdot u^p dx,$$

Then

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot udx + a\lambda\int_0^1 x\psi \cdot udx = ac_1 J_0(\sqrt{\lambda})\int_0^1 k_2(x)udx + \int_0^1 x\psi \cdot u^p dx,$$

Now, by applying Jensen inequality, we find

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot udx + a\lambda\int_0^1 x\psi \cdot udx \geq a\omega c_1 J_0(\sqrt{\lambda})\int_0^1 udx + (\int_0^1 x\psi dx)^{1-p}(\int_0^1 x\psi u)^p, \tag{3.39}$$

In the other hand, we have

$$\int_0^1 udx \geq \frac{1}{c_1}\int_0^1 x\psi \cdot udx, \text{ where } c_1 = \|x\psi\|_{L^\infty(0,1)}.$$

Then, the inequality (3.39) becomes

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot udx + a\lambda\int_0^1 x\psi \cdot udx \geq a\omega J_0(\sqrt{\lambda})\int_0^1 x\psi \cdot udx + (\int_0^1 x\psi dx)^{1-p}(\int_0^1 x\psi u)^p,$$

Finally, we obtain

$$\frac{\partial}{\partial t}\int_0^1 x\psi \cdot u dx + a\left(\lambda - \omega J_0(\sqrt{\lambda})\right)\int_0^1 x\psi \cdot u dx \geq (\int_0^1 x\psi dx)^{1-p}(\int_0^1 x\psi u)^p,$$

By putting

$$\Pi(t) = \int_0^1 x\psi \cdot u dx,$$

We find

$$\Pi'(t) + a\left(\lambda - \omega J_0(\sqrt{\lambda})\right)\Pi(t) \geq (\int_0^1 x\psi dx)^{1-p}\Pi^p(t), \tag{3.40}$$

By putting

$$r = (\int_0^1 x\psi dx)^{1-p} \text{ and } d = a\left(\lambda - \omega J_0(\sqrt{\lambda})\right)$$

The equality (3.40) becomes

$$\Pi'(t) + d\Pi(t) \geq r\Pi^p(t),$$

Let solve the following Bernoulli equation

$$\Pi'(t) + d\Pi(t) - r\Pi^p(t) = 0 \tag{3.41}$$

By putting

$$v = \Pi^{1-p}, \tag{3.42}$$

and replacing (3.42) in (3.41), we find

$$\frac{1}{1-p}v'v^{\frac{p}{1-p}} + dv^{\frac{1}{1-p}} - rv^{\frac{p}{1-p}} = 0, \tag{3.43}$$

multiply the equation (3.43) by $(1-p)v^{\frac{-p}{1-p}}$, we get

$$v' + Kv - r(1-p) = 0; \text{where } K = (1-p)\,d, \tag{3.44}$$

Firstly, we are going to solve the following homogeneous equation:

$$v' + Kv = 0,$$

Which have a known solution gives by:

$$v_h(t) = Be^{-Kt}$$

Now, we move for solving the non-homogeneous equation (3.44) by the method of constant variation, where we put

$$v_g(t) = B(t)e^{-Kt}, \tag{3.45}$$

so

$$v_g'(t) = B'(t)e^{-Kt} - KB(t)e^{-Kt}, \tag{3.46}$$

Combining (3.45) and (3.46) with (3.44) where we get

$$B'(t)e^{-Kt} = r(1-p),$$

then

$$v_g(t) = \frac{r(1-p)}{K}$$

So, the final solution gives by

$$\begin{aligned} v(t) &= v_h(t) + v_g(t) \\ &= Be^{-Kt} + \frac{r(1-p)}{K}, \end{aligned}$$

so

$$\Pi(t) = \left(Be^{-Kt} + \frac{r(1-p)}{K}\right)^{\dfrac{1}{1-p}}$$

For $t = 0$, we get

$$B = (\Pi(0))^{1-p} - \frac{r(1-p)}{K},$$

finally, we get

$$\Pi(t) = \left(\frac{1}{\left((\Pi(0))^{1-p} - \frac{r(1-p)}{K}\right)e^{-Kt} + \frac{r(1-p)}{K}}\right)^{\frac{1}{p-1}}$$

like $\dfrac{1}{p-1} > 0$, then

$$\Pi \to \infty \text{ if } \left((\Pi(0))^{1-p} - \frac{r(1-p)}{K}\right)e^{-Kt} + \frac{r(1-p)}{K} \to 0$$

So we get

$$T^* = \frac{1}{K}\ln\left(\frac{\frac{r(1-p)}{K}}{\left((\Pi(0))^{1-p} - \frac{r(1-p)}{K}\right)}\right).$$

Is the finite time blow up of the problem (P_7). ■

Chapter 4

Existence of a nonlinear reaction-diffusion equations with nonlinear boundary conditions

It is very well important that in this chapter to discussion a nonlinear problem with nonlinear conditions mixed between nonlinearity and integral condition to know the impaction of the nonlinear reaction term and boundary conditions on the existence of the main problem and to overcome these study we chose the Fadeo galarkin method.

4.1 Formulation of the problem

Consider the following initial boundary value problem for a nonlinear parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\Delta u = -bu^p, & \forall\, (x,t) \in Q \\ u\,(x,0) = \varphi\,(x)\,, & \forall x \in \Omega \\ \dfrac{\partial u}{\partial x}(0,t) = \int\limits_{\Omega} u^q(x,t)dx, & \forall t \in (0,T) \\ \dfrac{\partial u}{\partial x}(1,t) = u^q(1,t). & \forall t \in (0,T) \end{cases} \tag{P4}$$

Where Ω is a bounded domain in $\mathbb{R}$ with the smooth boundary Γ, such that $\Omega = (0,1)$, $Q = \Omega \times (0,T)$ and $\varphi \in L^2(\Omega)$.

Assuming that a, b, p and q are positive odd integers such that $2q \leq p+1$ and $q \geq 1$.

Whose parabolic equation is given as follows :

$$\mathcal{L}u = \frac{\partial u}{\partial t} - a\Delta u = -bu^p \tag{4.1}$$

With the initial condition

$$u(x,0) = \varphi\left(x\right), \qquad \forall x \in \left(0,1\right), \tag{4.2}$$

And the boundary conditions :

$$u_x(0,t) = \int_0^1 u^q(x,t), \quad t \in \left(0,T\right). \tag{4.3}$$

$$u_x(1,t) = u^q(1,t)dx, \quad t \in \left(0,T\right). \tag{4.4}$$

In what follows we study the existence of this problem by just following the three major steps of the compactness method . Where the purpose of this chapter is to study a function $u = u\left(x,t\right)$, $x \in \Omega$, $t \in [0,T]$; solution of the problem $(P4)$. In order to properly pose the problem, and to have the tools to solve it, we need to introduce some concepts and some functional spaces that we will use later.

We define space V by:

$$V = \left\{u \in H^1\left(\Omega\right) \cap L^{p+1}\left(\Omega\right) \quad \right\}.$$

Where the space V provided with the norm $\|v\|_V = \|v\|_{H^1(\Omega)} + \|v\|_{L^{p+1}(\Omega)}$ is a Hilbert space.

Definition 4.1 *The weak solution of the problem (P_1) is a function that verifies:*

(i) $u \in L^2\left(0,T;\ H^1\left(\Omega\right)\right) \cap L^\infty\left(0,T;\ H^1\left(\Omega\right)\right)$

(ii) *u admits a strong derivative* $\dfrac{\partial u}{\partial t} \in u' \in L^2(0,T;\ L^2(\Omega))$,

(iii) $u\left(0\right) = \varphi$.

(iv) *Identity*

$$(u_t,v) + a\left(u_x,v_x\right) + b\left(u^p,v\right) = u_x(l,t)v(l) - \ u_x(l,t)v(l) \ \ \forall v \in V; \qquad \forall v \in V \ \textit{and} \ \forall t \in \left[0,T\right].$$

4.2 Variational formulation

By multiplying the equation :

$$\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} = -bu^p, \tag{4.5}$$

By an element $v \in V$, by integrating it on Ω we obtain :

$$\int_{\Omega} \frac{\partial u}{\partial t} \cdot v dx - a \int_{\Omega} \frac{\partial^2 u}{\partial x^2} \cdot v dx = -b \int_{\Omega} u^p \cdot v dx. \tag{4.6}$$

By using the boundary conditions and using Green's formula, (4.6) becomes

$$(u_t, v) + a(u_x, v_x) + b(u^p, v) = u_x(l,t)v(l) - \ u_x(l,t)v(l) \quad \forall v \in V, \tag{4.7}$$

Where $(.,.)$ denotes the scalar product $L^2(\Omega)$.

4.3 Study of the existence of weak solution of the problem $(P4)$

The demonstration of the existence of the solution of the problem $(P4)$ is based on the Faedo-Galerkin method which consists of carrying out the following three steps:

Step one: "Construction of the approximate solutions"

The space V is separable, then there exists a sequence $w_1, w_2, ..., w_m$, having the following properties:

$$\begin{cases} w_i \in V, & \forall i, \\ \forall m, w_1, w_2, ..., w_m & \text{are linearly independent,} \\ V_m = \langle \{w_1, w_2, ..., w_m\} \rangle & \text{is dense in } V. \end{cases} \tag{4.8}$$

In particular :

$$\forall \varphi \in V \implies \exists (\alpha_{km})_m \in IN^*, \ \varphi_m = \sum_{k=1}^{m} \alpha_{km} w_k \longrightarrow \varphi \text{ when } m \longrightarrow +\infty. \tag{4.9}$$

Faedo Galerkin's approximation consists in searching for any integer $m \geq 1$, a function

$$t \mapsto u_m(x,t) = \sum_{i=1}^{m} g_{im}(t) w_i(x),$$

Existence of such α_{km} follows from $u_0 \in H^1(\Omega) \cap L^{P+1}(\Omega)$ and the fact that $\{w_k, \ k \in \mathbb{N}\}$ is the base in $H^1(\Omega) \cap L^{P+1}(\Omega)$.

Thus, (P_2) is reduced to the initial value problem for a system of first-order differential equations with respect to g_{im} :

$$\begin{cases} \sum_{i=1}^{m} (w_i, w_k) \frac{\partial g_{im}}{\partial t}(t) - a \sum_{i=1}^{m} g_{im}(t) (\Delta w_i, w_k) + b(u_m^p, w_k) = (f(t), w_k) \\ g_{km}(0) = \alpha_{km} \qquad \forall k = \overline{1,m}. \end{cases}, \tag{P_3}$$

Thus, we get

$$\begin{cases} \dfrac{\partial g_{km}}{\partial t} + a\lambda_k g_{km}(t) + G_k(g) = f_k(t) \\ g_{km}(0) = \alpha_{km} \qquad \forall k = \overline{1,m}. \end{cases}, \tag{4.10}$$

where

$$G_k(g) = ((bu_m^p, w_k))_{1\leq k\leq m}\,.$$

using the Carathéodory's existence theorem used for ordinary differential equations, we can conclude that there exists a t_m depends only on $|\alpha_{im}|$ such that in the interval $[0, t_m]$, the problem (4.10) admits a unique local solution $g_m(t) \in C[0, t_m]$ and $g_m^{'}(t) \in L^2[0, T]$.

Step two:" A priori estimate"

Now, We study the a priori estimates for the approximate solution $u_m(x,t)$ obtained in the previous step.

Theorem 4.1 *for all* $m \in \mathbb{N}^*$, $\frac{p}{2} \geq b$. *Supposing that* $\varphi \in H^1(\Omega) \cap L^{p+1}(\Omega)$, $f \in L^2(0, T,\ L^2(\Omega))$. *Then problem*$(P_1)$ *admits a solution* u *such that :*

$$u \in L^2\left(0, T;\ H^1(\Omega)\right) \cap L^\infty\left(0, T;\ H^1(\Omega)\right)$$

and

$$u' \in L^2(0, T;\ L^2(\Omega)).$$

Proof Multiplying both sides of equation in (P_2) by $g_{km}(t)$, and summing with respect to k, it yields

$$\sum_{k=1}^{m} ((u_m)_t, w_k)\cdot g_{km}(t) - a\sum_{k=1}^{m}(\Delta u_m, w_k)\cdot g_{km}(t) + \sum_{k=1}^{m}(bu_m^p, w_k)\cdot g_{km}(t)$$
$$= a\frac{\partial u_m}{\partial x}(1,t)\, u_m(1,t) - a\frac{\partial u_m}{\partial x}(0,t)\, u_m(0,t),$$

then, we find

$$\sum_{k=1}^{m}\int_\Omega (u_m)_t\cdot w_k\cdot g_{km}(t)dx + a\sum_{k=1}^{m}\int_\Omega \frac{\partial u_m}{\partial x}\cdot\frac{\partial w_k}{\partial x}\cdot g_{km}(t)dx + b\sum_{k=1}^{m}\int_\Omega u_m^p\cdot w_k\cdot g_{km}(t)dx$$
$$= au_m^q(x,t)u_m(1,t) - a\int_\Omega u_m^q(x,t)dx u_m(0,t),$$

Thus, after integrating again on Ω, we get

$$\frac{1}{2}\frac{\partial}{\partial t}\left\|u_m\right\|_{L^2(\Omega)}^2 + a\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2(\Omega)}^2 + b\left\|u_m\right\|_{L^{p+1}(\Omega)}^{p+1} = a\int_\Omega u_m^q(x,t)\left(u_m(1,t) - u_m(0,t)\right)$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left\|u_m\right\|_{L^2(\Omega)}^2 + a\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2(\Omega)}^2 + b\left\|u_m\right\|_{L^{p+1}(\Omega)}^{p+1} = a\left(\int_\Omega u_m^q(x,t)\right)\left(\int_\Omega \frac{\partial u_m}{\partial x}\right)$$

Using the property $|ab| \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\left\|u_m\right\|_{L^2(\Omega)}^2 + a\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2(\Omega)}^2 + b\left\|u_m\right\|_{L^{p+1}(\Omega)}^{p+1} &\leq \frac{\left(\int_\Omega u_m^q(x,t)\right)^2}{2\varepsilon} + \frac{\varepsilon\left(\int_\Omega \frac{\partial u_m}{\partial x}\right)^2}{2}\\ &\leq \frac{1}{2\varepsilon}\int_\Omega u_m^{2q}(x,t) + \frac{\varepsilon}{2}\int_\Omega\left(\frac{\partial u_m}{\partial x}\right)^2\end{aligned}$$

So, we have

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\left\|u_m\right\|_{L^2(\Omega)}^2 + \left(a - \frac{\varepsilon}{2}\right)\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2(\Omega)}^2 + b\left\|u_m\right\|_{L^{p+1}(\Omega)}^{p+1} &\leq \frac{1}{2\varepsilon}\int_\Omega u_m^{2q}(x,t)\\ &\leq \frac{1}{2\varepsilon}\left\|u_m\right\|_{L^{p+1}(\Omega)}^{2q}\end{aligned} \tag{4.11}$$

Under the posing that

$$1 \leq \left\|u_m\right\|_{L^{p+1}(\Omega)}$$

we find

$$\frac{1}{2}\frac{\partial}{\partial t}\left\|u_m\right\|_{L^2(\Omega)}^2 + \left(a - \frac{\varepsilon}{2}\right)\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2(\Omega)}^2 + b\left\|u_m\right\|_{L^{p+1}(\Omega)}^{p+1} \leq \frac{1}{2\varepsilon}\left\|u_m\right\|_{L^{p+1}(\Omega)}^{p+1},$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left\|u_m\right\|_{L^2(\Omega)}^2 + \left(a - \frac{\varepsilon}{2}\right)\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2(\Omega)}^2 + \left(b - \frac{1}{2\varepsilon}\right)\left\|u_m\right\|_{L^{p+1}(\Omega)}^{p+1} \leq 0,$$

integrating from 0 to t, we get

$$\frac{1}{2}\left\|u_m(t)\right\|_{L^2(\Omega)}^2 + \left(a - \frac{\varepsilon}{2}\right)\int_0^t\left\|\frac{\partial u_m}{\partial x}\right\|_{L^2(\Omega)}^2 d\tau + \left(b - \frac{1}{2\varepsilon}\right)\int_0^t\left\|u_m\right\|_{L^{p+1}(\Omega)}^{p+1} d\tau$$

$$\leq \frac{1}{2}\left\|\varphi_m\right\|_{L^2(\Omega)}^2.$$

By puting

$$C = \frac{1}{2} \left\| \varphi_m \right\|_{L^2(\Omega)}^2 . \tag{4.12}$$

We have

$$\frac{1}{2} \left\| u_m(t) \right\|_{L^2(\Omega)}^2 + \left(a - \frac{\varepsilon}{2} \right) \int_0^t \left\| \frac{\partial u_m}{\partial x} \right\|_{L^2(\Omega)}^2 d\tau + \left(b - \frac{1}{2\varepsilon} \right) \int_0^t \left\| u_m \right\|_{L^{p+1}(\Omega)}^{p+1} d\tau \leq C, \tag{4.13}$$

$$\frac{1}{2} \left\| u_m(t) \right\|_{L^\infty(0,T,L^2(\Omega))}^2 + \left(a - \frac{\varepsilon}{2} \right) \left\| \frac{\partial u_m}{\partial x} \right\|_{L^2(0,T,L^2(\Omega))}^2 + \left(b - \frac{1}{2\varepsilon} \right) \left\| u_m \right\|_{L^{p+1}(0,T,L^{p+1}(\Omega))}^{p+1} \leq C,$$

where C is a positive constant depending only on$\left\| \varphi_m \right\|_{L^2(\Omega)}^2$. It follows from (4.13) that

$$\left\| u_m(t) \right\|_{L^2(\Omega)}^2 \leq C.$$

This implies that the solution to the initial value problem for the system of ODE (4.10) can be extended to $[0,T]$, and on $[0,T]$, we have the following uniform a priori estimates :

$$\begin{cases} u_m \text{ uniformly bounded in } L^\infty(0,T;\ L^2(\Omega)) \\ u_m \text{ uniformly bounded in } L^2(0,T;\ H^1(\Omega)) \\ u_m \text{ uniformly bounded in } L^{p+1}(0,T;\ L^{p+1}(\Omega)) \end{cases} .$$

Now we would like to get more a priori estimates. In doing so, multiplying both sides of the equations in (4.10) by $g'_{km}(t)$, respectively,and then summing over k, we get

$$\sum_{k=1}^m \left((u_m)_t , w_k \right) \cdot g'_{km}(t) + \sum_{k=1}^m a \left(\Delta u_m, w_k \right) \cdot g'_{km}(t) + b \sum_{k=1}^m \left(u_m^p, w_k \right) \cdot g'_{km}(t)$$
$$= 0.$$

Then, we find

$$\sum_{k=1}^m \int_\Omega (u_m)_t \cdot w_k \cdot g'_{km}(t) dx + a \int_\Omega \frac{\partial u_m}{\partial x} \cdot \frac{\partial (u_m)_t}{\partial x} dx + b \sum_{k=1}^m \int_\Omega \left(u_m^p \right) \cdot w_k \cdot g'_{km}(t) dx$$
$$= 0.$$

So, it comes

$$\int_\Omega (u_m)_t^2 \, dx + a \int_\Omega \frac{\partial u_m}{\partial x} \cdot \frac{\partial (u_m)_t}{\partial x} dx + b \int_\Omega \left(u_m^p \right) \cdot (u_m)_t \, dx$$
$$= 0.$$

Thus, We obtain

$$\|(u_m)_t\|^2_{L^2(\Omega)} + a\int_\Omega \frac{\partial u_m}{\partial x}\cdot\frac{\partial (u_m)_t}{\partial x}dx + \frac{b}{p+1}\frac{\partial}{\partial t}\|u_m\|^{p+1}_{L^{p+1}(\Omega)} = (f + bu_m, (u_m)_t)\,.$$

On the other hand, we have

$$\begin{aligned} & a\int_\Omega \frac{\partial u_m}{\partial x}\cdot\frac{\partial (u_m)_t}{\partial x}dx \\ & = \frac{a}{2}\frac{\partial}{\partial t}\int_\Omega \left(\frac{\partial u_m}{\partial x}\right)^2 dx, \end{aligned}$$

from(3.44) , we have

$$\begin{aligned} & \|(u_m)_t\|^2_{L^2(\Omega)} + \frac{a}{2}\frac{\partial}{\partial t}\|(u_m)_x\|^2_{L^2(\Omega)} + \frac{b}{p+1}\frac{\partial}{\partial t}\|u_m\|^{p+1}_{L^{p+1}(\Omega)} \\ & \leq 0. \end{aligned}$$

It comes

$$\begin{aligned} & \|(u_m)_t\|^2_{L^2(\Omega)} + \frac{a}{2}\frac{\partial}{\partial t}\|(u_m)_x\|^2_{L^2(\Omega)} + \frac{b}{p+1}\frac{\partial}{\partial t}\|u_m\|^{p+1}_{L^{p+1}(\Omega)} \\ & \leq a u_m^q(x,t)\frac{\partial u_m}{\partial t}(1,t) - a\int_\Omega u_m^q(x,t)dx\frac{\partial u_m}{\partial t}(0,t)\,, \\ & \leq a\left(\int_\Omega u_m^q(x,t)dx\right)\left(\frac{\partial u_m}{\partial t}(1,t) - \frac{\partial u_m}{\partial t}(0,t)\right) \\ & \leq a\left(\int_\Omega u_m^q(x,t)dx\right)\left(\frac{\partial}{\partial t}\int_\Omega \frac{\partial u_m}{\partial x}\right) \\ & \leq a \max_{t\in[0,T]}\int_\Omega u_m^q(x,t)dx\left(\frac{\partial}{\partial t}\int_\Omega \frac{\partial u_m}{\partial x}\right) \end{aligned}$$

then

$$\begin{aligned} & \|(u_m)_t\|^2_{L^2(\Omega)} + \frac{a}{2}\frac{\partial}{\partial t}\|(u_m)_x\|^2_{L^2(\Omega)} + \frac{b}{p+1}\frac{\partial}{\partial t}\|u_m\|^{p+1}_{L^{p+1}(\Omega)} \\ & \leq a \max_{t\in[0,T]}\int_\Omega u_m^q(x,t)dx\left(\frac{\partial}{\partial t}\int_\Omega \frac{\partial u_m}{\partial x}\right) \end{aligned}$$

By integration on $(0,t)$, it comes :

$$\frac{1}{2}\int_0^t \|(u_m)_t\|^2_{L^2(\Omega)} + \frac{a}{2}\int_0^t \frac{\partial}{\partial t}\left(\|\nabla u_m\|^2_{L^2(\Omega)}\right) d\tau + \frac{b}{p+1}\int_0^t \frac{\partial}{\partial t}\|u_m(\tau)\|^{p+1}_{L^{p+1}(\Omega)} d\tau$$

$$\leq a \max_{t\in[0,T]}\int_\Omega u_m^q(x,t)dx\left(\int_\Omega \frac{\partial u_m}{\partial x} - \int_\Omega \frac{\partial \varphi_m}{\partial x}\right),$$

$$\leq \frac{a}{2\varepsilon}\left(\max_{t\in[0,T]}\int_\Omega u_m^q(x,t)dx\right)^2 + \frac{a\varepsilon}{2}\left(\int_\Omega \frac{\partial u_m}{\partial x} - \int_\Omega \frac{\partial \varphi_m}{\partial x}\right)^2$$

$$\leq a \max_{t\in[0,T]}\int_\Omega u_m^{2q}(x,t)dx + a\varepsilon\|\nabla u_m\|^2_{L^2(\Omega)} + a\varepsilon\|\nabla\varphi_m\|^2_{L^2(\Omega)}.$$

Which gives

$$\frac{1}{2}\|(u_m)_t\|^2_{L^2(0,T;\ L^2(\Omega))} + \left(\frac{a}{2} - a\varepsilon\right)\|\nabla u_m(t)\|^2_{L^\infty(0,T;\ L^2(\Omega))} + \left(\frac{b}{p+1} - a\varepsilon\right)\|u_m(t)\|^{p+1}_{L^\infty(0,T,L^{p+1}(\Omega))}$$

$$\leq \frac{b}{p+1}\|\varphi_m\|^{p+1}_{L^{p+1}(\Omega)} + \left(a\varepsilon - \frac{a}{2}\right)\|\nabla\varphi_m\|^2_{L^2(\Omega)}.$$

putting

$$C' = \frac{b}{p+1}\|\varphi_m(\tau)\|^{p+1}_{L^{p+1}(\Omega)} + \left(a\varepsilon - \frac{a}{2}\right)\|\nabla\varphi_m\|^2_{L^2(\Omega)}.$$

Finally, we obtain

$$\frac{1}{2}\|(u_m)_t\|^2_{L^2(0,T;\ L^2(\Omega))} + \left(\frac{a}{2} - a\varepsilon\right)\|\nabla u_m(t)\|^2_{L^\infty(0,T;\ L^2(\Omega))} + \left(\frac{b}{p+1} - a\varepsilon\right)\|u_m(t)\|^{p+1}_{L^\infty(0,T,L^{p+1}(\Omega))} \leq C'. \tag{4.14}$$

Hence, we have

$$\|(u_m)_t\|^2_{L^2(0,T;\ L^2(\Omega))} \leq C'_T. \tag{4.15}$$

Then, we get the following further a priori estimates :

$$\begin{cases} u_m \text{ uniformly bounded in } L^\infty(0,T;\ H^1(\Omega) \cap L^{p+1}(\Omega)) \\ u_m \text{ uniformly bounded in } L^2(0,T;\ H^1(\Omega)) \\ (u_m)_t \text{ uniformly bounded in } L^2(0,T;\ L^2(\Omega)) \end{cases}. \tag{4.16}$$

■

Step three:"Convergence and result of existence"

Thus, by Lemma (1.2) and Lemma (1.5), there is a subsequence of u_m, still denoted by u_m, such that

$$\begin{cases} u_m \longrightarrow u \text{ weakly star in } L^\infty(0,T;\ H^1(\Omega) \cap L^{p+1}(\Omega)) \\ u_m \longrightarrow u \text{ weakly in } L^2(0,T;\ H^1(\Omega)) \\ (u_m)_t \longrightarrow u_t \text{ weakly in } L^2(0,T;\ L^2(\Omega)) \end{cases}, \tag{4.17}$$

Taking

$$B_0 = H^2 \cap H^1,$$
$$B = H^1,$$
$$B_1 = L^2,$$

By Theorem (1.4), there is a subsequence of u_m, still denoted by u_m such that

$$u_m \longrightarrow u \text{ Strongly in } L^2\left(0, T;\ H^1(\Omega)\right)$$

By Lemma (1.3), there is a subsequence of u_m, still denoted by u_m such that u_m almost everywhere converges to u in $Q_T = \Omega \times [0, T]$. It turns out that

$$(u_m)^p \text{ almost everywhere converges to } u^p \text{ in } Q_T$$

On the other hand, (4.16) implies that $(u_m)^p$ is uniformly bounded in $L^{\frac{p+1}{p}}(Q_T)$. Therefore, we infer from Lemma (1.4) that

$$u_m^p \rightharpoonup u^p \qquad \text{weakly in } L^{\frac{p+1}{p}}\left(0, T,\ L^{\frac{p+1}{p}}(\Omega)\right).$$

It remains to demonstrate that $w = \frac{\partial u}{\partial t}$, for that we will demonstrate that

$$u(t) = \varphi + \int_0^t w(\tau) d\tau. \tag{4.18}$$

As

$$u_{m_k} \rightharpoonup u \qquad \text{in } L^2(0, T\ ;\ L^2(\Omega)) \quad ,$$

then, the demonstration of (4.18), is equivalent to demonstrating that

$$u_{m_k} \rightharpoonup \varphi + \chi \qquad \text{in } L^2(0, T;\ L^2(\Omega)) \quad ,$$

which means

$$\lim \left(u_{m_k} - \varphi - \chi, v\right)_{L^2(0,T\ ;\ L^2(\Omega))} = 0,\ \forall v \in L^2(0, T;\ L^2(\Omega)),$$

such as

$$\chi(t) = \int_0^t w(\tau) d\tau.$$

Using equality

$$u_{m_k} - \varphi_m = \int_0^t \frac{\partial u_{m_k}}{\partial \tau} d\tau,\ \forall\, t \in [0, T],$$

we have from $u_{m_k} \in L^2(0,T;V_{m_k})$ and $(u_{m_k})_t \in L^2(0,T;V_{m_k})$, that

$$\left(u_{m_k} - \varphi - \int_0^t w(\tau)d\tau, v\right)_{L^2(0,T\,;\,L^2(\Omega))}$$

$$= \left(u_{m_k} - \varphi_m - \int_0^t w(\tau)d\tau, v\right)_{L^2(0,\mathrm{T}\,;\,L^2(\Omega))} + (\varphi_m - \varphi, v)_{L^2(0,T\,;\,L^2(\Omega))},$$

$$= \left(\int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau)\right) d\tau, v\right)_{L^2(0,T;\,L^2(\Omega))} + (\varphi_m - \varphi, v)_{L^2(0,T\,;\,L^2(\Omega))}, \text{ for all } t \in [0,T],$$

which results from **(ii)** of lemma (1.6), that

$$\left(u_{m_k} - \varphi - \int_0^t w(\tau)d\tau, v\right)_{L^2(0,T\,;\,L^2(\Omega))}$$

$$= \int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau), v\right)_{L^2(0,T\,;\,L^2(\Omega))} d\tau + (\varphi_m - \varphi, v)_{L^2(0,T\,;\,L^2(\Omega))}, \text{ for all } t \in [0,T].$$

On the one hand, we have

$$\lim_{t \longrightarrow \infty} \int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau), v\right)_{L^2(0,T\,;\,L^2(\Omega))} d\tau = 0, \text{ for all } t \in [0,T]. \tag{4.19}$$

On the other hand according to (4.19), we have

$$\lim_{t \longrightarrow \infty} (\varphi_m - \varphi, v)_{L^2(0,T\,;\,L^2(\Omega))} = 0. \tag{4.20}$$

Finally, from (4.19) and (4.20) ,we get

$$\lim_{k \longrightarrow \infty} (u_{m_k} - \varphi - \chi, v)_{L^2(0,T\,;\,L^2(\Omega))} = 0, \quad \forall v \in L^2\left(0,T;L^2(\Omega)\right).$$

Now, passing to the limit in (P_2), since each term on the left side of (P_2) is weakly convergent in $L^{\frac{p+1}{p}}(\Omega)$, we obtain that the following holds in $L^{\frac{p+1}{p}}(\Omega)$.

Conclusion

In this work, we are interested in nonlinear problems for nonlinear parabolic partial differential equations with mixed boundary conditions between classical, nonclassical, linear, and nonlinear conditions

Where, after developing the methods of proving the existence of very complicated problems, we were able to establish these results:

* The existence and the uniqueness of a weak solution of a semi-linear parabolic problem with homogenous Neumann condition. The proof proceeds in two steps; using the energy inequality method for the solvability of the linear case and applying an iterative process and a priori estimate, we prove the existence and uniqueness of the weak solution of the semi-linear problem then by using the energy function method to estimate the finite time blow up.

* Study the solvability and blow-up of the weak solution for a semi-linear Bessel problem with Neumann integral condition, where we show the existence and uniqueness of the weak solution for the linear problem by devising it into two problems and solving them by using the separation method and the energy inequality method. Then, by applying an iterative process based on the results obtained for the linear problem, we prove the existence and the uniqueness of the weak solution of the semi-linear problem. After that by using the Kaplan's method to estimate the finite time blow-up.

* Finally, the last result is devoted to the study of the existence of the weak solution of the parabolic problem for a nonlinear parabolic equation with nonlinear Neumann condition and generalized integral condition of the second type and nonlinear condition.

It is important to note again that there does not yet exist for nonlinear problems a general theory analogous to that of classical problems.

This is due to the relative novelty of this theme on one hand and to the complexity of the questions that it raises on the other. Each problem then requires a specific treatment that underlines the topicality of the subject addressed in this thesis.

It is pointed out that many interesting problems to better enrich this study remain open, we cite a few here :

- The study of the solutions of the problems for a class of fractional partial differential equations with an generalized integral condition of second type and nonlinear conditions. This question seems very delicate and important, and it deserves to be studied.

- Also, for non-linear fractional PDEs with nonlinear classical and non-classical conditions seems clearly difficult and which certainly requires more precise hypotheses and a very difficult development of the classical methods.

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