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Asymptotic Analysis of Some Singular Perturbation Problems

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بسم الله الرحمن الرحيم

To my parents...

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Abstract

In the present work we focus on the analysis of the asymptotic behaviour of the solutions to anisotropic singular perturbation boundary value problems. A complete description of the asymptotic behaviour on the whole domain of definition is established. Two types of functions are constructed. The first type acts far away from the boundary layers to give the best possible approximation. The second one deals with the behaviour near the boundary layers to recover the complete approximation with a sharper rate of convergence. In fact, we go beyond the limit behaviour by considering the regular and the composite asymptotic expansions of arbitrary order. This allows to get an asymptotic approximation of a polynomial rate of convergence in arbitrary order or even an exponential one.

Keywords:

Anisotropic, singular perturbations, boundary layers, elliptic, linear problems, smoothness of the solutions, regular and composite asymptotic expansions, rate of convergence, exponential, correctors.

العُنْوَان: **التَّحْلِيلُ التَّقَارُبِي لِبَعْضِ مَسَائِلِ الاِضْطِرَابَاتِ المُتَبَايِنَةِ**

مُلَخَّص

لَقَدْ خُصِّصَ هَذَا العَمَلُ لِدِرَاسَةِ السُّلُوكِ المُقَارِبِ لِحُلُولِ مَسَائِلِ الاِضْطِرَابَاتِ المُتَبَايِنَةِ غَيْرِ المُتَمَاثِلَةِ. حَيْثُ تَمَّ تَقْدِيمَ وَصْفٍ كَامِلٍ لِلسُّلُوكِ المُقَارِبِ عَلَى كُلِ مَيْدَانِ التَّعْرِيفِ، وَبِنَاءًا عَلَيْهِ أَنْشَأْنَا نَوْعَيْنِ مِنَ الدَّوَالِ: اَلأُولَى تَعْمَلُ عَلَى إِعْطَاءِ أَفْضَلِ تَقْرِيبٍ مُمْكِنٍ بَعِيدًا عَنْ طَبَقَاتِ الحُدُودِ أَمَّا الثَّانِيَة فَتُعَالِجُ السُّلُوكَ المُقَارِبَ فِي جِوَارِ طَبَقَاتِ حُدُودِ مَيْدَانِ التَّعْرِيفِ لِإِسْتِعَادَةِ التَّقْرِيبِ الشَّامِلِ لِلْحَلِّ مَعَ تَوْضِيحِ مُعَدَلِ التَّقَارُبِ. فِي الوَاقِعِ ذَهَبْنَا أَبْعَدَ مِنْ النِّهَايَةِ العَادِيَّةِ بِإِسْتِعْمَالِ النُّشُورِ المُقَارِبَةِ المُنْتَظَمَةِ وَالمُرَكَّبَةَ مَا سَمَحَ لَنَا بِالحُصُولِ عَلَى تَقْرِيبٍ بِمُعَدَلٍ فَوْق خَطِّي بِدَرَجَةٍ كَيْفِيَّةٍ أَوْ حَتَى أُسِّي.

الكَلِمَات المِفْتَاحِيَة:

غَيْر مُتَمَاثِل، الاِضْطِرَابَاتُ المُتَبَايِنَةُ، الطَّبَقَاتُ الحُدُودِيَةُ، إِهْلِيجِي، مَسَائِلُ خَطِّيَةٌ، نِظَامِيَة الحُلُول، النُّشُورُ المُقَارِبَةُ المُنْتَظَمَةُ وَالمُرَكَّبَةُ، مُعدَل التَّقَارُب، أُسِّي، المُصَحِّحَاتُ.

Titre: **Analyse Asymptotique de Quelques Problèmes des Perturbations Singulières**

Résumé

Dans le présent travail, nous nous concentrons sur l'analyse du comportement asymptotique des solutions de problèmes aux limites des perturbations singuliers anisotropes. Une description complète du comportement asymptotique sur le domaine de définition tout entier est établie. Deux types de fonctions sont construits. Le premier type agit loin des couches limites pour obtenir la meilleure approximation possible. La deuxième famille de fonctions a affaire avec le comportement près des couches limites, afin de récupérer une approximation complète avec un taux de convergence plus précis. En fait, nous allons au-delà du comportement limite en considérant les développements asymptotiques réguliers et composites d'ordre arbitraire. Ceci permet d'obtenir une approximation d'un taux de convergence polynômial d'ordre arbitraire ou même exponentiel.

Mots Clés:

Anisotrope, perturbations singulières, couches limites, elliptique, problèmes linéaires, régularité des solutions, développements asymptotiques régulières et composites, fonctions de la couche limite, correcteurs.

Contents

Introduction

The singular perturbations theory plays an important role in mathematical analysis as well as in the experimental sciences (nature and life) from where the history of this topic started to describe distinctive phenomena in Biology, Chemistry and Physics ... These phenomena are characterized by small values with respect to other values that appear in the mathematical model. It is then natural to neglect them in order to get a simpler model ensuring, of course, the approximation between the solutions of the two models. This simplicity can be expressed as a reduction of the dimension, a substitution of the problem type, simplicity of the study framework and so on. In order to illustrate this clearly, let us consider a singular perturbed problem, including a small positive parameter known as ε, with a solution u_ε defined on a domain Ω. The unperturbed problem is defined for $\varepsilon = 0$ and its solution is denoted by u_0. In general the perturbed and the unperturbed problems are not from the same type which may cause a difference in the properties of the two solutions and by consequence the approximation may be violated. If this is not the case and the convergence of u_ε towards u_0 is guaranteed with respect to some norm, i.e.

$$\lim_{\varepsilon \to 0} |u_\varepsilon - u_0| = 0,$$

we say that the perturbation is *regular*, otherwise it is called *singular* (if for example the convergence is not ensured on the whole Ω).

As it is mentioned in [24], two main types of singular behaviour can be distinguished. In the first case the function u_ε is regular everywhere in Ω with the exception of small neighbourhoods of certain manifolds whose dimension is smaller than the initial domain Ω. This behaviour is usually called *boundary layer* behaviour. In the second case there are n–dimensional subsets $\Omega_0 \subset \Omega$, with non-empty interior, (which may coincide with Ω) such that the set of $x_0 \in \Omega_0$ for which the limit of the rescaled function $u\left(x_0;\varepsilon\right) = O_s(1)$ (as ε tends to zero) does not exist, is dense in Ω_0. This behaviour can be called *oscillatory.* The analysis for such functions with oscillatory behaviour has in general quite different techniques from the analysis of functions with boundary layer behaviour (see also [37]).

The *boundary layers* term is used firstly in the fluid dynamics domain, by Prandtl (1905) to give a description of the motion of viscous fluids near solid boundaries. It means a thin layer of fluid in which, due to viscosity, rapid variation of fluid-velocities takes place. This term was then used on a wide scale in the singular perturbation theory. Actually, the boundary layer appears in a small neighbourhood of the boundary or some part of the boundary, but it may also occur *inside* the domain or as a *free* boundary layers. In general, if the problem is well presented, it is relatively easy to find the location of the boundary layers, but it is not the case for the abstract mathematical models. For example, we consider an algebraic perturbed equation as $P(x;\varepsilon) = 0$ where P is a third order polynomial and $\varepsilon > 0$ is the coefficient of the main term. Suppose that this equation has three different solutions and, of course, the unperturbed equation ($\varepsilon = 0$) has at most two solutions. In this case, it is not easy to identify the lost solution among the three original solutions. But when it concerns with differential equations it is not even possible to keep all properties to be satisfied by the unperturbed solution when epsilon is the coefficient of the main term in the perturbed equation.

The theory of singular perturbations is, from the mathematical point of view, an extremely interesting subject because it is possible to apply with success the results of the more abstract theory of differential equations, in particular one needs a priori estimates of solutions. For more details, we refer the reader to [20, 36, 37, 38, 42, 44].

Many methods have been proposed to deal with singular perturbation problems. The more fundamental of them including the asymptotic expansion, the method of matched asymptotic expansions and WKB (Wentzel, Kramers and Brillouin) approximation for spatial problems and in time, the method of multiple scales, the Poincaré–Lindstedt method, and so on. An idea shared in almost all these methods is to correct or avoid the non-valid character, in an appropriate norm, of a first approximation. The successive complementary expansion method, which is the heart of this thesis, follows also this logic.

In this thesis we deal with the study of functions which are implicitly defined as solutions of differential equations satisfying some supplementary conditions, boundary conditions (in the weak sense). The considered elliptic operators are linear containing a small parameter ε, where the singularity only affects some directions. This is called anisotropic singular perturbation problems. The situation is here totally different from the isotropic singular perturbation (i.e. when the singularity depends on all directions). We will be obliged to work in some anisotropic Sobolev spaces (the anisotropy depends on the order of the derivatives) and of course this prevents the convergences of the derivatives in these directions to be hold. We have to point out that the dimension of the space where we look for the limit solution is reduced.

As a typical example, we consider on $\Omega = (0,1) \times (0,1)$ a diffusion problem where the diffusion velocity is very small in the x_1–direction

$$\begin{cases} -\varepsilon^2 \partial^2_{x_1} u_\varepsilon - \partial^2_{x_2} u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Of course $\varepsilon > 0$ is a small parameter and f represents the source term (for instance in Biology, f represents the density bacteria supplied from outside). We are interested in the limit behaviour of u_ε when ε approaches zero and the natural candidate is u_0 solution to

$$\begin{cases} -\partial^2_{x_2} u_0 = f & \text{in } (0,1)\,, \\ u_0 = 0 & \text{on } \{0,1\}\,. \end{cases}$$

Note that here the variable x_1 plays a parameter role. The aim of this theory is to study the limit of u_ε towards u_0 when ε tends to zero. We have of course to precise in what sense this convergence will take place. In this context the case of general elliptic as well as of parabolic and hyperbolic boundary value problems defined on arbitrary domains has been considered and different convergence results are shown in [7, 8, 10, 11, 13, 30, 31].

Our goal here is to determine an asymptotic approximation, or asymptotic expansion, of the solution as ε tends to zero by the successive complementary expansion method (SCEM) which consists of seeking a formal approximation of the solution by assuming that $u_0(x)$ is known. For instance, if $x \in \Omega \subset \mathbb{R}$, the formal approximation has the form

$$u_{as} = u_0\,(x) + w_0\,(\xi;\varepsilon) \quad \text{where } \xi = \frac{x}{\delta\,(\varepsilon)} \text{ with } \lim_{\varepsilon\to 0} \delta\,(\varepsilon) = 0.$$

The above approach is not usual since w_0 depends not only on ξ but also on ε. In the asymptotic method, the functions dependent on ε are clearly separated from the functions independent of ε. If the independence with respect to ε is required, SCEM is equivalent to the method of matched asymptotic expansion (MMAE), which consists of seeking regular approximations in different significant domains.

These approximations are matched to render the solution valid with respect to the considered norm. The useful norm is the uniform norm but this is not the case in this thesis. SCEM is not used very much, but in its regular form, it leads to the same results as MMAE without requiring the delicate notion of matching principle. In addition, the use of generalized expansions, which is an important feature of SCEM, enables to solve problems which are more difficult or even impossible with regular asymptotic expansions which is necessarily in the MMAE (see [21]).

This thesis can be roughly divided into three chapters. The first chapter is of an introductory nature where we give the essentials of the theory of the asymptotic expansions and the functional analysis of linear elliptic problems. Then we focus on some important results in the framework of the above typical example.

In the second chapter we investigate the regular asymptotic expansions in the weak sense for solutions of an anisotropic singular of elliptic type perturbation problems. After a position of the problem, in the first section, we define the coefficients of the development as solutions of an iterative sequence of elliptic problems in the second section. Since the perturbation is only taken in some directions we have to study the smoothness of the coefficients in order to complete the definition of the sequence of these elliptic problems. The third section is devoted to show the basic convergence theorem and in the last section we deal with different upgraded results.

Finally, the third chapter is devoted to the investigation of the composite asymptotic expansions in the weak sense. In particular we construct the boundary layer functions by the application of a stretching of the local coordinates.

Chapter 1

Basic Tools

1.1 Asymptotic definitions and properties

When we deal with the series expansions for solutions of equations, it is important to have a clear view on the meaning of these expansions. In fact we will in particular describe the orders of magnitude, the asymptotic sequences, the asymptotic approximations (expansions), the boundary layers and so on. More details about these concepts are well studied in [24].

We start this section by two subsections treating elementary real functions that only depend on the positive parameter ε.

1.1.1 Order symbols and order functions

Consider pair of real continuous functions $u(\varepsilon)$ and $v(\varepsilon)$, where $\varepsilon \in (0, \varepsilon_0]$. The behaviour of these functions, as ε tends to zero, can be compared by using the classical *Landau* order symbols "O" and "o", which are defined as follows.

Definition 1. *We say that*

1. $u(\varepsilon) = O\left(v(\varepsilon)\right)$ *as* $\varepsilon \to 0$, *if there exist positive constants* κ *and* γ *such that*

$$|u\left(\varepsilon\right)| \leq \kappa \left|v\left(\varepsilon\right)\right| \quad \textit{for all } 0 < \varepsilon < \gamma,$$

   here "$|.|$" denotes the usual absolute value.

2. $u(\varepsilon) = o\left(v(\varepsilon)\right)$ *as* $\varepsilon \to 0$, *if for every positive constant* κ, *there exists a constant* $\gamma > 0$ *(*γ *may depend on* κ*) such that*

$$|u\left(\varepsilon\right)| \leq \kappa \left|v\left(\varepsilon\right)\right| \quad \textit{for all } 0 < \varepsilon < \gamma.$$

An O–estimate can sometimes be improved to an o–estimate and some times it is not the case. Then, it is natural that sometimes we have to be more precise in our estimates. The following definition enables us to give a more precise information about the estimate.

Definition 2. *We write*

$$u\left(\varepsilon\right) = O_s\left(v\left(\varepsilon\right)\right) \quad \textit{as } \varepsilon \to 0,$$

if we have

$$u\left(\varepsilon\right) = O\left(v\left(\varepsilon\right)\right) \quad \textit{and} \quad u\left(\varepsilon\right) \neq o\left(v\left(\varepsilon\right)\right) \quad \textit{as } \varepsilon \to 0.$$

Remark 1.1. *It is interesting to note that the Landau order symbols enable also to compare functions that do not exist for* $\varepsilon = 0$*. For instance, the estimate*

$$\frac{1}{\varepsilon} = o\left(\frac{1}{\varepsilon^2}\right) \quad \text{as } \varepsilon \to 0,$$

informs us about the rate of divergence of the two functions $(1/\varepsilon)$ *and* $\left(1/\varepsilon^2\right)$ *as* ε *tends to zero.*

We shall now define a convenient set of simpler comparison functions, which will be called order function. These functions aim to describe the orders of magnitude of any continuous function.

Definition 3. *Let* Θ *be the set of functions* $\delta(\varepsilon)$ *defined on* $(0, \varepsilon_0]$ *that are real, positive, continuous, monotone and for which* $\lim_{\varepsilon\to 0} \delta(\varepsilon)$ *exists. The extended set* $\bar{\Theta}$ *consists of the functions* $\delta(\varepsilon)$ *with* $\delta \in \Theta$ *or* $(1/\delta) \in \Theta$*. An element* $\delta(\varepsilon)$ *of* $\bar{\Theta}$ *is called an order function.*

Example 1.1. *Well known examples of sequences of order functions are*

$$\delta_n(\varepsilon) = \varepsilon^n \quad \text{or} \quad \exp\left(\frac{-n}{\varepsilon}\right), \quad n = 1, 2, \dots$$

The following remarkable theorem allows to find, in principle, an estimate for any continuous function with respect to our order functions.

Theorem 1.1 (W. Eckhaus [24])**.** *Let* $u(\varepsilon)$ *be a real continuous function in* $(0, \varepsilon_0]$*, not identically equal to zero. Then there exists an order function* $\delta \in \bar{\Theta}$ *such that* $u = O_s(\delta)$*.*

1.1.2 Asymptotic sequences and series

Let us start by the asymptotic sequence.

Definition 4. *A sequence of order functions* $(\delta_n)_0^N$ *is an asymptotic sequence if for all* $n = 1, \dots, N$*,* $\delta_n = o(\delta_{n-1})$ *as* ε *tends to zero.*

In the above definition N may be finite or infinite. In addition, we remark that any asymptotic sequence of order functions is a totally ordered subset of the set $\bar{\Theta}$. It is useful to generalize this as follows.

Definition 5. *A sequence of continuous functions* $(u_n(\varepsilon))_0^N$*, in* $(0, \varepsilon_0]$ *is called asymptotic if one has* $u_n = O_s(\delta_n)$ *for all* $n = 0, \dots, N$*, where* $(\delta_n)_0^N$ *is an asymptotic sequence.*

Asymptotic sequences generate asymptotic series by the usual linear operations. We obtain

Definition 6. *A sum* $\sum_{n=0}^N a_n u_n(\varepsilon)$ *where* a_n *are constants, is called an asymptotic series if* $(u_n(\varepsilon))_0^N$ *is an asymptotic sequence.*

Remark 1.2. *Note that an asymptotic series may have a finite or infinite number of terms. Of course, the concept of an infinite asymptotic series is purely formal (nothing is implied about the convergence of such series). Furthermore, the question of convergence of an infinite asymptotic series is of no particular interest in the asymptotic theory.*

If we take a sequence of positive real numbers that converges towards zero, there exists no positive number smaller than all elements of this sequence. An analogous result does not hold for asymptotic sequences. More precisely we have

Theorem 1.2 (Du Bois–Reymond). *For any asymptotic sequence $(\delta_n(\varepsilon))_0^\infty$, there exist order functions $\delta(\varepsilon)$ such that for all $n \in \mathbb{N}$*

$$\delta = o(\delta_n) \quad \text{as } \varepsilon \to 0.$$

Example 1.2. *Let us consider the asymptotic sequence $\delta_n(\varepsilon) = \varepsilon^n$ as ε tends to zero, for $n = 0, 1, \ldots$ As a convenient choice here satisfying the above estimate we can take $\delta(\varepsilon) = \exp(-\eta/\varepsilon)$ for $\eta > 0$.*

Functions of the order of magnitude of δ are often called transcendentally small.

In modern developments of asymptotic analysis, motivated largely by the needs of the perturbations theory for differential equations, as we would do in this thesis, it is useful to introduce generalizations of the above classical concepts.

1.1.3 Orders of magnitude

Let here Ω be a domain of $\mathbb{R}^{\mathrm{n}}$. Consider a function parametrized by the small parameter ε, $u(x;\varepsilon) =: u_\varepsilon(x)$, $x \in \Omega$, i.e. u depends on more than one variable. The scalar variable denoted by ε, $\varepsilon \in (0, \varepsilon_0]$, will be considered as a small parameter, while the variable x will be interpreted as an *independent* variable in the domain Ω. Furthermore, these functions can be considered as maps from the interval $(0, \varepsilon_0]$ into $\mathcal{X}(\Omega)$, where $\mathcal{X}(\Omega)$ is a set of functions which will further specified according to the needs and the goals of the analysis. If $\mathcal{X}(\Omega)$ is a linear space on which a norm $|.|_{\mathcal{X}(\Omega)}$ is defined, then this norm is a natural instrument to evaluate the order of magnitude of functions. We thus arrive to the following definition.

Definition 7. *Consider the functions $u(x;\varepsilon) =: u_\varepsilon(x)$ where, $u_\varepsilon : \Omega \to \mathbb{R}^{\mathrm{n}}$, $\mathrm{n} \in \mathbb{N}$. Let the restrictions of u_ε to a subset $\Omega_0 \subset \Omega$ be elements of a normed linear space $\mathcal{X}(\Omega_0)$, with a given norm $|.|_{\mathcal{X}(\Omega_0)}$ and let δ be an order function belonging to Θ (or $\bar{\Theta}$). We always assume that the map $\varepsilon \mapsto |u_\varepsilon|_{\mathcal{X}(\Omega_0)}$ is continuous. Thus when $\varepsilon \to 0$,*

1. $u = O(\delta)$ *in* $\Omega_0 \subset \Omega$ *if* $|u_\varepsilon|_{\mathcal{X}(\Omega_0)} = O(\delta)$,
2. $u = o(\delta)$ *in* $\Omega_0 \subset \Omega$ *if* $|u_\varepsilon|_{\mathcal{X}(\Omega_0)} = o(\delta)$,
3. $u = O_s(\delta)$ *in* $\Omega_0 \subset \Omega$ *if* $|u_\varepsilon|_{\mathcal{X}(\Omega_0)} = O(\delta)$ *and* $|u_\varepsilon|_{\mathcal{X}(\Omega_0)} \neq o(\delta)$.

Remark 1.3. *The above definition also permits us to investigate orders of magnitude of u on the whole domain Ω by letting Ω_0 coincide with Ω.*

Naturally, different norms can be used according to the problem being studied. It must be recognized however that the orders of magnitude can be completely different when we change the choice of the norm. Thus, if we use the super-mum norm, then for any subset $\Omega_0 \subset \Omega$ the norm will be defined by

$$|u_\varepsilon|_{\mathcal{X}(\Omega_0)} = \sup_{x \in \Omega_0} |u_\varepsilon(x)|.$$

Similarly, if we use the L^2–norm (see Section 1.2), then

$$|u_\varepsilon|_{\mathcal{X}(\Omega_0)} = \left\{ \int_{\Omega_0} |u_\varepsilon(x)|^2 \, \mathrm{d}x \right\}^{\frac{1}{2}}.$$

Such definitions can be easily extended to norms associated with the super-mum norm or the L^2–norm and involving derivatives of functions, or to weighted norms etc. This is illustrated by the following well known example.

Example 1.3. *Consider* $u(x;\varepsilon) = \exp(-x/\varepsilon)$, $\Omega = [0,1]$. *Thus, for various choices of the definition of norm in* Ω, *we have*

1. *for* $|u|_{\mathcal{X}(\Omega_0)} = \sup_{x\in\Omega} |u(x;\varepsilon)|, \quad u = O_s(1) \quad$ *in* Ω,

2. *for* $|u|_{\mathcal{X}(\Omega_0)} = \sup_{x\in\Omega} |u(x;\varepsilon)| + \sup_{x\in\Omega} \left|\frac{\mathrm{d}u(x;\varepsilon)}{\mathrm{d}x}\right|, \quad u = O_s(1/\varepsilon) \quad$ *in* Ω,

3. *for* $|u|_{\mathcal{X}(\Omega_0)} = \left\{\int_0^1 [u(x;\varepsilon)]^2 \,\mathrm{d}x\right\}^{\frac{1}{2}}, \quad u = O_s(\sqrt{\varepsilon}) \quad$ *in* Ω,

4. *for* $|u|_{\mathcal{X}(\Omega_0)} = \left[\int_0^1 [u(x;\varepsilon)]^2 + \left[\frac{\mathrm{d}u(x;\varepsilon)}{\mathrm{d}x}\right]^2 \mathrm{d}x\right]^{\frac{1}{2}}, \quad u = O_s(1/\sqrt{\varepsilon}) \quad$ *in* Ω.

Next let $\Omega_0 \subset \Omega$ *be any compact subinterval of* Ω, *not containing the origin. Then it is not difficult to verify that the orders of magnitude of* u_ε *in* Ω_0 *are given by order functions of the type* $\exp(-\eta/\varepsilon)$, *where* η *is a positive constant.*

To conclude the discussion of the orders of magnitude, let us note that in applications need may arise for a further generalization of the concepts. Thus, in perturbation problems for differential equations it is sometimes necessary to introduce expressions of the type

$$|u(x;\varepsilon)|_{\mathcal{X}(\Omega_0)} + \varepsilon \sum_{i=1}^{k} \left|\frac{\partial u(x;\varepsilon)}{\partial x_i}\right|_{\mathcal{X}(\Omega_0)} \quad \text{with } k \leq \mathrm{n}$$

(in this thesis, we will deal with $1 \leq k < \mathrm{n}$, where $\mathrm{n} > 1$). In this context, we shall admit as a norm any mapping which, for every arbitrary but fixed value of ε, is a norm on $\mathcal{X}(\Omega_0)$, provided that for any function $u_\varepsilon(x) \in \mathcal{X}(\Omega_0)$, the expression $|u_\varepsilon|_{\mathcal{X}(\Omega_0)}$, is a continuous function of ε for $\varepsilon \in (0, \varepsilon_0]$.

We now state a result that will be used in the following when we want to improve the polynomial rate of convergence and show the exponential one.

Corollary 1.1. *Let there be given a non–trivial function* $u(x;\varepsilon)$, $x \in \Omega_0 \subset \mathbb{R}^{\mathrm{n}}$, *and a norm* $|.|_{\mathcal{X}(\Omega_0)}$, *such that* $|u_\varepsilon|_{\mathcal{X}(\Omega_0)}$ *is a continuous function of* ε. *Then there exists an order function* $\delta \in \bar{\Theta}$ *such that*

$$u_\varepsilon = O_s(\delta) \quad \text{as } \varepsilon \to 0.$$

Proof. We write $|u_\varepsilon|_{\mathcal{X}(\Omega_0)} = v(\varepsilon)$ and use the Eckhaus theorem. □

Naturally, all results and definitions also hold in the whole domain Ω, if one lets Ω_0 coincide with Ω.

1.1.4 Asymptotic expansion

It is clear from the Eckhaus theorem that any non–trivial continuous function $u_\varepsilon(x)$ can be estimated by an order function $\delta(\varepsilon)$ such that $u_\varepsilon(x) = O_s(\delta(\varepsilon))$. Consequently we can always rescale

$$u_\varepsilon(x) = \delta(\varepsilon)\,\bar{u}(x;\varepsilon)$$

with $\delta(\varepsilon)$ an order function and $\bar{u}(x;\varepsilon) = O_s(1)$. Rescaling is useful to define the *asymptotic approximations* in the sense that it is natural to rescale before estimating. This in particular means that we have to take into account the order of magnitude for each asymptotic approximation. If we omit this, we would have a definition such as $\tilde{u}$ is an asymptotic approximation of u if $u - \tilde{u} = o(1)$. This shows, for example, ε is an asymptotic approximation of $u_\varepsilon = \varepsilon^2 + \varepsilon^3$, while u_ε here is expected to be of ε^2–order of magnitude. We then have

Definition 8. *A function u_{as} is a non–trivial asymptotic approximation of u_ε in $\Omega_0 \subset \Omega$ if*

$$u_\varepsilon - u_{as} = o(\delta_0) \quad \text{in } \Omega_0$$

with $u_\varepsilon = O_s(\delta_0)$ where $\delta_0 \in \bar{\Theta}$, such that

$$u_{as} = O_s(\delta_0) \quad \text{in } \Omega_0.$$

Remark 1.4. *The explicit statement "non–trivial" in the above definition is introduced to distinguish the trivial case, which is not interesting that is to say when the order of magnitude is smaller than δ_0.*

Then we generalize the above definitions for sequences and series.

Definition 9. *A sequence of functions $(u_n(x;\varepsilon))_0^N$, $x \in \Omega_0 \subset \Omega$ is an asymptotic sequence if $(|u_n|_{\mathcal{X}(\Omega_0)})_0^N$ is an asymptotic sequence, that is to say if $|u_n|_{\mathcal{X}(\Omega_0)} = O_s(\delta_n)$ with $\delta_n = o(\delta_{n-1})$.*

A series $\sum_{n=0}^N u_n(x;\varepsilon)$ is an asymptotic series if $(u_n)_0^N$ is an asymptotic sequence.

Given as asymptotic approximation u_{as} of u_ε, one may attempt to construct a *better* asymptotic approximation. In this way, a non-trivial approximation u_{as} of u_ε to order δ_0 is obtained. Subsequently, we have

$$u_\varepsilon - \delta_0 \bar{u}(x;\varepsilon) = o(\delta_0)$$

and we set

$$u_{as} = \delta_0 \bar{u}(x;\varepsilon)$$

with

$$\bar{u}(x;\varepsilon) = O_s(1).$$

If a better approximation is desired, the above process can be repeated. It can be shown that

$$u_\varepsilon - \delta_0 \bar{u}(x;\varepsilon) = O_s(\delta_1).$$

If there exists a function $\bar{u}_1(x;\varepsilon) = O_s(1)$ such that

$$u_\varepsilon - \delta_0 \bar{u}(x;\varepsilon) = \delta_1 \bar{u}_1(x;\varepsilon) + O_s(\delta_2) \quad \text{with} \quad \delta_2 = o(\delta_1),$$

we can write

$$u_\varepsilon = \delta_0 \bar{u}(x;\varepsilon) + \delta_1 \bar{u}_1(x;\varepsilon) + O_s(\delta_2).$$

The process can be stopped without taking care of the neglected order of magnitude. Then, the following notation is used

$$u_\varepsilon = \delta_0 \bar{u}(x;\varepsilon) + \delta_1 \bar{u}_1(x;\varepsilon) + o(\delta_1).$$

The process can also be continued to some chosen order N. Furthermore, the definitions of an asymptotic series and an asymptotic approximation can be combined to study the asymptotic expansions, which are approximating series of the form

$$\sum_{n=0}^{N} \delta_n(\varepsilon)\, u_n(x,\varepsilon), \quad N \in \mathbb{N}, \tag{1.1.1}$$

where $(\delta_n)_0^N$ is an asymptotic sequence and $u_n(x;\varepsilon) = O_s(1)$.

The question of uniqueness

Even in the case of an infinite expansion, uniqueness of an asymptotic expansion, for any given function $u(x;\varepsilon)$, is not ensured. This can be see when we think about the different choices of the asymptotic sequence (δ_n). In addition, according to the Du Bois–Reymond theorem, any function $u^*(x;\varepsilon)$ has the same asymptotic expansion as $u(x;\varepsilon)$, to the considered order, if

$$u - u^* = O(\delta^*),$$

where δ^* is asymptotically identical to zero with respect to the asymptotic sequence (δ_n) of the considered order functions. This means that the previous asymptotic expansion can be slightly changed by adding δ^*–term. This is an other reason of the non–uniqueness of asymptotic expansions.

In applications one can use this property to modify an expansion to obtain some special property which shows that the lack of the uniqueness of expansions is not totally a disadvantage.

Taking into account the non-uniqueness of an asymptotic expansion, the number of terms is not a very characteristic feature. It is better to say that an asymptotic expansion to order δ_N is obtained. This expansion can be written more precisely as

$$u(x;\varepsilon) = \sum_{n=1}^{N} \delta_n(\varepsilon)\, u_n(x;\varepsilon) + O_s(\delta_{N+1}(\varepsilon)),$$

such that

$$u_n(x;\varepsilon) = O_s(1) \quad \text{and} \quad \delta_{n+1} = o(\delta_n) \quad n = 1, \ldots, N.$$

Convergence and accuracy

For a real valued function $u(\varepsilon)$ of class $\mathcal{C}^N[0,\varepsilon_0)$, where $N \in \mathbb{N}$, an elementary asymptotic expansion can be given from its Taylor series expansion when ε is small. Thus, an asymptotic expansion to order ε^N is

$$u(\varepsilon) = u(0) + \varepsilon u'(0) + \cdots + \varepsilon^N \frac{u^{(N)}(0)}{N!} + O_s\left(\varepsilon^{N+1}\right).$$

This asymptotic expansion deals with the approximation of $u(\varepsilon)$ by a polynomial of a fixed order N when ε tends to zero. Another type of convergence can be considered if we let N becomes infinite. Here we are concerning with the convergence of the series. However, we have to highlight that there is no link between the two types of convergence. A series has an infinite number of terms whereas an asymptotic expansion has a finite number of terms. An asymptotic expansion can have an infinite number of terms (in this case we have an asymptotic series) but the question of the convergence of the series has no connection with the behaviour of the function in the neighbourhood of $\varepsilon = 0$.

For example, the polynomial

$$\phi_m(\varepsilon) = \sum_{n=0}^{m} \frac{1}{n!}\varepsilon^n$$

is an asymptotic approximation of $u(\varepsilon) = \exp(\varepsilon) + \exp(-1/\varepsilon)$ when ε tends to zero, but we never have

$$\lim_{m\to+\infty} \phi_m(\varepsilon) = u(\varepsilon).$$

This allows, of course when ε tends to zero, to consider even the divergent series since we will take into account a finite number of terms. In fact only sometimes it is more important to consider the divergent series than the convergence ones in the asymptotic approximations (see [21, 33, 42]). Of course, for the divergent series it is better to take a small number of terms and this number may can be increased if ε becomes small and small. For more details about this concepts we refer the reader to [21, 23, 38, 43].

There is no contradiction between the different observations. The series is divergent because the limit of u_ε is taken when N tends to plus infinity for a fixed value of ε whereas the asymptotic expansion is valid for a fixed value of N as ε tends to zero.

For physical problems, the quality of an asymptotic expansion is not predictable. Sometimes, a good intuition can improve the result. If one writes

$$\sin\varepsilon = \varepsilon - \frac{\varepsilon^3}{6} + O\left(\varepsilon^5\right)$$

and

$$\sin\varepsilon = \frac{\varepsilon}{1+\varepsilon^2/6} + O\left(\varepsilon^5\right)$$

it is immediately seen that in the second case, with a single term, the same accuracy can be obtained as in the first case with two terms. This is due to the appropriate choice of the representative in the class of order functions. These convergence improvements are useful in practice.

Operations on the asymptotic expansions

According to the definition of the order functions, we can show that we can always construct a new asymptotic expansion from the addition of the previous asymptotic expansions. The same thing is possible for the subtraction, the multiplication and the division except may be the differentiation. From the application point of view, this allows to substitute the P.D.E. into a lot of equations where the asymptotic expansion of its solution can be deduce from those of the new equations using the above elementary operations.

In the following, we will study simpler expansions that called *regular* expansions, which are important tools in the asymptotic analysis. Note that in the literature the term *regular* can be used to express other senses.

1.1.5 Regular expansions and boundary layers

Regular expansions

Definition 10. *Suppose there exists an asymptotic sequence $(\delta_n(\varepsilon))_0^N$ and a sequence of non-trivial functions $u_n(x)$ independent of ε, such that the series*

$$u_{as}(x;\varepsilon) = \sum_{n=0}^{N} \delta_n(\varepsilon)\, u_n(x)$$

is an asymptotic expansion of $u(x;\varepsilon)$ in $\Omega_0 \subset \Omega$. Then u_{as} is a regular asymptotic expansion of u in $\Omega_0 \subset \Omega$.

Definition 11. *An asymptotic approximation of u_ε can then be written as*

$$u(x;\varepsilon) = \delta_0(\varepsilon)\, \bar{u}(x) + o(\delta_0).$$

A function u_ε having this property is said regular. It must be noted that this property is not necessarily valid to next orders.

Remark 1.5. *Obviously, regular asymptotic approximations (expansions) are special cases of the approximations (expansions) studied in the above subsection since the functions u_n are independent of ε. Regular expansions are sometimes called Poincaré, classical or straightforward asymptotic expansions, while expansions in the sense of (1.1.1) are called generalized expansions.*

Remark 1.6. *Note that the smoothness of the data play an important role in the order of the expansion since in each step when we want to increase the order of the asymptotic approximations and by consequence the number of terms, more smoothness assumptions are required.*

Now, we shall see that the uniqueness of regular expansions can be ensured by imposing further suitable restrictions on the order functions.

Gauge functions and the uniqueness of regular expansions

We shall now define sets that be called gauge sets, through their useful properties.

Definition 12. *[24] A gauge set $\bar{\Theta}_0$ is a subset of $\bar{\Theta}$ such that*

1. *For any two elements $\delta_1, \delta_2 \in \bar{\Theta}_0$ either $\delta_1 = o(\delta_2)$ or $\delta_2 = o(\delta_1)$ holds,*
2. *Given any $\delta \in \bar{\Theta}_0$, there exist elements $\delta_1, \delta_2 \in \bar{\Theta}_0$ such that*
$$\delta_1 = o(\delta) \text{ and } \delta = o(\delta_2),$$
3. *Given any pair $\delta_1, \delta_2 \in \bar{\Theta}_0$, with $\delta_1 = o(\delta_2)$, there exist elements $\delta_i \in \bar{\Theta}_0$ such that*
$$\delta_1 = o(\delta_i) \text{ and } \delta_i = o(\delta_2).$$

Remark 1.7. *We can say, in simple way, that a gauge function is an order function chosen as the representative of its class of equivalence with respect to the equivalence between functions "$\approx$" (see [21]).*

Using the gauge tools, the non-uniqueness of regular expansions can be removed and we can state this result as follows.

Lemma 1.1. *Let $\mathcal{X}_0(\Omega_0, \bar{\Theta}_0)$ be the set of functions of the structure*

$$(x, \varepsilon) \to \sum_n \delta_n(\varepsilon)\, u_n(x), \quad x \in \Omega_0, \quad \delta_n \in \bar{\Theta}_0,$$

where $\bar{\Theta}_0$ is some gauge set. If some given function u_ε has a regular expansion belonging to $\mathcal{X}_0(\Omega_0, \bar{\Theta}_0)$, then this expansion is unique in $\mathcal{X}_0(\Omega_0, \bar{\Theta}_0)$.

In general, the regular asymptotic expansion only takes place in some subsets of Ω which are in particular located far away from some parts of boundary, this leads us to deal with the boundary layer problems. In the following, we will give a qualitative description of the singular behaviour due to boundary layers.

The concept of a boundary layer

The boundary layer behaviour can be described as follows.

Definition 13. *[24] Let a function $u(.;\varepsilon)$, on $\Omega \subset \mathbb{R}^{\mathrm{n}}$, be such that there exists a manifold of lower dimension $S \subset \Omega$ with the following properties*

1. *u_ε is not regular in any n–dimensional, subset Ω_0 of Ω such that Ω_0 does not dependent on ε and $\Omega_0 \cap S \neq \varnothing$,*
2. *Any point $x \in \Omega - S$ belongs to some n–dimensional, subset of Ω, independent of ε, in which u_ε is regular.*

Then u_ε has its singularity on S, and we say that u exhibits boundary layers along S. A neighbourhood of S in Ω with some determined size, will be called a boundary layer of the function u_ε.

As it is said above the boundary layers often occur near parts of the boundary Ω but they are not limited to the boundary. One can also have *interior* or *free* boundary layers. This, of course, forces the study to be focussed near this boundary layer regions using the local coordinates tool, this is what we will use in Chapter 3, to illustrate this local coordinates techniques, we consider the function $u_\varepsilon(x)$ defined on $\Omega \subset \mathbb{R}$. Suppose there exists a point $x_0 = 0 \in \Omega$, with the property that $u_\varepsilon(x)$ has no regular expansion in each subset of Ω containing the point $x_0 = 0$.

Suppose that near the point x_0 the boundary layer is characterized in size by an order function $\delta(\varepsilon)$. We *rescale* or *Stretch* the variable x by introducing the transformation

$$\varsigma = \frac{x}{\delta_s(\varepsilon)}, \quad \delta_s = o(1).$$

The variable ς, for some choice of δ_s, will be called a local variable (ς is also often called a stretched variable or an inner variable).

The function $u_\varepsilon(x)$ transforms to

$$\begin{aligned} u_\varepsilon(x) &= u_\varepsilon(x_0 + \delta(\varepsilon)\varsigma) \\ &= w_\varepsilon(\varsigma). \end{aligned}$$

It is then natural to continue the local analysis by expanding the function w_ε with respect to the local variable ς; we hope to find again an asymptotic expansion.

Introducing into $u(x;\varepsilon)$ the change of variables x to ς produces a function of ς and ε, which will be denoted by $w(\varsigma;\varepsilon)$. We shall say that the change of variables x to ς induces the change of $u(x,\varepsilon)$ to $w(\varsigma;\varepsilon)$. The domain of w_ε will be specified when it is needed.

In order to investigate that a formal approximation is really an asymptotic approximation, the presentation of the used material is essentially.

1.2 Basic functional techniques

We recall in this section some basic notions of the functional analysis and some other properties as an introduction to linear elliptic problems.

1.2.1 L^q–spaces and distributions

We recall firstly some basic concepts and techniques regarding L^q–spaces.

Definition 14. *Let $q \geq 1$ be a real number, Ω an open subset of $\mathbb{R}^{\mathrm{n}}$,* $\mathrm{n} \geq 1$

$$L^q(\Omega) := \left\{ v : \Omega \to \mathbb{R},\ v \textit{ is measurable and } \int_\Omega |v(x)|^q \,\mathrm{d}x < \infty \right\},$$

and

$$L^\infty(\Omega) := \{ v : \Omega \to \mathbb{R},\ v \textit{ is measurable} \mid \exists \kappa \textit{ such that} |v(x)| \leq \kappa, \quad \textit{a.e. } x \in \Omega \}.$$

Equipped with the norm

$$|v|_{q,\Omega} = \left\{ \int_\Omega |v(x)|^q \,\mathrm{d}x \right\}^{\frac{1}{q}},$$

and

$$|v|_{\infty,\Omega} = \inf\{\kappa \textit{ such that } |v(x)| \leq \kappa, \quad \textit{a.e. } x \in \Omega\}$$

respectively, where $L^q(\Omega)$ and $L^\infty(\Omega)$ are Banach spaces.

In the following we recall some elementary properties of L^q–spaces that will be useful in the next chapters (in particular, the case where $q = 2$).

Theorem 1.3 (Dominated convergence theorem)**.** *Let (u_n) be a sequence in $L^q(\Omega)$ and let $u \in L^q(\Omega)$, such that $|u_n - u|_{q,\Omega}$ towards zero. Then, there exist a subsequence (u_{n_k}) and a non-negative function $v \in L^q(\Omega)$ such that*

1. $u_{n_k}(x) \to u(x) \quad$ *a.e. on* Ω,
2. *for all* k, $|u_{n_k}(x)| \leq v(x) \quad$ *a.e. on* Ω.

Theorem 1.4 (Fatou)**.** *Let (u_n) be a sequence of functions in $L^1(\Omega)$ that satisfies*

1. *for all* n, $u_n \geq 0 \quad$ *a.e. on* Ω,
2. $\sup_n \int_\Omega u_n \, \mathrm{d}x < \infty$.

For almost all $x \in \Omega$ we set $u(x) = \liminf_{n\to\infty} u_n(x) \leq +\infty$. Then $u \in L^1(\Omega)$, and

$$\int_\Omega u \, \mathrm{d}x \leq \liminf_{n\to+\infty} \int_\Omega u_n \, \mathrm{d}x.$$

Theorem 1.5 (Tonelli)**.** *Let $u(X_1, X_2) : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a measurable function satisfying*

1. $\int_{\Omega_2} u(X_1, X_2) \, \mathrm{d}X_2 < \infty \quad$ *a.e.* $X_1 \in \Omega_1$,
2. $\int_{\Omega_1} \mathrm{d}X_1 \int_{\Omega_2} u(X_1, X_2) \, \mathrm{d}X_2 < \infty$.

Then $u \in L^1(\Omega_1 \times \Omega_2)$.

Theorem 1.6 (Fubini)**.** *Assume that $u \in L^1(\Omega_1 \times \Omega_2)$. Then for a.e. $x \in \Omega_1$, $u(X_1, X_2) \in L^1(\Omega_2)$ and*

$$\int_{\Omega_2} u(X_1, X_2) \, \mathrm{d}X_2 \in L^1(\Omega_1).$$

Similarly, for a.e. $X_2 \in \Omega_2$,

$$u(X_1, X_2) \in L^1(\Omega_1) \quad \textit{and} \quad \int_{\Omega_1} u(X_1, X_2) \, \mathrm{d}X_1 \in L^1(\Omega_2).$$

Moreover, one has

$$\int_{\Omega_1} \mathrm{d}X_1 \int_{\Omega_2} u(X_1, X_2) \, \mathrm{d}X_2 = \int_{\Omega_2} \mathrm{d}X_2 \int_{\Omega_1} u(X_1, X_2) \, \mathrm{d}X_1 = \iint_{\Omega_1 \times \Omega_2} u(X_1, X_2) \, \mathrm{d}X_2 \, \mathrm{d}X_1.$$

We denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support in Ω where the support of a function ρ is defined as

$$\mathrm{supp}\rho = \text{the closure of the set } \{x \in \Omega \mid \rho(x) \neq 0\}.$$

Definition 15. *Let $\Omega \subset \mathbb{R}^{\mathrm{n}}$ be open and let $1 \leq q \leq \infty$. We say that a real value function u belongs to $L^q_{loc}(\Omega)$ if $u\chi \in L^q(\Omega)$ for every compact set K contained in Ω where χ is the characteristic function of K.*

Note that if $u \in L^q_{loc}(\Omega)$ then $u \in L^1_{loc}(\Omega)$. Then we have

Lemma 1.2. *Let $u \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega u\varphi \, \mathrm{d}x = 0, \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

Then $u = 0$ a.e. on Ω.

If $\mathcal{X}$ is a normed linear space, then the collection of bounded linear functional on $\mathcal{X}$ is called its dual space and is denoted by $\mathcal{X}'$. In particular, for any $1 \leq q < \infty$ the dual of $L^q(\Omega)$ can be identified with $L^{q'}(\Omega)$ where q' is the conjugate number of q which is defined by

$$\frac{1}{q} + \frac{1}{q'} = 1,$$

i.e., $q' = \frac{q}{q-1}$ with the convention that $q' = +\infty$ when $q = 1$ (in particular if $q = 2$ we also have $q' = 2$).

Theorem 1.7 (Holder's inequality)**.** *Assume that $u \in L^q(\Omega)$ and $v \in L^{q'}(\Omega)$ with $1 \leq q \leq \infty$. Then $uv \in L^1(\Omega)$ and*

$$\int_\Omega |uv| \, \mathrm{d}x \leq |u|_{q,\Omega} \, |v|_{q',\Omega}.$$

For $q = 2$, $q' = 2$, we have the Cauchy–Schwartz inequality in $L^2(\Omega)$. We also recall the classical form of Young inequality

$$\kappa_1\kappa_2 \leq \frac{1}{q}\kappa_1^q + \frac{1}{q'}\kappa_2^{q'} \quad \forall \kappa_1, \kappa_2 \geq 0$$

with $1 < q < \infty$. More general Young inequality can be written as

$$\kappa_1\kappa_2 \leq \frac{\varepsilon}{q}\kappa_1^q + \frac{1}{\sqrt[q-1]{\varepsilon}\,q'}\kappa_2^{q'} \quad \forall \kappa_1, \kappa_2 \geq 0$$

where ε is positive constant. Note that the last inequality can be derived from the previous one by replacing κ_1 and κ_2 by $\sqrt[q]{\varepsilon}\kappa_1$ and $\kappa_2/\sqrt[q]{\varepsilon}$ respectively.

In the following we would like to introduce the notion of distributions.

Definition 16. *A distribution T on Ω is a continuous linear functional on $\mathcal{D}(\Omega)$. We will denote by*

$$\langle T, \varphi \rangle := T(\varphi) \quad \forall \varphi \in \mathcal{D}(\Omega),$$

and by $\mathcal{D}'(\Omega)$ the space of all distributions on Ω.

In what follows, we use the abbreviation D^α, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\mathrm{n}) \in \mathbb{N}^\mathrm{n}$, the partial derivative given by

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_\mathrm{n}}^{\alpha_\mathrm{n}}}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_\mathrm{n}$. In the next we will clarify the preceding definition.

Definition 17. *Let $T \in \mathcal{D}'(\Omega)$ for some open set $\Omega \subset \mathbb{R}^\mathrm{n}$.*

1. *If α is any multi-index, then we define $D^\alpha T$, the α–weak partial derivative, by*

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle.$$

2. *Given* $\psi \in \mathcal{C}^{\infty}(\Omega)$, *we define* ψT *by*

$$\langle \psi T, \varphi \rangle = \langle T, \psi\varphi \rangle .$$

3. *If* $h \in \mathbb{R}^{\mathrm{n}}$ *and* u *is any function on* $\mathbb{R}^{\mathrm{n}}$ *then we define the shift of function by* h *as* $\tau_h u(x) = u(x+h)$. *We set*

$$\langle \tau_h u, \varphi \rangle = \langle u, \tau_{-h}\varphi \rangle .$$

4. *If* M *is an invertible linear transformation of* $\mathbb{R}^{\mathrm{n}}$, *then we set*

$$\langle T \circ M, \varphi \rangle = |\mathrm{detJ}_M|^{-1} \left\langle T, \varphi \circ M^{-1} \right\rangle ,$$

where detJ_M *denotes the Jacobian determinant, i.e. the determinant of the Jacobian matrix* $\mathrm{J}_M = (\partial M_i / \partial_{x_j})$.

5. $L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$ *in the following sense, for every* $f \in L^1_{loc}(\Omega)$

$$\langle Tf, \varphi \rangle = \int_{\Omega} fv \, \mathrm{d}x \quad \forall \varphi \in \mathcal{D}(\Omega) .$$

Remark 1.8. *One of the key feature of distributions is the fact that we can differentiate them - in some sense - as much as we wish. Indeed, if* u *is a function that is* k*–times differentiable in* Ω, $k \geq |\alpha|$, *then the distribution* $D^{\alpha}u$ *coincides with the function* $D^{\alpha}u$, *the partial derivative of* u *in the usual sense.*

We define now the convergence in $\mathcal{D}'(\Omega)$.

Definition 18. *Let* $(T_n)_{n\in\mathbb{N}}$ *be a sequence of distributions on* Ω. *We say that when* $n \to \infty$,

$$T_n \to T \quad \text{in } \mathcal{D}'(\Omega)$$

iff

$$\langle T_n, \varphi \rangle \to \langle T, \varphi \rangle , \quad \forall \varphi \in \mathcal{D}(\Omega) .$$

Now we recall the notion of *weak convergence.* A sequence (u_n), $n \in \mathbb{N}$, in a normed space $\mathcal{X}$, is called *weakly convergent* to an element $u \in \mathcal{X}$, if

$$f(u_n) \to f(u) \quad \text{as } n \to +\infty, \quad \forall f \in \mathcal{X}'.$$

In this case u is called a *weak limit* of the sequence and we write

$$u_n \rightharpoonup u \quad \text{as } n \to +\infty.$$

Note that

$$u_n \to v \quad \text{in } \mathcal{X} \Rightarrow u_n \rightharpoonup v \quad \text{in } \mathcal{X}, \quad \text{as } n \to +\infty.$$

The converse is not true in general. However, a weakly convergent sequence is bounded. Then we have

Proposition 1. *Let* $u_n, u \in L^q(\Omega)$. *If we suppose that when* $n \to +\infty$,

$$u_n \to u \quad \text{in } L^q(\Omega) , \quad (\text{respectively } u_n \rightharpoonup u \text{ in } L^q(\Omega)\,) \quad 1 \leq q < \infty,$$

then we have

$$u_n \to u \quad \text{in } \mathcal{D}'(\Omega) .$$

Proposition 2. *The operator D^α, $\alpha \in \mathbb{N}^{\mathrm{n}}$ is continuous on $\mathcal{D}'(\Omega)$, i.e.*

$$T_n \to T \text{ in } \mathcal{D}'(\Omega) \Rightarrow D^\alpha T_n \to D^\alpha T \text{ in } \mathcal{D}'(\Omega) \quad \text{as } n \to +\infty.$$

We conclude this subsection by the following results often used to prove the weak convergence of a whole sequence.

Theorem 1.8 (Weak compactness of balls)**.** *If (u_n) is a bounded sequence in H, there exists a subsequence (u_{n_k}) of (u_n) and $u \in H$ such that*

$$u_{n_k} \rightharpoonup u \quad \text{as } n_k \to +\infty.$$

Proposition 3. *Let $\mathcal{X}$ be a reflexive Banach space and (u_n) a bounded sequence in $\mathcal{X}$. We assume that there exists $u \in \mathcal{X}$ such that every weakly convergent subsequence of (u_n) has a limit equal to u; then the whole sequence (u_n) weakly converges to u.*

We also have

Theorem 1.9. *Let u_n, v_n be two sequences in H such that $u_n \to u$ and $v_n \rightharpoonup v$ when n tends to plus infinity. Then we have*

$$\langle u_n, v_n \rangle \to \langle u, v \rangle \quad \text{in } \mathbb{R}, \quad \text{as } n \to +\infty.$$

1.2.2 Lax–Milgram theorem

Our goal here is to introduce the Lax–Milgram theorem which is a key tool for solving elliptic partial differential equations. Instead of a scalar product one can consider more generally a continuous bilinear form. Suppose that $(H, \langle .,. \rangle_H)$ is a Hilbert space, $\mathcal{V} \subseteq H$ is a linear subspace of H, and $\langle .,. \rangle_{\mathcal{V}}$ is an inner product on $\mathcal{V}$ that turns $\mathcal{V}$ into a Hilbert space.

Moreover, we suppose that the embedding $\mathcal{V} \hookrightarrow H$ is continuous, i.e. there is a positive constant κ such that

$$|v|_H \le \kappa \, |v|_{\mathcal{V}} \quad \forall v \in \mathcal{V}. \tag{1.2.1}$$

Definition 19. *A bilinear form $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is said to be*

1. *continuous if there is a positive constant Λ such that $|a(u,v)| \le \Lambda \, |u|_{\mathcal{V}} \, |v|_{\mathcal{V}} \quad \forall u, v \in \mathcal{V}$,*
2. *coercive if there is a positive constant λ such that $a(v,v) \ge \lambda \, |v|_{\mathcal{V}}^2 \quad \forall v \in \mathcal{V}$.*

Then we have the following theorem.

Theorem 1.10 (Lax–Milgram)**.** *Assume that $a(.,.)$ is a continuous coercive bilinear form on $\mathcal{V}$. Then, given any $f \in \mathcal{V}'$, there exists a unique element $u \in \mathcal{V}$ such that*

$$a(u,v) = \langle f, v \rangle_{\mathcal{V}} \quad \forall v \in \mathcal{V}.$$

1.2.3 Essential features of the Sobolev spaces

The Sobolev spaces are very convenient tools to study the partial differential equations. In this subsection we concentrate ourselves to the most simple ones. Then, we firstly define $W^{1,q}(\Omega)$ as follows

$$W^{1,q}(\Omega) = \{ v \in L^q(\Omega) \mid \partial_{x_i} v \in L^q(\Omega), \quad i = 1, ..., \mathrm{n} \},$$

where $\partial_{x_i} v$ denotes the partial derivative of v in the weak sense. Equipped with the norm $|.|_{q,\Omega}$ is a Banach space and it is reflexive for $1 < q < \infty$. Then we give a simple characterization of difference quotients in L^q–version that we will use in the regularity results in the next chapters.

Proposition 4. *Let $u \in L^q(\Omega)$ with $1 < q \leq \infty$. The following properties are equivalent*

1. $u \in W^{1,q}(\Omega)$,

2. *there exists a constant κ such that for all $\Omega' \subset\subset \Omega$, and all $h \in \mathbb{R}^{\mathrm{n}}$ with $|h| < \operatorname{dist}(\Omega', \partial\Omega)$ we have*

$$|\tau_h u - u|_{q,\Omega'} \leq \kappa \, |h| \, , \tag{1.2.2}$$

where $\Omega' \subset\subset \Omega$ means that $\bar{\Omega}'$ is in Ω.

Remark 1.9. *Note that when $q = 1$, the implication $(1) \Rightarrow (2)$ remains true.*

In the next, we focus on the properties of Hilbert spaces. For $q = 2$, Ω be an open subset of $\mathbb{R}^{\mathrm{n}}$, $\mathrm{n} \geq 1$. We denote by $H^1(\Omega)$ the subset of $L^2(\Omega)$ defined by $H^1(\Omega) := W^{1,q}(\Omega)$ which is a Hilbert space equipped with the scalar product

$$(u, v)_{1,\Omega} = \int_\Omega \nabla u \cdot \nabla v + uv \, \mathrm{d}x,$$

where "∇u" denotes the Gradient vector $(\partial_{x_1} u, \partial_{x_2} u, \ldots, \partial_{x_{\mathrm{n}}} u)^T$. For more details about the space $H^1(\Omega)$ we refer the reader to [1, 5, 39]. However, here we can also state

Theorem 1.11. *Assume that Ω is a bounded Lipschitz domain of $\mathbb{R}^{\mathrm{n}}$, then the canonical embedding from $H^1(\Omega)$ into $L^2(\Omega)$ is compact.*

We use the above theorem in partial differential equations as follows. From any bounded sequence in $H^1(\Omega)$ we can extract a subsequence converges weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in Ω.

Now, let us denote by Γ_0 a subset of $\Gamma = \partial\Omega$ the boundary of Ω, such that

$$|\Gamma_0| > 0,$$

where $|\Gamma_0|$ denotes the measure of Γ_0. Define $\mathcal{C}_0^1\left(\bar{\Omega}, \Gamma_0\right)$ as the set of continuously differentiable functions on $\bar{\Omega}$ (Γ is a Lipschitz boundary). An important subspace of $H^1(\Omega)$ is $\mathcal{V}$ the space defined as

$$\mathcal{V} = \text{the closure of } \mathcal{C}_0^1\left(\bar{\Omega}, \Gamma_0\right) \text{ in } H^1(\Omega).$$

The mapping

$$|u|_{\mathcal{V}} = \left\{ \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x \right\}^{\frac{1}{2}} = |\nabla u|_{2,\Omega} \tag{1.2.3}$$

defines a norm on $\mathcal{V}$ which is equivalent the H^1–norm when Ω is bounded in some directions . This is an immediate consequence of the following lemma.

Lemma 1.3 (Poincaré's inequality on $\mathcal{V}$)**.** *Let Ω be a bounded open subset of $\mathbb{R}^{\mathrm{n}}$. There exists a positive constant κ such that*

$$|v|_{2,\Omega} \leq \kappa \, |v|_{\mathcal{V}} = \kappa \, |\nabla v|_{2,\Omega} \quad \forall v \in \mathcal{V}.$$

In particular when $\Gamma_0 = \Gamma = \partial\Omega$ we set $H_0^1(\Omega) := H_0^1(\Omega, \Gamma)$ which is the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. We denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$ and $\mathcal{V} = H_0^1(\Omega, \Gamma_0)$.

We will also consider higher order Sobolev spaces, for k a positive integer

$$H^k(\Omega) = \left\{ v \in L^2(\Omega) \mid D^\alpha v \in L^2(\Omega), \quad |\alpha| \leq k \right\}.$$

They are Hilbert spaces for the Hilbertian norm

$$|v|_{k,\Omega} = \left\{ \int_\Omega \sum_{|\alpha| \le k} |D^\alpha v|^2 \, \mathrm{d}x \right\}^{\frac{1}{2}} .$$

Without going into the details, we mention that on a Lipschitz surface or boundary Γ we have

Remark 1.10. *Note that the higher order spaces require more regularity of* Γ*.*

Remark 1.11. *Having the trace operator at hand we can now make an interpretation of the Dirichlet boundary condition. Studying the Poison equation with Dirichlet boundary condition*

$$- \triangle u = f, \quad u \mid_\Gamma = g$$

we conclude two things. First, the equation $u \mid_\Gamma = g$ means that $\gamma_0 u = g$ in $H^{1/2}(\Gamma)$ since the variational formulation of the Poison equation is posed in $H^1(\Omega)$ (subject to the Dirichlet condition). Second, the Dirichlet condition makes sense only for $g \in H^{1/2}(\Gamma)$. If $g \notin H^{1/2}(\Gamma)$ then there does not exist a solution $u \in H^1(\Omega)$ of the given boundary value problem. This is a conclusion of the surjectivity of the trace operator.

An other most important properties of the trace are the following

Theorem 1.12 (Green formula)**.** *For all $u, v \in H^1(\Omega)$, we have*

$$\int_\Omega v\,(x)\,\partial_{x_i} u\,(x)\;\mathrm{d}x = \int_\Omega \partial_{x_i} v\,(x)\,u\,(x)\;\mathrm{d}x - \int_{\partial\Omega} \gamma_0\,(v)\,\gamma_0\,(u)\,\nu_i\,\mathrm{d}\sigma$$

where $\nu = (\nu_1, ..., \nu_\mathrm{n})$ the outward unit normal to $\Gamma = \partial\Omega$ and γ_0 (respectively $\mathrm{d}\sigma$) denotes the operator trace (respectively the superficial measure on $\Gamma = \partial\Omega$).

Finally, when dealing with boundary value problems one wants to control not only the values of a function at the boundary, but also the values of its derivatives. For this using a *cut-off function* in $C_0^1[a, b]$ is not enough, one needs smooth *cut-off functions* as in the following theorem.

Theorem 1.13. *Given any finite interval $[a, b] \subseteq \mathbb{R}$ there are functions $\rho_n \in \mathcal{C}^\infty[a, b]$ satisfying*

1. $0 \le \rho_n \le 1$,
2. $\rho_n = 1$ *on* $[a + 1/n, b - 1/n]$,
3. $\rho_n = 0$ *on* $[a, a + 1/2n]$ *and on* $[b - 1/2n, b]$.

1.2.4 Elliptic problems in divergence form

In this subsection we apply the results of previous subsections to solve in the weak sense linear elliptic problems.

Let Ω be a bounded open subset of $\mathbb{R}^\mathrm{n}$ with a Lipschitz boundary Γ. Let a_{ij}, $i, j = 1, \ldots, \mathrm{n}$, be functions in $L^\infty(\Omega)$ such that for $\lambda > 0$ it holds that

$$a_{ij}\,(x)\,\xi_j\xi_i \ge \lambda\,|\xi|^2 \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^\mathrm{n}.$$

Then, for $u, v \in H^1(\Omega)$ we define

$$a\,(u, v) = \int_\Omega a_{ij}\,(x)\,\partial_{x_j} u \partial_{x_i} v\,\mathrm{d}x.$$

Note that since $a_{ij} \in L^\infty(\Omega)$ the above definition makes sense. Moreover we have

Theorem 1.14. *Under the above assumptions* $a(.,.)$ *is a continuous bilinear form on* $H^1(\Omega)$ *and coercive on* $\mathcal{V}$.

As a consequence of the Lax–Milgram theorem we can show

Theorem 1.15 (Mixed boundary conditions problem). *Let* $\mathcal{V} = H_0^1(\Omega, \Gamma_0)$ *with* $|\Gamma_0| > 0$. *Then, for every* $f \in \mathcal{V}'$ *there exists a unique* u *solution to*

$$\begin{cases} a(u,v) = \langle f, v\rangle & \forall v \in \mathcal{V}, \\ u \in \mathcal{V}. \end{cases} \tag{1.2.4}$$

A solution u to (1.2.4) is a weak solution to the mixed Dirichlet–Neumann boundary condition problems.

$$\begin{cases} -\sum\limits_{i,j=1}^{\mathrm{n}} \partial_{x_i}\left(a_{ij}(x)\,\partial_{x_j} u\right) = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ \partial\nu_A u = 0 & \text{on } \Gamma\setminus\Gamma_0 \end{cases}$$

where ν_A is the co-normal vector. Clearly if $\Gamma_0 = \Gamma$ we end up with the Dirichlet boundary value problem.

We now investigate the case of some non-homogeneous boundary conditions. For that we consider

$$g \in H^1(\Omega), \quad h \in L^2(\Gamma), \quad f \in L^2(\Omega) \tag{1.2.5}$$

when Ω is a Lipschitz open subset of $\mathbb{R}^{\mathrm{n}}$. Then we have

Theorem 1.16. *Let* $\mathcal{V}$ *be either* $H_0^1(\Omega)$ *or* $H_0^1(\Omega, \Gamma_0)$ *and under the above assumptions, there exists a unique* u *solution to*

$$\begin{cases} a(u,v) = \langle f, v\rangle & \forall v \in \mathcal{V} \\ u - g \in \mathcal{V} \end{cases} \tag{1.2.6}$$

Remark 1.12. *Clearly* u *solution to (1.2.6) is a weak solution to*

$$\begin{cases} -\sum\limits_{i,j=1}^{\mathrm{n}} \partial_{x_i}\left(a_{ij}(x)\,\partial_{x_j} u\right) = f \quad \textit{in } \Omega \\ u = g \quad \textit{on } \Gamma_0 \quad \partial\nu_A u = h \quad \textit{on } \Gamma\setminus\Gamma_0 \end{cases}$$

respectively where $\Gamma_0 = \Gamma$, u *is the weak solution to the non-homogeneous Dirichlet problem.*

1.3 Anisotropic singular perturbation problems

This section is concerned with some selected works on the subject of anisotropic singular perturbation problems. Then, in addition to the notation in the previous sections, we also add here some basic tools and notation which will used in this thesis. For Ω a bounded open subset of $\mathbb{R}^{\mathrm{n}}$, $\Omega_\kappa = (-\kappa,\kappa)^p \times \omega$ a cylinder of $\mathbb{R}^{\mathrm{n}}$, ω a bounded Lipschitz domain of $\mathbb{R}^{\mathrm{n}-p}$, p and n positive integers such that $p < \mathrm{n}$, $\kappa > 0$, we denote by $x = (x_1, \ldots, x_{\mathrm{n}}) = (X_1, X_2)$ the points in $\mathbb{R}^{\mathrm{n}}$ where $X_1 = (x_1, \ldots, x_p)$ and $X_2 = (x_{p+1}, \ldots, x_{\mathrm{n}})$, i.e. we split the coordinates into two parts. Let Π_1 and Π_2 be the orthogonal projections from $\mathbb{R}^{\mathrm{n}}$ onto the spaces $X_2 = 0$ and $X_1 = 0$ respectively. For any $X_1 \in \Pi_1(\Omega) := \Pi_\Omega$ and any $X_2 \in \Pi_2(\Omega)$ we denote by Ω_{X_1} and Ω_{X_2} the sections of Ω above X_1 and X_2 respectively, i.e. $\Omega_{X_1} = \{X_2 \mid (X_1, X_2) \in \Omega\}$ and $\Omega_{X_2} = \{X_1 \mid (X_1, X_2) \in \Omega\}$. With this notation, similarly we set

$$\nabla u = (\partial_{x_1} u, \ldots, \partial_{x_{\mathrm{n}}} u)^T = \begin{pmatrix} \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix},$$

and

$$\nabla_{X_1} u = (\partial_{x_1} u, \ldots, \partial_{x_p} u)^T, \quad \nabla_{X_2} u = (\partial_{x_{p+1}} u, \ldots, \partial_{x_\mathrm{n}} u)^T.$$

The Laplace operators $\Delta_{X_1}, \Delta_{X_2}, \Delta$, and Δ_{q_1,X_1}, Δ_{q_2,X_2}, Δ_q (the q_i–Laplace operators in X_1, X_2 and x respectively) are defined by

$$\Delta_{X_1} = \partial^2_{x_1} + \cdots + \partial^2_{x_p}, \quad \Delta_{X_2} = \partial^2_{x_{p+1}} + \cdots + \partial^2_{x_\mathrm{n}}, \quad \Delta = \Delta_{X_1} + \Delta_{X_2},$$

and

$$\Delta_{q_1,X_1} \cdot = \nabla_{X_1} \cdot \left(|\nabla_{X_1} \cdot|^{q_1-2} \nabla_{X_1} \cdot \right), \quad \Delta_{q_2,X_2} \cdot = \nabla_{X_2} \cdot \left(|\nabla_{X_2} \cdot|^{q_2-2} \nabla_{X_2} \cdot \right),$$
$$\Delta_q \cdot = \nabla \cdot \left(|\nabla \cdot|^{q-2} \nabla \cdot \right),$$

where q_1, q_2, $q > 1$ are real constants. For the evolution problems, we denote by ∂_t or $.'$ the partial derivative in the time which will belong to some time interval $(0, \text{Ŧ})$, where Ŧ is a positive constant. It will be also convenient to simplify the notation by setting

$$z = (t, x) \in \mathbb{R}^{\mathrm{n}+1}, \quad Q = (0, \text{Ŧ}) \times \Omega, \quad Q_\kappa = [0, \text{Ŧ}] \times \Omega_\kappa,$$
$$Q_\infty = [0, \text{Ŧ}] \times \omega, \quad Q_{X_1} = (0, \text{Ŧ}) \times \Omega_{X_1}, \quad Q_{X_2} = (0, \text{Ŧ}) \times \Omega_{X_2}.$$

Let $A = (a_{ij}(z))$ be a $\mathrm{n} \times \mathrm{n}$ matrix with

$$a_{ij} \in L^\infty(Q) \quad \forall i, j = 1, \ldots, \mathrm{n} \tag{1.3.1}$$

and such that, for some $\lambda > 0$, we have

$$A\xi \cdot \xi \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^\mathrm{n}, \quad \text{a.e. } z \in Q. \tag{1.3.2}$$

where "$\cdot$" denotes the canonical scalar product in $\mathbb{R}^\mathrm{n}$. We decompose A into four blocks by writing

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \partial_t A = \frac{\partial}{\partial t} A = \begin{pmatrix} \partial_t A_{11} & \partial_t A_{12} \\ \partial_t A_{21} & \partial_t A_{22} \end{pmatrix} \tag{1.3.3}$$

where A_{11}, A_{22} are respectively $p \times p$ and $(\mathrm{n} - p) \times (\mathrm{n} - p)$ matrices. We then set for $\varepsilon > 0$

$$A_\varepsilon = A_\varepsilon(z) = \begin{pmatrix} \varepsilon^2 A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{pmatrix}. \tag{1.3.4}$$

We have therefore, for a.e. $z \in Q$ and every $\xi \in \mathbb{R}^\mathrm{n}$,

$$A_\varepsilon \xi \cdot \xi = A\xi_\varepsilon \cdot \xi_\varepsilon \geq \lambda |\xi_\varepsilon|^2 = \lambda \{\varepsilon^2 |\bar{\xi}_1|^2 + |\bar{\xi}_2|^2\}, \tag{1.3.5}$$
$$A_{22} \bar{\xi}_2 \cdot \bar{\xi}_2 \geq \lambda |\bar{\xi}_2|^2,$$

where $\xi = \binom{\bar{\xi}_1}{\bar{\xi}_2}$ with $\bar{\xi}_1 = (\xi_1, \ldots, \xi_p)^T$, $\bar{\xi}_2 = (\xi_{p+1}, \ldots, \xi_\mathrm{n})^T$ and $\xi_\varepsilon = (\varepsilon \bar{\xi}_1, \bar{\xi}_2)^T$. Thus, we have

$$A_\varepsilon \xi \cdot \xi \geq \lambda (\varepsilon^2 \wedge 1) |\xi|^2 \quad \forall \xi \in \mathbb{R}^\mathrm{n}, \quad \text{a.e. } z \in Q, \tag{1.3.6}$$

where "$\wedge$" denotes the minimum of two numbers. It follows that A_ε and A_{22} are positive definite matrices.

The full and integral proofs of theorems of this section can be founded in the mentioned papers, here only the main results of some selected papers are given.

1.3.1 Linear elliptic anisotropic singular perturbation problems

In this subsection since we will deal with elliptic problems then we assume that the matrix A is independent of t, i.e. $a_{ij}(t,x) = a_{ij}(x) \in L^\infty(\Omega)$, $i,j = 1,\dots,$n. We analyse diffusion problems when the diffusion coefficients in certain directions are going toward zero, i.e. for a source term

$$f \in L^2(\Omega), \tag{1.3.7}$$

we consider the following elliptic problem

$$\begin{cases} u_\varepsilon \in H_0^1(\Omega), \\ \displaystyle\int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla v \,\mathrm{d}x = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega). \end{cases} \tag{1.3.8}$$

More precisely we are interested in determining the corresponding limit problem of (1.3.8) and the speed of convergence of the solution u_ε toward its limit. Obviously thanks to the Lax–Milgram theorem and the previous hypotheses there exists a unique $u_\varepsilon \in H_0^1(\Omega)$ solution to problem (1.3.8). If we formally pass to the limit in the above problem, we may see that the candidate limit of u_ε is $u_0 = u_0(X_1,.)$ defined for a.e. $X_1 \in \Pi_\Omega$ as a solution to

$$\begin{cases} u_0(X_1,.) \in H_0^1(\Omega_{X_1}), \\ \displaystyle\int_{\Omega_{X_1}} A_{22}\nabla_{X_2} u_0(X_1,.) \cdot \nabla_{X_2} v \,\mathrm{d}X_2 = \int_{\Omega_{X_1}} f(X_1,.) v \,\mathrm{d}X_2 \quad \forall v \in H_0^1(\Omega_{X_1}). \end{cases} \tag{1.3.9}$$

Note that there exists u_0 solution to the above elliptic problem set on the section Ω_{X_1} since A_{22} is definite positive and for a.e. $X_1 \in \Pi_\Omega$ we have $f(X_1,.) \in L^2(\Omega_{X_1})$. We have of course now to precise in what sense the convergence

$$u_\varepsilon \to u_0 \quad \text{as } \varepsilon \to 0, \tag{1.3.10}$$

will take place. Firstly we will establish the convergence for arbitrary domain Ω.

Asymptotic behaviour in arbitrary domains

Even, in general the convergence (1.3.10) in the space $H^1(\Omega)$ does not hold, the following theorem ensures some convergence results.

Theorem 1.17. *Under the assumptions above we have when $\varepsilon \longrightarrow 0$,*

$$u_\varepsilon \longrightarrow u_0, \quad \nabla_{X_2} u_\varepsilon \longrightarrow \nabla_{X_2} u_0, \qquad \varepsilon \nabla_{X_1} u_\varepsilon \longrightarrow 0 \qquad \text{in } L^2(\Omega) \tag{1.3.11}$$

where u_ε (respectively u_0) is the weak solution to (1.3.8) *(respectively* (1.3.9)*).*

(In the above convergences the vectorial convergence in $L^2(\Omega)$ means the convergence component by component.)

Proof. See [13, p. 181-184]. □

Now we are concerned with the special case where Ω is a cylinder and we give precise conditions which insure $u_0 \in H^1(\Omega)$ and estimate the rate of convergence of u_ε towards u_0 for different norms.

The rate of convergence in general cylindrical domains

Here we suppose that Ω is of a special type namely $\Omega = \omega_1 \times \omega_2$ where ω_1 and ω_2 are bounded Lipschitz domains of $\mathbb{R}^p$ and $\mathbb{R}^{\mathrm{n}-p}$ respectively. Then for any $X_1 \in \Pi_1(\Omega)$ and any $X_2 \in \Pi_2(\Omega)$ one has $\Omega_{X_1} = \omega_2$, $\Omega_{X_2} = \omega_1$ and the problem (1.3.9) can be written

$$\begin{cases} u_0(X_1, .) \in H_0^1(\omega_2), \\ \displaystyle\int_{\omega_2} A_{22} \nabla_{X_2} u_0(X_1, .) \cdot \nabla_{X_2} v \, \mathrm{d}X_2 = \int_{\omega_2} f(X_1, .) \, v \, \mathrm{d}X_2 \quad \forall v \in H_0^1(\omega_2). \end{cases} \tag{1.3.12}$$

In order to obtain a convergence of u_ε with respect to the H^1–norm far away from the boundary layers $\partial\omega_1 \times \omega_2$, we need to guarantee that $\nabla_{X_1} u_0$ belongs to $L^2(\Omega')$ for each open subset $\Omega' = \omega_1' \times \omega_2$ where $\omega_1' \subset\subset \omega_1$, which means that

$$u_0 \in H^1(\Omega'). \tag{1.3.13}$$

So if we assume that

$$\partial_{x_k} f \in L^2(\Omega), \quad \partial_{x_k} a_{ij} \in L^\infty(\Omega), \quad k = 1, \ldots, p, \quad i, j = p+1, \ldots, \mathrm{n} \tag{1.3.14}$$

we get

Proposition 5. *Under the assumptions (1.3.1), (1.3.2), (1.3.7) and (2.2.8) we have $u_0 \in H^1(\Omega)$.*

Proof. See [13, p. 185-186]. □

Remark 1.13. *Note that, in general $u_0 \notin H_0^1(\Omega)$, for example, in the particular case when $\Omega = (0,1) \times (0,1)$, $A = \mathrm{Id}$, and $f = f(x_2)$ then u_0 is independent of x_1 and is not identically equal to zero if $f \neq 0$. Thus the function $u_0 = u_0(x_2) \notin H_0^1(\Omega)$ and we cannot expect the convergence of u_ε towards u_0 in $H^1(\Omega)$ as ε tends to zero.*

Next, it is clear that for any open set ω_1' satisfying $\omega_1' \subset\subset \omega_1$, we can find another open subset ω_1'' such that $\omega_1' \subset\subset \omega_1'' \subset\subset \omega_1$. Then, we suppose that

$$\partial_{x_i} a_{ij}, \quad \partial_{x_j} a_{ij} \in L^\infty(\omega_1'' \times \omega_2), \qquad i = 1, \ldots, p, \quad j = p+1, \ldots, \mathrm{n}. \tag{1.3.15}$$

Now we can state the following theorem.

Theorem 1.18. *Under the assumptions of Proposition 5 and if we suppose that we have (1.3.15) then for any $\omega_1' \subset\subset \omega_1$,*

$$|u_\varepsilon - u_0|_{2,\Omega'}, \quad |\nabla_{X_2}(u_\varepsilon - u_0)|_{2,\Omega'} = O(\varepsilon),$$

and

$$\nabla_{X_1} u_\varepsilon \rightharpoonup \nabla_{X_1} u_0 \quad \text{in } L^2(\Omega').$$

Proof. See [13, p. 187-189]. □

We can improve the above result if we have some information on $\frac{1}{\varepsilon}(u_\varepsilon - u_0)$. More precisely we have

Theorem 1.19. *Given the assumptions of Theorem 1.18, the following claims are equivalent*

1. $\frac{1}{\varepsilon}(u_\varepsilon - u_0)$, $\frac{1}{\varepsilon}[\nabla_{X_2}(u_\varepsilon - u_0)] \to 0$ *strongly in* $L^2(\omega_1' \times \omega_2)$, $\quad \forall \omega_1' \subset\subset \omega_1$,
2. $\frac{1}{\varepsilon}(u_\varepsilon - u_0) \rightharpoonup 0$ *weakly in* $L^2(\omega_1' \times \omega_2)$, $\quad \forall \omega_1' \subset\subset \omega_1$.

Proof. See [13, p. 189-190]. □

The next theorems improve the convergence rate when the matrix A has a diagonal structure.

Theorem 1.20. *Under the assumptions of Theorem 1.18 and in addition we assume that $A_{12} = A_{21}^T = 0$, then we have as $\varepsilon \to 0$,*

$$|u_\varepsilon - u_0|_{2,\Omega'}, \quad |\nabla_{X_2}(u_\varepsilon - u_0)|_{2,\Omega'} = o(\varepsilon), \qquad |\nabla_{X_1}(u_\varepsilon - u_0)|_{2,\Omega'} = o(1).$$

Proof. See [13, p. 190-191]. □

Theorem 1.21. *Under the assumptions of Theorem 1.20 and if, moreover, the limit u_0 and the matrix A_{11} are smooth enough in the X_1–directions, i.e. they satisfy*

$$\nabla^2_{X_1} u_0 \in L^2(\Omega) \quad \text{and} \quad \nabla_{X_1} A_{11} \in L^\infty(\Omega), \tag{1.3.16}$$

we have when $\varepsilon \to 0$,

$$|u_\varepsilon - u_0|_{2,\Omega'}, \quad |\nabla_{X_2}(u_\varepsilon - u_0)|_{2,\Omega'} = O\left(\varepsilon^2\right), \qquad |\nabla_{X_1}(u_\varepsilon - u_0)|_{2,\Omega'} = O(\varepsilon)$$

($\nabla^2_{X_1} u_0 = \left(\partial_{x_i}\partial_{x_j} u_0\right)_{i,j=1,\dots,p}$ is the Hessian matrix in the directions X_1 and $\nabla_{X_1} A_{11} \in L^\infty(\Omega)$ means that $\nabla_{X_1} a_{ij} \in L^\infty(\Omega)$, $i, j = 1, \dots, p$.

Proof. See [13, p. 191]. □

Remark 1.14. *We can show that a necessary and sufficient condition to get the weak convergence*

$$\frac{1}{\varepsilon}(u_\varepsilon - u_0) \rightharpoonup 0 \quad \text{in } L^2\left(\omega_1' \times \omega_2\right),$$

for every $\omega_1' \subset\subset \omega_1$, is

$$\int_\Omega A_{12} \nabla_{X_2} u_0 \cdot \nabla_{X_1} v \, \mathrm{d}x + \int_\Omega A_{21} \nabla_{X_1} u_0 \cdot \nabla_{X_2} v \, \mathrm{d}x = 0 \quad \forall v \in H_0^1(\Omega). \tag{1.3.17}$$

For example, if A_{12} and A_{21} are constants, this is the case when $A_{12} = -A_{21}^T$. The following example is given to clarify the previous situation.

Example 1.4. *We take $\Omega = (0,1) \times (0,1)$ and*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In this case, we do not have the weak convergence

$$\frac{1}{\varepsilon}(u_\varepsilon - u_0) \rightharpoonup 0 \quad \text{in } L^2(\Omega).$$

Indeed, if we combine (1.3.8) and (2.2.10), we get

$$\int_\Omega \varepsilon \partial_{x_1} u_\varepsilon \partial_{x_1} v \, \mathrm{d}x + \int_\Omega \partial_{x_2} u_\varepsilon \partial_{x_1} v \, \mathrm{d}x + \int_\Omega \frac{1}{\varepsilon}(u_\varepsilon - u_0) \partial^2_{x_2} v \, \mathrm{d}x = 0 \quad \forall v \in \mathcal{D}(\Omega).$$

Supposing that we have $\frac{1}{\varepsilon}(u_\varepsilon - u_0)$ towards zero in $L^2(\Omega)$ and letting ε go to zero, then we deduce

$$\int_\Omega \partial_{x_2} u_0 \partial_{x_1} v \, \mathrm{d}x = 0.$$

Replacing v by $\partial_{x_2} v$ we derive

$$\int_\Omega \partial_{x_2}^2 u_0 \partial_{x_1} v \,\mathrm{d}x = -\int_\Omega \partial_{x_2} u_0 \partial_{x_1} \partial_{x_2} v \,\mathrm{d}x = 0.$$

We have therefore,

$$\int_\Omega \partial_{x_1} f v \,\mathrm{d}x = -\int_\Omega f \partial_{x_1} v \,\mathrm{d}x = 0,$$

which implies that f and u_0 are independent of x_1.

Remark 1.15. *In the case when the limit solution u_0 is independent of X_1, as in the example above, and in addition if the hypothesis (1.3.17), which will be reduced to*

$$\int_\Omega A_{12} \nabla_{X_2} u_0 \cdot \nabla_{X_1} v \,\mathrm{d}x = 0 \quad \forall v \in H_0^1(\Omega),$$

is true, we can show, using the iteration technique given in [19], that u_ε converges towards u_0 in $H^1(\omega_1' \times \omega_2)$ with an exponential rate of convergence, for any $\omega_1' \subset\subset \omega_1$, i.e.

$$|u_\varepsilon - u_0|_{H^1(\omega_1' \times \omega_2)} \leq \kappa \exp\left(-\frac{\eta}{\varepsilon}\right),$$

where the positive constants η, κ are independent of ε.

Correctors for some anisotropic singular perturbation problems

As it is shown above, the solution u_ε does not converge with respect to the H^1–norm on the whole domain, towards some u_0. Here we construct correctors (or boundary layer functions), to get good approximations of u_ε on the whole domain. Of course we will more interested in the behaviour of u_ε near the boundary layers since we already have the needed convergence far away from this part of the boundary. To do this, let us suppose that Ω is a cylindrical domain defined as $\Omega = (0,1) \times \omega$, i.e. here we take $\omega_1 = (0,1), \omega_2 = \omega$, and $A = \mathrm{Id}$. Then we do not have convergence, in the H^1–norm near $\{0\} \times \omega$ and $\{1\} \times \omega$. As in Section 1.1, the idea is to construct a corrector near $\{0\} \times \omega$ and another one near $\{1\} \times \omega$. For reason of simplicity we take a Neumann condition on $\{0\} \times \omega$ and by consequence we do not have to define a corrector near this part of the boundary. This means that we consider here the mixed boundary value problem

$$\begin{cases} -\varepsilon^2 \partial_{X_1}^2 u_\varepsilon - \Delta_{X_2} u_\varepsilon = f & \text{in } \Omega = (0,1) \times \omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \setminus \{0\} \times \omega, \\ \frac{\partial u_\varepsilon}{\partial X_1} = 0 & \text{on } \{0\} \times \omega. \end{cases} \tag{1.3.18}$$

For a.e. $X_1 \in (0,1)$ one can define u_0 the solution to

$$\begin{cases} -\Delta_{X_2} u_0 (X_1, .) = f (X_1, .) & \text{in } \omega, \\ u_0 (X_1, .) \in H_0^1(\omega). \end{cases} \tag{1.3.19}$$

The goal here is to *correct* $u_\varepsilon - u_0$ by a simple function θ_ε which describes the behaviour of $u_\varepsilon - u_0$ near $\{1\} \times \omega$ and is such that

$$u_\varepsilon - u_0 - \theta_\varepsilon \to 0 \quad \text{in } H_0^1(\Omega). \tag{1.3.20}$$

Note that there is no convergence problem for u_ε near $\{0\} \times \omega$ since if U_ε is solution to

$$\begin{cases} -\varepsilon^2 \partial_{X_1}^2 U_\varepsilon - \Delta_{X_2} U_\varepsilon = F & \text{in } (-1,1) \times \omega, \\ U_\varepsilon \in H_0^1((-1,1) \times \omega), \end{cases}$$

where F is an even function in X_1 and equals f on $(0,1)\times\omega$, we can show that (see [11, p. 273-274])

$$U_\varepsilon(-X_1,X_2)=U_\varepsilon(X_1,X_2)=u_\varepsilon(X_1,X_2)\quad \text{a.e. on } \Omega.$$

Of course thanks to the results of the previous sub-subsection, if u_0 is smooth enough, we have

$$u_\varepsilon \to u_0 \qquad \text{in } H^1((0,a)\times\omega), \quad \text{for all } 0<a<1.$$

Let now $\mathcal{V}$ be the space defined as

$$\mathcal{V}=\left\{v\in H^1(\Omega) \mid v=0 \text{ on } \partial\Omega\diagdown\{0\}\times\omega\right\}. \tag{1.3.21}$$

There exists a unique u_ε solution to (1.3.18) such that

$$\begin{cases} u_\varepsilon \in \mathcal{V}, \\ \displaystyle\int_\Omega \varepsilon^2 \partial_{X_1} u_\varepsilon \partial_{X_1} v + \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \quad \forall v\in\mathcal{V}. \end{cases} \tag{1.3.22}$$

Clearly (2.2.7) is the weak formulation of (1.3.18). We assume that $f\in L^2(\Omega)$ and the existence of a unique solution to (2.2.7) follows from the Lax–Milgram theorem. The weak formulation of (1.3.19) reads for a.e. $X_1\in(0,1)$

$$\begin{cases} u_0(X_1,.)\in H_0^1(\omega), \\ \displaystyle\int_\omega \nabla_{X_2} u_0(X_1,.)\cdot\nabla_{X_2} v \,\mathrm{d}X_2 = \int_\omega f(X_1,.)\, v\,\mathrm{d}X_2 \quad \forall v\in H_0^1(\omega). \end{cases} \tag{1.3.23}$$

In the case where

$$f, \quad \partial_{X_1} f \in L^2(\Omega), \tag{1.3.24}$$

one has $u_0\in H^1(\Omega)$, see the precedent sub-subsection.

In order to construct a corrector for u_ε we denote by S_κ the strip $S_\kappa=(\kappa,+\infty)\times\omega$, $\kappa>0$. Let $w\in H^1(S_0)$ be the weak solution to

$$\begin{cases} \Delta w=0 \quad \text{in } S_0, \\ w\mid_{\{0\}\times\omega} = -u_0\mid_{\{1\}\times\omega}, \quad w\mid_{(0,+\infty)\times\partial\omega}=0. \end{cases} \tag{1.3.25}$$

We denote by θ_ε the function defined as

$$\theta_\varepsilon(X_1,X_2)=w\left(\frac{1-X_1}{\varepsilon},X_2\right) \quad \text{a.e. on } \Omega. \tag{1.3.26}$$

We can show the following estimates

Lemma 1.4. *There exist positive constants κ and η independent of ε such that*

$$\int_{S_{\frac{1}{\varepsilon}}} |\nabla w|^2 \,\mathrm{d}x \le \kappa \exp\left(-\frac{\eta}{\varepsilon}\right). \tag{1.3.27}$$

Proof. See [11, p. 266-267]. □

The theorem below plays an important role to get the main result of this sub-subsection.

Theorem 1.22. *Let u_ε, u_0 be the solutions to (1.3.18), (1.3.19). Then under the assumption (1.3.24) there exist two positive constants κ and η independent of ε, such that*

$$\begin{aligned} \tfrac{3}{4}\int_\Omega \varepsilon^2 \left(\partial_{X_1}\left(u_\varepsilon - u_0 - \theta_\varepsilon\right)\right)^2 + \left|\nabla_{X_2}\left(u_\varepsilon - u_0 - \theta_\varepsilon\right)\right|^2 \mathrm{d}x \\ \leq \kappa \exp\left(-\frac{\eta}{\varepsilon}\right) - \varepsilon^2 \int_\Omega \partial_{X_1} u_0 \partial_{X_1}\left(u_\varepsilon - u_0 - \theta_\varepsilon\right) \mathrm{d}x. \end{aligned} \tag{1.3.28}$$

Proof. See [11, p. 267-269]. □

As a first application of Theorem 1.22, we have

Theorem 1.23. *The solution u_0 is a strong limit of the sequence $(u_\varepsilon - \theta_\varepsilon)$ in $H^1(\Omega)$ and the following error estimate is valid as $\varepsilon \to 0$,*

$$|u_\varepsilon - u_0 - \theta_\varepsilon|_{2,\Omega}, \quad \left|\nabla_{X_2}\left(u_\varepsilon - u_0 - \theta_\varepsilon\right)\right|_{2,\Omega} = o(\varepsilon), \qquad \left|\partial_{X_1}\left(u_\varepsilon - u_0 - \theta_\varepsilon\right)\right|_{2,\Omega} = o(1).$$

Proof. See [11, p. 269]. □

We can improve the rate of convergence above if we assume more smoothness of f as in the following theorem.

Theorem 1.24. *Under the assumptions of Theorem 1.22 and if*

$$\partial^2_{X_1} u_0 \in L^2(\Omega) \quad \text{and} \quad \partial_{X_1} u_0 = 0 \text{ on } \{0\} \times \omega, \tag{1.3.29}$$

then we have when $\varepsilon \to 0$

$$|u_\varepsilon - u_0 - \theta_\varepsilon|_{2,\Omega}, \quad \left|\nabla_{X_2}\left(u_\varepsilon - u_0 - \theta_\varepsilon\right)\right|_{2,\Omega} = O(\varepsilon^2), \qquad \left|\partial_{X_1}\left(u_\varepsilon - u_0 - \theta_\varepsilon\right)\right|_{2,\Omega} = O(\varepsilon).$$

Proof. See [11, p. 270]. □

Remark 1.16. *Note that*

1. *The second hypothesis in (1.3.29) means that for a.e. $X_2 \in \omega$ we have $\partial_{X_1} u_0(0, X_2) = 0$.*
2. *For instance, if f is smooth enough, we can show that the hypotheses $\partial^2_{X_1} f \in L^2(\Omega)$ and $\partial_{X_1} f = 0$ on $\{0\} \times \omega$ imply (1.3.29) using the representation formula*

   $$u_0(x) = \int_\omega f(X_1, y)\, G(X_2, y)\, \mathrm{d}y$$

   where G is the Green function, see [25].

Thanks to Theorem 1.22, if we assume that f is independent of X_1 we get an exponential rate of convergence.

Theorem 1.25. *Under the assumptions above and in addition if f is independent of X_1 then we have an exponential convergence of $u_\varepsilon - \theta_\varepsilon$ to u_0 in the whole domain Ω, i.e. there exist positive constants κ and η independent of ε such that*

$$\int_\Omega \left|\nabla\left(u_\varepsilon - u_0 - \theta_\varepsilon\right)\right|^2 \mathrm{d}x \leq \kappa \exp\left(-\frac{\eta}{\varepsilon}\right).$$

Proof. This is an immediate consequence of (1.3.28). □

1.3.2 Parabolic and hyperbolic problems

This subsection is devoted to the study of anisotropic singular perturbation theory for linear parabolic as well as hyperbolic problems. A description of the asymptotic behaviour of the solution, as ε tends to zero, is given and in the case of cylindrical domains, as for the elliptic problems, we can improve the rate of convergence in regions far away from the boundary layers. However, some difficulties arise, especially for the strong convergence.

Parabolic problems

We would like to consider the family of parabolic problems

$$\begin{cases} \partial_t u_\varepsilon - \nabla \cdot A_\varepsilon \nabla u_\varepsilon = f & \text{in } Q, \\ u_\varepsilon = 0 & \text{on } (0,T) \times \partial\Omega, \\ u_\varepsilon(0,.) = u^0 & \text{in } \Omega. \end{cases} \tag{1.3.30}$$

As in the case of elliptic problems we are interesting to analyse the behaviour of u_ε when ε tends to zero. If we let ε go to zero formally - we see that a reasonable candidate for the limit of u_ε is given by $u_0 = u_0(t,x)$ satisfies, for a.e. $X_1 \in \Pi_\Omega$,

$$\begin{cases} \partial_t u_0(X_1,.) - \nabla_{X_2} \cdot (A_{22} \nabla_{X_2} u_0)(X_1,.) = f(X_1,.) & \text{in } Q_{X_1}, \\ u_0(X_1,.) = 0 & \text{on } (0,T) \times \partial\Omega_{X_1}, \\ u_0(0,X_1,.) = u^0(X_1,.) & \text{in } \Omega_{X_1}. \end{cases} \tag{1.3.31}$$

We suppose here that

$$f \in L^2(Q), \quad u^0 \in L^2(\Omega). \tag{1.3.32}$$

The existence and the uniqueness of the solution u_ε is ensured in the weak sense.

Now in our case of the linear parabolic problems (1.3.30) the solution u_ε is unique, then we have the following

Theorem 1.26. *The following convergences hold for the whole sequence*

$$u_\varepsilon \longrightarrow u_0, \quad \nabla_{X_2} u_\varepsilon \longrightarrow \nabla_{X_2} u_0, \quad \varepsilon \nabla_{X_1} u_\varepsilon \longrightarrow 0 \quad \text{in } L^2(Q),$$
$$\partial_t u_\varepsilon \longrightarrow \partial_t u_0 \quad \text{in } L^2\left(0,T; H^{-1}(\Omega)\right).$$

Proof. This proof run on the same way as the proves in [10, Theorem 3.1 and 4.2, p. 1778-1788] by taking into account that $A(z,s) = A(z)$ for all $z \in Q$. □

Remark 1.17. *We can also have, for all $t \in (0,T)$*

$$u_\varepsilon(t,.) \longrightarrow u_0(t,.) \quad \text{in } L^2(\Omega), \quad \text{as } \varepsilon \to 0. \tag{1.3.33}$$

Hyperbolic problems

We would like now to consider the following hyperbolic problem

$$\begin{cases} u_\varepsilon'' - \nabla \cdot (A_\varepsilon \nabla u_\varepsilon) = f & \text{in } Q, \\ u_\varepsilon = 0 & \text{on } (0,T) \times \partial\Omega, \\ u_\varepsilon(0) = u_\varepsilon^0, \quad u_\varepsilon'(0) = u_\varepsilon^1 & \text{in } \Omega \end{cases} \tag{1.3.34}$$

and make here the following assumptions that insure the existence and the uniqueness of the solution to the above problem

$$u^0, u^0_\varepsilon \in H^1_0(\Omega), \quad u^1, u^1_\varepsilon \in L^2(\Omega), \quad f \in L^2(Q), \tag{1.3.35}$$

$$a_{ij} \in \mathcal{C}^1(\bar{Q}), \quad a_{ij} = a_{ji}, \qquad \forall i,j = 1,\dots,\mathrm{n}. \tag{1.3.36}$$

Of course as it is shown in [27], the convergence of u_ε requires a limit of the initial conditions as

$$u^0_\varepsilon \to u^0 \quad \text{in } H^1_0(\Omega), \quad u^1_\varepsilon \to u^1 \quad \text{in } L^2(\Omega). \tag{1.3.37}$$

(The subscript should not be confused with the power notation.) As in the precedent problems the natural candidate limit of u_ε is u_0 defined for a.e. $X_1 \in \Pi_1(\Omega)$ as a solution to

$$\begin{cases} u_0''(X_1,.) - \nabla_{X_2} \cdot (A_{22} \nabla_{X_2} u_0(X_1,.)) = f(X_1,.) & \text{in } Q_{X_1}, \\ u_0(X_1,.) = 0 & \text{on } (0,T) \times \partial\Omega_{X_1}, \\ u_0\left(0, X_1, \cdot\right) = u^0\left(X_1,.\right) \quad \text{and} \quad u_0'\left(0, X_1,.\right) = u^1\left(X_1,.\right) & \text{in } \Omega_{X_1}. \end{cases} \tag{1.3.38}$$

Note that X_1 plays a parameter role in this problem. Taking into account (1.3.35), it follows that -see [13]-

$$u^0\left(X_1,.\right) \in H^1_0(\Omega_{X_1}), \quad u^1\left(X_1,.\right) \in L^2(\Omega_{X_1}) \quad \text{and} \quad f\left(X_1,.\right) \in L^2\left(Q_{X_1}\right)$$

for a.e. $X_1 \in \Pi_1(\Omega)$. Then (1.3.38) has a unique solution in the weak sense.

The main result here, that deals with the limit behaviour of u_ε is given in the following

Theorem 1.27. *Under the assumptions above, we have as $\varepsilon \to 0$, for every $t \in [0,T]$*

$$\begin{aligned} &u_\varepsilon\left(t\right) \to u_0\left(t\right), \quad \nabla_{X_2} u_\varepsilon\left(t\right) \to \nabla_{X_2} u_0\left(t\right), \\ &\varepsilon \nabla_{X_1} u_\varepsilon\left(t\right) \to 0, \quad u_\varepsilon'\left(t\right) \to u_0'\left(t\right), \end{aligned} \tag{1.3.39}$$

in $L^2\left(\Omega\right)$ where u_ε and u_0 are the weak solutions of (1.3.34) and (1.3.38) respectively.

Proof. See [31, p. 8986-8988]. □

Note that it is shown in [31] that $u_0 \in \mathcal{C}([0,T]; L^2\left(\Omega\right))$, $\nabla_{X_2} u_0 \in \mathcal{C}([0,T]; \left[L^2\left(\Omega\right)\right]^{\mathrm{n}-p})$ and $u' \in \mathcal{C}([0,T]; L^2\left(\Omega\right))$ which justifies the point-wise limits with respect to t in (1.3.39). As a consequence of Theorem 1.27 and the following estimates

$$\left(\nabla_{X_2} u_\varepsilon\right)_\varepsilon, \quad \left(u_\varepsilon\right)_\varepsilon, \quad \left(u_\varepsilon'\right)_\varepsilon \text{ and } \left(\varepsilon \nabla_{X_1} u_\varepsilon\right)_\varepsilon \text{ are bounded in } L^\infty(0,T; L^2\left(\Omega\right)). \tag{1.3.40}$$

Then, we have

Corollary 1.2. *For every $1 \le q < \infty$ we have*

$$\begin{aligned} \nabla_{X_2} u_\varepsilon \to \nabla_{X_2} u_0,\ \varepsilon u_\varepsilon \to u_0,\ \nabla_{X_1} u_\varepsilon \to 0,\ u_\varepsilon' \to u_0' \quad &\text{in } L^q\left(0,T; L^2\left(\Omega\right)\right), \\ u_\varepsilon'' - u_0'' \to 0 \quad &\text{in } L^q(0,T; H^{-1}(\Omega)). \end{aligned} \tag{1.3.41}$$

Proof. See [31, p. 8989]. □

Remark 1.18. *Note that*

1. *We can replace the condition (1.3.37) by the following weaker ones*

$$\nabla_{X_2} u^0_\varepsilon \to \nabla_{X_2} u^0, \quad u^0_\varepsilon \to u^0, \quad \varepsilon \nabla_{X_1} u^0_\varepsilon \to 0, \quad u^1_\varepsilon \to u^1 \ \text{in } L^2(\Omega).$$

In this case we only have to suppose that $u^0, \nabla_{X_2} u^0 \in L^2\left(\Omega\right)$. This is the reason why $u^0_\varepsilon, u^1_\varepsilon$ are chosen depending on ε in this sub-subsection.

2. *The point-wise convergences shown in Theorem 1.27 are indispensable to study strong limits as in (1.3.41).*

Next in order to deal with the rate of convergence we assume that the domain Ω is cylindrical, i.e. $\Omega = \omega_1 \times \omega_2$ and that the matrix A is independent of time, i.e.

$$A\left(t,x\right) = A\left(x\right), \quad \text{a.e. } x \in \Omega. \tag{1.3.42}$$

For a.e. $X_1 \in \omega_1$, in the problem (1.3.38) we replace Ω_{X_1} by ω_2. Our aim is now to improve the convergence u_ε towards u_0, in particular the convergence of $\nabla_{X_1} u_\varepsilon$. For this reason, as a necessary condition we need to assume more regularity hypothesis, i.e. $u_0\left(t\right) \in H^1\left(\Omega\right)$ for a.e. $t \in (0, \text{Ŧ})$ and of course the regularity of u_0 in the X_1–directions depends on the regularity of the data in the same direction. We can show that if (see [31, p. 8989-8992])

$$\partial_{x_i} u^1 \in L^2\left(\omega_1; H^{-1}\left(\omega_2\right)\right), \quad \partial_{x_i} f \in L^2\left(\left(0, \text{Ŧ}\right) \times \omega_1; H^{-1}\left(\omega_2\right)\right), \quad i = 1, \ldots, p, \tag{1.3.43}$$

then

$$u_0 \in L^\infty\left(0, \text{Ŧ}; H^1\left(\Omega\right)\right), \quad \partial_{x_i} u_0' \in L^\infty\left(0, \text{Ŧ}; L^2\left(\omega_1; H^{-1}\left(\omega_2\right)\right)\right), \quad i = 1, \ldots, p. \tag{1.3.44}$$

Note that although we ignore if $u_0 \in \mathcal{C}\left(0, \text{Ŧ}; H^1\left(\Omega\right)\right)$ or not we have

$$u_0\left(t\right) \in H_0^1\left(\Omega, \omega_1 \times \partial\omega_2\right) := \left\{v \in H^1\left(\Omega\right) \mid v|_{\omega_1 \times \partial\omega_2} = 0\right\}, \quad \forall t \in [0, \text{Ŧ}]. \tag{1.3.45}$$

Let ω_1, ω_2 be the two open subsets defined in Subsection 1.3.1 ($\omega_1' \subset\subset \omega_1'' \subset\subset \omega_1$). With this notation we set $\Omega' = \omega_1' \times \omega_2, \Omega'' = \omega_1'' \times \omega_2$ and $Q' = (0, \text{Ŧ}) \times \Omega', Q'' = (0, \text{Ŧ}) \times \Omega''$. For every $t \in [0, \text{Ŧ}]$, we define the functions

$$U_0\left(t\right) = \int_0^t u_0\left(\sigma\right) \mathrm{d}\sigma \quad \text{and} \quad U_\varepsilon\left(t\right) = \int_0^t u_\varepsilon\left(\sigma\right) \mathrm{d}\sigma. \tag{1.3.46}$$

Then we have

Theorem 1.28. *Under the above hypotheses, we assume in addition that when $\varepsilon \to 0$,*

$$\left|u_\varepsilon^0 - u^0\right|_{2,\Omega} = O\left(\varepsilon\right) \quad \text{and} \quad \left|u_\varepsilon^1 - u^1\right|_{L^2\left(\omega_1; H^{-1}\left(\omega_2\right)\right)} = O\left(\varepsilon\right), \tag{1.3.47}$$

then we have

$$\sup_{t \in [0, \text{Ŧ}]} \left|\left(u_\varepsilon' - u_0'\right)\left(t\right)\right|_{H^{-1}\left(\Omega'\right)}, \quad \sup_{t \in [0, \text{Ŧ}]} \left|\left(u_\varepsilon - u_0\right)\left(t\right)\right|_{2,\Omega'} = O\left(\varepsilon\right),$$

$$\sup_{t \in [0, \text{Ŧ}]} \left|\nabla_{X_2}\left(U_\varepsilon - U_0\right)\left(t\right)\right|_{2,\Omega'} = O\left(\varepsilon\right), \quad \nabla_{X_1} U_\varepsilon\left(t\right) \rightharpoonup \nabla_{X_1} U_0\left(t\right) \ \text{in } L^2\left(\Omega'\right), \ \forall t \in [0, \text{Ŧ}].$$

Proof. See [31, p. 8993-8995]. □

When the matrix A has a diagonal structure we can improve this result as in the elliptic case.

Corollary 1.3 (Diagonal matrix)**.** *Under the assumptions of Theorem 1.28, in addition we suppose that $A_{12} = A_{21}^T = 0$, $\left|u_\varepsilon^0 - u^0\right|_{2,\Omega} = o\left(\varepsilon\right)$ and $\left|u_\varepsilon^1 - u^1\right|_{L^2\left(\omega_1; H^{-1}\left(\omega_2\right)\right)} = o\left(\varepsilon\right)$, then when $\varepsilon \to 0$*

$$\sup_{t \in [0, \text{Ŧ}]} \left|\left(u_\varepsilon' - u_0'\right)\left(t\right)\right|_{H^{-1}\left(\Omega'\right)}, \quad \sup_{t \in [0, \text{Ŧ}]} \left|\left(u_\varepsilon - u_0\right)\left(t\right)\right|_{2,\Omega'} = o\left(\varepsilon\right),$$

$$\sup_{t \in [0, \text{Ŧ}]} \left|\nabla_{X_2}\left(U_\varepsilon - U_0\right)\left(t\right)\right|_{2,\Omega'} = o\left(\varepsilon\right), \ \sup_{t \in [0, \text{Ŧ}]} \left|\nabla_{X_1}\left(U_\varepsilon - U_0\right)\left(t\right)\right|_{2,\Omega'} = o\left(1\right).$$

Proof. See [31, p. 8995]. □

Remark 1.19. *As a trivial consequence of the corollary above we have*

$$\sup_{t\in[0,T]} |U_\varepsilon(t) - U_0(t)|_{H^1(\Omega')} \to 0 \quad \text{as } \varepsilon \to 0.$$

We turn now to present some results on the non-linear anisotropic singular perturbation problems.

1.3.3 Abstract singular perturbation problems

In this subsection we deal with singular perturbations of variational inequalities involving some non-linear operators and depending on a small positive parameter ε. Here the results are very general but we have more particularly in mind anisotropic cases where ε only acts on some variables of a domain $\Omega \subset \mathbb{R}^n$. First we consider the abstract theory of singular perturbation problems, defined in Banach spaces, then we describe the asymptotic behaviour of the solutions when ε tends to zero. These abstract results are applied to some boundary value problems. This approach has the advantage to include in a short theory a wide class of problems spread in the literature.

Singular perturbations of variational inequalities

Let V and W be two reflexive separable Banach spaces equipped with the norms $|.|_V$ and $|.|_W$ respectively. We suppose that the space $V\cap W$ is dense in V and W, and is equipped with the norm $|.|_{V\cap W} = |.|_V + |.|_W$. Of course the $V\cap W$ is a Banach space equipped with the previous norm. It is clear that $V\cap W \subset V$, W and $V', W' \subset (V\cap W)'$. Moreover one can check that $(V\cap W)' = V' + W'$. We consider two non-linear operators $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} : V \to V'$ and $\mathcal{B} : W \to W'$. We suppose that $\mathcal{A}, \mathcal{B}$ are monotone, that is to say that

$$\langle \mathcal{A}u - \mathcal{A}v, u - v\rangle_V \geq 0, \quad \forall u, v \in V, \quad \langle \mathcal{B}u - \mathcal{B}v, u - v\rangle_W \geq 0, \quad \forall u, v \in W. \tag{1.3.48}$$

We denote by $K \neq \varnothing$ a closed convex set of $V\cap W$ and for $\mathcal{A}$ and $\mathcal{B}$ we make the following coerciveness assumption. We suppose that for some $v_0 \in K$ one has

$$\frac{\langle \mathcal{A}u - \mathcal{A}v_0, u - v_0\rangle_V}{|u - v_0|_V} \to +\infty \quad \text{when } |u - v_0|_V \to +\infty, \quad u \in K, \tag{1.3.49}$$

$$\frac{\langle \mathcal{B}u - \mathcal{B}v_0, u - v_0\rangle_W}{|u - v_0|_W} \to +\infty \quad \text{when } |u - v_0|_W \to +\infty, \quad u \in K. \tag{1.3.50}$$

Of course if K is bounded in V (respectively in W) we will not need the assumption (1.3.49) (respectively (1.3.50)). In addition we assume that $\mathcal{A}$, $\mathcal{B}$ are bounded and hemi-continuous. Under the assumptions above, for $f \in (V\cap W)'$ and $\varepsilon > 0$ there exists u_ε solution to the problem

$$\begin{cases} \varepsilon\langle \mathcal{A}u_\varepsilon, v - u_\varepsilon\rangle_V + \langle \mathcal{B}u_\varepsilon, v - u_\varepsilon\rangle_W \geq \langle f, v - u_\varepsilon\rangle_{V\cap W}, \quad \forall v \in K, \\ u_\varepsilon \in K, \end{cases} \tag{1.3.51}$$

(see [14, p. 3-4]). Moreover if $\mathcal{A}$ or $\mathcal{B}$ is strictly monotone, i.e. if one of the inequalities (1.3.48) is strict for $u \neq v$, the solution is unique.

Remark 1.20. *When $K = V\cap W$ one sees by taking $v = u_\varepsilon \pm v, v \in K$ that u_ε is solution to*

$$\begin{cases} \varepsilon\mathcal{A}u_\varepsilon + \mathcal{B}u_\varepsilon = f, \\ u_\varepsilon \in V\cap W. \end{cases} \tag{1.3.52}$$

We are now interested in studying the behaviour of u_ε when ε tends to zero. Note that it is not possible here to get an explicit estimates as we got in the previous subsection because to the weak coerciveness assumptions (1.3.49), (1.3.50) and the essential convergences are given as follows

Theorem 1.29. *Suppose that $f \in W'$ and let u_ε be solution to (1.3.51). Then we have as $\varepsilon \to 0$*

$$u_\varepsilon \text{ is bounded in } W \text{ independently of } \varepsilon, \tag{1.3.53}$$

$$\varepsilon u_\varepsilon \to 0 \quad \text{in } V, \tag{1.3.54}$$

$$\varepsilon \mathcal{A} u_\varepsilon \to 0 \quad \text{in } V', \tag{1.3.55}$$

$$\langle \varepsilon \mathcal{A} u_\varepsilon, u_\varepsilon \rangle_V \to 0. \tag{1.3.56}$$

Proof. See [14, p. 5-6]. □

Remark 1.21. *In the case where $K = V \cap W$, from the equation (1.3.52) one derives that*

$$\mathcal{B} u_\varepsilon - f \to 0 \quad \text{in } V', \quad \text{as } \varepsilon \to 0. \tag{1.3.57}$$

In addition we have

Theorem 1.30. *Suppose that for some sequence $\varepsilon_k \to 0$ one has $u_{\varepsilon_k} \rightharpoonup u_0$ in W. Then u_0 is a solution to the variational inequality*

$$\begin{cases} \langle \mathcal{B} u_0, v - u_0 \rangle_W \geq \langle f, v - u_0 \rangle_W, & \forall v \in \bar{K}^W \\ u_0 \in \bar{K}^W. \end{cases} \tag{1.3.58}$$

Moreover one has

$$\mathcal{B} u_{\varepsilon_k} \rightharpoonup \mathcal{B} u_0 \text{ in } W', \qquad \langle \mathcal{B} u_{\varepsilon_k}, u_{\varepsilon_k} \rangle_W \to \langle \mathcal{B} u_0, u_0 \rangle_W. \tag{1.3.59}$$

Proof. See [14, p. 7]. □

Remark 1.22. *Note that*

1. *It is shown that the only possible limits for the sub-sequences of $(u_\varepsilon)_\varepsilon$ are solutions of the variational inequality (1.3.58). In particular if the solution is unique one has $u_\varepsilon \rightharpoonup u_0$ in W, and $\mathcal{B} u_\varepsilon \rightharpoonup \mathcal{B} u_0$ in W'. This is the case when $\mathcal{B}$ is strictly monotone for instance.*

2. *In the case where $K = V \cap W$, we have $\bar{K}^W = W$ and u_0 is a solution to the equation $\mathcal{B} u_0 = f$.*

As a corollary we have

Corollary 1.4. *It is interesting to note that*

1. *If we suppose that $\mathcal{A}$ is strongly coercive in the sense that*

$$\langle \mathcal{A} v, v \rangle_V \geq \lambda |v|_V^\gamma, \qquad \forall v \in V, \tag{1.3.60}$$

for some positive constants λ and γ, then one has

$$\varepsilon^{1/\gamma} u_\varepsilon \to 0 \quad \text{in } V, \quad \text{as } \varepsilon \to 0. \tag{1.3.61}$$

2. *If $\mathcal{B}$ is strongly monotone in the sense that for some $\mu > 0$ and $\eta > 1$*

$$\langle \mathcal{B} u - \mathcal{B} v, u - v \rangle_W \geq \mu |u - v|_W^\eta, \qquad \forall v, u \in W \tag{1.3.62}$$

then the solution u_0 of (1.3.58) is unique and one has the convergence of u_ε towards u_0 in W (see [14, p. 8]).

Remark 1.23. *Assuming only the basic coerciveness (1.3.49) of $\mathcal{A}$, the convergence result (1.3.54) is sharp since if γ approaches one in (1.3.61) the exponent of ε tends to one.*

Next we pay attention to more regular problems, i.e. when some solutions of (1.3.58) are in V.

Corollary 1.5. *If the variational inequality (1.3.58) has a solution $\hat{u} \in K$ satisfying*

$$\liminf \langle \mathcal{A}u, u - \hat{u} \rangle_V > 0 \quad \text{when } |u|_V \to \infty, \quad u \in K, \tag{1.3.63}$$

then u_ε is bounded in V and there exists always a sequence u_{ε_k} such that

$$u_{\varepsilon_k} \rightharpoonup u_0 \quad \text{in } V \text{ and } W \tag{1.3.64}$$

where $u_0 \in K$ is solution to (1.3.58), i.e. the accumulation points of $(u_\varepsilon)_\varepsilon$ are all in K and solutions to (1.3.58) (see [14, p. 9]). In addition if $\mathcal{B}$ satisfies (1.3.62), one has

$$|u_\varepsilon - u_0|_W = o\left(\varepsilon^{1/\eta}\right). \tag{1.3.65}$$

Some applications

It is interesting to note that, using a priory estimates in the above sub-subsection, there is no need to have some compactness assumptions to pass to the limit in non-linear terms, which effectively do not exist. In order to illustrate this we will consider here three non-linear elliptic boundary value problems as examples of the abstract theory above. We will apply the theory to some anisotropic singular perturbation problems in the last two examples. To also see the power of this abstract analysis in general, we consider a very classical case of non-linear obstacle problems.

Non-linear obstacle problems We denote by $a(\xi) = (a_i(\xi))$ a continuous vector field in $\mathbb{R}^n$. We suppose that a is such that for some $\lambda, \Lambda > 0$ and $\kappa \in \mathbb{R}$

$$a(\xi) \cdot \xi \geq \lambda |\xi|^2 + \kappa, \quad |a(\xi)| \leq \Lambda |\xi|, \quad \forall \xi \in \mathbb{R}^n \tag{1.3.66}$$

$$(a(\xi) - a(\zeta)) . (\xi - \zeta) \geq 0, \quad \forall \xi, \zeta \in \mathbb{R}^n. \tag{1.3.67}$$

Then, for $f \in L^2(\Omega)$ there exists a unique u_ε solution to

$$\begin{cases} u_\varepsilon \in K_0 = \{v \in H_0^1(\Omega) \,|\, v(x) \geq 0, \quad \text{a.e. } x \in \Omega\}, \\ \varepsilon \int_\Omega a(\nabla u_\varepsilon) \cdot \nabla (v - u_\varepsilon)\, dx + \int_\Omega u_\varepsilon \quad (v - u_\varepsilon)\, dx \\ \qquad\qquad \geq \int_\Omega f \quad (v - u_\varepsilon)\, dx, \ \forall v \in K_0, \end{cases} \tag{1.3.68}$$

where Ω is a bounded open subset in $\mathbb{R}^n$. Then setting $V = H_0^1(\Omega)$, $W = L^2(\Omega)$, $\mathcal{A}u = -\operatorname{div}(a(\nabla u))$, and $\mathcal{B} = \mathrm{Id}$, the above results apply and we get that u_ε towards f^+ in $L^2(\Omega)$ where f^+ (f^-) denotes the positive (negative) part of f respectively. Indeed, thanks to Theorems 1.29, 1.30, and Corollary 1.4 we see that u_ε towards u_0 in $L^2(\Omega)$ where u_0 is the unique solution to

$$\begin{cases} u_0 \in \bar{K}_0 = \left\{v \in L^2(\Omega) \,|\, v(x) \geq 0, \ \text{a.e. } x \in \Omega\right\}, \\ \int_\Omega u_0 (v - u_0)\, dx \geq \int_\Omega f(v - u_0)\, dx, \quad \forall v \in \bar{K}_0. \end{cases} \tag{1.3.69}$$

But clearly

$$\int_\Omega f^+ \left(v - f^+\right) \mathrm{d}x = \int_\Omega \left(f + f^-\right)\left(v - f^+\right) \mathrm{d}x$$
$$= \int_\Omega f\left(v - f^+\right) \mathrm{d}x + \int_\Omega f^- v \,\mathrm{d}x \geq \int_\Omega f\left(v - f^+\right) \mathrm{d}x, \quad \forall v \in \bar{K}_0$$

and $u_0 = f^+$. As a corollary of Theorems 1.29, 1.30 and Corollary 1.4, we can state the following.

Corollary 1.6. *When ε tends to zero we have*

$$u_\varepsilon \to f^+ \quad \textit{in } L^2(\Omega), \quad \varepsilon u_\varepsilon \to 0 \quad \textit{in } H_0^1(\Omega),$$
$$\varepsilon \int_\Omega a(\nabla u_\varepsilon) \cdot \nabla u_\varepsilon \,\mathrm{d}x \to 0, \quad \varepsilon \partial_{x_i}(a(\nabla u_\varepsilon)) \to 0 \quad \textit{in } H^{-1}(\Omega), \quad i = 1, \ldots, \mathrm{n}.$$

Remark 1.24. *Note that, as in (1.3.66), we may add a constant $\kappa \in \mathbb{R}$ in (1.3.60) since it will be neglected once it is multiplied by ε, i.e. $\langle \mathcal{A}v, v\rangle_V \geq \lambda |v|_V^\gamma + \kappa$ for all v in V. Of course, here the strong convergence of $\sqrt{\varepsilon}\nabla u_\varepsilon$ comes from the last convergence in the above corollary, i.e. $\sqrt{\varepsilon}\nabla u_\varepsilon$ towards zero in $L^2(\Omega)$.*

Semi-linear elliptic problems We consider the following semi-linear elliptic problem

$$\begin{cases} -\varepsilon \Delta_{X_1} u_\varepsilon - \Delta_{X_2} u_\varepsilon + g(x, u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon \in H_0^1(\Omega) \cap L^q(\Omega), \end{cases} \tag{1.3.70}$$

where $f \in L^2(\Omega) + L^{q'}(\Omega)$, for $q > 1$ (q' is the conjugate of q). In order to apply the abstract approach we assume that $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathédeory function and non-decreasing in the second variable and there exist $\kappa, \kappa' \geq 0$, such that

$$|g(x,t)| \leq \kappa |t|^{q-1} + \kappa', \quad g(x,t)\,t \geq |t|^q, \ \forall t \in \mathbb{R}, \quad \text{a.e. } x \in \Omega.$$

It is clear that if $u \in L^q(\Omega)$ then $g(., u(.)) \in L^{q'}(\Omega)$. So g defines an operator (still labelled by g) from $L^q(\Omega)$ into $L^{q'}(\Omega)$ by

$$u \mapsto g(., u(.)), \tag{1.3.71}$$

which is bounded, monotone and hemi-continuous. Then we choose the suitable Banach spaces

$$V = \left\{u \in L^2(\Omega) \mid \nabla_{X_1} u \in \left[L^2(\Omega)\right]^p, \ u(., X_2) \in H_0^1(\Omega_{X_2}), \text{ a.e. } X_2 \in \Pi_2(\Omega)\right\},$$
$$W = \left\{u \in L^2(\Omega) \cap L^q(\Omega) \mid \nabla_{X_2} u \in \left[L^2(\Omega)\right]^{\mathrm{n}-p}, \ u(X_1, .) \in H_0^1(\Omega_{X_1}), \text{ a.e. } X_1 \in \Pi_1(\Omega)\right\}$$

equipped with the norms $|v|_V := |\nabla_{X_1} v|_{2,\Omega}$ and $|v|_W := |\nabla_{X_2} v|_{2,\Omega} + |v|_{q,\Omega}$ respectively. We can easily check that V and W are separable reflexive Banach spaces. Next we set $\mathcal{A} = -\Delta_{X_1}$ and $\mathcal{B} = -\Delta_{X_2} + g(x,.)$. Then the operator $\mathcal{A} : V \to V'$ is linear, bounded and coercive. Since the operator $\mathcal{B} : W \to W'$ is a sum of a linear operator, satisfying the same properties as $\mathcal{A}$, and the operator defined in (1.3.71), it is bounded, monotone and coercive. In this example the limit problem is defined for a.e. $X_1 \in \Pi_1(\Omega)$ as

$$\begin{cases} -\Delta_{X_2} u_0(X_1, .) + g((X_1, .), u_0(X_1, .)) = f(X_1, .) & \text{in } \Omega_{X_1}, \\ u_0(X_1, .) = 0 & \text{on } \partial\Omega_{X_1}. \end{cases} \tag{1.3.72}$$

Moreover we can show that if the boundary of Ω is smooth then $V \cap W = H_0^1(\Omega) \cap L^{\bar{p}}(\Omega)$, see[14, p. 12,13]. As it is known, we need a point-wise convergence to pass to the limit in the non-linear term $g(.,u_\varepsilon)$. But the estimates that one has, i.e. $|\nabla_{X_2}u_\varepsilon|_{2,\Omega}$, $|u_\varepsilon|_{\bar{p},\Omega}$ are bounded, are not sufficient to get the point-wise limit of $(u_\varepsilon)_\varepsilon$ since the embedding $W \subset L^2(\Omega)$ is not compact. So in this case the monotonicity hypothesis is necessary and as an obvious consequence of Theorems 1.29, 1.30 and Corollary 1.4 we have

Corollary 1.7. *When ε tends to zero, we have*

$$u_\varepsilon \to u_0,\ \nabla_{X_2}u_\varepsilon \to \nabla_{X_2}u_0 \quad \textit{and} \quad \sqrt{\varepsilon}\nabla_{X_1}u_\varepsilon \to 0 \quad \textit{in } L^2(\Omega).$$

Moreover if g is strongly monotone then we obtain.

$$u_\varepsilon \to u_0 \quad \textit{in } L^q(\Omega).$$

Remark 1.25. *Note that, even if $\mathcal{B}$ is not strongly monotone, the first two convergences hold strongly. This is due to the uniform monotonicity of $-\Delta_{X_2}$, i.e.*

$$\langle \Delta_{X_2}v - \Delta_{X_2}u, v-u\rangle_W + \int_\Omega (g(x,v) - g(x,u))(v-u)\,\mathrm{d}x \geq |\nabla_{X_2}(v-u)|^2_{2,\Omega}, \quad \forall v,u \in W.$$

q–Laplacian type problem The second application of the abstract theory, in the anisotropic case, is the following quasilinear elliptic problem

$$\begin{cases} -\varepsilon\Delta_{q_1,X_1}u_\varepsilon - \Delta_{q_2,X_2}u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3.73}$$

where $q_1, q_2 > 1$ are real constants. We assume that $f \in L^{q_2'}(\Omega)$ - q_2' is the conjugate of q_2. In this case we set

$$V = \left\{u \in L^{q_1}(\Omega) \mid \nabla_{X_1}u \in [L^{q_1}(\Omega)]^p, u(\cdot,X_2) \in W_0^{1,q_1}(\Omega_{X_2}),\ \text{a.e. } X_2 \in \Pi_2(\Omega)\right\},$$
$$W = \left\{u \in L^{q_2}(\Omega) \mid \nabla_{X_2}u \in [L^{q_2}(\Omega)]^{\mathrm{n}-p}, u(X_1,\cdot) \in W_0^{1,q_2}(\Omega_{X_1}),\ \text{a.e. } X_1 \in \Pi_1(\Omega)\right\}$$

equipped with the norms $|v|_V = |\nabla_{X_1}v|_{q_1,\Omega}$ and $|v|_W = |\nabla_{X_2}v|_{q_2,\Omega}$ respectively. We can easily show that V and W are separable reflexive Banach spaces. Then we define the operators $\mathcal{A}: V \to V'$ and $\mathcal{B}: W \to W'$ as $\mathcal{A} = -\Delta_{q_1,X_1}$ and $\mathcal{B} = -\Delta_{q_2,X_2}$ respectively. It is easy to see that $\mathcal{A}$ and $\mathcal{B}$ are coercive, bounded and hemi-continuous. The monotonicity of $\mathcal{A}$ and $\mathcal{B}$ is ensured by the following monotonicity inequalities

$$\left(|\xi|^{q-2}\xi - |\eta|^{q-2}\eta\right)\cdot(\xi-\eta) \geq \kappa_q\{|\xi|+|\eta|\}^{q-2}|\xi-\eta|^2, \quad \forall q>1, \quad \forall \xi,\eta \in \mathbb{R}^{\mathrm{n}},$$
$$\left(|\xi|^{q-2}\xi - |\eta|^{q-2}\eta\right)\cdot(\xi-\eta) \geq \kappa_q|\xi-\eta|^q, \quad \forall q \geq 2, \quad \forall \xi,\eta \in \mathbb{R}^{\mathrm{n}},$$

where κ_q is a positive constant (see [7, 36]).

Thus the operator $\mathcal{A}$ ($\mathcal{B}$) is strictly monotone for all $q_1 > 1$ ($q_2 > 1$) and strongly monotone if $q_1 \geq 2$ ($q_2 \geq 2$) respectively. The limit problem is defined, for a.e. $X_1 \in \Pi_1(\Omega)$, as

$$\begin{cases} -\Delta_{q_2,X_2}u_0(X_1,\cdot) = f(X_1,\cdot) & \text{in } \Omega_{X_1}, \\ u_0(X_1,\cdot) = 0 & \text{on } \partial\Omega_{X_1}. \end{cases} \tag{1.3.74}$$

Finally and more precisely we have

$$V \cap W = \left\{u \in L^{\max(q_1,q_2)}(\Omega),\ \nabla_{X_1}u \in [L^{q_1}(\Omega)]^p,\ \nabla_{X_2}u \in [L^{q_2}(\Omega)]^{\mathrm{n}-p}, u|_{\partial\Omega} = 0\right\},$$

which gives a sense to the boundary conditions. Then by Theorems 1.29, 1.30 and Corollary 1.4, we have

Corollary 1.8. *For all* $q_1, q_2 > 1$,

$$u_\varepsilon \rightharpoonup u_0 \quad in\ W, \quad \varepsilon \nabla_{X_1} u_\varepsilon \to 0 \quad in\ L^{q_1}(\Omega), \tag{1.3.75}$$

$$\varepsilon \Delta_{q_1, X_1} u_\varepsilon \to 0 \quad in\ V', \quad \Delta_{q_2, X_2} u_\varepsilon \rightharpoonup f \quad in\ W' \tag{1.3.76}$$

where u_ε *and* u_0 *are the solutions of (1.3.73) and (1.3.74) respectively. Moreover if* $q_1 \geq 2$ *then*

$$\varepsilon^{1/q_1} \nabla_{X_1} u_\varepsilon \to 0 \quad in\ L^{q_1}(\Omega), \tag{1.3.77}$$

and if $q_2 \geq 2$ *then*

$$u_\varepsilon \to u_0, \ \nabla_{X_2} u_\varepsilon \to \nabla_{X_2} u_0 \quad in\ L^{q_2}(\Omega). \tag{1.3.78}$$

Chapter 2

Regular Asymptotic Expansion

In this chapter we construct a regular asymptotic expansion to the weak solution of anisotropic singular perturbation problems of elliptic type when the main idea aims to go deep in the study of the rate of asymptotic convergence. In fact the analysis of the asymptotic expansion strongly helps in understanding what is, and what is not, needed to improve the rate and the type of asymptotic convergence. In order to define the coefficients of the development some smoothness basic properties (with respect to the parameters) of elliptic problems are established.

2.1 Position of the problem

In order to describe the class of problems that we would like to address, we first recall some basic notation and hypotheses. Let Ω be a bounded open cylinder in $\mathbb{R}^{\mathrm{n}}$, i.e.

$$\Omega = \omega_1 \times \omega_2,$$

where ω_1, ω_2 are bounded Lipschitz domains of $\mathbb{R}^p$ and $\mathbb{R}^{\mathrm{n}-p}$ respectively $(\mathrm{n} > p \geq 1)$.

Let $A = \big(a_{ij}(x)\big)$ be a $\mathrm{n} \times \mathrm{n}$ matrix such that

$$a_{ij} \in L^\infty(\Omega), \qquad \forall i, j = 1, \dots, \mathrm{n} \tag{2.1.1}$$

and for some constant $\lambda > 0$, it satisfies the ellipticity condition

$$A\zeta \cdot \zeta \geq \lambda \left|\zeta\right|^2 \qquad \forall \zeta \in \mathbb{R}^{\mathrm{n}}, \qquad \text{a.e. on } \Omega. \tag{2.1.2}$$

We decompose A into four blocks by writing

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11}, A_{22} are respectively $p \times p$ and $(\mathrm{n}-p) \times (\mathrm{n}-p)$ matrices. We then define for $\varepsilon > 0$, the perturbed matrix,

$$A_\varepsilon = \begin{pmatrix} \varepsilon^2 A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{pmatrix}.$$

Therefore we have, for a.e. $x \in \Omega$ and every $\zeta \in \mathbb{R}^{\mathrm{n}}$

$$A_\varepsilon \zeta \cdot \zeta = A\zeta_\varepsilon \cdot \zeta_\varepsilon \geq \lambda \left|\zeta_\varepsilon\right|^2 = \lambda \left\{ \varepsilon^2 \left|\bar{\zeta}_1\right|^2 + \left|\bar{\zeta}_2\right|^2 \right\}$$

where $\zeta = \begin{pmatrix} \bar{\zeta}_1 \\ \bar{\zeta}_2 \end{pmatrix}$, $\bar{\zeta}_1 = (\zeta_1, \ldots, \zeta_p)^T$, $\bar{\zeta}_2 = (\zeta_{p+1}, \ldots, \zeta_{\mathrm{n}})^T$ and $\zeta_\varepsilon = \begin{pmatrix} \varepsilon\bar{\zeta}_1 \\ \bar{\zeta}_2 \end{pmatrix}$. Thus we have

$$\begin{aligned} A_\varepsilon\zeta \cdot \zeta &\geq \lambda \left(\varepsilon^2 \wedge 1\right) |\zeta|^2 \qquad \forall \zeta \in \mathbb{R}^{\mathrm{n}}, \qquad \text{a.e. on } \Omega, \\ A_{22}\bar{\zeta}_2 \cdot \bar{\zeta}_2 &\geq \lambda \left|\bar{\zeta}_2\right|^2 \qquad \forall \bar{\zeta}_2 \in \mathbb{R}^{\mathrm{n}-p}, \qquad \text{a.e. on } \Omega. \end{aligned} \tag{2.1.3}$$

It follows that A_ε and A_{22} are positive definite matrices. For a function

$$f \in L^2(\Omega),$$

we ensure the existence and the uniqueness of a weak solution u_ε to

$$\begin{cases} -\nabla \cdot A_\varepsilon \nabla u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1.4}$$

in the following sense

$$\begin{cases} u_\varepsilon \in H_0^1(\Omega), \\ \varepsilon^2 \displaystyle\int_\Omega A_{11}\nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v \,\mathrm{d}x \quad +\varepsilon \displaystyle\int_\Omega A_{12}\nabla_{X_2} u_\varepsilon \cdot \nabla_{X_1} v \,\mathrm{d}x \\ \qquad\qquad +\varepsilon \displaystyle\int_\Omega A_{21}\nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} v \,\mathrm{d}x \\ \qquad\qquad + \displaystyle\int_\Omega A_{22}\nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v \,\mathrm{d}x \quad = \displaystyle\int_\Omega f v \,\mathrm{d}x, \qquad \forall v \in H_0^1(\Omega). \end{cases} \tag{2.1.5}$$

In the present chapter, we aim to go beyond the limit in describing the asymptotic behaviour of u_ε where we focus on finding an upgraded asymptotic approximation simply depending on ε, i.e. a regular asymptotic approximation. It is useful and common to approach u_ε by a finite partial sum of terms of its possible series expansion and insure a sharper quantitative estimate of the error. This allows to go deeper in the analysis of the asymptotic behaviour of u_ε and to understand what happens if we want to improve the rate of asymptotic convergence and under which optimal conditions these improvements take place. For example we will see that the difference between the rates of asymptotic convergence of $\nabla_{X_2} u_\varepsilon$ and $\nabla_{X_1} u_\varepsilon$, established in [13] (or see for instance Sub-section 1.3.1) for the limit as an asymptotic approximation, is more attached to a smoothness statement of the asymptotic approximation. Another more interesting example is dealt with concerning the exponential rate of asymptotic convergence that is always limited to some symmetries as sufficient conditions (see [13]). Here we will see sufficient and necessary conditions that guarantee an exponential error of the asymptotic development and give a concrete example without any symmetry. Note that the asymptotic convergence results are ensured in a Sobolev type space on sub-domains located far away from the boundary layers.

2.2 Formal asymptotic expansion and some regularity results

As it is shown in [13] the limit u_0 of u_ε is the unique solution to the following lower dimension problem, for a.e. $X_1 \in \omega_1$,

$$\begin{cases} -\nabla_{X_2} \cdot A_{22}\nabla_{X_2} u_0(X_1, .) = f(X_1, .) & \text{in } \omega_2, \\ u_0(X_1, .) = 0 & \text{on } \partial\omega_2, \end{cases} \tag{2.2.1}$$

in the following sense, for a.e. $X_1 \in \omega_1$,

$$\begin{cases} u_0(X_1,.) \in H_0^1(\omega_2), \\ \displaystyle\int_{\omega_2} A_{22}\nabla_{X_2}u_0(X_1,.)\cdot\nabla_{X_2}v\,\mathrm{d}X_2 = \int_{\omega_2} f(X_1,.)v\,\mathrm{d}X_2 \qquad \forall v \in H_0^1(\omega_2). \end{cases} \tag{2.2.2}$$

The existence and the uniqueness of u_0 are followed from the Lax–Milgram theorem since we have (2.1.1), (2.1.3) and $f(X_1,.) \in L^2(\omega_2)$ for a.e. $X_1 \in \omega_1$. The asymptotic convergence holds out to the boundary of Ω, but with respect to topologies weaker than those of the existence space of u_ε; $H_0^1(\Omega)$. We mean in the following functional space

$$\mathcal{V}(\Omega) = \left\{v \in L^2(\Omega) \mid \ \nabla_{X_2}v \in L^2(\Omega)\right\},$$

equipped with the norm

$$|v|^2_{\mathcal{V}(\Omega)} = \int_\Omega |v(x)|^2 + |\nabla_{X_2}v(x)|^2\,\mathrm{d}x.$$

Note that u_0 is also the unique solution to (2.2.1) in $\mathcal{V}_0(\Omega)$, the closure of $\mathcal{D}(\Omega)$ in $\mathcal{V}(\Omega)$, and there is an equivalent weak formulation to (2.2.2) defined on $\mathcal{V}_0(\Omega)$ (see [13]). The improvements related to the asymptotic convergence of u_ε towards u_0 investigated on one hand the topology type by considering the standard Sobolev space on domains located far away from the boundary layers $\partial\omega_1 \times \omega_2$ and on the other hand the rate of asymptotic convergence. Unfortunately, these improvements are limited by the nature of the problem. It can go until an exponential rate of asymptotic convergence if A_{12}, A_{22} and f are independent of X_1, however, a rate of asymptotic convergence as

$$u_\varepsilon - u_0 = o(\varepsilon) \quad \text{in } L^2\left(\omega_1' \times \omega_2\right), \quad \omega_1' \subset\subset \omega_1$$

can not take place if

$$\nabla_{X_1}\cdot A_{12}\nabla_{X_2}u_0 + \nabla_{X_2}\cdot A_{21}\nabla_{X_1}u_0 \neq 0 \quad \text{in } \Omega.$$

Now, in order to reduce the approximation error we have no choice than to consider a regular asymptotic approximation U_ε of u_ε depending on ε in a simple way. Arguing as in many singular perturbation (isotropic) works, we can propose an asymptotic development of u_ε, i.e. it should be expressed as a power series of ε in the form

$$u_\varepsilon = u_0 + \varepsilon u_1 + \cdots. \tag{2.2.3}$$

Consequently, this allows to chose U_ε as a polynomial in ε, i.e.

$$U_\varepsilon^N = u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N, \tag{2.2.4}$$

where N be a positive integer and u_i be independent of ε, $i = 0, \ldots, N$. However the main goal of this chapter is to deal with the limit behaviour of the error

$$R_\varepsilon^N = u_\varepsilon - U_\varepsilon^N$$

and evaluate its rate of asymptotic convergence with respect to different norms.

Formally, if we substitute the asymptotic expansion of (2.2.3) into (2.1.4) and expand the left-hand side in powers of ε, we then deduce, after equating the coefficients of equal powers of ε, that the coefficients u_i, $i = 0, \ldots, N$, are solutions of the following system of boundary value problems, defined on the section ω_2 for a.e. $X_1 \in \omega_1$,

$$\begin{cases} -\nabla_{X_2}\cdot A_{22}\nabla_{X_2}u_0(X_1,.) = f(X_1,.) \qquad \text{in } \omega_2, \\ u_0(X_1,.) \in H_0^1(\omega_2), \end{cases} \tag{2.2.5}$$

$$\begin{cases} -\nabla_{X_2} \cdot A_{22} \nabla_{X_2} u_1(X_1, .) = \nabla_{X_1} \cdot A_{12} \nabla_{X_2} u_0(X_1, .) + \nabla_{X_2} \cdot A_{21} \nabla_{X_1} u_0(X_1, .) \quad \text{in } \omega_2, \\ u_1(X_1, .) \in H_0^1(\omega_2), \end{cases} \tag{2.2.6}$$

and for $N \geq 2$,

$$\begin{cases} -\nabla_{X_2} \cdot A_{22} \nabla_{X_2} u_N(X_1, .) & = & \nabla_{X_1} \cdot A_{11} \nabla_{X_1} u_{N-2}(X_1, .) \\ & & +\nabla_{X_1} \cdot A_{12} \nabla_{X_2} u_{N-1}(X_1, .) \\ & & +\nabla_{X_2} \cdot A_{21} \nabla_{X_1} u_{N-1}(X_1, .) \qquad \text{in } \omega_2, \\ u_N(X_1, .) \in H_0^1(\omega_2). \end{cases} \tag{2.2.7}$$

Our perturbed problem is now reduced to the sequence of the elliptic boundary value problems (2.2.5), (2.2.6) and (2.2.7) which can be easily solved iteratively once the solution of (2.2.5) has been constructed and has, in addition to A_{11}, A_{12} and A_{21}, the necessary smoothness. In fact, since the solution u_ε remains in the Sobolev space $H_0^1(\Omega)$, its regular asymptotic approximation U_ε^N is envisaged at least in a Sobolev type space as $\mathcal{V}_0(\Omega)$. Moreover, according to the expressions of the left-hand side in the equations (2.2.5), (2.2.6) and (2.2.7), reliable hypotheses have to be ensured on the data to make the suitable purpose workable.

Let us now deal with the regularity of the solutions to problems occurred above in a general context.

Theorem 2.1. *Under the above assumptions on A_{22}, let u be a solution (in the weak sense) to the following problem, for a.e. $X_1 \in \omega_1$,*

$$\begin{cases} -\nabla_{X_2} \cdot A_{22} \nabla_{X_2} u(X_1, .) & = & g(X_1, .) + \nabla_{X_2} \cdot G(X_1, .) \qquad \text{in } \omega_2, \\ u(X_1, .) \in H_0^1(\omega_2), \end{cases} \tag{2.2.8}$$

where $g \in L^2(\Omega)$, $G \in \left[L^2(\Omega)\right]^{\mathrm{n}-p}$. In addition, we assume, for a multi-index $\alpha \in \mathbb{N}^p \times \{0\}^{\mathrm{n}-p}$, that

$$D^\beta a_{ij} \in L^\infty(\Omega), \qquad \forall i, j = p+1, \dots, \mathrm{n} \tag{2.2.9}$$

and

$$D^\beta g, \quad D^\beta G \in L^2(\Omega), \tag{2.2.10}$$

for all $\beta \in \mathbb{N}^p \times \{0\}^{\mathrm{n}-p}$ with $\beta \leq \alpha$. Then the solution u has the maximal regularity in the X_1–directions according to that of g, G and A_{22}, i.e. we have

$$D^\beta u, \quad D^\beta\left(\nabla_{X_2} u\right) \in L^2(\Omega) \quad \text{for } \beta \leq \alpha. \tag{2.2.11}$$

(The comparison "$\leq$" between the multi-indices is meant component by component.)

To show this and more let us introduce the higher order difference quotients which are the main tool to deal with the smoothness of the solution. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_{\mathrm{n}}) \in \mathbb{N}^{\mathrm{n}}$ and a step $h > 0$, we define the difference quotients of order 1, i.e. $\alpha = e_i$ where e_i denotes the unit coordinate vector in the x_i–direction, by

$$\triangle_h^\alpha v(x) := \triangle_{i,h} v(x) = \frac{v\left(x + he_i\right) - v\left(x\right)}{h},$$

and the higher order differences by

$$\triangle_h^\alpha v = \triangle_h^{\alpha_1} \triangle_h^{\alpha_2} \cdots \triangle_h^{\alpha_{\mathrm{n}}} v,$$

where

$$\triangle_h^{\alpha_k} v = \triangle_{k,h} \triangle_h^{\alpha_k - 1} v.$$

The following lemma shows the link between the weak derivatives and the difference quotients (see also [3] for more properties).

Lemma 2.1. *i) Let $v \in W^{m,p}(\Omega)$, $m \in \mathbb{N}$, $1 \le p \le \infty$, and $\alpha \in \mathbb{N}^n$ be a multi-index such that $|\alpha| \le m$. Then $\triangle_h^\alpha v \in L^p(\Omega')$ for any $\Omega' \subset\subset \Omega$ satisfying $0 < h < \frac{1}{|\alpha|} \operatorname{dist}(\Omega', \partial\Omega)$, and we have*

$$|\triangle_h^\alpha v|_{p,\Omega'} \le |D^\alpha v|_{p,\Omega}.$$

ii) Let $v \in L^p(\Omega)$ with $1 < p \le \infty$, and we suppose that for a multi-index $\alpha \in \mathbb{N}^n$, there exists a constant $\kappa > 0$ such that $\triangle_h^\alpha v \in L^p(\Omega')$ and $|\triangle_h^\alpha v|_{p,\Omega'} \le \kappa$ for any $\Omega' \subset\subset \Omega$ and all h satisfying $0 < h < \frac{1}{|\alpha|} \operatorname{dist}(\Omega', \partial\Omega)$. Then the α^{th}–weak derivative $D^\alpha v$ exists in $L^p(\Omega)$ and also satisfies

$$|D^\alpha v|_{p,\Omega} \le \kappa.$$

Proof of the lemma. In fact this lemma is an extension of [5, Proposition IX.3] shown for $m = 1$. For the first assertion, it is enough to apply [5, Proposition IX.3] $|\alpha|$ times. It then holds, for any choice $\beta^1, \beta^2, \dots, \beta^{|\alpha|}$ with $\beta^1 \le \beta^2 \le \cdots \le \beta^{|\alpha|} := \alpha$ and $\left|\beta^i\right| + 1 = \left|\beta^{i+1}\right|$, $i = 1, \dots, |\alpha| - 1$,

$$\begin{aligned}
|\triangle_h^\alpha v|_{p,\Omega'} &\le \left|\triangle_h^{\alpha-\beta^1} D^{\beta^1} v\right|_{p,\Omega'_h} \le \left|\triangle_h^{\alpha-\beta^2} D^{\beta^2} v\right|_{p,\Omega'_{2h}} \\
&\le \cdots \\
&\le \left|\triangle_h^{\alpha-\beta^{|\alpha|}} D^{\beta^{|\alpha|}} v\right|_{p,\Omega'_{|\alpha|h}} \le |D^\alpha v|_{p,\Omega},
\end{aligned}$$

where Ω'_τ denotes a $\tau-$neighborhood of Ω' for a positive number τ.

Now we pass to the second assertion. Assume, for some constant κ, that $|\triangle_h^\alpha v|_{p,\Omega'} \le \kappa$ holds for any $\Omega' \subset\subset \Omega$ and all $0 < h < \frac{1}{|\alpha|} \operatorname{dist}(\Omega', \partial\Omega)$. Therefore, there exist a function $w \in L^p(\Omega')$ and a subsequence $h_k \longrightarrow 0$ such that

$$\triangle_{h_k}^\alpha v \rightharpoonup w \qquad \text{in } L^p(\Omega') \quad \text{for } 1 < p < \infty,$$

$$\triangle_{h_k}^\alpha v \overset{*}{\rightharpoonup} w \qquad \text{in } L^\infty(\Omega') \quad \text{if } p = \infty.$$

Then, it follows that for each $\varphi \in \mathcal{D}(\Omega')$ and h_k small enough

$$\begin{aligned}
\int_{\Omega'} v D^\alpha \varphi \, \mathrm{d}x &= \int_\Omega v D^\alpha \varphi \, \mathrm{d}x \\
&= \lim_{h_k \longrightarrow 0} \int_\Omega v \triangle_{-h_k}^\alpha \varphi \, \mathrm{d}x \\
&= (-1)^{|\alpha|} \lim_{h_k \longrightarrow 0} \int_\Omega \triangle_{h_k}^\alpha v \varphi \, \mathrm{d}x \\
&= (-1)^{|\alpha|} \int_\Omega w \varphi \, \mathrm{d}x.
\end{aligned}$$

This means that $w = D^\alpha v$ on Ω' and since the norm is lower semi-continuous we derive

$$|D^\alpha v|_{p,\Omega'} \le \liminf_{h_k \longrightarrow 0} \left|\triangle_{h_k}^\alpha v\right|_{p,\Omega'} \le \kappa.$$

Of course the above inequality holds on the whole domain Ω since Ω' is arbitrary in Ω. $\square$

Proof of Theorem 2.1. The theorem 2.1 is an extension of [13, Proposition 1], where it was shown for $|\alpha| = 1$. We then proceed by induction on α with $|\alpha| \geq 2$. Suppose that (2.2.11) holds for all $\beta \in \mathbb{N}^{\mathrm{n}}$ with $\beta < \alpha$. Let ω_1' be a bounded subset such that $\omega_1' \subset\subset \omega_1$ and h be a positive constant such that $0 < |\alpha|\, h < \operatorname{dist}(\omega_1', \partial\omega_1)$. For a.e. $X_1 \in \omega_1'$, if we apply the difference quotient operator $\triangle_h^\alpha$, with $\alpha = (\alpha_1, \dots, \alpha_p, 0, \dots, 0) \in \mathbb{N}^p \times \{0\}^{\mathrm{n}-p}$, to the integral formulation of problem (2.2.8),

$$\int_{\omega_2} A_{22} \nabla_{X_2} u(X_1, .) \cdot \nabla_{X_2} v \, \mathrm{d}X_2 = \int_{\omega_2} g(X_1, .) v \, \mathrm{d}X_2 - \int_{\omega_2} G(X_1, .) \cdot \nabla_{X_2} v \, \mathrm{d}X_2, \quad \forall v \in H_0^1(\omega_2),$$

we get, using [3, Lemma 3.3] to develop the difference quotient of a product,

$$\int\limits_{\omega_2} \sum_{i,j=p+1}^{\mathrm{n}} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \triangle_h^\beta a_{ij}\big(x + (\alpha - \beta)h\big) \triangle_h^{\alpha-\beta} \Big(\partial_{x_j} u\Big) \partial_{x_i} v \, \mathrm{d}X_2 = \int\limits_{\omega_2} \triangle_h^\alpha g v - \triangle_h^\alpha G \cdot \nabla_{X_2} v \, \mathrm{d}X_2,$$

where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$. Permuting the derivatives and the difference quotient and taking, for a.e. $X_1 \in \omega_1'$, $v = \triangle_h^\alpha u\,(X_1, .) \in H_0^1(\omega_2)$ as a test function yield

$$\begin{aligned}
&\int\limits_{\omega_2} \sum_{i,j=p+1}^{\mathrm{n}} a_{ij}(x + \alpha h) \partial_{x_j} \Big(\triangle_h^\alpha u\Big) \partial_{x_i} \Big(\triangle_h^\alpha u\Big) \, \mathrm{d}X_2 \\
&\qquad = \int\limits_{\omega_2} \Big(\triangle_h^\alpha g\Big)\Big(\triangle_h^\alpha u\Big) - \triangle_h^\alpha G \cdot \nabla_{X_2} \Big(\triangle_h^\alpha u\Big) \, \mathrm{d}X_2 \\
&\qquad\quad - \int\limits_{\omega_2} \sum_{i,j=p+1}^{\mathrm{n}} \sum_{0<\beta \leq \alpha} \binom{\alpha}{\beta} \triangle_h^\beta a_{ij}\big(x + (\alpha - \beta)h\big) \partial_{x_j} \Big(\triangle_h^{\alpha-\beta} u\Big) \partial_{x_i} \Big(\triangle_h^\alpha u\Big) \, \mathrm{d}X_2.
\end{aligned}$$

Integrating now on ω_1' and using the ellipticity assumption, we derive

$$\begin{aligned}
&\lambda \left|\nabla_{X_2} \left(\triangle_h^\alpha u\right)\right|_{2,\omega_1' \times \omega_2}^2 \\
&\qquad \leq \left|\triangle_h^\alpha g\right|_{2,\omega_1' \times \omega_2} \left|\triangle_h^\alpha u\right|_{2,\omega_1' \times \omega_2} + \left|\triangle_h^\alpha G\right|_{2,\omega_1' \times \omega_2} \left|\triangle_h^\alpha \left(\nabla_{X_2} u\right)\right|_{2,\omega_1' \times \omega_2} \\
&\qquad\quad + \sum_{0<\beta \leq \alpha} \binom{\alpha}{\beta} \left|\triangle_h^\beta A_{22}\right|_{\infty,\omega_1' \times \omega_2} \left|\triangle_h^{\alpha-\beta} \left(\nabla_{X_2} u\right)\right|_{2,\omega_1' \times \omega_2} \left|\nabla_{X_2} \left(\triangle_h^\alpha u\right)\right|_{2,\omega_1' \times \omega_2}.
\end{aligned}$$

Applying the Poincaré inequality in the X_2–directions on the norm $\left|\triangle_h^\alpha u\right|_{2,\omega_1' \times \omega_2}$, then Young's inequality, we end up with

$$\left|\nabla_{X_2} \left(\triangle_h^\alpha u\right)\right|_{2,\omega_1' \times \omega_2}^2 \leq \kappa \left|\triangle_h^\alpha g\right|_{2,\omega_1' \times \omega_2}^2 + \kappa \left|\triangle_h^\alpha G\right|_{2,\omega_1' \times \omega_2}^2 + \kappa \sum_{0<\beta \leq \alpha} \left|\triangle_h^{\alpha-\beta} \left(\nabla_{X_2} u\right)\right|_{2,\omega_1' \times \omega_2}^2.$$

Of course, thanks to Lemma 2.1 *i)* with (2.2.9), (2.2.10) and the inductive hypotheses, we can easily deduce that the right-hand side of the above inequality is bounded independently of h, i.e.

$$\left|\triangle_h^\alpha u\right|_{2,\omega_1' \times \omega_2}, \quad \left|\nabla_{X_2} \left(\triangle_h^\alpha u\right)\right|_{2,\omega_1' \times \omega_2} \leq \kappa. \tag{2.2.12}$$

We applied the Poincaré inequality in the X_2-direction to get the first above estimate from the second one. Finally, (2.2.11) follows from (2.2.12) and Lemma 2.1 *ii)*. □

Remark 2.1. *Theorem 2.1 remains valid if we suppose that the domain ω_2 is only bounded in some directions (not all) to apply the Poincaré inequality.*

2.3 Asymptotic convergence results

As it is mentioned above, the problem (2.2.7) will be solved iteratively. That is to say, in order to define u_N, for $N \geq 2$, as a solution of (2.2.7) in the following weak sense, for a.e. $X_1 \in \omega_1$,

$$\begin{cases} u_N(X_1,.) \in H_0^1(\omega_2), \\ \displaystyle\int_{\omega_2} A_{22}\nabla_{X_2}u_N(X_1,.)\cdot\nabla_{X_2}v\,\mathrm{d}X_2 = \displaystyle\int_{\omega_2}\nabla_{X_1}\cdot\big(A_{11}\nabla_{X_1}u_{N-2}(X_1,.)\big)v\,\mathrm{d}X_2 \\ \qquad\qquad + \displaystyle\int_{\omega_2}\nabla_{X_1}\cdot\big(A_{12}\nabla_{X_2}u_{N-1}(X_1,.)\big)v\,\mathrm{d}X_2 \\ \qquad\qquad - \displaystyle\int_{\omega_2}\big(A_{21}\nabla_{X_1}u_{N-1}(X_1,.)\big)\cdot\nabla_{X_2}v\,\mathrm{d}X_2 \qquad \forall v \in H_0^1(\omega_2), \end{cases} \tag{2.3.1}$$

we need to ensure the smoothness of u_{N-1}, u_{N-2}, A_{11} and A_{12} in the following sense

$$D^1_{X_1}A_{11}, \quad D^1_{X_1}A_{12} \in L^\infty(\Omega), \quad D^2_{X_1}u_{N-2}, \quad D^1_{X_1}u_{N-1}, \quad D^1_{X_1}\nabla_{X_2}u_{N-1} \in L^2(\Omega). \tag{2.3.2}$$

($D^m_{X_i}$ denotes the derivatives in the direction X_i of order up to m.) Again we have to ensure the existence and the uniqueness of u_{N-1}, u_{N-2}, of course as solutions of the same problem (2.3.1) replacing N by $N-1$, $N-2$ respectively, as well as their smoothness in (2.3.2). So on, this leads to require, taking into account Theorem 2.1, the following hypotheses on u_0, u_1 that appear in the left-hand side of the equation defining u_2

$$\begin{gathered} D^{N-1}_{X_1}A_{11}, \quad D^{N-1}_{X_1}A_{12}, \quad D^{N-2}_{X_1}A_{21}, \quad D^{N-2}_{X_1}A_{22} \in L^\infty(\Omega), \\ D^N_{X_1}u_0, \quad D^{N-1}_{X_1}u_1, \quad D^{N-1}_{X_1}\nabla_{X_2}u_1 \in L^2(\Omega). \end{gathered} \tag{2.3.3}$$

Now, if we require a solution u_1 satisfying $D^{N-1}_{X_1}u_1$, $D^{N-1}_{X_1}\nabla_{X_2}u_1 \in L^2(\Omega)$ we have to assume, taking into account (2.3.3) and (2.2.6), that

$$\begin{gathered} D^{N-1}_{X_1}A_{11}, \quad D^{N}_{X_1}A_{12}, \quad D^{N-1}_{X_1}A_{21}, \quad D^{N-1}_{X_1}A_{22} \in L^\infty(\Omega), \\ D^N_{X_1}u_0, \quad D^{N}_{X_1}\nabla_{X_2}u_0 \in L^2(\Omega), \end{gathered} \tag{2.3.4}$$

of course this is always according to Theorem 2.1. Note that we can also consider that (2.3.1) is the weak formulation of (2.2.6) with $N=1$ and $u_{-1}=0$. Finally, the smoothness requested on u_0 solution to (2.2.5) with (2.3.4) leads to state, using again Theorem 2.1, the following result.

Corollary 2.1. *The existence and the uniqueness of $u_1,\ldots,u_N \in \mathcal{V}_0(\Omega)$ as solutions of the elliptic system (2.2.6), (2.2.7) are ensured if we assume that*

$$D^{N-1}_{X_1}A_{11}, \quad D^{N}_{X_1}A_{12}, \quad D^{N-1}_{X_1}A_{21}, \quad D^{N}_{X_1}A_{22} \in L^\infty(\Omega), \quad D^N_{X_1}f \in L^2(\Omega). \tag{2.3.5}$$

Remark 2.2. *If we assume that the smoothness hypotheses (2.3.5) are satisfied for $K+N$ instead of N, where K is a positive integer, we have*

$$u_N \in H^K\big(\omega_1; H^1_0(\omega_2)\big). \tag{2.3.6}$$

The above remark shows the regularity of the coefficients which allowed to consider U^N_ε as an asymptotic approximation in a standard Sobolev space. In fact if we take $K=1$ and $N=0$, we may covert the hypotheses assumed in [13, Theorem 3] to get

$$u_\varepsilon - u_0,\, \nabla_{X_2}(u_\varepsilon - u_0) = O(\varepsilon), \qquad \nabla_{X_1}u_\varepsilon \rightharpoonup \nabla_{X_1}u_0 \quad \text{in } L^2\left(\omega_1' \times \omega_2\right), \tag{2.3.7}$$

where $\omega_1' \subset\subset \omega_1$. Now we pass to the main regular asymptotic expansion result improving (2.3.7) and we state

Theorem 2.2 (Regular Asymptotic Expansion of Higher Order)**.** *Under the assumptions (2.1.1), (2.1.2) and (2.3.5) satisfied for $N+1$, it holds that, when $\varepsilon \to 0$,*

$$\begin{aligned} u_\varepsilon - \left(u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N\right) = O\left(\varepsilon^{N+1}\right) \quad &\text{in } \mathcal{V}\left(\omega_1' \times \omega_2\right), \\ \tfrac{1}{\varepsilon^N} \nabla_{X_1}\left[u_\varepsilon - \left(u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N\right)\right] \rightharpoonup 0 \quad &\text{weakly in } L^2\left(\omega_1' \times \omega_2\right), \end{aligned} \tag{2.3.8}$$

for any $\omega_1' \subset\subset \omega_1$. (The vectorial convergence is meant component by component.)

Proof. First, we notice that under the above assumptions and according to Remark 2.2, we have

$$u_k \in H^1(\Omega), \quad \text{for } k = 0, \ldots, N.$$

We can also see that for $N = 0$, the same results of this theorem are shown in [13, Theorem 3]. For $N > 0$, we proceed by induction on N and suppose that the estimates (2.3.8) take place for any $N' < N$, and we will prove this result for N. Starting from (2.3.1) written for $k = 1, \ldots, N$ and multiplying each k–identity by ε^k then summing up over $k = 1, \ldots, N$, we obtain, for $v \in H_0^1(\Omega)$

$$\begin{aligned} \varepsilon^2 \int_\Omega A_{11} \nabla_{X_1} U_\varepsilon^{N-2} \cdot \nabla_{X_1} v \,\mathrm{d}x + \varepsilon \int_\Omega A_{12} \nabla_{X_2} U_\varepsilon^{N-1} \cdot \nabla_{X_1} v \,\mathrm{d}x \\ + \varepsilon \int_\Omega A_{21} \nabla_{X_1} U_\varepsilon^{N-1} \cdot \nabla_{X_2} v \,\mathrm{d}x + \int_\Omega A_{22} \nabla_{X_2} U_\varepsilon^{N} \cdot \nabla_{X_2} v \,\mathrm{d}x = \int_\Omega f\, v \,\mathrm{d}x, \end{aligned}$$

where $U_\varepsilon^{-1} = 0$. Of course the identity (2.2.2) is added and we integrated over ω_1. Then subtracting the above identity from (2.1.5), we get

$$\begin{aligned} \varepsilon^2 \int_\Omega A_{11} \nabla_{X_1} R_\varepsilon^{N-2} \cdot \nabla_{X_1} v \,\mathrm{d}x + \varepsilon \int_\Omega A_{12} \nabla_{X_2} R_\varepsilon^{N-1} \cdot \nabla_{X_1} v \,\mathrm{d}x \\ + \varepsilon \int_\Omega A_{21} \nabla_{X_1} R_\varepsilon^{N-1} \cdot \nabla_{X_2} v \,\mathrm{d}x + \int_\Omega A_{22} \nabla_{X_2} R_\varepsilon^{N} \cdot \nabla_{X_2} v \,\mathrm{d}x = 0. \end{aligned}$$

Thus

$$\begin{aligned} \varepsilon^2 \int_\Omega A_{11} \nabla_{X_1} R_\varepsilon^{N} \cdot \nabla_{X_1} v \,\mathrm{d}x + \varepsilon \int_\Omega A_{12} \nabla_{X_2} R_\varepsilon^{N} \cdot \nabla_{X_1} v \,\mathrm{d}x \\ + \varepsilon \int_\Omega A_{21} \nabla_{X_1} R_\varepsilon^{N} \cdot \nabla_{X_2} v \,\mathrm{d}x + \int_\Omega A_{22} \nabla_{X_2} R_\varepsilon^{N} \cdot \nabla_{X_2} v \,\mathrm{d}x \\ = -\,\varepsilon^{N+1} \int_\Omega A_{11} \nabla_{X_1} u_{N-1} \cdot \nabla_{X_1} v \,\mathrm{d}x - \varepsilon^{N+2} \int_\Omega A_{11} \nabla_{X_1} u_N \cdot \nabla_{X_1} v \,\mathrm{d}x \\ -\,\varepsilon^{N+1} \int_\Omega A_{12} \nabla_{X_2} u_N \cdot \nabla_{X_1} v \,\mathrm{d}x - \varepsilon^{N+1} \int_\Omega A_{21} \nabla_{X_1} u_N \cdot \nabla_{X_2} v \,\mathrm{d}x. \end{aligned} \tag{2.3.9}$$

Next, we consider a smooth function $\rho = \rho(X_1)$ supported in ω_1'' such that $\omega_1' \subset\subset \omega_1'' \subset\subset \omega_1$, $\rho = 1$ on ω_1' and $0 \le \rho \le 1$. This allows to take $v = R_\varepsilon^N(.)\rho^2(X_1) \in H_0^1(\Omega)$ as a test function in (2.3.9) that leads to

$$\begin{aligned}
&\int_\Omega \rho^2 A_\varepsilon \nabla R_\varepsilon^N \cdot \nabla R_\varepsilon^N \,\mathrm{d}x \\
&\quad = -2\varepsilon^2 \int_\Omega A_{11} \nabla_{X_1} R_\varepsilon^{N-1} \cdot (\nabla_{X_1}\rho)\rho R_\varepsilon^N \,\mathrm{d}x - 2\varepsilon^{N+1} \int_\Omega A_{11} \nabla_{X_1} u_{N-1} \cdot (\nabla_{X_1}\rho)\rho R_\varepsilon^N \,\mathrm{d}x \\
&\qquad - 2\varepsilon \int_\Omega A_{12} \nabla_{X_2} R_\varepsilon^{N-1} \cdot (\nabla_{X_1}\rho)\rho R_\varepsilon^N \,\mathrm{d}x - \varepsilon^{N+2} \int_\Omega \rho^2 A_{11} \nabla_{X_1} u_N \cdot \nabla_{X_1} R_\varepsilon^N \,\mathrm{d}x \\
&\qquad - \varepsilon^{N+1} \int_\Omega \rho^2 A_{21} \nabla_{X_1} u_N \cdot \nabla_{X_2} R_\varepsilon^N \,\mathrm{d}x - \varepsilon^{N+1} \int_\Omega \rho^2 A_{11} \nabla_{X_1} u_{N-1} \cdot \nabla_{X_1} R_\varepsilon^N \,\mathrm{d}x \\
&\qquad - \varepsilon^{N+1} \int_\Omega \rho^2 A_{12} \nabla_{X_2} u_N \cdot \nabla_{X_1} R_\varepsilon^N \,\mathrm{d}x. \qquad (2.3.10)
\end{aligned}$$

Next applying the ellipticity assumption, the Poincaré and the Cauchy-Schwartz inequalities for the first five terms of the right-hand side of the above inequality, we get

$$\begin{aligned}
&\varepsilon^2 \left|\rho \nabla_{X_1} R_\varepsilon^N\right|_{2,\Omega}^2 + \left|\rho \nabla_{X_2} R_\varepsilon^N\right|_{2,\Omega}^2 \\
\leq &\tfrac{2\varepsilon^2 \sqrt{\kappa_{\omega_2}}}{\lambda} \left|A_{11}\right|_{\infty,\omega_1''\times\omega_2} \left|\nabla_{X_1}\rho\right|_{\infty,\omega_1''} \left|\nabla_{X_1} R_\varepsilon^{N-1}\right|_{2,\omega_1''\times\omega_2} \left|\rho\nabla_{X_2} R_\varepsilon^N\right|_{2,\Omega} \\
&+ \tfrac{2\varepsilon^{N+1}\sqrt{\kappa_{\omega_2}}}{\lambda} \left|A_{11}\right|_{\infty,\omega_1''\times\omega_2} \left|\nabla_{X_1}\rho\right|_{\infty,\omega_1''} \left|\nabla_{X_1} u_{N-1}\right|_{2,\Omega} \left|\rho\nabla_{X_2} R_\varepsilon^N\right|_{2,\Omega} \\
&+ \tfrac{2\varepsilon\sqrt{\kappa_{\omega_2}}}{\lambda} \left|A_{12}\right|_{\infty,\omega_1''\times\omega_2} \left|\nabla_{X_1}\rho\right|_{\infty,\omega_1''} \left|\nabla_{X_2} R_\varepsilon^{N-1}\right|_{2,\omega_1''\times\omega_2} \left|\rho\nabla_{X_2} R_\varepsilon^N\right|_{2,\Omega} \\
&+ \tfrac{\varepsilon^{N+1}}{\lambda} \left|A_{11}\right|_{\infty,\omega_1''\times\omega_2} \left|\nabla_{X_1} u_N\right|_{2,\Omega} \varepsilon \left|\rho\nabla_{X_1} R_\varepsilon^N\right|_{2,\Omega} + \tfrac{\varepsilon^{N+1}}{\lambda} \left|A_{21}\right|_{\infty,\omega_1''\times\omega_2} \left|\nabla_{X_1} u_N\right|_{2,\Omega} \left|\rho\nabla_{X_2} R_\varepsilon^N\right|_{2,\Omega} \\
&- \tfrac{\varepsilon^{N+1}}{\lambda} \int_\Omega \rho^2 A_{11} \nabla_{X_1} u_{N-1} \cdot \nabla_{X_1} R_\varepsilon^N \,\mathrm{d}x - \tfrac{\varepsilon^{N+1}}{\lambda} \int_\Omega \rho^2 A_{12} \nabla_{X_2} u_N \cdot \nabla_{X_1} R_\varepsilon^N \,\mathrm{d}x,
\end{aligned}$$

where κ_{ω_2} is the Poincaré constant. Then, using Young inequality and the convergences (2.3.8) on $\omega_1'' \times \omega_2$ for $N-1$, we derive

$$\begin{aligned}
\varepsilon^2 \left|\rho \nabla_{X_1} R_\varepsilon^N\right|_{2,\Omega}^2 + \left|\rho \nabla_{X_2} R_\varepsilon^N\right|_{2,\Omega}^2 \quad \leq \quad & \kappa \varepsilon^{2N+2} - \tfrac{2\varepsilon^{N+1}}{\lambda} \int_\Omega \rho^2 A_{11} \nabla_{X_1} u_{N-1} \cdot \nabla_{X_1} R_\varepsilon^N \,\mathrm{d}x \\
& - \tfrac{2\varepsilon^{N+1}}{\lambda} \int_\Omega \rho^2 A_{12} \nabla_{X_2} u_N \cdot \nabla_{X_1} R_\varepsilon^N \,\mathrm{d}x. \qquad (2.3.11)
\end{aligned}$$

Now we treat the last two integrals of the above inequality separately. For the first one we have

$$\begin{aligned}
\int_\Omega \rho^2 A_{11} \nabla_{X_1} u_{N-1} \cdot \nabla_{X_1} R_\varepsilon^N \,\mathrm{d}x \quad = \quad & \sum_{i,j=1}^{p} \int_\Omega \rho^2 a_{ij} \partial_{x_j} u_{N-1} \partial_{x_i} R_\varepsilon^N \,\mathrm{d}x \\
= \quad & -\sum_{i,j=1}^{p} \int_\Omega \left(\partial_{x_i} a_{ij}\rho + 2a_{ij}\partial_{x_i}\rho\right) \rho R_\varepsilon^N \partial_{x_j} u_{N-1} \,\mathrm{d}x \\
& -\sum_{i,j=1}^{p} \int_\Omega a_{ij} \rho^2 R_\varepsilon^N \partial_{x_i}\partial_{x_j} u_{N-1} \,\mathrm{d}x.
\end{aligned}$$

Of course the last integral is well defined since, by Remark 2.2, $u_{N-1} \in H^2\big(\omega_1; H_0^1(\omega_2)\big)$. Also, the last integral in (2.3.11) can be rewritten as

$$\begin{aligned}\int_\Omega \rho^2 A_{12}\nabla_{X_2}u_N\cdot\nabla_{X_1}R_\varepsilon^N\,\mathrm{d}x &= \sum_{i=1}^{p}\sum_{j=p+1}^{\mathrm{n}}\int_\Omega \rho^2 a_{ij}\partial_{x_j}u_N\partial_{x_i}R_\varepsilon^N\,\mathrm{d}x\\ &= -\sum_{i=1}^{p}\sum_{j=p+1}^{\mathrm{n}}\int_\Omega\left(\partial_{x_i}a_{ij}\rho+2a_{ij}\partial_{x_i}\rho\right)\rho R_\varepsilon^N\partial_{x_j}u_N\,\mathrm{d}x\\ &\quad-\sum_{i=1}^{p}\sum_{j=p+1}^{\mathrm{n}}\int_\Omega a_{ij}\rho^2R_\varepsilon^N\partial_{x_i}\partial_{x_j}u_N\,\mathrm{d}x.\end{aligned}$$

Here also, thanks to Remark 2.2, we have $u_N\in H^1\big(\omega_1;H_0^1(\omega_2)\big)$. Now if we use the above new forms of the last integrals in (2.3.11) and arguing as we did above, we get

$$\varepsilon^2\left|\rho\nabla_{X_1}R_\varepsilon^N\right|_{2,\Omega}^2+\left|\rho\nabla_{X_2}R_\varepsilon^N\right|_{2,\Omega}^2\le\kappa\varepsilon^{2N+2}. \tag{2.3.12}$$

Then, according to the boundedness of $\frac{1}{\varepsilon^N}\left|\nabla_{X_1}R_\varepsilon^N\right|_{2,\omega_1'\times\omega_2}$ from the above estimate (2.3.12), we can extract a weakly converging subsequence from $\frac{1}{\varepsilon^N}\nabla_{X_1}R_\varepsilon^N$ in $L^2\left(\omega_1'\times\omega_2\right)$. It follows then that the whole sequence converges weakly to zero, i.e.

$$\tfrac{1}{\varepsilon^N}\nabla_{X_1}\left[u_\varepsilon-\left(u_0+\varepsilon u_1+\cdots+\varepsilon^N u_N\right)\right]\rightharpoonup 0\quad\text{weakly in } L^2\left(\omega_1'\times\omega_2\right).$$

Of course, this is thanks to the density of $\mathcal{D}\left(\omega_1'\times\omega_2\right)$ in $L^2\left(\omega_1'\times\omega_2\right)$ and the continuity of the derivation in $\mathcal{D}'\left(\omega_1'\times\omega_2\right)$ which allows to use the convergence of the sequence $\frac{1}{\varepsilon^N}R_\varepsilon^N$ in $L^2\left(\omega_1'\times\omega_2\right)$ to zero. More details about the argument can be found in [10]. This completes the proof of the theorem. □

Remark 2.3. *If we look for (2.3.8) to be satisfied on some given $\omega_1'\subset\subset\omega_1$ we can replace ω_1 in (2.3.5) and (3.4.13) by any ω_1'' such that $\omega_1'\subset\subset\omega_1''\subset\subset\omega_1$. Nevertheless we keep the hypotheses (2.3.5), (3.4.13) as they are since they also serve for the existence of the coefficients u_N for arbitrary $\omega_1'\subset\subset\omega_1$.*

Remark 2.4. *We can easily see from (2.3.1) that (2.3.2) can be weakened as follows*

$$A_{11}',\,A_{12}'\in L^\infty(\Omega),\quad D_{X_1}^2u_{N-2},\,D_{X_1}^1u_{N-1},\,D_{X_1}^1\nabla_{X_2}u_{N-1}\in L^2(\Omega),$$

where $A_{11}'=\left(\partial_{x_i}a_{ij}\right)_{i,j=1,\dots,p}$ and $A_{12}'=\left(\partial_{x_i}a_{ij}\right)_{i=1,\dots,p,\ j=p+1,\dots,\mathrm{n}}$. This also allows to weaken (2.3.5) and replace it by

$$A_{12}',\,D_{X_1}^1A_{22}\in L^\infty(\Omega),\quad D_{X_1}^1f\in L^2(\Omega) \tag{2.3.13}$$

to guarantee the existence of u_1 in $\mathcal{V}_0(\Omega)$, and for u_N $(N>1)$ we may assume

$$D_{X_1}^{N-2}A_{11}',\,D_{X_1}^{N-1}A_{12}',\,D_{X_1}^{N-1}A_{21},\,D_{X_1}^{N}A_{22}\in L^\infty(\Omega),\quad D_{X_1}^Nf\in L^2(\Omega). \tag{2.3.14}$$

In fact, even if the particular case $N=0$ is shown in [13, Theorem 3], it is possible to improve the results of such theorem if we take into account Theorem 2.2 for $N=1$. Indeed, let us show it for any N. Suppose that (2.3.8) holds for $N+1$, then in particular we have

$$\left|\nabla_{X_1}R_\varepsilon^{N+1}\right|_{2,\omega_1'\times\omega_2}=O\left(\varepsilon^{N+1}\right).$$

This implies

$$\begin{aligned}
\left|\nabla_{X_1} R_\varepsilon^N\right|_{2,\omega_1'\times\omega_2} &= \left|\nabla_{X_1} R_\varepsilon^{N+1} + \varepsilon^{N+1}\nabla_{X_1} u_{N+1}\right|_{2,\omega_1'\times\omega_2} \\
&\le \left|\nabla_{X_1} R_\varepsilon^{N+1}\right|_{2,\omega_1'\times\omega_2} + \varepsilon^{N+1}\left|\nabla_{X_1} u_{N+1}\right|_{2,\omega_1'\times\omega_2} \\
&\le \kappa\varepsilon^{N+1}.
\end{aligned}$$

Then we end up with

Corollary 2.2. *Under the assumptions (2.1.1), (2.1.2) and (2.3.13) for $N=0$ or (2.3.14) (satisfied for $N+2$) if $N>0$, we have*

$$u_\varepsilon - (u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N) = O(\varepsilon^{N+1}) \quad \textit{in } H^1(\omega_1'\times\omega_2), \quad \forall \omega_1' \subset\subset \omega_1.$$

2.4 Sharper asymptotic convergence and estimate results

2.4.1 General view

An expected natural improvement of the convergence rate (2.3.8) may be expressed, for any $\omega_1' \subset\subset \omega_1$, as

$$\tfrac{1}{\varepsilon^{N+1}} R_\varepsilon^N \rightharpoonup 0 \quad \text{in } L^2\left(\omega_1'\times\omega_2\right). \tag{2.4.1}$$

In fact this means, by (2.3.9), that

$$\nabla_{X_1}\cdot A_{11}\nabla_{X_1} u_{N-1} + \nabla_{X_1}\cdot A_{12}\nabla_{X_2} u_N + \nabla_{X_2}\cdot A_{21}\nabla_{X_1} u_N = 0 \quad \text{in } \Omega, \tag{2.4.2}$$

where $u_{-1}=0$. Indeed, in addition to the weak convergence (2.4.1) we also have

$$\tfrac{1}{\varepsilon^{N+1}}\nabla_{X_2} R_\varepsilon^N \rightharpoonup 0 \quad \text{in } L^2\left(\omega_1'\times\omega_2\right), \tag{2.4.3}$$

thanks to the boundedness of $\frac{1}{\varepsilon^{N+1}}\left|\nabla_{X_2} R_\varepsilon^N\right|_{2,\omega_1'\times\omega_2}$ (see (2.3.8)). Now rewriting the identity (2.3.9), for $v\in\mathcal{D}(\omega_1'\times\omega_2)$, as

$$\begin{aligned}
\tfrac{1}{\varepsilon^{N-1}}\int_\Omega A_{11}\nabla_{X_1} R_\varepsilon^N\cdot\nabla_{X_1} v\,\mathrm{d}x &+ \tfrac{1}{\varepsilon^N}\int_\Omega A_{12}\nabla_{X_2} R_\varepsilon^N\cdot\nabla_{X_1} v\,\mathrm{d}x \\
&+ \tfrac{1}{\varepsilon^N}\int_\Omega A_{21}\nabla_{X_1} R_\varepsilon^N\cdot\nabla_{X_2} v\,\mathrm{d}x \\
&+ \tfrac{1}{\varepsilon^{N+1}}\int_\Omega A_{22}\nabla_{X_2} R_\varepsilon^N\cdot\nabla_{X_2} v\,\mathrm{d}x \\
&= -\,\varepsilon\int_\Omega A_{11}\nabla_{X_1} u_N\cdot\nabla_{X_1} v\,\mathrm{d}x - \int_\Omega A_{11}\nabla_{X_1} u_{N-1}\cdot\nabla_{X_1} v\,\mathrm{d}x \\
&\quad - \int_\Omega A_{12}\nabla_{X_2} u_N\cdot\nabla_{X_1} v\,\mathrm{d}x - \int_\Omega A_{21}\nabla_{X_1} u_N\cdot\nabla_{X_2} v\,\mathrm{d}x.
\end{aligned}$$

Using (2.3.8) and (2.4.3), then passing to the limit in each term of the above identity we end up with (2.4.2). By consequence, we deduce, from (2.3.1) and (2.4.2), that $u_{N+1}=0$ iff we have (2.4.2). This means that the existence of u_{N+1} solution to (2.3.1) is ensured by the above identity (2.4.2) in the distributional sense and we do not need to pass by Corollary 2.1 to ensure this existence.

Now, if we expect a little bit more as

$$\tfrac{1}{\varepsilon^{N+2}}\nabla_{X_2} R_\varepsilon^N \rightharpoonup 0 \quad \text{in } L^2\left(\omega_1'\times\omega_2\right),$$

we likewise deduce

$$u_{N+1} = 0 \quad \text{and} \quad u_{N+2} = 0.$$

As above, this holds iff we have (2.4.2) and

$$\nabla_{X_1} \cdot A_{11} \nabla_{X_1} u_N = 0 \quad \text{in } \mathcal{D}'(\Omega). \tag{2.4.4}$$

Here also, u_{N+2} exists and is null since the right-hand side of (2.3.1) is vanishing.

Nevertheless the last expectations are not little, we can easily see from (2.3.1), that

$$u_k = 0, \quad \forall k > N.$$

Actually, it follows that R_ε^N goes to zero in $H^1(\omega_1' \times \omega_2)$ faster than any power of ε, i.e. for any positive constant κ

$$R_\varepsilon^N = O(\varepsilon^\kappa) \quad \text{in } H^1(\omega_1' \times \omega_2).$$

Of course, this lets us to think about a sharper rate of asymptotic convergence as it is summarized in the following theorem.

Theorem 2.3. *Given the assumptions of Theorem 2.2 and consider the following assertions, for any* $\omega_1' \subset\subset \omega_1$,

(i) $u_{N+1} = 0$,

(ii) the condition (2.4.2) holds,

(iii) $\frac{1}{\varepsilon^{N+1}} \left[u_\varepsilon - (u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N) \right] \rightharpoonup 0$ *in* $L^2\left(\omega_1' \times \omega_2\right)$,

(iv) $u_\varepsilon - (u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N) = o(\varepsilon^{N+1})$ *in* $\mathcal{V}\left(\omega_1' \times \omega_2\right)$,

(v) $u_{N+1} = 0$ *and* $u_{N+2} = 0$,

(vi) the conditions (2.4.2) and (2.4.4) hold,

(vii) $u_k = 0, \quad \forall k > N$,

(viii) $\frac{1}{\varepsilon^{N+2}} \nabla_{X_2} \left[u_\varepsilon - (u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N) \right] \rightharpoonup 0$ *in* $L^2\left(\omega_1' \times \omega_2\right)$,

(ix) $u_\varepsilon - (u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N) = O\left(\exp(\frac{-\eta}{\varepsilon})\right)$ *in* $H^1(\omega_1' \times \omega_2)$, *for some* $\eta > 0$.

Then we have, as $\varepsilon \longrightarrow 0$,

$$(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftarrow (v) \Leftrightarrow (vi) \Leftrightarrow (vii) \Leftrightarrow (viii) \Leftrightarrow (ix).$$

Proof. We can easily see, from what is done above, that

$$(i) \Leftrightarrow (ii) \Leftarrow (iii) \Leftarrow (iv).$$

Then let us show, for example, how *(ii)* ensures the strong convergence *(iv)* which also guarantees the equivalence between the first four assertions. Going back to (2.3.9) and taking into account (2.4.2), we get for $v \in H_0^1(\Omega)$

$$\begin{aligned}
\varepsilon^2 \int_\Omega A_{11} \nabla_{X_1} R_\varepsilon^N \cdot \nabla_{X_1} v \, dx + \varepsilon \int_\Omega A_{12} \nabla_{X_2} R_\varepsilon^N \cdot \nabla_{X_1} v \, dx& \\
+ \varepsilon \int_\Omega A_{21} \nabla_{X_1} R_\varepsilon^N \cdot \nabla_{X_2} v \, dx + \int_\Omega A_{22} \nabla_{X_2} R_\varepsilon^N \cdot \nabla_{X_2} v \, dx& \\
= - \varepsilon^{N+2} \int_\Omega A_{11} \nabla_{X_1} u_N \cdot \nabla_{X_1} v \, dx.&
\end{aligned}$$

As above, we take as a test function $v = R_\varepsilon^N(.)\rho^2(X_1) \in H_0^1(\Omega)$, we obtain

$$\begin{aligned}
&\tfrac{1}{\varepsilon^{2N+2}} \int_\Omega \rho^2 A_\varepsilon \nabla R_\varepsilon^N \cdot \nabla R_\varepsilon^N \,\mathrm{d}x \\
&\qquad = -\tfrac{2}{\varepsilon^{2N}} \int_\Omega A_{11} \nabla_{X_1} R_\varepsilon^N \cdot (\nabla_{X_1}\rho)\rho R_\varepsilon^N \,\mathrm{d}x - \tfrac{2}{\varepsilon^{2N+1}} \int_\Omega A_{12} \nabla_{X_2} R_\varepsilon^N \cdot (\nabla_{X_1}\rho)\rho R_\varepsilon^N \,\mathrm{d}x \\
&\qquad - \tfrac{1}{\varepsilon^{N}} \int_\Omega \rho^2 A_{11} \nabla_{X_1} u_N \cdot \nabla_{X_1} R_\varepsilon^N \,\mathrm{d}x - \tfrac{2}{\varepsilon^{N}} \int_\Omega A_{11} \nabla_{X_1} u_N \cdot (\nabla_{X_1}\rho)\rho R_\varepsilon^N \,\mathrm{d}x. \qquad (2.4.5)
\end{aligned}$$

Applying the ellipticity assumption and taking into account the convergences appearing in (2.3.8), we get

$$\left|\nabla_{X_1} R_\varepsilon^N\right|_{2,\omega_1'\times\omega_2} = o\left(\varepsilon^N\right) \quad \text{and} \quad \left|\nabla_{X_2} R_\varepsilon^N\right|_{2,\omega_1'\times\omega_2} = o\left(\varepsilon^{N+1}\right). \qquad (2.4.6)$$

Note that, in the first two integrals of the right-hand side of (2.4.5), we use (2.3.8) to pass to the limit in a scalar product of weakly and strongly converging sequences. Then applying the Poincaré inequality in X_2-directions in the last identity we end up with *(iv)*. Similarly, we can also see that

$$(v) \Leftrightarrow (vi) \Leftrightarrow (vii) \Leftarrow (viii) \Leftarrow (ix).$$

Now, we have just to prove for example $(v) \Rightarrow (ix)$ to ensure the equivalence between the last five assertions and by consequence to end the proof. We proceed as in [19] or [31]. Without loss of generality, we assume that $\operatorname{dist}(\omega_1', \omega_1\backslash\omega_1'') > 1$ where $\omega_1' \subset\subset \omega_1'' \subset\subset \omega_1$. For ε small enough we can always construct a sequence $(D_k)_{0\le k\le m+1}$ with $m = \left[\frac{1}{\varepsilon}\right]$, of strictly increasing sets such that

$$\omega_1' =: D_0 \subset\subset D_1 \subset\subset \cdots \subset\subset D_m \subset\subset D_{m+1} := \omega_1'', \quad \operatorname{dist}(D_k, \omega_1\backslash D_{k+1}) \ge \varepsilon, \quad k = 0,\ldots,m.$$

Let $(\rho_k)_{0\le k\le m}$ be a family of functions depending only on X_1 such that $\operatorname{supp}\rho_k \subset \overline{D_{k+1}}$, $\rho_k = 1$ on D_k, $0 \le \rho_k \le 1$ and $|\nabla_{X_1}\rho_k|_{\infty,\mathbb{R}^p} \le \frac{\mathfrak{c}}{\varepsilon}$ for some constant $\mathfrak{c}$ independent of ε. Next, rewriting (2.3.10) for $N+2$, replacing ρ by ρ_k and taking into account the fact that $u_{N+1} = u_{N+2} = 0$, we get

$$\begin{aligned}
\int_\Omega \rho_k^2 A_\varepsilon \nabla R_\varepsilon^N \cdot \nabla R_\varepsilon^N \,\mathrm{d}x = -2\varepsilon^2 &\int_\Omega A_{11} \nabla_{X_1} R_\varepsilon^N \cdot (\nabla_{X_1}\rho_k)\rho_k R_\varepsilon^N \,\mathrm{d}x \\
&- 2\varepsilon \int_\Omega A_{12} \nabla_{X_2} R_\varepsilon^N \cdot (\nabla_{X_1}\rho_k)\rho_k R_\varepsilon^N \,\mathrm{d}x.
\end{aligned}$$

Applying the ellipticity assumption, the Young and the Poincaré inequalities yield

$$\begin{aligned}
\left|\nabla_\varepsilon R_\varepsilon^N\right|_{2,D_k\times\omega_2}^2 &\le \frac{\left(|A_{11}|_{\infty,\Omega}^2 + |A_{12}|_{\infty,\Omega}^2\right)\mathfrak{c}^2\kappa_{\omega_2}}{\lambda^2} \left|\nabla_\varepsilon R_\varepsilon^N\right|_{2,(D_{k+1}\backslash D_k)\times\omega_2}^2 \\
&= \kappa \left|\nabla_\varepsilon R_\varepsilon^N\right|_{2,D_{k+1}\times\omega_2}^2 - \kappa \left|\nabla_\varepsilon R_\varepsilon^N\right|_{2,D_k\times\omega_2}^2,
\end{aligned}$$

where $\nabla_\varepsilon \cdot = \begin{pmatrix} \varepsilon\nabla_{X_1}\cdot \\ \nabla_{X_2}\cdot \end{pmatrix}$. We used the identity $\nabla_{X_1}\rho_k = 0$ on D_k and $\varepsilon\left|\nabla_{X_1}\rho_k\right|_{\infty,\mathbb{R}^p} \le \mathfrak{c}$. Thus

$$\left|\nabla_\varepsilon R_\varepsilon^N\right|_{2,D_k\times\omega_2}^2 \le r \left|\nabla_\varepsilon R_\varepsilon^N\right|_{2,D_{k+1}\times\omega_2}^2,$$

where $r = \frac{\kappa}{1+\kappa} < 1$, $\kappa = \frac{\left(|A_{11}|_{\infty,\Omega}^2 + |A_{12}|_{\infty,\Omega}^2\right)\mathfrak{c}^2\kappa_{\omega_2}}{\lambda^2}$. Iterating this inequality for $k = 0,\ldots,m$, we derive

$$\left|\nabla_\varepsilon R_\varepsilon^N\right|_{2,\omega_1'\times\omega_2}^2 \le r^{[\frac{1}{\varepsilon}]+1} \left|\nabla_\varepsilon R_\varepsilon^N\right|_{2,\omega_1''\times\omega_2}^2,$$

i.e.

$$\left|\nabla_\varepsilon R_\varepsilon^N\right|_{2,\omega_1'\times\omega_2}^2 \le \kappa \exp\left(\tfrac{\ln r}{\varepsilon}\right).$$

This completes the proof by setting $\eta = -\ln r$. □

Remark 2.5. *It is clear from (2.4.6) that the assertion (iii) also implies an asymptotic convergence faster than ε^N ensured in a standard Sobolev space, as $\varepsilon \longrightarrow 0$, i.e.*

$$u_\varepsilon - \left(u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N\right) = o\left(\varepsilon^N\right) \quad \text{in } H^1\left(\omega_1' \times \omega_2\right) \quad \text{for any } \omega_1' \subset\subset \omega_1.$$

Remark 2.6. *Keeping the same results of the above theorem, the weak convergence (viii) can be weakened as follows*

$$\tfrac{1}{\varepsilon^{N+2}} R_\varepsilon^N \rightharpoonup 0 \quad \text{in } L^2\left(\omega_1' \times \omega_2\right) \quad \text{for any } \omega_1' \subset\subset \omega_1, \tag{2.4.7}$$

if we assume more smoothness as

$$\left(A_{21}^T\right)' = \left(\partial_{x_j} a_{ij}\right)^T_{j=1,\ldots,p,\ i=p+1,\ldots,\mathrm{n}}, \ \left(A_{22}^T\right)' = \left(\partial_{x_j} a_{ij}\right)^T_{j,i=p+1,\ldots,\mathrm{n}} \in L^\infty(\Omega).$$

Indeed, thanks to the Poincaré inequality, we deduce, from (viii), that the sequence $\frac{1}{\varepsilon^{N+2}} R_\varepsilon^N$ is bounded in $L^2\left(\omega_1' \times \omega_2\right)$. Then for $\varphi \in \mathcal{D}\left(\omega_1' \times \omega_2\right)$ extended by zero on $\omega_1' \times \mathbb{R}^{\mathrm{n}-p}$, we have

$$\begin{aligned}\int_\Omega \tfrac{1}{\varepsilon^{N+2}} R_\varepsilon^N \varphi \mathrm{d}x &= \int_\Omega \tfrac{1}{\varepsilon^{N+2}} R_\varepsilon^N \partial_{x_i} \int_{-\infty}^{x_i} \varphi(\ldots, s, \ldots) \, \mathrm{d}s\mathrm{d}x \\ &= -\int_\Omega \tfrac{1}{\varepsilon^{N+2}} \partial_{x_i} R_\varepsilon^N \int_{-\infty}^{x_i} \varphi(\ldots, s, \ldots) \, \mathrm{d}s\mathrm{d}x \to 0,\end{aligned}$$

where $i = p+1, \ldots, \mathrm{n}$. This means that (viii) implies (3.4.18). It is now enough to show, for instance, that (3.4.18) implies (2.4.2) and (2.4.4). Of course, (2.4.2) is clearly guaranteed since (3.4.18) implies (iii). Now to achieve the second integral identity, we rewrite (2.3.9) as

$$\begin{aligned}&\varepsilon^{-N} \int_\Omega A_{11} \nabla_{X_1} R_\varepsilon^N \cdot \nabla_{X_1} v \, \mathrm{d}x + \varepsilon^{-N-1} \int_\Omega A_{12} \nabla_{X_2} R_\varepsilon^N \cdot \nabla_{X_1} v \, \mathrm{d}x \\ &- \varepsilon \int_\Omega \varepsilon^{-N-2} R_\varepsilon^N \nabla_{X_1} \cdot \left(A_{21}^T \nabla_{X_2} v\right) \mathrm{d}x - \int_\Omega \varepsilon^{-N-2} R_\varepsilon^N \nabla_{X_2} \cdot \left(A_{22}^T \nabla_{X_2} v\right) \mathrm{d}x \\ &= -\int_\Omega A_{11} \nabla_{X_1} u_N \cdot \nabla_{X_1} v \, \mathrm{d}x\end{aligned}$$

where "$.^T$" denotes the transpose of the matrix. Of course, (2.4.2) is taken into account and the integrals are well defined since $\left(A_{21}^T\right)', \left(A_{22}^T\right)' \in L^\infty(\Omega)$. Finally, we take into account (2.3.8) and (iv) (which is implied by (2.4.2)) in the first line and (3.4.18) in the second line to pass to the limit in the above identity and end up with (2.4.4).

Remark 2.7. *Assuming (i), it hints at a sharper convergence than what is mentioned in (iv) since we have $R_\varepsilon^N = R_\varepsilon^{N+1}$. This allows to think about*

$$u_\varepsilon - \left(u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N\right) = O(\varepsilon^{N+2}) \quad \text{in } \mathcal{V}\left(\omega_1' \times \omega_2\right), \tag{2.4.8}$$

for any $\omega_1' \subset\subset \omega_1$. Here also the smoothness of the data may improve this as it is expected and done above many times. However, if we assume in addition that (2.3.5) is satisfied for $N+2$, the above theorem remains true if we replace (iv) by (2.4.8). Always, thanks to (2.4.8), we also have

$$\tfrac{1}{\varepsilon^{N+1}} \nabla_{X_1} \left[u_\varepsilon - \left(u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N\right)\right] \rightharpoonup 0 \quad \text{weakly in } L^2\left(\omega_1' \times \omega_2\right),$$

for any $\omega_1' \subset\subset \omega_1$, and moreover if (2.3.5) is satisfied for $N+3$, it holds

$$u_\varepsilon - \left(u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N\right) = O(\varepsilon^{N+2}) \quad \text{in } H^1\left(\omega_1' \times \omega_2\right)$$

for any $\omega_1' \subset\subset \omega_1$.

In [13], an exponential rate of asymptotic convergence is shown under the condition (2.4.2) satisfied for $N = 0$ and the independence of u_0 of X_1 which implies (2.4.4) for $N = 0$. They were sufficient conditions, however, we are here faced with sufficient and necessary conditions. In the following we formulate the above analysis in terms of data.

2.4.2 Diagonal block matrices

We here assume that

$$A_{12} = A_{21}^T = 0. \tag{2.4.9}$$

It is clear that (2.4.2) holds for $N = 0$ which implies that $u_1 = 0$. Then the condition (2.4.2) written for $N = 2$ is reduced to

$$\nabla_{X_1} \cdot A_{12} \nabla_{X_2} u_2 + \nabla_{X_2} \cdot A_{21} \nabla_{X_1} u_2 = 0 \quad \text{in } \Omega,$$

which is also held thanks to (2.4.9) and by consequence $u_3 = 0$. Following the same argument we end up with

$$u_{2k+1} = 0, \quad \forall k \in \mathbb{N},$$

and consequently the even order coefficients u_{2k}, $k \in \mathbb{N} \setminus \{0\}$ are solutions to the system

$$\begin{cases} -\nabla_{X_2} \cdot A_{22} \nabla_{X_2} u_{2k}(X_1, .) = \nabla_{X_1} \cdot A_{11} \nabla_{X_1} u_{2k-2}(X_1, .) & \text{in } \omega_2, \\ u_{2k}(X_1, .) \in H_0^1(\omega_2), \end{cases}$$

with assumptions (2.3.14) for $N = 2k$ $(k > 0)$ reduced to

$$D_{X_1}^{N-2} A_{11}', \quad D_{X_1}^N A_{22} \in L^\infty(\Omega), \quad D_{X_1}^N f \in L^2(\Omega). \tag{2.4.10}$$

Then, under the assumptions (2.1.2), (2.4.9), (2.4.10) for $N = 2k+1$ $(k > 0)$ and thanks to Theorem 2.3, we have for any $\omega_1' \subset\subset \omega_1$, as $\varepsilon \to 0$,

$$u_\varepsilon - (u_0 + \varepsilon^2 u_2 + \cdots + \varepsilon^{2k} u_{2k}) = o(\varepsilon^{2k+1}) \quad \text{in } \mathcal{V}\left(\omega_1' \times \omega_2\right),$$

$$\nabla_{X_1} \left[u_\varepsilon - (u_0 + \varepsilon^2 u_2 + \cdots + \varepsilon^{2k} u_{2k})\right] = o(\varepsilon^{2k}) \quad \text{in } L^2\left(\omega_1' \times \omega_2\right).$$

Moreover if (2.4.10) is assumed to be held for $N = 2k+2$ $(k > 0)$ (respectively for $N = 2k+3$), then in view of Remark 2.7, we have for any $\omega_1' \subset\subset \omega_1$,

$$u_\varepsilon - (u_0 + \varepsilon^2 u_2 + \cdots + \varepsilon^{2k} u_{2k}) = O(\varepsilon^{2k+2}) \quad \text{in } \mathcal{V}\left(\omega_1' \times \omega_2\right),$$

$$\tfrac{1}{\varepsilon^{2k+1}} \nabla_{X_1} \left[u_\varepsilon - (u_0 + \varepsilon^2 u_2 + \cdots + \varepsilon^{2k} u_{2k})\right] \rightharpoonup 0 \quad \text{in } L^2\left(\omega_1' \times \omega_2\right),$$

respectively

$$u_\varepsilon - (u_0 + \varepsilon^2 u_2 + \cdots + \varepsilon^{2k} u_{2k}) = O(\varepsilon^{2k+2}) \quad \text{in } H^1\left(\omega_1' \times \omega_2\right).$$

Remark 2.8. *Since we have*

$$\nabla_{X_1} \cdot A_{12} \nabla_{X_2} v + \nabla_{X_2} \cdot A_{21} \nabla_{X_1} v = 0, \quad \forall v \in \mathcal{D}\left(\Omega\right),$$

if the matrix A satisfies

$$A_{12} = -A_{21}^T, \quad A_{12}' = 0,$$

the matrices A and $\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$ define the same problem (2.1.4) and by consequence the subsection result maintains for the above case.

2.4.3 Polynomial data

In the present subsection we consider a polynomial structure of the data as follows. We suppose that the matrix A still satisfied the hypotheses (2.1.1), (2.1.2) and moreover

$$A = \begin{pmatrix} A_{11}(X_1, X_2) & A_{12}(X_2) \\ A_{21}(X_2) & A_{22}(X_2) \end{pmatrix} \tag{2.4.11}$$

where A_{11} is a first order polynomial in X_1 and f is a polynomial of degree $k \in \mathbb{N} \setminus \{0\}$ in X_1, i.e.

$$f(X_1, X_2) = \sum_{|\alpha| \leq k} X_1^{\alpha} f_{\alpha}(X_2), \quad f_{\alpha} \in L^2(\omega_2), \tag{2.4.12}$$

where $X_1^{\alpha} := x_1^{\alpha_1} \cdots x_p^{\alpha_p}$, for some multi-index $\alpha \in \mathbb{N}^p$. Let u_0^{α} be the unique solution to the following elliptic problem,

$$\begin{cases} -\nabla_{X_2} \cdot A_{22} \nabla_{X_2} u_0^{\alpha} = f_{\alpha} & \text{in } \omega_2, \\ u_0^{\alpha} = 0 & \text{on } \partial\omega_2. \end{cases}$$

Thanks to the linearity of the problems, the solution of problem (2.2.1) can be expressed as

$$u_0(X_1, X_2) = \sum_{|\alpha| \leq k} X_1^{\alpha} u_0^{\alpha}(X_2), \tag{2.4.13}$$

which is also a polynomial of degree k in X_1. This means that if the right-hand side of the problem (2.2.1) is a polynomial of degree k in X_1 the solution is also a polynomial of the same degree in X_1. Going back now to problem (2.2.6), for u_0 defined in (2.4.13), we deduce that u_1 is a polynomial of degree at most $k-1$ in X_1. Next regarding (2.2.7) rewritten for $N = 2$, u_2 is a polynomial of degree at most $k-1$ also, i.e. we cannot guarantee the reduction of the polynomial degree in this step. Applying again the same argument we can see that u_3 and u_4 are polynomials of degree at most $k-2$ in X_1. For a general statement we can show that u_{2m-1} and u_{2m} are polynomials of degree at most $k-m$ in X_1 for $m = 1, \dots, k$. This leads to end with two coefficients u_{2k-1} and u_{2k} of (2.2.4) that are independent of X_1. Consequently it follows that $u_{2k+1} = u_{2k+2} = 0$ and thanks to Theorem 2.3 we end up with an exponential rate of asymptotic convergence. To summarize we have

Corollary 2.3. *Under the assumptions (2.1.1), (2.1.2), (2.4.11) and (2.4.12), there exists a constant $\eta > 0$ independent of $\varepsilon > 0$ such that*

$$u_{\varepsilon} - (u_0 + \varepsilon u_1 + \cdots + \varepsilon^{2k} u_{2k}) = O\big(\exp(\tfrac{-\eta}{\varepsilon})\big) \quad \text{in } H^1(\omega_1' \times \omega_2),$$

for any $\omega_1' \subset\subset \omega_1$.

Remark 2.9. *If the matrix A is independent of X_1 the above development will be reduced as*

$$u_{\varepsilon} - (u_0 + \varepsilon u_1 + \cdots + \varepsilon^{k} u_{k}) = O\big(\exp(\tfrac{-\eta}{\varepsilon})\big) \quad \text{in } H^1(\omega_1' \times \omega_2).$$

Remark 2.10. *In the case where the function f is independent of X_1 and the matrix A satisfies (2.1.1), (2.1.2) and*

$$A = \begin{pmatrix} A_{11}(X_1, X_2) & A_{12}(X_1, X_2) \\ A_{21}(X_1, X_2) & A_{22}(X_2) \end{pmatrix}$$

we can get the following exponential rate of asymptotic convergence, when $\varepsilon \longrightarrow 0$,

$$u_{\varepsilon} - u_0 = O\big(\exp(\tfrac{-\eta}{\varepsilon})\big) \quad \text{in } H^1(\omega_1' \times \omega_2),$$

for some constant $\eta > 0$ independent of $\varepsilon > 0$ iff the limit u_0, which is independent of X_1, satisfies

$$\nabla_{X_1} \cdot A_{12} \nabla_{X_2} u_0 = 0 \quad \text{in } \Omega.$$

The last hypothesis holds for example if $A_{12}' = 0$.

2.4.4 Special example

In the last two cases as well as in many works in the bibliography, some symmetries properties are always the unique concrete examples for which the significant improvement (as the exponential rate of asymptotic convergence) takes place. The following example shows that the exponential rate of asymptotic convergence can be maintained for a different class of problems. We take,

$$\Omega = \omega_1 \times \omega_2 = (0,1) \times (0,\pi)\,, \quad X_1 = x_1, \quad X_2 = x_2,$$

$$A = \begin{pmatrix} \dfrac{me^{-x_1}\cos\frac{x_2}{4}}{\cos\left(\frac{1}{2}e^{x_1}\sin x_2\right)} & \dfrac{x_1 e^{-x_1}}{\cos\left(\frac{1}{2}e^{x_1}\sin x_2\right)} \\ \dfrac{-e^{-x_1}}{\cos\left(\frac{1}{2}e^{x_1}\sin x_2\right)} & 2m\left(1+x_1+x_2\right) \end{pmatrix},$$

$$A_{11} = \frac{me^{-x_1}\cos\frac{x_2}{4}}{\cos\left(\frac{1}{2}e^{x_1}\sin x_2\right)}, \quad A_{12} = \frac{x_1 e^{-x_1}}{\cos\left(\frac{1}{2}e^{x_1}\sin x_2\right)}, \quad A_{21} = \frac{-e^{-x_1}}{\cos\left(\frac{1}{2}e^{x_1}\sin x_2\right)},$$

$$A_{22} = 2m\Big(1+x_1+x_2\Big).$$

$$\begin{aligned} f = me^{x_1}\Bigg\{&\Big[\Big(1+x_1+x_2\Big)\sin x_2 - \cos x_2\Big]\cos\left(\tfrac{1}{2}e^{x_1}\sin x_2\right) \\ &+\tfrac{1}{2}e^{x_1}\Big(1+x_1+x_2\Big)\cos^2 x_2 \sin\left(\tfrac{1}{2}e^{x_1}\sin x_2\right)\Bigg\}. \end{aligned}$$

For a sufficient large positive integer m, the matrix A satisfies the ellipticity assumption (2.1.2) and of course there exists a unique $u_\varepsilon \in H_0^1(\Omega)$ solution to (2.1.4). The limit solution to (2.2.1) can be found explicitly, i.e.

$$u_0 = \sin\left(\tfrac{1}{2}e^{x_1}\sin x_2\right) \in H_0^1(\omega_2)\,, \quad \text{for a.e. } x_1 \in (0,1)\,.$$

For this choice the identities (2.4.2) and (2.4.4) are fulfilled for $N = 0$, and thanks to Theorem 2.3 we end up with the exponential rate of asymptotic convergence

$$u_\varepsilon - u_0 = O\left(\exp\left(\tfrac{-\eta}{\varepsilon}\right)\right) \quad \text{in } H^1(\omega_1' \times \omega_2)$$

for some $\eta > 0$.

Chapter 3

Composite Asymptotic Expansion

The rate of asymptotic convergence in the previous chapter is obtained and showed far away from the boundary layers. In the present chapter we will give an equivalent asymptotic expansion of the boundary layer functions to get a composite development on the whole domain for the same type of problems.

3.1 Position of the problem

Let now Ω be a bounded open cylinder in $\mathbb{R}^{\mathrm{n}}$, i.e. we take

$$\Omega = (-1,1)^2 \times \omega_2,$$

where ω_2 is a bounded Lipschitz domain of $\mathbb{R}^{\mathrm{n}-2}$. We denote by $x = (x_1, \dots, x_{\mathrm{n}}) = (X_1, X_2)$ the point of $\mathbb{R}^{\mathrm{n}}$ where

$$X_1 = (x_1, x_2), \quad X_2 = (x_3, \dots, x_{\mathrm{n}}),$$

i.e. we split the coordinates into two parts. With this notation we set

$$\nabla u = (\partial_{x_1} u, \dots, \partial_{x_{\mathrm{n}}} u)^T = \begin{pmatrix} \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix},$$

where

$$\nabla_{X_1} u = (\partial_{x_1} u, \partial_{x_2} u)^T, \quad \nabla_{X_2} u = (\partial_{x_3} u, \dots, \partial_{x_{\mathrm{n}}} u)^T.$$

Let $A = (a_{ij}(x))$ be a $\mathrm{n} \times \mathrm{n}$ matrix such that

$$a_{ij} \in L^\infty(\Omega), \qquad \forall i, j = 1, \dots, \mathrm{n} \tag{3.1.1}$$

and for some constant $\lambda > 0$, it satisfies the ellipticity condition

$$A\zeta \cdot \zeta \geq \lambda |\zeta|^2 \qquad \forall \zeta \in \mathbb{R}^{\mathrm{n}}, \qquad \text{a.e. on } \Omega. \tag{3.1.2}$$

We decompose A into four blocks by writing

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_{11} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A_{12} = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} c_1 & c_2 \end{pmatrix}$$

where A_{22} is a $(\mathrm{n}-2) \times (\mathrm{n}-2)$ matrix and the lines of A_{12} and the columns of A_{21} are defined as

$$l_i = (a_{i3} \dots a_{i\mathrm{n}}), \quad c_i = (a_{3i}, \dots, a_{\mathrm{n}i})^T, \quad i = 1, 2.$$

We also set

$$\check{A} = \begin{pmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{pmatrix}.$$

We then define for $\varepsilon > 0$, the perturbed matrix and gradient, i.e.

$$A_\varepsilon = \begin{pmatrix} \varepsilon^2 A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{pmatrix}, \quad \nabla_\varepsilon u = \begin{pmatrix} \varepsilon \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix}.$$

Therefore we have, for a.e. $x \in \Omega$ and every $\zeta \in \mathbb{R}^{\mathrm{n}}$

$$A_\varepsilon \zeta \cdot \zeta = A\zeta_\varepsilon \cdot \zeta_\varepsilon \geq \lambda |\zeta_\varepsilon|^2 = \lambda \left\{ \varepsilon^2 \left|\bar{\zeta}_1\right|^2 + \left|\bar{\zeta}_2\right|^2 \right\},$$

where we set $\zeta = \begin{pmatrix} \bar{\zeta}_1 \\ \bar{\zeta}_2 \end{pmatrix}$, $\bar{\zeta}_1 = (\zeta_1, \zeta_2)^T$, $\bar{\zeta}_2 = (\zeta_3, \dots, \zeta_{\mathrm{n}})^T$ and $\zeta_\varepsilon = \left(\varepsilon\bar{\zeta}_1, \bar{\zeta}_2\right)^T$. Thus we have

$$\begin{aligned} A_\varepsilon \zeta \cdot \zeta &\geq \lambda \left(\varepsilon^2 \wedge 1\right) |\zeta|^2 \qquad \forall \zeta \in \mathbb{R}^{\mathrm{n}}, \qquad \text{a.e. on } \Omega, \\ \check{A}\zeta \cdot \zeta &\geq \lambda |\zeta|^2 \qquad \forall \zeta \in \mathbb{R}^{\mathrm{n}}, \qquad \text{a.e. on } \Omega, \\ A_{22}\bar{\zeta}_2 \cdot \bar{\zeta}_2 &\geq \lambda \left|\bar{\zeta}_2\right|^2 \qquad \forall \bar{\zeta}_2 \in \mathbb{R}^{\mathrm{n}-2}, \qquad \text{a.e. on } \Omega. \end{aligned} \tag{3.1.3}$$

It follows that A_ε, $\check{A}$ and A_{22} are positive definite matrices. For a function

$$f \in L^2(\Omega), \tag{3.1.4}$$

we can ensure the existence and the uniqueness of a weak solution u_ε to

$$\begin{cases} -\nabla \cdot A_\varepsilon \nabla u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

in the following sense

$$\begin{cases} u_\varepsilon \in H_0^1(\Omega), \\ \varepsilon^2 \displaystyle\int_\Omega A_{11} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v \,\mathrm{d}x \quad + \varepsilon \displaystyle\int_\Omega A_{12} \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_1} v \,\mathrm{d}x \\ \qquad + \varepsilon \displaystyle\int_\Omega A_{21} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} v \,\mathrm{d}x \\ \qquad + \displaystyle\int_\Omega A_{22} \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x, \quad \forall v \in H_0^1(\Omega). \end{cases} \tag{3.1.5}$$

As it is shown in [13] that the limit u_0 of u_ε is the unique solution, for a.e. $X_1 \in (-1,1)^2$, to the following lower dimension problem

$$\begin{cases} -\nabla_{X_2} \cdot A_{22} \nabla_{X_2} u_0(X_1, .) = f(X_1, .) & \text{in } \omega_2, \\ u_0(X_1, .) = 0 & \text{on } \partial\omega_2, \end{cases}$$

in the following sense

$$\begin{cases} u_0(X_1, .) \in H_0^1(\omega_2), \\ \displaystyle\int_{\omega_2} A_{22} \nabla_{X_2} u_0(X_1, .) \cdot \nabla_{X_2} v \,\mathrm{d}X_2 = \int_{\omega_2} f(X_1, .) v \,\mathrm{d}X_2, \quad \forall v \in H_0^1(\omega_2). \end{cases} \tag{3.1.6}$$

Note that X_1 here plays a parameter role and the existence and the uniqueness of u_0, in the Sobolev space $H_0^1(\omega_2)$, is followed from the Lax–Milgram theorem, since for a.e. $X_1 \in (-1,1)^2$, $f(X_1, .) \in L^2(\omega_2)$ and we have (3.1.3).

The convergences of u_ε and $\nabla_{X_2} u_\varepsilon$ hold on the whole domain Ω with respect to the L^2–norm. However, the convergence $u_\varepsilon \to u_0$ ceases to be hold in ∇_ε–norm except if $u_0 \in H_0^1(\Omega)$ which is not the case in general because of the occurring boundary layer near $\left\{\partial(-1,1)^2\right\} \times \omega_2$. Nevertheless, if u_0 is smooth (belongs to $H^1(\Omega)$) we recover the convergence for the ∇_ε–norm or even for the H^1–norm but for regions located far away from the boundary layer $\left\{\partial(-1,1)^2\right\} \times \omega_2$.

In order to get a complete description of the asymptotic behaviour of the solution u_ε on the whole domain Ω, we focus our study near the boundary layers. This leads to introduce the boundary layer functions, i.e. we look for a function θ_ε, simply defined, such that

$$u_\varepsilon - \theta_\varepsilon \to u_0 \qquad \text{for the } \nabla_\varepsilon\text{–norm on the whole domain } \Omega.$$

In this context and if X_1 is one dimensional coordinate a construction of these boundary layer functions is done for a mixed Newmann–Dirichlet conditions in [11] (see also [16]).

3.2 First composite asymptotic approximation

From what is done above, u_0 is a first regular asymptotic approximation for u_ε. In this section we will deal with the first composite asymptotic approximation on the whole domain for u_ε.

3.2.1 Formal boundary layer functions

Of course we will follow the same idea used in the unidimensional case (see [11]). This leads to correct near some parts of $\partial\Omega$ of the lower dimension n $-$ 1, i.e. near the faces $\{1\} \times (-1,1) \times \omega_2$, $\{-1\} \times (-1,1) \times \omega_2$, $(-1,1) \times \{1\} \times \omega_2$ and $(-1,1) \times \{-1\} \times \omega_2$. It is clear that a such correction operates twice near the four parts of $\partial\Omega$ of a lower dimension n $-$ 2, i.e. near the edges $\{i\} \times \{j\} \times \omega_2$ such that $i, j = \pm 1$ (or simply near the intersection of every two adjacent faces defined above). This requires a new correction in the neighbourhood of these parts. Of course, the face and the edge notions have sense in $\mathbb{R}^3$, here we adapt these notions for higher dimension spaces for the parts of $\partial\Omega$ of lower dimensions n $-$ 1 and n $-$ 2 respectively. Then, it seems that the boundary layer functions (or *correctors*) will be defined in two steps according to the dimension of X_1–directions (here $p = 2$). In general this construction will depend on the smoothness of the domain in some sense.

Formal correctors near the faces

Let us begin by the first correctors level, i.e. near the faces $x_1 = \pm 1$ or $x_2 = \pm 1$.

Case $\alpha = (1,0)$. We start by the formal corrector operating near the face $x_1 = 1$. We take $\alpha = (1,0)$ which means that we are only concerned with the face $x_1 = 1$. We define the function $w^\alpha(., x_2, .)$ as the weak solution, for a.e. $x_2 \in (-1,1)$, to

$$\left\{ \begin{array}{l} \nabla \cdot \mathfrak{A}^\alpha \nabla w^\alpha = 0 \quad \text{in } I_0^\alpha, \\ w^\alpha(., x_2, .)\,|_{(0,+\infty)\times\partial\omega_2} = 0, \quad w^\alpha(., x_2, .)\,|_{x_1=0} = -u_0(., x_2, .)\,|_{x_1=1} \end{array} \right. \tag{3.2.1}$$

where $I_0^\alpha = (0,+\infty) \times \omega_2$, $\nabla := \nabla^\alpha = \binom{\partial_{x_1}}{\nabla_{X_2}}$ is here the gradient in $\mathbb{R}_{x_1} \times \omega_2$ and $\mathfrak{A}^\alpha$ is a generic square matrix depends on the choice of α. Here for $\alpha = (1,0)$, we set

$$\Omega_\kappa^\alpha = (0,\kappa) \times (-1,1) \times \omega_2, \quad \kappa > 0$$

and $\mathfrak{A}^\alpha$ is defined on $\Omega^\alpha_{\frac{1}{\varepsilon}}$, as a square sub-matrix obtained from $\breve{A}$ by removing the second line and the second column (since $\alpha_2 = 0$ here), then using the following change of variable

$$\varsigma_\alpha : \mathbb{R}_{x_1} \to \mathbb{R}_{x_1},\ x_1 \longmapsto \varsigma_\alpha(x_1) = \frac{1-x_1}{\varepsilon}.$$

This also means that ς_α only operates on x_1 if there is more than one component, for example $\varsigma_\alpha(x) = \left(\frac{1-x_1}{\varepsilon}, x_2, X_2\right)$. Outside $\Omega^\alpha_{\frac{1}{\varepsilon}}$, $\mathfrak{A}^\alpha$ is the identity matrix. That is to say

$$\mathfrak{A}^\alpha = \begin{pmatrix} a_{11}\circ\varsigma_\alpha^{-1}(x) & -l_1\circ\varsigma_\alpha^{-1}(x) \\ -c_1\circ\varsigma_\alpha^{-1}(x) & A_{22}\circ\varsigma_\alpha^{-1}(x) \end{pmatrix} \quad \text{a.e. } x\in\Omega^\alpha_{\frac{1}{\varepsilon}}, \quad \mathfrak{A}^\alpha = \mathrm{Id} \quad \text{a.e. } x\in S^\alpha_0\backslash\Omega^\alpha_{\frac{1}{\varepsilon}}$$

where $\varsigma_\alpha^{-1}(x) = (1-\varepsilon x_1, x_2, X_2)$ and $S^\alpha_0 = (0,+\infty)\times(-1,1)\times\omega_2$. Then, we set

$$\theta^\alpha(x;\varepsilon) = w^\alpha\circ\varsigma_\alpha(x) = w^\alpha\Big(\frac{1-x_1}{\varepsilon}, x_2, X_2\Big) \quad \text{a.e. } x\in\Omega. \tag{3.2.2}$$

Case $\alpha = (-1,0)$. Near the face $x_1 = -1$, we take $\alpha = (-1,0)$. We define the function $w^\alpha(.,x_2,.)$ as the weak solution, for a.e. $x_2\in(-1,1)$, to

$$\begin{cases} \nabla\cdot\mathfrak{A}^\alpha\nabla w^\alpha = 0 \quad \text{in } I^\alpha_0, \\ w^\alpha(.,x_2,.)\mid_{(-\infty,0)\times\partial\omega_2} = 0, \quad w^\alpha(.,x_2,.)\mid_{x_1=0} = -u_0(.,x_2,.)\mid_{x_1=-1} \end{cases} \tag{3.2.3}$$

where $I^\alpha_0 = (-\infty,0)\times\omega_2$, $\nabla := \nabla^\alpha$ is also the gradient in $\mathbb{R}_{x_1}\times\omega_2$ and $\mathfrak{A}^\alpha$ is defined as above and just here we apply the corresponding change of variable

$$\varsigma_\alpha : \mathbb{R}_{x_1} \to \mathbb{R}_{x_1},\ x_1 \longmapsto \varsigma_\alpha(x_1) = \frac{-1-x_1}{\varepsilon},$$

i.e.

$$\mathfrak{A}^\alpha = \begin{pmatrix} a_{11}\circ\varsigma_\alpha^{-1}(x) & -l_1\circ\varsigma_\alpha^{-1}(x) \\ -c_1\circ\varsigma_\alpha^{-1}(x) & A_{22}\circ\varsigma_\alpha^{-1}(x) \end{pmatrix} \quad \text{a.e. } x\in\Omega^\alpha_{\frac{1}{\varepsilon}}, \quad \mathfrak{A}^\alpha = \mathrm{Id} \quad \text{a.e. } x\in S^\alpha_0\backslash\Omega^\alpha_{\frac{1}{\varepsilon}}$$

where $\varsigma_\alpha^{-1}(x) = (-1-\varepsilon x_1, x_2, X_2)$ and

$$\Omega^\alpha_\kappa = (-\kappa,0)\times(-1,1)\times\omega_2, \quad S^\alpha_0 = (-\infty,0)\times(-1,1)\times\omega_2.$$

Then, we set

$$\theta^\alpha(x;\varepsilon) = w^\alpha\circ\varsigma_\alpha(x) = w^\alpha\Big(\frac{-1-x_1}{\varepsilon}, x_2, X_2\Big) \quad \text{a.e. } x\in\Omega. \tag{3.2.4}$$

Case $\alpha = (0,1)$. Then, near the face $x_2 = 1$ we take $\alpha = (0,1)$. As in the previous cases, we define the function $w^\alpha(x_1,.)$ as the weak solution, for a.e. $x_1\in(-1,1)$, to

$$\begin{cases} \nabla\cdot\mathfrak{A}^\alpha\nabla w^\alpha = 0 \quad \text{in } I^\alpha_0, \\ w^\alpha(x_1,.)\mid_{(0,+\infty)\times\partial\omega_2} = 0, \quad w^\alpha(x_1,.)\mid_{x_2=0} = -u_0(x_1,.)\mid_{x_2=1} \end{cases} \tag{3.2.5}$$

where the notation is adapted here for $\alpha = (0,1)$. For example $I^\alpha_0 \subset \mathbb{R}_{x_2}\times\omega_2$ is defined as for the case $\alpha = (1,0)$, ∇ is the gradient in $\mathbb{R}_{x_2}\times\omega_2$ and $\mathfrak{A}^\alpha$ is defined as above we just remove this time the fist line and the first column (since $\alpha_1 = 0$ here) then we make the following change of variable

$$\varsigma_\alpha : \mathbb{R}_{x_2} \to \mathbb{R}_{x_2},\ x_2 \longmapsto \varsigma_\alpha(x_2) = \frac{1-x_2}{\varepsilon},$$

i.e.

$$\mathfrak{A}^\alpha = \begin{pmatrix} a_{22}\circ\varsigma_\alpha^{-1}(x) & -l_2\circ\varsigma_\alpha^{-1}(x) \\ -c_2\circ\varsigma_\alpha^{-1}(x) & A_{22}\circ\varsigma_\alpha^{-1}(x) \end{pmatrix} \quad \text{a.e. } x\in\Omega^\alpha_{\frac{1}{\varepsilon}}, \quad \mathfrak{A}^\alpha = \mathrm{Id} \quad \text{a.e. } x\in S_0^\alpha\backslash\Omega^\alpha_{\frac{1}{\varepsilon}}$$

where $\varsigma_\alpha^{-1}(x) = (x_1, 1-\varepsilon x_2, X_2)$, $\Omega_\kappa^\alpha = (-1,1)\times(0,\kappa)\times\omega_2$ and $S_0^\alpha = (-1,1)\times(0,+\infty)\times\omega_2$. Here also we take

$$\theta^\alpha(x;\varepsilon) = w^\alpha\circ\varsigma_\alpha(x) = w^\alpha(x_1, \frac{1-x_2}{\varepsilon}, X_2) \quad \text{a.e. } x\in\Omega. \tag{3.2.6}$$

Case $\alpha = (0,-1)$. Near the face $x_2=-1$, we take $\alpha=(0,-1)$. As in the previous cases, we define the function $w^\alpha(x_1,.)$ as the weak solution, for a.e. $x_1\in(-1,1)$, to

$$\begin{cases} \nabla\cdot\mathfrak{A}^\alpha\nabla w^\alpha = 0 \quad \text{in } I_0^\alpha, \\ w^\alpha(x_1,.)\,|_{(-\infty,0)\times\partial\omega_2} = 0, \quad w^\alpha(x_1,.)\,|_{x_2=0} = -u_0(x_1,.)\,|_{x_2=-1} \end{cases} \tag{3.2.7}$$

where the notation is adapted here for $\alpha=(0,-1)$. For example $I_0^\alpha\subset\mathbb{R}_{x_2}\times\omega_2$ is defined as for the case $\alpha=(-1,0)$, ∇ is the gradient in $\mathbb{R}_{x_2}\times\omega_2$ and $\mathfrak{A}^\alpha$ is defined as above we have just to make now the change of variable

$$\varsigma_\alpha : \mathbb{R}_{x_2}\to\mathbb{R}_{x_2},\ x_2\longmapsto\varsigma_\alpha(x_2) = \frac{-1-x_2}{\varepsilon},$$

i.e.

$$\mathfrak{A}^\alpha = \begin{pmatrix} a_{22}\circ\varsigma_\alpha^{-1}(x) & -l_2\circ\varsigma_\alpha^{-1}(x) \\ -c_2\circ\varsigma_\alpha^{-1}(x) & A_{22}\circ\varsigma_\alpha^{-1}(x) \end{pmatrix} \quad \text{a.e. } x\in\Omega^\alpha_{\frac{1}{\varepsilon}}, \quad \mathfrak{A}^\alpha = \mathrm{Id} \quad \text{a.e. } x\in S_0^\alpha\backslash\Omega^\alpha_{\frac{1}{\varepsilon}}$$

where $\varsigma_\alpha^{-1}(x) = (x_1, -1-\varepsilon x_2, X_2)$, $\Omega_\kappa^\alpha = (-1,1)\times(-\kappa,0)\times\omega_2$ and $S_0^\alpha = (-1,1)\times(-\infty,0)\times\omega_2$. Here also we set

$$\theta^\alpha(x;\varepsilon) = w^\alpha\circ\varsigma_\alpha(x) = w^\alpha(x_1, \frac{-1-x_2}{\varepsilon}, X_2) \quad \text{a.e. } x\in\Omega. \tag{3.2.8}$$

The existence of the functions w^α in the Sobolev spaces $H^1(I_0^\alpha)$ for $\alpha=(\pm1,0)$ or $(0,\pm1)$ is ensured by the Lax–Milgram theorem. For example, we easily see that

$$w^{(1,0)}(x) + \rho(x_1)\,u_0(1-x_1,x_2,X_2)\in H^1\left(I_0^{(1,0)}\right) \text{ for a.e. } x_2\in(-1,1),$$

(where $\rho(0)=1, \rho(x)=0$ for $x\in(1,+\infty)$ and it is linear on $(0,1)$) is a solution to a non-homogeneous elliptic equation defined in I_0^α equipped with homogeneous Dirichlet conditions. This means that we do not need to add any assumption in this level except those mentioned in [2]. Indeed, since u_0 appears in the boundary conditions of (3.2.1), (3.2.3), (3.2.5) and (3.2.7), this requires $\partial_{x_1}u_0$, $\partial_{x_2}u_0\in L^2(\Omega)$. That is to say that $u_0\in H^1(\Omega)$ which is ensured whenever

$$\nabla_{X_1}A_{22}\in L^\infty(\Omega), \quad \nabla_{X_1}f\in L^2(\Omega). \tag{3.2.9}$$

Formal correctors near the edges

As it is mentioned above, there is a double correction near the edges $(x_1,x_2)=\alpha=(\pm1,\pm1)$. For example on $x_1=1$ and near $(x_1,x_2)=(1,1)$ there is also another non-suitable effect of $\theta^{(0,1)}(.;\varepsilon)$ which, in principal, have to operate only near $x_2=1$ and vice-versa. Thus we have to re-correct near these edges.

Case $\alpha = (1,1)$**.** Near the edge $(x_1, x_2) = (1,1)$, we take $\alpha = (1,1)$. We define the function w^α as the weak solution to

$$\begin{cases} \nabla \cdot \mathfrak{A}^\alpha \nabla w^\alpha = 0 \quad \text{in } S_0^\alpha, \\ w^\alpha \mid_{(0,+\infty)\times(0,+\infty)\times\partial\omega_2} = 0, \quad w^\alpha \mid_{x_1=0} = -w^{(0,1)} \mid_{x_1=1}, \quad w^\alpha \mid_{x_2=0} = -w^{(1,0)} \mid_{x_2=1} \end{cases} \tag{3.2.10}$$

where $S_0^\alpha = (0,+\infty) \times (0,+\infty) \times \omega_2$, ∇ is the gradient in $\mathbb{R}^2 \times \omega_2$. This time we do not have to remove any line or column from $\breve{A}$ to define $\mathfrak{A}^\alpha$ (since $|\alpha| = 2$ here), i.e.

$$\mathfrak{A}^\alpha = \breve{A} \circ \varsigma_\alpha^{-1}(x) \quad \text{a.e. } x \in \Omega_{\frac{1}{\varepsilon}}^\alpha, \quad \mathfrak{A}^\alpha = \mathrm{Id} \quad \text{a.e. } x \in S_0^\alpha \backslash \Omega_{\frac{1}{\varepsilon}}^\alpha \tag{3.2.11}$$

where

$$\varsigma_\alpha^{-1}(x) = \varsigma_{(1,0)}^{-1} \circ \varsigma_{(0,1)}^{-1}(x) = (1 - \varepsilon x_1, 1 - \varepsilon x_2, X_2), \quad \Omega_\kappa^\alpha = (0,\kappa) \times (0,\kappa) \times \omega_2.$$

Then we take

$$\theta^\alpha(x;\varepsilon) = w^\alpha \circ \varsigma_\alpha(x) = w^\alpha(\frac{1-x_1}{\varepsilon}, \frac{1-x_2}{\varepsilon}, X_2) \quad \text{a.e. } x \in \Omega. \tag{3.2.12}$$

Case $\alpha = (1,-1)$**.** Near the edge $(x_1, x_2) = (1,-1)$ (i.e. $\alpha = (1,-1)$), we define the function w^α as the weak solution to

$$\begin{cases} \nabla \cdot \mathfrak{A}^\alpha \nabla w^\alpha = 0 \quad \text{in } S_0^\alpha, \\ w^\alpha \mid_{(0,+\infty)\times(-\infty,0)\times\partial\omega_2} = 0, \quad w^\alpha \mid_{x_1=0} = -w^{(0,-1)} \mid_{x_1=1}, \quad w^\alpha \mid_{x_2=0} = -w^{(1,0)} \mid_{x_2=-1} \end{cases} \tag{3.2.13}$$

where $S_0^\alpha = (0,+\infty) \times (-\infty,0) \times \omega_2$ and $\mathfrak{A}^\alpha$ is also defined as in (3.2.11) with

$$\varsigma_\alpha^{-1}(x) = \varsigma_{(1,0)}^{-1} \circ \varsigma_{(0,-1)}^{-1}(x) = (1 - \varepsilon x_1, -1 - \varepsilon x_2, X_2), \quad \Omega_\kappa^\alpha = (0,\kappa) \times (-\kappa,0) \times \omega_2.$$

Then we take

$$\theta^\alpha(x;\varepsilon) = w^\alpha \circ \varsigma_\alpha(x) = w^\alpha(\frac{1-x_1}{\varepsilon}, \frac{-1-x_2}{\varepsilon}, X_2) \quad \text{a.e. } x \in \Omega. \tag{3.2.14}$$

Case $\alpha = (-1,1)$**.** Near the edge $(x_1, x_2) = (-1,1)$, we take $\alpha = (-1,1)$. We define the function w^α as the weak solution to

$$\begin{cases} \nabla \cdot \mathfrak{A}^\alpha \nabla w^\alpha = 0 \quad \text{in } S_0^\alpha, \\ w^\alpha \mid_{(-\infty,0)\times(0,+\infty)\times\partial\omega_2} = 0, \quad w^\alpha \mid_{x_1=0} = -w^{(0,1)} \mid_{x_1=-1}, \quad w^\alpha \mid_{x_2=0} = -w^{(-1,0)} \mid_{x_2=1} \end{cases} \tag{3.2.15}$$

where $S_0^\alpha = (-\infty,0) \times (0,+\infty) \times \omega_2$ and $\mathfrak{A}^\alpha$ is also defined as in (3.2.11) with

$$\varsigma_\alpha^{-1}(x) = \varsigma_{(-1,0)}^{-1} \circ \varsigma_{(0,1)}^{-1}(x) = (-1 - \varepsilon x_1, 1 - \varepsilon x_2, X_2), \quad \Omega_\kappa^\alpha = (-\kappa,0) \times (0,\kappa) \times \omega_2.$$

Then we take

$$\theta^\alpha(x;\varepsilon) = w^\alpha \circ \varsigma_\alpha(x) = w^\alpha(\frac{-1-x_1}{\varepsilon}, \frac{1-x_2}{\varepsilon}, X_2) \quad \text{a.e. } x \in \Omega. \tag{3.2.16}$$

Case $\alpha = (-1,-1)$**.** Near the edge $(x_1, x_2) = (-1,-1)$, we take $\alpha = (-1,-1)$. We define the function w^α as the weak solution to

$$\begin{cases} \nabla \cdot \mathfrak{A}^\alpha \nabla w^\alpha = 0 \quad \text{in } S_0^\alpha, \\ w^\alpha \mid_{(-\infty,0)\times(-\infty,0)\times\partial\omega_2} = 0, \; w^\alpha \mid_{x_1=0} = -w^{(0,-1)} \mid_{x_1=-1}, \; w^\alpha \mid_{x_2=0} = -w^{(-1,0)} \mid_{x_2=-1} \end{cases} \tag{3.2.17}$$

where $S_0^\alpha = (-\infty, 0) \times (-\infty, 0) \times \omega_2$ and $\mathfrak{A}^\alpha$ is also defined as in (3.2.11) with

$$\varsigma_\alpha^{-1}(x) = \varsigma_{(-1,0)}^{-1} \circ \varsigma_{(0,-1)}^{-1}(x) = (-1 - \varepsilon x_1, -1 - \varepsilon x_2, X_2), \quad \Omega_\kappa^\alpha = (-\kappa, 0) \times (-\kappa, 0) \times \omega_2.$$

Then we take

$$\theta^\alpha(x; \varepsilon) = w^\alpha \circ \varsigma_\alpha(x) = w^\alpha\left(\frac{-1 - x_1}{\varepsilon}, \frac{-1 - x_2}{\varepsilon}, X_2\right) \quad \text{a.e. } x \in \Omega. \tag{3.2.18}$$

For the existence of the functions w^α in the Sobolev spaces $H^1(S_0^\alpha)$ for $\alpha = (\pm 1, \pm 1)$, we need to guarantee the smoothness of the functions w^α for $|\alpha| = 1$ with the compatibility conditions (see [32]). Formally if the functions w^α, $\alpha = (1, 0)$ and $(0, 1)$, are very smooth on S_0^α (u_0 is also smooth), it is clear that the compatibility condition

$$w^{(1,0)}(1, 1, X_2) = w^{(0,1)}(1, 1, X_2) = u_0(1, 1, X_2)$$

is satisfied which ensures the existence of $w^{(1,1)}$ in $H^1\left(S_0^{(1,1)}\right)$. Concerning the smoothness, for instance for $\alpha = (1, 1)$, we need to ensure that

$$\partial_{x_2} w^{(1,0)} \in L^2\left(S_0^{(1,0)}\right) \quad \text{and} \quad \partial_{x_1} w^{(0,1)} \in L^2\left(S_0^{(0,1)}\right).$$

If we come back to the problems satisfied by $w^{(1,0)}$ and $w^{(0,1)}$ where x_2 and x_1 play a parameter role, and using the smoothness theorem given in Chapter 2, we have to assume, in addition to (3.2.9), that

$$\begin{array}{ccc} \partial_{x_2} a_{11}, \quad \partial_{x_1} a_{22} \in L^\infty(\Omega), & & \\ \partial_{x_2} l_1, \quad \partial_{x_1} l_2 \in L^\infty(\Omega) & \text{and} & \partial_{x_2} c_1, \quad \partial_{x_1} c_2 \in L^\infty(\Omega), \\ \partial_{x_1 x_2} A_{22} \in L^\infty(\Omega) & \text{and} & \partial_{x_1 x_2} f \in L^2(\Omega). \end{array}$$

Similarly, for α $(0 \le |\alpha| \le 2)$ arbitrary we assume that

$$\begin{array}{l} a_{ii}, \ \partial_{x_k} a_{ii}, \quad l_i, \ \partial_{x_k} l_i, \quad c_i, \ \partial_{x_k} c_i \in L^\infty(\Omega), \quad i, k = 1, 2 \quad \text{with } i \ne k \\ a_{12}, \quad a_{21} \in L^\infty(\Omega), \\ A_{22}, \ \partial_{x_1} A_{22}, \quad \partial_{x_2} A_{22}, \quad \partial_{x_1 x_2} A_{22} \in L^\infty(\Omega) \quad \text{and} \quad f, \ \partial_{x_1} f, \quad \partial_{x_2} f, \quad \partial_{x_1 x_2} f \in L^2(\Omega). \end{array} \tag{3.2.19}$$

Remark 3.1. *Note that under the above assumptions we also have*

$$\partial_{x_i} w^\alpha, \quad \partial_{x_i}(\nabla^\alpha w^\alpha), \quad \nabla^\alpha w^\alpha \in L^2(S_0^\alpha) \quad \textit{for } |\alpha| = 1 \textit{ whenever } \alpha_i = 0, \ i = 1, 2,$$

$$\nabla w^\alpha \in L^2(S_0^\alpha) \quad \textit{for } |\alpha| = 2$$

and

$$\partial_{x_1} u_0, \quad \partial_{x_2} u_0, \quad \partial_{x_1}(\nabla_{X_2} u_0), \quad \partial_{x_2}(\nabla_{X_2} u_0), \quad \partial_{x_1 x_2}(\nabla_{X_2} u_0) \in L^2(\Omega).$$

Remark 3.2. *Note that the condition of the compatibility of the boundary conditions given in (3.2.10), (3.2.13), (3.2.15), and (3.2.17) is ensured from*

$$\partial_{x_1 x_2}(\nabla_{X_2} u_0) \in L^2(\Omega).$$

Remark 3.3. *If $A = \mathrm{Id}$ then $\mathfrak{A}^\alpha$ is always the corresponding identity matrix.*

Remark 3.4. *If $A = A(X_2)$, we do not have to make the change of variable ς_α at all, since $\mathfrak{A} = \breve{A} \circ \varsigma_\alpha^{-1}(x) = \breve{A} = \breve{A}(X_2)$ on S_0^α.*

3.2.2 Main properties of the formal correctors

Let us now explore the main properties of the above formal correctors.

Lemma 3.1. *For $\alpha \in E$ ($|\alpha| \neq 0$), the following identity holds for every $v \in H_0^1(\Omega)$*

$$\int_{\Omega^\alpha} A\nabla_\varepsilon \theta^\alpha(.;\varepsilon) \cdot \nabla_\varepsilon v \,\mathrm{d}x = -\int_{\Omega \setminus \Omega^\alpha} \nabla_\varepsilon \theta^\alpha(.;\varepsilon) \cdot \nabla_\varepsilon v \,\mathrm{d}x, \tag{3.2.20}$$

where $E = \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 = -1, 0, 1\}$, $\Omega^\alpha := \Omega_1^\alpha$, $\nabla_\varepsilon := \nabla_\varepsilon^\alpha$ is defined, as ∇, according to the choice of α, i.e. the i^{th} component is removed whenever $\alpha_i = 0$ and $A := A^\alpha$ is a square sub-matrix obtained from A by removing the i^{th} line and the i^{th} column whenever $\alpha_i = 0$. If there is no ambiguity we keep the same notation of A and we omit the subscript.

Proof. Even if the matrix A depends on x which is not the case in [11] we can apply the same argument. Let us do it for $|\alpha| = 2$. Note that if $v \in H_0^1(\Omega)$ then $v \circ \varsigma_\alpha^{-1} \in H_0^1(\Omega^\alpha_{\frac{2}{\varepsilon}})$ which allows to test the equation satisfied by w^α (for example the equation in (3.2.10) for $\alpha = (1,1)$) and we obtain

$$\int_{\Omega^\alpha_{\frac{2}{\varepsilon}}} \mathfrak{A}\nabla w^\alpha(x) \cdot \nabla \left(v \circ \varsigma_\alpha^{-1}(x)\right) \mathrm{d}x = 0,$$

where $\mathfrak{A} := \mathfrak{A}^\alpha$. Note that if $|\alpha| = 1$ we have just to integrate on $(-1, 1)$, in the x_i–direction for which $\alpha_i = 0$, to get the above identity that can also be written as

$$\int_{\Omega^\alpha_{\frac{1}{\varepsilon}}} \mathfrak{A}\nabla w^\alpha(x) \cdot \nabla \left(v \circ \varsigma_\alpha^{-1}(x)\right) \mathrm{d}x = -\int_{\Omega^\alpha_{\frac{2}{\varepsilon}} \setminus \Omega^\alpha_{\frac{1}{\varepsilon}}} \nabla w^\alpha(x) \cdot \nabla \left(v \circ \varsigma_\alpha^{-1}(x)\right) \mathrm{d}x.$$

Making the change of variable $x \to \varsigma_\alpha(x)$ in the above identity, we end up with (3.2.20). □

Theorem 3.1. *For every $\alpha \in E$ ($|\alpha| \neq 0$), there exist positive constants κ and η independent of ε such that*

Case $|\alpha| = 1$. *we have*

$$|\partial_{x_i} w^\alpha(.;\varepsilon)|_{2, S_0^\alpha \setminus \Omega^\alpha_{\frac{1}{\varepsilon}}}, \quad |\partial_{x_i}\left(\nabla^\alpha w^\alpha(.;\varepsilon)\right)|_{2, S_0^\alpha \setminus \Omega^\alpha_{\frac{1}{\varepsilon}}}, \quad |\nabla^\alpha w^\alpha(.;\varepsilon)|_{2, S_0^\alpha \setminus \Omega^\alpha_{\frac{1}{\varepsilon}}} \leq \kappa \exp\left(-\eta/\varepsilon\right)$$

whenever $\alpha_i = 0$, $i = 1, 2$.

Case $|\alpha| = 2$.

$$|\nabla w^\alpha(.;\varepsilon)|_{2, S_0^\alpha \setminus \Omega^\alpha_{\frac{1}{\varepsilon}}} \leq \kappa \exp\left(-\eta/\varepsilon\right).$$

Proof. The proof of this theorem is running as what is done in [11, Theorem 1, p. 135-137]. □

Corollary 3.1. *For every $\alpha \in E$ ($|\alpha| \neq 0$), there exists a positive constant η independent of ε such that*

Case $|\alpha| = 1$. *we have*

$$|\partial_{x_i} \theta^\alpha(.;\varepsilon)|_{2, \Omega \setminus \Omega^\alpha}, \quad |\partial_{x_i} \nabla^\alpha \theta^\alpha(.;\varepsilon)|_{2, \Omega \setminus \Omega^\alpha}, \quad |\nabla^\alpha \theta^\alpha(.;\varepsilon)|_{2, \Omega \setminus \Omega^\alpha} \leq \kappa \exp\left(-\eta/\varepsilon\right)$$

whenever $\alpha_i = 0$, $i = 1, 2$.

Case $|\alpha| = 2$.

$$|\nabla \theta^\alpha(.;\varepsilon)|_{2, \Omega \setminus \Omega^\alpha} \leq \kappa \exp\left(-\eta/\varepsilon\right).$$

Proof. The above corollary is an immediate consequence if Theorem 3.1 by making the change of variable $x \to \varsigma_\alpha^{-1}$. □

3.2.3 Global formal corrector

Since indices α indicate the region where the boundary layer function is acting and in order to substantiate this, we define the function $\varphi_\alpha(X_1) = \varphi(\alpha_1 x_1)\varphi(\alpha_2 x_2)$ where $\varphi : [-1,1] \to [0,1]$ is a continuous function such that $\varphi = 1$ on $(0,1)$, $\varphi(-1) = 0$ and is linear on $(-1,0)$. This means that $\varphi_\alpha(X_1) = 1$ whenever $x = (X_1, X_2) \in \Omega^\alpha$ and is vanishing on the boundary parts of Ω except the adjacent ones to the considered correction region. Then we define our global corrector as

$$\theta_\varepsilon = \sum_{|\alpha|\neq 0} \varphi_\alpha \theta^\alpha(.;\varepsilon).$$

Before to state the first main asymptotic convergence result we need to the following

Theorem 3.2. *Under the above assumptions (3.1.1), (3.2.9), (3.2.19) and for $\mu > 0$ arbitrary, there exist two positive constants $\kappa = \kappa(\mu)$ and η independent of ε, such that for every $v \in H_0^1(\Omega)$*

$$\begin{aligned}
&\int_\Omega A\nabla_\varepsilon(u_\varepsilon - u_0 - \theta_\varepsilon)\cdot\nabla_\varepsilon v\,\mathrm{d}x - \mu|\nabla_\varepsilon v|_{2,\Omega}^2 \\
\leq\; & \kappa\exp\left(\frac{-\eta}{\varepsilon}\right) - \varepsilon^2\int_\Omega A_{11}\nabla_{X_1}u_0\cdot\nabla_{X_1}v\,\mathrm{d}x - \varepsilon\int_\Omega A_{12}\nabla_{X_2}u_0\cdot\nabla_{X_1}v\,\mathrm{d}x - \varepsilon\int_\Omega A_{21}\nabla_{X_1}u_0\cdot\nabla_{X_2}v\,\mathrm{d}x \\
&-\varepsilon^2\sum_{i=1}^{2}\sum_{\substack{|\alpha|=1\\ \alpha_i=0}}\int_\Omega a_{12}\partial_{x_2}[\varphi_\alpha\theta^\alpha(.;\varepsilon)]\,\partial_{x_1}v + a_{21}\partial_{x_1}[\varphi_\alpha\theta^\alpha(.;\varepsilon)]\,\partial_{x_2}v + a_{ii}\partial_{x_i}[\varphi_\alpha\theta^\alpha(.;\varepsilon)]\,\partial_{x_i}v\,\mathrm{d}x \\
&-\varepsilon\sum_{i=1}^{2}\sum_{\substack{|\alpha|=1\\ \alpha_i=0}}\int_\Omega l_i\nabla_{X_2}[\varphi_\alpha\theta^\alpha(.;\varepsilon)]\,\partial_{x_i}v + c_i\partial_{x_i}[\varphi_\alpha\theta^\alpha(.;\varepsilon)]\cdot\nabla_{X_2}v\,\mathrm{d}x.
\end{aligned} \tag{3.2.21}$$

Proof. Let us first taking $v \in H_0^1(\Omega)$ as a test function in (3.1.6) and integrating it on $\omega_1 = (-1,1)^2$ then comparing the produced identity with (3.1.5), we get

$$\begin{aligned}
&\int_\Omega A\nabla_\varepsilon(u_\varepsilon - u_0 - \theta_\varepsilon)\cdot\nabla_\varepsilon v\,\mathrm{d}x \\
=\; & -\varepsilon^2\int_\Omega A_{11}\nabla_{X_1}u_0\cdot\nabla_{X_1}v\,\mathrm{d}x - \varepsilon\int_\Omega A_{12}\nabla_{X_2}u_0\cdot\nabla_{X_1}v\,\mathrm{d}x - \varepsilon\int_\Omega A_{21}\nabla_{X_1}u_0\cdot\nabla_{X_2}v\,\mathrm{d}x \\
&-\sum_{|\alpha|\neq 0}\int_\Omega A\nabla_\varepsilon[\varphi_\alpha\theta^\alpha(.;\varepsilon)]\cdot\nabla_\varepsilon v\,\mathrm{d}x.
\end{aligned} \tag{3.2.22}$$

The last sum can also be written as

$$\begin{aligned}
&\sum_{|\alpha|\neq 0}\int_\Omega A\nabla_\varepsilon[\varphi_\alpha\theta^\alpha(.;\varepsilon)]\cdot\nabla_\varepsilon v\,\mathrm{d}x \\
=\; & \sum_{|\alpha|=1}\int_\Omega A\nabla_\varepsilon[\varphi_\alpha\theta^\alpha(.;\varepsilon)]\cdot\nabla_\varepsilon v\,\mathrm{d}x + \sum_{|\alpha|=2}\int_\Omega A\nabla_\varepsilon[\varphi_\alpha\theta^\alpha(.;\varepsilon)]\cdot\nabla_\varepsilon v\,\mathrm{d}x
\end{aligned}$$

then

$$\begin{aligned}
&\sum_{|\alpha|\neq 0}\int_\Omega A\nabla_\varepsilon\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\cdot\nabla_\varepsilon v\,\mathrm{d}x\\
=\;&\varepsilon^2\sum_{i=1}^{2}\sum_{\substack{|\alpha|=1\\ \alpha_i=0}}\int_\Omega a_{12}\partial_{x_2}\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\partial_{x_1}v+a_{21}\partial_{x_1}\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\partial_{x_2}v+a_{ii}\partial_{x_i}\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\partial_{x_i}v\,\mathrm{d}x\\
&+\varepsilon\sum_{i=1}^{2}\sum_{\substack{|\alpha|=1\\ \alpha_i=0}}\int_\Omega l_i\nabla_{X_2}\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\partial_{x_i}v+c_i\partial_{x_i}\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\cdot\nabla_{X_2}v\,\mathrm{d}x\\
&+\sum_{|\alpha|=2}\int_\Omega A\nabla_\varepsilon\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\cdot\nabla_\varepsilon v\,\mathrm{d}x+\sum_{|\alpha|=1}\int_\Omega A^\alpha\nabla_\varepsilon^\alpha\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x. \qquad (3.2.23)
\end{aligned}$$

We rewrite for every α, the integrals in the last line of the above identity over Ω^α and $\Omega\backslash\Omega^\alpha$ separately and using the definition of the function φ_α, we obtain

$$\begin{aligned}
&\sum_{|\alpha|=2}\int_\Omega A\nabla_\varepsilon\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\cdot\nabla_\varepsilon v\,\mathrm{d}x+\sum_{|\alpha|=1}\int_\Omega A^\alpha\nabla_\varepsilon^\alpha\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x\\
=\;&\sum_{|\alpha|=2}\int_{\Omega^\alpha}A\nabla_\varepsilon\theta^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon v\,\mathrm{d}x+\sum_{|\alpha|=2}\int_{\Omega\backslash\Omega^\alpha}A\nabla_\varepsilon\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\cdot\nabla_\varepsilon v\,\mathrm{d}x\\
&+\sum_{|\alpha|=1}\int_{\Omega^\alpha}A^\alpha\nabla_\varepsilon^\alpha\theta^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x+\sum_{|\alpha|=1}\int_{\Omega\backslash\Omega^\alpha}A^\alpha\nabla_\varepsilon^\alpha\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x.
\end{aligned}$$

For the Ω^α–integrals we have just to apply Lemma 3.1, we derive

$$\begin{aligned}
&\sum_{|\alpha|=2}\int_\Omega A\nabla_\varepsilon\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\cdot\nabla_\varepsilon v\,\mathrm{d}x+\sum_{|\alpha|=1}\int_\Omega A^\alpha\nabla_\varepsilon^\alpha\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x\\
=\;&-\sum_{|\alpha|=2}\int_{\Omega\backslash\Omega^\alpha}\nabla_\varepsilon\theta^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon v\,\mathrm{d}x+\sum_{|\alpha|=2}\int_{\Omega\backslash\Omega^\alpha}\varphi_\alpha A\nabla_\varepsilon\theta^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon v\,\mathrm{d}x\\
&+\sum_{|\alpha|=2}\int_{\Omega\backslash\Omega^\alpha}\theta^\alpha(.;\varepsilon)A\nabla_\varepsilon\varphi_\alpha\cdot\nabla_\varepsilon v\,\mathrm{d}x-\sum_{|\alpha|=1}\int_{\Omega\backslash\Omega^\alpha}\nabla_\varepsilon^\alpha\theta^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x\\
&+\sum_{|\alpha|=1}\int_{\Omega\backslash\Omega^\alpha}\varphi_\alpha A^\alpha\nabla_\varepsilon^\alpha\theta^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x+\sum_{|\alpha|=1}\int_{\Omega\backslash\Omega^\alpha}\theta^\alpha(.;\varepsilon)A^\alpha\nabla_\varepsilon^\alpha\varphi_\alpha\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x.
\end{aligned}$$

We apply the Cauchy-Schwartz and Young inequalities to separate the integrals of v and those of $\theta^\alpha(.;\varepsilon)$. Then, in each $\Omega\backslash\Omega^\alpha$–integral of $\theta^\alpha(.;\varepsilon)$ we make the corresponding change of variables $x\to\varsigma_\alpha^{-1}(x)$, it follows that

$$\begin{aligned}
&\sum_{|\alpha|=2}\int_\Omega A\nabla_\varepsilon\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\cdot\nabla_\varepsilon v\,\mathrm{d}x+\sum_{|\alpha|=1}\int_\Omega A^\alpha\nabla_\varepsilon^\alpha\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x\\
&\qquad\leq\kappa\frac{1}{\varepsilon^2}\sum_{|\alpha|\neq 0}\left|\nabla^\alpha w^\alpha(.;\varepsilon)\right|^2_{2,S_0^\alpha\backslash\Omega^\alpha_{\frac{1}{\varepsilon}}}+\mu\left|\nabla_\varepsilon v\right|^2_{2,\Omega}
\end{aligned}$$

where μ is a constant that we will determine later. Note that we also used the Poincaré inequality on ω_2. Thus, thanks to Theorem 3.1, we get

$$\sum_{|\alpha|=2} \int_\Omega A\nabla_\varepsilon \left[\varphi_\alpha \theta^\alpha(.;\varepsilon)\right] \cdot \nabla_\varepsilon v \,\mathrm{d}x + \sum_{|\alpha|=1} \int_\Omega A^\alpha \nabla_\varepsilon^\alpha \left[\varphi_\alpha \theta^\alpha(.;\varepsilon)\right] \cdot \nabla_\varepsilon^\alpha v \,\mathrm{d}x \\ \leq \kappa \exp\left(\frac{-\eta}{\varepsilon}\right) + \mu \left|\nabla_\varepsilon v\right|_{2,\Omega}^2 . \tag{3.2.24}$$

Finally, substituting (3.2.24) in (3.2.23) and then (3.2.23) in (3.2.22), we obtain

$$\begin{aligned} & \int_\Omega A\nabla_\varepsilon \left(u_\varepsilon - u_0 - \theta_\varepsilon\right) \cdot \nabla_\varepsilon v \,\mathrm{d}x \\ \leq\ & -\varepsilon^2 \int_\Omega A_{11} \nabla_{X_1} u_0 \cdot \nabla_{X_1} v \,\mathrm{d}x - \varepsilon \int_\Omega A_{12} \nabla_{X_2} u_0 \cdot \nabla_{X_1} v \,\mathrm{d}x - \varepsilon \int_\Omega A_{21} \nabla_{X_1} u_0 \cdot \nabla_{X_2} v \,\mathrm{d}x \\ & -\varepsilon^2 \sum_{i=1}^2 \sum_{\substack{|\alpha|=1 \\ \alpha_i=0}} \int_\Omega a_{12} \partial_{x_2} \left[\varphi_\alpha \theta^\alpha(.;\varepsilon)\right] \partial_{x_1} v + a_{21} \partial_{x_1} \left[\varphi_\alpha \theta^\alpha(.;\varepsilon)\right] \partial_{x_2} v + a_{ii} \partial_{x_i} \left[\varphi_\alpha \theta^\alpha(.;\varepsilon)\right] \partial_{x_i} v \,\mathrm{d}x \\ & -\varepsilon \sum_{i=1}^2 \sum_{\substack{|\alpha|=1 \\ \alpha_i=0}} \int_\Omega l_i \nabla_{X_2} \left[\varphi_\alpha \theta^\alpha(.;\varepsilon)\right] \partial_{x_i} v + c_i \partial_{x_i} \left[\varphi_\alpha \theta^\alpha(.;\varepsilon)\right] \cdot \nabla_{X_2} v \,\mathrm{d}x + \kappa \exp\left(\frac{-\eta}{\varepsilon}\right) + \mu \left|\nabla_\varepsilon v\right|_{2,\Omega}^2 . \end{aligned}$$

This completes the proof of the theorem. □

The lemma bellow allows us to choose

$$u_\varepsilon - u_0 - \theta_\varepsilon$$

as a test function in (3.2.21), which permits to show the following asymptotic convergence and evaluate its rate,

$$\left|\nabla_\varepsilon \left(u_\varepsilon - u_0 - \theta_\varepsilon\right)\right|_{2,\Omega} \to 0, \quad \text{as } \varepsilon \to 0.$$

Lemma 3.2. *Under the assumptions of Theorem 3.2 and according to the above definitions of the formal correctors $\theta^\alpha(.;\varepsilon)$, $\alpha \in E$ $(|\alpha| \neq 0)$, it holds that, for $\varepsilon > 0$*

$$u_0 + \theta_\varepsilon \in H_0^1(\Omega).$$

Proof. We will use the trace of function in $H^1(\Omega)$ to show that $u_0 + \theta_\varepsilon = 0$ on each face which also means that $u_0 + \theta_\varepsilon = 0$ on $\partial\Omega$, i.e. $u_0 + \theta_\varepsilon \in H_0^1(\Omega)$. Let us consider the face $x_1 = 1$. Then we have, according to the above definition of θ_ε,

$$\begin{aligned} & \left(u_0 + \theta_\varepsilon\right)|_{x_1=1} \\ & \quad = u_0(1, x_2, X_2) + \theta^{(1,0)}(x;\varepsilon)|_{x_1=1} + \varphi(x_2)\, \theta^{(0,1)}(x;\varepsilon)|_{x_1=1} \\ & \quad + \varphi(-x_2)\, \theta^{(0,-1)}(x;\varepsilon)|_{x_1=1} + \varphi(x_2)\, \theta^{(1,1)}(x;\varepsilon)|_{x_1=1} + \varphi(-x_2)\, \theta^{(1,-1)}(x;\varepsilon)|_{x_1=1} \\ & \quad = u_0(1, x_2, X_2) + w^{(1,0)}(0, x_2, X_2) + \varphi(x_2)\, w^{(0,1)}(1, \frac{1-x_2}{\varepsilon}, X_2) \\ & \quad + \varphi(-x_2)\, w^{(0,-1)}(1, \frac{-1-x_2}{\varepsilon}, X_2) + \varphi(x_2)\, w^{(1,1)}(0, \frac{1-x_2}{\varepsilon}, X_2) \\ & \quad + \varphi(-x_2)\, w^{(1,-1)}(0, \frac{-1-x_2}{\varepsilon}, X_2). \end{aligned}$$

Taking into account the boundary conditions in (3.2.1), (3.2.5), (3.2.7), (3.2.10) and (3.2.13), we derive

$$\left(u_0 + \theta_\varepsilon\right)|_{x_1=1} = 0.$$

Note that the other boundary layer functions do not appear above since $\varphi(-1) = 0$. The traces on the other faces can be established using the same argument as above. □

3.2.4 Asymptotic convergence results

In this subsection, we will apply Theorem 3.2, but we have first to suppose that

$$\partial_{x_1} a_{12}, \quad \partial_{x_2} a_{21} \in L^\infty(\Omega) \quad \text{and} \quad \partial_{x_i} l_i \in L^\infty(\Omega) \quad i = 1, 2. \tag{3.2.25}$$

Then, we can prove the following theorem.

Theorem 3.3. *Under the assumptions (3.1.1), (3.1.2), (3.1.4), (3.2.19) and (3.2.25), it holds that, when $\varepsilon \to 0$,*

$$\begin{gathered} |u_\varepsilon - u_0 - \theta_\varepsilon|_{2,\Omega}, \quad |\nabla_{X_2}(u_\varepsilon - u_0 - \theta_\varepsilon)|_{2,\Omega} = O(\varepsilon), \\ \nabla_{X_1}(u_\varepsilon - u_0 - \theta_\varepsilon) \rightharpoonup 0 \quad \text{in } L^2(\Omega). \end{gathered} \tag{3.2.26}$$

Proof. Firstly, thanks to Theorem 3.2 and Lemma 3.2, testing (3.2.21) by $v = u_\varepsilon - u_0 - \theta_\varepsilon \in H_0^1(\Omega)$, then using the ellipticity condition (3.1.2), we get

$$\begin{aligned} &(\lambda - \mu)\, |\nabla_\varepsilon (u_\varepsilon - u_0 - \theta_\varepsilon)|^2_{2,\Omega} \le \kappa \exp(-\eta/\varepsilon) \\ &-\varepsilon^2 \int_\Omega A_{11} \nabla_{X_1} u_0 \cdot \nabla_{X_1}(u_\varepsilon - u_0 - \theta_\varepsilon)\, \mathrm{d}x - \varepsilon^2 \sum_{i=1}^{2} \sum_{\substack{|\alpha|=1 \\ \alpha_i = 0}} \int_\Omega a_{ii} \partial_{x_i} [\varphi_\alpha \theta^\alpha(.;\varepsilon)]\, \partial_{x_i}(u_\varepsilon - u_0 - \theta_\varepsilon)\, \mathrm{d}x \\ &-\varepsilon \int_\Omega A_{21} \nabla_{X_1} u_0 \cdot \nabla_{X_2}(u_\varepsilon - u_0 - \theta_\varepsilon)\, \mathrm{d}x - \varepsilon \sum_{i=1}^{2} \sum_{\substack{|\alpha|=1 \\ \alpha_i = 0}} \int_\Omega c_i \partial_{x_i} [\varphi_\alpha \theta^\alpha(.;\varepsilon)] \cdot \nabla_{X_2}(u_\varepsilon - u_0 - \theta_\varepsilon)\, \mathrm{d}x \\ &-\varepsilon^2 \sum_{|\alpha|=1} \int_\Omega a_{12} \partial_{x_2} [\varphi_\alpha \theta^\alpha(.;\varepsilon)]\, \partial_{x_1}(u_\varepsilon - u_0 - \theta_\varepsilon) + a_{21} \partial_{x_1} [\varphi_\alpha \theta^\alpha(.;\varepsilon)]\, \partial_{x_2}(u_\varepsilon - u_0 - \theta_\varepsilon)\, \mathrm{d}x \\ &-\varepsilon \int_\Omega A_{12} \nabla_{X_2} u_0 \cdot \nabla_{X_1}(u_\varepsilon - u_0 - \theta_\varepsilon)\, \mathrm{d}x - \varepsilon \sum_{i=1}^{2} \sum_{\substack{|\alpha|=1 \\ \alpha_i = 0}} \int_\Omega l_i \nabla_{X_2} [\varphi_\alpha \theta^\alpha(.;\varepsilon)]\, \partial_{x_i}(u_\varepsilon - u_0 - \theta_\varepsilon)\, \mathrm{d}x. \end{aligned} \tag{3.2.27}$$

Note that, under the assumptions of this Theorem 3.3, we can have

$$\begin{array}{rcll} \nabla_{X_1} \cdot A_{12} \nabla_{X_2} u_0 & \in & L^2(\Omega) & \\ \partial_{x_i}(l_i \nabla_{X_2} [\varphi_\alpha \theta^\alpha(.;\varepsilon)]) & \in & L^2(\Omega) & \text{for } |\alpha| = 1 \text{ with } \alpha_i = 0, \quad i = 1, 2 \\ \partial_{x_1}(a_{12} \partial_{x_2} [\varphi_\alpha \theta^\alpha(.;\varepsilon)]) & \in & L^2(\Omega) & \text{for } |\alpha| = 1 \text{ with } \alpha_1 = 0, \\ \partial_{x_2}(a_{21} \partial_{x_1} [\varphi_\alpha \theta^\alpha(.;\varepsilon)]) & \in & L^2(\Omega) & \text{for } |\alpha| = 1 \text{ with } \alpha_2 = 0. \end{array}$$

Using the Green's formula in the x_i–direction, $i = 1, 2$, in the last two lines of 3.2.27, then we choose $\mu = \frac{\lambda}{4}$, to get

$$\begin{aligned} &\frac{3\lambda}{4} |\nabla_\varepsilon [u_\varepsilon - u_0 - \theta_\varepsilon]|^2_{2,\Omega} \le \kappa \exp\left(\frac{-\eta}{\varepsilon}\right) \\ &-\varepsilon^2 \int_\Omega A_{11} \nabla_{X_1} u_0 \cdot \nabla_{X_1}(u_\varepsilon - u_0 - \theta_\varepsilon)\, \mathrm{d}x + \varepsilon^2 \sum_{\substack{|\alpha|=1 \\ \alpha_1 = 0}} \int_\Omega \partial_{x_1}(a_{12} \partial_{x_2} [\varphi_\alpha \theta^\alpha(.;\varepsilon)])\, (u_\varepsilon - u_0 - \theta_\varepsilon)\, \mathrm{d}x \end{aligned}$$

$$-\varepsilon\int_{\Omega}A_{21}\nabla_{X_1}u_0\cdot\nabla_{X_2}\left(u_\varepsilon-u_0-\theta_\varepsilon\right)\,\mathrm{d}x+\varepsilon^2\sum_{\substack{|\alpha|=1\\ \alpha_2=0}}\int_{\Omega}\partial_{x_2}\left(a_{21}\partial_{x_1}\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\right)\left(u_\varepsilon-u_0-\theta_\varepsilon\right)\,\mathrm{d}x$$

$$-\varepsilon\sum_{i=1}^{2}\sum_{\substack{|\alpha|=1\\ \alpha_i=0}}\int_{\Omega}c_i\partial_{x_i}\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\cdot\nabla_{X_2}\left(u_\varepsilon-u_0-\theta_\varepsilon\right)\,\mathrm{d}x$$

$$-\varepsilon^2\sum_{i=1}^{2}\sum_{\substack{|\alpha|=1\\ \alpha_i=0}}\int_{\Omega}a_{ii}\partial_{x_i}\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\partial_{x_i}\left(u_\varepsilon-u_0-\theta_\varepsilon\right)\,\mathrm{d}x+\varepsilon\int_{\Omega}\nabla_{X_1}\cdot\left(A_{12}\nabla_{X_2}u_0\right)\left(u_\varepsilon-u_0-\theta_\varepsilon\right)\,\mathrm{d}x$$

$$+\varepsilon\sum_{i=1}^{2}\sum_{\substack{|\alpha|=1\\ \alpha_i=0}}\int_{\Omega}\partial_{x_i}\left(l_i\cdot\nabla_{X_2}\left[\varphi_\alpha\theta^\alpha(.;\varepsilon)\right]\right)\left(u_\varepsilon-u_0-\theta_\varepsilon\right)\,\mathrm{d}x.$$

Using the Cauchy–Schwartz, the Poincaré on ω_2, and Young inequalities then applying the change of variable $x\to\varsigma_\alpha^{-1}(x)$ and taking into account Theorem 3.1, we obtain

$$\varepsilon^2\left|\nabla_{X_1}\left[u_\varepsilon-u_0-\theta_\varepsilon\right]\right|_{2,\Omega}^2+\left|\nabla_{X_2}\left[u_\varepsilon-u_0-\theta_\varepsilon\right]\right|_{2,\Omega}^2\leq\kappa\exp\left(-\eta/\varepsilon\right)+\kappa\varepsilon^2.$$

Thus

$$\begin{gathered}\left|u_\varepsilon-u_0-\theta_\varepsilon\right|_{2,\Omega},\quad\left|\nabla_{X_2}\left[u_\varepsilon-u_0-\theta_\varepsilon\right]\right|_{2,\Omega}\leq\kappa\varepsilon,\\ \left|\nabla_{X_1}\left[u_\varepsilon-u_0-\theta_\varepsilon\right]\right|_{2,\Omega}\leq\kappa,\end{gathered}\tag{3.2.28}$$

since $\exp\left(\frac{-\eta}{\varepsilon}\right)=O(\varepsilon^2)$. Then, according to the above estimate (3.2.28), we can extract a weakly converging subsequence from $\nabla_{X_1}\left(u_\varepsilon-u_0-\theta_\varepsilon\right)$ in $L^2(\Omega)$. It follows that the whole sequence converges weakly to zero, i.e.

$$\nabla_{X_1}\left[u_\varepsilon-u_0-\theta_\varepsilon\right]\rightharpoonup 0\quad\text{weakly in } L^2(\Omega).$$

Of course, this is thanks to the density of $\mathcal{D}'(\Omega)$ in $L^2(\Omega)$ and the continuity of the derivation in $\mathcal{D}'(\Omega)$. This completes the proof of the theorem. □

Note that we also have

$$\begin{aligned}\left|\nabla_\varepsilon[u_\varepsilon-u_0-\sum_{|\alpha|\neq0}\theta^\alpha(.;\varepsilon)]\right|_{2,\Omega}^2&=\left|\nabla_\varepsilon\left[u_\varepsilon-u_0-\theta_\varepsilon\right]-\sum_{|\alpha|\neq0}\nabla_\varepsilon[(1-\varphi_\alpha)\,\theta^\alpha(.;\varepsilon)]\right|_{2,\Omega}^2\\ &\leq|E|\left|\nabla_\varepsilon\left[u_\varepsilon-u_0-\theta_\varepsilon\right]\right|_{2,\Omega}^2+|E|\sum_{|\alpha|\neq0}\left|\nabla_\varepsilon[(1-\varphi_\alpha)\,\theta^\alpha(.;\varepsilon)]\right|_{2,\Omega}^2.\end{aligned}$$

Since $\operatorname{supp}(1-\varphi_\alpha)\subset\Omega\backslash\Omega^\alpha$ and $|1-\varphi_\alpha|\leq1$, where $|E|$ is the cardinal of E, it holds that

$$\left|\nabla_\varepsilon[u_\varepsilon-u_0-\sum_{|\alpha|\neq0}\theta^\alpha(.;\varepsilon)]\right|_{2,\Omega}^2\leq|E|\left|\nabla_\varepsilon\left[u_\varepsilon-u_0-\theta_\varepsilon\right]\right|_{2,\Omega}^2+|E|\sum_{|\alpha|\neq0}\left|\nabla_\varepsilon\theta^\alpha(.;\varepsilon)\right|_{2,\Omega\backslash\Omega^\alpha}^2$$

Making the change of variable $x\to\varsigma_\alpha^{-1}(x)$ on $\Omega\backslash\Omega^\alpha$, using Theorem 3.1 and (3.2.26), we get

$$\begin{aligned}\left|\nabla_\varepsilon[u_\varepsilon-u_0-\sum_{|\alpha|\neq0}\theta^\alpha(.;\varepsilon)]\right|_{2,\Omega}^2&\leq\kappa\varepsilon^2+\kappa\exp\left(-\eta/\varepsilon\right)\\ &\leq\kappa\varepsilon^2,\end{aligned}$$

for ε small enough. By consequence, using the continuity of the derivative operator in X_1–directions as above, we get

Corollary 3.2. *Under the assumptions of Theorem 3.3, we have*

$$\begin{gathered}\left| u_\varepsilon - u_0 - \sum_{|\alpha|\neq 0} \theta^\alpha(.;\varepsilon) \right|_{2,\Omega}, \quad \left| \nabla_{X_2}[u_\varepsilon - u_0 - \sum_{|\alpha|\neq 0} \theta^\alpha(.;\varepsilon)] \right|_{2,\Omega} = O(\varepsilon), \\ \nabla_{X_1}[u_\varepsilon - u_0 - \sum_{|\alpha|\neq 0} \theta^\alpha(.;\varepsilon)] \rightharpoonup 0 \quad \text{in } L^2(\Omega).\end{gathered} \tag{3.2.29}$$

3.3 Composite asymptotic expansion of arbitrary order

Of course we will follow the same idea used in the previous section to construct the formal boundary layer functions. This leads to correct near the boundaries of lower dimension n − 1 and n − 2, i.e. near the faces and the edges mentioned above.

3.3.1 Formal construction of boundary layer functions

We start by the construction of the second order formal correctors in two steps as above.

Formal correctors of second order

Here we first recall by the second coefficient of the regular asymptotic expansion, u_1, which is defined, for a.e. $X_1 \in \omega_1 = (-1,1)^2$,

$$\begin{cases} -\nabla_{X_2} \cdot A_{22} \nabla_{X_2} u_1 = \nabla_{X_1} \cdot A_{12} \nabla_{X_2} u_0 + \nabla_{X_2} \cdot A_{21} \nabla_{X_1} u_0 & \text{in } \omega_2, \\ u_1(X_1,.) \in H_0^1(\omega_2), \end{cases} \tag{3.3.1}$$

and we have $u_1 \in H^1(\Omega)$ under the following assumptions, see Chapter 2,

$$D^1_{X_1} A'_{12}, \quad D^1_{X_1} A_{21}, \quad D^2_{X_1} A_{22} \in L^\infty(\Omega), \quad D^2_{X_1} f \in L^2(\Omega). \tag{3.3.2}$$

Formal correctors near the faces Let us start by the first correctors level, i.e. near the faces $x_1 = \pm 1$ or $x_2 = \pm 1$.

Cases $\alpha = (\pm 1, 0)$. For the second coefficients of the local asymptotic expansion operating near the faces $x_1 = \pm 1$. We define the function $w_1^\alpha(.,x_2,.)$ as the weak solution, for a.e. $x_2 \in (-1,1)$, to

$$\begin{cases} \nabla \cdot \mathfrak{A}^\alpha \nabla w_1^\alpha = \partial_{x_2} (\mathfrak{L}_\alpha \nabla w_0^\alpha) + \nabla \cdot \mathfrak{C}_\alpha \partial_{x_2} w_0^\alpha & \text{in } I_0^\alpha, \\ w_1^\alpha(x_1,x_2,.)\,|_{\partial\omega_2} = 0, \quad w_1^\alpha(.,x_2,.)\,|_{x_1=0} = -u_1(.,x_2,.)\,|_{x_1=\alpha_1} \end{cases} \tag{3.3.3}$$

where I_0^α, Ω_κ^α, S_0^α, $\nabla := \nabla^\alpha$, $\mathfrak{A}^\alpha$ are defined as in the above section and $w_0^\alpha := w^\alpha$. Of course here, they are corresponding to the cases $\alpha = (\pm 1, 0)$, and the new coefficients are defined as

$$\mathfrak{L}_\alpha = \begin{pmatrix} a_{21} \circ \varsigma_\alpha^{-1}(x) & -l_2 \circ \varsigma_\alpha^{-1}(x) \end{pmatrix}, \qquad \mathfrak{C}_\alpha = \begin{pmatrix} a_{12} \circ \varsigma_\alpha^{-1}(x) \\ -c_2 \circ \varsigma_\alpha^{-1}(x) \end{pmatrix} \quad \text{a.e. } x \in S_0^\alpha,$$

where $\varsigma_\alpha^{-1}(x) = (\alpha_1 - \varepsilon x_1, x_2, X_2)$. When we extract $\mathfrak{A}^\alpha$ from $\check{A} \circ \varsigma_\alpha^{-1}$ on $\Omega^\alpha_{\frac{1}{\varepsilon}}$, the blocks $\mathfrak{L}_\alpha$ and $\mathfrak{C}_\alpha$ are defined as the remaining line and column respectively without the diagonal element $\mathfrak{D}_\alpha = a_{22} \circ \varsigma_\alpha^{-1}$ here (since $\alpha_2 = 0$). Then, we set

$$\theta_1^\alpha(x;\varepsilon) = w_1^\alpha \circ \varsigma_\alpha(x) = w_1^\alpha \left(\frac{\alpha_1 - x_1}{\varepsilon}, x_2, X_2 \right) \quad \text{a.e. } x \in \Omega. \tag{3.3.4}$$

Cases $\alpha = (0, \pm 1)$. Then near the faces $x_2 = \pm 1$, as in the previous case, we define the function $w_1^\alpha(x_1, .)$ as the weak solution, for a.e. $x_1 \in (-1, 1)$, to

$$\begin{cases} \nabla \cdot \mathfrak{A}^\alpha \nabla w_1^\alpha = \partial_{x_1}\left(\mathfrak{L}_\alpha \nabla w_0^\alpha\right) + \nabla \cdot \mathfrak{C}_\alpha \partial_{x_1} w_0^\alpha & \text{in } I_0^\alpha, \\ w_1^\alpha(x_1, x_2, .) \mid_{\partial\omega_2} = 0, \ w_1^\alpha(x_1, .) \mid_{x_2=0} = -u_1(x_1, .) \mid_{x_2=\alpha_2} \end{cases} \tag{3.3.5}$$

where the notation is adapted for $\alpha = (0, \pm 1)$, except the blocks of the right-hand side are defined by

$$\mathfrak{L}_\alpha = \begin{pmatrix} a_{12} \circ \varsigma_\alpha^{-1}(x) & -l_1 \circ \varsigma_\alpha^{-1}(x) \end{pmatrix}, \qquad \mathfrak{C}_\alpha = \begin{pmatrix} a_{21} \circ \varsigma_\alpha^{-1}(x) \\ -c_1 \circ \varsigma_\alpha^{-1}(x) \end{pmatrix} \quad \text{a.e. } x \in S_0^\alpha,$$

since $\alpha_1 = 0$, and $\varsigma_\alpha^{-1}(x) = (x_1, \alpha_2 - \varepsilon x_2, X_2)$. Then we take

$$\theta_1^\alpha(x; \varepsilon) = w_1^\alpha \circ \varsigma_\alpha(x) = w_1^\alpha\left(x_1, \frac{\alpha_2 - x_2}{\varepsilon}, X_2\right) \quad \text{a.e. } x \in \Omega. \tag{3.3.6}$$

The existence of the functions $w_1^\alpha(., x_2, .) \in H^1(I_0^\alpha)$ for $\alpha = (\pm 1, 0)$ and a.e. $x_2 \in (-1, 1)$ or $w_1^\alpha(x_1, .) \in H^1(I_0^\alpha)$ for $\alpha = (0, \pm 1)$ and a.e. $x_1 \in (-1, 1)$ is ensured by the Lax–Milgram theorem as in the previous section. For example, we easily see that $w_1^{(1,0)}(x) + \rho(x_1) u_1(1 - x_1, x_2, X_2) \in H^1\left(I_0^{(1,0)}\right)$ is a solution to a non-homogeneous elliptic equation equipped with homogeneous Dirichlet boundary conditions. This means that we do not need to add any assumption to (3.3.2), i.e. when $|\alpha| = 1$.

Formal correctors near the edges Now, in order to define the second coefficients of the local asymptotic expansion operating near the edges, we have just to change the boundary conditions in each problem satisfied by the corresponding formal correctors for the cases when $|\alpha| = 2$, in the above section. That is to say we have just to add the index 1 to each w^α in the problems (3.2.10), (3.2.13), (3.2.15) and (3.2.17), then the correctors θ_1^α are also defined as the correctors θ^α replacing w^α by w_1^α. For example, when $\alpha = (1, 1)$ we define w_1^α as the weak solution to (3.2.10) replacing the boundary conditions by

$$w_1^\alpha \mid_{\partial\omega_2} = 0, \quad w_1^\alpha \mid_{x_1=0} = -w_1^{(0,1)} \mid_{x_1=1}, \quad w_1^\alpha \mid_{x_2=0} = -w_1^{(1,0)} \mid_{x_2=1},$$

and the corresponding formal corrector is defined as in (3.2.12), i.e

$$\theta_1^\alpha(x; \varepsilon) = w_1^\alpha \circ \varsigma_\alpha(x) = w_1^\alpha\left(\frac{1 - x_1}{\varepsilon}, \frac{1 - x_2}{\varepsilon}, X_2\right) \quad \text{a.e. } x \in \Omega.$$

For the three remaining formal correctors, i.e. for $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| = 2$, we define w_1^α as the weak solution to (3.2.13), (3.2.15) and (3.2.17), replacing the boundary conditions by

$$w_1^\alpha \mid_{\partial\omega_2} = 0, \quad w_1^\alpha \mid_{x_1=0} = -w_1^{(0,\alpha_2)} \mid_{x_1=\alpha_1}, \quad w_1^\alpha \mid_{x_2=0} = -w_1^{(\alpha_1,0)} \mid_{x_2=\alpha_2}, \tag{3.3.7}$$

with respect to the values of α. The corresponding formal corrector is defined as

$$\theta_1^\alpha(x; \varepsilon) = w_1^\alpha \circ \varsigma_\alpha(x) = w_1^\alpha\left(\frac{\alpha_1 - x_1}{\varepsilon}, \frac{\alpha_2 - x_2}{\varepsilon}, X_2\right) \quad \text{a.e. } x \in \Omega. \tag{3.3.8}$$

For the existence of the functions w_1^α for $\alpha = (\pm 1, \pm 1)$, as in the previous section, we have to take care of the smoothness of the function w_1^α with the compatibility of the boundary conditions. By using the smoothness theorem given in Chapter 2, we can ensure, under the following assumptions

$$\begin{cases} \partial_{x_j} a_{ii}, \quad \partial^2_{x_j} a_{ii} \quad \partial_{x_j} a_{ij}, \quad \partial^2_{x_i} a_{ij} \in L^\infty(\Omega), \quad i,j=1,2, \quad j \neq i, \\ D^\beta \nabla_{X_1} a_{ij}, \quad D^\beta \partial^2_{x_j} a_{ii} \in L^\infty(\Omega), \quad i=1,2, j=1,\dots,\mathrm{n}, \quad j \neq i, \\ D^\beta \nabla_{X_1} a_{ij} \in L^\infty(\Omega), \quad j=1,2, i=1,\dots,\mathrm{n}, \quad j\neq i, \\ D^\beta \nabla_{X_1} A_{22} \in L^\infty(\Omega), \qquad D^\beta \nabla_{X_1} f \in L^2(\Omega), \end{cases} \tag{3.3.9}$$

and in addition to (3.3.2), the existence of w_1^α in $H^1(S_0^\alpha)$ for all $\alpha \in E$.

Formal correctors of higher order

Firstly, we recall the coefficients of the regular asymptotic expansion, u_N $(N \geq 2)$, as the weak solutions of the following system, for a.e. $X_1 \in \omega_1 = (-1,1)^2$,

$$\begin{cases} u_N(X_1,.) \in H^1_0(\omega_2), \\ -\nabla_{X_2} \cdot A_{22} \nabla_{X_2} u_N = \nabla_{X_1} \cdot A_{11} \nabla_{X_1} u_{N-2} + \nabla_{X_1} \cdot A_{12} \nabla_{X_2} u_{N-1} + \nabla_{X_2} \cdot A_{21} \nabla_{X_1} u_{N-1} \quad \text{in } \omega_2, \end{cases} \tag{3.3.10}$$

and we have $u_N \in H^1(\Omega)$ under the following assumptions

$$D^{N-1}_{X_1} A'_{11}, \quad D^N_{X_1} A'_{12}, \quad D^N_{X_1} A_{21}, \quad D^{N+1}_{X_1} A_{22} \in L^\infty(\Omega), \quad D^{N+1}_{X_1} f \in L^2(\Omega). \tag{3.3.11}$$

Then, we suppose the following notation

$$X_{\bar\alpha} := x_i \text{ if } |\alpha| = 1 \text{ whenever } \alpha_i = 0, \; i=1,2 \qquad \text{and} \quad X_{\bar\alpha} := X_1 \text{ if } |\alpha| = 0.$$

By consequence $\nabla_{X_{\bar\alpha}} := \partial_{x_i}$ for $|\alpha| = 1$ whenever $\alpha_i = 0$, $i = 1,2$ and $\nabla_{X_{\bar\alpha}} := \nabla_{X_1}$ for $|\alpha| = 0$.

We aim here to construct the formal higher order correctors w_N^α for $N \geqslant 2$ that will be also run in two levels as above.

Case $|\alpha| = 1$. Near the faces $x_1 = \pm 1$ or $x_2 = \pm 1$, we take for example $\alpha = (\alpha_1, 0)$ and we define $w_N^\alpha(X_{\bar\alpha},.) := w_N^\alpha(.,x_2,.)$ as the weak solution, for a.e. $X_{\bar\alpha} := x_2 \in (-1,1)$ since $\alpha_2 = 0$, to

$$\begin{cases} \nabla \cdot \mathfrak{A}^\alpha \nabla w_N^\alpha = -\nabla_{X_{\bar\alpha}} \cdot \mathfrak{D}_\alpha \nabla_{X_{\bar\alpha}} w^\alpha_{N-2} + \nabla_{X_{\bar\alpha}} \cdot \mathfrak{L}_\alpha \nabla^\alpha w^\alpha_{N-1} + \nabla^\alpha \cdot \mathfrak{C}_\alpha \nabla_{X_{\bar\alpha}} w^\alpha_{N-1} \quad \text{in } I_0^\alpha, \\ w_N^\alpha(X_1,.)\mid_{\partial\omega_2} = 0, \quad w_N^\alpha(X_{\bar\alpha},.)\mid_{x_1=0} = -u_N(X_{\bar\alpha},.)\mid_{x_1=\alpha_1}, \end{cases} \tag{3.3.12}$$

where $\nabla_{X_{\bar\alpha}} := \partial_{x_2}$. Of course the notation here is adapted for the indices $\alpha = (\alpha_1, 0) = (\pm 1, 0)$. Then, we denote by $\theta_N^\alpha(.;\varepsilon)$ the function defined as

$$\theta_N^\alpha(x;\varepsilon) = w_N^\alpha \circ \varsigma_\alpha(x) = w_N^\alpha(\frac{\alpha_1 - x_1}{\varepsilon}, x_2, X_2) \quad \text{a.e. } x \in \Omega. \tag{3.3.13}$$

Of course we follow the same rule to define $w_N^\alpha(X_{\bar\alpha},.)$ for the other indices α whenever $|\alpha| = 1$.

Case $|\alpha| = 2$. Near the edges $(x_1, x_2) = (\pm 1, \pm 1)$, we take $\alpha = (\alpha_1, \alpha_2)$, then we define w_N^α as the weak solution to

$$\begin{cases} \nabla \cdot \mathfrak{A}^\alpha \nabla w_N^\alpha = 0 \quad \text{in } S_0^\alpha, \\ w_N^\alpha(X_1,.)\mid_{\partial\omega_2} = 0, \quad w_N^\alpha\mid_{x_1=0} = -w_N^{(0,\alpha_2)}\mid_{x_1=\alpha_1}, \quad w_N^\alpha\mid_{x_2=0} = -w_N^{(\alpha_1,0)}\mid_{x_2=\alpha_2}. \end{cases} \tag{3.3.14}$$

Then, we denote by $\theta_N^\alpha(.;\varepsilon)$ the function defined as

$$\theta_N^\alpha(x;\varepsilon) = w_N^\alpha \circ \varsigma_\alpha(x) = w_N^\alpha(\frac{\alpha_1 - x_1}{\varepsilon}, \frac{\alpha_2 - x_2}{\varepsilon}, X_2) \quad \text{a.e. } x \in \Omega. \tag{3.3.15}$$

Note that (3.3.3) can be considered in (3.3.12) by setting $w^\alpha_{-2} = w^\alpha_{-1} = 0$ in S_0^α, for all $\alpha \in E$ and $w_0^\alpha = w^\alpha$. Also (3.3.14) may define all w_N^α for $N \geqslant 0$ and $|\alpha| = 2$, using the same corresponding notation.

Arguing as above and by recurrence, we derive the following assumptions on the data to ensure the existence of w_N^α in $H^1(S_0^\alpha)$, for $N \geq 2$ and for all $\alpha \in E$,

$$\begin{cases} D^\beta D_{X_1}^N a_{ij}, \qquad D^\beta D_{X_1}^{N-1} \partial_{x_i}^2 a_{ij} \in L^\infty(\Omega), \quad i = 1, 2, \quad j = 1, \dots, \mathrm{n} \\ D^\beta D_{X_1}^N a_{ij} \in L^\infty(\Omega), \quad i, j = 3, \dots, \mathrm{n}, \\ D^\beta D_{X_1}^N A_{22} \in L^\infty(\Omega), \qquad D^\beta D_{X_1}^N f \in L^2(\Omega), \end{cases} \tag{3.3.16}$$

for all $\beta \in Q$ where $Q = \{0,1\}^2$ and D^β denotes the mixed partial derivative in X_1–directions, however $D_{X_1}^N$, it is also in the same sense as D^β but up to order N.

3.3.2 Main properties of the formal boundary layer functions

Before to deal with the asymptotic convergence we need the lemmas below.

Lemma 3.3. *For all $N \in \mathbb{N}$ and $\alpha \in E$ $(|\alpha| \neq 0)$, the following identities hold for every $v \in H_0^1(\Omega)$.*

Case $|\alpha| = 1$.

$$\begin{aligned} \int_{\Omega^\alpha} A^\alpha \nabla_\varepsilon^\alpha \theta_N^\alpha(.;\varepsilon) \cdot \nabla_\varepsilon^\alpha v \,\mathrm{d}x \quad &+ \int_{\Omega^\alpha} L_\alpha \nabla_\varepsilon^\alpha \theta_{N-1}^\alpha(.;\varepsilon) \cdot \nabla_{X_{\bar\alpha}} v \,\mathrm{d}x \\ &+ \int_{\Omega^\alpha} C_\alpha \nabla_{X_{\bar\alpha}} \theta_{N-1}^\alpha(.;\varepsilon) \cdot \nabla_\varepsilon^\alpha v \,\mathrm{d}x \\ &+ \int_{\Omega^\alpha} D_\alpha \nabla_{X_{\bar\alpha}} \theta_{N-2}^\alpha(.;\varepsilon) \cdot \nabla_{X_{\bar\alpha}} v \,\mathrm{d}x \\ = \quad &- \int_{\Omega \setminus \Omega^\alpha} \nabla_\varepsilon^\alpha \theta_N^\alpha(.;\varepsilon) \cdot \nabla_\varepsilon^\alpha v + \nabla_{X_{\bar\alpha}} \theta_{N-2}^\alpha(.;\varepsilon) \cdot \nabla_{X_{\bar\alpha}} v \,\mathrm{d}x, \end{aligned} \tag{3.3.17}$$

where L_α, C_α and D_α are the remainder line, column and diagonal element respectively when we extract the block A^α from the matrix A. Note also that $\theta_{-2}^\alpha = \theta_{-1}^\alpha = 0$ for all $\alpha \in E$.

Case $|\alpha| = 2$.

$$\int_{\Omega^\alpha} A \nabla_\varepsilon \theta_N^\alpha(.;\varepsilon) \cdot \nabla_\varepsilon v \,\mathrm{d}x = - \int_{\Omega \setminus \Omega^\alpha} \nabla_\varepsilon \theta_N^\alpha(.;\varepsilon) \cdot \nabla_\varepsilon v \,\mathrm{d}x. \tag{3.3.18}$$

Remark 3.5. *Note that when $N = 0$, we recover the results of Lemma 3.1.*

PROOF OF LEMMA 3.3. First note that for $\alpha \in E$ $(|\alpha| \neq 0)$, $v \in H_0^1(\Omega)$, we have $v \circ \varsigma_\alpha^{-1}(x) \in H_0^1(\Omega_{\frac{2}{\varepsilon}}^\alpha)$ and we will follow the same argument as in the proof of Lemma 3.1. We replace the test function in the weak formulation of (3.3.12) by $v \circ \varsigma_\alpha^{-1}(x)$.

Case $|\alpha| = 1$. Whenever $\alpha_i = 0$ $(i = 1, 2)$ we integrate over $(-1, 1)$ in the x_i–direction, we obtain

$$\begin{aligned} &\int_{\Omega_{\frac{2}{\varepsilon}}^\alpha} \mathfrak{A}^\alpha \nabla^\alpha w_N^\alpha(x) \cdot \nabla^\alpha \left(v \circ \varsigma_\alpha^{-1}(x)\right) \mathrm{d}x - \int_{\Omega_{\frac{2}{\varepsilon}}^\alpha} \mathfrak{L}_\alpha \nabla^\alpha w_{N-1}^\alpha(x) \cdot \nabla_{X_{\bar\alpha}} \left(v \circ \varsigma_\alpha^{-1}(x)\right) \mathrm{d}x \\ - &\int_{\Omega_{\frac{2}{\varepsilon}}^\alpha} \mathfrak{C}_\alpha \nabla_{X_{\bar\alpha}} w_{N-1}^\alpha(x) \cdot \nabla^\alpha \left(v \circ \varsigma_\alpha^{-1}(x)\right) \mathrm{d}x + \int_{\Omega_{\frac{2}{\varepsilon}}^\alpha} \mathfrak{D}_\alpha \nabla_{X_{\bar\alpha}} w_{N-2}^\alpha(x) \cdot \nabla_{X_{\bar\alpha}} \left(v \circ \varsigma_\alpha^{-1}(x)\right) \mathrm{d}x = 0 \end{aligned}$$

whence

$$\begin{aligned}
&\int_{\Omega^\alpha_{\frac{1}{\varepsilon}}} \mathfrak{A}^\alpha \nabla^\alpha w^\alpha_N(x)\cdot\nabla^\alpha\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x - \int_{\Omega^\alpha_{\frac{1}{\varepsilon}}} \mathfrak{L}_\alpha\nabla^\alpha w^\alpha_{N-1}(x)\cdot\nabla_{X_{\bar\alpha}}\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x\\
&-\int_{\Omega^\alpha_{\frac{1}{\varepsilon}}} \mathfrak{C}_\alpha\nabla_{X_{\bar\alpha}} w^\alpha_{N-1}(x)\cdot\nabla^\alpha\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x + \int_{\Omega^\alpha_{\frac{1}{\varepsilon}}} \mathfrak{D}_\alpha\nabla_{X_{\bar\alpha}} w^\alpha_{N-2}(x)\cdot\nabla_{X_{\bar\alpha}}\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x\\
&=-\int_{\Omega^\alpha_{\frac{2}{\varepsilon}}\setminus\Omega^\alpha_{\frac{1}{\varepsilon}}} \nabla^\alpha w^\alpha_N(x)\cdot\nabla^\alpha\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x - \int_{\Omega^\alpha_{\frac{2}{\varepsilon}}\setminus\Omega^\alpha_{\frac{1}{\varepsilon}}} \nabla_{X_{\bar\alpha}} w^\alpha_{N-2}(x)\cdot\nabla_{X_{\bar\alpha}}\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x.
\end{aligned} \tag{3.3.19}$$

We rewrite the blocks $\mathfrak{A}^\alpha$, $\mathfrak{L}_\alpha$, $\mathfrak{C}_\alpha$ and $\mathfrak{D}_\alpha$ of the left-hand side in terms of A^α, L_α, C_α and D_α respectively. Making the change of variable $x\to\varsigma_\alpha(x)$ in the obtained equation, we can rewrite the right-hand side of (3.3.19) as

$$\begin{aligned}
&\int_{\Omega^\alpha_{\frac{1}{\varepsilon}}} \mathfrak{A}^\alpha \nabla^\alpha w^\alpha_N(x)\cdot\nabla^\alpha\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x - \int_{\Omega^\alpha_{\frac{1}{\varepsilon}}} \mathfrak{L}_\alpha\nabla^\alpha w^\alpha_{N-1}(x)\cdot\nabla_{X_{\bar\alpha}}\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x\\
&-\int_{\Omega^\alpha_{\frac{1}{\varepsilon}}} \mathfrak{C}_\alpha\nabla_{X_{\bar\alpha}} w^\alpha_{N-1}(x)\cdot\nabla^\alpha\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x + \int_{\Omega^\alpha_{\frac{1}{\varepsilon}}} \mathfrak{D}_\alpha\nabla_{X_{\bar\alpha}} w^\alpha_{N-2}(x)\cdot\nabla_{X_{\bar\alpha}}\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x\\
=\;&\int_{\Omega^\alpha}\left[A^\alpha\nabla^\alpha_\varepsilon\theta^\alpha_N(x;\varepsilon)\cdot\nabla^\alpha_\varepsilon v + L_\alpha\nabla^\alpha_\varepsilon\theta^\alpha_{N-1}(x;\varepsilon)\cdot\nabla_{X_{\bar\alpha}}v\right]|\det J_\alpha|\,\mathrm{d}x\\
&+\int_{\Omega^\alpha}\left[C_\alpha\nabla_{X_{\bar\alpha}}\theta^\alpha_{N-1}(x;\varepsilon)\cdot\nabla^\alpha_\varepsilon v + D_\alpha\nabla_{X_{\bar\alpha}}\theta^\alpha_{N-2}(x;\varepsilon)\cdot\nabla_{X_{\bar\alpha}}v\right]|\det J_\alpha|\,\mathrm{d}x,
\end{aligned}$$

where $|\det J_\alpha| = \left|\left(\frac{-1}{\varepsilon}\right)^{|\alpha|}\right| = \frac{1}{\varepsilon}$ is the Jacobian adapted here for the α–change of variable ($|\alpha|=1$). Then

$$\begin{aligned}
&\int_{\Omega^\alpha_{\frac{1}{\varepsilon}}} \mathfrak{A}^\alpha \nabla^\alpha w^\alpha_N(x)\cdot\nabla^\alpha\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x - \int_{\Omega^\alpha_{\frac{1}{\varepsilon}}} \mathfrak{L}_\alpha\nabla^\alpha w^\alpha_{N-1}(x)\cdot\nabla_{X_{\bar\alpha}}\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x\\
&-\int_{\Omega^\alpha_{\frac{1}{\varepsilon}}} \mathfrak{C}_\alpha\nabla_{X_{\bar\alpha}} w^\alpha_{N-1}(x)\cdot\nabla^\alpha\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x + \int_{\Omega^\alpha_{\frac{1}{\varepsilon}}} \mathfrak{D}_\alpha\nabla_{X_{\bar\alpha}} w^\alpha_{N-2}(x)\cdot\nabla_{X_{\bar\alpha}}\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x\\
=\;&\frac{1}{\varepsilon}\int_{\Omega^\alpha} A^\alpha\nabla^\alpha_\varepsilon\theta^\alpha_N(x;\varepsilon)\cdot\nabla^\alpha_\varepsilon v + L_\alpha\nabla^\alpha_\varepsilon\theta^\alpha_{N-1}(x;\varepsilon)\cdot\nabla_{X_{\bar\alpha}}v\mathrm{d}x\\
&+\frac{1}{\varepsilon}\int_{\Omega^\alpha} C_\alpha\nabla_{X_{\bar\alpha}}\theta^\alpha_{N-1}(x;\varepsilon)\cdot\nabla^\alpha_\varepsilon v + D_\alpha\nabla_{X_{\bar\alpha}}\theta^\alpha_{N-2}(x;\varepsilon)\cdot\nabla_{X_{\bar\alpha}}v\mathrm{d}x.
\end{aligned} \tag{3.3.20}$$

For the left-hand side of the equation (3.3.19), we also make the corresponding change of variable $x\to\varsigma_\alpha(x)$, we obtain

$$\begin{aligned}
&-\int_{\Omega^\alpha_{\frac{2}{\varepsilon}}\setminus\Omega^\alpha_{\frac{1}{\varepsilon}}} \nabla^\alpha w^\alpha_N(x)\cdot\nabla^\alpha\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x - \int_{\Omega^\alpha_{\frac{2}{\varepsilon}}\setminus\Omega^\alpha_{\frac{1}{\varepsilon}}} \nabla_{X_{\bar\alpha}} w^\alpha_{N-2}(x)\cdot\nabla_{X_{\bar\alpha}}\left(v\circ\varsigma^{-1}_\alpha(x)\right)\mathrm{d}x\\
=\;&-\int_{\Omega\setminus\Omega^\alpha}\left[\nabla^\alpha_\varepsilon\theta^\alpha_N(x;\varepsilon)\cdot\nabla^\alpha_\varepsilon v + \nabla_{X_{\bar\alpha}}\theta^\alpha_{N-2}(x;\varepsilon)\cdot\nabla_{X_{\bar\alpha}}v\right]|\det J_\alpha|\,\mathrm{d}x\\
=\;&-\frac{1}{\varepsilon}\int_{\Omega\setminus\Omega^\alpha}\nabla^\alpha_\varepsilon\theta^\alpha_N(x;\varepsilon)\cdot\nabla^\alpha_\varepsilon v + \nabla_{X_{\bar\alpha}}\theta^\alpha_{N-2}(x;\varepsilon)\cdot\nabla_{X_{\bar\alpha}}v\,\mathrm{d}x.
\end{aligned} \tag{3.3.21}$$

We compensate (3.3.20) and (3.3.21) in (3.3.19), we complete the proof of (3.3.17).

Case $|\alpha| = 2$. We follow the same argument as above. Then, firstly, we have

$$\int_{\Omega^{\alpha}_{\frac{2}{\varepsilon}}} \mathfrak{A}\nabla w^{\alpha}_{N}(x)\cdot\nabla\left(v\circ\varsigma^{-1}_{\alpha}(x)\right)\mathrm{d}x = 0$$

whence

$$\int_{\Omega^{\alpha}_{\frac{1}{\varepsilon}}} \mathfrak{A}\nabla w^{\alpha}_{N}(x)\cdot\nabla\left(v\circ\varsigma^{-1}_{\alpha}(x)\right)\mathrm{d}x = -\int_{\Omega^{\alpha}_{\frac{2}{\varepsilon}}\setminus\Omega^{\alpha}_{\frac{1}{\varepsilon}}} \nabla w^{\alpha}_{N}(x)\cdot\nabla\left(v\circ\varsigma^{-1}_{\alpha}(x)\right)\mathrm{d}x. \tag{3.3.22}$$

Recall that $\mathfrak{A} := \breve{A}\circ\varsigma^{-1}_{\alpha}$ on $\Omega^{\alpha}_{\frac{1}{\varepsilon}}$ and outside of $\Omega^{\alpha}_{\frac{1}{\varepsilon}}$, $\mathfrak{A} := \mathrm{Id}$. Making the corresponding change of variable $x \to \varsigma_{\alpha}(x)$, on the right-hand side of (3.3.22) gives

$$\int_{\Omega^{\alpha}_{\frac{1}{\varepsilon}}} \mathfrak{A}\nabla w^{\alpha}_{N}(x)\cdot\nabla\left(v\circ\varsigma^{-1}_{\alpha}(x)\right)\mathrm{d}x = \int_{\Omega^{\alpha}} A\nabla_{\varepsilon}\theta^{\alpha}_{N}(x;\varepsilon)\cdot\nabla_{\varepsilon}v\,|\det J_{\alpha}|\,\mathrm{d}x,$$

where $|\det J_{\alpha}| = \left|\left(\frac{-1}{\varepsilon}\right)^{|\alpha|}\right| = \left(\frac{1}{\varepsilon}\right)^{2}$ is the Jacobian. Then

$$\int_{\Omega^{\alpha}_{\frac{1}{\varepsilon}}} \mathfrak{A}\nabla w^{\alpha}_{N}(x)\cdot\nabla\left(v\circ\varsigma^{-1}_{\alpha}(x)\right)\,\mathrm{d}x = \left(\frac{1}{\varepsilon}\right)^{2}\int_{\Omega^{\alpha}} A\nabla_{\varepsilon}\theta^{\alpha}_{N}(x;\varepsilon)\cdot\nabla_{\varepsilon}v\,\mathrm{d}x. \tag{3.3.23}$$

For the left-hand side of the equation (3.3.22), we get

$$\begin{aligned}
-\int_{\Omega^{\alpha}_{\frac{2}{\varepsilon}}\setminus\Omega^{\alpha}_{\frac{1}{\varepsilon}}} \nabla w^{\alpha}_{N}(x)\cdot\nabla\left(v\circ\varsigma^{-1}_{\alpha}(x)\right)\mathrm{d}x &= -\int_{\Omega\setminus\Omega^{\alpha}} \nabla_{\varepsilon}\theta^{\alpha}_{N}(x;\varepsilon)\cdot\nabla_{\varepsilon}v\,|\det J_{\alpha}|\,\mathrm{d}x \\
&= -\left(\frac{1}{\varepsilon}\right)^{2}\int_{\Omega\setminus\Omega^{\alpha}} \nabla_{\varepsilon}\theta^{\alpha}_{N}(x;\varepsilon)\cdot\nabla_{\varepsilon}v\,\mathrm{d}x. \quad (3.3.24)
\end{aligned}$$

We compensate (3.3.23) and (3.3.24) in (3.3.22), we end up with (3.3.18).

This completes the proof of the lemma. □

Lemma 3.4. *For every $N \in \mathbb{N}$ and $\alpha \in E$ ($|\alpha| \neq 0$), there exist positive constants κ and η independent of ε such that*

Case $|\alpha| = 1$. *Whenever $\alpha_i = 0$, $i = 1, 2$, we have*

$$\left|\partial^{N-k+1}_{x_i} w^{\alpha}_{k}\right|_{S^{\alpha}_{0}\setminus\Omega^{\alpha}_{\frac{1}{\varepsilon}}},\quad \left|\partial^{N-k+1}_{x_i}\left(\nabla^{\alpha} w^{\alpha}_{k}\right)\right|_{S^{\alpha}_{0}\setminus\Omega^{\alpha}_{\frac{1}{\varepsilon}}},\quad |\nabla^{\alpha} w^{\alpha}_{k}|_{S^{\alpha}_{0}\setminus\Omega^{\alpha}_{\frac{1}{\varepsilon}}} \le \kappa\exp\left(\frac{-\eta}{\varepsilon}\right).$$

Case $|\alpha| = 2$. *It holds that*

$$|\nabla w^{\alpha}_{k}|_{S^{\alpha}_{0}\setminus\Omega^{\alpha}_{\frac{1}{\varepsilon}}} \le \kappa\exp\left(\frac{-\eta}{\varepsilon}\right),$$

where $k = 0, \ldots, N$.

Proof. We argue as we did for the proof of the smoothness theorem in Chapter 2. □

Now, we can show the theorem below, which will play an important role in the following.

Theorem 3.4. *Let u_ε, u_0, u_i $(i \geq 1)$, be the weak solutions of (3.1.5), (3.1.6), (3.3.10) respectively, and $\theta_i^\alpha(.;\varepsilon)$, $\alpha \in E$ $(|\alpha| \neq 0)$, given by (3.3.13) and (3.3.15). Then, under the assumptions (3.1.1), (3.3.9) and (3.3.16) satisfying for $N \in \mathbb{N}$, there exist two constants κ and $\eta > 0$ independent of ε, such that*

$$\begin{aligned} \int_\Omega A\nabla_\varepsilon RW_N(.;\varepsilon) \quad & \cdot \quad \nabla_\varepsilon v \, \mathrm{d}x - \frac{\lambda}{4}|\nabla_\varepsilon v|^2_{2,\Omega} \\ & \leq \quad \kappa \exp\left(\frac{-\eta}{\varepsilon}\right) - \varepsilon^{N+2}\sum_{|\alpha|<2}\int_\Omega D_\alpha \nabla_{X_{\bar\alpha}}\left[\varphi_\alpha \theta_N^\alpha(.;\varepsilon)\right]\cdot\nabla_{X_{\bar\alpha}} v\,\mathrm{d}x \\ & \quad\; -\varepsilon^{N+1}\sum_{|\alpha|<2}\int_\Omega C_\alpha \nabla_{X_{\bar\alpha}}\left[\varphi_\alpha \theta_N^\alpha(.;\varepsilon)\right]\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x \\ & \quad\; -\varepsilon^{N+1}\sum_{|\alpha|<2}\int_\Omega D_\alpha \nabla_{X_{\bar\alpha}}\left[\varphi_\alpha \theta_{N-1}^\alpha(.;\varepsilon)\right]\cdot\nabla_{X_{\bar\alpha}} v\,\mathrm{d}x \\ & \quad\; -\varepsilon^{N+1}\sum_{|\alpha|<2}\int_\Omega L_\alpha \nabla_\varepsilon^\alpha\left[\varphi_\alpha \theta_N^\alpha(.;\varepsilon)\right]\cdot\nabla_{X_{\bar\alpha}} v\,\mathrm{d}x, \end{aligned} \tag{3.3.25}$$

where $RW_N(.;\varepsilon) = u_\varepsilon - \sum_{i=0}^{N}\sum_{\alpha\in E}\varepsilon^i\left[\varphi_\alpha\theta_i^\alpha(.;\varepsilon)\right]$, by taking into account that $\theta_i^{(0,0)} = u_i$, for all $i \in \mathbb{N}$, and

$$D_\alpha := A_{11}, \quad C_\alpha := A_{21}, \quad L_\alpha := A_{12} \quad \text{and} \quad \nabla_{X_{\bar\alpha}} = \nabla_{X_1} \quad \text{for } \alpha = (0,0). \tag{3.3.26}$$

Proof. Following the same argument as in Chapter 2, then we get for $v \in H_0^1(\Omega)$,

$$\begin{aligned} &\varepsilon^2\int_\Omega A_{11}\nabla_{X_1}R_{N-2}(.;\varepsilon)\cdot\nabla_{X_1}v\,\mathrm{d}x + \varepsilon\int_\Omega A_{12}\nabla_{X_2}R_{N-1}(.;\varepsilon)\cdot\nabla_{X_1}v\,\mathrm{d}x \\ &+\varepsilon\int_\Omega A_{21}\nabla_{X_1}R_{N-1}(.;\varepsilon)\cdot\nabla_{X_2}v\,\mathrm{d}x + \int_\Omega A_{22}\nabla_{X_2}R_N(.;\varepsilon)\cdot\nabla_{X_2}v\,\mathrm{d}x = 0, \end{aligned}$$

where $R_N(.;\varepsilon) = u_\varepsilon - \sum_{i=0}^{N}\varepsilon^i u_i$. This can be written as an integral equation of $RW_N(.;\varepsilon)$ in the following form

$$\begin{aligned} \int_\Omega A\nabla_\varepsilon RW_N(.;\varepsilon)\cdot\nabla_\varepsilon v\,\mathrm{d}x \quad = \quad & -\varepsilon^{N+1}\int_\Omega A_{11}\nabla_{X_1}(u_{N-1}+\varepsilon u_N)\cdot\nabla_{X_1}v\,\mathrm{d}x \\ & -\varepsilon^{N+1}\int_\Omega A_{12}\nabla_{X_2}u_N\cdot\nabla_{X_1}v\,\mathrm{d}x - \varepsilon^{N+1}\int_\Omega A_{21}\nabla_{X_1}u_N\cdot\nabla_{X_2}v\,\mathrm{d}x \\ & -\sum_{i=0}^{N}\sum_{|\alpha|\neq 0}\varepsilon^i\int_\Omega A\nabla_\varepsilon\left[\varphi_\alpha\theta_i^\alpha(.;\varepsilon)\right]\cdot\nabla_\varepsilon v\,\mathrm{d}x. \end{aligned}$$

We rewrite, for every $\alpha \in E$ $(|\alpha| \neq 0)$, each integral in the last line of the above identity as a sum of two integrals over Ω^α and $\Omega\backslash\Omega^\alpha$, we derive

$$\begin{aligned} \int_\Omega A\nabla_\varepsilon RW_N(.;\varepsilon)\cdot \quad & \nabla_\varepsilon v\,\mathrm{d}x = -\varepsilon^{N+1}\int_\Omega A_{11}\nabla_{X_1}(u_{N-1}+\varepsilon u_N)\cdot\nabla_{X_1}v\,\mathrm{d}x \\ & -\varepsilon^{N+1}\int_\Omega A_{12}\nabla_{X_2}u_N\cdot\nabla_{X_1}v\,\mathrm{d}x - \varepsilon^{N+1}\int_\Omega A_{21}\nabla_{X_1}u_N\cdot\nabla_{X_2}v\,\mathrm{d}x \\ & -\sum_{i=0}^{N}\sum_{|\alpha|\neq 0}\varepsilon^i\int_{\Omega^\alpha} A\nabla_\varepsilon\theta_i^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon v\,\mathrm{d}x \\ & -\sum_{i=0}^{N}\sum_{|\alpha|\neq 0}\varepsilon^i\int_{\Omega\backslash\Omega^\alpha} A\nabla_\varepsilon\left[\varphi_\alpha\theta_i^\alpha(.;\varepsilon)\right]\cdot\nabla_\varepsilon v\,\mathrm{d}x. \end{aligned} \tag{3.3.27}$$

We treat the last two sums separately. For the first one, we have

$$\begin{aligned} T_1 &= \sum_{i=0}^{N} \sum_{|\alpha|\neq 0} \varepsilon^i \int_{\Omega^\alpha} A\nabla_\varepsilon \theta_i^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon v\,\mathrm{d}x \\ &= \sum_{i=0}^{N} \sum_{|\alpha|=2} \varepsilon^i \int_{\Omega^\alpha} A\nabla_\varepsilon \theta_i^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon v\,\mathrm{d}x + \sum_{i=0}^{N} \sum_{|\alpha|=1} \varepsilon^i \int_{\Omega^\alpha} A\nabla_\varepsilon \theta_i^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon v\,\mathrm{d}x. \end{aligned}$$

Then, for each $\alpha \in E$ with $|\alpha| = 1$, we factorize the matrix A in terms of the blocks A^α, L_α, C_α and D_α as follows

$$\begin{aligned} T_1 &= \sum_{i=0}^{N} \sum_{|\alpha|=2} \varepsilon^i \int_{\Omega^\alpha} A\nabla_\varepsilon \theta_i^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon v\,\mathrm{d}x \\ &+ \sum_{i=0}^{N} \sum_{|\alpha|=1} \varepsilon^i \left[\int_{\Omega^\alpha} A^\alpha\nabla_\varepsilon^\alpha \theta_i^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x + \varepsilon\int_{\Omega^\alpha} L_\alpha\nabla_\varepsilon^\alpha \theta_i^\alpha(.;\varepsilon)\cdot\nabla_{X_{\bar\alpha}} v\,\mathrm{d}x\right] \\ &+ \sum_{i=0}^{N} \sum_{|\alpha|=1} \varepsilon^i \left[\varepsilon\int_{\Omega^\alpha} C_\alpha\nabla_{X_{\bar\alpha}} \theta_i^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x + \varepsilon^2\int_{\Omega^\alpha} D_\alpha\nabla_{X_{\bar\alpha}} \theta_i^\alpha(.;\varepsilon)\cdot\nabla_{X_{\bar\alpha}} v\,\mathrm{d}x\right]. \end{aligned}$$

We reorder the terms of the above identity as powers of ε and we take into account the fact that $\theta_{-2}^\alpha = \theta_{-1}^\alpha = 0$, for all $\alpha \in E$, then applying Lemma 3.3, we get

$$\begin{aligned} T_1 &= \varepsilon^{N+1} \sum_{|\alpha|=1} \int_{\Omega^\alpha} D_\alpha\nabla_{X_{\bar\alpha}}\left[\theta_{N-1}^\alpha(.;\varepsilon) + \varepsilon\theta_N^\alpha(.;\varepsilon)\right]\cdot\nabla_{X_{\bar\alpha}} v\,\mathrm{d}x \\ &+ \varepsilon^{N+1} \sum_{|\alpha|=1} \left[\int_{\Omega^\alpha} L_\alpha\nabla_\varepsilon^\alpha \theta_N^\alpha(.;\varepsilon)\cdot\nabla_{X_{\bar\alpha}} v\,\mathrm{d}x + \int_{\Omega^\alpha} C_\alpha\nabla_{X_{\bar\alpha}} \theta_N^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x\right] \\ &- \sum_{i=0}^{N} \sum_{|\alpha|=1} \varepsilon^i \left[\int_{\Omega\setminus\Omega^\alpha} \nabla_\varepsilon^\alpha \theta_i^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x + \int_{\Omega\setminus\Omega^\alpha} \nabla_{X_{\bar\alpha}} \theta_{i-2}^\alpha(.;\varepsilon)\cdot\nabla_{X_{\bar\alpha}} v\,\mathrm{d}x\right] \\ &- \sum_{i=0}^{N} \sum_{|\alpha|=2} \varepsilon^i \int_{\Omega\setminus\Omega^\alpha} \nabla_\varepsilon \theta_i^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon v\,\mathrm{d}x. \end{aligned}$$

We reorder again the terms of the last two lines, we obtain

$$\begin{aligned} -T_1 &= -\varepsilon^{N+1} \sum_{|\alpha|=1} \int_{\Omega^\alpha} D_\alpha\nabla_{X_{\bar\alpha}}\left[\theta_{N-1}^\alpha(.;\varepsilon) + \varepsilon\theta_N^\alpha(.;\varepsilon)\right]\cdot\nabla_{X_{\bar\alpha}} v\,\mathrm{d}x \\ &- \varepsilon^{N+1} \sum_{|\alpha|=1} \int_{\Omega^\alpha} L_\alpha\nabla_\varepsilon^\alpha \theta_N^\alpha(.;\varepsilon)\cdot\nabla_{X_{\bar\alpha}} v\,\mathrm{d}x - \varepsilon^{N+1} \sum_{|\alpha|=1} \int_{\Omega^\alpha} C_\alpha\nabla_{X_{\bar\alpha}} \theta_N^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x \\ &+ \sum_{|\alpha|\neq 0} \sum_{i=0}^{N-2} \varepsilon^i \int_{\Omega\setminus\Omega^\alpha} \nabla_\varepsilon \theta_i^\alpha(.;\varepsilon)\cdot\nabla_\varepsilon v\,\mathrm{d}x \\ &+ \varepsilon^{N-1} \sum_{|\alpha|\neq 0} \int_{\Omega\setminus\Omega^\alpha} \nabla_\varepsilon^\alpha\left[\theta_{N-1}^\alpha(.;\varepsilon) + \varepsilon\theta_N^\alpha(.;\varepsilon)\right]\cdot\nabla_\varepsilon^\alpha v\,\mathrm{d}x. \end{aligned} \tag{3.3.28}$$

Using the Cauchy-Schwartz and Young inequalities, we derive

$$\begin{aligned}-T_1 \quad \leq \quad & -\varepsilon^{N+1}\sum_{|\alpha|=1}\int_{\Omega^\alpha} D_\alpha \nabla_{X_{\bar{\alpha}}}\left[\theta^\alpha_{N-1}(.;\varepsilon)+\varepsilon\theta^\alpha_N(.;\varepsilon)\right]\cdot\nabla_{X_{\bar{\alpha}}}v\,\mathrm{d}x \\ & -\varepsilon^{N+1}\sum_{|\alpha|=1}\int_{\Omega^\alpha} L_\alpha\nabla^\alpha_\varepsilon\theta^\alpha_N(.;\varepsilon)\cdot\nabla_{X_{\bar{\alpha}}}v\,\mathrm{d}x-\varepsilon^{N+1}\sum_{|\alpha|=1}\int_{\Omega^\alpha} C_\alpha\nabla_{X_{\bar{\alpha}}}\theta^\alpha_N(.;\varepsilon)\cdot\nabla^\alpha_\varepsilon v\,\mathrm{d}x \\ & +\sum_{|\alpha|\neq 0}\sum_{i=0}^{N}\mu\left|\nabla_\varepsilon\theta^\alpha_i(.;\varepsilon)\right|^2_{2,\Omega\backslash\Omega^\alpha}+\frac{(|E|-1)(N+1)}{4\mu}\left|\nabla_\varepsilon v\right|^2_{2,\Omega}.\end{aligned}$$

Choosing $\mu=\frac{2(|E|-1)(N+1)}{\lambda}$ and making the change of variable $x\to\varsigma_\alpha(x)$ in the last line over $\Omega\backslash\Omega^\alpha$, we get

$$\begin{aligned}-T_1 \leq \quad & \kappa\exp\left(\frac{-\eta}{\varepsilon}\right)+\frac{\lambda}{8}\left|\nabla_\varepsilon v\right|^2_{2,\Omega} \\ & -\varepsilon^{N+1}\sum_{|\alpha|=1}\int_{\Omega^\alpha} D_\alpha\nabla_{X_{\bar{\alpha}}}\left[\theta^\alpha_{N-1}(.;\varepsilon)+\varepsilon\theta^\alpha_N(.;\varepsilon)\right]\cdot\nabla_{X_{\bar{\alpha}}}v\,\mathrm{d}x \\ & -\varepsilon^{N+1}\sum_{|\alpha|=1}\int_{\Omega^\alpha} L_\alpha\nabla^\alpha_\varepsilon\theta^\alpha_N(.;\varepsilon)\cdot\nabla_{X_{\bar{\alpha}}}v\,\mathrm{d}x \\ & -\varepsilon^{N+1}\sum_{|\alpha|=1}\int_{\Omega^\alpha} C_\alpha\nabla_{X_{\bar{\alpha}}}\theta^\alpha_N(.;\varepsilon)\cdot\nabla^\alpha_\varepsilon v\,\mathrm{d}x.\end{aligned} \tag{3.3.29}$$

For the last sum T_2 in (3.3.27), we do not need to use Lemma 3.3 since the integrals are taken on $\Omega\backslash\Omega^\alpha$ and they will be transformed by the appropriate change of variable to $S^\alpha_0\backslash\Omega^\alpha_{\frac{1}{\varepsilon}}$–integral terms. The correctors on this last sub-domains are decaying exponentially. Thus we have

$$\begin{aligned}T_2 = \quad & \sum_{i=0}^{N}\sum_{|\alpha|\neq 0}\varepsilon^i\int_{\Omega\backslash\Omega^\alpha} A\nabla_\varepsilon\left[\varphi_\alpha\theta^\alpha_i(.;\varepsilon)\right]\cdot\nabla_\varepsilon v\,\mathrm{d}x \\ = \quad & \sum_{i=0}^{N-2}\sum_{|\alpha|\neq 0}\varepsilon^i\int_{\Omega\backslash\Omega^\alpha} A\nabla_\varepsilon\left[\varphi_\alpha\theta^\alpha_i(.;\varepsilon)\right]\cdot\nabla_\varepsilon v\,\mathrm{d}x \\ & +\varepsilon^{N-1}\sum_{|\alpha|\neq 0}\int_{\Omega\backslash\Omega^\alpha} A\nabla_\varepsilon\left[\varphi_\alpha\theta^\alpha_{N-1}(.;\varepsilon)\right]\cdot\nabla_\varepsilon v\,\mathrm{d}x \\ & +\varepsilon^{N}\sum_{|\alpha|\neq 0}\int_{\Omega\backslash\Omega^\alpha} A\nabla_\varepsilon\left[\varphi_\alpha\theta^\alpha_N(.;\varepsilon)\right]\cdot\nabla_\varepsilon v\,\mathrm{d}x.\end{aligned} \tag{3.3.30}$$

We follow the same argument as above, we deduce

$$\begin{aligned}-T_2 \leq \quad & -\varepsilon^{N+1}\sum_{|\alpha|=1}\int_{\Omega\backslash\Omega^\alpha} D_\alpha\nabla_{X_{\bar{\alpha}}}\left[\varphi_\alpha\left(\theta^\alpha_{N-1}(.;\varepsilon)+\varepsilon\theta^\alpha_N(.;\varepsilon)\right)\right]\cdot\nabla_{X_{\bar{\alpha}}}v\,\mathrm{d}x \\ & -\varepsilon^{N+1}\sum_{|\alpha|=1}\int_{\Omega\backslash\Omega^\alpha} L_\alpha\nabla^\alpha_\varepsilon\left[\varphi_\alpha\theta^\alpha_N(.;\varepsilon)\right]\cdot\nabla_{X_{\bar{\alpha}}}v\,\mathrm{d}x \\ & -\varepsilon^{N+1}\sum_{|\alpha|=1}\int_{\Omega\backslash\Omega^\alpha} C_\alpha\nabla_{X_{\bar{\alpha}}}\left[\varphi_\alpha\theta^\alpha_N(.;\varepsilon)\right]\cdot\nabla^\alpha_\varepsilon v\,\mathrm{d}x \\ & +\sum_{i=0}^{N}\sum_{|\alpha|\neq 0}|A|_{\infty,\Omega}\left(\kappa_{\omega_2}+1\right)\mu\left|\nabla_\varepsilon\theta^\alpha_i(.;\varepsilon)\right|^2_{2,\Omega\backslash\Omega^\alpha} \\ & +\frac{|A|_{\infty,\Omega}\left(\kappa_{\omega_2}+1\right)(|E|-1)(N+1)}{4\mu}\left|\nabla_\varepsilon v\right|^2_{2,\Omega}.\end{aligned} \tag{3.3.31}$$

Choosing $\mu = \frac{2|A|_{\infty,\Omega}\left(\kappa_{\omega_2}+1\right)(|E|-1)(N+1)}{\lambda}$ and making the change of variable $x \to \varsigma_\alpha(x)$ in the last line over $\Omega \backslash \Omega^\alpha$, we get

$$\begin{aligned}
-T_2 \leq\ & \frac{\lambda}{8}\left|\nabla_\varepsilon v\right|_{2,\Omega}^2 + \kappa \exp\left(\frac{-\eta}{\varepsilon}\right) \\
& -\varepsilon^{N+1} \sum_{|\alpha|=1} \int_{\Omega\backslash\Omega^\alpha} D_\alpha \nabla_{X_{\bar{\alpha}}}\left[\varphi_\alpha\left(\theta_{N-1}^\alpha(.;\varepsilon) + \varepsilon\theta_N^\alpha(.;\varepsilon)\right)\right] \cdot \nabla_{X_{\bar{\alpha}}} v \,\mathrm{d}x \\
& -\varepsilon^{N+1} \sum_{|\alpha|=1} \int_{\Omega\backslash\Omega^\alpha} L_\alpha \nabla_\varepsilon^\alpha \left[\varphi_\alpha \theta_N^\alpha(.;\varepsilon)\right] \cdot \nabla_{X_{\bar{\alpha}}} v \,\mathrm{d}x \\
& -\varepsilon^{N+1} \sum_{|\alpha|=1} \int_{\Omega\backslash\Omega^\alpha} C_\alpha \nabla_{X_{\bar{\alpha}}} \left[\varphi_\alpha \theta_N^\alpha(.;\varepsilon)\right] \cdot \nabla_\varepsilon^\alpha v \,\mathrm{d}x.
\end{aligned} \tag{3.3.32}$$

Substitute the estimates (3.3.29) and (3.3.32) in (3.3.27), taking into account that $\theta_N^{(0,0)} = u_N$, $\theta_{-1}^\alpha = 0$ (for all $\alpha \in E$) and (3.3.26), we obtain

$$\begin{aligned}
& \int_\Omega A\nabla_\varepsilon RW_N(.;\varepsilon) \cdot \nabla_\varepsilon v \,\mathrm{d}x - \frac{\lambda}{4}\left|\nabla_\varepsilon v\right|_{2,\Omega}^2 \\
\leq\ & \kappa \exp\left(\frac{-\eta}{\varepsilon}\right) - \varepsilon^{N+2} \sum_{|\alpha|<2} \int_\Omega D_\alpha \nabla_{X_{\bar{\alpha}}} \left[\varphi_\alpha \theta_N^\alpha(.;\varepsilon)\right] \cdot \nabla_{X_{\bar{\alpha}}} v \,\mathrm{d}x \\
& -\varepsilon^{N+1} \sum_{|\alpha|<2} \int_\Omega D_\alpha \nabla_{X_{\bar{\alpha}}} \left[\varphi_\alpha \theta_{N-1}^\alpha(.;\varepsilon)\right] \cdot \nabla_{X_{\bar{\alpha}}} v \,\mathrm{d}x \\
& -\varepsilon^{N+1} \sum_{|\alpha|<2} \int_\Omega L_\alpha \nabla_\varepsilon^\alpha \left[\varphi_\alpha \theta_N^\alpha(.;\varepsilon)\right] \cdot \nabla_{X_{\bar{\alpha}}} v \,\mathrm{d}x \\
& -\varepsilon^{N+1} \sum_{|\alpha|<2} \int_\Omega C_\alpha \nabla_{X_{\bar{\alpha}}} \left[\varphi_\alpha \theta_N^\alpha(.;\varepsilon)\right] \cdot \nabla_\varepsilon^\alpha v \,\mathrm{d}x.
\end{aligned}$$

This completes the proof of the theorem. □

As in the previous section (see Lemma 3.2), we can easily show that

$$v = RW_N(.;\varepsilon) \in H_0^1(\Omega). \tag{3.3.33}$$

Indeed, following the same argument as in the proof of Lemma 3.2, i.e. using the trace to show that $RW_N(.;\varepsilon) = 0$ on each face which also means that $RW_N(.;\varepsilon) = 0$ on $\partial\Omega$. Then it is enough to ensure that $u_i - \sum_{|\alpha|\neq 0} \varphi_\alpha \theta_i^\alpha(.;\varepsilon) = 0$ for all $i = 1, \ldots, N$ on each face as for $u_0 - \theta_\varepsilon$.

3.3.3 Asymptotic convergence results

In this subsection, we will see the results deduced from the first application of Theorem 3.4, but first we add the following assumptions

$$\partial_{x_1} a_{11}, \quad \partial_{x_2} a_{22} \in L^\infty(\Omega). \tag{3.3.34}$$

Then, we can prove

Theorem 3.5. *Under the assumptions of Theorem 3.3 and (3.3.16) satisfying for $N \in \mathbb{N}$, taking into account the addition of (3.3.34) whenever $N = 1$, it holds that, as $\varepsilon \to 0$,*

$$\begin{aligned} &\left| u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha \in E} \varepsilon^i \theta_i^\alpha (.;\varepsilon) \right|_{2,\Omega}, \quad \left| \nabla_{X_2} [u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha \in E} \varepsilon^i \theta_i^\alpha (.;\varepsilon)] \right|_{2,\Omega} = O(\varepsilon^{N+1}), \\ &\frac{1}{\varepsilon^N} \nabla_{X_1} [u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha \in E} \varepsilon^i \theta_i^\alpha (.;\varepsilon)] \rightharpoonup 0 \quad \text{in } L^2(\Omega). \end{aligned} \tag{3.3.35}$$

Proof. First, we take $v = RW_N (.;\varepsilon) \in H_0^1(\Omega)$ as a test function in (3.3.25) then using the ellipticity assumption (3.1.2), we get

$$\begin{aligned} &\frac{3\lambda}{4} |\nabla_\varepsilon RW_N (.;\varepsilon)|_{2,\Omega}^2 \\ \leq\ & \kappa \exp\left(\frac{-\eta}{\varepsilon}\right) - \varepsilon^{N+2} \sum_{|\alpha|<2} \int_\Omega D_\alpha \nabla_{X_{\bar\alpha}} [\varphi_\alpha \theta_N^\alpha (.;\varepsilon)] \cdot \nabla_{X_{\bar\alpha}} RW_N (.;\varepsilon) \, \mathrm{d}x \\ & -\varepsilon^{N+1} \sum_{|\alpha|<2} \int_\Omega C_\alpha \nabla_{X_{\bar\alpha}} [\varphi_\alpha \theta_N^\alpha (.;\varepsilon)] \cdot \nabla_\varepsilon^\alpha RW_N (.;\varepsilon) \, \mathrm{d}x \\ & -\varepsilon^{N+1} \sum_{|\alpha|<2} \int_\Omega D_\alpha \nabla_{X_{\bar\alpha}} \left[\varphi_\alpha \theta_{N-1}^\alpha (.;\varepsilon)\right] \cdot \nabla_{X_{\bar\alpha}} RW_N (.;\varepsilon) \, \mathrm{d}x \\ & -\varepsilon^{N+1} \sum_{|\alpha|<2} \int_\Omega L_\alpha \nabla_\varepsilon^\alpha [\varphi_\alpha \theta_N^\alpha (.;\varepsilon)] \cdot \nabla_{X_{\bar\alpha}} RW_N (.;\varepsilon) \, \mathrm{d}x. \end{aligned} \tag{3.3.36}$$

Note that under the assumptions of Theorem 3.5 satisfied for N, we have for all $\alpha \in E$ with $|\alpha| \neq 2$,

$$\nabla_{X_{\bar\alpha}} \cdot D_\alpha \nabla_{X_{\bar\alpha}} \left[\varphi_\alpha \theta_{N-1}^\alpha (.;\varepsilon)\right], \quad \nabla_{X_{\bar\alpha}} \cdot L_\alpha \nabla_\varepsilon^\alpha [\varphi_\alpha \theta_N^\alpha (.;\varepsilon)] \in L^2(\Omega).$$

Applying Green's formula in $X_{\bar\alpha}$–directions to the last two sums in (3.3.36), we obtain

$$\begin{aligned} &\frac{3\lambda}{4} |\nabla_\varepsilon RW_N (.;\varepsilon)|_{2,\Omega}^2 \\ \leq\ & \kappa \exp\left(\frac{-\eta}{\varepsilon}\right) - \varepsilon^{N+2} \sum_{|\alpha|<2} \int_\Omega D_\alpha \nabla_{X_{\bar\alpha}} [\varphi_\alpha \theta_N^\alpha (.;\varepsilon)] \cdot \nabla_{X_{\bar\alpha}} RW_N (.;\varepsilon) \, \mathrm{d}x \\ & -\varepsilon^{N+1} \sum_{|\alpha|<2} \int_\Omega C_\alpha \nabla_{X_{\bar\alpha}} [\varphi_\alpha \theta_N^\alpha (.;\varepsilon)] \cdot \nabla_\varepsilon^\alpha RW_N (.;\varepsilon) \, \mathrm{d}x \\ & +\varepsilon^{N+1} \sum_{|\alpha|<2} \int_\Omega \nabla_{X_{\bar\alpha}} \cdot \left(D_\alpha \nabla_{X_{\bar\alpha}} \left[\varphi_\alpha \theta_{N-1}^\alpha (.;\varepsilon)\right]\right) RW_N (.;\varepsilon) \, \mathrm{d}x \\ & +\varepsilon^{N+1} \sum_{|\alpha|<2} \int_\Omega \nabla_{X_{\bar\alpha}} \cdot \left(L_\alpha \nabla_\varepsilon^\alpha [\varphi_\alpha \theta_N^\alpha (.;\varepsilon)]\right) RW_N (.;\varepsilon) \, \mathrm{d}x, \end{aligned}$$

since $RW_N(.;\varepsilon) \in H_0^1(\Omega)$. Using the inequalities of Cauchy-Schwartz and Poincaré on ω_2, we get

$$\begin{aligned}
&\frac{3\lambda}{4} \left|\nabla_\varepsilon RW_N\left(.;\varepsilon\right)\right|_{2,\Omega}^2 \\
\leq\ & \kappa \exp\left(\frac{-\eta}{\varepsilon}\right) + \varepsilon^{N+1} \sum_{|\alpha|<2} \left|D_\alpha\right|_{\infty,\Omega} \left|\nabla_{X_{\bar\alpha}} \theta_N^\alpha\left(.;\varepsilon\right)\right|_{2,\Omega} \left|\varepsilon \nabla_{X_{\bar\alpha}} RW_N\left(.;\varepsilon\right)\right|_{2,\Omega} \\
&+\varepsilon^{N+1} \sum_{|\alpha|<2} \left|C_\alpha\right|_{\infty,\Omega} \left|\nabla_{X_{\bar\alpha}} \theta_N^\alpha\left(.;\varepsilon\right)\right|_{2,\Omega} \left|\nabla_\varepsilon^\alpha RW_N\left(.;\varepsilon\right)\right|_{2,\Omega} \\
&+\varepsilon^{N+1} \sum_{|\alpha|<2} \left|\nabla_{X_{\bar\alpha}} D^\alpha\right|_{\infty,\Omega} \sqrt{\kappa_{\omega_2}} \left|\nabla_{X_{\bar\alpha}} \theta_{N-1}^\alpha\left(.;\varepsilon\right)\right|_{2,\Omega} \left|\nabla_{X_2} RW_N\left(.;\varepsilon\right)\right|_{2,\Omega} \\
&+\varepsilon^{N+1} \sum_{|\alpha|<2} \left|D_\alpha\right|_{\infty,\Omega} \sqrt{\kappa_{\omega_2}} \left|D_{X_{\bar\alpha}}^2 \theta_{N-1}^\alpha\left(.;\varepsilon\right)\right|_{2,\Omega} \left|\nabla_{X_2} RW_N\left(.;\varepsilon\right)\right|_{2,\Omega} \\
&+\varepsilon^{N+1} \sum_{|\alpha|<2} \left|\nabla_{X_{\bar\alpha}} L_\alpha\right|_{\infty,\Omega} \sqrt{\kappa_{\omega_2}} \left|\nabla_\varepsilon^\alpha \theta_N^\alpha\left(.;\varepsilon\right)\right|_{2,\Omega} \left|\nabla_{X_2} RW_N\left(.;\varepsilon\right)\right|_{2,\Omega} \\
&+\varepsilon^{N+1} \sum_{|\alpha|<2} \left|L_\alpha\right|_{\infty,\Omega} \sqrt{\kappa_{\omega_2}} \left|\nabla_{X_{\bar\alpha}} \left[\nabla_\varepsilon^\alpha \theta_N^\alpha\left(.;\varepsilon\right)\right]\right|_{2,\Omega} \left|\nabla_{X_2} RW_N\left(.;\varepsilon\right)\right|_{2,\Omega} \\
&+\varepsilon^{N+2} \sum_{|\alpha|<2} \left|\nabla_{X_{\bar\alpha}} L_\alpha\right|_{\infty,\Omega} \sqrt{\kappa_{\omega_2}} \left|\theta_N^\alpha\left(.;\varepsilon\right)\right|_{2,\Omega} \left|\nabla_{X_2} RW_N\left(.;\varepsilon\right)\right|_{2,\Omega} \\
&+\varepsilon^{N+2} \sum_{|\alpha|<2} \left|L_\alpha\right|_{\infty,\Omega} \sqrt{\kappa_{\omega_2}} \left|\nabla_{X_{\bar\alpha}} \theta_N^\alpha\left(.;\varepsilon\right)\right|_{2,\Omega} \left|\nabla_{X_2} RW_N\left(.;\varepsilon\right)\right|_{2,\Omega}.
\end{aligned}$$

Again using Young's inequality and the boundedness of $\nabla_\varepsilon^\alpha \theta_i^\alpha\left(.;\varepsilon\right)$ and $\nabla_{X_{\bar\alpha}} \left[\nabla_\varepsilon^\alpha \theta_N^\alpha\left(.;\varepsilon\right)\right]$ in $L^2(\Omega)$, we obtain

$$\left|\nabla_\varepsilon RW_N(.;\varepsilon)\right|_{2,\Omega}^2 \leq \kappa \exp\left(\frac{-\eta}{\varepsilon}\right) + \kappa\varepsilon^{2N+2}. \tag{3.3.37}$$

From an other side we have,

$$u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha\in E} \varepsilon^i \theta_i^\alpha(.;\varepsilon) = RW_N(.;\varepsilon) - \sum_{i=0}^{N} \sum_{|\alpha|\neq 0} \varepsilon^i \left(1-\varphi_\alpha\right) \theta_i^\alpha(.;\varepsilon).$$

Recall that $1-\varphi_\alpha\left(X_1\right) = 0$ on Ω^α, i.e. $\operatorname{supp}\left(1-\varphi_\alpha\left(X_1\right)\right) \subset \Omega \backslash \Omega^\alpha$. Thus, if we take the L^2–norm of $\nabla_\varepsilon[u_\varepsilon - \sum\limits_{i=0}^{N} \sum\limits_{\alpha\in E} \varepsilon^i \theta_i^\alpha(.;\varepsilon)]$ on Ω, we get

$$\begin{aligned}
&\left|\nabla_\varepsilon[u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha\in E} \varepsilon^i \theta_i^\alpha(.;\varepsilon)]\right|_{2,\Omega}^2 \\
=\ & \left|\nabla_\varepsilon RW_N(.;\varepsilon) - \sum_{i=0}^{N} \sum_{|\alpha|\neq 0} \varepsilon^i \nabla_\varepsilon[\left(1-\varphi_\alpha\right) \theta_i^\alpha(.;\varepsilon)]\right|_{2,\Omega}^2 \\
\leq\ & 3\left|\nabla_\varepsilon RW_N(.;\varepsilon)\right|_{2,\Omega}^2 + 3\left|E\right|\left(N+1\right) \sum_{i=0}^{N} \sum_{\alpha\neq 0} \left|\nabla_\varepsilon \theta_i^\alpha(.;\varepsilon)\right|_{2,\Omega\backslash\Omega^\alpha}^2 \\
&+3\left|E\right|\left(N+1\right) \sum_{i=0}^{N} \sum_{|\alpha|\neq 0} \left|\theta_i^\alpha(.;\varepsilon)\right|_{2,\Omega\backslash\Omega^\alpha}^2.
\end{aligned}$$

Using (3.3.37), making the change of variable $x \to \varsigma_\alpha\left(x\right)$ on $\Omega\backslash\Omega^\alpha$, and Lemma 3.4 we obtain

$$\varepsilon^2 \left|\nabla_{X_1}[u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha\in E} \varepsilon^i \theta_i^\alpha(.;\varepsilon)]\right|_{2,\Omega}^2 + \left|\nabla_{X_2}[u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha\in E} \varepsilon^i \theta_i^\alpha(.;\varepsilon)]\right|_{2,\Omega}^2 \leq \kappa \exp\left(\frac{-\eta}{\varepsilon}\right) + \kappa\varepsilon^{2N+2}.$$

Thus

$$\begin{aligned}
\left| u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha\in E} \varepsilon^i \theta_i^\alpha \right|_{2,\Omega}, \quad \left| \nabla_{X_2}[u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha\in E} \varepsilon^i \theta_i^\alpha(.;\varepsilon)] \right|_{2,\Omega} &\leq \kappa\varepsilon^{N+1}, \\
\frac{1}{\varepsilon^N} \left| \nabla_{X_1}[u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha\in E} \varepsilon^i \theta_i^\alpha(.;\varepsilon)] \right|_{2,\Omega} &\leq \kappa,
\end{aligned} \tag{3.3.38}$$

since $\exp\left(\frac{-\eta}{\varepsilon}\right) = O(\varepsilon^{2N+2})$. Then, according to the above estimate (3.3.38), we can extract a weakly converging subsequence from $\frac{1}{\varepsilon^N}\nabla_{X_1}[u_\varepsilon - \sum\limits_{i=0}^{N} \sum\limits_{\alpha\in E} \varepsilon^i \theta_i^\alpha(.;\varepsilon)]$ in $L^2(\Omega)$. It follows, thanks to the uniqueness of the limit, that the whole sequence converge weakly to zero, i.e.

$$\frac{1}{\varepsilon^N}\nabla_{X_1}[u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha\in E} \varepsilon^i \theta_i^\alpha(.;\varepsilon)] \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega).$$

Of course, this is thanks to the density of $\mathcal{D}'(\Omega)$ in $L^2(\Omega)$ and the continuity of the derivation in $\mathcal{D}'(\Omega)$. This completes the proof of the theorem. □

Note that under the assumptions of Theorem 3.5 satisfied for $N+1$, we have

$$\begin{aligned}
&\left| \nabla_{X_1}[u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha\in E} \varepsilon^i \theta_i^\alpha(.;\varepsilon)] \right|_{2,\Omega}^2 \\
=& \left| \nabla_{X_1}[u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha\in E} \varepsilon^i \theta_i^\alpha(.;\varepsilon)] \pm \varepsilon^{N+1} \sum_{\alpha\in E} \nabla_{X_1}\theta_{N+1}^\alpha(.;\varepsilon) \right|_{2,\Omega}^2 \\
=& \left| \nabla_{X_1}[u_\varepsilon - \sum_{i=0}^{N+1} \sum_{\alpha\in E} \varepsilon^i \theta_i^\alpha(.;\varepsilon)] + \varepsilon^{N+1}\nabla_{X_1}u_{N+1} + \varepsilon^{N+1} \sum_{|\alpha|\neq 0} \nabla_{X_1}\theta_{N+1}^\alpha(.;\varepsilon) \right|_{2,\Omega}^2,
\end{aligned}$$

thus

$$\begin{aligned}
&\left| \nabla_{X_1}[u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha\in E} \varepsilon^i \theta_i^\alpha(.;\varepsilon)] \right|_{2,\Omega}^2 \\
\leq\; & 3\left| \nabla_{X_1}[u_\varepsilon - \sum_{i=0}^{N+1} \sum_{\alpha\in E} \varepsilon^i \theta_i^\alpha(.;\varepsilon)] \right|_{2,\Omega}^2 + 3\varepsilon^{2N+2}\left|\nabla_{X_1}u_{N+1}\right|_{2,\Omega}^2 + 3\varepsilon^{2N+2}\left| \sum_{|\alpha|\neq 0} \nabla_{X_1}\theta_{N+1}^\alpha(.;\varepsilon) \right|_{2,\Omega}^2 \\
\leq\; & 3\left| \nabla_{X_1}[u_\varepsilon - \sum_{i=0}^{N+1} \sum_{\alpha\in E} \varepsilon^i \theta_i^\alpha(.;\varepsilon)] \right|_{2,\Omega}^2 + 3\varepsilon^{2N+2}\left|\nabla_{X_1}u_{N+1}\right|_{2,\Omega}^2 \\
&+6\varepsilon^{2N+2}\kappa_{|E|} \sum_{|\alpha|=1} \left|\left(\nabla_{X_{\bar\alpha}}\theta_{N+1}^\alpha(.;\varepsilon)\right)\right|_{2,\Omega}^2 + 6\varepsilon^{2N+2}\kappa_{|E|} \sum_{|\alpha|\neq 0} \left|\nabla^\alpha\theta_{N+1}^\alpha(.;\varepsilon)\right|_{2,\Omega}^2 \\
\leq\; & \kappa\varepsilon^{2N},
\end{aligned}$$

since $\left|\nabla^\alpha\theta_{N+1}^\alpha(.;\varepsilon)\right|_{2,\Omega}^2 \leq \frac{\kappa}{\varepsilon^2}$. Then we can not conclude that there is an improvement for the lower order (ε^N)–composite asymptotic approximations in its convergence rate if we get the (ε^{N+1})-order composite asymptotic approximations as in Chapter 2.

Remark 3.6. *For $p = 1$, i.e. $X_1 \in (-1, 1)$, we do not add any assumptions for the construction of the boundary layer functions, i.e. under the same assumptions supposed in the study of the regular asymptotic expansion we construct the correctors.*

3.4 Sharper asymptotic convergence and estimate results

An expected natural improvement of the asymptotic convergence rate (3.3.35) may be expressed, as follows

$$\tfrac{1}{\varepsilon^{N+1}} RW_N\left(.;\varepsilon\right) \rightharpoonup 0 \quad \text{in } L^2\left(\Omega\right), \tag{3.4.1}$$

or simply

$$\tfrac{1}{\varepsilon^{N+1}}[u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha\in E} \varepsilon^i \theta_i^\alpha\left(.;\varepsilon\right)] \rightharpoonup 0 \quad \text{in } L^2\left(\Omega\right). \tag{3.4.2}$$

In this case, thanks to Theorem 3.5, we can easily show that $\left|\nabla_\varepsilon RW_N\left(.;\varepsilon\right)\right|_{2,\Omega} = O\left(\varepsilon^{N+1}\right)$ and we deduce in particular that

$$\tfrac{1}{\varepsilon^{N+1}} \nabla_\varepsilon RW_N\left(.;\varepsilon\right) \rightharpoonup 0 \quad \text{in } L^2\left(\Omega\right). \tag{3.4.3}$$

By consequence, we derive

$$\tfrac{1}{\varepsilon^{N+1}} \nabla_{X_2} RW_N\left(.;\varepsilon\right) \rightharpoonup 0 \quad \text{in } L^2\left(\Omega\right). \tag{3.4.4}$$

Note that we used the continuity of the derivative operators on $\mathcal{D}'(\Omega)$. In fact this means, by (3.3.27), that for all $|\alpha| = 1$, when $\varepsilon \to 0$,

$$\nabla_{X_1} \cdot A_{11} \nabla_{X_1} u_{N-1} + \nabla_{X_1} \cdot A_{12} \nabla_{X_2} u_N + \nabla_{X_2} \cdot A_{21} \nabla_{X_1} u_N = 0 \quad \text{a.e. on } \Omega \tag{3.4.5}$$

and

$$\left[\nabla_{X_{\bar\alpha}} \cdot D_\alpha \nabla_{X_{\bar\alpha}} \theta_{N-1}^\alpha + \nabla_{X_{\bar\alpha}} \cdot L_\alpha \nabla_\varepsilon^\alpha \theta_N^\alpha + \nabla_\varepsilon^\alpha \cdot C_\alpha \nabla_{X_{\bar\alpha}} \theta_N^\alpha\right]\left(.;\varepsilon\right) = o\left(1\right) \quad \text{a.e. on } \Omega^\alpha. \tag{3.4.6}$$

Taking into account Lemma 3.4 when we make the change of variable $x \to \varsigma_\alpha$, (3.4.5) and (3.4.6) simply mean, as $\varepsilon \to 0$, that

$$\nabla_{X_1} \cdot A_{11} \nabla_{X_1} u_{N-1} + \nabla_{X_1} \cdot A_{12} \nabla_{X_2} u_N + \nabla_{X_2} \cdot A_{21} \nabla_{X_1} u_N = 0 \quad \text{a.e. on } \Omega \tag{3.4.7}$$

and

$$\left[\nabla_{X_{\bar\alpha}} \cdot \mathfrak{D}_\alpha \nabla_{X_{\bar\alpha}} w_{N-1}^\alpha - \nabla_{X_{\bar\alpha}} \cdot \mathfrak{L}_\alpha \nabla^\alpha w_N^\alpha - \nabla^\alpha \cdot \mathfrak{C}_\alpha \nabla_{X_{\bar\alpha}} w_N^\alpha\right]\left(.;\varepsilon\right) = o\left(1\right) \tag{3.4.8}$$

a.e. on S_0^α for all $|\alpha| = 1$ where $w_{-1}^\alpha = 0$ for all $\alpha \in E$.

Indeed, rewriting the identity (3.3.27) and taking into account (3.3.28) and (3.3.30), we get for every $v \in H_0^1(\Omega)$,

$$\begin{aligned}
& \frac{1}{\varepsilon^{N+1}} \int_\Omega A \nabla_\varepsilon RW_N\left(.;\varepsilon\right) \cdot \nabla_\varepsilon v \,\mathrm{d}x \\
= \; & -\int_\Omega \left[\nabla_{X_1} \cdot A_{11} \nabla_{X_1} u_{N-1} + \nabla_{X_1} \cdot A_{12} \nabla_{X_2} u_N + \nabla_{X_2} \cdot A_{21} \nabla_{X_1} u_N\right] v \,\mathrm{d}x \\
& -\sum_{|\alpha|=1} \int_{\Omega^\alpha} \left[\nabla_{X_{\bar\alpha}} \cdot D_\alpha \nabla_{X_{\bar\alpha}} \theta_{N-1}^\alpha + \nabla_{X_{\bar\alpha}} \cdot L_\alpha \nabla_\varepsilon^\alpha \theta_N^\alpha + \nabla_\varepsilon^\alpha \cdot C_\alpha \nabla_{X_{\bar\alpha}} \theta_N^\alpha\right]\left(.;\varepsilon\right) v \,\mathrm{d}x \\
& -\varepsilon \int_\Omega A_{11} \nabla_{X_1} u_N \cdot \nabla_{X_1} v \,\mathrm{d}x - \varepsilon \sum_{|\alpha|=1} \int_{\Omega^\alpha} D_\alpha \nabla_{X_{\bar\alpha}} \theta_N^\alpha\left(.;\varepsilon\right) \cdot \nabla_{X_{\bar\alpha}} v \,\mathrm{d}x \\
& +\sum_{i=0}^{N-2} \sum_{|\alpha|\neq 0} \varepsilon^{i-N-1} \int_{\Omega\setminus\Omega^\alpha} \nabla_\varepsilon \theta_i^\alpha\left(.;\varepsilon\right) \cdot \nabla_\varepsilon v \,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon^{-2}\sum_{|\alpha|\neq 0}\int_{\Omega\backslash\Omega^{\alpha}}\nabla_{\varepsilon}^{\alpha}\left[\theta_{N-1}^{\alpha}\left(.;\varepsilon\right)+\varepsilon\theta_{N}^{\alpha}\left(.;\varepsilon\right)\right]\cdot\nabla_{\varepsilon}^{\alpha}v\,\mathrm{d}x\\
&-\sum_{i=0}^{N-2}\sum_{|\alpha|\neq 0}\varepsilon^{i-N-1}\int_{\Omega\backslash\Omega^{\alpha}}A\nabla_{\varepsilon}\left[\varphi_{\alpha}\theta_{i}^{\alpha}\left(.;\varepsilon\right)\right]\cdot\nabla_{\varepsilon}v\,\mathrm{d}x\\
&-\varepsilon^{-2}\sum_{|\alpha|\neq 0}\int_{\Omega\backslash\Omega^{\alpha}}A\nabla_{\varepsilon}\left[\varphi_{\alpha}\theta_{N-1}^{\alpha}\left(.;\varepsilon\right)\right]\cdot\nabla_{\varepsilon}v\,\mathrm{d}x-\varepsilon^{-1}\sum_{|\alpha|\neq 0}\int_{\Omega\backslash\Omega^{\alpha}}A\nabla_{\varepsilon}\left[\varphi_{\alpha}\theta_{N}^{\alpha}\left(.;\varepsilon\right)\right]\cdot\nabla_{\varepsilon}v\,\mathrm{d}x.
\end{aligned}
\tag{3.4.9}
$$

Note that

$$
\begin{aligned}
&\frac{1}{\varepsilon^{N+1}}\int_{\Omega}A\nabla_{\varepsilon}RW_{N}\left(.;\varepsilon\right)\cdot\nabla_{\varepsilon}v\,\mathrm{d}x\\
=\;&\varepsilon\int_{\Omega}\frac{1}{\varepsilon^{N}}A_{11}\nabla_{X_1}RW_{N}\left(.;\varepsilon\right)\cdot\nabla_{X_1}v\,\mathrm{d}x+\int_{\Omega}\frac{1}{\varepsilon^{N}}A_{12}\nabla_{X_2}RW_{N}\left(.;\varepsilon\right)\cdot\nabla_{X_1}v\,\mathrm{d}x\\
&+\int_{\Omega}\frac{1}{\varepsilon^{N}}A_{21}\nabla_{X_1}RW_{N}\left(.;\varepsilon\right)\cdot\nabla_{X_2}v\,\mathrm{d}x+\int_{\Omega}\frac{1}{\varepsilon^{N+1}}A_{22}\nabla_{X_2}RW_{N}\left(.;\varepsilon\right)\cdot\nabla_{X_2}v\,\mathrm{d}x.
\end{aligned}
$$

When we pass to the limit in the identity (3.4.9), using (3.3.35), (3.4.4), and what we have as exponential convergences (we mean the behaviour of the correctors on the sub-domains $\Omega\backslash\Omega^{\alpha}$ whenever $|\alpha|\neq 0$, which is decaying exponentially to zero) we have just to replace by $o\,(1)$ the left-hand side and all terms after the second line in the right-hand side. This implies, for every $v\in H_0^1(\Omega)$,

$$
\begin{aligned}
&\int_{\Omega}\left[\nabla_{X_1}\cdot A_{11}\nabla_{X_1}u_{N-1}+\nabla_{X_1}\cdot A_{12}\nabla_{X_2}u_{N}+\nabla_{X_2}\cdot A_{21}\nabla_{X_1}u_{N}\right]v\,\mathrm{d}x\\
=\;&o\,(1)-\lim_{\varepsilon\to 0}\sum_{|\alpha|=1}\int_{\Omega^{\alpha}}\left[\nabla_{X_{\bar\alpha}}\cdot D_{\alpha}\nabla_{X_{\bar\alpha}}\theta_{N-1}^{\alpha}+\nabla_{X_{\bar\alpha}}\cdot L_{\alpha}\nabla_{\varepsilon}^{\alpha}\theta_{N}^{\alpha}+\nabla_{\varepsilon}^{\alpha}\cdot C_{\alpha}\nabla_{X_{\bar\alpha}}\theta_{N}^{\alpha}\right]\left(.;\varepsilon\right)v\,\mathrm{d}x.
\end{aligned}
\tag{3.4.10}
$$

Note that for $v\in\mathcal{D}(\Omega)$, the support of v is located far away from all the boundary layers (faces and edges). This means all correctors θ_i^{α}, $|\alpha|\neq 0$ and $i=0,\ldots,N$, have no considerable effect since each of them is decaying exponentially to zero far away from its principal boundary layer region. Then we also can replace the right-hand side of (3.4.10) by $o\,(1)$ for this choice of a test function in $\mathcal{D}(\Omega)$ and we deduce that

$$
\int_{\Omega}\left[\nabla_{X_1}\cdot A_{11}\nabla_{X_1}u_{N-1}+\nabla_{X_1}\cdot A_{12}\nabla_{X_2}u_{N}+\nabla_{X_2}\cdot A_{21}\nabla_{X_1}u_{N}\right]v\,\mathrm{d}x=0\quad\forall v\in\mathcal{D}(\Omega).\tag{3.4.11}
$$

By consequence, we deduce from (3.3.10) and (3.4.11), that $u_{N+1}=0$ iff we have (3.4.11) (see Chapter 2). This also means that the existence of u_{N+1} solution of the problem (3.3.10) is ensured by the identity (3.4.11) in the distributional sense and we do not need to pass by the assumptions (3.3.11) to ensure this existence.

For the second member of (3.4.10), for example taking $\alpha=(1,0)$ and testing by $v\in H_0^1(\Omega_{(1,\kappa)}^{\alpha})$, $\Omega_{(1,\kappa)}^{\alpha}=(0,1)\times(-\kappa,\kappa)\times\omega_2$ $(0<\kappa<1)$. Note that on these sub-domains $\Omega_{(1,\kappa)}^{(1,0)}$, only the correctors $\theta_{N-1}^{(1,0)}$ and $\theta_{N}^{(1,0)}$ have a significant effect, however the others have no considerable effect on these sub-domains, i.e. θ_{N-1}^{α}, θ_{N}^{α} for $\alpha=(-1,0)$, $(0,1)$, $(0,-1)$, and more they are decaying exponentially to zero. Using (3.4.11), we derive from (3.4.10), when $\varepsilon\to 0$, that

$$
\int_{\Omega_{(1,\kappa)}^{(1,0)}}\left[\partial_{x_2}(a_{22}\partial_{x_2}\theta_{N-1}^{(1,0)})+\partial_{x_2}(L_{(1,0)}\nabla_{\varepsilon}^{(1,0)}\theta_{N}^{(1,0)})+\nabla_{\varepsilon}^{(1,0)}\cdot C_{(1,0)}\partial_{x_2}\theta_{N}^{(1,0)}\right]\left(.;\varepsilon\right)v\,\mathrm{d}x=o\,(1)\,.
$$

By consequence, making the change of variable $x \to \varsigma_\alpha$ and using Lemma 3.4, we deduce from (3.3.12), when $\varepsilon \to 0$, that

$$\int_{S^{(1,0)}_{(0,\kappa)}} [\nabla_{X_{\bar{\alpha}}} \cdot \mathfrak{D}_\alpha \nabla_{X_{\bar{\alpha}}} w^{(1,0)}_{N-1} - \nabla_{X_{\bar{\alpha}}} \cdot \mathfrak{L}_\alpha \nabla^\alpha w^{(1,0)}_N - \nabla^\alpha \cdot \mathfrak{C}_\alpha \nabla_{X_{\bar{\alpha}}} w^{(1,0)}_N] (.; \varepsilon)\, v \,\mathrm{d}x = o\,(1)$$

where $S^\alpha_{(0,\kappa)} = (0, +\infty) \times (-\kappa, \kappa) \times \omega_2$. Thanks to Lemma 3.4, we get from (3.3.12) that

$$\left|\nabla^\alpha w^{(1,0)}_{N+1}\right|_{2, S^{(1,0)}_0} = o\,(1) \quad \text{as } \varepsilon \to 0, \quad \text{on } S^{(1,0)}_0.$$

Arguing as above for the other three remaining indices $\alpha = (-1, 0)$, $(0, 1)$, $(0, -1)$, we can establish that, when $\varepsilon \to 0$, we have

$$\left|\nabla^\alpha w^\alpha_{N+1}\right|_{2, S^{(1,0)}_0} = o\,(1)\,, \quad \text{for } |\alpha| = 1. \tag{3.4.12}$$

Remark 3.7. *Note here that we do not have $w^\alpha_{N+1}(.; \varepsilon) = 0$ on $S^{(1,0)}_0$ (as in (3.4.11)), and we have just $w^\alpha_{N+1}(.; \varepsilon) \in H^1(I^\alpha_0)$, for $|\alpha| = 1$, and (3.4.12). We do not also have the existence of $w^\alpha_{N+1}(.; \varepsilon)$ in $H^1(S^\alpha_0)$. For the last one we need the assumptions (3.3.16) to be satisfied for $N+1$. More details about this point will be given below.*

Now, if we expect a little bit more as

$$\tfrac{1}{\varepsilon^{N+2}} \nabla_{X_2} RW_N\,(.; \varepsilon) \rightharpoonup 0 \quad \text{in } L^2(\Omega)$$

or simply

$$\tfrac{1}{\varepsilon^{N+2}} \nabla_{X_2} [u_\varepsilon - \sum_{i=0}^{N} \sum_{\alpha \in E} \varepsilon^i \theta^\alpha_i\,(.; \varepsilon)] \rightharpoonup 0 \quad \text{in } L^2(\Omega).$$

As above, this holds iff we have (3.4.5), (3.4.6) satisfied with an $o\,(\varepsilon)$–order of magnitude and

$$\begin{cases} \nabla_{X_1} \cdot A_{11} \nabla_{X_1} u_N = 0 \quad \text{in } D'\,(\Omega) \\ \nabla_{X_{\bar{\alpha}}} \cdot D_\alpha \nabla_{X_{\bar{\alpha}}} \theta^\alpha_N\,(.; \varepsilon) = o\,(1) \quad \text{in } D'\,(\Omega^\alpha) \text{ as } \varepsilon \longrightarrow 0, \text{ for all } |\alpha| = 1, \end{cases} \tag{3.4.13}$$

or simply, iff we have (3.4.7), (3.4.8) satisfied with an $o\,(\varepsilon)$–estimate with

$$\nabla_{X_1} \cdot A_{11} \nabla_{X_1} u_N = 0 \quad \text{in } D'\,(\Omega) \tag{3.4.14}$$

and

$$\nabla_{X_{\bar{\alpha}}} \cdot \mathfrak{D}_\alpha \nabla_{X_{\bar{\alpha}}} w^\alpha_N\,(.; \varepsilon) = o\,(1) \quad \text{in } \mathcal{D}'\,(S^\alpha_0) \quad \text{as } \varepsilon \longrightarrow 0, \quad \text{for all } |\alpha| = 1. \tag{3.4.15}$$

We likewise deduce, u_{N+2} exists and is null since the right-hand side of (3.3.10) is vanishing. However for $w^\alpha_{N+2}(.; \varepsilon)$ when $|\alpha| = 1$, we do not have the sufficient hypotheses to be well defined in $H^1(I^\alpha_0)$. We will treat this point in the next.

Nevertheless the last expectations are not little, we can easily see from (3.3.10), that

$$u_k = 0, \quad \forall k > N$$

and $\nabla_\varepsilon RW_N\,(.; \varepsilon)$ has the following order of magnitude

$$\nabla_\varepsilon RW_N\,(.; \varepsilon) = o\left(\varepsilon^{N+2}\right) \quad \text{in } L^2(\Omega).$$

Actually, it follows that if we have (3.4.7), (3.4.8) and (3.4.14), (3.4.15) with $O\left(\varepsilon^{\nu+1}\right)$– and $O\left(\varepsilon^{\nu}\right)$–estimate respectively ($\nu > 0$), we derive

$$\frac{1}{\varepsilon^{N+2}} \nabla_{\varepsilon} RW_N\left(.;\varepsilon\right) = O(\varepsilon^{\nu}) \quad \text{in } L^2(\Omega).$$

By consequence, we get

$$\frac{1}{\varepsilon^{N+2}} \nabla_{\varepsilon}[u_{\varepsilon} - \sum_{i=0}^{N} \sum_{\alpha \in E} \varepsilon^{i} \theta_i^{\alpha}\left(.;\varepsilon\right)] = O(\varepsilon^{\nu}) \quad \text{in } L^2(\Omega).$$

Of course, this lets us to think about a sharper rate of asymptotic convergence as it is summarized in the following theorem.

Theorem 3.6. *Given the assumptions of Theorem 3.5 and consider the following assertions,*

(i) $\nabla^{\alpha} w_{N+1}^{\alpha}\left(.;\varepsilon\right) = o\left(1\right)$ *in* $L^2\left(S_0^{\alpha}\right)$, *for all* $|\alpha| < 2$,

(ii) The conditions (3.4.7) and (3.4.8) hold, with an $o\left(1\right)$*–estimate,*

(iii) $\frac{1}{\varepsilon^{N+1}}[u_{\varepsilon} - \sum_{i=0}^{N} \sum_{\alpha \in E} \varepsilon^{i} \theta_i^{\alpha}\left(.;\varepsilon\right)] \rightharpoonup 0$ *in* $L^2(\Omega)$,

(iv) $u_{\varepsilon} - \sum_{i=0}^{N} \sum_{\alpha \in E} \varepsilon^{i} \theta_i^{\alpha}\left(.;\varepsilon\right) = o(\varepsilon^{N+1})$ *in* $\mathcal{V}\left(\Omega\right)$,

(v) The conditions (3.4.7), (3.4.8) and (3.4.14), (3.4.15) hold, with an $o\left(\varepsilon\right)$*– and* $o\left(1\right)$*–estimate respectively,*

(vi) $\frac{1}{\varepsilon^{N+2}} \nabla_{X_2}[u_{\varepsilon} - \sum_{i=0}^{N} \sum_{\alpha \in E} \varepsilon^{i} \theta_i^{\alpha}\left(.;\varepsilon\right)] \rightharpoonup 0$ *in* $L^2\left(\Omega\right)$,

(vii) $u_{\varepsilon} - \sum_{i=0}^{N} \sum_{\alpha \in E} \varepsilon^{i} \theta_i^{\alpha}\left(.;\varepsilon\right) = o\left(\varepsilon^{N+2}\right)$ *in* $\mathcal{V}\left(\Omega\right)$.

Then we have, as $\varepsilon \longrightarrow 0$

$$(vii) \Leftrightarrow (vi) \Leftrightarrow (v) \Rightarrow (iv) \Leftrightarrow (iii) \Leftrightarrow (ii) \Leftrightarrow (i).$$

Proof. We can easily see, from what is done above, that

$$(i) \Leftrightarrow (ii) \Leftarrow (iii) \Leftarrow (iv).$$

Then let us start by showing how *(ii)* ensures the strong convergence *(iv)* which also guarantees the equivalence between the first four assertions. Going back to (3.3.25) testing with $v = RW_N\left(x;\varepsilon\right) \in H_0^1\left(\Omega\right)$, we get

$$\begin{aligned} & \frac{3\lambda}{4}\left|\nabla_{\varepsilon} RW_N\left(.;\varepsilon\right)\right|_{2,\Omega}^2 \\ \leq \; & \kappa \exp\left(\frac{-\eta}{\varepsilon}\right) - \varepsilon^{2N+2} \int_{\Omega} A_{11} \nabla_{X_1} u_N \cdot \left[\varepsilon^{-N} \nabla_{X_1} RW_N\left(x;\varepsilon\right)\right] \mathrm{d}x \\ & -\varepsilon^{2N+2} \sum_{|\alpha|=1} \int_{\Omega} D_{\alpha} \nabla_{X_{\bar{\alpha}}} \theta_N^{\alpha} \cdot [\varepsilon^{-N} \nabla_{X_{\bar{\alpha}}} RW_N\left(x;\varepsilon\right)] \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&-\varepsilon^{2N+2}\int_{\Omega}\left[\nabla_{X_1}\cdot A_{11}\nabla_{X_1}u_{N-1}+\nabla_{X_1}\cdot A_{12}\nabla_{X_2}u_N+\nabla_{X_2}\cdot A_{21}\nabla_{X_1}u_N\right]\varepsilon^{-N-1}RW_N\left(.;\varepsilon\right)\mathrm{d}x\\
&-\varepsilon^{2N+2}\sum_{|\alpha|=1}\int_{\Omega}\left[\nabla_{\varepsilon}^{\alpha}\cdot C_{\alpha}\nabla_{X_{\bar\alpha}}\theta_N^{\alpha}\left(.;\varepsilon\right)+\nabla_{X_{\bar\alpha}}\cdot D_{\alpha}\nabla_{X_{\bar\alpha}}\theta_{N-1}^{\alpha}\left(.;\varepsilon\right)\right]\varepsilon^{-N-1}RW_N\left(.;\varepsilon\right)\mathrm{d}x\\
&-\varepsilon^{2N+2}\sum_{|\alpha|=1}\int_{\Omega}\nabla_{X_{\bar\alpha}}\cdot L_{\alpha}\nabla_{\varepsilon}^{\alpha}\theta_N^{\alpha}\left(.;\varepsilon\right)\varepsilon^{-N-1}RW_N\left(.;\varepsilon\right)\mathrm{d}x. \qquad (3.4.16)
\end{aligned}
$$

Using (3.3.35), what we have as exponential estimates and the assertion *(ii)*, we obtain

$$|\nabla_{\varepsilon}RW_N\left(.;\varepsilon\right)|_{2,\Omega}=o(\varepsilon^{N+1})$$

and by consequence,

$$\left|\nabla_{\varepsilon}[u_{\varepsilon}-\sum_{i=0}^{N}\sum_{\alpha\in E}\varepsilon^i\theta_i^{\alpha}\left(.;\varepsilon\right)]\right|_{2,\Omega}=o(\varepsilon^{N+1}).$$

This means

$$|\nabla_{X_1}RW_N\left(.;\varepsilon\right)|_{2,\Omega}=o\left(\varepsilon^N\right)\quad\text{and}\quad|\nabla_{X_2}RW_N\left(.;\varepsilon\right)|_{2,\Omega}=o\left(\varepsilon^{N+1}\right)$$

and

$$\left|\nabla_{X_1}[u_{\varepsilon}-\sum_{i=0}^{N}\sum_{\alpha\in E}\varepsilon^i\theta_i^{\alpha}\left(.;\varepsilon\right)]\right|_{2,\Omega}=o\left(\varepsilon^N\right)\quad\text{and}\quad\left|\nabla_{X_2}[u_{\varepsilon}-\sum_{i=0}^{N}\sum_{\alpha\in E}\varepsilon^i\theta_i^{\alpha}\left(.;\varepsilon\right)]\right|_{2,\Omega}=o\left(\varepsilon^{N+1}\right). \qquad (3.4.17)$$

Then applying the Poincaré inequality in the X_2–directions in the last identity we end up with *(iv)*. Now everything is easy from what is done above, i.e. we have just to prove for example *(v)* $\Rightarrow$ *(vii)*. Going back to (3.4.16) and taking into account *(v)*, and what we have as exponential estimates, we obtain directly

$$|\nabla_{\varepsilon}RW_N\left(.;\varepsilon\right)|_{2,\Omega}=o\left(\varepsilon^{N+2}\right)$$

and by consequence

$$\left|\nabla_{\varepsilon}[u_{\varepsilon}-\sum_{i=0}^{N}\sum_{\alpha\in E}\varepsilon^i\theta_i^{\alpha}\left(.;\varepsilon\right)]\right|_{2,\Omega}=o\left(\varepsilon^{N+2}\right).$$

This completes the proof of the theorem. □

Remark 3.8. *It is clear, from (3.4.17), that the assertion (iii) also implies a convergence faster than* ε^N *ensured in a standard Sobolev space, as* $\varepsilon\longrightarrow 0$*, i.e.*

$$u_{\varepsilon}-\sum_{i=0}^{N}\sum_{\alpha\in E}\varepsilon^i\theta_i^{\alpha}\left(.;\varepsilon\right)=o\left(\varepsilon^N\right)\quad in\ H^1\left(\Omega\right).$$

Remark 3.9. *In the case when we have the weak convergence (vi), we can show that*

$$\tfrac{1}{\varepsilon^{N+2}}[u_{\varepsilon}-\sum_{i=0}^{N}\sum_{\alpha\in E}\varepsilon^i\theta_i^{\alpha}\left(.;\varepsilon\right)]\rightharpoonup 0\quad in\ L^2\left(\Omega\right). \qquad (3.4.18)$$

Of course, because to the weak convergence (vi), we have

$$\tfrac{1}{\varepsilon^{N+2}}\left|\nabla_{X_2}[u_{\varepsilon}-\sum_{i=0}^{N}\sum_{\alpha\in E}\varepsilon^i\theta_i^{\alpha}\left(.;\varepsilon\right)]\right|_{2,\Omega}\leq\kappa.$$

Applying the Poincaré inequality in the X_2–directions, we deduce the boundedness of the sequence

$$\tfrac{1}{\varepsilon^{N+2}}\left|u_\varepsilon - \sum_{i=0}^{N}\sum_{\alpha\in E}\varepsilon^i\theta_i^\alpha\left(.;\varepsilon\right)\right|_{2,\Omega}.$$

Then, we use again the density of $\mathcal{D}'\left(\Omega\right)$ in $L^2\left(\Omega\right)$ and the continuity of the derivation in $\mathcal{D}'\left(\Omega\right)$ we end up with (3.4.18) (without adding any assumptions on the data).

Remark 3.10. *Keeping the same results of the above theorem, the weak convergence (vi) can be weakened as follows*

$$\tfrac{1}{\varepsilon^{N+2}}[u_\varepsilon - \sum_{i=0}^{N}\sum_{\alpha\in E}\varepsilon^i\theta_i^\alpha\left(.;\varepsilon\right)] \rightharpoonup 0 \quad \text{in } L^2\left(\Omega\right) \tag{3.4.19}$$

and by consequence, we also have

$$\tfrac{1}{\varepsilon^{N+2}}RW_N\left(.;\varepsilon\right) \rightharpoonup 0 \quad \text{in } L^2\left(\Omega\right),$$

if we assume more smoothness as

$$\partial_{x_2}a_{12},\quad \partial_{x_1}a_{21},\quad \left(A_{21}^T\right)' = \left(\partial_{x_j}a_{ij}\right)^T_{i=2+1,\dots,\mathrm{n},\ j=1,\dots,2},\ \left(A_{22}^T\right)' = \left(\partial_{x_j}a_{ij}\right)^T_{i,j=2+1,\dots,\mathrm{n}} \in L^\infty(\Omega).$$

where "$.^T$" denotes the transpose of the matrix.

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