Chapter 20
A Method of Solution for Integro-Differential Parabolic Equation with Purely Integral Conditions

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Abstract The objective of this paper is to prove existence, uniqueness, and continuous dependence upon the data of solution to integro-differential parabolic equation with purely integral conditions. The proofs are based on a priory estimates and Laplace transform method. Finally, we obtain the solution by using a numerical technique for inverting the Laplace transforms.

20.1 Introduction

In this paper we are concerned with the following parabolic Integro-differential equation,

$$\frac{\partial v}{\partial t}(x,t)-\frac{\partial^2 v}{\partial x^2}(x,t)=g(x,t)+\int_0^t a(t-s)\,v(x,s)\,ds,\ 0<x<1,\ 0<t\leq T, \tag{20.1}$$

subject to the initial condition

$$v(x,0)=\Phi(x),\ \ 0<x<1, \tag{20.2}$$

and the integral conditions

$$\int_0^1 v(x,t)\,dx=r(t),\ \ 0<t\leq T, \tag{20.3}$$

$$\int_0^1 xv(x,t)\,dx=q(t),\ \ 0<t\leq T, \tag{20.4}$$

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where v is an unknown function, r, q, and $\Phi(x)$ are given functions supposed to be sufficiently regular, a is suitably defined function satisfying certain conditions to be specified later, and T is a positive constant. Certain problems of modern physics and technology can be effectively described in terms of nonlocal problems for partial differential equations [3–7, 9–13, 15, 16, 20, 21, 23–27]. Ang [2] has considered a one-dimensional heat equation with nonlocal (integral) conditions. The author has taken the Laplace transform of the problem and then used numerical technique for the inverse Laplace transform to obtain the numerical solution.

This paper is organized as follows. In Sect. 20.2, we begin introducing certain function spaces which are used in the next sections, and we reduce the posed problem to one with homogeneous integral conditions. In Sect. 20.3, we first establish the existence of solution by the Laplace transform. In Sect. 20.4, we establish a priory estimates, which give the uniqueness and continuous dependence upon the data.

20.2 Statement of the Problem and Notation

Since integral conditions are inhomogeneous, it is convenient to convert problem (20.1)–(20.2) to an equivalent problem with homogenous integral conditions. For this, we introduce a new function $u(x,t)$ representing the deviation of the function $v(x,t)$ from the function

$$u(x,t) = v(x,t) - u_1(x,t), \quad 0 < x < 1,\ 0 < t \leq T, \tag{20.5}$$

where

$$u_1(x,t) = 6\left(2q(t) - r(t)\right)x - 2\left(3q(t) - 2r(t)\right). \tag{20.6}$$

Problem (20.1)–(20.2) with inhomogeneous integral conditions (20.3), (20.4) can be equivalently reduced to the problem of finding a function u satisfying

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t) + \int_0^t a(t-s)\, u(x,s)\, ds,\ 0 < x < 1,\ 0 < t \leq T, \tag{20.7}$$

$$u(x,0) = \varphi(x),\ \ 0 < x < 1, \tag{20.8}$$

$$\int_0^1 u(x,t)\, dx = 0,\ \ 0 < t \leq T, \tag{20.9}$$

$$\int_0^1 x u(x,t)\, dx = 0,\ \ 0 < t \leq T \tag{20.10}$$

where

$$f(x,t) = g(x,t) - \left(\frac{\partial u_1}{\partial t}(x,t) - \frac{\partial^2 u_1}{\partial x^2}(x,t) - \int_0^t a(t-s)\, u_1(x,s)\, ds\right) \tag{20.11}$$

and

$$\varphi(x) = \Phi(x) - u_1(x,0) \tag{20.12}$$

Hence, instead of solving for v, we simply look for u. The solution of problem (20.1)–(20.4) will be obtained by the relation (20.5), (20.6). We introduce the appropriate function spaces that will be used in the rest of the note. Let H be a Hilbert space with a norm $\|.\|_H$.

Let $L^2(0,1)$ be the standard function space.

Definition 20.1. (i) Denote by $L^2(0,T,H)$ the set of all measurable abstract functions $u(.,t)$ from $(0,T)$ into H equipped with the norm

$$\|u\|_{L^2(0,T,H)} = \left(\int_0^T \|u(.,t)\|_H^2 \, dt \right)^{1/2} < \infty \tag{20.13}$$

(ii) Let $C(0,T,H)$ be the set of all continuous functions $u(.,t) : (0,T) \longrightarrow H$ with

$$\|u\|_{C(0,T,H)} = \max_{0 \le t \le T} \|u(.,t)\|_H < \infty \tag{20.14}$$

(iii) We denote by $C_0(0,1)$ the vector space of continuous functions with compact support in $(0,1)$. Since such function are Lebesgue integrable with respect to dx, we can define on $C_0(0,1)$ the bilinear form given by

$$((u,w)) = \int_0^1 J_x^m u . J_x^m w dx, \quad m \ge 1 \tag{20.15}$$

where

$$J_x^m u = \int_0^x \frac{(x-\zeta)^{m-1}}{(m-1)!} u(\zeta,t) \, d\zeta; \quad \text{for } m \ge 1 \tag{20.16}$$

The bilinear form (20.15) is considered as a scalar product on $C_0(0,1)$ is not complete.

Definition 20.2. Denote by $B_2^m(0,1)$, the completion of $C_0(0,1)$ for the scalar product (20.15), which is denoted $(.,.)_{B_2^m(0,1)}$, introduced by [5]. By the norm of function u from $B_2^m(0,1)$, $m \ge 1$, we understand the nonnegative number:

$$\|u\|_{B_2^m(0,1)} = \left(\int_0^1 (J_x^m u)^2 \, dx \right)^{1/2} = \|J_x^m u\|; \text{ for } m \ge 1 \tag{20.17}$$

Lemma 20.3. *For all $m \in \mathbb{N}^*$, the following inequality holds:*

$$\|u\|^2_{B_2^m(0,1)} \le \frac{1}{2} \|u\|^2_{B_2^{m-1}(0,1)} . \tag{20.18}$$

Proof. See [5]. □

Corollary 20.4. *For all* $m \in \mathbb{N}^*$, *we have the elementary inequality*

$$\|u\|^2_{B_2^m(0,1)} \leq \left(\frac{1}{2}\right)^m \|u\|^2_{L^2(0,1)} . \tag{20.19}$$

Definition 20.5. We denote by $L^2(0,T;B_2^m(0,1))$ the space of functions which are square integrable in the Bochner sense, with the scalar product

$$(u,w)_{L^2(0,T;B_2^m(0,1))} = \int_0^T (u(.,t),w(.,t))_{B_2^m(0,1)}\, dt. \tag{20.20}$$

Since the space $B_2^m(0,1)$ is a Hilbert space, it can be shown that $L^2(0,T;B_2^m(0,1))$ is a Hilbert space as well. The set of all continuous abstract functions in $[0,T]$ equipped with the norm

$$\sup_{0\leq t\leq T} \|u(.,t)\|_{B_2^m(0,1)}$$

is denoted $C(0,T;B_2^m(0,1))$.

Corollary 20.6. *For every* $u \in L^2(0,1)$, *from which we deduce the continuity of the imbedding* $L^2(0,1) \longrightarrow B_2^m(0,1)$, *for* $m \geq 1$.

Lemma 20.7. (*Gronwall Lemma*) *Let* $f_1(t)$, $f_2(t) \geq 0$ *be two integrable functions on* $[0,T]$, $f_2(t)$ *is nondecreasing. If*

$$f_1(\tau) \leq f_2(\tau) + c\int_0^\tau f_1(t)\, dt, \qquad \forall \tau \in [0,T], \tag{20.21}$$

where $c \in \mathbb{R}^+$, *then*

$$f_1(t) \leq f_2(t)\exp(ct), \qquad \forall t \in [0,T]. \tag{20.22}$$

Proof. The proof is the same as that of Lemma 1.3.19 in [19]. □

20.3 Existence of the Solution

In this section we shall apply the Laplace transform technique to find solutions of partial differential equations; we have the Laplace transform

$$V(x,s) = \mathscr{L}\{v(x,t); t \longrightarrow s\} = \int_0^\infty v(x,t)\exp(-st)\, dt, \tag{20.23}$$

where s is positive reel parameter. Taking the Laplace transforms on both sides of (20.1), we have

$$(s - A(s))V(x,s) - \frac{d^2}{dx^2}V(x,s) = G(x,s) + s\Phi(x), \tag{20.24}$$

where $G(x,s)=\mathscr{L}\{g(x,t);t\longrightarrow s\}$. Similarly, we have

$$\int_0^1 V(x,s)\,dx = R(s), \tag{20.25}$$

$$\int_0^1 xV(x,s)\,dx = Q(s), \tag{20.26}$$

where

$$R(s)=\mathscr{L}\{r(t);t\longrightarrow s\}$$

and

$$Q(s)=\mathscr{L}\{q(t);t\longrightarrow s\}.$$

Now, we have the following cases:

Case 1: If $s-A(s)>0$

Case 2: If $s-A(s)<0$

Case 3: If $s-A(s)=0$

We only consider cases 2 and 3, as case 1 can be dealt with similarly as in [2]. For $(s-A(s))=0$, we have

$$\frac{d^2}{dx^2}V(x,s)=-G(x,s)-s\Phi(x), \tag{20.27}$$

The general solution for case 3 is given by

$$V(x,s)=-\int_0^x\int_0^y[G(x,s)+s\Phi(x)]\,dzdy+C_1(s)x+C_2(s), \tag{20.28}$$

Putting the integral conditions $(3.3),(3.4)$ in (3.6) we get

$$\begin{aligned}&\frac{1}{2}C_1(s)+C_2(s)\\&=\int_0^1\int_0^x\int_0^y[G(x,s)+s\Phi(x)]\,dzdy+R(s),\end{aligned} \tag{20.29}$$

$$\begin{aligned}&\frac{1}{3}C_1(s)+\frac{1}{2}C_2(s)\\&=\int_0^1\int_0^x\int_0^y x[G(x,s)+s\Phi(x)]\,dzdy+Q(s),\end{aligned} \tag{20.30}$$

where

$$\begin{aligned}C_1(s)=\;&12\int_0^1\int_0^x\int_0^y x[G(x,s)+s\Phi(x)]\,dzdy-\\&6\int_0^1\int_0^x\int_0^y[G(x,s)+s\Phi(x)]\,dzdy+\\&12Q(s)-6R(s),\end{aligned} \tag{20.31}$$

$$\begin{aligned}C_2(s) = 4\int_0^1\int_0^x\int_0^y [G(x,s)+s\Phi(x)]\,dzdy - \\ 6\int_0^1\int_0^x\int_0^y x[G(x,s)+s\Phi(x)]\,dzdy - \\ 6Q(s)+4R(s).\end{aligned} \tag{20.32}$$

For case 2, that is, $(s - A(s)) < 0$,using the method of variation of parameter, we have the general solution as

$$\begin{aligned}V(x,s) = \frac{1}{\sqrt{A(s)-s}}\int_0^x (G(x,s)+s\Phi(x))\sin\left(\sqrt{A(s)-s}\right)(x-\tau)\,d\tau \\ +d_1(s)\cos\sqrt{(A(s)-s)}x + d_2(s)\sin\sqrt{(A(s)-s)}x\end{aligned} \tag{20.33}$$

From the integral conditions (20.25), (20.26) we get

$$\begin{aligned}&d_1(s)\int_0^1 \cos\sqrt{(A(s)-s)}xdx + d_2(s)\int_0^1 \sin\sqrt{(A(s)-s)}xdx = \\ &R(s) - \frac{1}{\sqrt{A(s)-s}}\int_0^1\int_0^x [(G(x,s)+s\Phi(x)) \\ &\sin\left(\sqrt{A(s)-s}\right)(x-\tau)\Big]\,d\tau dx,\end{aligned} \tag{20.34}$$

$$\begin{aligned}&d_1(s)\int_0^1 x\cos\sqrt{(A(s)-s)}xdx + d_2(s)\int_0^1 x\sin\sqrt{(A(s)-s)}xdx = \\ &Q(s) - \frac{1}{\sqrt{A(s)-s}}\int_0^1\int_0^x [x(G(x,s)+s\Phi(x)) \\ &\sin\left(\sqrt{A(s)-s}\right)(x-\tau)\Big]\,d\tau dx.\end{aligned} \tag{20.35}$$

Thus d_1, d_2 are given by

$$\begin{pmatrix} d_1(s) \\ d_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \times \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}, \tag{20.36}$$

and

$$\begin{aligned}a_{11}(s) &= \int_0^1 \cos\sqrt{(A(s)-s)}xdx, \\ a_{12}(s) &= \int_0^1 \sin\sqrt{(A(s)-s)}xdx, \\ a_{21}(s) &= \int_0^1 x\cos\sqrt{(A(s)-s)}xdx, \\ a_{22}(s) &= \int_0^1 x\sin\sqrt{(A(s)-s)}xdx,\end{aligned}$$

$$\begin{aligned} b_1\left(s\right) &= R(s) - \frac{1}{\sqrt{A\left(s\right)-s}} \int_0^1 \int_0^x \left(G\left(x,s\right)+s\Phi\left(x\right)\right) \\ &\quad \times \sin\left(\sqrt{A\left(s\right)-s}\right)\left(x-\tau\right)d\tau dx, \\ b_2\left(s\right) &= Q(s) - \frac{1}{\sqrt{A\left(s\right)-s}} \int_0^1 \int_0^x \left[x\left(G\left(x,s\right)+s\Phi\left(x\right)\right)\right. \\ &\quad \left.\sin\left(\sqrt{A\left(s\right)-s}\right)\left(x-\tau\right)\right] d\tau dx. \end{aligned} \tag{20.37}$$

If it is not possible to calculate the integrals directly, then we calculate it numerically. We approximate similarly as given in [2]. If the Laplace inversion is possible directly for (20.28) and (20.33), in this case we shall get our solution. In another case we use the suitable approximate method and then use the numerical inversion of the Laplace transform. Considering $A\left(s\right)-s=k\left(s\right)$ and using Gauss's formula given in [1] we have the following approximations of the integrals:

$$\begin{aligned} &\int_0^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \cos\sqrt{k\left(s\right)}x dx \\ &\simeq \frac{1}{2}\sum_{i=1}^{N} w_i \begin{pmatrix} 1 \\ \frac{1}{2}\left[x_i+1\right] \end{pmatrix} \cos\left(\sqrt{k\left(s\right)}\frac{1}{2}\left[x_i+1\right]\right), \end{aligned} \tag{20.38}$$

$$\begin{aligned} &\int_0^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \sin\sqrt{k\left(s\right)}x dx \\ &\simeq \frac{1}{2}\sum_{i=1}^{N} w_i \begin{pmatrix} 1 \\ \frac{1}{2}\left[x_i+1\right] \end{pmatrix} \sin\left(\sqrt{k\left(s\right)}\frac{1}{2}\left[x_i+1\right]\right), \end{aligned} \tag{20.39}$$

$$\begin{aligned} &\int_0^x \left(G\left(x,s\right)+s\Phi\left(x\right)\right)\sin\left(\sqrt{k\left(s\right)}\right)\left(x-\tau\right)d\tau \\ &\simeq \frac{x}{2}\sum_{i=1}^{N} w_i \left[G\left(\frac{x}{2}\left[x_i+1\right];s\right)+s\Phi\left(\frac{x}{2}\left[x_i+1\right]\right)\right] \\ &\quad \times \sin\left(\sqrt{k\left(s\right)}\left[x-\frac{x}{2}\left[x_i+1\right]\right]\right), \end{aligned} \tag{20.40}$$

$$\begin{aligned} &\int_0^1 \left[\left[G\left(\tau,s\right)+s\Phi\left(\tau\right)\right]\int_\tau^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \sin\left(\sqrt{k\left(s\right)}\right)\left(x-\tau\right)dx\right]d\tau \\ &\simeq \frac{1}{2}\sum_{i=1}^{N} w_i \left[G\left(\frac{1}{2}\left[x_i+1\right];s\right)+s\Phi\left(\frac{1}{2}\left[x_i+1\right]\right)\right] \times \\ &\left(\frac{1-\frac{1}{2}\left[x_i+1\right]}{2}\right) \times \sum_{i=1}^{N} w_j \left(\begin{matrix} 1 \\ \frac{1-\frac{1}{2}\left[x_i+1\right]}{2}x_j + \frac{1-\frac{1}{2}\left[x_i+1\right]}{2} \end{matrix}\right) \times \end{aligned}$$

$$\sin\left[\sqrt{k(s)}\times\left(\frac{1-\frac{1}{2}[x_i+1]}{2}x_j+\frac{1+\frac{1}{2}[x_i+1]}{2}-\frac{1}{2}(x_i+1)\right)\right], \tag{20.41}$$

where x_i and w_i are the abscissa and weights, defined as

$$x_i : i^{th} \text{ zero of } P_n(x), \qquad \omega_i = 2/\left(1-x_i^2\right)\left[P_n'(x)\right]^2.$$

Their tabulated values can be found in [1] for different values of N.

20.3.1 Numerical Inversion of Laplace Transform

Sometimes, an analytical inversion of a Laplace domain solution is difficult to obtain [28]; therefore, a numerical inversion method must be used. A nice comparison of four frequently used numerical Laplace inversion algorithms is given by Hassan Hassanzadeh, Mehran Pooladi-Darvish [18]. In this work we use the Stehfest's algorithm [29] that is easy to implement. This numerical technique was first introduced by Graver [17] and its algorithm then offered by [29]. Stehfest's algorithm approximates the time domain solution as

$$v(x,t)\approx\frac{\ln 2}{t}\sum_{n=1}^{2m}\beta_n V\left(x;\frac{n\ln 2}{t}\right), \tag{20.42}$$

where, m is the positive integer,

$$\beta_n=(-1)^{n+m}\sum_{k=\left[\frac{n+1}{2}\right]}^{\min(n,m)}\frac{k^m(2k)!}{(m-k)!k!(k-1)!(n-k)!(2k-n)!}, \tag{20.43}$$

and $[q]$ denotes the integer part of the real number q.

20.4 Uniqueness and Continuous Dependence of the Solution

We establish an a priori estimate; the uniqueness and continuous dependence of the solution with respect to the data are immediate consequences.

Theorem 20.8. *If* $u(x,t)$ *is a solution of problem* (20.7)–(20.10) *and* $f\in C\left(\overline{D}\right)$, *then we have a priory estimates:*

$$\begin{aligned}&\|u(.,\tau)\|^2_{L^2(0,1)}\\ &\leq c_1\left(\|f(.,t)\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)}+\|\varphi\|^2_{L^2(0,1)}\right)\end{aligned} \tag{20.44}$$

$$\left\| \frac{\partial u(.,\tau)}{\partial t} \right\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)}$$

$$\leq c_2 \left(\|f(.,t)\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)} + \|\varphi\|^2_{L^2(0,1)} \right) \tag{20.45}$$

where $c_1 = \exp(a_0 T)$, $c_2 = \frac{\exp(a_0 T)}{1-a_0}$, $1 < a(x,t) < a_0$, *and* $0 \leq \tau \leq T$.

Proof. Taking the scalar product in $B_2^1(0,1)$ of equation (20.7) and $\frac{\partial u}{\partial t}$ and integrating over $(0,\tau)$, we have

$$\int_0^\tau \left(\frac{\partial u(.,t)}{\partial t}, \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt -$$

$$\int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial x^2}, \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt$$

$$= \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt +$$

$$\int_0^\tau \left(\int\limits_0^t a(t-s)\, u(x,s)\, ds, \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt \tag{20.46}$$

By integrating by parts, the first and second terms in the left-hand side of (20.46) we obtain

$$\left\| \frac{\partial u(.,t)}{\partial t} \right\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)} +$$

$$\frac{1}{2} \|u(.,\tau)\|^2_{L^2(0,1)} - \frac{1}{2} \|\varphi\|^2_{L^2(0,1)}$$

$$= \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt +$$

$$\int_0^\tau \left(\int\limits_0^t a(t-s)\, u(x,s)\, ds, \frac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt \tag{20.47}$$

By the **Cauchy inequality**, the first term in the right-hand side of (20.46) is bounded by

$$\frac{1}{2} \|f(.,t)\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)} + \frac{1}{2} \left\| \frac{\partial u(.,t)}{\partial t} \right\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)} \tag{20.48}$$

and second term in the right-hand side of (20.46) is bounded by

$$\frac{a_0}{2} \int\limits_0^t \|u(x,s)\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)} ds + \frac{a_0}{2} \left\| \frac{\partial u(.,t)}{\partial t} \right\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)} \tag{20.49}$$

Substitution of (20.48), (20.49) into (20.47) yields

$$(1-a_0)\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)}+\|u(.,\tau)\|^2_{L^2(0,1)}\leq$$

$$\left(\|f(.,t)\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)}+\|\varphi\|^2_{L^2(0,1)}\right)+\frac{a_0}{2}\int_0^t\|u(x,s)\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)}ds. \qquad (20.50)$$

By Gronwall Lemma, we have

$$(1-a_0)\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)}+\|u(.,\tau)\|^2_{L^2(0,1)}$$
$$\leq\exp(a_0T)\left(\|f(.,t)\|^2_{L^2\left(0,T;\,B_2^1(0,1)\right)}+\|\varphi\|^2_{L^2(0,1)}\right). \qquad (20.51)$$

From (20.51), we obtain estimates (20.44) and (20.45). □

Corollary 20.9. *If problem* (20.7)–(20.10) *has a solution, then this solution is unique and depends continuously on* (f,φ).

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