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Thanks

In the name of Allah, the Most Merciful, the Most Merciful.

<< Glory to You! We don't have to know that You taught us. Certainly, you are the Omniscient, the Wise >>. Sura 2 verse 32.

O Allah! Send peace and blessing on Abraham, Ishmael, Isaac, Jacob, Moses, Jesus, on Mohamed the seal of the prophets, his companions, his family and all those who follow them on the right path until the last day. Amen.

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Résumé

Dans ce travail on a étudié deux problèmes directe et inverse pour des équations aux dérivées partielles fractionnaires avec des conditions de Dirichlet.

On a débuté par des rappels de certaines notions préliminaires fondamentales et les outils nécessaires dans ce travail.

Le deuxième chapitre voué à étudier l'existence et l'unicité d'une solution forte d'un problème pour une classe d'équation aux dérivées partielles fractionnaires avec des conditions de Dirichlet.

Enfin, le troisième chapitre est destiné à la solvabilité de la solution pour un problème inverse pour une classe d'équations aux dérivées partielles fractionnaires avec une information supplémentaire sous forme de condition intégrale.

Mots clés : Equations paraboliques fractionnaires, Inégalités d'énergie, Espaces fonctionnelles, Conditions intégrales, Théorème de point fixe.

Abstract

In this work we have studied two classical and fractional linear parabolic problems with boundary conditions of Dirichlet type.

We started with reminders of some fundamental preliminary concepts and tools needed in this work.

The second chapter is devoted to studying the existence and uniqueness of a strong solution of a fractional linear parabolic problem with Dirichlet condition.

Finally, the third chapter is intended for the solvability of the solution for an inverse problem for a class of fractional partial differential equations with additional information in the form of an integral condition.

Keywords: Fractional parabolic equations, Energy inequality, Sobolev spaces, Integral condition, Fixed point theorem.

ملخص

يهدف هذه العمل إلى دراسة مسألتين حديتين من المعادلات التفاضلية الجزئية الكافئتي الكسرية والعادية المقرونة بشروط حدية من نوع ديركلي.

بدأنا بتذكير بعض المفاهيم والأدوات الأولية الأساسية اللازمة في هذا العمل.

الفصل الثاني مخصص لدراسة وجود وتفرد حل قوي لمشكلة قطع مكافئ كسري خطي مع وجود شرط حدي من نوع ديركلي.

أخيرًا ، الفصل الثالث مخصص لدراسة الوجود والوحدانية لحل مشكلة عكسية لفئة من المعادلات التفاضلية الجزئية الكسرية مع معلومات إضافية في شكل شرط متكامل.

الكلمات المفتاحية:

معادلة تكافئية كسرية، طريقة التقديرات القبلية، فضاء سوبوليف، شرط حدي من نوع تكامل ، نظرية النقطة الثابتة

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Introduction

Fractional differential equations (FDEs) are obtained by generalizing differential equations to an arbitrary order. Since fractional differential equations are used to model complex phenomena, they play a crucial role in engineering, physics and applied mathematics. Therefore they have been generating increasing interest from engineers and scientists in recent years. Since FDEs have memory, nonlocal relations in space and time, complex phenomena can be modeled by using these equations. Due to this fact, materials with memory and hereditary effects, through strongly anomalous media. Indeed, we can find numerous applications in viscoelasticity, electro-chemistry, signal processing, control theory, porous media, fluid flow, rheology, diffusive transport, electrical networks, electromagnetic theory and probability, signal processing, and many other physical processes are diverse applications of FDEs [1, 19, 20, 22, 23, 35].

Recently, there has been a significant development in fractional differential and partial differential equations; see the monographs of Kilbas et al. [35], Miller and Ross [9], Samko et al. [10] and the papers of Agarwal et al. [40], Anguraj A. and Karthikeyan P. [47], Belmekki et al. [41], Daftardar-Gejji and Jafari [36], Furati and Tatar [29, 34], Kaufmann and Mboumi [37], Kilbas and Marzan [31], Yu and Gao [33], Oussaeif [54], and also the

general references in Baleanu et al. [48], and the references therein.

The study of existence and uniqueness, periodicity, asymptotic behavior, stability, and methods of analytic and numerical solutions of fractional differential equations have been studied extensively in a large cycle works. But there are not many works in the fractional field of partial differential equation, this is due to the difficulty of applying classical theories and methods to a field of fractional partial differential equation.

On the other hand; Inverse parabolic equation problems occur naturally in many fields like:

- Petroleum engineering (seismic and magnetic prospecting, determination of permeability in reservoirs)
- Medical imaging (ultrasound, scanners, x-rays)
- Hydrogeology (determining hydraulic permeability)
- Chemistry (determining reaction constants)
- Radar and underwater acoustics (obstacle determination)
- Image processing (recovery of blurry images).

And there is extensive literature on inverse heat equation problems (see [27, 32, 55, 56], and references therein).

In this study the supplementary or additional information about the solution to the inverse problem comes in the form of the integral condition.

In engineering and physics, the parameter recognition in a partial differential equation from the data of the integral overdetermination condition plays an important role

[7, 8, 24, 39, 51, 53]. From a physical point of view, these conditions can be interpreted by a system averaging the domain of spatial variables as measurements of the temperature solution $u(x, t)$.

Note that inverse problems related with integral overdetermination [5, 52]. Studies have shown that when we deal with these kinds of nonclassical problems, classical approaches sometimes do not work [2, 15]. To date, different methods for addressing problems resulting from nonlocal inverse problem have been suggested. The choice of approach depends on the form of nonlocal boundary value that are involved.

We note that several authors have studied the inverse parabolic problem with condition of type integral and its special solubility (see, for example, [3, 4, 6, 12, 55 – 57]). There are also several articles dedicated to the study of the existence and uniqueness of inverse problem solutions for different parabolic equations with unknown source functions. Inverse problems related by determining unknown function in source term of a parabolic equation with overdetermination condition studied in [17, 21].

Motivated by this, we conducted a detailed and thorough study in this field to see the behavior of the solution to fractional partial differential linear parabolic problems using and developing the classic energy estimate method. Also, a new research on the inverse problem of a fractional parabolic equation is discussed.

Chapter 1

Preliminary concept

This chapter is devoted to essential reminders of the basic notions and concepts of analysis used throughout this work, for permanent use in the next chapters. These important notions are stated in the form of definitions, theorems, corollaries and lemmas. For more details, references to the literature will be systematically given.

1.1 Unbounded linear operators

Definition 1 [60] *Let E and F be two vector spaces. An operator T is an application from E to F :*

$$T : E \rightarrow F.$$

Any linear operator T is completely defined by its graph $G(T)$ which is a vector subspace of $E \times F$ defined by:

$$G(T) = \{(u, Tu), u \in D(T)\}$$

where $D(T)$ is the domain of definition of the operator T .

Definition 2 [60] *An operator T from E to F is said to be linear if and only if:*

$$\forall u_1, u_2 \in E, \forall \mu, \lambda \in \mathbb{C} \text{ we have : } T(\lambda u_1 + \mu u_2) = \lambda T(u_1) + \mu T(u_2),$$

where \mathbb{C} is the field of scalars of E and F .

Definition 3 [60] *We say that S is an extension of T if $D(T) \subset D(S)$ and $Tu = Su$ for all $u \in D(T)$. In other words, $G(T) \subset G(S)$.*

Remark 4 *It is not true that every subspace of $E \times F$ is the graph of an operator.*

Definition 5 [60] *T is said to be closed if its graph $G(T)$ is closed from $E \times F$.*

Definition 6 [60] *A linear operator T is said to be closable in E if it admits a closed extension.*

In other words T is closable if and only if for any sequence $(u_n)_{n \in \mathbb{N}} \subset D(T)$ such that $u_n \rightarrow 0$ and $Tu_n \rightarrow v$, then $v = 0$.

1.2 Relationship between orthogonality and density in Hilbert spaces

Definition 7 [60] *Let M be a vector subspace of the Hilbert space F , we define M^\perp the orthogonal of M , by:*

$$M^\perp = \{f \in F, \langle f, g \rangle_F = 0, \forall g \in M\}$$

Proposition 8 [60] *Let M be a vector subspace of the Hilbert space F . Then M is dense in F if and only if $M^\perp = \{0\}$.*

Proof. Suppose first that M is dense in F . Let $f \in M^\perp \subset F$, let $(f_n)_{n \in \mathbb{N}}$ be a sequence of elements of M which converges to f . We have $\langle f, f_n \rangle_F = 0$ for all $n \in \mathbb{N}$. Passing to the limit, we conclude that $\|f\|_F = 0$. So $f = 0$, which gives $M^\perp = \{0\}$.

Conversely, suppose $M^\perp = \{0\}$. Then we have $(M^\perp)^\perp = \{0\}^\perp = F$, and since $M \subset \overline{M}$ it follows that $(\overline{M})^\perp \subset M^\perp$, and so $(M^\perp)^\perp \subset ((\overline{M})^\perp)^\perp$, month \overline{M} is a closed one, then $((\overline{M})^\perp)^\perp = \overline{M}$, then we find $(M^\perp)^\perp \subset \overline{M} \implies F \subset \overline{M}$. Hence $F = \overline{M}$. ■

1.3 Fractional derivation

Fractional derivation is a concept of generalization of the (classical) derivation to a non-integer order. It is also introduced naturally into the mechanical modeling of materials which retain the memory of past transformations. Hence the particular interest in calculus and fractional analysis during the last decades. Although classical differential calculus provides powerful tools for modeling a large number of phenomena studied by applied sciences, these tools do not make it possible to take into account the abnormal dynamics presented by certain complex systems encountered in nature or in society interactions. Experimental results show that many processes related to complex systems have non-local dynamics involving long-term effects.

The history of the derivative of non-integer order spreads out from the end of the 17th century until our days. scholars agree to trace its beginning to the end of 1695 when **L'Hospital** raised a question to **Leibniz** wondering about the meaning of $\frac{d^n y}{dx^n}$ when $n = \frac{1}{2}$. The first serious attempt to give a logical definition for the fractional derivative is due to Liouville who published nine papers in this subject between 1832 and 1837. Independently,

Riemann proposed an approach which turned out to be essentially that of Liouville, and it is since it bears the name "**Riemann-Liouville approach**". Later, other theories appeared such as that of **Grunwald-Leitnikov**, **Weyl** and **Caputo**. At that time there were almost no practical applications of this theory, and it is for this reason that it was considered an abstract containing only mathematical manipulations of little use. The transition from pure mathematical formulations to applications began to emerge from the 1990s, when fractional differential equations appeared in several fields such as physics, engineering, biology, mechanics...

1.3.1 Special function

One of the basic functions of fractional calculus is the **Gamma** function $\Gamma(x)$, which plays a very important role in fractional calculus theory.

Definition 9 [35] *We call Eulerian Gamma function (or Eulerian unequal of the second kind) the function denoted Γ defined for any complex number x such that $\text{Re}(x) > 0$ by:*

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt,$$

this integral is convergent for $\text{Re}(x) > 0$.

Proposition 10 [35] *For all $x \in \mathbb{R}_*^+$ we have:*

$$\Gamma(x+1) = x\Gamma(x).$$

Proof. We have by definition 9

$$\Gamma(x+1) = \int_0^{+\infty} e^{-t} t^x dt,$$

we use the integral by parts, we get

$$\begin{aligned}\Gamma(x+1) &= -e^{-tx} \Big|_{t=0}^{t=\infty} + x \int_0^{+\infty} e^{-tx} t^{x-1} dt \\ &= x \int_0^{+\infty} e^{-tx} t^{x-1} dt\end{aligned}$$

so

$$\Gamma(x+1) = x\Gamma(x).$$

■

1.3.2 Riemann Liouville approach

Definition 11 [46] *Let $\alpha \in \mathbb{R}^+$ and v be a locally integral function defined on $[0, T]$. The derivative of order α of v is defined by:*

1. *Derivative in the sense of **Riemann Liouville on the left***

$${}^R D_t^\alpha v(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{v(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau. \quad (1.1)$$

2. *Derivative in the sense of **Riemann Liouville on the right***

$${}^R D_t^\alpha v(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^T \frac{v(\tau)}{(\tau-t)^{\alpha-n+1}} d\tau, \quad (1.2)$$

where the integer n is chosen such that: $n-1 < \alpha < n$.

1.3.3 Fractional derivatives in the sense of Caputo

Although the fractional derivation in the sense of **Riemann-Liouville** has played an important role in the development of fractional calculus, several authors including Caputo

(1967 – 1969) have realized that this definition needs to be revised because the problems applied in visco- elasticity, solid mechanics and rheology, require initial conditions that can be physically interpreted by classical derivatives, which is not the case in modeling by the **Riemann-Liouville** approach which requires knowledge of the initial conditions of fractional derivatives.

Definition 12 [46] *Let $\alpha \in \mathbb{R}^+$ and v be a locally integral function defined on $[0, T]$. The derivative of order α of v is defined by:*

1. *Derivative in the sense of **Caputo on the left***

$${}^C D_t^\alpha v(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{v^{(n)}(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau, \quad (1.3)$$

2. *Derivative in the sense of **Caputo on the right***

$${}^C D_t^\alpha v(t) = \frac{(-1)^n}{\Gamma(n - \alpha)} \int_t^T \frac{v^{(n)}(\tau)}{(\tau - t)^{\alpha - n + 1}} d\tau, \quad (1.4)$$

where n is a positive integer satisfying the inequality: $n - 1 < \alpha < n$.

1.3.4 Fractional integration

Let $\alpha \in \mathbb{R}_+$ and v be an integrable function defined on $[a, b]$. The fractional integration of order α of v is defined by: [8]

$$\begin{aligned} {}_a I_t^\alpha v(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t \frac{v(\tau)}{(t - \tau)^{1 - \alpha}} d\tau, \\ {}_t I_b^\alpha v(t) &= \frac{1}{\Gamma(\alpha)} \int_t^b \frac{v(\tau)}{(\tau - t)^{1 - \alpha}} d\tau, \end{aligned}$$

1.3.5 Relationship between Riemann-Liouville, Caputo derivative and fractional integration

Let $\alpha \in \mathbb{R}^+$ with $n - 1 < \alpha < n$, ($n \in \mathbb{N}^*$). Suppose that v has the fractional derivatives, then we have

$$\begin{aligned}
{}^R D_t^\alpha v(t) &= {}^C D_t^\alpha v(t) + \sum_{i=0}^{n-1} \frac{v^{(i)}(0)(t)^{i-\alpha}}{\Gamma(i-\alpha+1)}, \\
{}^R D_t^\alpha v(t) &= {}^C D_t^\alpha v(t) + \sum_{i=0}^{n-1} \frac{v^{(i)}(T)(T-t)^{i-\alpha}}{\Gamma(i-\alpha+1)}. \\
{}^C D_t^\alpha I_t^\alpha v(t) &= v(t), \\
I_t^\alpha ({}^C D_t^\alpha v(t)) &= v(t) - \sum_{i=0}^{n-1} \frac{t^i}{i!} \frac{d^i}{dt^i} v(0).
\end{aligned} \tag{1.5}$$

For $n = 1$, we have:

$${}^R D_t^\alpha v(t) = {}^C D_t^\alpha v(t) + \frac{v(0)}{\Gamma(1-\alpha)t^\alpha}, \tag{1.6}$$

$${}^R D_t^\alpha v(t) = {}^C D_t^\alpha v(t) + \frac{v(T)}{\Gamma(1-\alpha)(T-t)^\alpha},$$

$$I_t^\alpha ({}^C D_t^\alpha v(t)) = v(t) - v(0). \tag{1.7}$$

If $v^{(i)}(0) = 0$ with $i = 0, 1, \dots, n - 1$, then the fractional derivative of **Riemann-Liouville** and **Caputo** are coincident, i.e.

$${}^R D_t^\alpha v(t) = {}^C D_t^\alpha v(t).$$

If $\alpha > 0$, we have

$${}^R D_t^\alpha ({}^R D_t^{-\alpha} v(t)) = v(t),$$

which means that the fractional differentiation operator in the sense of **Riemann-Liouville** is a left inverse of the fractional integration operator in the sense of **Riemann-Liouville** of the same order.

1.3.6 Some properties of fractional derivation

1. Linearity

Similar to integer order differentiation, fractional differentiation in the sense of **Riemann-Liouville** is a linear operation.

Theorem 13 [10] *Let v and w be two functions whose fractional **Riemann-Liouville** derivatives of order α exist. Then for $\lambda, \mu \in \mathbb{R}$, ${}^R D_t^\alpha(\lambda v + \mu w)$ exists and we have:*

$${}^R D_t^\alpha(\lambda v + \mu w)(t) = \lambda {}^R D_t^\alpha v(t) + \mu {}^R D_t^\alpha w(t).$$

Proof. Let $\alpha \in \mathbb{R}^+$ with $n - 1 < \alpha < n$ we have:

$$\begin{aligned} {}^R D_t^\alpha(\lambda v + \mu w)(t) &= \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{(\lambda v + \mu w)(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau \\ &= \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \left(\int_0^t \frac{\lambda v(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau + \int_0^t \frac{\mu w(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau \right) \\ &= \frac{\lambda}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{v(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau + \frac{\mu}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{w(\tau)}{(t - \tau)^{\alpha - n + 1}} d\tau \\ &= \lambda {}^R D_t^\alpha v(t) + \mu {}^R D_t^\alpha w(t) \end{aligned}$$

■

2. Noncommutativity

Proposition 14 [10] *Let v be the function such that $v^{(k)}(0) = 0$, $k = 0, 1, \dots, n - 1$, then the two fractional derivatives of **Riemann-Liouville** and of **Caputo** are commuted with the derivative of order n , $n \in \mathbb{N}$:*

$${}^R D_t^n {}^R D_t^\alpha v(t) = {}^R D_t^{\alpha + n} v(t) = {}^R D_t^{\alpha R} D_t^n v(t),$$

and

$${}^C D_t^n {}^C D_t^\alpha v(t) = {}^C D_t^{\alpha+n} v(t) = {}^C D_t^\alpha {}^C D_t^n v(t).$$

Lemma 15 [10] *We suppose that $n - 1 < \alpha < n$, $m - 1 < \beta < m$ and be the function v such that ${}^R D_t^\alpha v$ exists, then*

$${}^R D_t^\alpha \left({}^R D_t^\beta v(t) \right) = {}^R D_t^{\alpha+\beta} v(t) \neq {}^R D_t^\beta \left({}^R D_t^\alpha v(t) \right).$$

Proof. By using the definition of the fractional derivative in the sense of Riemann-Liouville and the composition with derivatives of integer order, we will have

$$\begin{aligned} {}^R D_t^\alpha \left({}^R D_t^\beta v(t) \right) &= {}^R D_t^n \left[{}^R D_t^{-(n-\alpha)} \left({}^R D_t^\beta v(t) \right) \right] \\ &= {}^R D_t^n \left[{}^R D_t^{\alpha+\beta-n} \left({}^R D_t^\beta v(t) \right) - \sum_{i=1}^n \left[{}^R D_t^{\beta-i} v(t) \right]_{t=0} \frac{t^{n-\alpha-i}}{\Gamma(1+n-\alpha-i)} \right] \\ &= {}^R D_t^{\alpha+\beta} v(t) - \sum_{i=1}^n \left[{}^R D_t^{\beta-i} v(t) \right]_{t=0} \frac{t^{-\alpha-i}}{\Gamma(1-\alpha-i)}. \end{aligned}$$

By interchanging α and β (and therefore n and m), we can write

$${}^R D_t^\beta \left({}^R D_t^\alpha v(t) \right) = {}^R D_t^{\alpha+\beta} v(t) - \sum_{i=1}^n \left[{}^R D_t^{\alpha-i} v(t) \right]_{t=0} \frac{t^{-\beta-i}}{\Gamma(1-\beta-i)}.$$

So we have

$${}^R D_t^\alpha \left({}^R D_t^\beta v(t) \right) = {}^R D_t^{\alpha+\beta} v(t) \neq {}^R D_t^\beta \left({}^R D_t^\alpha v(t) \right).$$

■

Remark 16 *If $\left[{}^R D_t^{\alpha-i} v(t) \right]_{t=0}$ and $\left[{}^R D_t^{\beta-i} v(t) \right]_{t=0}$ for all $i = 1, 2, \dots, n$, or $\alpha = \beta$, we have*

$${}^R D_t^\alpha \left({}^R D_t^\beta v(t) \right) = {}^R D_t^{\alpha+\beta} v(t) = {}^R D_t^\beta \left({}^R D_t^\alpha v(t) \right).$$

3. Leibniz's rule

Corollary 17 [10] *Let $t > 0$ and $n - 1 < \alpha < n$. If v and w and all its derivatives are continuous on $[0, T]$, then:*

$${}^R D_t^\alpha (vw)(t) = \sum_{i=0}^{\infty} \frac{\alpha!}{i! (\alpha - i)!} ({}^R D_t^{\alpha - i} v(t)) w^{(i)}(t).$$

4. Fourier Transform [46]

For all $\alpha \in \mathbb{R}^+$ and $v \in C_0^\infty(\mathbb{R})$, we have

$$\mathcal{F}({}^R D_t^\alpha v(t)) = (i\omega)^\alpha \mathcal{F}(v(t)) \omega, \quad (1.8)$$

$$\mathcal{F}({}_t^R D^\alpha v(t)) = (-i\omega)^\alpha \mathcal{F}(v(t)) \omega. \quad (1.9)$$

Remark 18 *According to the relation between the fractional derivatives of Riemann-Liouville and Caputo, linearity, non-commutativity remain true and for Leibniz's rule we have:*

$${}^C D_t^\alpha (vw)(t) = \sum_{i=0}^{\infty} \frac{\alpha!}{i! (\alpha - i)!} ({}^R D_t^{\alpha - i} v(t)) w^{(i)}(t) - \sum_{i=0}^{n-1} \frac{(t)^{i-\alpha}}{\Gamma(i - \alpha + 1)} \left((vw)^{(i)}(0) \right).$$

1.3.7 Comparison between Riemann-Liouville and Caputo fractional derivatives

Lemma 19 [10] *Let the function v such that ${}^R D_t^\alpha v$ and ${}^C D_t^\alpha v$ exists, with $n - 1 < \alpha < n$, then we have:*

$${}^R D_t^\alpha v(t) \neq {}^C D_t^\alpha v(t).$$

Example 20 *The fractional derivative of a constant function in the sense of Riemann-*

Liouville is neither zero nor constant. in effect :

$$\begin{aligned}
{}^R D_t^\alpha C &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{C}{(t-\tau)^{\alpha-n+1}} d\tau \\
&= \frac{C}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha+1} d\tau \\
&= \frac{C}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left. -\frac{(t-\tau)^{n-\alpha}}{n-\alpha} \right|_{\tau=0}^{\tau=t} \\
&= \frac{C}{\Gamma(n-\alpha)(n-\alpha)} \frac{d^n}{dt^n} t^{n-\alpha} \\
&= \frac{C(n-\alpha)(n-\alpha-1)\dots(n-\alpha-(n-1))}{\Gamma(1-\alpha)(n-\alpha)(n-\alpha-1)\dots(n-\alpha-(n-1))} t^{-\alpha} \\
&= \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}$$

And for the fractional derivative of a constant function in the sense of Caputo, we have:

$$\begin{aligned}
{}^C D_t^\alpha C &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{C^{(n)}}{(t-\tau)^{\alpha-n+1}} d\tau \\
&= 0.
\end{aligned}$$

Proposition 21 [46] *Let $n-1 < \alpha < n$, then:*

$$\lim_{\alpha \rightarrow n} {}^R D_t^\alpha v(t) = \lim_{\alpha \rightarrow n} {}^C D_t^\alpha v(t) = v^{(n)}(t).$$

Proof. We use integration by parts and proposition 10, we get:

$$\begin{aligned}
{}^R D_t^\alpha v(t) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{v(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \\
&= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left(-v(\tau) \frac{(t-\tau)^{n-\alpha}}{n-\alpha} \Big|_{\tau=0}^{\tau=t} + \int_0^t v'(\tau) \frac{(t-\tau)^{n-\alpha}}{n-\alpha} d\tau \right) \\
&= \frac{1}{\Gamma(n-\alpha+1)} \frac{d^n}{dt^n} \left(v(0)t^{n-\alpha} + \int_0^t v'(\tau) (t-\tau)^{n-\alpha} d\tau \right) \tag{1.10}
\end{aligned}$$

And

$$\begin{aligned}
{}^C D_t^\alpha v(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{v^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \\
&= \frac{1}{\Gamma(n-\alpha)} \left(-v^{(n)} \frac{(t-\tau)^{n-\alpha}}{n-\alpha} \Big|_{\tau=0}^{\tau=t} + \int_0^t v^{(n+1)}(\tau) \frac{(t-\tau)^{n-\alpha}}{n-\alpha} d\tau \right) \\
&= \frac{1}{\Gamma(n-\alpha+1)} \left(v^{(n)}(0)t^{n-\alpha} + \int_0^t v^{(n+1)}(\tau) (t-\tau)^{n-\alpha} d\tau \right) \quad (1.11)
\end{aligned}$$

By taking the limit $\alpha \rightarrow n$ on (1.10) and (1.11) we have:

$$\begin{aligned}
\lim_{\alpha \rightarrow n} {}^R D_t^\alpha v(t) &= \frac{d^n}{dt^n} \left(v(0) + \int_0^t v'(\tau) d\tau \right) \\
&= v^{(n)}(t).
\end{aligned}$$

$$\begin{aligned}
\lim_{\alpha \rightarrow n} {}^C D_t^\alpha v(t) &= \left(v^{(n)}(0) + \int_0^t v^{(n+1)}(\tau) d\tau \right) \\
&= v^{(n)}(t).
\end{aligned}$$

From where

$$\lim_{\alpha \rightarrow n} {}^R D_t^\alpha v(t) = \lim_{\alpha \rightarrow n} {}^C D_t^\alpha v(t) = v^{(n)}(t).$$

■

1.4 Functional spaces

1.4.1 $L^2(\Omega)$ space

For the study of some problems, we need to recall some functional spaces. Let $L^2(0, d)$, $d \in \mathbb{R}_+^*$, be the usual Hilbert space endowed with a star product noted $(\cdot, \cdot)_{L^2(0, d)}$ and an associated norm $\|\cdot\|_{L^2(0, d)}$. The Hilbert space $L^2(\Omega) = L^2((0, T), L^2(0, d))$ ($\Omega =$

$(0, d) \times (0, T)$) consists of (classes of) definite functions and squares integrable in Ω . The scalar product in $L^2(\Omega)$ is denoted $(\cdot, \cdot)_{L^2(\Omega)}$ defined by:

$$(u, v)_{L^2(\Omega)} = (u, v)_{L^2(\Omega)} = \int_0^d (u(x, \cdot), v(x, \cdot))_{L^2(0, T)} dx$$

and an associated norm denoted $\|\cdot\|_{L^2(\Omega)}$ defined by:

$$\|u\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega)} = \left(\int_0^d \|u(x, \cdot)\|_{L^2(0, T)}^2 dx \right)^{1/2}.$$

1.4.2 Fractional spaces

Let the domain $Q = \Omega \times I$ such that Ω, I are two intervals of \mathbb{R} .

Let $C^\infty(I)$ denote the space of indefinitely differentiable functions on I and $C_0^\infty(I)$

denote the space of indefinitely differentiable functions with compact support in I .

The Sobolev space $H^\alpha(\mathbb{R})$

Definition 22 For all $\alpha \in \mathbb{R}_+$, we define:

$$H^\alpha(\mathbb{R}) = \left\{ u / u \in L^2(\mathbb{R}) ; (1 + |x|^2)^{\frac{\alpha}{2}} \mathcal{F}(u)(x) \in L^2(\mathbb{R}), \forall x \in \mathbb{R}^n \right\},$$

with the norm defined by:

$$\|u\|_{H^\alpha(\mathbb{R})} = \left\| (1 + |x|^2)^{\frac{\alpha}{2}} \mathcal{F}(u)(x) \right\|_{L^2(\mathbb{R})}$$

which is induced by the scalar product:

$$(u, v)_{H^\alpha(\mathbb{R})} = \left((1 + |x|^2)^{\frac{\alpha}{2}} \mathcal{F}(u), (1 + |x|^2)^{\frac{\alpha}{2}} \mathcal{F}(v) \right)_{L^2(\mathbb{R})},$$

where $\mathcal{F}(u)$ denotes the Fourier transform of u defined by:

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}} u(t) \exp(-2\pi i t \xi) dt.$$

Definition 23 For I denotes a bounded interval in \mathbb{R} , we define the space $H^\alpha(I)$ by:

$$H^\alpha(I) = \{u \in L^2(I) / \exists \tilde{u} \in H^\alpha(\mathbb{R}) \text{ such that } \tilde{u}|_I = u\},$$

with the norm defined by:

$$\|u\|_{H^\alpha(I)} = \inf_{\tilde{v} \in H^\alpha(\mathbb{R}), \tilde{v}|_I = u} \|\tilde{u}\|_{H^\alpha(\mathbb{R})}.$$

Definition 24 Let $\alpha \in \mathbb{R}_+^*$, we define

$$H_0^\alpha(I) = \left\{ u; \|u\|_{H_0^\alpha(I)} < \infty \right\},$$

with the norm defined by

$$\|u\|_{H_0^\alpha(I)} = (\|u\|_{L^2(I)}^2 + |u|_{H_0^\alpha(I)}^2)^{1/2},$$

with

$$|u|_{H_0^\alpha(I)} = \left| \frac{({}^R D_t^\alpha u, {}^R D^\alpha u)_{L^2(I)}}{\cos(\alpha\pi)} \right|^{1/2},$$

defines a semi-norm on $H_0^\alpha(I)$. We therefore define the space $H_0^\alpha(I)$ as a completion of the space $C_0^\infty(I)$ by the norm $\|\cdot\|_{H_0^\alpha(I)}$.

Remark 25 If $\alpha = 1$ and I a bounded interval in \mathbb{R} , the expression

$$(u, v)_{H_0^1(I)} = \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \right)_{L^2(I)},$$

is a scalar product inducing the norm

$$\|u\|_{H_0^1(I)} = \left\| \frac{\partial u}{\partial x} \right\|_{L^2(I)}.$$

The spaces ${}^lH^\alpha(\mathbf{I})$, ${}^rH^\alpha(\mathbf{I})$ and ${}^cH^\alpha(\mathbf{I})$

Definition 26 Let $\alpha \in \mathbb{R}_+^*$, we define

$${}^lH^\alpha(I) = \left\{ u; \|u\|_{{}^lH^\alpha(I)} < \infty \right\},$$

with the norm defined by

$$\|u\|_{{}^lH^\alpha(I)} = (\|u\|_{L^2(I)}^2 + |u|_{{}^lH^\alpha(I)}^2)^{1/2},$$

with

$$|u|_{{}^lH^\alpha(I)} = \left\| {}^R D_t^\alpha u \right\|_{L^2(I)},$$

defines a semi-norm on ${}^lH^\alpha(I)$. We therefore define the space ${}^lH^\alpha(I)$ as a completion of the space $C_0^\infty(I)$ with the norm $\|\cdot\|_{{}^lH^\alpha(I)}$.

Remark 27 First, we have $|u|_{{}^lH^\alpha(I)}$ is a semi-norm and is not a norm because, if we put:

$$u(\theta) = (t - \theta)^\alpha (t - 2\theta),$$

for $n = 1$, we have

$$\begin{aligned} |u|_{{}^lH^\alpha(I)} &= \left(\int_I \left(\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{(t-\tau)^\alpha (t-2\tau)}{(t-\tau)^\alpha} d\tau \right)^2 dt \right)^{\frac{1}{2}} \\ &= \left(\int_I \left(\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-2\tau) d\tau \right)^2 dt \right)^{\frac{1}{2}} \\ &= 0. \end{aligned}$$

Then, we find $u \neq 0$, although $|u|_{{}^lH^\alpha(I)} = 0$.

Second, it suffices to apply the definition of a norm, and to verify the three essential properties. The main difficulty is the triangle inequality, and for the norm $\|u\|_{{}^lH^\alpha(I)}$, we use the **Minkowski** inequality.

Definition 28 Let $\alpha \in \mathbb{R}_+^*$, we define

$${}^r H^\alpha(I) = \left\{ u; \|u\|_{rH^\alpha(I)} < \infty \right\},$$

with the norm defined by

$$\|u\|_{rH^\alpha(I)} = (\|u\|_{L^2(I)}^2 + |u|_{rH^\alpha(I)}^2)^{1/2},$$

with

$$|u|_{rH^\alpha(I)} = \left\| {}^R D_t^\alpha u \right\|_{L^2(I)},$$

defines a semi-norm on ${}^r H^\alpha(I)$. We therefore define the space ${}^r H^\alpha(I)$ as a completion of the space $C_0^\infty(I)$ with the norm $\|\cdot\|_{rH^\alpha(I)}$.

Definition 29 Let $\alpha \in \mathbb{R}_+^*$, for $\alpha \neq n + \frac{1}{2}$ we define

$${}^c H^\alpha(I) = \left\{ u; \|u\|_{rH^\alpha(I)} < \infty \right\},$$

with the norm defined by

$$\|u\|_{cH^\alpha(I)} = (\|u\|_{L^2(I)}^2 + |u|_{cH^\alpha(I)}^2)^{1/2},$$

with

$$|u|_{cH^\alpha(I)} = \left| \left({}^R D_t^\alpha u, {}^R D_t^\alpha u \right)_{L^2(I)} \right|^{1/2},$$

defines a semi-norm on ${}^c H^\alpha(I)$. We therefore define the space ${}^c H^\alpha(I)$ as a completion of the space $C_0^\infty(I)$ with the norm $\|\cdot\|_{cH^\alpha(I)}$.

1.5 Energy-inequality method

the method of energy inequalities is an effective technique for studying the existence and uniqueness of the solution of partial differential equations, it is also called the

method of functional analysis or the method of a priori estimates. this method has a superior character that we can derive the existence theorem from the solution of the problem posed, from the uniqueness theorem. The difficult points of this method lie in the choice of the functional spaces E and F and in the choice of the multiplier Mu . The scheme of the method can be summarized as follows:

1. First we write the problem in the form of an operational equation:

$$Lu = \mathcal{F}, \quad u \in D(L);$$

where the operator L is considered from a Banach space E into a suitably chosen Hilbert space F .

2. Then we establish the a priori estimate for the operator L .
3. Then we prove the density of the set of values of this operator in the space F .

More precisely, we follow in this work the following scheme:

We prove the energy inequality of the type

$$\|u\|_B \leq c \|Lu\|_F. \tag{1.12}$$

This type of a priori estimate is obtained by multiplying the equation considered by an integro-differential operator Mu (containing the function u or its derivatives) defined on the domain Q_T .

The choice of the operator Mu is fundamental, it is dictated by the equation and the boundary conditions.

Next, we show that the operator L from B to F admits a closure \bar{L} , hence the solution of the operational equation:

$$\bar{L}u = \mathcal{F}, \quad u \in D(\bar{L}), \quad (1.13)$$

is called the generalized strong solution of the considered problem.

By passing to the limit, the estimate (1.12) will be extended to \bar{L} , i.e.:

$$\|u\|_B \leq c \|\bar{L}u\|_F.$$

Thus, we deduce the uniqueness of the solution of equation (1.13).

As the image of operator \bar{L} is closed in F and $R(\bar{L}) = \overline{R(L)}$, establishing the density of the set $R(\bar{L})$ in F guarantees the existence of the strong solution of the problem (1.13).

1.6 Some useful inequalities

Let $\Omega \subset \mathbb{R}^n$:

*** Cauchy inequality:**

$$\forall u, v \in L^2(\Omega), \left| \int_{\Omega} uv dx \right| \leq \left(\int_{\Omega} |u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v|^2 dx \right)^{\frac{1}{2}}.$$

*** Cauchy inequality with ε :**

Also called ε -inequality

$$|xy| \leq \frac{\varepsilon}{2} |x|^2 + \frac{1}{2\varepsilon} |y|^2, \text{ for all } \varepsilon > 0 \text{ and arbitrary } x, y \text{ (real numbers).}$$

*** Cauchy-Schwarz inequality:**

Let V be a Hilbert space.

$$\forall u, v \in V, |\langle u, v \rangle_V| \leq \|u\|_V \|v\|_V.$$

*** Minkowski inequality:**

Let $p \in [1, \infty[$. For $u, v \in L^p(\Omega)$, we have

$$\left(\int_{\Omega} |u + v|^p dx \right)^{\frac{1}{p}} \leq \left(\int_{\Omega} |u|^p dx \right)^{\frac{1}{p}} + \left(\int_{\Omega} |v|^p dx \right)^{\frac{1}{p}}.$$

Chapter 2

Unique solvability of a dirichlet problem for a fractional parabolic equation using energy-inequality method

2.1 Introduction and Preliminaries

The present chapter is devoted to the study of initial-boundary value problem for a parabolic equation with time-fractional derivative with Dirichlet condition, which has not been studied so far [58].

Let $\Omega = [0, T]$ be a finite interval of the real numbers \mathbb{R} and $\Gamma(\cdot)$ denote the gamma function. For any positive integer $0 < \alpha < 1$, the Caputo derivative are the Riemann

Liouville derivative are, respectively, defined as follows: Let $\Gamma(\cdot)$ denote the gamma function. For any positive integer $0 < \alpha < 1$, the Caputo derivative are the Riemann Liouville derivative are, respectively, defined as follows:

(1) The left Caputo derivatives:

$${}^C D_t^\alpha u(x, t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, \tau)}{\partial \tau} \frac{1}{(t-\tau)^\alpha} d\tau. \quad (2.1)$$

(2) The left Riemann-Liouville derivatives:

$${}^R D_t^\alpha u(x, t) := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(x, \tau)}{(t-\tau)^\alpha} d\tau. \quad (2.2)$$

(3) The right Caputo derivatives:

$${}^C D_t^\alpha u(x, t) := \frac{-1}{\Gamma(1-\alpha)} \int_t^T \frac{\partial u(x, \tau)}{\partial \tau} \frac{1}{(\tau-t)^\alpha} d\tau. \quad (2.3)$$

(4) The right Riemann-Liouville derivatives:

$${}^R D_t^\alpha u(x, t) = \frac{-1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_t^T \frac{u(x, \tau)}{(\tau-t)^\alpha} d\tau, \quad (2.4)$$

Many authors think that the Caputo's version is more natural because it allows the handling of inhomogeneous initial conditions in a easier way. Then the two definitions (2.1) and (2.2) are linked by the following relationship, which can be verified by a direct calculation:

$${}^R D_t^\alpha u(x, t) = {}^C D_t^\alpha u(x, t) + \frac{u(x, 0)}{\Gamma(1-\alpha) t^\alpha}. \quad (2.5)$$

Definition 30 [46] *For any real $\sigma > 0$ and finite interval $[a, b]$ of the real axis \mathbb{R} , we define the semi-norm:*

$$|u|_{H^\sigma(\Omega)}^2 := \| {}^R D_t^\sigma u \|_{L_2(\Omega)}^2,$$

and norm:

$$\|u\|_{lH^\sigma(\Omega)} := \left(\|u\|_{L^2(\Omega)}^2 + |u|_{lH^\sigma(\Omega)}^2 \right)^{\frac{1}{2}}, \quad (2.6)$$

we then define ${}^lH_0^\sigma(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{lH^\sigma(\Omega)}$.

Definition 31 [46] For any real $\sigma > 0$, we define the semi-norm:

$$|u|_{rH^\sigma(\Omega)}^2 := \left\| {}^R D_t^\sigma u \right\|_{L^2(\Omega)}^2,$$

and norm:

$$\|u\|_{rH^\sigma(\Omega)} := \left(\|u\|_{L^2(\Omega)}^2 + |u|_{rH^\sigma(\Omega)}^2 \right)^{\frac{1}{2}}, \quad (2.7)$$

we then define ${}^RH_0^\sigma(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{rH^\sigma(\Omega)}$.

Definition 32 For any real $\sigma > 0$, we define the semi-norm:

$$|u|_{cH^\sigma(\Omega)} = \left| \frac{({}^R D_t^\sigma u, {}^R D_t^\sigma u)_{L^2(\Omega)}}{\cos(\sigma\pi)} \right|^{1/2}$$

and norm:

$$\|u\|_{cH^\sigma(\Omega)} = (\|u\|_{L^2(\Omega)}^2 + |u|_{cH^\sigma(\Omega)}^2)^{1/2}.$$

Lemma 33 [46, 49] For any real $\sigma \in \mathbb{R}_+$, if $u \in {}^lH^\alpha(\Omega)$ and $v \in C_0^\infty(\Omega)$, then

$$({}^R D_t^\sigma u(t), v(t))_{L^2(\Omega)} = (u(t), {}^R D_t^\sigma v(t))_{L^2(\Omega)}.$$

Lemma 34 [46, 49] For $0 < \sigma < 2$, $\sigma \neq 1$, $u \in H_0^{\frac{\sigma}{2}}(\Omega)$, on a :

$${}^R D_t^\sigma u(t) = {}^R D_t^{\frac{\sigma}{2}} {}^R D_t^{\frac{\sigma}{2}} u(t).$$

Lemma 35 [46, 49] For $\sigma \in \mathbb{R}_+$, $\sigma \neq n + \frac{1}{2}$, the semi-norms $|\cdot|_{lH^\sigma(\Omega)}$, $|\cdot|_{rH^\sigma(\Omega)}$ and $|\cdot|_{cH^\sigma(\Omega)}$

are equivalent. Then we pose

$$|\cdot|_{lH^\sigma(\Omega)} \cong |\cdot|_{rH^\sigma(\Omega)} \cong |\cdot|_{cH^\sigma(\Omega)}.$$

Lemma 36 [46, 49] *For any real $\sigma > 0$, the space ${}^R H_0^\sigma(\Omega)$ with respect to the norm (2.7) is complete.*

Definition 37 *We denote by $L_2(0, T, L_2(0, 1)) := L_2(Q)$ the space of functions which are square integrable in the Bochner sense, with the scalar product*

$$(u, w)_{L_2(0, T, L_2(0, 1))} = \int_0^T ((u, \cdot), (w, \cdot))_{L_2(0, 1)} dt. \quad (2.8)$$

Since the space $L_2(0, 1)$ is a Hilbert space, it can be shown that $L_2(0, T, L_2(0, 1))$ is a Hilbert space as well. Let $C^\infty(0, T)$ denote the space of infinitely differentiable functions on $(0, T)$ and $C_0^\infty(0, T)$ denote the space of infinitely differentiable functions with compact support in $(0, T)$.

2.2 Position of the problem

In the rectangular domain $Q = (0, d) \times (0, T)$, with $d, T < \infty$ and $0 < \alpha < 1$, we shall study the existence and uniqueness of solutions $u = u(x, t)$ to the following fractional parabolic problem :

$$\left\{ \begin{array}{ll} {}^C D_t^\alpha u(x, t) - \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial u(x, t)}{\partial x} \right) + b(x, t)u(x, t) = \tilde{f}(x, t) & \text{in } Q; \\ u(x, 0) = \varphi(x) & \forall x \in (0, d), \\ u(0, t) = u(d, t) = 0 & \forall t \in (0, T). \end{array} \right. \quad (P)$$

We consider the following fractional parabolic equation of the type:

$$\mathcal{L}u = {}^C D_t^\alpha u - \frac{\partial}{\partial x} \left(a \frac{\partial u}{\partial x} \right) + bu = \tilde{f},$$

with the initial condition :

$$\ell u = u(x, 0) = \varphi(x), \quad \forall x \in (0, d),$$

and Dirichlet condition :

$$u(0, t) = u(d, t) = 0, \quad \forall t \in (0, T).$$

Where a, b, \tilde{f} and φ are known functions.

We shall assume that the function φ satisfies a compatibility conditions, i.e.,

$$\varphi(0) = \varphi(d) = 0.$$

And the functions a, b verify:

$$0 < a_0 \leq a(x, t) \leq a_1,$$

$$0 < b_0 \leq b(x, t) \leq b_1.$$

Now, we shall introduce a new function :

$$v(x, t) = u(x, t) - U(x) \implies u(x, t) = v(x, t) + U(x),$$

where

$$\varphi(x) = U(x).$$

So, we get :

$$\left\{ \begin{array}{ll} {}^C D_t^\alpha v(x, t) - \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v(x, t)}{\partial x} \right) + b(x, t)v(x, t) = \tilde{f}(x, t) - \mathcal{L}\varphi(x) = f(x, t) & \text{in } Q; \\ v(x, 0) = 0 & \forall x \in (0, d), \\ v(0, t) = v(d, t) = 0 & \forall t \in (0, T). \end{array} \right.$$

Such that :

$${}^C D_t^\alpha v(x, t) - \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v(x, t)}{\partial x} \right) + b(x, t)v(x, t) = f(x, t) \quad (2.9)$$

with the initial condition :

$$\ell v = v(x, 0) = 0, \quad \forall x \in (0, d), \quad (2.10)$$

the boundary condition of Dirichlet type :

$$v(0, t) = v(d, t) = 0, \quad \forall t \in (0, T) \quad (2.11)$$

where

$$f(x, t) = \tilde{f}(x, t) + \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial \varphi(x)}{\partial x} \right) - b(x, t) \varphi(x).$$

2.3 A priori estimate

The method used here is one of the most efficient functional analysis methods and important techniques for solving partial differential equations with integral conditions, which has been successfully used in investigating the existence, uniqueness, and continuous dependence of the solutions of PDE's, the so-called a priori estimate method or the energy-inequality method. This method is essentially based on the construction of multipliers for each specific given problem, which provides the a priori estimate from which it is possible to establish the solvability of the posed problem. More precisely, the proof is based on an energy inequality and the density of the range of the operator generated by the abstract formulation of the stated problem, so to investigate the posed problem, we introduce the needed function spaces. In this paper, we prove the existence and the uniqueness for solution of the problem (2.9) – (2.11) as a solution of the operator equation

$$Lv = \mathcal{F}, \quad (2.12)$$

Where $L = (\mathcal{L}, \ell)$, with domain of definition B consisting of functions $v \in L^2(Q)$, such that $v, {}^C D_t^\alpha v, \frac{\partial v}{\partial x} \in L^2(Q)$ and v satisfies the condition (2.11).

The operator L is considered from B to F , where B is the Banach space consisting of all functions $v(x, t)$ having a finite norm

$$\|v\|_B^2 = \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2 + \left\| \frac{\partial v}{\partial x} \right\|_{L^2(Q)}^2,$$

and F is the Hilbert space consisting of all elements $\mathcal{F} = (f, 0)$ for which the norm $L^2(Q)$ is finite.

Theorem 38 *For any function $u \in B$, we have the inequality*

$$\|v\|_B \leq k \|Lv\|_{L^2(Q)} \quad (2.13)$$

where k is a positive constant independent of v .

Proof. Multiplying the equation (2.9) by the following function :

$$Mv = v(x, t),$$

and integrating over $Q = (0, d) \times (0, T)$, we get :

$$\begin{aligned} & \int_Q \mathcal{L}v \cdot Mv dx dt \\ &= \int_Q {}^C D_t^\alpha v(x, t) \cdot v(x, t) dx dt - \int_Q \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v(x, t)}{\partial x} \right) v(x, t) dx dt \\ & \quad + \int_Q b(x, t) \cdot v^2(x, t) dx dt \\ &= \int_Q f(x, t) \cdot v(x, t) dx dt. \end{aligned}$$

As $v(x, 0) = 0$, we have ${}^C D_t^\alpha v(x, t) = {}^R D_t^\alpha v(x, t)$, so by applying the lemmas 33,34 et 35, we find:

$$\begin{aligned}
& \int_Q {}^C D_t^\alpha v(x, t) \cdot v(x, t) dx dt \\
&= ({}^C D_t^\alpha v(x, t), v(x, t))_{L^2(Q)} \\
&= ({}^C D_t^{\frac{\alpha}{2}} {}^C D_t^{\frac{\alpha}{2}} v(x, t), v(x, t))_{L^2(Q)} \quad (\text{According to Lemma 34}) \\
&= ({}^C D_t^{\frac{\alpha}{2}} v(x, t), {}^C D_t^{\frac{\alpha}{2}} v(x, t))_{L^2(Q)} \quad (\text{According to Lemma 33}) \\
&= \cos\left(\frac{\alpha}{2}\pi\right) |v|_{cH^{\frac{\alpha}{2}}(Q)}^2 \quad (\text{According to Definition 32})
\end{aligned}$$

then, according to Lemma 35, we get that

$$\begin{aligned}
& \int_Q {}^C D_t^\alpha v(x, t) \cdot v(x, t) dx dt \\
&= \cos\left(\frac{\alpha}{2}\pi\right) |v|_{cH^{\frac{\alpha}{2}}(Q)}^2 \\
&\cong \cos\left(\frac{\alpha}{2}\pi\right) |v|_{lH^{\frac{\alpha}{2}}(Q)}^2 \\
&= \cos\left(\frac{\alpha}{2}\pi\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2.
\end{aligned}$$

and by integration by parts over $(0, d)$; we get

$$\begin{aligned}
& - \int_Q \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v(x, t)}{\partial x} \right) v(x, t) dx dt \\
&= - \int_0^T \int_0^d \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v(x, t)}{\partial x} \right) v(x, t) dx dt \\
&= - \int_0^T a(x, t) \frac{\partial v(x, t)}{\partial x} v(x, t) \Big|_{x=0}^{x=d} dt + \int_Q a(x, t) \left(\frac{\partial v(x, t)}{\partial x} \right)^2 dx dt \\
&= \int_Q a(x, t) \left(\frac{\partial v(x, t)}{\partial x} \right)^2 dx dt.
\end{aligned}$$

By using the Cauchy inequality with ε , for $\varepsilon < 2b_0$; and because of the equivalent of the

semi-norms $|\cdot|_{H^\sigma(Q)}$ and $|\cdot|_{cH^\sigma(Q)}$, there is a positive constant m such that

$$m |\cdot|_{H^\sigma(Q)} \leq |\cdot|_{cH^\sigma(Q)}.$$

So, we have

$$m^2 |\cdot|_{H^\sigma(Q)}^2 \leq |\cdot|_{cH^\sigma(Q)}^2,$$

which gives that

$$\begin{aligned} & \int_Q {}^C D_t^\alpha v(x, t) \cdot v(x, t) dx dt \\ &= \cos\left(\frac{\alpha}{2}\pi\right) |v|_{cH^{\frac{\alpha}{2}}(Q)}^2 \\ &\geq m^2 \cos\left(\frac{\alpha}{2}\pi\right) |v|_{H^{\frac{\alpha}{2}}(Q)}^2 = m^2 \cos\left(\frac{\alpha}{2}\pi\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2. \end{aligned}$$

Hence, we get

$$\begin{aligned} & m^2 \cos\left(\frac{\alpha}{2}\pi\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + a_0 \int_Q \left(\frac{\partial v(x, t)}{\partial x}\right)^2 dx dt \\ &+ b_0 \int_Q v^2(x, t) dx dt \\ &\leq \int_Q {}^C D_t^\alpha v(x, t) \cdot v(x, t) dx dt + \int_Q a(x, t) \left(\frac{\partial v(x, t)}{\partial x}\right)^2 dx dt \\ &+ \int_Q b(x, t) v^2(x, t) dx dt \\ &\leq \frac{1}{2\varepsilon} \int_Q |f(x, t)|^2 dx dt + \frac{\varepsilon}{2} \int_Q |v(x, t)|^2 dx dt. \end{aligned}$$

So, we obtain

$$\begin{aligned}
& m^2 \cos\left(\frac{\alpha}{2}\pi\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \int_Q a_0 \left(\frac{\partial v(x,t)}{\partial x} \right)^2 dx dt \\
& + \int_Q \left(b_0 - \frac{\varepsilon}{2} \right) v^2(x,t) dx dt \\
& \leq \frac{1}{2\varepsilon} \int_Q |f(x,t)|^2 dx dt.
\end{aligned}$$

As all the terms are positive, we have

$$\begin{aligned}
& \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \left\| \frac{\partial v}{\partial x} \right\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2 \\
& \leq \frac{1}{2\varepsilon} \left(\frac{1}{\min\{a_0, m^2 \cos\left(\frac{\alpha}{2}\pi\right), (b_0 - \frac{\varepsilon}{2})\}} \right) \|f\|_{L^2(Q)}^2.
\end{aligned}$$

Finally, it follows that

$$\left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \left\| \frac{\partial v}{\partial x} \right\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2 \leq C \|f\|_{L^2(Q)}^2,$$

with

$$C = \frac{1}{2\varepsilon} \left(\frac{1}{\min\{a_0, m^2 \cos\left(\frac{\alpha}{2}\pi\right), (b_0 - \frac{\varepsilon}{2})\}} \right).$$

Therefore, we obtain that

$$\|v\|_B \leq k \|Lv\|_F, \quad \text{where } k = \sqrt{C}.$$

Hence the uniqueness of the solution. ■

Remark 39 *This inequality $\|v\|_B \leq k \|Lv\|_F$ gives the uniqueness of the solution, indeed:*

Let v_1 and v_2 two solutions, so

$$\begin{cases} Lv_1 = \mathcal{F} \\ Lv_2 = \mathcal{F} \end{cases} \implies L(v_1 - v_2) = 0$$

then

$$\begin{aligned} \|v_1 - v_2\|_B &\leq k \|0\|_F \\ \implies \|v_1 - v_2\|_B \leq 0 &\implies v_1 - v_2 = 0 \end{aligned}$$

which gives the uniqueness of the solution.

Proposition 40 *The operator L from B to F admits a closure.*

Proof. Let $(v_n)_{n \in \mathbb{N}} \subset D(L)$ a sequence such that:

$$v_n \rightarrow 0 \quad \text{in } B, \tag{2.14}$$

and

$$Lv_n \rightarrow \mathcal{F} \quad \text{in } F, \tag{2.15}$$

it must be shown that

$$f \equiv 0.$$

The convergence of v_n toward 0 in B entails that :

$$v_n \rightarrow 0 \quad \text{in } (C_0^\infty(Q))'. \tag{2.16}$$

As the continuity of the fractional derivation and the derivation of the first order (as a particular case of the fractional derivative) of $(C_0^\infty(Q))'$ in $(C_0^\infty(Q))'$, then (2.15) and (2.16) implies :

$$\mathcal{L}u_n \rightarrow 0 \quad \text{in } (C_0^\infty(Q))'. \tag{2.17}$$

On the other hand the convergence of $\mathcal{L}v_n$ to f in $F = L^2(Q)$ implies that :

$$\mathcal{L}u_n \rightarrow f \quad \text{in } (C_0^\infty(Q))'. \tag{2.18}$$

By virtue of the uniqueness of the limit in $(C_0^\infty(Q))'$, we conclude from (2.17) and (2.18) that

$$f \equiv 0.$$

Hence, the operator L is closable. ■

Definition 41 *Let \bar{L} the closure of L and $D(\bar{L})$ the definition domain of \bar{L} . The solution of the equation*

$$\bar{L}v = \mathcal{F}$$

is called generalized strong solution of the problem (2.9) – (2.11).

Theorem 38 is valid for a generalized strong solution, ie we have the following inequality:

$$\|v\|_B \leq k \|\bar{L}v\|_F, \quad \forall v \in D(\bar{L}). \quad (2.19)$$

Consequently, this last inequality entails the following corollaries :

Corollary 42 *The strong solution of the problem (2.9) – (2.11) is unique, if it exists and depends continuously on $f \in F$.*

Corollary 43 *The range $R(\bar{L})$ of the operator \bar{L} is equal to the closure $\overline{R(L)}$ of $R(L)$.*

Proof. Let $z \in \overline{R(L)}$, then there exists a Cauchy sequence $(z_n)_n$ in F consists of the elements of the set $R(L)$ such that

$$\lim_{n \rightarrow +\infty} z_n = z.$$

So there is a corresponding sequence $(v_n)_n \subset D(L)$ such that

$$Lv_n = z_n.$$

From the estimate (2.13), we obtain:

$$\|v_p - v_q\|_B \leq k \|Lv_p - Lv_q\|_F \rightarrow 0, \text{ when } p, q \rightarrow +\infty. \quad (2.20)$$

We can deduce that $(v_n)_n$ is a Cauchy sequence in B , so there is $v \in B$:

$$\lim_{n \rightarrow +\infty} v_n = v \text{ in } B.$$

By virtue of the definition of \bar{L} ($\lim_{n \rightarrow +\infty} v_n = v$ in B ; if $\lim_{n \rightarrow +\infty} Lv_n = \lim_{n \rightarrow +\infty} z_n = z$, so $\lim_{n \rightarrow +\infty} \bar{L}v_n = z$ and as \bar{L} is closed so $\bar{L}v = z$), the function v verify that:

$$v \in D(\bar{L}), \quad \bar{L}v = z.$$

thus $z \in R(\bar{L})$, then

$$\overline{R(L)} \subset R(\bar{L}).$$

So we conclude here that $R(\bar{L})$ is closed because it is complete (any complete subspace of a metric space (not necessarily complete) is closed).

It remains to show the opposite inclusion.

Let $z \in R(\bar{L})$, then there is a sequence of $(z_n)_n$ in F consists of the elements of the set $R(\bar{L})$ such that

$$\lim_{n \rightarrow +\infty} z_n = z.$$

where $z \in R(\bar{L})$, because $R(\bar{L})$ is closed subset of a complete space F , then $R(\bar{L})$ is complete.

So there is a corresponding sequence $(v_n)_n \subset D(\bar{L})$ such that

$$\bar{L}v_n = z_n.$$

From the estimate (2.19), we obtain:

$$\|v_p - v_q\|_B \leq k \|\bar{L}v_p - \bar{L}v_q\|_F \rightarrow 0, \text{ if } p, q \rightarrow +\infty.$$

We can deduce that $(v_n)_n$ is a Cauchy sequence in B , so there is $v \in B$:

$$\lim_{n \rightarrow +\infty} v_n = v \text{ in } B.$$

Once more, there is a corresponding sequence $(L(v_n))_n \in R(L)$ such that

$$Lv_n = \bar{L}v_n \text{ over } R(L), \forall n.$$

Then

$$\lim_{n \rightarrow +\infty} Lv_n = z,$$

Consequently $z \in \overline{R(L)}$, and then we conclude that

$$R(\bar{L}) \subset \overline{R(L)}.$$

■

2.4 Existence of the solution

To show the existence of solutions, we prove that $R(L)$ is dense in F for all $u \in B$ and for arbitrary $\mathcal{F} = (f, 0) \in F$.

Theorem 44 *For $w \in L^2(Q)$ and for all $v \in B$, we have*

$$\int_Q \mathcal{L}v \cdot w dx dt = 0, \tag{2.21}$$

then w vanishes almost everywhere in Q , this implies that the problem (2.9) – (2.11) admits a unique solution.

Proof. The idea of the proof of the theorem is choose $w \in R(L)^\perp$ (exactly $w \in R(L)^\perp \subset L^2(Q)$) and for all $v \in B$, we have and demonstrate that $R(L)^\perp = \{0\}$ which give $\overline{R(L)} = F$.

The scalar product of F is defined by

$$(Lv, W)_F = \int_Q \mathcal{L}v \cdot w dxdt, \quad \text{where } W = (w, 0) \in D(L). \quad (2.22)$$

The equality (2.21) can be written as follows :

$$\int_Q \left({}^C D_t^\alpha v(x, t) - \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v(x, t)}{\partial x} \right) + b(x, t)v(x, t) \right) \cdot w(x, t) dxdt = 0 \quad (2.23)$$

where ${}^C D_t^\alpha v$, $\frac{\partial v}{\partial x}$, $v \in L^2(Q)$, with v satisfies the boundary conditions of (2.11). From (2.23), we get the equality:

$$\int_Q \left[{}^C D_t^\alpha v(x, t) \cdot w(x, t) - \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial v(x, t)}{\partial x} \right) \cdot w(x, t) + b(x, t)v(x, t) \cdot w(x, t) \right] dxdt = 0 \quad (2.24)$$

And from the equality (2.24), we give the function w in terms of v as follows:

$$w = v, \quad w(x, 0) = 0; \quad (2.25)$$

then $w \in L^2(Q)$.

Replacing w in (2.24) by its representation (2.25) and integrating by parts each term of (2.24) and by taking the condition of v , we obtain

$$\int_Q {}^C D_t^\alpha v(x, t) \cdot v(x, t) dxdt + \int_Q b(x, t)v^2(x, t) dxdt + \int_Q a(x, t) \left(\frac{\partial v(x, t)}{\partial x} \right)^2 dxdt = 0. \quad (2.26)$$

According to Lemma 33, 34, 35 and definition 32, follow the same steps in the previous section, we have that (2.26) becomes

$$\begin{aligned} & m^2 \cos\left(\frac{\alpha}{2}\pi\right) \int_Q \left({}^C D_t^{\frac{\alpha}{2}} v(x, t)\right)^2 dxdt + \int_Q b(x, t)v^2(x, t)dxdt + \int_Q a(x, t) \left(\frac{\partial v(x, t)}{\partial x}\right)^2 dxdt \\ & \leq 0. \end{aligned}$$

So, we obtain

$$\begin{aligned} & m^2 \cos\left(\frac{\alpha}{2}\pi\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \int_Q b(x, t)v^2(x, t)dxdt \\ & \leq - \int_Q a(x, t) \left(\frac{\partial v(x, t)}{\partial x}\right)^2 dxdt \\ & \leq -a_0 \int_Q \left(\frac{\partial v(x, t)}{\partial x}\right)^2 dxdt \\ & \leq 0, \end{aligned}$$

then

$$m^2 \cos\left(\frac{\alpha}{2}\pi\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + b_0 \|v\|_{L^2(Q)}^2 \leq 0,$$

Hence

$$\|v\|_{L^2(Q)} = 0$$

And thus $v = 0$ in Q which gives $w = 0$ in Q . So $R(L)^\perp = \{0\}$. This proves Theorem 44.

So $\overline{R(L)} = F$. ■

Chapter 3

Existence and uniqueness of the solution for an inverse problem of a fractional diffusion equation with integral condition

3.1 Introduction

This chapter devoted to study the solvability of a pair of functions $\{u, f\}$ satisfying the following fractional parabolic

problem:

$${}^C D_t^\alpha u - \Delta u + \beta u = f(t)g(x, t); \quad x \in \Omega, t \in [0, T], \quad (3.1)$$

with the initial condition

$$u(x, 0) = \varphi(x), \quad x \in \Omega, \quad (3.2)$$

the boundary condition

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T], \quad (3.3)$$

and the nonlocal condition

$$\int_{\Omega} v(x) u(x, t) dx = \theta(t), \quad t \in [0, T]. \quad (3.4)$$

Here, Ω is a bounded domain in \mathbb{R}^n with smooth boundary $\partial\Omega$. The functions g , φ , and θ are known functions and β is a positive constant. And $\Gamma(\cdot)$ denote the gamma function.

For any positive integer $0 < \alpha < 1$, the left Caputo derivative is defined as

$${}^C D_t^\alpha u(x, t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, \tau)}{\partial \tau} \frac{1}{(t-\tau)^\alpha} d\tau. \quad (3.5)$$

Namely, in the present chapter, a new research on the inverse problem of a fractional parabolic equation is discussed [59], for which the solvability of the problem (3.1) – (3.4) is reduced to the concept of a fixed point technique. This work is divided into four sections; we start with an introduction then we give some definitions of function space and important lemmas. The third section is devoted to studying the solvability of the direct fractional parabolic problem.

Finally, in the last section, we prove the existence and uniqueness of the solution to the main problem.

3.2 Functional space

Definition 45 *Let us introduce certain notations used below, we set*

$$g^*(t) = \int_{\Omega} v(x) g(x, t) dx, \quad Q_T = \Omega \times [0, T]. \quad (3.6)$$

The spaces $W_2^1(\Omega)$, $C((0, T), L_2(\Omega))$ and $W_2^1(Q_T)$ with related norms are given as follows:

The Banach space $W_2^1(\Omega)$ with the following norm

$$\|u\|_{W_2^1(\Omega)} = \left(\|u\|_{L_2(\Omega)}^2 + \|u_x\|_{L_2(\Omega)}^2 \right)^{\frac{1}{2}},$$

We denote by $C((0, T), L_2(\Omega))$ the space is composed of all continuous functions on $(0, T)$

with values in $L_2(\Omega)$. The corresponding norm is defined by

$$\|u\|_{C((0, T), L_2(\Omega))} = \max_{(0, T)} \|u\|_{L_2(\Omega)} < \infty.$$

The Sobolev space $W_2^1(Q_T)$ of functions $u(x, t)$ with finite norm

$$\|u\|_{W_2^1(Q_T)} = \left(\|u\|_{L_2(Q_T)}^2 + \|u_x\|_{L_2(Q_T)}^2 \right)^{\frac{1}{2}}$$

where

$$\|u\| \equiv \|u\|_{L_2(\Omega)},$$

The weighted arithmetic-geometric mean inequality, (Cauchy's ε -inequality) is:

$$2|ab| \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \quad \text{for } \varepsilon > 0.$$

Also, we have

$$\|\nabla u\| = \left(\int_{\Omega} \sum_{i=1}^n u_{x_i}^2 dx \right)^{\frac{1}{2}} \quad \text{and} \quad \|\Delta u\| = \left(\int_{\Omega} \sum_{i,j=1}^n u_{x_i x_j}^2 dx \right)^{\frac{1}{2}}.$$

For any $0 < \alpha < 1$, The Caputo and Riemann Liouville derivatives are defined, respectively, as follows:

(i) The left Caputo derivatives:

$${}_0^C D_t^\alpha u(x, t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, \tau)}{\partial \tau} \frac{1}{(t-\tau)^\alpha} d\tau. \quad (3.7)$$

(ii) The left Riemann-Liouville derivatives:

$${}_0^R D_t^\alpha u(x, t) := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(x, \tau)}{(t-\tau)^\alpha} d\tau. \quad (3.8)$$

(iii) The right Riemann-Liouville derivatives:

$${}_t^R D_T^\alpha u(x, t) := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_t^T \frac{u(x, \tau)}{(t-\tau)^\alpha} d\tau. \quad (3.9)$$

Many authors believe that the Caputo version is more natural because it makes it easier to manage inhomogeneous initial conditions. Then, the following relationship is related to the two concepts (3.7) and (3.8), which can be checked by a direct calculation:

$${}_0^R D_t^\alpha u(x, t) = {}_0^C D_t^\alpha u(x, t) + \frac{u(x, 0)}{\Gamma(1-\alpha) t^\alpha}. \quad (3.10)$$

Definition 46 [46] For any real $\sigma > 0$, we define the space ${}^l H_0^\sigma(I)$ as the closure of $C_0^\infty(I)$

with respect to the following norm $\|\cdot\|_{{}^l H_0^\sigma(I)}$:

$$\|u\|_{{}^l H_0^\sigma(I)} := \left(\|u\|_{L_2(I)}^2 + |u|_{{}^l H_0^\sigma(I)}^2 \right)^{\frac{1}{2}}, \quad (3.11)$$

where

$$|u|_{{}^l H_0^\sigma(I)}^2 := \left\| {}_0^R D_t^\sigma u \right\|_{L_2(I)}^2, \quad (3.12)$$

Definition 47 For any real $\sigma > 0$, we define the space ${}^r H_0^\sigma(I)$ as the closure of $C_0^\infty(I)$

with respect to the following norm $\|\cdot\|_{{}^r H_0^\sigma(I)}$:

$$\|u\|_{{}^r H_0^\sigma(I)} := \left(\|u\|_{L_2(I)}^2 + |u|_{{}^r H_0^\sigma(I)}^2 \right)^{\frac{1}{2}}, \quad (3.13)$$

where

$$|u|_{rH_0^\sigma(I)}^2 := \left\| {}^R_t \partial_T^\sigma u \right\|_{L^2(I)}^2, \quad (3.14)$$

Definition 48 For any real $\sigma > 0$, we define the semi-norm:

$$|u|_{cH^\sigma(I)} = \left| \frac{({}^R D_t^\sigma u, {}^R D_t^\sigma u)_{L^2(I)}}{\cos(\alpha\pi)} \right|^{1/2}$$

and norm:

$$\|u\|_{cH^\sigma(I)} = (\|u\|_{L^2(I)}^2 + |u|_{cH^\sigma(I)}^2)^{1/2}.$$

Lemma 49 [46, 49] For any real $\sigma \in \mathbb{R}_+$, if $u \in {}^l H^\alpha(I)$ and $v \in C_0^\infty(I)$, then

$$({}^R D_t^\sigma u(t), v(t))_{L^2(I)} = (u(t), {}^R D_t^\sigma v(t))_{L^2(I)}. \quad (3.15)$$

Lemma 50 [46, 49] For $0 < \sigma < 2$, $\sigma \neq 1$, $u \in H_0^{\frac{\sigma}{2}}(I)$, we have :

$${}^R D_t^\sigma u(t) = {}^R D_t^{\frac{\sigma}{2}} {}^R D_t^{\frac{\sigma}{2}} u(t). \quad (3.16)$$

Lemma 51 [46, 49] For $\sigma \in \mathbb{R}_+$, $\sigma \neq n + \frac{1}{2}$, the semi-norms $|\cdot|_{lH^\sigma(I)}$, $|\cdot|_{rH^\sigma(I)}$ and $|\cdot|_{cH^\sigma(I)}$ are equivalent. Then, we pose

$$|\cdot|_{lH^\sigma(I)} \cong |\cdot|_{rH^\sigma(I)} \cong |\cdot|_{cH^\sigma(I)}. \quad (3.17)$$

Lemma 52 [46] For any real $\sigma > 0$, the space ${}^l H_0^\sigma(I)$ with respect to the norm (3.10) is complete.

Definition 53 We denote by $L_2(0, T, L_2(0, d)) := L_2(Q)$ the space of square functions, integrated with the scalar product in the Bochner sense,

$$(u, w)_{L_2(0, T, L_2(0, d))} = \int_0^T ((u, \cdot), (w, \cdot))_{L_2(0, d)} dt. \quad (3.18)$$

Since the space $L_2(0, d)$ is a Hilbert space, it can be shown that $L_2(0, T, L_2(0, d))$ is a Hilbert space as well. Let $C^\infty(0, T)$ denote the space of infinitely differentiable functions on $(0, T)$ and $C_0^\infty(0, T)$ denote the space of infinitely differentiable functions with compact support in $(0, T)$.

3.3 Solvability of the direct fractional parabolic problem

3.3.1 Position of the problem

In the rectangular domain $Q = (0, d) \times (0, T)$, with $d, T < \infty$ and $0 < \alpha < 1$, we shall study the existence and uniqueness of solutions $u = u(x, t)$ to the following fractional parabolic problem :

$$\left\{ \begin{array}{ll} {}^C D_t^\alpha u(x, t) - \left(\frac{\partial^2 u(x, t)}{\partial x^2} \right) + bu(x, t) = \tilde{f}(x, t) & \text{in } Q; \\ u(x, 0) = \varphi(x) & \forall x \in (0, d), \\ u(0, t) = u(d, t) = 0 & \forall t \in (0, T). \end{array} \right. \quad (3.19)$$

We consider the following fractional parabolic equation of the type:

$$\mathcal{L}u = {}^C D_t^\alpha u - \frac{\partial^2 u}{\partial x^2} + bu = \tilde{f}, \quad (3.20)$$

with the initial condition :

$$\ell u = u(x, 0) = \varphi(x), \quad \forall x \in (0, d), \quad (3.21)$$

and Dirichlet condition :

$$u(0, t) = u(d, t) = 0, \quad \forall t \in (0, T). \quad (3.22)$$

where $b \in \mathbb{R}_*^+$; \tilde{f} and φ are known functions.

We shall assume that the function φ satisfies a compatibility conditions, i.e.,

$$\varphi(0) = \varphi(d) = 0. \quad (3.23)$$

Now, introducing a new function :

$$v(x, t) = u(x, t) - U(x) \implies u(x, t) = v(x, t) + U(x), \quad (3.24)$$

where

$$\varphi(x) = U(x). \quad (3.25)$$

So, we get :

$$\left\{ \begin{array}{ll} {}^C D_t^\alpha v(x, t) - \left(\frac{\partial^2 v(x, t)}{\partial x^2} \right) + bv(x, t) = \tilde{f}(x, t) - \mathcal{L}\varphi(x) = f(x, t) & \text{in } Q; \\ v(x, 0) = 0 & \forall x \in (0, d), \\ v(0, t) = v(d, t) = 0 & \forall t \in (0, T). \end{array} \right. \quad (3.26)$$

Such that :

$${}^C D_t^\alpha v(x, t) - \frac{\partial^2 v(x, t)}{\partial x^2} + bv(x, t) = f(x, t) \quad (3.27)$$

with the initial condition :

$$\ell v = v(x, 0) = 0, \quad \forall x \in (0, d), \quad (3.28)$$

the boundary condition of Dirichlet type :

$$v(0, t) = v(d, t) = 0, \quad \forall t \in (0, T) \quad (3.29)$$

where

$$f(x, t) = \tilde{f}(x, t) + \frac{\partial^2 \varphi(x)}{\partial x^2} - b\varphi(x). \quad (3.30)$$

3.3.2 A priori estimate

In this section, we illustrate the existence and uniqueness of the problem's solution.

(3.27) – (3.29) as a solution of the operator equation

$$Lv = \mathcal{F}, \quad (3.31)$$

where $L = (\mathcal{L}, \ell)$, with domain of definition B consisting of functions $v \in L^2(Q)$, such that $v, {}^C D_t^\alpha v, \frac{\partial v}{\partial x} \in L^2(Q)$ and v verify (3.29).

The operator L is considered from B to F , where B is the Banach space consisting of all functions $v(x, t)$ having a finite norm

$$\|v\|_B^2 = \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \|v\|_{L^2(Q_T)}^2 + \left\| \frac{\partial v}{\partial x} \right\|_{L^2(Q)}^2, \quad (3.32)$$

and F is the Hilbert space consisting of all elements $\mathcal{F} = (f, 0)$ for which the norm $L^2(Q)$ is finite.

Theorem 54 *For any function $u \in B$, we have the inequality*

$$\|v\|_B \leq k \|Lv\|_{L^2(Q)} \quad (3.33)$$

where k is a positive constant independent of v .

Proof. Multiplying equation (3.27) by the following function :

$$Mv = v(x, t), \quad (3.34)$$

and integrating over $Q = (0, d) \times (0, T)$, we get :

$$\begin{aligned}
& \int_Q \mathcal{L}v \cdot Mv dxdt \tag{3.35} \\
&= \int_Q {}^C D_t^\alpha v(x, t) \cdot v(x, t) dxdt - \int_Q \frac{\partial^2 v(x, t)}{\partial x^2} v(x, t) dxdt \\
&\quad + \int_Q b \cdot v^2(x, t) dxdt \\
&= \int_Q f(x, t) \cdot v(x, t) dxdt.
\end{aligned}$$

As $v(x, 0) = 0$, so by applying Lemmas 49,50 and 51, becomes

$$\begin{aligned}
& \int_Q {}^C D_t^\alpha v(x, t) \cdot v(x, t) dxdt \\
&= ({}^C D_t^\alpha v(x, t), v(x, t))_{L^2(Q)} \\
&= ({}^R D_t^{\frac{\alpha}{2}} {}^R D_t^{\frac{\alpha}{2}} v(x, t), v(x, t))_{L^2(Q)} \tag{According to Lemma 50} \tag{3.36} \\
&= ({}^R D_t^{\frac{\alpha}{2}} v(x, t), {}^R D_t^{\frac{\alpha}{2}} v(x, t))_{L^2(Q)} \tag{According to Lemma 49} \\
&= |u|_{cH^\alpha(Q)}^2 \cong |u|_{iH^\alpha(Q)}^2 = \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2, \tag{According to Lemma 51}
\end{aligned}$$

and by integration by parts over $(0, d)$, we get

$$- \int_Q \frac{\partial^2 v(x, t)}{\partial x^2} v(x, t) dxdt = \int_Q \left(\frac{\partial v(x, t)}{\partial x} \right)^2 dxdt. \tag{3.37}$$

So, we obtain

$$\begin{aligned}
& \int_Q \left({}^R D_t^\alpha v(x, t) - \frac{\partial^2 v(x, t)}{\partial x^2} + bv(x, t) \right) \cdot Mv dxdt \tag{3.38} \\
&\cong \left\| {}^R D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \int_Q \left(\frac{\partial v(x, t)}{\partial x} \right)^2 dxdt + \int_Q bv^2(x, t) dxdt \\
&\leq \frac{1}{2\varepsilon} \int_Q |f(x, t)|^2 dxdt + \frac{\varepsilon}{2} \int_Q |v(x, t)|^2 dxdt.
\end{aligned}$$

So, we get

$$\begin{aligned}
& \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \int_Q \left(\frac{\partial v(x,t)}{\partial x} \right)^2 dxdt \\
& + \int_Q bv^2(x,t) dxdt \\
& \leq \frac{1}{2\varepsilon} \int_Q |f(x,t)|^2 dxdt + \frac{\varepsilon}{2} \int_Q |v(x,t)|^2 dxdt.
\end{aligned} \tag{3.39}$$

which give

$$\begin{aligned}
& \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \int_Q \left(\frac{\partial v(x,t)}{\partial x} \right)^2 dxdt \\
& + \int_Q \left(b - \frac{\varepsilon}{2} \right) v^2(x,t) dxdt \\
& \leq \frac{1}{2\varepsilon} \int_Q |f(x,t)|^2 dxdt,
\end{aligned} \tag{3.40}$$

So, we have

$$\left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 \leq \frac{1}{2\varepsilon} \|f\|_{L^2(Q)}^2, \tag{3.41}$$

On the other hand, we have

$$\left\| \frac{\partial v}{\partial x} \right\|_{L^2(Q)}^2 \leq \frac{1}{2\varepsilon} \|f\|_{L^2(Q)}^2. \tag{3.42}$$

Also, we have

$$\|v\|_{L^2(Q)}^2 \leq \frac{1}{2\varepsilon \left(b - \frac{\varepsilon}{2} \right)} \|f\|_{L^2(Q)}^2. \tag{3.43}$$

By combining (3.41), (3.42) and (3.43), for $\varepsilon < \frac{b}{2}$, we get

$$\begin{aligned}
& \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \left\| \frac{\partial v}{\partial x} \right\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2 \\
& \leq \frac{1}{2\varepsilon} \left(1 + \frac{1}{\left(b - \frac{\varepsilon}{2} \right)} \right) \|f\|_{L^2(Q)}^2.
\end{aligned} \tag{3.44}$$

Finally, it follows that

$$\left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \left\| \frac{\partial v}{\partial x} \right\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2 \leq C \|f\|_{L^2(Q)}^2, \quad (3.45)$$

with

$$C = \frac{1}{2\varepsilon} \left(1 + \frac{1}{(b - \frac{\varepsilon}{2})} \right). \quad (3.46)$$

Therefore, we obtain that

$$\|v\|_B \leq k \|Lv\|_F \quad \text{where } k = \sqrt{C}. \quad (3.47)$$

Hence, the uniqueness of the solution. ■

Remark 55 *This inequality $\|v\|_B \leq k \|Lv\|_F$ gives the uniqueness of the solution, indeed:*

Let v_1 and v_2 two solutions, so

$$\begin{cases} Lv_1 = \mathcal{F} \\ Lv_2 = \mathcal{F} \end{cases} \implies L(v_1 - v_2) = 0 \quad (3.48)$$

then

$$\|v_1 - v_2\|_B \leq k \|0\|_F \implies \|v_1 - v_2\|_B \leq 0 \implies v_1 - v_2 = 0 \quad (3.49)$$

which gives the uniqueness of the solution.

Proposition 56 *The operator L from B to F admits a closure.*

Proof. Let $(v_n)_{n \in \mathbb{N}} \subset D(L)$ be a sequence such that:

$$v_n \rightarrow 0 \quad \text{in } B, \quad (3.50)$$

$$Lv_n \rightarrow \mathcal{F} \quad \text{dans } F,$$

it must be shown that

$$f \equiv 0. \quad (3.51)$$

The convergence of v_n toward 0 in B entails that

$$v_n \rightarrow 0 \text{ in } (C_0^\infty(Q_T))'. \quad (3.52)$$

As the continuity of the fractional derivation (3.2) and the derivation of the first order (as a particular case of the fractional derivative) of $(C_0^\infty(Q_T))'$ in $(C_0^\infty(Q_T))'$, then (3.52) implies :

$$\mathcal{L}u_n \rightarrow 0 \text{ in } (C_0^\infty(Q_T))'. \quad (3.53)$$

On the other hand the convergence of $\mathcal{L}v_n$ to f in $F = L^2(Q_T)$ implies that

$$\mathcal{L}u_n \rightarrow f \text{ in } (C_0^\infty(Q_T))'. \quad (3.54)$$

By virtue of the uniqueness of the limit in $(C_0^\infty(Q_T))'$, we conclude between (3.53) and (3.54) that

$$f \equiv 0. \quad (3.55)$$

Hence, the operator L is closable. ■

Definition 57 *Let \bar{L} the closure of L and $D(\bar{L})$ the definition domain of \bar{L} . The solution of the equation*

$$\bar{L}v = \mathcal{F} \quad (3.56)$$

is called generalized strong solution of the problem (3.27) – (3.29).

Theorem 54 is valid for a generalized strong solution, i.e, we have the following inequality:

$$\|v\|_B \leq k \|\bar{L}v\|_F, \forall v \in D(\bar{L}). \quad (3.57)$$

Consequently, this last inequality entails the following corollaries :

Corollary 58 *The strong solution of the problem (3.27) – (3.29) is unique and depends continuously on $f \in F$.*

Corollary 59 *The range $R(\bar{L})$ of the operator \bar{L} is equal to the closure $\overline{R(L)}$ of $R(L)$.*

Proof. Let $z \in \overline{R(L)}$, then there exists a Cauchy sequence $(z_n)_n$ in F consists of the elements of the set $R(L)$ such that

$$\lim_{n \rightarrow +\infty} z_n = z. \quad (3.58)$$

So there is a corresponding sequence $(v_n)_n \subset D(L)$ such that

$$Lv_n = z_n. \quad (3.59)$$

From the estimate (3.41), we obtain

$$\|v_p - v_q\|_B \leq k \|Lv_p - Lv_q\|_F \rightarrow 0, \text{ when } p, q \rightarrow +\infty. \quad (3.60)$$

We can deduce that $(v_n)_n$ is a Cauchy sequence in B , so there is $v \in B$

$$\lim_{n \rightarrow +\infty} v_n = v \text{ in } B. \quad (3.61)$$

By virtue of the definition of \bar{L} ($\lim_{n \rightarrow +\infty} v_n = v$ in B ; if $\lim_{n \rightarrow +\infty} Lv_n = \lim_{n \rightarrow +\infty} z_n = z$,

so $\lim_{n \rightarrow +\infty} \bar{L}v_n = z$ and as \bar{L} is closed so $\bar{L}v = z$), the function v verifies that

$$v \in D(\bar{L}), \quad \bar{L}v = z. \quad (3.62)$$

Thus, $z \in R(\bar{L})$, then

$$\overline{R(L)} \subset R(\bar{L}). \quad (3.63)$$

So we conclude here that $R(\bar{L})$ is closed because it is complete (any complete subspace of a metric space (not necessarily complete) is closed).

It remains to show the opposite inclusion.

Let $z \in R(\bar{L})$, then there is a sequence of $(z_n)_n$ in F consists of the elements of the set $R(\bar{L})$ such that

$$\lim_{n \rightarrow +\infty} z_n = z. \quad (3.64)$$

where $z \in R(\bar{L})$, because $R(\bar{L})$ is closed subset of a complete space F , then, $R(\bar{L})$ is complete.

So there is a corresponding sequence $(v_n)_n \subset D(\bar{L})$ such that

$$\bar{L}v_n = z_n. \quad (3.65)$$

From the estimate (3.57), we obtain

$$\|v_p - v_q\|_B \leq k \|\bar{L}v_p - \bar{L}v_q\|_F \rightarrow 0, \text{ if } p, q \rightarrow +\infty. \quad (3.66)$$

We can deduce that $(v_n)_n$ is a Cauchy sequence in B , so there is $v \in B$

$$\lim_{n \rightarrow +\infty} v_n = v \text{ in } B. \quad (3.67)$$

Once more, there is a corresponding sequence $(L(v_n))_n \in R(L)$ such that

$$Lv_n = \bar{L}v_n \text{ over } R(L), \forall n. \quad (3.68)$$

Then

$$\lim_{n \rightarrow +\infty} Lv_n = z, \quad (3.69)$$

Consequently, $z \in \overline{R(L)}$, and then, we conclude that

$$R(\overline{L}) \subset \overline{R(L)}. \quad (3.70)$$

■

3.3.3 Existence of solution

To show the existence of solutions, we prove that $R(L)$ is dense in F for all $u \in B$ and for arbitrary $\mathcal{F} = (f, 0) \in F$.

Theorem 60 *The problem (3.27) – (3.29) admits a solution.*

Proof. The scalar product of F is defined by

$$(Lv, W)_F = \int_{Q_T} \mathcal{L}v \cdot w dxdt, \quad \text{where } W = (w, 0) \in D(L). \quad (3.71)$$

If we put $w \in R(L)^\perp$, we have

$$\int_{Q_T} \left({}^C D_t^\alpha v(x, t) - \frac{\partial^2 v(x, t)}{\partial x^2} + bv(x, t) \right) \cdot w(x, t) dxdt = 0, \quad (3.72)$$

where ${}^C D_t^\alpha v$, $\frac{\partial v}{\partial x}$, $v \in L^2(Q_T)$, with v satisfies the boundary conditions of (3.27) – (3.29).

From (3.72), we get the equality

$$\begin{aligned} & \int_{Q_T} {}^C D_t^\alpha v(x, t) \cdot w(x, t) dxdt \\ & - \int_{Q_T} \frac{\partial^2 v(x, t)}{\partial x^2} \cdot w(x, t) dxdt \\ & + b \int_{Q_T} v(x, t) \cdot w(x, t) dxdt \\ & = 0 \end{aligned} \quad (3.73)$$

And from the equality (3.73), we give the function w in terms of v as follows:

$$w = v \quad (3.74)$$

then $w \in L^2(Q_T)$.

Replacing w in (3.73) by its representation (3.74) and integrating by parts each term of (3.73) and by taking the condition of v , we obtain

$$\begin{aligned} & \int_{Q_T} \left({}^C D_t^{\frac{\alpha}{2}} v(x, t) \right)^2 dxdt + \int_{Q_T} bv^2(x, t) dxdt \\ & \leq - \int_{Q_T} \left(\frac{\partial v(x, t)}{\partial x} \right)^2 dxdt \\ & \leq 0, \end{aligned} \quad (3.75)$$

then

$$\left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q_T)}^2 + b \|v\|_{L^2(Q_T)}^2 \leq 0, \quad (3.76)$$

Hence

$$\|v\|_{L^2(Q_T)} = 0 \quad (3.77)$$

And thus, $v = 0$ in Q_T which gives $w = 0$ in Q_T . This proves Theorem 60. So

$\overline{R(L)} = F$. ■

3.4 Existence and uniqueness of the solution of main problem

We are finding a solution in the form of the original inverse problem. $\{u, f\} = \{z, f\} + \{y, 0\}$ where y is the solution of the direct problem

$${}^C D_t^\alpha y - \Delta y + \beta y = 0. \quad (x, t) \in Q_T, \quad (3.78)$$

$$y(x, 0) = \varphi(x), \quad x \in \Omega, \quad (3.79)$$

$$y(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T], \quad (3.80)$$

while the pair $\{z, f\}$ is the solution of the inverse problem

$${}^C D_t^\alpha z - \Delta z + \beta z = f(t) g(x, t). \quad (x, t) \in Q, \quad (3.81)$$

$$z(x, 0) = 0, \quad x \in \Omega, \quad (3.82)$$

$$z(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T], \quad (3.83)$$

$$\int_{\Omega} v(x) z(x, t) dx = E(t), \quad t \in [0, T], \quad (3.84)$$

where

$$E(t) = \theta(t) - \int_{\Omega} v(x) y(x, t) dx. \quad (3.85)$$

We will assume that the functions that appear in the problem data are measurable and fulfill the following conditions:

$$\left\{ \begin{array}{l} g \in C((0, T), L_2(\Omega)), \quad v \in W_2^1(\Omega), \quad E \in W_2^2(0, T), \\ \|g(x, t)\| \leq m; \quad |g^*(t)| \geq p > 0, \quad \text{for } p \in \mathbb{R}, \quad (x, t) \in Q_T, \\ \varphi(x) \in W_2^1(\Omega) \quad \text{where } g^* \text{ is defined in (3.6)}. \end{array} \right. \quad (3.86)$$

The correspondence between f and z can be seen as one way of defining the linear operator.

$$A : L_2(0, T) \rightarrow L_2(0, T). \quad (3.87)$$

with the values

$$(Af)(t) = \frac{1}{g^*} \left\{ \int_{\Omega} \nabla z \nabla v dx \right\}. \quad (3.88)$$

In this view, the linear equation of the second form for the function is rational to refer to f over the space $L_2(0, T)$:

$$f = Af + W, \quad (3.89)$$

where

$$W = \frac{{}^C D_t^\alpha E + \beta E}{g^*}. \quad (3.90)$$

Remark 61 As $\{u, f\} = \{z, f\} + \{y, 0\}$ where y is the solution of the direct problem (3.78)–(3.80). Obviously, the previous section implies that y exists and is unique, but instead of demonstrating the solvability of the initial problem (3.1)–(3.4), we demonstrate the existence and uniqueness of the inverse problem (3.81) – (3.84) solution.

Theorem 62 Suppose the input of the inverse problem data (3.81) – (3.84) satisfies (3.86). Then the following assertions are valid: (i) if the inverse problem (3.81) – (3.84) is solvable, then so is equation (3.89). And (ii) if equation (3.89) has a solution and the condition of compatibility has

$$E(0) = 0, \quad (3.91)$$

holds, then a solution to the inverse problem exists.

Proof. (i) Suppose that the inverse problem (3.81) – (3.84) is solvable. We denote its solution by $\{z, f\}$. Multiplying the function v scalarly in $L_2(\Omega)$ both sides of (3.81) , we get

$${}^C D_t^\alpha \int_{\Omega} z v dx + \int_{\Omega} \nabla z \nabla v dx + \beta \int_{\Omega} z v dx = f(t) g^*(x, t). \quad (3.92)$$

With (3.84) and (3.88), from (3.92), it follows that

$$f = Af + \frac{{}^C D_t^\alpha E + \beta E}{g^*}.$$

This gives that f solves equation (3.89).

(ii) Equation (3.89) has a solution in space, according to the assumption, $L_2(0, T)$, say f .

The resulting relationship (3.81) – (3.83) can be viewed as a direct problem with a unique solution $z \in W_2^1(Q_T)$ when inserting this function in (3.81). Let us show that the z function also satisfies the condition of integral overdetermination (3.84). By equation (3.92), the function z is subject to the following relation

$${}^C D_t^\alpha E + \beta E + \int_{\Omega} \nabla z \nabla v dx = f(t) g^*(x, t). \quad (3.93)$$

Subtracting equation (3.92) from equation (3.93), we get

$${}^C D_t^\alpha \int_{\Omega} z v dx + \beta \int_{\Omega} z v dx = {}^C D_t^\alpha E + \beta E. \quad (3.94)$$

Integrating the preceding differential equation and taking into account the compatibility condition (3.89), we find that the overdetermination condition (3.84) is satisfied by z and the function pair $\{z, f\}$ is a solution to the inverse problem (3.81) – (3.84).

This completes the theorem's proof. ■

Now, we are touching on some properties of operator A .

Lemma 63 *Let the condition (3.86) hold. Then, there exist a positive ε for which A is a contracting operator in $L_2(0, T)$.*

Proof. Obviously, (3.88) yields the estimate

$$\|Af\|_{L_2(0, T)} \leq \frac{k}{p} \left(\int_0^T \|\nabla z(\cdot, \tau)\|_{L_2(\Omega)}^2 d\tau \right)^{\frac{1}{2}}, \quad (3.95)$$

where

$$k = \|\nabla v\|_{L_2(\Omega)}. \quad (3.96)$$

Multiplying both sides of (3.81) by z scalarly in $L_2(Q_T)$ and integrating the resulting by parts with use of (3.82), we get

$$\left\| {}^C D_t^{\frac{\alpha}{2}} z \right\|_{L_2(Q_T)}^2 + \|\nabla z\|_{L_2(Q_T)}^2 + \beta \|z\|_{L_2(Q_T)}^2 = \int_0^T \left(f(t) \int_{\Omega} g(x,t) z dx \right) dt. \quad (3.97)$$

Thus, by using the Cauchy's ε -inequality, we obtain

$$\begin{aligned} & \left\| {}^C D_t^{\frac{\alpha}{2}} z \right\|_{L_2(Q_T)}^2 + \|\nabla z\|_{L_2(Q_T)}^2 + \beta \|z\|_{L_2(Q_T)}^2 \\ & \leq \frac{m|\Omega|}{2\varepsilon} \int_0^T |f(t)|^2 dt + \frac{\varepsilon}{2} \|z\|_{L_2(Q_T)}^2, \end{aligned} \quad (3.98)$$

Choosing $0 < \varepsilon < 2\beta$, we get

$$\begin{aligned} & \left\| {}^C D_t^{\frac{\alpha}{2}} z \right\|_{L_2(Q_T)}^2 + \|\nabla z\|_{L_2(Q_T)}^2 + \left(\beta - \frac{\varepsilon}{2} \right) \|z\|_{L_2(Q_T)}^2 \\ & \leq \frac{m|\Omega|}{2\varepsilon} \int_0^T |f(\tau)|^2 dt. \end{aligned} \quad (3.99)$$

Omitting some terms on the left-hand side (3.99) leads to

$$\|\nabla z\|_{L_2(Q_T)}^2 = \int_0^T \|\nabla z(\cdot, \tau)\|_{L_2(\Omega)}^2 d\tau \leq \frac{m|\Omega|}{2\varepsilon} \int_0^T |f(\tau)|^2 dt. \quad (3.100)$$

According to (3.95) and (3.100), we can obtain the following estimate:

$$\|Af\|_{L_2(0,T)} \leq \delta \int_0^T |f(\tau)|^2 dt, \quad 0 \leq t \leq T, \quad (3.101)$$

where

$$\delta = \frac{k\sqrt{m}|\Omega|}{p\sqrt{2\varepsilon}}. \quad (3.102)$$

So, we obtain

$$\|Af\|_{L_2(0,T)} \leq \delta \|f\|_{L_2(0,T)}. \quad (3.103)$$

It is obvious from the above that there is positive ε such that

$$\delta < 1. \quad (3.104)$$

Inequality (3.103) shows that the operator A is a contracting mapping on $L_2(0, T)$.

■

The following result may be useful with respect to the particular solvability of the inverse problem concerned.

Theorem 64 *Let the compatibility condition (3.91) and the condition (3.86) hold. Then, the inverse problem (3.81) – (3.84) has a unique solution $\{z, f\}$.*

Proof. This means that the equation (3.89) has a unique solution f in $L_2(0, T)$.

The existence of a solution to the inverse problem (3.81) – (3.84) is verified, according to Lemma 51.

The uniqueness of this solution has yet to be proven.

Suppose the contrary that there are two distinct solutions $\{z_1, f_1\}$ and $\{z_2, f_2\}$ of the under consideration inverse problem.

Also, the linear operator A is contracting on $L_2(0, T)$ from Lemma 63, which gives that $f_1 = f_2$, then, by the theorem of the uniqueness of the solution of main direct problem (3.78) – (3.80), we will just have $z_1 = z_2$. ■

Corollary 65 *The solution f to equation (3.91) depends continuously, under the conditions of Theorem 64, on the data W .*

Proof. Let V_1 and V_2 two sets of data that satisfy Theorem 64's assumptions.

Let f and g be solutions of the equation (3.89) corresponding to the data V_1 and V_2 , respectively. According to (3.103), we have

$$f = Af + V_1, \quad (3.105)$$

$$g = Ag + V_2.$$

Let us estimate the difference first, $f - g$. It is easy to see with the use of (3.103), that

$$\begin{aligned} \|f - g\|_{L_2(0,T)} &= \|(Af + V_1) - (Ag + V_2)\|_{L_2(0,T)} & (3.106) \\ &= \|A(f - g) + (V_1 - V_2)\|_{L_2(0,T)} \\ &\leq \delta \|f - g\|_{L_2(0,T)} + \|V_1 - V_2\|_{L_2(0,T)}, \end{aligned}$$

so, we get

$$\|f - g\|_{L_2(0,T)} \leq \frac{1}{(1 - \delta)} \|V_1 - V_2\|_{L_2(0,T)}. \quad (3.107)$$

■

Conclusion and Perspectives

* In the second work we are interested in fractional parabolic problems with boundary conditions. Where, through the use of the energy inequality method, it has been possible to establish the existence and uniqueness of a strong solution of the Dirichlet problem for a fractional parabolic equation

It is important to note again that there does not yet exist for non-local problems a general theory analogous to that of classical problems.

This is due to the relative novelty of this theme on the one hand and to the complexity of the questions it raises on the other. Each problem then requires a specific treatment, which underlines the topicality of the subject approached in this thesis.

It is pointed out that many interesting problems in this study remain open, we cite a few here:

- The study of the solutions of the fractional problems for a class of PDEs (parabolic and hyperbolic) with an integral condition of type 2 or nonlocal conditions.

- Thus, many interesting perspectives for numerical analysis could allow to continue the work undertaken in this study, especially on the development side of efficient numerical methods, in order to be compatible with non-local conditions of integral type.

* The third chapter contains a new inverse problem by investigating the fractional derivatives where we develop the method of fixed point and energy inequality method for proving the solvability of an inverse fractional problem. We note that our work extends to the existence of open problems as a study of the nonlinear case of inverse problems and the numerical part.

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