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NUMERICAL CALCULUS OF SOME PDE MODELS WITH NON-LOCAL CONDITIONS OF INTEGRAL TYPES

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TABLE OF CONTENTS

NOTATIONS

$\pounds$	Differential Operator
$\Im_x^m f$	The operator integral of order m with respect to the space variable x.
B_2^m	Bouziani space
I_0^α	Fractional integral in order α
${}_0^c\partial_t^\alpha$	Fractionnal derivative in order α in sens of Caputo
${}_0^{RL}D_x^\beta$	Fractionnal Derivative in order β in the sens of Reimann-Liouville
${}_0^{GL}D_x^\beta$	Fractionnal Derivative in order β in the sens of Grünewald-Letnikov
PDE	Partial Derivative Equation.
FDE	Fractional Derivative Equation
FDM	Finite Difference Method
$FPDE$	Fractional Partial Derivative Equation.
$D(L)$	Domain of definition of the operator L
$R(L)$	Runge of the operator L
H	Hilbert space
B	Banach space
Q	Q is the domain $(0;1) \times (0;T)$
$L_2(\Omega)$	Space of measurable square integrable functions in Ω
$\binom{m}{n}$	Binomial coefficient.
$C_0(\Omega)$	Space of continuous functions with compact support in Ω
$L^2(B_2^m, \Omega)$	Hilbert space of measurable square integrable functions on Ω taking values in B_2^m
$G(L)$	Graph of the operator L.

GENERAL INTRODUCTION

Introduction

Fractional Partial Differential Equations **(FPDEs)** have become very important in recent years due to their use in several mathematical models. (FPDE) considered as the generalization of (PDE) of an integer order of an arbitrary order. These generalizations play an essential role in engineering, physics and applied mathematics. Due to the properties of Fractional Differential Equations(FDE), a differents models are created for complex phenomena using FPDEs, for example in electroanalytical chemistry, viscoelasticity [1, 2, 35], porous environment, fluid flow, thermodynamic [3, 4], diffusion transport, rheology [5, 6, 7, 22, 23, 37], electromagnetism, signal processing [8], electrical network [9] and others [10, 11, 12, 39]. Some relevant applications of fractional differential equations in the modeling of tribo-fatigue systems and new materials can be mentioned as, a method for the experimental study of friction in an active system [13], volumetric damage state of the tribofatigue system [14], TheTribo-Fatigue Damage Transition and Mapping for Wheel Material under Rolling-Sliding Contact Condition [15]; this study is based on construction a tribo-fatigue damage map of high-speed railway wheel material under different tangential forces and contact pressure conditions through JD-1testing equipment. Several problems have been mentioned in modern physics and technology using partial differential equations (PDEs) where the nonlocal conditions are described by integrals as $\int_0^1 v\left(x,t\right)dx$, $\int_0^1 \varphi(x)v\left(x,t\right)dx$. These integrals conditions are of great interest due to their applications in many fields, in population dynamics, heat diffusion-advection, models of blood circulation, chemical engineering thermoelasticity [16]. The existence and uniqueness of the solution of these problems have been studied by many authors [17, 18, 19, 20]. Some results have been obtained by construction of a variational formulation which depends on the choice of spaces and their norms, Lax-Milgram theorem, Poincaré theorem, fixed point theory, Laplace transforms. For the numerical study of (FPDE) with **classical boundary non-local conditions**, we can cite the works of A. Alikhanov [21, 22, 23], M.M, Meerschaert, S. Shen and F Liu [24, 25], El-Nabulsi, R.A [26, 27, 28] and others [29, 30, 31, 32, 33]. Among these authors we can cite Yuriy Povstenko [29, 34] who studied the time fractional diffusion-wave equation with **classic boundary conditions**.Taki and Bouziani [18, 27] have been study a problem of (FPDE) having boundary condition of integral type $\int_0^1 v(x,t)\,dx$ with respect time derivative of order α $(0 < \alpha < 1)$.

In this thesis, we are interested in a new problem of (FPDE) with **boundary conditions of integrals types** $\int_0^1 v(x,t)\,dx$, $\int_0^1 x^n v(x,t)\,dx$. We consider the time fractional advection-diffusion equation, obtained by replacing the second-order time derivative in standard wave equation with a fractional derivative of order α $(1 < \alpha < 2$ and $0 < \alpha < 1)$,and classical boundary conditions with integral boundary conditions [27, 28]. The physical interpretation of the fractional derivative is that it represents a degree of memory in the diffusing material. For the theoretical study in third chapitre, we use the energy inequalities method to prove the existence and the uniqueness. The numerical study is based on the application of a combination of the finite difference method with a numerical integration method to obtain an approximate solution of the proposed problem. We use a uniform space-time discretization. The Caputo fractional operator of order α $(1 < \alpha < 2$ and $0 < \alpha < 1)$ is approximated by a scheme called $L2$ and $L2$, and the integer-order differential operators are approximated by central and advanced numerical schemes. Stability and convergence of the numerical scheme obtained show that the method used is conditionally stable and convergent. Numerical tests carried out give very satisfactory results that is the values of the approximate solution are very close to the exact solution. All numerical and graphical results obtained are produced using MATLAB software.

The main objective of this thesis is the study of some problems for (FPDE) with non-local conditions of integral type by developing a numerical method by applying the finite difference method and numerical integration to obtain approximate solutions to the proposed continuous problems.

Structure of the thesis

The thesis consists of introduction, three chapters and conclusion as follows

- General introduction.

- The first chapter is devoted to reminders of some fundamental preliminary notions and the necessary tools namely unbounded linear operators, orthogonality and density in Hibert spaces, important inequalities, fractional calculation, finite difference method.
- The second chapter contains a theoretical and numerical study of a fractional problem with two boundary conditions of integral type.
- The third chapter is the subject of a Numerical Resolution of a Fractional Parabolic Differential Equation with boundary condition of integral type.
- Conclusion and outlook.

In the second chapter the theoretical study can be summarized as follows.

1. First we write the problem posed in the form of an operational equation

$$Lu = \mathcal{F}, \quad u \in D(L); \tag{1}$$

where the operator L is considered from a Banach space B in a suitably chosen Hilbert space F.

2. We establish the a priori estimates for the operator L.
3. We prove the density of the set of values of this operator in the space F.

then we study the existence and uniqueness of the solution of poblem, more precisely, we prove the energy inequality of the type

$$\|u\|_B \leq c \|Lu\|_F. \tag{2}$$

The choice of the operator Mu is imposed by the equation and the boundary conditions, finally we show that the operator L of B in F admits a closure L, therefore the solution of the operational equation

$$\overline{L}u = \mathcal{F}, \quad u \in D\left(\overline{L}\right), \tag{3}$$

is called strong generalized solution of the considered problem. By crossing the limit, the estimate (2)will be extended to $\overline{L}$, that is

$$\|u\|_B \leq c \left\|\overline{L}u\right\|_F. \tag{4}$$

Thus, we deduce the uniqueness of the solution from equation (4). As the image of the operator L is closed in F and that $R\left(\overline{L}\right) = \overline{R\,(L)}$, the establishment of the density of the set $R(L)$ in F guarantees the existence of the strong solution of equation (1).

1.0 CHAPTER 1. NOTIONS AND PRELIMINARIES

First, we need to reminder of some fundamental preliminary notions and necessary tools namely unbounded linear operators, orthogonality and density in Hibert spaces, some importantes inequalities, fractional derivative and integration of Caputo, Riemann and Grundwald, finite difference method.

1.1 UNBOUNDED LINEAR OPERATORS

Let E and F be normed spaces vectorial.

Definition 1. *An operator L from E to F is called linear if and only if*

$$\forall\ u_1, u_2\ \in E,\ \forall\ \mu, \lambda \in \mathbb{k},\quad L\left(\lambda u_1 + \mu u_2\right) = \lambda L\left(u_1\right) + \mu L\left(u_2\right),$$

where K is the field of the scalars of E and F.

Definition 2. (*Closed operator*). *The operator $L : D(L) \subset E \longrightarrow F$ is closed if and only if its graph $G(L)$ is closed in $E\times\ F$, where $G\left(L\right)$ is a subspace defined in $E \times F$ by $G\left(L\right) = \left\{\left(u, Lu\right),\ \ u \in D\left(L\right)\right\}$, and $D\left(L\right)$ is the domain of definition of the operator L.*

Remark 1. *A closed operator may also be defined by for any sequence $(u_n) \subset D(L)$ such that. $u_n \to u$ and $Lu_n \to v$, we have $Lu = v$.*

Definition 3. *An operator S is an extension of L if $D(T) \subset D(S)$ and $Lu = Su$ for all $u \in D(L)$, that is $G\left(L\right) \subset G\left(S\right)$.*

Proposition 1. *A subspace $G \subset E \times F$ is the graph of a linear operator if and only if*

$$(0, y) \in G \Rightarrow y = 0. \tag{1.1}$$

Definition 4. (*Closable operator*). *A linear operator* $L : D(L) \subset H$ *from its domain* $D(L)$ *into a Hilbert space* H *is* ***closable*** *if it has a closed extension.*

Remark 2. *To prove that a linear operator* L *is closable, we often prove the following, for any sequence* $(u_n) \subset D(L)$ *such that* $u_n \to 0$ *and* $Lu_n \to v$*, we have* $v = 0$.

Remark 3. T *is Closable in* E *if and only if the adherence* $\overline{G(T)}$ *is a graph. The close operator* $\overline{L}$ *such* $G\left(\overline{L}\right) = \overline{G(L)}$ *is called closure of* L.

Theorem 4. *Let* E *and* F *two spaces of Banach.*

Let E and F two spaces of Banach.

1. If T is bijectif continu linear operator from E into F, we have T^{-1} continu from F into E.
2. We assume $G(T)$ is close in $E \times F$ we have T continu.

1.1.1 Orthogonality and density in Hilbert spaces

Let M be a subspace of a real Hilbert space H.

Definition 5. *The* ***orthogonal*** *of* M *is defined by*

$$M^{\perp} = \{f \in F, \ \langle f, \ g\rangle_F = 0, \ \forall g \in M\}.$$

Proposition 2. M *is dense in* H *if and only if* $M^{\perp} = \{0\}$.

Proof. We assume M dense in H. Let $f \in M^{\perp} \subset F$, and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of elements of M which converges to f.

so $\langle f, \ f_n\rangle_F = 0$ for all $n \in \mathbb{N}$. by passing to the limit we conclude that $\|f\|_F = 0$, so $f = 0$, therefore $M^{\perp} = \{0\}$.

Conversely, suppose that $M^{\perp} = \{0\}$, so we have $\left(M^{\perp}\right)^{\perp} = \{0\}^{\perp} = F$, hence $M \subset \overline{M}$ then $\left(\overline{M}\right)^{\perp} \subset M^{\perp}$

therefore $\left(M^{\perp}\right)^{\perp} \subset \left(\left(\overline{M}\right)^{\perp}\right)^{\perp}$, $\overline{M}$ is closed, then $\left(\left(\overline{M}\right)^{\perp}\right)^{\perp} = \overline{M}$, and $\left(M^{\perp}\right)^{\perp} \subset \overline{M}$ $\Rightarrow F \subset \overline{M}$. so $F = \overline{M}$. □

1.2 SOME USEFUL INEQUALITIES

1.2.1 Cauchy-Schwarz Inequality

For $u,\ v \in L^2(\Omega)$, we have the following inequality

$$\int_\Omega u(x).v(x)\,dx \leq \left(\int_\Omega u^2(x)\,dx\right)^{\frac{1}{2}}\left(\int_\Omega v^2(x)\,dx\right)^{\frac{1}{2}}, \tag{1.2}$$

1.2.2 Holder Inequality

For $u,\ v \in L^p(\Omega)$, we have

$$\int_\Omega u(x).v(x)\,dx \leq \left(\int_\Omega u^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_\Omega v^p(x)\,dx\right)^{\frac{1}{p}},\ p>1. \tag{1.3}$$

1.2.3 Cauchy Inequality with ε

For all $\varepsilon > 0$,and $a,\ b$ in $\mathbb{R}$, we have

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2. \tag{1.4}$$

1.2.4 Young Inequality with ε

For all $\varepsilon > 0$,and $a,\ b$ in $\mathbb{R}$, we have the following inequalite

$$|ab| \leq \frac{1}{p}|\varepsilon a|^p + \frac{p-1}{p}\left|\frac{b}{\varepsilon}\right|^{\frac{p}{p-1}} \text{ For all } p>1, \tag{1.5}$$

which is the generalization of the Cauchy inequality with ε.

Definition 6. (*See* [34]) *. Let us denote by* $C_0(0,1)$ *the space of continuous fonctions with compact support in* $(0,1)$ *, and its bilinear form is given by*

$$((u,w)) = \int_0^1 \Im_x^m u.\Im_x^m w dx \qquad (m \in \mathbb{N}^*), \tag{1.6}$$

where

$$\Im_x^m u(x,t) = \begin{cases} \Im_x^0 u = u \qquad For\ m=0 \\ \Im_x^1 u = \Im_x u = \int\limits_0^x u(\xi,t)\,d\xi \qquad For\ m=1 \\ \Im_t^1 u = \Im_t u = \int\limits_0^t u(x,\tau)\,d\tau. \qquad For\ m=1 \\ \int\limits_0^x \frac{(x-\xi)^{m-1}}{(m-1)!} u(\xi,t)\,d\xi \qquad m\in\mathbb{N}^* \end{cases}$$

The bilinear form (1.6) is considered as scalar product on $C_0(0,1)$ when is not complete.

Definition 7. *(See [34]). We denote by*

$$B_2^m(0,1) = \begin{cases} L^2(0,1) & for\ m=0 \\ u\ mesurable\ /\Im_x^m u \in L^2(0,1) & for\ m\in\mathbb{N}^*, \end{cases}$$

the Bouziani space witch is the completion of $C_0(0,1)$ for the scalar product defined by (1.6). The associated norm to the scalar product is

$$\|u\|_{B_2^m(0,1)} = \|\Im_x^m u\|_{L^2(0,1)} = \left(\int\limits_0^1 (\Im_x^m u)^2\,dx\right)^{\frac{1}{2}}$$

1.2.5 Poincaré Inequality

Lemma 1. *For $u\in L^2(0,l)$, we have this estimate*

$$\|\Im_x u\|_{L^2(0,l)} \le \frac{l}{\sqrt{2}}\|u\|_{L^2(0,l)}. \tag{1.7}$$

Proof. Cauchy-Schwarz inequality leads

$$\begin{aligned}(\Im_x u)^2 &= \left[\int_x^l u(\xi,t)\,d\xi\right]^2 \le \left[\int_x^l d\xi\right]\left[\int_x^l (u(\xi,t))^2\,d\xi\right] \\ &\le (l-x)\int_x^l (u(\xi,t))^2\,d\xi \le (l-x)\int_0^l (u(x,t))^2\,dx,\end{aligned}$$

we integrate on $(0,l)$, we obtain

$$\begin{aligned}\int_0^l (\Im_x u)^2\,dx &\leq \left(\int_0^l (l-x)\,dx\right)\left(\int_0^l (u(x,t))^2\,dx\right)\\ &\leq \frac{l^2}{2}\int_0^l (u(x,t))^2\,dx.\end{aligned}$$

therfore

$$\|\Im_x u\|_{L^2(0,l)} \leq \frac{l}{\sqrt{2}}\|u\|_{L^2(0,l)}. \tag{1.8}$$

□

Lemma 2. *(See [12]). For all $m \in \mathbb{N}^*$, we have*

$$\|u\|_{B_2^m(0,1)} \leq \left(\frac{1}{\sqrt{2}}\right)^m \|u\|_{L^2(0,1)} \tag{1.9}$$

Definition 8. *On the rectangle $Q=(0,1)\times(0,T)$, we define $L^2\left(B_2^m(0,1),(0,T)\right)$ as the space of measurable functions on $(0,T)$ taking values in $B_2^m(0,1)$*

This space also has a structure of a Hilbert space inherited from that of Bouziani space $B_2^m(0,1)$. Its scalar product is given by

$$(u,v)_{L^2\left(B_2^m(0,1),(0,T)\right)} = \int_0^T (u,v)_{B_2^m(0,1)}\,dt,$$

thus its norm is

$$\|u\|_{L^2\left(B_2^m(0,1),(0,T)\right)} = \left(\int_0^T (u,v)_{B_2^m(0,1)}\,dt\right)^{\frac{1}{2}}$$

1.3 FRACTIONAL CALCULUS

1.3.1 Introduction

The Riemann-Liouville-type fractional derivation has played an important role in the development of the theory of derivatives and fractional integrals with a view to their applications in pure mathematics (solutions of differential equations of non-integer order, definition of new classes of functions, ...). Many works have appeared, especially in the theory of elasticity viscosity of solid mechanics, where fractional derivatives are used to give a good description of the properties of materials. Mathematical modeling based on rheological models naturally leads to fractional order differential equations. M. Caputo (in the sixties) proposed another definition which he adapted with Mainardi in the structure of the theory of viscoelasticity, which is more restrictive than that of Riemann-Liouville.

Definition 9. *We call the Gamma function, the function denoted* Γ *defined by:*

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt,$$

where z *is a complex number with* $Re(z) > 0$.

Proposition 3. *The Gamma function has a following proprieties*

1. $\Gamma(1) = \Gamma(2) = 1.$
2. $\Gamma(z+1) = z\Gamma(z).$
3. $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}^*$.

Definition 10. (*See* [2]). *The fractional integral of order* α *of the function* $f \in L^1[a,b]$ *is defined by*

$$I_0^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds; \ t > 0$$

1.3.2 Riemann-Liouville Derivative

Definition 11. *Let $\alpha \in \mathbb{R}^{+}$, $n \in \mathbb{N}$, and f is function integrable on $[0,T]$ such $n-1<\alpha<n$.*

1. *The left **Riemann-Liouville** derivative of order α is defined by*

$$ {}_{0}^{R}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau. \tag{1.10} $$

2. *The right **Riemann-Liouville** derivative for $n-1<\alpha<n$ can be expressed as*

$$ {}_{t}^{R}D_{T}^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_t^T \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau. \tag{1.11} $$

1.3.3 Caputo derivative

Definition 12. *The left Caputo derivative for $1<\alpha<2$ can be expressed as*

$$ {}_{0}^{c}\partial_{t}^{\alpha} f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{f''(s)}{(t-s)^{\alpha-1}} ds;\ \ t>0 $$

Lemma 3. (*See* [12])*. For all real $1<\alpha<2$ we have the inequality*

$$ \int_0^1 {}_{0}^{c}\partial_{t}^{\alpha} (\Im_x u)^2 \, dx \leq 2 \int_0^1 \left({}_{0}^{c}\partial_{t}^{\alpha} \Im_x u\right)(\Im_x u) \, dx \tag{1.12} $$

Lemma 4. (*See* [27])*. For all real $1<\alpha<2$ we have the inequality*

$$ \int_Q \left({}_{0}^{c}\partial_{t}^{\alpha} u\right)(\Im_x u) \, dxdt = \int_Q \left({}_{0}^{c}\partial_{t}^{\frac{\alpha}{2}} \Im_x u\right)^2 dxdt \tag{1.13} $$

Proposition 4. (*See* [12]) *If $0<p<1,\ 0<q<1,\ f(0)=0$ and $t>0$ we have*

$$ {}_{0}^{R}D_{t}^{p+q} f(t) = {}_{0}^{R}D_{t}^{p}\, {}_{0}^{R}\partial_{t}^{q} f(t) = {}_{0}^{R}D_{t}^{q}\, {}_{0}^{R}\partial_{t}^{p} f(t). $$

1.3.4 Relation between Riemann-Liouville and Caputo derivatives

Let $\alpha \in \mathbb{R}^{+}$ with $n-1<\alpha<n$, $(n \in \mathbb{N}^{*})$, and f is function such ${}_{0}^{C}\partial_{t}^{\alpha} f(t)$, ${}_{0}^{R}D_{t}^{\alpha} f(t)$ exists, so

$$ {}_{0}^{R}D_{t}^{\alpha} f(t) = {}_{0}^{C}\partial_{t}^{\alpha} f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(0)\, t^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{1.14} $$

For $n=2$ and $f'(0)=f(0)=0$ we have

$$ {}_{0}^{R}D_{t}^{\alpha} f(t) = {}_{0}^{C}\partial_{t}^{\alpha} f(t). \tag{1.15} $$

1.3.5 Composition with fractional integration operator

$$ {}_{0}^{C}\partial_{t}^{\alpha}\left(I_{0}^{\alpha}f\left(t\right)\right)=f\left(t\right), \tag{1.16}$$

and

$$ I_{0}^{\alpha}\left({}_{0}^{C}\partial_{t}^{\alpha}f\left(t\right)\right)=f\left(t\right)-\sum_{k=0}^{n-1}\frac{f^{(k)}\left(0\right)}{k!}t^{k}. \tag{1.17}$$

for $n=2$, we obtain

$$ I_{0}^{\alpha}\left({}_{0}^{C}\partial_{t}^{\alpha}f\left(t\right)\right)=f\left(t\right)-f'\left(0\right)t-f(0). \tag{1.18}$$

1.3.6 Grünewald-Letnikov Derivative

The fractional Grünewald-Letnikov derivative involves the limits of finite differences of fractional order. This approach allows to obtain very important formulas necessary for the construction of numerical schemes by the discretization of operators of non-integer order .

The **Grunwald-Letnikov** fractional derivative of order β $(n-1<\beta<n)$ of the function f in $[a,b]$ is defined by

$$ {}_{0}^{GL}D_{x}^{\beta}f(x)=\lim_{\substack{h\to0\\x=kh}}\frac{1}{h^{\beta}}\sum_{j=0}^{k}(-1)^{j}\binom{\beta}{j}\,f(x-jh),\ \text{where}\ \binom{\beta}{j}=C_{\beta}^{\,j}=\frac{j!}{\beta!\,(j-\beta)!}=\frac{\Gamma\left(j-\beta\right)}{\Gamma\left(j+1\right)}$$

then

$$ {}_{0}^{GL}D_{x}^{\beta}f(x)=\lim_{h\to0}\frac{h^{-\alpha}}{\Gamma\left(-\alpha\right)}\sum_{j=0}^{k}\frac{\Gamma\left(j-\beta\right)}{\Gamma\left(j+1\right)}f\left(x-jh\right) \tag{1.19}$$

1.3.7 Grunwald Scheme

The grunwald numerical scheme is based on the formula (**??**)by the following approximation

$$ {}_{0}^{GL}D_{x}^{\beta}f(x)\simeq\frac{h^{-\alpha}}{\Gamma\left(-\alpha\right)}\sum_{j=0}^{k}\frac{\Gamma\left(j-\beta\right)}{\Gamma\left(j+1\right)}f\left(x-jh\right).$$

1.4 FINITE DIFFERENCE METHOD

In this part we present the essential outils for Finite Difference Method. The general principle of thIS method is based on Taylor's formula.

Let u be a function of variable $x \in I$. We consider a uniform subdivision of interval I as follow

$$x_i = ih;\ i = 0, ..., N\ ;$$

$$u(x+h) = u(x) + u(x)'h + u(x)''\frac{h^2}{2!} + u(x)'''\frac{h^3}{3!} + ... \tag{1.20}$$

we put

$$u(x_i) = u_i,\ u(x_i+h) = u_{i+1},\ u(x_i-h) = u_{i-1},\ u\prime(x_i) = u_i^{'} \tag{1.21}$$

$$u_{i+1} = u_i + u_i'h + u_i''\frac{h^2}{2!} + u_i'''\frac{h^3}{3!} + u_i^{(4)}\frac{h^4}{4!} + ... \tag{1.22}$$

$$u_{i-1} = u_i - u_i'h + u_i''\frac{h^2}{2!} - u_i'''\frac{h^3}{3!} + u_i^{(4)}\frac{h^4}{4!} + \dots \tag{1.23}$$

By the formulas (1.22) (1.23) we can obtain the following schemes.

The scheme **"forward"** for u_i' of order 1 is

$$u_i' = \frac{u_{i+1} - u_i}{h} + 0(h) \tag{1.24}$$

The scheme **"backward"** for u_i' of order 1 is

$$u_i' = \frac{u_i - u_{i-1}}{h} + 0(h) \tag{1.25}$$

The scheme **"centered"** u_i' of order 2 is

$$u_i' = \frac{u_{i+1} - u_{i-1}}{2h} + 0(h^2) \tag{1.26}$$

The scheme **"forward"** for $u_i^{''}$ of order 1 is

$$u_i^{''} = \frac{u_{i+2} - 2u_i + u_{i+1}}{h^2} + 0(h) \tag{1.27}$$

The scheme **"backward"** for $u_i^{''}$ of order 1 is

$$u_i^{''} = \frac{u_i - 2u_{i-1} + u_{i-2}}{h^2} + 0(h) \tag{1.28}$$

The scheme **"centered"** for $u_i^{''}$ of order 2 is

$$u_i^{''} = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + 0(h^2) \tag{1.29}$$

More generally, the **nth** order forward, backward, and central differences respectively are given by,

Forward

$$u^{(n)}(x) = \frac{\sum_{i=0}^{n} (-1)^{n-j} \binom{n}{j} u(x + jh)}{h^n} \tag{1.30}$$

Fackward

$$u^{(n)}(x) = \frac{\sum_{i=0}^{n} (-1)^{n-j} \binom{n}{j} u(x - jh)}{h^n} \tag{1.31}$$

Centred

$$u^{(n)}(x) = \frac{\sum_{i=0}^{n} (-1)^{n-j} \binom{n}{j} u\left(x + \left(\frac{n}{2} - j\right) h\right)}{h^n} \tag{1.32}$$

1.5 FRACTIONAL FINITE DIFFERENCE METHOD

In this part we take the definition of Caputo fractional derivation.

1.5.1 Numerical scheme L_1

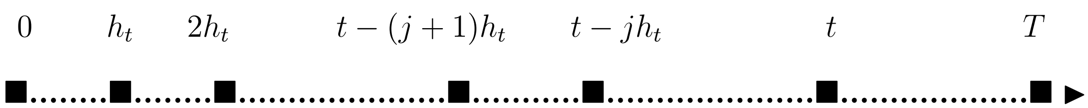
For $0 < \alpha < 1$, $t \in [0, T]$

$$ {}_0^C\partial_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau. \tag{1.33}$$

We consider a uniform subdivision of intervals $[0, T]$ as follows

$$t_k = kh_t; \ \ k = 0, ..., M.$$

$$t_k = kh_t, \ k = \overline{1, M}, \ f_{j+1} = f(t_j + h_t), \ f_j = f(t_j); \ f'(t_j) = \frac{f_{j+1} - f_j}{h_t} + 0(h_t)$$



$$f'(t-(j+1)h_t) = \frac{f(t-jh_t) - f(t-(j+1)h_t)}{h_t} + o(h) \tag{1.34}$$

from the formula of Caputo derivative we have

$$ {}_0^C\partial_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau. \tag{1.35}$$

we put

$$t - \tau = z \ \text{ then } t - z = \tau. \tag{1.36}$$

then

$$ {}_0^C\partial_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(t-z)}{z^\alpha} dz. \tag{1.37}$$

using the last formula we find

$$
\begin{aligned}
{}_{0}^{C}\partial_t^{\alpha} f(t) &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k-1} \frac{(f(t-jh_t) - f(t-(j+1)h_t))}{h_t} \int_{jh_t}^{(j+1)h_t} \frac{dz}{z^{\alpha}} \\
&= \frac{h_t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k-1} (f(t-jh_t) - f(t-(j+1)h_t)) \left((j+1)^{1-\alpha} - j^{1-\alpha}\right)
\end{aligned}
$$

then

$$
{}_{0}^{C}\partial_t^{\alpha} f(t) = \frac{h_t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k-1} \left(f_{k-j} - f_{k-(j+1)}\right) d_j;\ d_j = (j+1)^{1-\alpha} - j^{1-\alpha}\ ;\ 0 < \alpha < 1 \quad (1.38)
$$

the last formula called scheme L_1.

1.5.1.1 Application and illustration

We take $f(x) = x,\ \alpha = 0.5,\ n = 1$

$$
\begin{aligned}
{}_{0}^{C}\partial_x^{\alpha} f(x) &= \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau \\
&= \frac{1}{\Gamma(0.5)} \int_0^x \frac{1}{(x-\tau)^{0.5}} d\tau \quad (1.39) \\
&= \frac{1}{\sqrt{\pi}} \int_0^x \frac{1}{\sqrt{x-\tau}} d\tau = \frac{2\sqrt{x}}{\sqrt{\pi}}. \quad (1.40)
\end{aligned}
$$

For $x = 1$ we have

$$
{}_{0}^{C}\partial_t^{\alpha} f(t) = \frac{1}{\Gamma(1.5)} = \frac{2}{\sqrt{\pi}}.
$$

By the scheme L_1 we obtain

$$
\begin{aligned}
{}_{0}^{C}\partial_t^{\alpha} f(t) &= \frac{h_t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k-1} \left(f_{k-j} - f_{k-(j+1)}\right) d_j \\
&= \frac{h_t^{-\alpha}}{\Gamma\left(\frac{3}{2}\right)} \sum_{j=0}^{k-1} (f(t-jh_t) - f(t-(j+1)h_t))\, d_j \quad (1.41)
\end{aligned}
$$

if we take

$$
k = 100;\ t = kh_t,\ h_t = \frac{1}{100},\ \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}
$$

So

$$\left.{}_{0}^{C}\partial_t^{\alpha}f(t)\right|_{t=1} = \frac{20}{\sqrt{\pi}}\sum_{j=0}^{99}\left(\left(1-j\frac{1}{100}\right)-\left(1-(j+1)\frac{1}{100}\right)\right)\left((j+1)^{0.5}-j^{0.5}\right)$$
$$= \mathbf{1.128\,4} \simeq \frac{2}{\sqrt{\pi}} \tag{1.42}$$

For $\alpha = 0.99$ and $x = 1$

$${}_{0}^{C}\partial_t^{\alpha}f(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} = \frac{1}{\Gamma(2-0.99)} \simeq 1.005706528500385.$$

By scheme L1

$$\left.{}_{0}^{C}\partial_t^{\alpha}f(t)\right|_{t=1} = \frac{(0.01)^{-0.99}}{\Gamma(2-0.99)}\sum_{j=0}^{99}\left(\left(1-j\frac{1}{100}\right)-\left(1-(j+1)\frac{1}{100}\right)\right)\left((j+1)^{0.01}-j^{0.01}\right)$$
$$\simeq 1.0057065285004 \tag{1.43}$$

For $\alpha = 0.01$ and $x = 1$

$${}_{0}^{C}\partial_x^{\alpha}f(x) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} = \frac{1}{\Gamma(2-0.01)} \simeq 0.994162299216064$$

By scheme L1

$$\left.{}_{0}^{C}\partial_x^{\alpha}f(x)\right|_{t=1} = \frac{(0.01)^{-0.01}}{\Gamma(2-0.01)}\sum_{j=0}^{99}\left(\left(1-j\frac{1}{100}\right)-\left(1-(j+1)\frac{1}{100}\right)\right)\left((j+1)^{0.99}-j^{0.99}\right)$$
$$\simeq 0.9941622992161 \tag{1.44}$$

For x=1

values of α	${}_{0}^{C}\partial_x^{\alpha}f(x)$ by definition	${}_{0}^{C}\partial_x^{\alpha}f(x)$ by scheme L1
0.5	$\frac{2}{\sqrt{\pi}}$	**1.128 4**
0.99	1.005706528500385.	1.0057065285004
0.01	0.994162299216064	0.9941622992161

For $x = 4$, $f(x) = x$

values of α	${}_{0}^{C}\partial_x^{\alpha}f(x)$ by definition	${}_{0}^{C}\partial_x^{\alpha}f(x)$ by scheme L1
0.5	$\frac{4}{\sqrt{\pi}}$	2.256 8
0.9999	1.000196362054837	1.0003350283239
0.01	3.961516658833110	3.9615166588331

1.5.2 Numerical scheme L_2

We take $1 \prec \alpha \prec 2$, $t \in [0, T]$

$$\begin{aligned}
{}_0^C\partial_x^\alpha f(x) &= \frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{f''(\tau)}{(x-\tau)^{\alpha-1}} d\tau = \frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{f''(x-z)}{z^{\alpha-1}} dz \\
&= \frac{1}{\Gamma(3-\alpha)} \sum_{j=0}^{k-1} \left(\frac{f(x-(j+1)h) - 2f(x-jh) - f(x-(j-1)h)}{h^2} \right) \int_{jh}^{(j+1)h} \frac{dz}{z^{\alpha-1}} \\
&= \frac{1}{\Gamma(3-\alpha)} \sum_{j=0}^{k-1} \left(\frac{f(t-(j+1)h_t) - 2f(t-jh_t) - f(t-(j-1)h_t)}{h^2} \right) \left((j+1)^{2-\alpha} - j^{2-\alpha} \right)
\end{aligned}$$

then

$$ {}_0^C\partial_x^\alpha f(x) = \frac{h^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{k-1} \left(\left(f_{k-(j+1)} - 2f_{k-j} + f_{k-(j-1)} \right) \right) d_j; \quad d_j = (j+1)^{2-\alpha} - j^{2-\alpha} \quad (1.45)$$

The last formula (1.45) is called scheme L_2.

1.5.2.1 Example

We take $f(x) = x^2$, $\alpha = 1.5$

$$ {}_0^C\partial_t^\alpha f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{f''(\tau)}{(t-\tau)^{\alpha-1}} d\tau = \frac{4\sqrt{t}}{\sqrt{\pi}}$$

then

$$ {}_0^C\partial_t^\alpha f(t) = \frac{4\sqrt{2}}{\sqrt{\pi}}$$

For $x = 2$, $k = 100$

By the scheme L_2 and $x = 2$, $k = 100$, $h = \frac{2}{100}$

$$\begin{aligned}
& {}_0^C\partial_x^\alpha f(x)\Big|_{t=2} \\
= & \frac{(0.02)^{-1.99}}{\Gamma(3-1.99)} \sum_{j=0}^{99} \left((2-(j+1)\frac{2}{100})^2 - 2(2-j\frac{2}{100})^2 + (2-(j-1)\frac{2}{100})^2 \right) \left((j+1)^{2-1.99} - j^{2-199} \right)
\end{aligned}$$

then

$$ {}_0^C\partial_x^\alpha f(x)\Big|_{x=2} = 3.1915 \simeq \frac{4\sqrt{2}}{\sqrt{\pi}} \quad (1.46)$$

Table of derivative for different values α

For $x = 2$

values of α	${}_0^C\partial_x^\alpha f\left(x\right)$ by definition	${}_0^C\partial_x^\alpha f\left(x\right)$ by scheme L_2
1.5	$\frac{4\sqrt{2}}{\sqrt{\pi}}$	3. 191 5
1.99	2.025403541203477	2.0254035412035
1.01	3.989071186254150	3.9890711862542

We take

$$f(x) = x^2, n = 2,\ \alpha = 1.0001,\ x = 3$$

For $x = 3$

values of α	${}_0^C\partial_x^\alpha f\left(x\right)$ by definition	${}_0^C\partial_x^\alpha f\left(x\right)$ by scheme L_2
1.0001	5.999594497582733	5.9995944975827

2.0 CHAPTER 2. THEORETICAL AND NUMERICAL ASPECT OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH PURELY INTEGRAL CONDITIONS.

In this part, we are interested in the study of a Caputo time fractional advection–diffusion equation with non homogeneous boundary conditions of integral types $\int_0^1 v\left(x,t\right)dx$ and $\int_0^1 xv\left(x,t\right)dx$. The existence and uniqueness of the given problem solution is proved by using the method of the energy inequalities known as the "a priori estimate" method relying on the range density of the operator generated by the considered problem. The approximate solution for this problem with these new kinds of boundary conditions is established by using a combination of the finite difference method and the numerical integration. Finally, we give some numerical tests to illustrate the usefulness of the obtained results.

2.1 THEORETICAL STUDY

In this section, we prove the existence and uniqueness of the strong solution and its dependence on the data of a problem of fractional partial differential equations (FPDE) with boundary conditions of integral type.

2.1.1 Position of the problem

In the rectangular domain

$$Q = \left\{(x,t) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < t < T\right\},\ \text{ where } T > 0,$$

we consider the fractional differential equation

$$\pounds v ={}_0^c \partial_t^\alpha v + a(x,t)\frac{\partial^2 v}{\partial x^2} + b(x,t)\frac{\partial v}{\partial x} + c\,(x,t)\,v = g\,(x,t)\,,\ \text{where } 1<\alpha<2, \tag{2.1}$$

to the equation (2.1), we associate the initial conditions

$$\begin{cases} \ell v = v(x,0) = \Phi(x), & x\in(0,1), \\ qv = \frac{\partial v(x,0)}{\partial t} = \Psi(x), & x\in(0,1), \end{cases} \tag{2.2}$$

and the purely integrals conditions

$$\begin{cases} \int_0^1 v\,(x,t)\,dx = \mu(t), & t\in(0,T), \\ \int_0^1 xv\,(x,t)\,dx = E(t), & t\in(0,T), \end{cases} \tag{2.3}$$

where Φ,Ψ,μ,E,a,b,c et g are known continuous functions.

Assumptions:

1) for all $x,t\in\overline{Q}$, we assume that

$$\sup_Q a(x,t)\le 0, \sup_Q \frac{\partial^4 a(x,t)}{\partial x^4}\ge 0,\ \inf_Q \frac{\partial^3 b(x,t)}{\partial x^3}\le 0,\ c(x,t)\ge 0,\ \sup_Q \frac{\partial^2 c(x,t)}{\partial x^2}\ge 0, \tag{2.4}$$

2) For all $x,t\in\overline{Q}$ there exist $M>0$ and $\varepsilon>0$ such that

$$\begin{aligned} 0 \ < \ & M \le 4\frac{\partial^2 a(x,t)}{\partial x^2} - 4\sup_Q a(x,t) - \frac{1}{2}\sup_Q \frac{\partial^4 a(x,t)}{\partial x^4} + \frac{1}{2}\inf_Q \frac{\partial^3 b(x,t)}{\partial x^3} \\ & -\frac{1}{2}\sup_Q \frac{\partial^2 c(x,t)}{\partial x^2} - 3\frac{\partial b(x,t)}{\partial x} + 2c\,(x,t) - \frac{1}{2\varepsilon}. \end{aligned} \tag{2.5}$$

this hypothesis is equivalent to the following one

There exists $M>0$ such that : every positif number $M'>0$ could be written as $M+\frac{1}{2\varepsilon}$ with $M>0$ and $\varepsilon>0$

$$0 \ < \ M' \le 4\frac{\partial^2 a(x,t)}{\partial x^2} - 4\sup_Q a(x,t) - \frac{1}{2}\sup_Q \frac{\partial^4 a(x,t)}{\partial x^4} + \frac{1}{2}\inf_Q \frac{\partial^3 b(x,t)}{\partial x^3} \tag{2.6}$$

$$-\frac{1}{2}\sup_Q \frac{\partial^2 c(x,t)}{\partial x^2} - 3\frac{\partial b(x,t)}{\partial x} + 2c\,(x,t)\,. \tag{2.7}$$

3) The functions $\Phi(x)$ and $\Psi(x)$ satisfy the following compatibility conditions

$$\int_0^1 \Phi dx = \mu(0),\ \int_0^1 x\Phi dx = E(0),\ \int_0^1 \Psi dx = \mu'(0),\ \int_0^1 x\Psi dx = E'(0). \tag{2.8}$$

Our proof consists of three essential steps:

1. Reformulation of the problem into a problem with homogeneous conditions.
2. The uniqueness of the solution to the problem using the a priori estimate method.
3. The existence of the solution of the problem based on the density of the range of the operator generated by the abstract formulation of the problem.

2.1.2 Reformulation of the problem

We transform a problem (2.1) – (2.3) with nonhomegenous integral conditions to the equivalent problem with homegenous integral conditions,by introducing a new unknown function $\widetilde{u}$ defined by

$$v(x,t) = \widetilde{u}(x,t) + U(x,t), \tag{2.9}$$

where

$$U(x,t) = 2(2-3x)\mu(t) + 6(2x-1)E(t). \tag{2.10}$$

Now we study a new problem with homegeouns integral conditions

$$\begin{cases} \pounds\widetilde{u} = {}_0^c\partial_t^\alpha \widetilde{u} + a(x,t)\frac{\partial^2 \widetilde{u}}{\partial x^2} + b(x,t)\frac{\partial \widetilde{u}}{\partial x} + c\,(x,t)\,\widetilde{u} = h\,(x,t)\,, \\ \ell v = \widetilde{u}(x,0) = \varphi(x), \qquad x \in (0,1), \\ qv = \frac{\partial \widetilde{u}(x,0)}{\partial t} = \psi(x), \qquad x \in (0,1), \\ \int_0^1 \widetilde{u}\,(x,t)\,dx = 0, \qquad t \in (0,T), \\ \int_0^1 x\widetilde{u}\,(x,t)\,dx = 0, \qquad t \in (0,T), \end{cases} \tag{2.11}$$

where

$$\begin{aligned} h(x,t) &= g(x,t) - \pounds U(x,t), \\ \varphi(x) &= \Phi(x) - \ell U, \\ \psi(x) &= \Psi(x) - qU. \end{aligned}$$

and

$$\int_0^1 \varphi(x)dx = 0, \int_0^1 x\varphi(x)dx = 0, \int_0^1 \psi(x) = 0, \int_0^1 x\psi(x) = 0.$$

Again we introduce a new function u defined by

$$u(x,t)=\widetilde{u}(x,t)-\psi\left(x\right)t-\varphi\left(x\right), \tag{2.12}$$

therefore a problem (2.11) is given as follow

$$\begin{cases} \pounds u =_{0}^{c}\partial_{t}^{\alpha}u+a(x,t)\frac{\partial^{2}u}{\partial x^{2}}+b(x,t)\frac{\partial u}{\partial x}+c\left(x,t\right)u=f\left(x,t\right), \\ \ell u=u(x,0)=0, \qquad x\in(0,1), \\ qu=\frac{\partial u(x,0)}{\partial t}=0, \qquad x\in(0,1), \\ \int_{0}^{1}u(x,t)dx=0, \qquad t\in(0,T), \\ \int_{0}^{1}xu(x,t)dx=0, \qquad t\in(0,T), \end{cases} \tag{2.13}$$

Thus, instead of seeking the solution v of the problem(2.1) $-$ (2.3), we establish the existence and the uniqueness of the solution u of the problem (2.13)

The solution v will simply be given by:

$$v(x,t)=\widetilde{u}(x,t)+U(x,t) \tag{2.14}$$

2.1.3 Energy inequality method and consequences

To prove the existence of the solution, we use the energy inequality method known also as the "a priori estimate" method, which consists mainly to reformulate the problem (2.13) in an equivalent operational form

$$Lu=\mathcal{F},$$

where the operator $L=(\pounds,\ell,q)$ acts from B to F, whith B is a Banach space of functions $u\in L^{2}(Q)$, whose finite norm

$$\left\Vert u\right\Vert_{B}=\left(\int_{Q}\left({}_{0}^{c}\partial_{t}^{\frac{\alpha}{2}}\left(\Im_{x}u\right)\right)^{2}dxdt+\int_{Q}\left(\Im_{x}u\right)^{2}dxdt\right)^{\frac{1}{2}}, \tag{2.15}$$

and F is a Hilbert space consisting of all the elements $\mathcal{F}=(f,0,0)$ whose finite norm

$$\left\Vert\mathcal{F}\right\Vert_{F}=\left(\int_{Q}f^{\,2}dxdt\right)^{\frac{1}{2}}. \tag{2.16}$$

Let $D(L)$, the domain of the operator L, be the set of all functions u such that $\Im_{x}u$, $\Im_{x}\left({}_{0}^{c}\partial_{t}^{\alpha}u\right)$, $\Im_{x}\frac{\partial u}{\partial x}$, $\Im_{x}\frac{\partial^{2}u}{\partial x^{2}}\in L^{2}(Q)$ and u satisfies the integral conditions (2.3).

Theorem 5. *Under assumptions* (2.4)-(2.5)*, for all* $u \in D(L)$ *we have the estimate*

$$\|u\|_B \leq C \|Lu\|_F , \tag{2.17}$$

where C *is a positive constant independent of* u*.*

Proof. Multiplying the fractional differential equation in the problem(2.13) by $Mu = -2\Im_x^2 u$ and integrating it on Q we find

$$\begin{aligned} & -2\int_Q \left({}_0^c\partial_t^\alpha u\right)\Im_x^2 u dx dt - 2\int_Q a(x,t)\frac{\partial^2 u}{\partial x^2}\Im_x^2 u dx dt \\ & -2\int_Q b(x,t)\frac{\partial u}{\partial x}\Im_x^2 u dx dt - 2\int_Q c\,(x,t)\,u\,\Im_x^2 u dx dt \\ = & -2\int_Q f\ \Im_x^2 u dx dt. \end{aligned} \tag{2.18}$$

Integrating by parts the four integrals in the left side of(2.18), we obtain

$$-2\int_Q \left({}_0^c\partial_t^\alpha u\right)\Im_x^2 u dx dt = 2\int_Q \left({}_0^c\partial_t^\alpha \Im_x u\right)\left(\Im_x u\right) dx dt, \tag{2.19}$$

$$\begin{aligned} -2\int_Q a(x,t)\frac{\partial^2 u}{\partial x^2}\Im_x^2 u dx dt &= 4\int_Q \frac{\partial^2 a}{\partial x^2}\left(\Im_x u\right)^2 dx - 2\int_Q a u^2 dx dt \\ & \quad -\int_Q \frac{\partial a^4}{\partial x^4}\left(\Im_x^2 u\right)^2 dx, \end{aligned} \tag{2.20}$$

$$-2\int_Q b(x,t)\frac{\partial u}{\partial x}\Im_x^2 u dx = \int_Q \frac{\partial^3 b}{\partial x^3}\left(\Im_x^2 u\right)^2 dx - 3\int_Q \frac{\partial b}{\partial x}\left(\Im_x u\right)^2 dx, \tag{2.21}$$

$$-2\int_Q c\,(x,t)\,u\,\Im_x^2 u dx = -\int_Q \frac{\partial^2 c}{\partial x^2}\left(\Im_x^2 u\right)^2 dx + 2\int_Q c\left(\Im_x u\right)^2 dx \tag{2.22}$$

Substuting (2.19) − (2.22) in (2.18) yields

$$\begin{aligned}
& 2\int_Q^c {}_0\partial_t^\alpha \left(\Im_x u\right)^2 dx + 4\int_Q \frac{\partial^2 a}{\partial x^2}\left(\Im_x u\right)^2 dx - 2\int_Q a u^2 dx \\
& -\int_Q \frac{\partial a^4}{\partial x^4}\left(\Im_x^2 u\right)^2 dx + \int_Q \frac{\partial^3 b}{\partial x^3}\left(\Im_x^2 u\right)^2 dx - 3\int_Q \frac{\partial b}{\partial x}\left(\Im_x u\right)^2 dx \\
& -\int_Q \frac{\partial^2 c}{\partial x^2}\left(\Im_x^2 u\right)^2 dx + 2\int_Q c\left(\Im_x u\right)^2 dx \\
= & -2\int_Q f\ \Im_x^2 u dx. \qquad (2.23)
\end{aligned}$$

By the elementary inequalities in lemma (1.12)−(1.13) and assumptions (2.4) − (2.5) we obtain

$$\begin{aligned}
& 2\int_Q \left({}_0^c\partial_t^{\frac{\alpha}{2}}\left(\Im_x u\right)\right)^2 dxdt + \int_Q (4\frac{\partial^2 a}{\partial x^2} - 4\sup a \\
& -\frac{1}{2}\frac{\partial a^4}{\partial x^4} + \frac{1}{2}\inf\frac{\partial^3 b}{\partial x^3} - 3\frac{\partial b}{\partial x} \\
& -\frac{1}{2}\sup\frac{\partial^2 c}{\partial x^2} + 2c)\left(\Im_x u\right)^2 dxdt \\
\leq & -2\int_Q f\ \Im_x^2 u dxdt. \qquad (2.24)
\end{aligned}$$

The estimate of the right side of (2.24) gives:

$$\begin{aligned}
& \int_Q \left({}_0^c\partial_t^\alpha \left(\Im_x u\right)^2\right) dxdt + \int_Q (4\frac{\partial^2 a}{\partial x^2} - 4\sup a \\
& -\frac{1}{2}\frac{\partial a^4}{\partial x^4} dxdt + \frac{1}{2}\inf\frac{\partial^3 b}{\partial x^3} - 3\frac{\partial b}{\partial x} \\
& -2\sup\frac{\partial^2 c}{\partial x^2} + 2c - \frac{1}{2\varepsilon})\left(\Im_x u\right)^2 dxdt \\
\leq & \ \varepsilon\int_Q f^2 dxdt. \qquad (2.25)
\end{aligned}$$

So, by using the assumptions (2.4) − (2.5) we find

$$\begin{aligned}
& 2\int_Q \left({}_0^c\partial_t^\alpha \left(\Im_x u\right)^2\right) dxdt + M\int_Q \left(\Im_x u\right)^2 dxdt \\
\leq & \ \varepsilon\int_Q f^{\ 2} dxdt \qquad (2.26)
\end{aligned}$$

Finally, we obtain the a priori estimate

$$\|u\|_B \leq C\,\|Lu\|_F\,, \qquad (2.27)$$

where

$$C=\left(\frac{\varepsilon}{\min\left(2,M\right)}\right)^{\frac{1}{2}}.$$

□

We proved the uniqueness of solution in case of existence, and we have

Corollary 1. *The operator L from B to F has a closure $\overline{L}$.*

Proof. See [18]. □

The a priori estimate (2.17) can be extended to cover strong solution of the problem (2.13) by passing to the limit.

Corollary 2. *The range of the operator $\overline{L}$ is closed in F and $R(\overline{L})=\overline{R(L)}$.*

Consequently, the strong solution of the problem is unique if it exists, and depends continuously on $\mathcal{F}=(f,0,0)$.

2.1.4 Existence of the solution

To prove the existence, it suffices to prove that $R(L)$ is dense in F, that is its orthogonal is reduced to the singleton $\{0\}$.

Theorem 6. *Let us suppose that the assumptions (2.4) – (2.5) and integral conditions (3:3) are filled, and for $\omega\in L^2(Q)$ and for all $u\in D(L)$, we have*

$$\int_Q \pounds u\omega dxdt=0 \tag{2.28}$$

then ω is almost every where in Q.

Proof. We can rewrite equation (2.28)as follows

$$\begin{aligned}\int_Q \left({}_0^c\partial_t^\alpha u\omega\right)dxdt &= -\int_Q a\left(x,t\right)\frac{\partial^2 u}{\partial x^2}\omega dxdt-\int_Q b\left(x,t\right)\frac{\partial u}{\partial x}\omega dxdt\\ &\quad -\int_Q c\left(x,t\right)u\omega dxdt,\end{aligned} \tag{2.29}$$

We express the function ω in terms of u as follows

$$\omega = -2\Im_x^2 u \tag{2.30}$$

Substiting ω by its representation (2.30) in (2.29), integrating by parts, and taking into account the conditions (2.3) and the assumptions (2.4) − (2.5), we obtain:

$$2\int_Q \left({}_0^c\partial_t^\alpha \Im_x u\right) \Im_x u dxdt = -4\int_Q \frac{\partial^2 a}{\partial x^2} \left(\Im_x u\right)^2 dxdt + 2\int_Q au^2 dxdt + \int_Q \frac{\partial^4 a}{\partial x^4} \left(\Im_x u\right)^2 dxdt$$

$$-\int_Q \frac{\partial^3 b}{\partial x^3} \left(\Im_x u\right)^2 dxdt + 3\int_Q \frac{\partial b}{\partial x} \left(\Im_x u\right)^2 dxdt + \int_Q \frac{\partial^2 c}{\partial x^2} \left(\Im_x u\right)^2 dxdt - 2\int_Q c \left(\Im_x u\right)^2 dxdt,$$

□

under assumptions (2.4) − (2.5) and conditions (2.3), we obtain

$$\begin{aligned} 2\int_Q \left({}_0^c\partial_t^\alpha \Im_x u\right) \Im_x u dxdt = &-\int_Q (4\frac{\partial^2 a}{\partial x^2} + 4\sup a \\ &+\frac{1}{2}\frac{\partial^4 a}{\partial x^4} - \frac{1}{2}\inf\frac{\partial^3 b}{\partial x^3} + 3\frac{\partial b}{\partial x} + 2\sup\frac{\partial^2 c}{\partial x^2} - 2c)\left(\Im_x u\right)^2 dxdt, \end{aligned}$$

using the condition (2.3) under assumptions, we find

$$2\int_Q \left({}_0^c\partial_t^\alpha \Im_x u\right) \Im_x u dxdt \leq -\left(\frac{1}{2\varepsilon} + M\right)\int_Q \left(\Im_x u\right)^2 dxdt,$$

By **lemma** (1.9), (1.12) and (1.13) we obtain

$$2\int_Q \left({}_0^c\partial_t^{\frac{\alpha}{2}} \left(\Im_x u\right)\right)^2 dxdt \leq -\left(\frac{1}{2\varepsilon} + M\right)\int_Q \left(\Im_x u\right)^2 dxdt.$$

Then

$$\left(\Im_x u\right)^2 = 0 \tag{2.31}$$

we obtain

$$u = 0.$$

So $u = 0$ in Ω wich gives $\omega = 0$ in $L^2(Q)$.

2.2 NUMERICAL STUDY

In this section we present the numerical technique besed on the (FDM) that we will apply to the problem considered above, and we illustrate the schemes obtained with well-chosen applications.

2.2.1 Discretization of the problem

We consider a uniform subdivision of intervals $[0,1]$ and $[0,T]$ as follows

$$x_i = ih;\ \ i = 0, ..., N \text{ and } t_k = kh_t;\ \ k = 0, ..., M.$$

We denote by v_i^k the approximate solution of $v(x_i, t_k)$ at point (x_i, t_k), and L the operator defined by

$$L = a\frac{\partial^2}{\partial x^2} + b\frac{\partial}{\partial x} + c,\ \ L(.)_i^k = a_i^k \frac{\partial^2 (.)}{\partial x^2} + b_i^k \frac{\partial (.)}{\partial x} + c_i^k (.) \tag{2.32}$$

where

$$a_i^k = a(x_i, t_k), \quad b_i^k = b(x_i, t_k), \quad c_i^k = c(x_i, t_k),$$

From the Taylor devlopment of function v at the point (x_i, t_k) we have

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_i^k = \frac{1}{h^2}\left(v_{i-1}^k - 2v_i^k + v_{i+1}^k\right) + O\left(h^2\right),\ \ \left(\frac{\partial v}{\partial x}\right)_i^k = \frac{v_{i+1}^k - v_i^k}{h} + O(h) \tag{2.33}$$

Substiting (2.33) in the operateur L_i^k expressed in(2.32) gives

$$Lv_i^{k+1} = \left(\frac{a_i^{k+1}}{h^2} + \frac{b_i^{k+1}}{h}\right) v_{i+1}^{k+1} + \left(c_i^{k+1} - 2\frac{a_i^{k+1}}{h^2} - \frac{b_i^{k+1}}{h}\right) v_i^{k+1} + \frac{a_i^{k+1}}{h^2} v_{i-1}^{k+1} \tag{2.34}$$

The approximate of Caputo derivative fractional operator ${}_0^c\partial_t^\alpha v$ with $1 < \alpha < 2$ defined by (1.45)

$$\left({}_0^c\partial_t^\alpha v\right)_i^{k+1} \simeq \gamma \sum_{j=0}^{k} \left(v_i^{k-j-1} - 2v_i^{k-j} + v_i^{k-j+1}\right) d_j\ . \tag{2.35}$$

$$\text{where} \begin{cases} d_j = (j+1)^{2-\alpha} - j^{2-\alpha} \\ d_0 = 1; k = 1, ..., M \end{cases}, \quad \gamma = \frac{h_t^{-\alpha}}{\Gamma(3-\alpha)}$$

Writing fractional differential equation (2.1) in point $(ih, (k+1)\,h_t)$, we find

$$\gamma \sum_{j\,=\,0}^{k} \left(v_{_i}^{\,k-j-1} - 2v_{_i}^{\,k-j} + v_{_i}^{\,k-j+1}\right) d_j + Lv_i^{k+1} = g_i^{\,k+1},\ i = \overline{1, N-1} \tag{2.36}$$

then

$$F_i^{k+1}v_{i-1}^{k+1}+A_i^{k+1}v_i^{k+1}+B_i^{k+1}v_{i+1}^{k+1}-2\gamma d_k v_i^k+\gamma d_k v_i^{k-1}+\gamma\sum_{j=1}^{k-1}S_j d_j+\gamma\left(v_i^{-1}-2v_i^0+v_i^1\right)d_k = g_i^{k+1} \tag{2.37}$$

where

$$\begin{aligned} A_i^{k+1} &= \gamma + c_i^{k+1} - 2\frac{a_i^{k+1}}{h^2} - \frac{b_i^{k+1}}{h}, \quad B_i^{k+1} = \frac{a_i^{k+1}}{h^2} + \frac{b_i^{k+1}}{h}, \\ F_i^{k+1} &= \frac{a_i^{k+1}}{h^2}, \quad S_j = v_{_i}^{\,k-j-1} - 2v_{_i}^{\,k-j} + v_{_i}^{\,k-j+1}. \end{aligned}$$

To eliminate v_i^{-1}, we use initial condition (2.2), we find

$$\left(\frac{\partial v}{\partial t}\right)_i^n \simeq \frac{v_i^n - v_i^{n-1}}{h_t}$$

therefore

$$v_i^{-1} \simeq \Phi_i - h_t\Psi_i = v_i^0 - h_t\Psi_i, \quad i = \overline{1, N-1} \tag{2.38}$$

Subsititing (2.38) in (2.37) , we obtain

$$F_i^{k+1}v_{i-1}^{k+1} + A_i^{k+1}v_i^{k+1} + B_i^{k+1}v_{i+1}^{k+1} - 2\gamma d_k v_i^k + \gamma d_k v_i^{k-1} + \gamma\sum_{j=1}^{k-1} S_j d_j$$

$$= d_k\gamma v_i^0 + d_k\gamma h_t\Psi_i - d_k\gamma v_i^1 + g_i^{k+1}. \tag{2.39}$$

For $k = 0$, relation (2.39)gives

$$F_i^1 v_{i-1}^1 + A_i^1 v_i^1 + B_i^1 v_{i+1}^1 = \gamma v_i^0 + \gamma h_t\Psi_i + g_i^1 \quad \text{with} \quad i = \overline{1, N-1}. \tag{2.40}$$

For $i = 1$, $i = N-1$, v_0^1 and v_N^1 are not defined, by the conditions (3.3) , and trapeze method we obtain, we have

$$\int_0^1 xv\left(x,t\right)dx = \frac{h}{2}\left[x_0 v_0^1 + 2\sum_{j=1}^{N-1} x_j v_j^1 + x_N v_N^1\right] = E\left(h_t\right)$$

then

$$v_N^1 = \frac{2E\left(h_t\right)}{h} - 2\sum_{j=1}^{N-1} jhv_j^1 \tag{2.41}$$

and

$$\begin{aligned}\int_0^1 v\left(x,t\right)dx &= \frac{h}{2}\left[v_0^1 + 2\sum_{j=1}^{N-1} v_j^1 + v_N^1\right] = \mu\left(h_t\right)\\ v_0^1 &= \frac{2\mu\left(h_t\right)}{h} - 2\sum_{j=1}^{N-1} v_j^1 - \left(\frac{2E\left(h_t\right)}{h} - 2\sum_{j=1}^{N-1} jhv_i^1\right)\end{aligned}$$

then

$$v_0^1 = \frac{2\mu\left(h_t\right) - 2E\left(h_t\right)}{h} + 2\sum_{j=1}^{N-1}\left(jh-1\right)v_j^1 \tag{2.42}$$

For $i = 1$

$$\left(A_1^1 + 2F_1^1\left(h-1\right)\right)v_1^1 + \left(B_1^1 + 2F_1^1\left(2h-1\right)\right)v_2^1 + 2F_1^1\sum_{j=3}^{N-1}\left(jh-1\right)v_j^1$$

$$= \gamma v_1^0 + \gamma h_t \Psi_1 + g_1^1 + \frac{2F_1^1}{h}\left(E\left(h_t\right) - \mu\left(h_t\right)\right) \tag{2.43}$$

For $i = N-1$

$$-2B_{N-1}^1\sum_{j=1}^{N-3} jhv_j^1 + \left(F_{N-1}^1 - 2B_{N-1}^1\left(N-2\right)h\right)v_{N-2}^1 + \left(A_{N-1}^1 - 2B_{N-1}^1\left(N-1\right)h\right)v_{N-1}^1$$

$$= \gamma v_{N-1}^0 + \gamma h_t \Psi_{N-1} + g_{N-1}^1 - \frac{2B_{N-1}^1}{h}E\left(h_t\right) \tag{2.44}$$

2.2.2 Matrix's form

We denote by

$$w_i = \gamma v_i^0 + \gamma h_t \Psi_i + g_i^1, \quad y_1^1 = \frac{2F_1^1}{h}\left(E\left(h_t\right) - \mu\left(h_t\right)\right), \quad z_{N-1}^1 = -\frac{2B_{N-1}^1}{h} E\left(h_t\right),$$

$$\begin{aligned}
P^1 &= \left(l_{i,j}\right)_{N-1,N-1} \text{ is square matrix defined by} \\
l_{1,1} &= A_1^1 + 2F_1^1\left(h-1\right), \ l_{1,2} = B_1^1 + 2F_1^1\left(2h-1\right), \\
l_{N-1,N-2} &= F_{N-1}^1 - 2B_{N-1}^1\left(N-2\right)h, \ l_{N-1,N-1} = A_{N-1}^1 - 2B_{N-1}^1\left(N-1\right)h \ ,
\end{aligned}$$

$$l_{i,j} = \begin{cases} 2F_1^1\left(jh-1\right) & \text{when} \quad i=1, \ j=\overline{3,N-1} \\ 0 & \text{when} \quad |i-j| \geq 2 \ , \ i=\overline{2,N-2} \\ A_i^1 & \text{when} \quad i=j, \ i=\overline{2,N-2} \\ B_i^1 & \text{when} \quad i=j-1, \ i=\overline{1,N-2} \\ F_i^1 & \text{when} \quad i=j+1, \ i=\overline{2,N-1} \\ -2B_{N-1}^1 jh & \text{when} \quad i=N-1, \ j=\overline{1,N-3} \end{cases}$$

Taking into account (2.40) for $i=\overline{2,N-2}$, (2.43) and (2.44), we obtain the matricial system

$$P^1.V^1 = H^1 \tag{2.45}$$

where

$$H^1 = W^{\ 1} + R^1, \ W^{\ 1} = \left(w_1^1, w_2^1, ..., w_{N-1}^1\right)^T, \ R^1 = \left(y_1^1, 0, 0, ..., 0, z_{N-1}^1\right)^T.$$

To solve the system (2.45) we can apply one of direct methods.

It is readily checked that, *for* $k \geq 1$

$$\sum_{j=1}^{k-1} S_j d_j = (d_2 - 2d_1)v_i^{k-1} + d_1 v_i^k + d_{k-1}v_i^0 + \left(d_{k-2} - 2d_{k-1}\right)v_i^1 + \sum_{m=2}^{k-2} \sigma_m v_i^{k-m} \tag{2.46}$$

$$\text{where} \quad \sigma_m = d_{m-1} - 2d_m + d_{m+1}, \ m = \overline{2,k-2}$$

Lemma 5. *If* $k \geq 1$; *the numerical scheme* (2.39) *is equivalent to*

$$F_i^{k+1} v_{i-1}^{k+1} + A_i^{k+1} v_i^{k+1} + B_i^{k+1} v_{i+1}^{k+1} = -\gamma \sum_{m=1}^{k-1} \sigma_m v_i^{k-m} + \gamma \left(2 - d_1\right) v_i^k + \gamma \left(d_k - d_{k-1}\right) v_i^0$$

$$+\gamma d_k h_t \Psi_i + g_i^{k+1}, \quad \textit{for} \quad i = 1, \ldots, N-1 \tag{2.47}$$

Proof. From the scheme (2.39), we have

$$F_i^{k+1} v_{i-1}^{k+1} + A_i^{k+1} v_i^{k+1} + B_i^{k+1} v_{i+1}^{k+1} - 2\gamma d_k v_i^k + \gamma d_k v_i^{k-1} + \gamma \sum_{j=1}^{k-1} S_j d_j$$

$$= d_k \gamma v_i^0 + d_k \gamma h_t \Psi_i - d_k \gamma v_i^1 + g_i^{k+1}$$

□

so

$$F_i^{k+1} v_{i-1}^{k+1} + A_i^{k+1} v_i^{k+1} + B_i^{k+1} v_{i+1}^{k+1} + \gamma \sum_{j=2}^{k-2} S_j d_i + \gamma (v_i^{k+1} - 2v_i^k + v_i^{k-1})\ d_0$$

$$+\gamma (v_i^1 - 2v_i^0 + v_i^{-1}) d_k = g_i^{k+1} \tag{2.48}$$

using (2.46) we obtain

$$F_i^{k+1} v_{i-1}^{k+1} + A_i^{k+1} v_i^{k+1} + B_i^{k+1} v_{i+1}^{k+1} = -\gamma \sum_{m=1}^{k-1} \sigma_m v_i^{k-m} + \gamma \left(2 - d_1\right) v_i^k + \gamma \left(d_k - d_{k-1}\right) v_i^0$$

$$+\gamma d_k h_t \Psi_i + g_i^{k+1}, \text{ for } \quad i = 1, \ldots, N-1 \tag{2.49}$$

Using conditions (2.3) , and by trapeze method we obtain,

For $i = 1$

$$\left(A_1^{k+1} + 2F_1^{k+1} \left(h - 1\right)\right) v_1^{k+1} + \left(B_1^{k+1} + 2F_1^{k+1} \left(2h - 1\right)\right) v_2^{k+1} + 2F_1^{k+1} \sum_{j=3}^{N-1} \left(jh - 1\right) v_j^{k+1}$$

$$= \frac{2F_1^{k+1}}{h}\left(E\left((k+1)h_t\right) - \mu\left((k+1)h_t\right)\right) - \gamma\sum_{m=1}^{k-1}\sigma_m v_1^{k-m} + \gamma\left(d_k - d_{k-1}\right)v_1^0 + \gamma d_k h_t \Psi_1 + g_1^{k+1} \tag{2.50}$$

For $i = N-1$

$$-2B_{N-1}^{k+1}\sum_{j=1}^{N-3} jhv_j^{k+1} + \left(F_{N-1}^{k+1} - 2B_{N-1}^{k+1}(N-2)h\right)v_{N-2}^{k+1} + \left(A_{N-1}^{k+1} - 2B_{N-1}^{k+1}(N-1)h\right)v_{N-1}^{k+1}$$

$$\begin{aligned} = & -\frac{2B_{N-1}^{k+1}}{h}E\left((k+1)h_t\right) - \gamma\sum_{m=1}^{k-1}\sigma_m v_{N-1}^{k-m} + \gamma\left(2-d_1\right)v_{N-1}^k \\ & +\gamma\left(d_k - d_{k-1}\right)v_{N-1}^0 + \gamma d_k h_t \Psi_{N-1} + g_{N-1}^{k+1} \end{aligned} \tag{2.51}$$

Matrix's form

We take expression (2.49) for $i = \overline{2, N-2}$ and equations (2.50), (2.51) to formulate the matrix systems:

$$\begin{cases} P^{k+1}V^{k+1} = H^{k+1}; \quad k \geq 1 \\ \\ V^0, \ V^1 \text{ are known} \end{cases} \tag{2.52}$$

where

$$\begin{aligned} P^{k+1} &= \left(l_{i,j}^{k+1}\right)_{N-1,N-1} \text{ is square matrix defined by} \\ l_{1,1}^{k+1} &= A_1^{k+1} + 2F_1^{k+1}(h-1), \ l_{1,2}^{k+1} = B_1^{k+1} + 2F_1^{k+1}(2h-1), \\ l_{N-1,N-2}^{k+1} &= F_{N-1}^{k+1} - 2B_{N-1}^{k+1}(N-2)h, \ l_{N-1,N-1}^{k+1} = A_{N-1}^{k+1} - 2B_{N-1}^{k+1}(N-1)h \ , \end{aligned}$$

$$l_{i,j}^{k+1} = \begin{cases} 2F_1^{k+1}(jh-1) & \text{when} \quad i=1, \ j=\overline{3,N-1} \\ 0 & \text{when} \quad |i-j| \geq 2 \ , \ i=\overline{2,N-2} \\ A_i^{k+1} & \text{when} \quad i=j, \ i=\overline{2,N-2} \\ B_i^{k+1} & \text{when} \quad i=j-1, \ i=\overline{1,N-2} \\ F_i^{k+1} & \text{when} \quad i=j+1, \ i=\overline{2,N-1} \\ -2B_{N-1}^{k+1}jh & \text{when} \quad i=N-1, \ j=\overline{1,N-3} \end{cases}$$

and

$$
\begin{aligned}
V^{\ k+1} &= \left(v_1^{k+1}, ..., v_{N-1}^{k+1}\right)^T; \ H^{k+1} = -\gamma \sum_{m=1}^{k-1} \sigma_m V^{\ k-m} + W^{\ k+1} + R^{\ k+1}; \ k \geq 1 \\
W^{\ k+1} &= \left(w_1^{k+1}, w_2^{k+1}, ..., w_{N-1}^{k+1}\right)^T, \ R^{k+1} = \left(y_1^{k+1}, 0, 0, ..., 0, z_{N-1}^{k+1}\right)^T, \\
w_i^{k+1} &= \gamma \left(2 - d_1\right) v_i^k + \gamma \left(d_k - 2d_{k-1}\right) v_i^0 + \gamma d_k h_t \Psi_i + g_i^{k+1}, \\
y_1^{k+1} &= \frac{2F_1^{k+1}}{h} \left(E\left(\left(k+1\right) h_t\right) - \mu\left(\left(k+1\right) h_t\right)\right); \ z_{N-1}^{k+1} = -\frac{2B_{N-1}^{k+1}}{h} E\left(\left(k+1\right) h_t\right).
\end{aligned}
$$

In order to prove system (2.52) has a unique solution we denote ρ as an eigenvalue of the matrix P^k, and $X = (x_1, x_2, ..., x_{N-1})^T$ is an nonzero eigenvector corresponding to ρ

We choose i such as

$$|x_i| = max\{|x_j| : j = 1; ...; N-1\}.$$

then

$$\sum_{j=1}^{N-1} l_{i,j} x_j = \rho x_i; \ i = \overline{1; N-1}$$

therefore

$$\rho = l_{i,i} + \sum_{\substack{j=1 \\ j \neq i}}^{N-1} l_{i,\, j} \frac{x_j}{x_i} \tag{2.53}$$

Substituting the values of $l_{i,j}$ into (2.53) , and taking into account that F_i^k, a_i^k are negative and $\left|\frac{x_j}{x_i}\right| \leq 1$ we get

For $i = 1$

$$
\begin{aligned}
\rho &= \left(A_1^{k+1} + 2F_1^{k+1}\left(h-1\right)\right) + \left(B_1^{k+1} + 2F_1^{k+1}\left(2h-1\right)\right) \frac{x_2}{x_1} + 2F_1^{k+1} \sum_{j=3}^{N-1} \left(jh-1\right) \frac{x_j}{x_1} \\
&= \gamma + c_i^{k+1} - F_1^{k+1} - B_1^{k+1} \left(1 - \frac{x_2}{x_1}\right) + 2F_1^{k+1} \sum_{j=2}^{N-1} \left(jh-1\right) \frac{x_j}{x_1}.
\end{aligned}
\tag{2.54}
$$

For $i = N-1$

$$
\begin{aligned}
\rho &= l_{i,i} + \sum_{\substack{j=1\\ j\neq i}}^{N-1} l_{i,\,j} \frac{x_j}{x_i} \\
&= A_{N-1}^{k+1} - 2B_{N-1}^{k+1}\left(N-1\right)h + \left(F_{N-1}^{k+1} - 2B_{N-1}^{k+1}\left(N-2\right)h\right)\left(\frac{x_{N-2}}{x_{N-1}}\right) - 2B_{N-1}^{k+1}\sum_{j=1}^{N-3} jh\frac{x_j}{x_{N-1}} \\
&= \gamma + c_{N-1}^{k+1} - B_{N-1}^{k+1} + F_{N-1}^{k+1}\left(\frac{x_{N-2}}{x_{N-1}} - 1\right) - 2B_{N-1}^{k+1}\left(N-1\right)h - 2B_{N-1}^{k+1}\sum_{j=1}^{N-2} jh\frac{x_j}{x_{N-1}}.
\end{aligned}
\tag{2.55}
$$

For $i = \overline{2, N-2}$

$$
\begin{aligned}
\rho &= l_{i,i} + \sum_{\substack{j=1\\ j\neq i}}^{N-1} l_{i,j} \frac{x_j}{x_i} \\
&= A_i^{k+1} + F_i^{k+1}\frac{x_{i-1}}{x_i} + B_i^{k+1}\frac{x_{i+1}}{x_i} \\
&= \gamma + c_i^{k+1} - B_i^{k+1} - F_i^{k+1} + F_i^{k+1}\frac{x_{i-1}}{x_i} + B_i^{k+1}\frac{x_{i+1}}{x_i} \\
&= \gamma + c_i^{k+1} + F_i^{k+1}\left(\frac{x_{i-1}}{x_i} - 1\right) + \frac{a_i^{k+1} + hb_i^{k+1}}{h^2}\left(\frac{x_{i+1}}{x_i} - 1\right).
\end{aligned}
\tag{2.56}
$$

From the above we conclude for $i = \overline{1, N-1}$,

If $b_i^{k+1} < 0$, $B_{N-1}^{k+1} < 0$ then $\rho > 0$.

If $b_i^{k+1} > 0$ and $h \leq \min_{1\leq i\leq N-1}\left(\frac{-a_i^{k+1}}{b_i^{k+1}}\right)$, $\rho > 0$, then all eigenvalues of matrix P^{k+1} are strictly positive, therefore P^{k+1} is inversible.

2.2.3 Stability and Convergence

2.2.3.1 Stability We have

$$F_i^{k+1} + A_i^{k+1} + B_i^{k+1} = \gamma + c_i^{k+1},\ F_i^{k+1} \leq 0,\ A_i^{k+1} + B_i^{k+1} \geq 0$$

Let u_i^{k+1}be the approximate solution of (2.49) , and e_i^{k+1} the error at point (x_i, t_{k+1}) defined by

$$v_i^{k+1} - u_i^{k+1} = e_i^{k+1},\ \text{and}\ \left\| E^{\,k} \right\| = \underset{1 \,\leq\, i \,\leq\, N-1}{\text{Max}} |e_i^k|,\ E^{\,k} = \left(e_1^k, ..., e_{N-1}^k\right)^T$$

for $k = 0$ we apply (2.40) we get

$$\begin{aligned}
\left\| E^1 \right\| &\leq \left(\gamma + c_i^1\right) \left\| E^1 \right\| = \left(F_i^1 + A_i^1 + B_i^1\right) \left\| E^1 \right\| \\
&= F_i^1 \left\| E^1 \right\| + \left(A_i^1 + B_i^1\right) \left\| E^1 \right\| \\
&\leq \left(\left(A_i^1 + B_i^1\right) \left\| E^1 \right\| + F_i^1 \left| e_{i-1}^1 \right|\right) \\
&\leq \underset{1 \,\leq\, i \,\leq\, N-1}{\text{Max}} \left| F_i^1 e_{i-1}^1 + A_i^1 e_i^1 + B_i^1 e_{i+1}^1 \right| = \gamma \left\| E^0 \right\|
\end{aligned}$$

so

$$\left\| E^1 \right\| \preceq \frac{\gamma}{\gamma + c_i^1} \left\| E^0 \right\| \preceq \left\| E^0 \right\|. \tag{2.57}$$

Therefore the method is stable.

Lemma 6. *For $k \geq 1$ the scheme (2.48) is stable and we have*

$$\left\| E^{k+1} \right\| \leq C \left\| E^0 \right\|,\ C > 0, \textit{ for all } k \geq 1$$

Proof. We use mathematical induction. ☐

We assume $\|E^j\| \leq c_j \|E^0\|$, and $C_{\max} = \max c_j$; where $c_j \succ 0,\ j = \overline{1,k}$.

from (2.49) we get

$$\begin{aligned} & F_i^{k+1} e_{i-1}^{k+1} + A_i^{k+1} e_i^{k+1} + B_i^{k+1} e_{i+1}^{k+1} \\ = \ & -\gamma \sum_{m=1}^{k-1} \sigma_m e_i^{k-m} + \gamma (2 - d_1) e_i^k + \gamma (d_k - d_{k-1}) e_i^0,\ i = \overline{1, N-1} \end{aligned} \tag{2.58}$$

so

$$\begin{aligned} \left(\gamma + c_i^{k+1}\right) \left\|E^{k+1}\right\| & \leq \left(\left(A_i^{k+1} + B_i^{k+1}\right) \left\|E^{k+1}\right\| + F_i^{k+1} \left|e_{i-1}^{k+1}\right|\right) \\ & \leq \left\| -\gamma \sum_{m=1}^{k-1} \sigma_m e_i^{k-m} + \gamma (2 - d_1) e_i^k + \gamma (d_k - d_{k-1}) e_i^0 \right\| \\ & \leq \gamma \left(\sum_{m=1}^{k-1} |\sigma_m| \left\|E_i^{k-m}\right\| + (2 - d_1) \left\|e_i^k\right\| + (d_{k-1} - d_k) \left\|e_i^0\right\| \right) \\ & \leq \gamma C_{\max} \left(\sum_{m=1}^{k-1} |\sigma_m| + 2 - d_1 + d_{k-1} - d_k \right) \left\|E^0\right\| \\ & \leq \gamma C_{\max} \left(5 - 2^{3-\alpha}\right) \left\|E^0\right\|. \end{aligned}$$

where

$$\sum_{m=1}^{k-1} |\sigma_m| + 2 - d_1 + d_{k-1} - d_k = 5 - 2^{3-\alpha},\ 0 < \ \sigma_m < 1,\ \ -1 < d_k - d_{k-1} < 0,\ 1 < 2 - d_1 < 2$$

then

$$\left\|E^{k+1}\right\| \leq C \left\|E^0\right\|;\ C = C_{\max} \left(5 - 2^{3-\alpha}\right). \tag{2.59}$$

Therefore the method is stable.

2.2.3.2 Convergence Let $v(x_i;t_{k+1})$ as the exact solution and v_i^{k+1} is the approximate solution of scheme (2.36), we put $v(x_i;t_{k+1}) - v_i^{k+1} = \epsilon_i^{k+1}$; for $i = \overline{1, N-1}$, $k = \overline{1, M-1}$.

The scheme L_2 defined on (2.35)verifed ([25])

$$\left| \frac{\partial^\alpha v}{\partial t^\alpha} - \left(\frac{\partial^\alpha v}{\partial t^\alpha} \right)_{L_2} \right| \leq O\left(h_t\right) \tag{2.60}$$

substitution in to (2.36) and using (2.33), (2.60) leads to

$$\gamma \sum_{j=0}^{k} \left(v(x_i;t_{k-j-1}) - \epsilon_i^{k-j-1} - 2\left(v(x_i;t_{k-j}) - \epsilon_i^{k-j} \right) + \left(v(x_i;t_{k-j+1}) - \epsilon_i^{k-j+1} \right) \right) d_j$$

$$+L\left(v(x_i;t_{k+1}) - \epsilon_i^{k+1} \right) = g_i^{k+1}.$$

then

$$\gamma \sum_{j=0}^{k} \left(v(x_i;t_{k-j-1}) - 2v(x_i;t_{k-j}) + \left(v(x_i;t_{k-j+1}) \right) \right) d_j + Lv(x_i;t_{k+1})$$

$$-\gamma \sum_{j=0}^{k} \left(\epsilon_i^{k-j-1} - 2\epsilon_i^{k-j} + \epsilon_i^{k-j+1} \right) d_j - L\epsilon_i^{k+1} = g_i^{k+1}.$$

so

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} + O(h_t) + Lv(x;t) + O(h) - \gamma \sum_{j=0}^{k} \left(\epsilon_i^{k-j-1} - 2\epsilon_i^{k-j} + \epsilon_i^{k-j+1} \right) d_j - L\epsilon_i^{k+1} = g_i^{k+1}.$$

hence

$$\gamma \sum_{j=0}^{k} \left(\epsilon_i^{k-j-1} - 2\epsilon_i^{k-j} + \epsilon_i^{k-j+1} \right) d_j + L\epsilon_i^{k+1} = O(h + h_t). \tag{2.61}$$

Taking

$$\left| \epsilon_l^k \right| = \left\| \epsilon^k \right\| = \underset{1 \leq i \leq N-1}{\text{Max}} \left| \epsilon_i^k \right|; \epsilon^k = \left(\epsilon_1^k, , ..., \epsilon_{N-1}^k \right)^T; \ \left\| \epsilon_i^0 \right\| = 0$$

for $k = 0$ we get

$$F_i^1 \epsilon_{i-1}^1 + A_i^1 \epsilon_i^1 + B_i^1 \epsilon_{i+1}^1 = \gamma \epsilon_i^0 + O(h + h_t) \quad \text{with} \quad i = \overline{1, N-1}. \tag{2.62}$$

we have

$$\begin{aligned}
\left\|\epsilon^1\right\| &= \left|\epsilon_l^1\right| \leq \left(F_i^1 + A_i^1 + B_i^1\right)\left|\epsilon_l^1\right| \\
&\leq \left(\left(A_i^1 + B_i^1\right)\left|\epsilon_l^1\right| + F_i^1\left|\epsilon_l^1\right|\right) \\
&\leq \underset{1\,\leq\, i\,\leq\, N-1}{\text{Max}}\left|F_i^1\epsilon_{i-1}^1 + A_i^1\epsilon_l^1 + B_i^1\epsilon_l^1\right| = O(h+h_t).
\end{aligned}$$

hence

$$\left\|\epsilon^1\right\| \leq O(h+h_t). \tag{2.63}$$

We assume : $\left|\epsilon_l^j\right| \leq O(h+h_t);\ j=\overline{1,k}$

from (2.61) we get

$$F_i^{k+1}\epsilon_{i-1}^{k+1} + A_i^{k+1}\epsilon_i^{k+1} + B_i^{k+1}\epsilon_{i+1}^{k+1} = -\gamma\sum_{m=1}^{k-1}\sigma_m\epsilon_i^{k-m} + \gamma\left(2-d_1\right)\epsilon_i^k + O(h+h_t) \tag{2.64}$$

we have

$$\begin{aligned}
\left\|\epsilon^{k+1}\right\| &\leq \left(\gamma + c_i^{k+1}\right)\left|\epsilon_l^{k+1}\right| = \left(F_i^{k+1} + A_i^{k+1} + B_i^{k+1}\right)\left|\epsilon_l^{k+1}\right| \\
&\leq \left(F_i^{k+1}\left|\epsilon_{i-1}^{k+1}\right| + \left(A_i^{k+1} + B_i^{k+1}\right)\left|\epsilon_l^{k+1}\right|\right) \\
&\leq \underset{1\,\leq\, i\,\leq\, N-1}{\text{Max}}\left|-\gamma\sum_{m=1}^{k-1}\sigma_m\epsilon_i^{k-m} + \gamma\left(2-d_1\right)\epsilon_i^k + O(h+h_t)\right| \\
&\leq \gamma\sum_{m=1}^{k-1}\sigma_m\left\|\epsilon^{k-m}\right\| + \gamma\left(2-d_1\right)\left\|\epsilon^k\right\| + O(h+h_t) \\
&\leq \gamma\left(\sum_{m=1}^{k-1}\sigma_m + (2-d_1)\right)O(h+h_t) + O(h+h_t)
\end{aligned}$$

hence

$$\left\|\epsilon^{k+1}\right\| \leq \frac{\gamma}{\gamma + c_i^{k+1}}O(h+h_t) + \frac{1}{\gamma + c_i^{k+1}}O(h+h_t) \leq O(h+h_t). \tag{2.65}$$

Therefore, the method is convergent.

2.3 APPLICATIONS

In this section, we give some numerical investigation tests.

2.3.1 Example1

We consider the problem $(2.1-2.3)$ with

$$\begin{aligned} \alpha &= \frac{3}{2}, \quad a(x,t) = -x-t, b(x,t) = x+t, \quad c(x) = 2, \phi(x) = \psi(x) = 0, \quad (2.66) \\ g(x,t) &= (\frac{3}{4}\sqrt{\pi} + 2t\sqrt{t})e^{x}, \ \mu(t) = (e-1)t^{\frac{3}{2}}, \quad E(t) = t^{\frac{3}{2}}. \end{aligned}$$

The analytical solution is given by $v(x,t) = t^{\frac{3}{2}}e^{x}$.

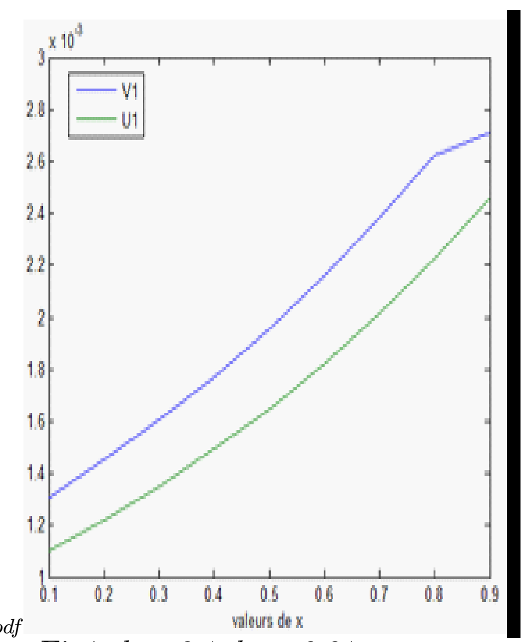
The approximate solution $u(x,t)$ with **A.E** is the absolute error.

For $h = 0.1$; $h_t = 0.01$

x_i	$v^1(x,t)$	$u^1(x,t)$	**A.E**
0.1	$1.3042e-03$	$1.1052e-03$	$1.99e-04$
0.2	$1.4523e-03$	$1.2214e-03$	$2.30e-04$
0.3	$1.6038e-03$	$1.3499e-03$	$2.53e-04$
0.4	$1.7710e-03$	$1.4918e-03$	$2.79e-04$
0.5	$1.9558e-03$	$1.6487e-03$	$3.07e-04$
0.6	$2.1600e-03$	$1.8221e-03$	$3.38e-04$
0.7	$2.3851e-03$	$2.0138e-03$	$3.71e-04$
0.8	$2.6235e-03$	$2.2255e-03$	$3.98e-04$
0.9	$2.7079e-03$	$2.4596e-03$	$2.48e-04$

Table1. $h = 0.1; h_t = 0.01$

1



$_1.pdf$

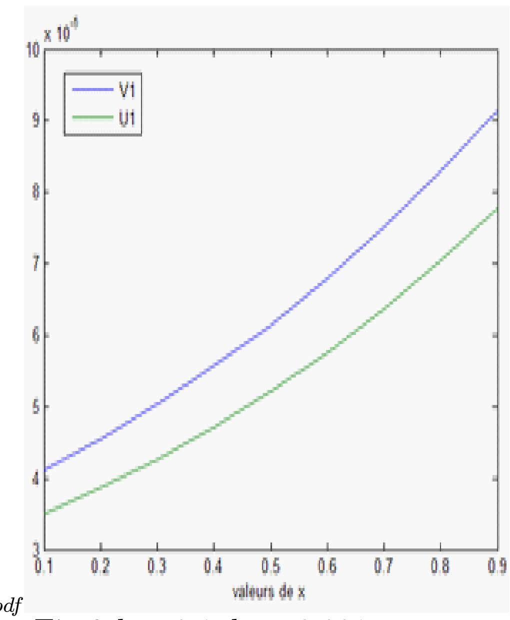
Fig1. $h = 0.1; h_t = 0.01$

For.$h = 0.1; h_t = 0.001$

x_i	$u^1(x,t)$	$v^1(x,t)$	**A.E**
0.1	$3.4949e-05$	$4.1175e-05$	$6.23e-06$
0.2	$3.8624e-05$	$4.5515e-05$	$6.89e-06$
0.3	$4.2686e-05$	$5.0301e-05$	$7.61e-06$
0.4	$4.7176e-05$	$5.5590e-05$	$8.41e-06$
0.5	$5.2137e-05$	$6.1435e-05$	$9.30e-06$
0.6	$5.7620e-05$	$6.7895e-05$	$1.03e-05$
0.7	$6.3680e-05$	$7.5034e-05$	$1.14e-05$
0.8	$7.0378e-05$	$8.2923e-05$	$1.25e-05$
0.9	$7.7802e-05$	$9.1415e-05$	$1.36e-05$

Table2.$h = 0.1; h_t = 0.001$

2



2.pdf

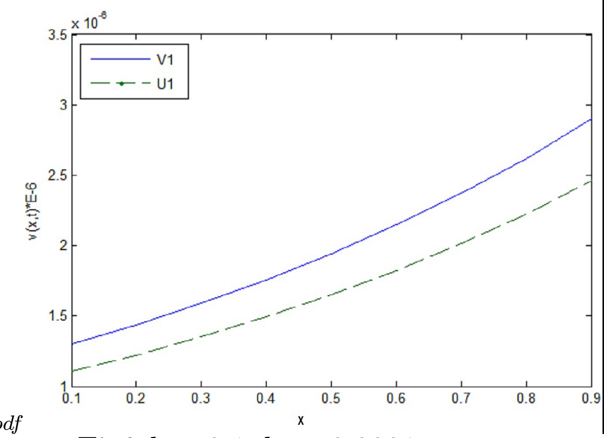
Fig 2.$h = 0.1; h_t = 0.001$

For.$h = 0.1; h_t = 0.0001$

x_i	$u^1(x,t)$	$v^1(x,t)$	**A.E**
0.1	$1.1052e-06$	$1.3020e-06$	$1e-07$
0.2	$1.2214e-06$	$1.4389e-06$	$2e-07$
0.3	$1.3499e-06$	$1.5903e-06$	$2e-07$
0.4	$1.4918e-06$	$1.7575e-06$	$2e-07$
0.5	$1.6487e-06$	$1.9424e-06$	$2e-07$
0.6	$1.8221e-06$	$2.1466e-06$	$3e-07$
0.7	$2.0138e-06$	$2.3724e-06$	$3e-07$
0.8	$2.2255e-06$	$2.6219e-06$	$3e-07$
0.9	$2.4596e-06$	$2.8974e-06$	$4e-07$

Table 3.$h = 0.1; h_t = 0.0001$

3



3.pdf

Fig3.$h = 0.1; h_t = 0.0001$

We see in Figures 1, 2 and 3 that the absolute error A.E decreases when the step h_t takes small values very close to zero. that is, for $h_t = 0.01$, $h_t = 0.001$, $h_t = 0.0001$ A.E decreases towards zero and the approximate solution tends towards the exact solution with convergence order of $O(h + h_t)$.

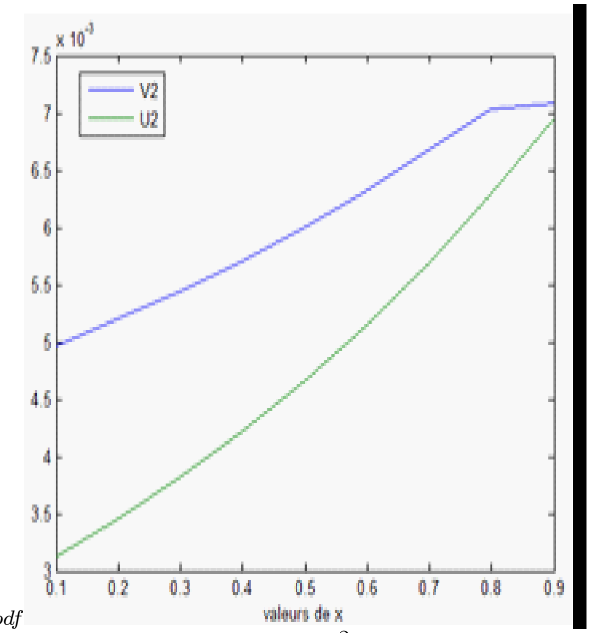
For $k = 1$ (second iteration)

Table $4, 5, 6$ show the absolute error for space step $h = 0.1$

x_i	$\mathbf{A.E}$ For $h_t = \mathbf{10^{-2}}$
0.1	$1.84e-03$
0.2	$1.74e-03$
0.3	$1.62e-03$
0.4	$1.49e-03$
0.5	$1.33e-03$
0.6	$1.17e-03$
0.7	$9.82e-04$
0.8	$7.40e-04$
0.9	$1.26e-04$

Table. 4

4



4.pdf

Fig. 4; $h_t = \mathbf{10^{-2}}$

5

x_i	**A.E** for $h_t = \mathbf{10^{-3}}$
0.1	$5.80e-05$
0.2	$5.45e-05$
0.3	$5.06e-05$
0.4	$4.63e-05$
0.5	$4.15e-05$
0.6	$3.63e-05$
0.7	$3.05e-05$
0.8	$2.40e-05$
0.9	$1.64e-05$

Table 5 $h_t = \mathbf{10^{-3}}$

5.pdf

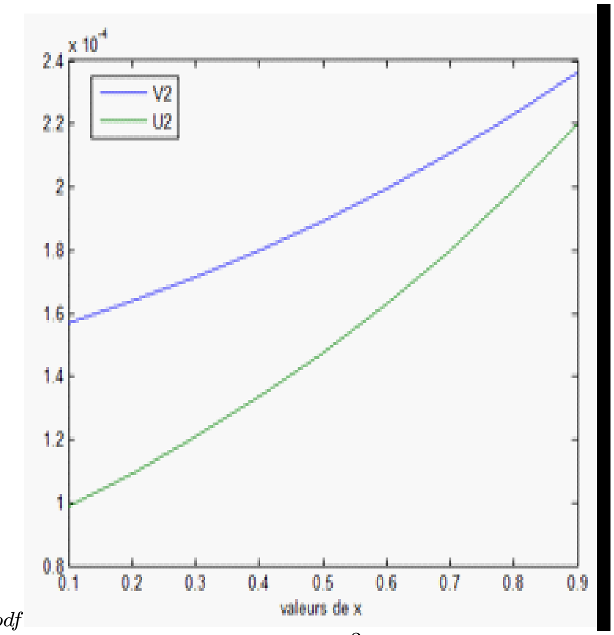


Fig. 5; $h_t = \mathbf{10^{-3}}$

x_i	**A.E** for $h_t = \mathbf{10^{-5}}$
0.1	$5.80e-08$
0.2	$5.45e-08$
0.3	$5.06e-08$
0.4	$4.63e-08$
0.5	$4.16e-08$
0.6	$3.63e-08$
0.7	$3.05e-08$
0.8	$2.41e-08$
0.9	$1.69e-08$

Table 6 $h_t = \mathbf{10^{-5}}$

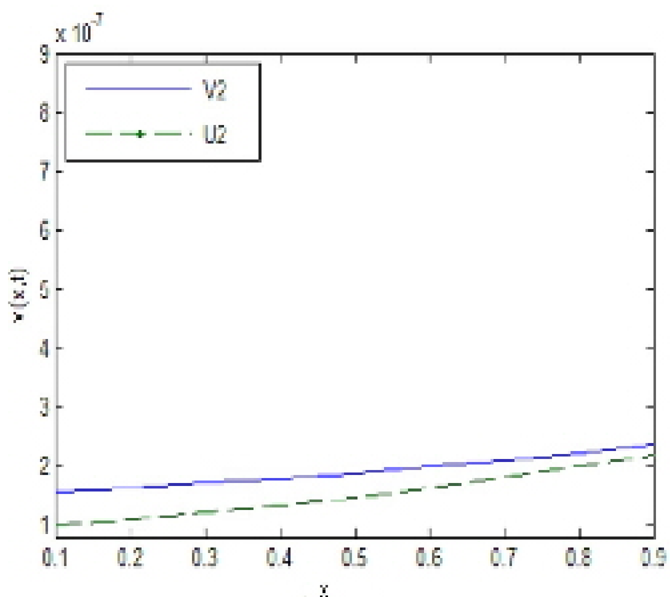


Fig.6 $h_t = \mathbf{10^{-5}}$

Table 4,5, and 6 show the absolute error decrease to zero and Figures 4,5, and 6 show the approximate solution u^2 after two steps $2h_t$ tends towards the exact solution when h_t close to zero, with convergence

order $O(h + h_t)$.

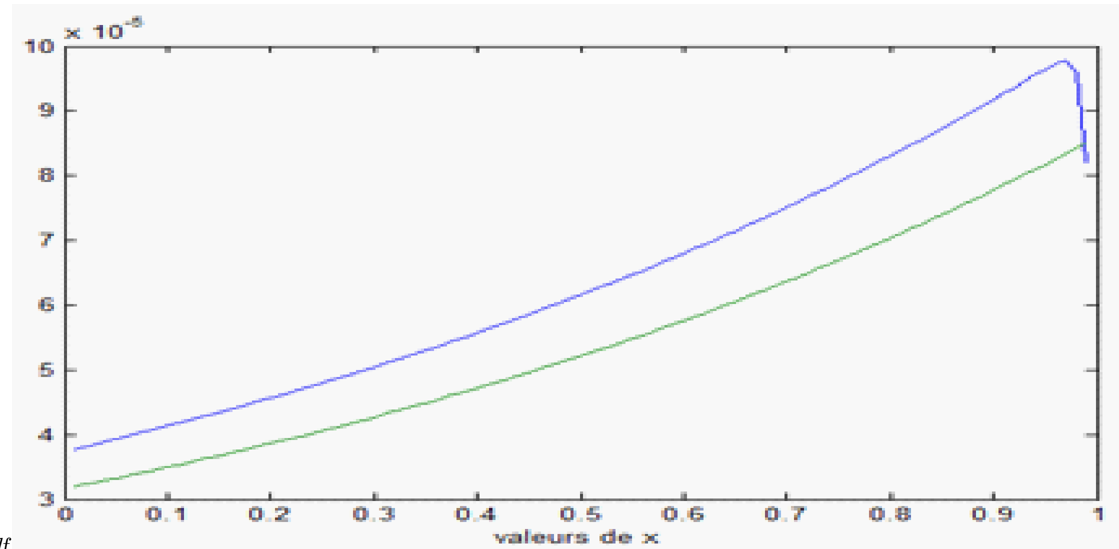
For space step $h = \mathbf{0.01}$ in example1 (**2.66**)

From tables $7, 8$ and Figures $7, 8$ with space step $h = \mathbf{0.01}$, we see that the approximate solution u^1 tends to the exact solution v^1 when h_t ($h_t = 10^{-3}, h_t = 10^{-5}$) takes values close to zero, with convergence order $O(h + h_t)$.

$\mathbf{i} = \overline{1,9}$	$\overline{10,18}$	$\overline{19,27}$	$\overline{28,36}$	$\overline{37,45}$	$\overline{46,54}$	$\overline{55,63}$	$\overline{64,72}$	$\overline{73,81}$	$\overline{82,89}$	$\overline{90,99}$
$5 * 10^{-6}$	$6 * 10^{-6}$	$6 * 10^{-6}$	$7 * 10^{-6}$	$8 * 10^{-6}$	$9 * 10^{-6}$	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
$5 * 10^{-6}$	$6 * 10^{-6}$	$7 * 10^{-6}$	$7 * 10^{-6}$	$8 * 10^{-6}$	$9 * 10^{-6}$	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
$5 * 10^{-6}$	$6 * 10^{-6}$	$7 * 10^{-6}$	$7 * 10^{-6}$	$8 * 10^{-6}$	$9 * 10^{-6}$	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
$6 * 10^{-6}$	$6 * 10^{-6}$	$7 * 10^{-6}$	$7 * 10^{-6}$	$8 * 10^{-6}$	$9 * 10^{-6}$	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
$6 * 10^{-6}$	$6 * 10^{-6}$	$7 * 10^{-6}$	$7 * 10^{-6}$	$8 * 10^{-6}$	$9 * 10^{-6}$	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
$6 * 10^{-6}$	$6 * 10^{-6}$	$7 * 10^{-6}$	$7 * 10^{-6}$	$8 * 10^{-6}$	$9 * 10^{-6}$	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
$6 * 10^{-6}$	$6 * 10^{-6}$	$7 * 10^{-6}$	$8 * 10^{-6}$	$8 * 10^{-6}$	$9 * 10^{-6}$	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
$6 * 10^{-6}$	$6 * 10^{-6}$	$7 * 10^{-6}$	$8 * 10^{-6}$	$8 * 10^{-6}$	$9 * 10^{-6}$	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}
$6 * 10^{-6}$	$6 * 10^{-6}$	$7 * 10^{-6}$	$8 * 10^{-6}$	$8 * 10^{-6}$	$9 * 10^{-6}$	10^{-5}	10^{-5}	10^{-5}	10^{-5}	10^{-5}

Table 7. The absolute error for h= $\mathbf{0.01}$; $h_t = \mathbf{10^{-3}}$

7



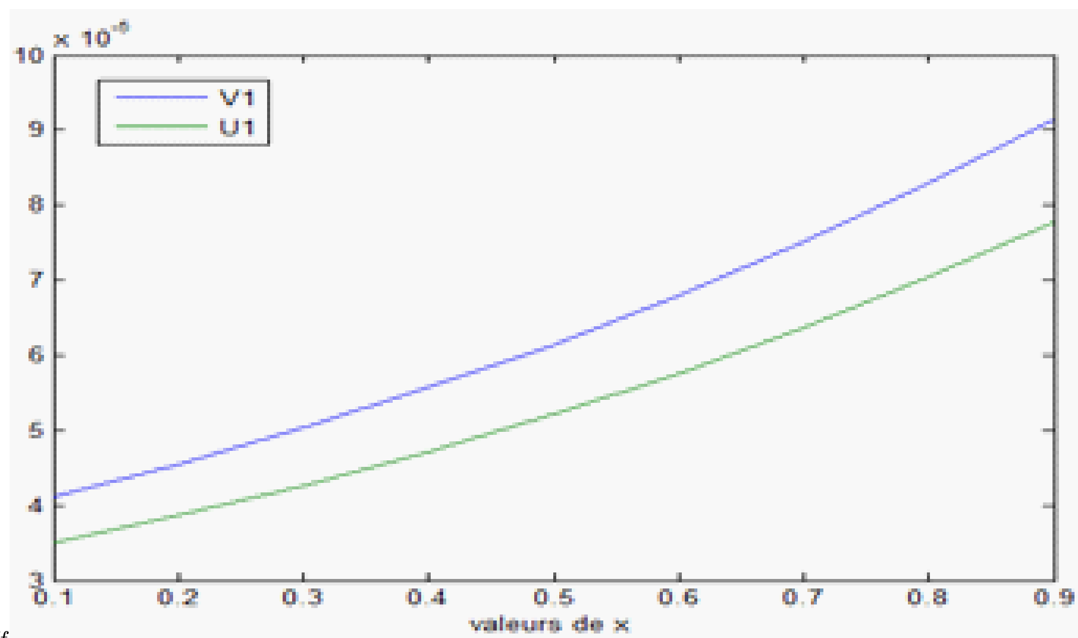
$_7$.pdf

Fig. 7 h=**0.01,** $h_t = \mathbf{10^{-3}}$

$\mathbf{i} =\overline{1,9}$	$\overline{10,18}$	$\overline{19,27}$	$\overline{28,36}$	$\overline{37,45}$	$\overline{46,54}$	$\overline{55,63}$	$\overline{64,72}$	$\overline{73,81}$	$\overline{82,89}$	$\overline{90,99}$
$5*10^{-9}$	$6*10^{-9}$	$6*10^{-9}$	$7*10^{-9}$	$8*10^{-9}$	$8*10^{-9}$	$9*10^{-9}$	10^{-8}	10^{-8}	10^{-8}	10^{-8}
$5*10^{-9}$	$6*10^{-9}$	$6*10^{-9}$	$7*10^{-9}$	$8*10^{-9}$	$9*10^{-9}$	$9*10^{-9}$	10^{-8}	10^{-8}	10^{-8}	10^{-8}
$5*10^{-9}$	$6*10^{-9}$	$6*10^{-9}$	$7*10^{-9}$	$8*10^{-9}$	$9*10^{-9}$	$9*10^{-9}$	10^{-8}	10^{-8}	10^{-8}	10^{-8}
$5*10^{-9}$	$6*10^{-9}$	$7*10^{-9}$	$7*10^{-9}$	$8*10^{-9}$	$9*10^{-9}$	10^{-9}	10^{-8}	10^{-8}	10^{-8}	10^{-8}
$5*10^{-9}$	$6*10^{-9}$	$7*10^{-9}$	$7*10^{-9}$	$8*10^{-9}$	$9*10^{-9}$	10^{-9}	10^{-8}	10^{-8}	10^{-8}	10^{-8}
$5*10^{-9}$	$6*10^{-9}$	$7*10^{-9}$	$7*10^{-9}$	$8*10^{-9}$	$9*10^{-9}$	10^{-9}	10^{-8}	10^{-8}	10^{-8}	10^{-8}
$6*10^{-9}$	$6*10^{-9}$	$7*10^{-9}$	$7*10^{-9}$	$8*10^{-9}$	$9*10^{-9}$	10^{-9}	10^{-8}	10^{-8}	10^{-8}	10^{-8}
$6*10^{-9}$	$6*10^{-9}$	$7*10^{-9}$	$7*10^{-9}$	$8*10^{-9}$	$9*10^{-9}$	10^{-9}	10^{-8}	10^{-8}	10^{-8}	10^{-8}
$6*10^{-9}$	$6*10^{-9}$	$7*10^{-9}$	$8*10^{-9}$	$8*10^{-9}$	$9*10^{-9}$	10^{-9}	10^{-8}	10^{-8}	10^{-8}	10^{-8}

Table 8. The absolute error for $\mathbf{h} = \mathbf{0.01}$; $\mathbf{h}_t = \mathbf{10}^{-5}$

2



$_8$.pdf

Fig. 8 $h = \mathbf{0.01}$, $h_t = \mathbf{10}^{-5}$

2.3.2 Example2

$\alpha = \frac{3}{2}$, $\quad a(x,t) = -x^2 - t$, $\quad b(x,t) = x - t$, $\quad c(x) = x + 2t$, $\quad \mu(t) = (t+1)^2$

$g(x,t) = (4\sqrt{t} + (t+1)^2(x^2+2t)e^x$, $\Phi(x) = e^x$; $\quad \psi(x) = 2e^x$, $\quad E(t) = (t+1)^2$.

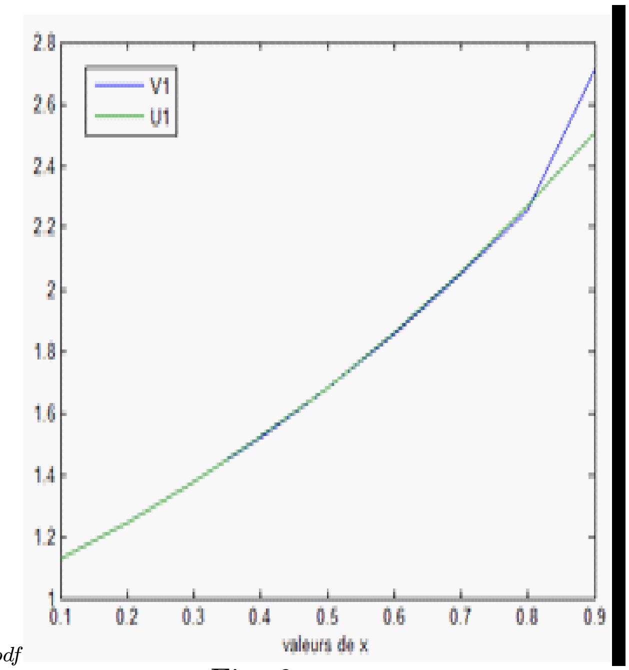
The analytical solution of this problem is given by $v(x,t) = (t+1)^2 e^x$.

The tables 9, 10 and 11 show the values of the absolute error.

x_i	$A.E$
0.1	$6.72e-04$
0.2	$2.47e-03$
0.3	$2.44e-03$
0.4	$2.41e-03$
0.5	$2.39e-03$
0.6	$2.38e-03$
0.7	$1.84e-03$
0.8	$1.40e-02$
0.9	$2.06e-01$

Table 9.$h = 0.1; h_t = 0.01$

8



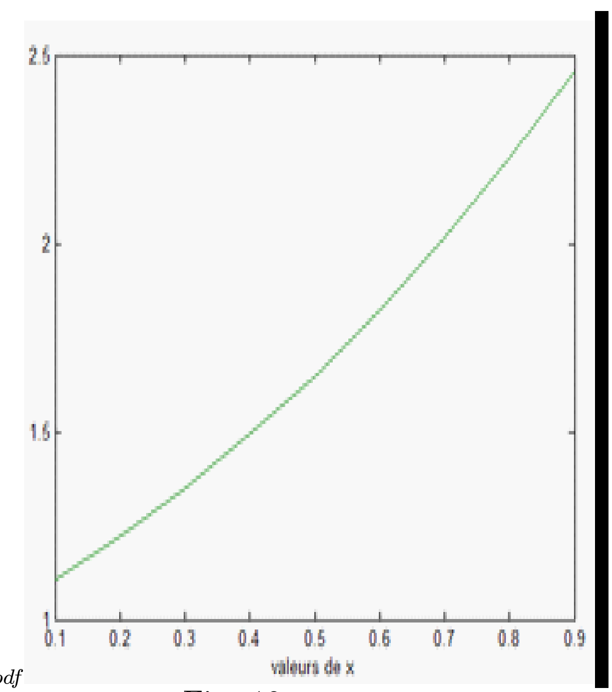
9.pdf

Fig. 9

x_i	$A.E$
0.1	$1.22e-05$
0.2	$4.28e-05$
0.3	$4.24e-05$
0.4	$4.20e-05$
0.5	$4.15e-05$
0.6	$4.10e-05$
0.7	$4.03e-05$
0.8	$4.83e-05$
0.9	$5.48e-03$

Table 10. h= $\mathbf{0.1}$, $h_t = \mathbf{10^{-3}}$

9



10.pdf

Fig. 10

x_i	$A.E$
0.1	$3.75e-07$
0.2	$1.26e-06$
0.3	$1.24e-06$
0.4	$1.25e-06$
0.5	$1.23e-06$
0.6	$1.22e-06$
0.7	$1.20e-06$
0.8	$1.19e-06$
0.9	$1.72e-04$

Table11.$h = 0.1,\ h_t = 10^{-4}$

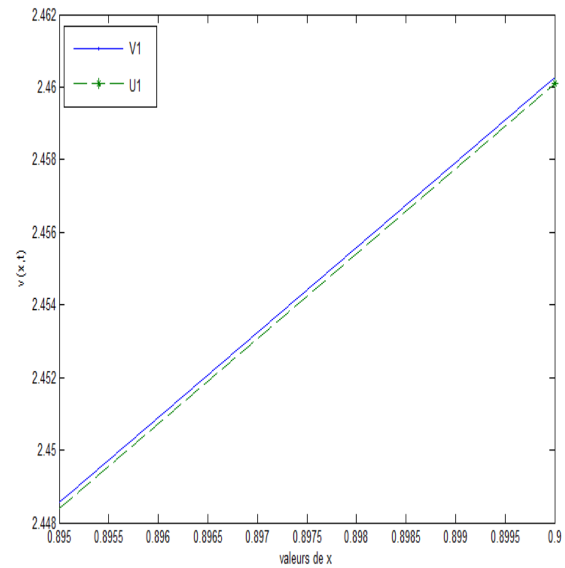


Fig 11. $x \in [0.8965, 0.9000]$

In this example from tables $9, 10, 11$ and Figures $9, 10, 11$ we see again for space step $h = 0.1$ the absolute error tends to zero, when the time step h_t ($\mathbf{10^{-2}}$, $\mathbf{10^{-3}}$, $\mathbf{10^{-4}}$) takes a values close to zero, with convergence order $O(h + h_t)$.In figure 11 we take into account $x \in [0.8965, 0.9000]$ to see the variation of error because it's very close to zero when $x \in [0.1, 0.8]$.

Table A shows for h= $\mathbf{0.1}$ the error norm $\|E^k\|_\infty$ for defferent value of α defined by

$$\|E^k\|_\infty = \underset{1 \le i \le N-1}{\text{Max}} \sum_{i=1}^{N-1} |e_i|\,, \text{ where } E^k = V^{\ k} - U^{\ k} = \left(e_1^k, ..., e_{N-1}^k\right)^T$$

values of h_t		h_t=$\mathbf{10^{-3}}$	h_t=$\mathbf{10^{-5}}$	h_t=$\mathbf{10^{-7}}$
. $\|E^1\|_\infty$ for	$\boldsymbol{\alpha} = \mathbf{1.2}$	$9.5736e * 10^{-4}$	$1.3196 * 10^{-6}$	$5.2768 * 10^{-9}$
	$\boldsymbol{\alpha} = \mathbf{1.4}$	$1.1294 * 10^{-4}$	$1.2671 * 10^{-7}$	$2.0154 * 10^{-10}$
	$\boldsymbol{\alpha} = \mathbf{1.6}$	$2.3162 * 10^{-5}$	$1.2692 * 10^{-8}$	$7.9794 * 10^{-12}$
	$\boldsymbol{\alpha} = \mathbf{1.8}$	$1.53 * 10^{-4}$	$1.4449 * 10^{-9}$	$3.4062 * 10^{-13}$
	$\boldsymbol{\alpha} = \mathbf{1.9}$	$4.7963 * 10^{-6}$	$6.2306 * 10^{-10}$	$8.6153 * 10^{-14}$

Table A , h= $\mathbf{0.1}$

We see in table A, for the space step $h = 0.1$, and for the different values of α, the error keeps the same behavior, that is the error norm tends towards zero when the time step h_t takes values close to zeros, with an order of convergence $O(h + h_t)$. for $\alpha = 1.2$ this value close to 1 the error is greater compared to the case $\alpha = 1.9$ value close to 2 due of the fractional operator is approximated by the formula called $L2$.

For $h = \mathbf{0.01}$, $\boldsymbol{\alpha} = \mathbf{1.5}$

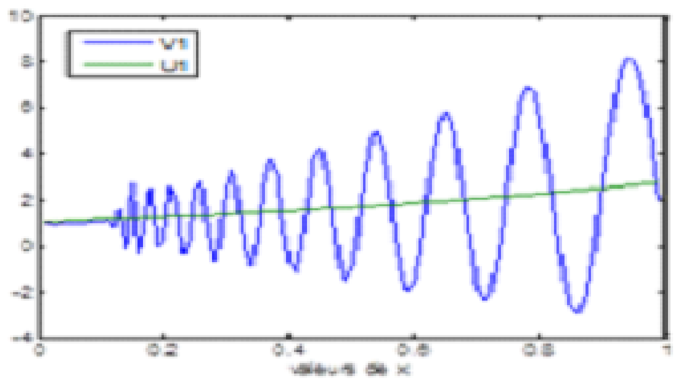


Fig. 12 $h_t = 0.01$

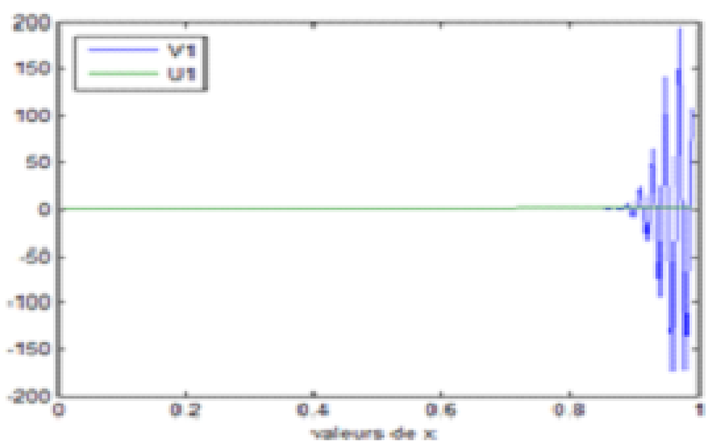


Fig. 13 $h_t = 0.001$

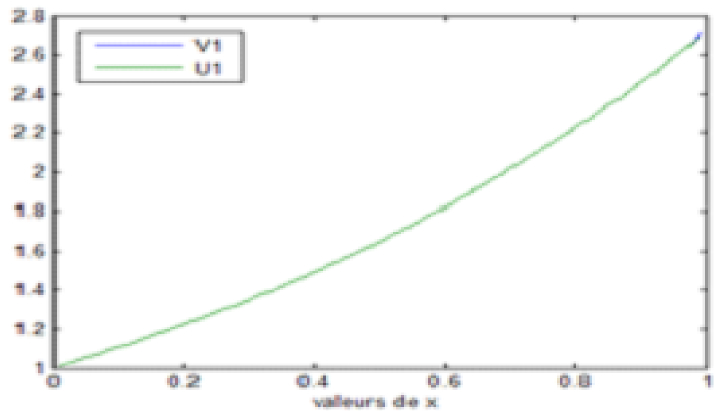


Fig. 14 $h_t = 0.0001$

The Figures 12 ,13 and 14 show where the space step is fixed at *h=0.01* and the time step h_t decreases towards zero ($h_t = 0.01$, $h_t = 0.001$, $h_t = 0.0001$), the approximate solution u^1 tends to the exact solution v^1, in the case where $h_t = 0.0001$ we see that the two curves of u^1 and v^1 are almost identical.

3.0 CHAPTER 3. NUMERICAL RESOLUTION OF PARABOLIC FRACTIONAL DIFFERENTIAL EQUATION WITH INTEGRALS CONDITIONS

Introduction

In this part, we are interested in a fractional problem with boundary conditions of integral type $\int_0^1 u(x,t)\,dx$, we consider the time fraction of the parabolic fractional differential equations, obtained by replacing the time derivative of the second order in standard parabolic equation by a fractional derivative of order α $(0 < \alpha < 1)$, and classical boundary condition with integral boundary condition.The theoretical study, is carried out by T-Oussif and A-Bouziani [27] where proved the existence and uniqueness by using the inequalities of energy method. Our contribution is on the numerical study of problem with boundary **conditions of integral type** $\int_0^1 u(x,t)\,dx$. The study of stability and convergence are proved. Some numerical tests give a very satisfactory results.

3.1 POSITION OF THE PROBLEM

We consider the fractional differential equation defined in domain Q by

$$Lu =_0^c \partial_t^\alpha u(x,t) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial}{\partial x}u(x,t)\right) = f(x,t)\,, \text{ where } 0 < \alpha < 1, \tag{3.1}$$

and

$$Q = \left\{(x,t) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < t < T\right\}, \text{ where } T > 0,$$

to the equation (3.1), we associate the initial condition

$$u(x,0) = \phi(x), \qquad 0 < x < 1, \tag{3.2}$$

and Newman conditions

$$\left\{ \frac{\partial}{\partial x}u(x,t)\Big|_{x=0} = \mu(t), \qquad 0 < t \leq T \right. \tag{3.3}$$

and the purely integral condition

$$\int_0^1 u\left(x,t\right)dx = m(t), \qquad 0 < t \leq T, \tag{3.4}$$

where f, ϕ, μ, a and m are known continuous functions and a, $\frac{\partial a}{\partial x}$ are positive functions.

We shall assume that the function ϕ satisfies a compatibility conditions with (3.3) and (3.4), i.e, ([27]) .

$$\frac{\partial}{\partial x}u(0,t) = \mu(0), \quad \int_0^1 \phi\left(x\right)dx = m(0)$$

We consider a uniform subdivision of the intervals $[0,1]$ and $[0,T]$ as follows

$$x_i = ih; \ i = 0,1,...,N \quad \text{and } t_k = kh_t; \ k = 0,1,...,M.$$

$$u\left(ih, kh_t\right) = u_i^k = u_{i,k}$$

such that $u_{i,k}$ represent the value of solution u in point (x_i, t_k).

$$a_i^k = a\left(x_i, t_k\right), \quad \varphi_i^k = \varphi\left(x_i, t_k\right), \ \phi(x_i) = \phi_i, \quad f_i^k = f\left(x_i, t_k\right)$$

The Caputo derivative fractional of operator ${}_0^cD_t^\alpha$ in point (x_i, t_{k+1}) called L_1 with $0 < \alpha < 1$ is defined by(1.38) .

$$\left({}_0^cD_t^\alpha u\left(x_i, t_{k+1}\right)\right)_{L_1} = \left({}_0^cD_t^\alpha u\right)_i^{k+1} \simeq \lambda \sum_{j=0}^{k} \left(u_i^{k-j+1} - u_i^{k-j}\right) d_j \tag{3.5}$$

$$\text{where} \left\{ \begin{array}{l} d_j = (j+1)^{1-\alpha} - j^{1-\alpha} \\ d_0 = 1; k = 1,...,M \end{array} \right. , \ \lambda = \frac{h_t^{-\alpha}}{\Gamma\left(2-\alpha\right)}.$$

let Λ be the operator defined by

$$\Lambda = \frac{\partial a}{\partial x}\frac{\partial}{\partial x} + a\frac{\partial^2}{\partial x^2} \tag{3.6}$$

From Taylor developpement we have

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i^{k+1} = \frac{1}{h^2}\left(u_{i-1}^{k+1} - 2u_i^{k+1} + u_{i+1}^{k+1}\right) + O(h^2), \quad \left(\frac{\partial u}{\partial x}\right)_i^{k+1} = \frac{u_{i+1}^{k+1} - u_i^{k+1}}{h} + O(h) \quad (3.7)$$

and

$$(\Lambda u)_i^{\ k+1} \simeq G_i^{k+1}\left(\frac{u_{i+1}^{k+1} - u_i^{k+1}}{h}\right) + \frac{a_i^{k+1}}{h^2}\left(u_{i-1}^{k+1} - 2u_i^{k+1} + u_{i+1}^{k+1}\right) \text{ where } \frac{\partial a}{\partial x} = G \quad (3.8)$$

We write the equation (3.1) at point (x_i, t_{k+1}) we find

$$\left({}_0^c D_t^\alpha u\right)_i^{k+1} - (\Lambda u)_i^{\ k+1} = f_i^{\ k+1} \quad (3.9)$$

then

$$\lambda \sum_{j=0}^{k}\left(u_i^{\ k-j+1} - u_i^{\ k-j}\right) d_j - G_i^{k+1}\left(\frac{u_{i+1}^{k+1} - u_i^{k+1}}{h}\right) - \frac{a_i^{k+1}}{h^2}\left(u_{i-1}^{k+1} - 2u_i^{k+1} + u_{i+1}^{k+1}\right) = f_i^{\ k+1}$$

then

$$\left(\lambda + \frac{G_i^{k+1}}{h} + 2\frac{a_i^{k+1}}{h^2}\right) u_i^{\ k+1} - \left(\frac{G_i^{k+1}}{h} + \frac{a_i^{k+1}}{h^2}\right) u_{i+1}^{k+1} - \frac{a_i^{k+1}}{h^2} u_{i-1}^{k+1}$$

$$= \lambda u_i^{\ k} - \lambda \sum_{j=1}^{k}\left(u_i^{\ k-j+1} - u_i^{\ k-j}\right) d_j + f_i^{\ k+1}$$

we have

$$\sum_{j=1}^{k}\left(u_i^{\ k-j+1} - u_i^{\ k-j}\right) d_j = d_1 u_i^{\ k} + \sum_{m=1}^{k-1}(d_{m+1} - d_m)\, u_i^{\ k-m} - d_k u_i^{\ 0}$$

then

$$-\frac{a_i^{k+1}}{h^2} u_{i-1}^{k+1} + \left(\lambda + \frac{G_i^{k+1}}{h} + 2\frac{a_i^{k+1}}{h^2}\right) u_i^{\ k+1} - \left(\frac{G_i^{k+1}}{h} + \frac{a_i^{k+1}}{h^2}\right) u_{i+1}^{k+1}$$

$$= \lambda(1 - d_1)\, u_i^k - \lambda \sum_{m=1}^{k-1}(d_{m+1} - d_m)\, u_i^{k-m} + \lambda d_k u_i^0 + f_i^{\ k+1}$$

therefore

$$D_i^{k+1} u_{i-1}^{k+1} + A_i^{k+1} u_i^{k+1} + \ B_i^{k+1} u_{i+1}^{k+1}$$

$$= \lambda \left(1 - d_1\right) u_i^k - \lambda \sum_{m=1}^{k-1} \left(d_{m+1} - d_m\right) u_i^{k-m} + \lambda d_k u_i^0 + f_i^{\,k+1} \tag{3.10}$$

where

$$D_i^{k+1} = -\frac{a_i^{k+1}}{h^2},\ A_i^{k+1} = \lambda + \frac{G_i^{k+1}}{h} + 2\frac{a_i^{k+1}}{h^2},\ B_i^{k+1} = -\frac{G_i^{k+1}}{h} - \frac{a_i^{k+1}}{h^2},$$

For $k = 0$ we get

$$D_i^1 u_{i-1}^1 + A_i^1 u_i^1 + \ B_i^1 u_{i+1}^1 = \lambda \left(2 - d_1\right) u_i^0 + f_i^{\,1} \tag{3.11}$$

Formula (3.11) is valable for $2 \leq i \leq N - 2$

For $i = 1$

$$D_1^1 u_0^1 + A_1^1 u_1^1 + \ B_1^1 u_2^1 = \lambda u_1^{\,0} + f_1^{\,1}$$

with condition (3.3) we find

$$\frac{u_i^{k+1} - u_{i-1}^{k+1}}{h} = \mu^{k+1} \text{ so } u_0^{k+1} = u_1^{k+1} - h\mu^{k+1} \text{ and } u_0^1 = u_1^1 - h\mu^1 \tag{3.12}$$

then

$$\left(A_1^1 + D_1^1\right) u_1^{\,1} + \ B_1^1 u_2^1 = \lambda u_1^{\,0} + D_1^1 h\mu^1 + f_1^{\,1} \tag{3.13}$$

For $i = N - 1$

$$D_{N-1}^1 u_{N-2}^1 + A_{N-1}^1 u_{N-1}^1 + \ B_{N-1}^1 u_N^1 = \lambda u_{N-1}^{\,0} + f_{N-1}^{\,1} \tag{3.14}$$

To eliminate u_N^1 we use conditions (3.4) and trapeze formula we get

$$\frac{h}{2}\left[u_0^{k+1} + 2\sum_{i=1}^{N-1} u_i^{k+1} + u_N^{k+1}\right] = m(t_{k+1})$$

then

$$u_N^{k+1} = \frac{2}{h} m(t_{k+1}) - u_0^{k+1} - 2\sum_{i=1}^{N-1} u_i^{k+1} \text{ so } u_N^{k+1} = \frac{2}{h} m(t_{k+1}) - 3u_1^{k+1} + h\mu^{k+1} - 2\sum_{i=2}^{N-1} u_i^{k+1}$$

and

$$u_N^1 = \frac{2}{h} m(h_t) - 3u_1^1 + h\mu^1 - 2\sum_{i=2}^{N-1} u_i^1 \tag{3.15}$$

we substitute (3.15) in (3.11) we get

$$D_{N-1}^1 u_{N-2}^1 + A_{N-1}^1 u_{N-1}^1 + B_{N-1}^1 \left(\frac{2}{h} m(h_t) - 3u_1^1 + h\mu(h_t) - 2\sum_{i=2}^{N-1} u_i^1 \right) = \lambda u_{N-1}^{\,0} + f_{N-1}^{\,1}$$

hence

$$-3B_{N-1}^1 u_1^1 - 2B_{N-1}^1 \sum_{i=2}^{N-3} u_i^1 + \left(D_{N-1}^1 - 2B_{N-1}^1\right) u_{N-2}^1 + \left(A_{N-1}^1 - 2B_{N-1}^1\right) u_{N-1}^1$$

$$= -B_{N-1}^1 \left(h\mu(h_t) + \frac{2}{h} m(h_t) \right) + \lambda u_{N-1}^{\,0} + f_{N-1}^{\,1} \tag{3.16}$$

we denote by

$$\begin{aligned} U^{\,k} &= \left(u_1^k, u_2^k, ..., u_{N-1}^k\right)^T, \ U^{\,0} = \left(\phi_1, \phi_2, ..., \phi_{N-1}\right)^T, \ F^{\,k} = \left(f_1^{\,k}, f_2^{\,k}, ..., f_{N-1}^{\,k}\right)^T. \\ R^k &= \left(hD_1^k \mu(h_t), 0, ..., 0, -B_{N-1}^1 \left(h\mu(h_t) + \frac{2}{h} m(h_t) \right) \right)^T; \ H^1 = \lambda U^{\,0} \end{aligned}$$

Taking account (3.16), (3.13), and (3.11) we obtain the matrix system

$$P^1.U^1 = M^1; \ M^1 = H^1 + R^1 + F^{\,1} \tag{3.17}$$

where

$$\begin{aligned} P^1 &= (l_{i,j})_{N-1,N-1} \text{ is square matrix defined by} \\ l_{1,1} &= A_1^1 + D_1^1, \ l_{1,2} = B_1^1, \ l_{N-1,N-2} = D_{N-1}^1 - 2B_{N-1}^1, \\ l_{N-1,N-1} &= A_{N-1}^1 - 2B_{N-1}^1, \ l_{N-1,1} = -3B_{N-1}^1, \end{aligned}$$

$$l_{i,j} = \begin{cases} 0 & \text{when} \quad |i-j| \geq 2 \ , \ i = \overline{2, N-2} \\ A_i^1 & \text{when} \quad i = j, \ i = \overline{2, N-2} \\ D_i^1 & \text{when} \quad i = j+1, \ i = \overline{2, N-2} \\ B_i^1 & \text{when} \quad i = j-1, \ i = \overline{2, N-1} \\ -2B_{N-1}^1 & \text{when} \quad i = N-1, \ j = \overline{2, N-3} \end{cases}$$

3.1.1 General case

For $k \geq 1$ we have

$$D_i^{k+1} u_{i-1}^{k+1} + A_i^{k+1} u_i^{k+1} + B_i^{k+1} u_{i+1}^{k+1}$$

$$= \lambda (1 - d_1) u_i^k - \lambda \sum_{m=1}^{k-1} (d_{m+1} - d_m) u_i^{k-m} + \lambda d_k u_i^0 + f_i^{\,k+1}; \; i = \overline{2, N-2} \tag{3.18}$$

For $i = 1$ and from (3.12) we get

$$\left(A_1^{k+1} + D_1^{k+1}\right) u_1^{k+1} + B_1^{k+1} u_2^{k+1}$$

$$= -\lambda \sum_{m=0}^{k-1} (d_{m+1} - d_m) u_1^{k-m} + \lambda d_k u_1^0 + D_1^{k+1} h \mu^{k+1} + f_1^{\,k+1} \tag{3.19}$$

For $i = N - 1$ and from trapeze formula we get

$$D_{N-1}^{k+1} u_{N-2}^{k+1} + A_{N-1}^{k+1} u_{N-1}^{k+1} + B_{N-1}^{k+1} u_N^{k+1}$$

$$= \lambda (1 - d_1) u_{N-1}^k - \lambda \sum_{m=1}^{k-1} (d_{m+1} - d_m) u_{N-1}^{k-m} + \lambda d_k u_{N-1}^0 + f_{N-1}^{\,k+1}$$

then

$$D_{N-1}^{k+1} u_{N-2}^{k+1} + A_{N-1}^{k+1} u_{N-1}^{k+1} + B_{N-1}^{k+1} \left(\frac{2}{h} m(t_{k+1}) - 3u_1^{k+1} + h\mu^{k+1} - 2 \sum_{i=2}^{N-1} u_i^{k+1} \right)$$

$$= \lambda (1 - d_1) u_{N-1}^k - \lambda \sum_{m=1}^{k-1} (d_{m+1} - d_m) u_{N-1}^{k-m} + \lambda d_k u_{N-1}^0 + f_{N-1}^{\,k+1}$$

hence

$$-3 B_{N-1}^{k+1} u_1^{k+1} - 2 B_{N-1}^{k+1} \sum_{i=2}^{N-3} u_i^{k+1} + \left(D_{N-1}^{k+1} - 2 B_{N-1}^{k+1}\right) u_{N-2}^{k+1} + \left(A_{N-1}^{k+1} - 2 B_{N-1}^{k+1}\right) u_{N-1}^{k+1}$$

$$= - B_{N-1}^{k+1} \frac{2}{h} m(t_{k+1}) + h B_{N-1}^{k+1} \mu^{k+1} - \lambda \sum_{m=0}^{k-1} (d_{m+1} - d_m) u_{N-1}^{k-m} + \lambda d_k u_{N-1}^0 + f_{N-1}^{\,k+1} \tag{3.20}$$

Taking account (3.18) , (3.19) , and (3.20) we obtain the matrix system

$$P^{k+1}.U^{k+1} = \lambda U^{\ 0} + R^{k+1} + F^{\ K+1} \tag{3.21}$$

$$P^{k+1}.U^{k+1} = M^{k+1};\ M^{k+1} = H^{k+1} + R^{k+1} + F^{\ K+1} \tag{3.22}$$

where

$$\begin{aligned} P^{k+1} &= (l_{i,j})_{N-1,N-1} \text{ is square matrix defined by} \\ l_{1,1} &= A_1^{k+1} + D_1^{k+1},\ l_{1,2} = B_1^{k+1},\ l_{N-1,N-2} = D_{N-1}^{k+1} - 2B_{N-1}^{k+1}, \\ l_{N-1,N-1} &= A_{N-1}^{k+1} - 2B_{N-1}^{k+1},\ l_{N-1,1} = -3B_{N-1}^{k+1}, \end{aligned}$$

$$l_{i,j} = \begin{cases} 0 & \text{when} \quad |i-j| \geq 2\ ,\ i = \overline{2, N-2} \\ A_i^{k+1} & \text{when} \quad i = j,\ i = \overline{2, N-2} \\ D_i^{k+1} & \text{when} \quad i = j+1,\ i = \overline{2, N-2} \\ B_i^{k+1} & \text{when} \quad i = j-1,\ i = \overline{2, N-1} \\ -2B_{N-1}^{k+1} & \text{when} \quad i = N-1,\ j = \overline{2, N-3} \end{cases}$$

To solve the system (3.22) we can apply one of direct methods.

In order to prove system (3.22) has a unique solution we denote ρ as an eigenvalue of the matrix P^k, and $X = (x_1, x_2, ..., x_{N-1})^T$ is an nonzero eigenvector corresponding to ρ

We choose i such as

$$|x_i| = max\{|x_j| : j = 1; ...; N-1\}.$$

then

$$\sum_{j=1}^{N-1} l_{i,j} x_j = \rho x_i;\ i = \overline{1; N-1}$$

therefore

$$\rho = l_{i,i} + \sum_{\substack{j=1 \\ j \neq i}}^{N-1} l_{i,\ j} \frac{x_j}{x_i} \tag{3.23}$$

Substituting the values of $l_{i,j}$ into (3.23) , and taking into account that G_i^k, a_i^k are positive and $\left|\frac{x_j}{x_i}\right| \leq 1$ we get

If $i = 1$

$$\begin{aligned} D_i^{k+1} &= -\frac{a_i^{k+1}}{h^2},\ A_i^{k+1} = \lambda + \frac{G_i^{k+1}}{h} + 2\frac{a_i^{k+1}}{h^2},\ B_i^{k+1} = -\frac{G_i^{k+1}}{h} - \frac{a_i^{k+1}}{h^2} \\ \lambda &= A_i^{k+1} + B_i^{k+1} + D_i^{k+1},\ A_i^{k+1} + B_i^{k+1} = \lambda - D_i^{k+1} > 0 \end{aligned}$$

$$\rho = A_1^{k+1} + D_1^{k+1} + B_1^{k+1}\frac{x_1}{x_2} = \lambda + B_1^{k+1}\left(\frac{x_1}{x_2} - 1\right) \tag{3.24}$$

If $i = N-1$

$$\begin{aligned} \rho &= A_{N-1}^{k+1} - 2\ B_{N-1}^{k+1} + \left(D_{N-1}^{k+1} - 2\ B_{N-1}^{k+1}\right)\frac{x_{N-2}}{x_{N-1}} - 2\ B_{N-1}^{k+1}\sum_{j=2}^{N-3}\frac{x_j}{x_{N-1}} - 3\ B_{N-1}^{k+1}\frac{x_1}{x_{N-1}} \\ &= \lambda - 3B_{N-1}^{k+1}\left(1 + \frac{x_1}{x_{N-1}}\right) + D_{N-1}^{k+1}\left(\frac{x_{N-2}}{x_{N-1}} - 1\right) - 2\ B_{N-1}^{k+1}\sum_{j=2}^{N-2}\frac{x_j}{x_{N-1}}. \end{aligned} \tag{3.25}$$

If $i = \overline{2, N-2}$ we get

$$\begin{aligned} \rho &= A_i^{k+1} +\ B_i^{k+1}\frac{x_{i+1}}{x_i} + D_i^{k+1}\frac{x_{i-1}}{x_i} \\ &= \lambda +\ B_i^{k+1}\left(\frac{x_{i+1}}{x_i} - 1\right) + D_i^{k+1}\left(\frac{x_{i-1}}{x_i} - 1\right) \end{aligned} \tag{3.26}$$

Since B_i^{k+1} and D_i^{k+1} are negatives then all eigen values of P^{k+1} are strictly positive and P^k is inversible.

3.1.2 Stability

We have

$$A_i^{k+1} + B_i^{k+1} + D_i^{k+1} = \lambda >> 1, \quad D_i^{k+1} \leq 0, \quad B_i^{k+1} \leq 0$$

Let u_i^{k+1} be the approximate solution of (3.10), and e_i^{k+1} the error at point (x_i, t_{k+1}) defined by

$$v_i^{k+1} - u_i^{k+1} = e_i^{k+1}, \text{ and } \left\| E^{\,k} \right\| = \underset{1 \leq i \leq N-1}{\text{Max}} |e_i^k|, \ E^{\,k} = \left(e_1^k, ..., e_{N-1}^k\right)^T$$

for $k = 0$ we apply (3.11) we get

$$D_i^1 e_{i-1}^1 + A_i^1 e_i^1 + B_i^1 u_{i+1}^1 = \lambda \left(2 - d_1\right) e_i^0$$

then

$$\begin{aligned} \lambda \left\| E^1 \right\| &= \left(D_i^1 + A_i^1 + B_i^1\right) \left\| E^1 \right\| \\ &\leq \left(\left(A_i^1 + B_i^1\right) \left\| E^1 \right\| + D_i^1 \left| e_{i-1}^1 \right|\right) \\ &\leq \underset{1 \leq i \leq N-1}{\text{Max}} \left| D_i^1 e_{i-1}^1 + A_i^1 e_i^1 + B_i^1 e_{i+1}^1 \right| = \lambda \left(2 - d_1\right) \left\| E^0 \right\| \end{aligned}$$

so

$$\left\| E^1 \right\| \preceq C_1 \left\| E^0 \right\|; \ C_1 = |2 - d_1| = \left|3 - 2^{1-\alpha}\right| \tag{3.27}$$

Therefore the method is stable.

Lemma 7. *For $k \geq 1$ the scheme (3.18) is stable and we have*

$$\left\| E^{k+1} \right\| \leq C \left\| E^0 \right\|, \ C > 0, \textit{ for all } k \geq 1$$

Proof. We use mathematical induction. □

We assume $\|E^j\| \leq c_j \|E^0\|$, and $C_{\max} = \max c_j$; where $c_j \succ 0,\ j = \overline{1, k}$.

from (3.18) we get

$$D_i^{k+1} e_{i-1}^{k+1} + A_i^{k+1} e_i^{k+1} + \ B_i^{k+1} e_{i+1}^{k+1}$$

$$= \lambda (1 - d_1) e_i^k - \lambda \sum_{m=1}^{k-1} (d_{m+1} - d_m) e_i^{k-m} + \lambda d_k e_i^0;\ i = \overline{1, N-1} \tag{3.28}$$

so

$$\begin{aligned}
\lambda \left\|E^{k+1}\right\| &= \left(D_i^{k+1} + A_i^{k+1} + \ B_i^{k+1}\right) \left\|E^{k+1}\right\| \\
&\leq \left(D_i^{k+1} \left\|E^{k+1}\right\| + A_i^{k+1} \left\|E^{k+1}\right\| + \ B_i^{k+1} \left\|E^{k+1}\right\|\right) \\
&\leq A_i^{k+1} \left\|E^{k+1}\right\| + \ B_i^{k+1} \left|e_{i+1}^{k+1}\right| + D_i^{k+1} \left|e_{i-1}^{k+1}\right| \\
&\leq \underset{1 \leq i \leq N-1}{\text{Max}} \left|\lambda (1 - d_1) e_i^k - \lambda \sum_{m=1}^{k-1} (d_{m+1} - d_m) e_i^{k-m} + \lambda d_k e_i^0\right| \\
&\leq \lambda \left((1 - d_1) \left\|E^k\right\| + \sum_{m=1}^{k-1} (d_{m+1} - d_m) \left\|E^{k-m}\right\| + d_k \left\|E^0\right\|\right) \\
&\leq \lambda \left((1 - d_1) C_k + C_{k-m} (d_k - d_1) + 1\right) \left\|E^0\right\| \\
&\leq \lambda C_{\max} (2 - 2d_1 + d_k) \left\|E^0\right\|
\end{aligned}$$

It is readily checked that, for $k \geq 1$

$$d_k \leq 1 \text{ and } 2 - 2d_1 + d_k \leq 2$$

then

$$\left\|E^{k+1}\right\| \leq C \left\|E^0\right\|;\ C = 2C_{\max} \tag{3.29}$$

Therefore the method is stable.

3.1.3 Convergence

Let $u(x_i;t_{k+1})$ as the exact solution and u_i^{k+1} is the approximate solution of scheme (3.10), we put $u(x_i;t_{k+1}) - u_i^{k+1} = \epsilon_i^{k+1}$; for $i = \overline{1, N-1}$, $k = \overline{1, M-1}$.

The scheme L_1 defined on (3.5)verifed ([23])

$$\left| \frac{\partial^\alpha u}{\partial t^\alpha} - \left(\frac{\partial^\alpha u}{\partial t^\alpha} \right)_{L_1} \right| \leq O\left(h_t\right) \tag{3.30}$$

Substituting in to (3.9) and using (3.7), (3.30) leads to

$${}_0^cD_t^\alpha \left(u(x_i;t_{k+1}) - \epsilon_i^{k+1} \right) - \Lambda \left(u(x_i;t_{k+1}) - \epsilon_i^{k+1} \right) = f_i^{\,k+1} \tag{3.31}$$

then

$$\lambda \sum_{j=0}^{k} \left(u(x_i;t_{k-j+1}) - \epsilon_i^{k-j+1} - u(x_i;t_{k-j}) + \epsilon_i^{k-j} \right) d_j$$

$$-G_i^{k+1} \left(\frac{u(x_{i+1};t_{k+1}) - \epsilon_{i+1}^{k+1} - u(x_i;t_{k+1}) + \epsilon_i^{k+1}}{h} \right)$$

$$-\frac{a_i^{k+1}}{h^2} \left(u(x_{i-1};t_{k+1}) - \epsilon_{i-1}^{k+1} - 2\left(u(x_i;t_{k+1}) - \epsilon_i^{k+1} \right) + u(x_{i+1};t_{k+1}) - \epsilon_{i+1}^{k+1} \right) \quad = f_i^{\,k+1} \tag{3.32}$$

so

$$\lambda \sum_{j=0}^{k} \left(u(x_i;t_{k-j+1}) - u(x_i;t_{k-j}) \right) d_j - \lambda \sum_{j=0}^{k} \left(\epsilon_i^{k-j+1} - \epsilon_i^{k-j} \right) d_j$$

$$-G_i^{k+1} \left(\frac{u(x_{i+1};t_{k+1}) - u(x_i;t_{k+1})}{h} \right) + G_i^{k+1} \left(\frac{\epsilon_{i+1}^{k+1} - \epsilon_i^{k+1}}{h} \right)$$

$$-\frac{a_i^{k+1}}{h^2} \left(u(x_{i-1};t_{k+1}) - 2u(x_i;t_{k+1}) + u(x_{i+1};t_{k+1}) \right)$$

$$+\frac{a_i^{k+1}}{h^2} \left(\epsilon_{i-1}^{k+1} - 2\epsilon_i^{k+1} + \epsilon_{i+1}^{k+1} \right) \quad = f_i^{\,k+1} \tag{3.33}$$

then

$$-\lambda \sum_{j=0}^{k} \left(\epsilon_i^{k-j+1} - \epsilon_i^{k-j}\right) d_j + G_i^{k+1} \left(\frac{\epsilon_{i+1}^{k+1} - \epsilon_i^{k+1}}{h}\right) + \frac{a_i^{k+1}}{h^2} \left(\epsilon_{i-1}^{k+1} - 2\epsilon_i^{k+1} + \epsilon_{i+1}^{k+1}\right) \quad = O(h_t + h) \tag{3.34}$$

so

$$D_i^{k+1}\epsilon_{i-1}^{k+1} + A_i^{k+1}\epsilon_i^{k+1} + \ B_i^{k+1}\epsilon_{i+1}^{k+1}$$

$$= \lambda \left(1 - d_1\right) \epsilon_i^k - \lambda \sum_{m=1}^{k-1} \left(d_{m+1} - d_m\right) \epsilon_i^{k-m} + \lambda d_k \epsilon_i^0 + O(h_t + h) \tag{3.35}$$

Taking

$$\left|\epsilon_l^k\right| = \left\|\epsilon^k\right\| = \underset{1 \,\le\, i \,\le\, N-1}{\mathrm{Max}} \left|\epsilon_i^k\right| ; \epsilon^k = \left(\epsilon_1^k, , ..., \epsilon_{N-1}^k\right)^T ; \ \ \left\|\epsilon_i^0\right\| = 0$$

For $k = 0$ we get

$$D_i^1 \ \epsilon_{i-1}^1 + A_i^1 \epsilon_i^1 + \ B_i^1 \epsilon_{i+1}^1 = \lambda \left(2 - d_1\right) \epsilon_i^0 \ + O(h + h_t) \quad \text{with} \quad i = \overline{1, N-1} \tag{3.36}$$

we have

$$\begin{aligned} \lambda \left\|\epsilon^1\right\| \ &= \ \left(D_i^1 + A_i^1 + B_i^1\right) \left\|\epsilon^1\right\| \\ &= \ \left(D_i^1 \left\|\epsilon^1\right\| + A_i^1 \left\|\epsilon^1\right\| + B_i^1 \left\|\epsilon^1\right\|\right) \\ &\le \ \left(A_i^1 \left\|\epsilon^1\right\| + B_i^1 \left|e_{i+1}^1\right| + D_i^1 \left|e_{i-1}^1\right|\right) \\ &\le \ \underset{1 \,\le\, i \,\le\, N-1}{\mathrm{Max}} \left|D_i^1 e_{i-1}^1 + A_i^1 e_i^1 + B_i^1 e_{i+1}^1\right| = \gamma \left(2 - d_1\right) \left\|\epsilon^0\right\| + O(h + h_t) \end{aligned}$$

so

$$\left\|\epsilon^1\right\| \le O(h + h_t) \tag{3.37}$$

For $k \ge 1$ we use (3.35)

We assume : $\left|\epsilon_l^j\right| \leq O(h+h_t);\ j=\overline{1,k}$

$$\begin{aligned}
\lambda \left\|\epsilon^{k+1}\right\| &= \left(D_i^{k+1}+A_i^{k+1}+\ B_i^{k+1}\right)\left\|\epsilon^{k+1}\right\| \\
&= \left(D_i^{1}\left\|\epsilon^{k+1}\right\|+A_i^{1}\left\|\epsilon^{k+1}\right\|+B_i^{1}\left\|\epsilon^{k+1}\right\|\right) \\
&\leq A_i^1\left\|\epsilon^1\right\|+B_i^1\left|e_{i+1}^1\right|+D_i^1\left|e_{i-1}^1\right| \\
&\leq \underset{1\,\leq\, i\,\leq\, N-1}{\text{Max}}\left|\lambda\left(1-d_1\right)\epsilon_i^k-\lambda\sum_{m=1}^{k-1}\left(d_{m+1}-d_m\right)\epsilon_i^{k-m}+\lambda d_k\epsilon_i^0+O(h_t+h)\right| \\
&\leq \lambda\left(1-d_1\right)\left\|\epsilon_i^k\right\|+\left\|\lambda\sum_{m=1}^{k-1}\left(d_{m+1}-d_m\right)\epsilon_i^{k-m}\right\|+\left\|\lambda d_k\epsilon_i^0\right\|+O(h_t+h) \\
&\leq \lambda\left(1-d_1\right)O(h_t+h)+\lambda O(h_t+h)\sum_{m=1}^{k-1}\left(d_{m+1}-d_m\right)+O(h_t+h) \\
&\leq \lambda\left(2-2d_1+d_k\right)O(h_t+h)
\end{aligned}$$

so

$$\left\|\epsilon^{k+1}\right\| \leq CO(h_t+h);\ C\leq 2 \tag{3.38}$$

therfore the method is convergent with order $O(h+h_t)$.

3.1.4 Application

In this section, we give some numerical investigation tests.

3.1.4.1 Example

We consider a problem $(3.1-3.3)$ with

$$\alpha=\frac{1}{2},\quad a(x,t)=\frac{1}{2}\left(x+t\right)\quad,\quad f(x,t)=\frac{2t^{2-\alpha}}{\Gamma\left(3-\alpha\right)}\left(x+1\right)^2-\left(t^2+1\right)\left(2x+t+1\right),$$

$$\phi(x)=(x+1)^2,\quad \mu(t)=2(t^2+1),\quad m(t)=\frac{7}{3}(t^2+1).$$

The analytical solution is given by

$$v(x,t)=(x+1)^2(t^2+1).$$

The approximate solution is $u(x,t)$, **A.E** is the absolute error.

h	$u^1(x,t)$	$v^1(x,t)$	**A.E**
0.1	1.2263	1.2100	$1.6328e-02$
0.2	1.5374	1.4400	$9.7355e-02$
0.3	1.7970	1.6900	$1.0701e-01$
0.4	2.0675	1.9600	$1.0750e-01$
0.5	2.3535	2.2500	$1.0350e-01$
0.6	2.6552	2.5600	$9.5177e-02$
0.7	2.9755	2.8900	$8.5472e-02$
0.8	3.3578	3.2400	$1.1783e-01$
0.9	4.3233	3.6100	$7.1325e-01$

Table1. $h=0.1; h_t=0.001, \alpha=0.9$

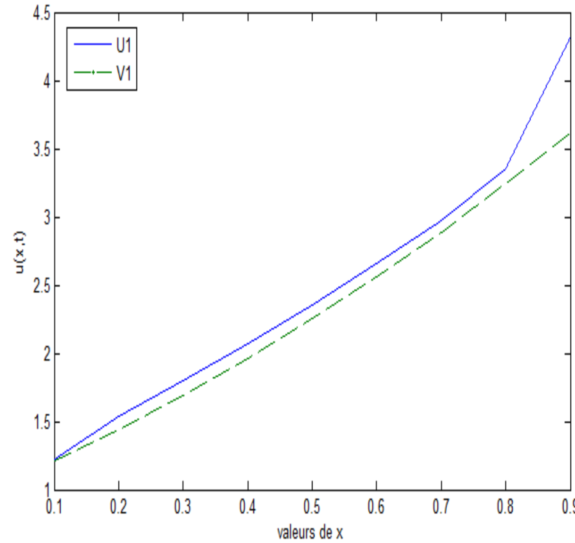


Fig.1 $\alpha=0.9$, h_t=0.001

h	$u^1(x,t)$	$v^1(x,t)$	**A.E**
0.1	1.2118	1.2100	$1.7838e-03$
0.2	1.4522	1.4400	$1.2159e-02$
0.3	1.7028	1.6900	$1.2778e-02$
0.4	1.9729	1.9600	$1.2885e-02$
0.5	2.2625	2.2500	$1.2509e-02$
0.6	2.5716	2.5600	$1.1581e-02$
0.7	2.9000	2.8900	$1.0044e-02$
0.8	3.2492	3.2400	$9.2016e-03$
0.9	3.7357	3.6100	$1.2570e-01$

Table 2. $h=0.1; h_t=0.0001$

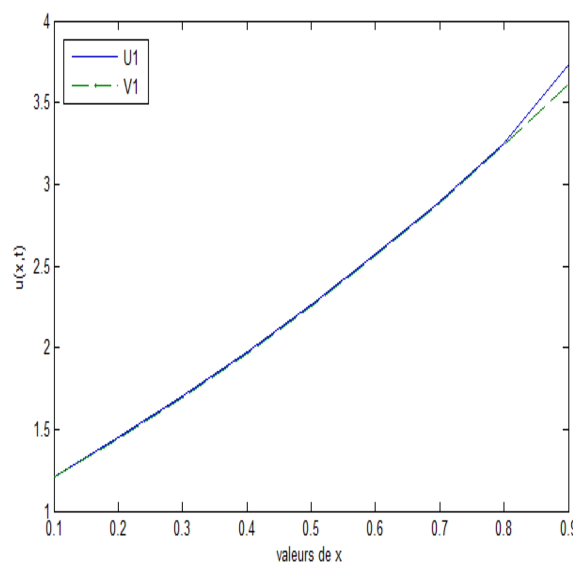


Fig.2 $\alpha=0.9$, h_t=0.0001

h	$u^1(x,t)$	$v^1(x,t)$	**A.E**
0.1	1.2102	1.2100	$2.2024e-04$
0.2	1.4415	1.4400	$1.5299e-03$
0.3	1.6916	1.6900	$1.5976e-03$
0.4	1.9616	1.9600	$1.6124e-03$
0.5	2.2516	2.2500	$1.5670e-03$
0.6	2.5615	2.5600	$1.4524e-03$
0.7	2.8913	2.8900	$1.2595e-03$
0.8	3.2410	3.2400	$1.0033e-03$
0.9	3.6266	3.6100	$1.6585e-02$

Table 3. $h = 0.1; h_t = 0.00001$

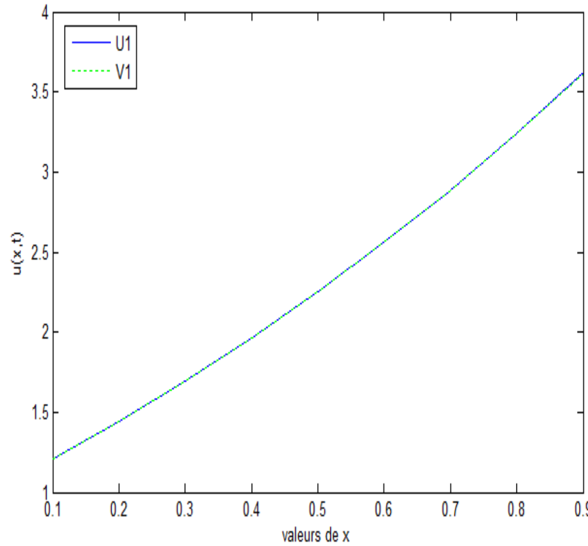


Fig.3 $\alpha = 0.9$, h_t=0.00001

We see in Figures 1, 2 and 3 that the absolute error **A.E** decreases when the step h_t takes small values very close to zero. that is, for $h_t = 0.001, 0.0001, 0.00001$ A.E decreases towards zero and the approximate solution tends towards the exact solution with convergence order of $O(h + h_t)$.

h	$u^1(x,t)$	$v^1(x,t)$	**A.E**
0.1	1.3570	1.3373	0.0198
0.2	1.4940	1.4779	0.0161
0.3	1.6462	1.6333	0.0129
0.4	1.8142	1.8051	0.0091
0.5	1.9992	1.9950	0.0042
0.6	2.2027	2.2048	0.0021
0.7	2.4264	2.4366	0.0102
0.8	2.6722	2.6929	0.0207
0.9	2.9421	2.9761	0.0340

Table 4. $h = 0.1; \ h_t = 0.1$

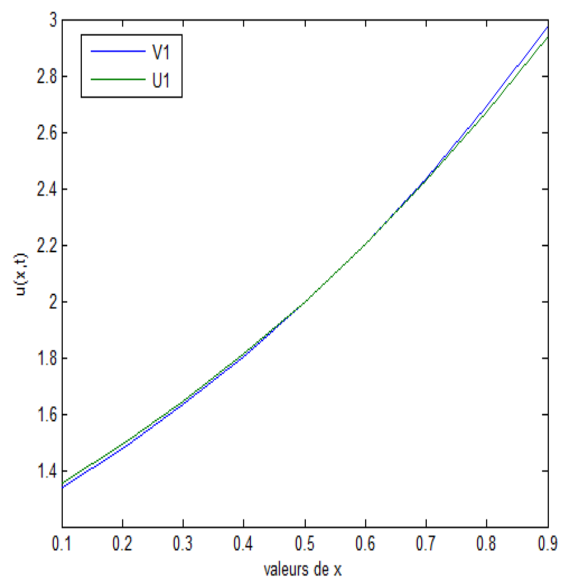


Fig.4 $\alpha = 0.8$, h_t=0.1

h	$u^1(x,t)$	$v^1(x,t)$	**A.E**
0.1	1.1298	1.1274	0.0025
0.2	1.2478	1.2460	0.0018
0.3	1.3787	1.3770	0.0017
0.4	1.5235	1.5218	0.0017
0.5	1.6833	1.6819	0.0014
0.6	1.8594	1.8587	0.0006
0.7	2.0532	2.0542	0.0010
0.8	2.2658	2.2703	0.0045
0.9	2.4980	2.5090	0.0111

Table 5. $h = 0.1; h_t = 0.01$

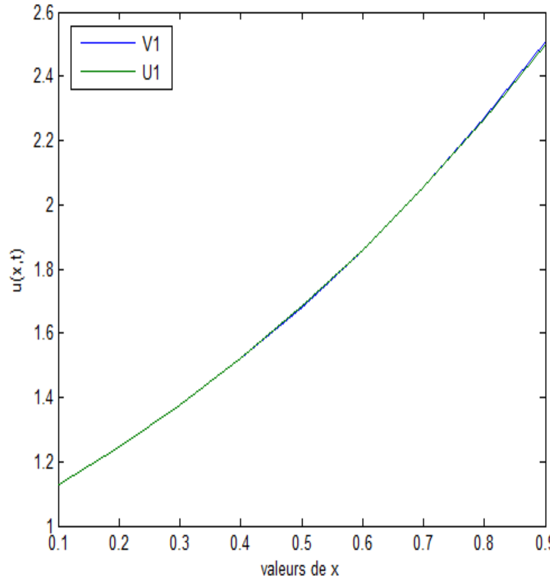


Fig.5 $\alpha = 0.8$, h_t=0.01

h	$u^1(x,t)$	$v^1(x,t)$	**A.E**
0.1	1.1078	1.1074	0.0004
0.2	1.2241	1.2238	0.0002
0.3	1.3528	1.3526	0.0003
0.4	1.4951	1.4948	0.0003
0.5	1.6523	1.6520	0.0003
0.6	1.8261	1.8258	0.0003
0.7	2.0179	2.0178	0.0001
0.8	2.2292	2.2300	0.0008
0.9	2.4591	2.4645	0.0054

Table 6. $h = 0.1; h_t = 0.001$

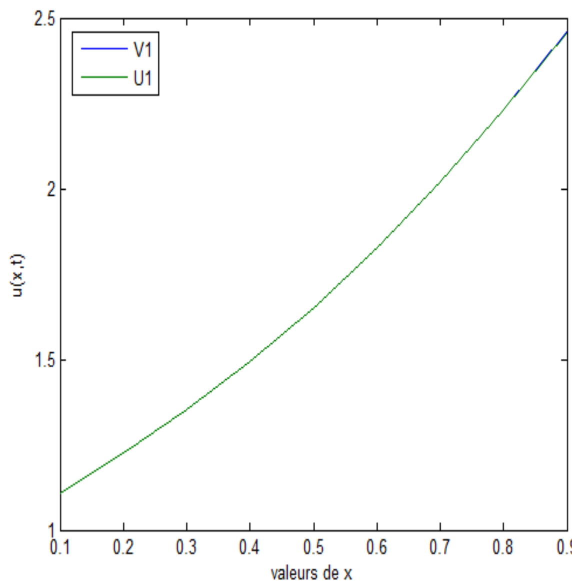


Fig.6 $\alpha = 0.8$, h_t=0.001

Figures and tables 4, 5 and 6 show that the absolute error **A.E** decreases when the step h_t takes small values very close to zero with convergence order of $O(h + h_t)$.

Table 7 shows the error norm $\left\|E^k\right\|_\infty$ for defferent value of α

Table B, $h = 0.1$

h_t		10^{-3}	10^{-5}	10^{-7}
$\cdot\ \|E^1\|_\infty$ for	$\alpha = 0.4$	$1.41e-02$	$7.8e-03$	$3.1e-03$
	$\alpha = 0.6$	$9.1e-03$	$2.2e-03$	$1.7857e-04$
	$\alpha = 0.8$	$5.4e-03$	$2.8938e-04$	$7.4928e-06$
	$\alpha = 0.9$	$3.8e-03$	$9.6813e-05$	$1.5500e-07$

We see in the table B, for the space step $h = 0.1$, and for the defferent values of α, the error norm tends to zeros when the time step h_t takes values close to zeros.

Conclusion and Outlook

In this thesis, we study a problems with fractional derivatives with **boundary conditions of integral types**. The study concerns a Caputo-type fractional parabolic equation where the fractional order derivative α with respect to time with $(1 < \alpha < 2$ and $0 < \alpha < 1)$. The numerical study of problems based on the finite difference method with a numerical integration method . Applications on certain examples clearly show that the numerical results obtained are very satisfactory, where we see the approximate solution u tends towards the exact solution for different values of α.

As further research directions and perspectiv, we aim at studying the same problems using the compact finite difference method in order to obtain more precise results, change the sense of fractional derivatives and/or extend the study to time–space fractional derivatives, systems of **FPDE** and Non-linear Problems.

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ملخص

في هذه الرسالة ، نهتم بدراسة مسائل المشتقات الجزئية الكسرية بشروط حدية غير متجانسة من نوع $\int_0^1 u(x,t)\,dx$ و $\int_0^1 \varphi(x)u(x,t)\,dx$. يتم إثبات وجود و وحدانية الحل للمسائل المقترحة باستخدام طريقة المتراجحات الطاقوية المعروفة بالتقديرالمسبق، وتستند الطريقة إلى كثافة المؤثر المتولد عن المسألة المقترحة. يتم إجراء الدراسة العددية للمشكلات المتعلقة بهذه الأنواع الجديدة من الشروط الحدية باستخدام مزيج من طريقة الفروق المحدودة والتكامل العددي. وأخيراً نجري بعض الاختبارات العددية لتوضيح النتائج التي تم الحصول عليها.

الكلمات المفتاحية :

المشتقات الكسرية-المشتقة بمفهوم كابيتو- معادلة الانتشار الحراري الكسري- طريقة المتراجحات الطاقوية - طريقة الفروق المنتهية- الشروط الحدية التكاملية.

Abstract

In this thesis, we are interested in the study of Fractional Partial Derivative Problems with non-homogeneous boundary conditions of integral types $\int_0^1 u(x,t)\,dx$ and $\int_0^1 \varphi(x)u(x,t)\,dx$. The existence and uniqueness of the given problem solution is proved using the method of the energy inequalities known as the a priori estimate method relying on the range density of the operator generated by the considered problem. The numerical study of problems with these new types of boundary conditions is carried out using a combination of the finite difference method and numerical integration. Finally, we give some numerical tests to illustrate the usefulness of the obtained results.

Keywords:

Fractional derivatives; Caputo derivative; fractional advection diffusion equation; energy inequalities; finite difference schemes; integrals conditions.

Résumé

Dans cette thèse, nous nous intéressons à l'étude de problèmes aux dérivées partielles fractionnaires avec conditions aux limites non homogènes de types intégrales $\int_0^1 u(x,t)\,dx$ et $\int_0^1 \varphi(x)u(x,t)\,dx$. L'existence et l'unicité de la solution des problèmes donnés sont prouvées à l'aide de la méthode des inégalités énergétique connue par l'estimation a priori, cette méthode se base sur la densité de l'opérateur généré par le problème considéré. L'étude numérique des problèmes avec ces nouveaux types de conditions aux limites est effectuée en utilisant une combinaison de la méthode des différences finies et de l'intégration numérique. Enfin, nous donnons quelques tests numériques pour illustrer des résultats obtenus.

Mots clés:

Dérivées fractionnaires; Dérivée au sens de Caputo; Equation de diffusion- advection fractionnaire; Inégalités d'énergie; schémas de différences finies; conditions intégrales.