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Solvability of Evolution Problem With Integral Condition By Energy Inequality Method

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Dedication

I dedicate my dissertation work to :

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To all the teachers of the Department of Mathematics.

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Contents

Notations

$\mathbb{N}$	: Set of natural numbers.
$\mathbb{R}$	: Set of real numbers.
$\parallel v \parallel$	: Norm in E.
$(.,.)_E$	: The inner product in E.
L	: Operator.
D (L)	: The domaine of L.
R (L)	: The range of L.
G (L)	: The graph of L.
$\overline{L}$	: Closed Linear operator.
Ω	: A bounded and regular domain of $\mathbb{R}$.
$D(\Omega)$	: A rectangular domain of $\mathbb{R}$.
$L^2(\Omega)$	: Space square-integrable functions v on Ω.
$\frac{\partial v}{\partial t}; v_t$	: The first partial derivative of v with respect to t.
$\frac{\partial v}{\partial x}; v_x$	: The first partial derivative of v with respect to x.
${}^C\partial_t^\alpha v$	: Caputo's fractional derivative of order α.
D^α	: Riemann-Liouville's fractional integral of order α.
$\Gamma(.)$	: Euler gamma function.
$\mathcal{M}v$	: Multiplicator of a function v.
$C(\Omega)$	: The space of all continuous functions from Ω into R.
$\Im_x^n$	: n^{th} Primitive of function v.
C_i	: Constant independent of v.

Introduction

Fractional-order differential equations have been used to study models of many phenomena in various fields of science and engineering, such as viscoelasticity, fluid mechanics, electrochemistry, control, porous media, mathematical biology, diffusive transport akin to diffusion, probability, statistics and electromagnetic bioengineering. In addition, recent investigations have shown that sometimes physical systems can be modelled more accurately using fractional derivative formulations. Their application has occupied various scientific fields, such as optimal control theory, chemistry, physics, mathematics, biology, finance, engineering, etc. [3, 21, 26, 27, 28, 29, 31]. The classical fractional calculus is based on several definitions for integration operators and the differentiation of arbitrary order [1, 40]. Among the various definitions of fractional differentiation, the Riemann–Liouville and Caputo fractional derivatives are widely used in the literature.

Integrodifferential equations are a combination of derivatives and integrals. They have attracted many researchers and scientists to their applications in many areas [9, 11]. Many mathematical formulations of physical phenomena include integrodifferential equations, which may arise in biological models, fluid dynamics. [7, 8, 12, 20].

Boundary conditions of integral type can be used when it is impossible to measure the desired quantity directly on the border where its total or average value is known. More precisely, the standard conditions (Dirichlet, Neumann, etc.) prescribed punctually are not always adequate because they hang from the physical context where the data can be measured at the domain's boundary studied. In some cases, it is not possible to prescribe the solution (pressure, temperature) punctually at the border because the solution's value can be measured along the border or on the part of this one. The fundamental reason for interest in this kind of problem is the basic physical meaning of the integral conditions (total energy, average temperature, the total mass of impurities, to total, moments, etc.). It is important to establish effective methods to solve FDEs. Recently, a great deal of attention was dedicated to FDEs solutions utilizing different methods: The Adomian decomposition method [5, 39, 43], the Laplace transform method [6, 23], exponential differential operators [1], the F-expansion method [42], Reproducing kernel space method [18, 22],...The search for exact or analytical solutions of the FDEs required much mathematical effort and remained problems. The applicability of most techniques becomes difficult with the presence of integral condition. The Energy Inequality method is a useful tool for studying nonlocal fractional and classical problems. Compared with other techniques, this one has an essential role in establishing the solution's existence and uniqueness proof. It depends on density arguments and certain a priori bounds.

A few works related to nonlinear fractional partial equations employing the mentioned method, citing, for example [25, 34]. Furthermore, for partial differential equations with classical order, many results deal with this method [15, 30, 32, 38, 41]. The aim of this thesis is to prove the well posedness of some posed nonlinear mixed problems by using the functional analysis method, called also the energy inequality method. Moreover, we demonstrate the solution's uniqueness, existence, and dependence on the given data [36] .

To investigate the posed problem, we will follow the underlined scheme: We first write the associated linear problem in the operator form

$$Lv = \mathcal{F}, \tag{1}$$

The operator L is considered from a Banach space E into a Hilbert space F, which are conveniently chosen. Then, we establish a priori estimate (energy inequality) for the operator L of the form:

$$\| v \|_E \leq C \| Lv \|_F, \quad \forall v \in D(L), \tag{2}$$

where $D(L)$ is the domain of definition of the operator L .

The estimate (2) can be obtained by choosing a certain multiplier $\mathcal{M}v$ which is in general an integro-differential operator and using appropriate integrations by parts. We should mention here that up to now there is no systematic way for constructing such multiplier operators $\mathcal{M}v$. We then show that the operator L is closable, and we denote by $\overline{L}$ its closure. We define a strong solution of the considered problem as the solution of the operational equation :

$$\overline{L}v = \mathcal{F}, \quad \forall v \in D(\overline{L}).$$

We extend estimate (2) to the set $v \in D(\overline{L})$ of solutions (strong solutions) by passing to the limit; we then have the priori estimate

$$\| v \|_E \leq C \| \overline{L}v \|_F, \quad \forall v \in D(\overline{L}). \tag{3}$$

Thus we establish the uniqueness of a strong solution and closedness of the range $R(L)$ of the operator $\overline{L}$ in the space F and $R(\overline{L}) = \overline{R(L)}$. Then, we prove the density of the range $R(L)$ of the operator L in F, hence the existence of a strong solution of the associated linear problem.

Finally, by applying an iterative process based on the results obtained for the linear problem, we establish the existence, the uniqueness of the weak solution of the nonlinear problem.

This thesis is outlined as follows:

In the first chapter, we give the necessary tools and some notions on the theory of the used function spaces and the theory of operators and some important inequalities.

The second chapter proves the existence and uniqueness of solution of a linear fractional Integro-Differential Equation with integral and a Neumann conditions. After writing this latter problem on its operator form, we establish an a priori bound from which we deduce the uniqueness of the strong solution. Furthermore, for the solvability, we prove that the range of the operator generated by the considered problem is dense.

The last chapter is preserved to study a nonlinear nonlocal fractional Integro-Differential Equation. First, we establish the well-posedness of the posed problem. Then, based on the obtained results of the linear problem in the precedent chapter, we apply an iterative process to prove the existence and uniqueness of solutions to the nonlinear problem.

Finally, we give a conclusion and a complete bibliography mainly on the treated subject and related ones.

Chapter 1

Basic Concepts and Preliminaries

This chapter will be devoted to the basic definitions, such as fractional derivation, fractional integration, fractional-order operator definitions and other notions that we will need in the rest of our work. Also we give some main theorems in functional analysis such as unbounded operator. Also we give some basic properties of Lebesgue and Sobolav spaces . We refer to [2, 10, 13, 16, 19, 24, 35, 37].

1.1 Functional spaces

1.1.1 Normed space

Definition 1.1 (Norm) *[19] A norm on a linear space E is a real-valued function, whose value at x is denoted by $\| x \|$ and has the properties:*

1. $\| x_1 + x_2 \| \leq \| x_1 \| + \| x_2 \|$,

2. $\| \alpha x \| = |\alpha| \| x \|$, *$\alpha$ is scalar,*

3. $\| x \| \geq 0$,

4. $\| x \| \neq 0$ *if* $x \neq 0$.

Definition 1.2 (Normed linear space) *[19] A linear space on which a norm is defined is called a normed linear space.*

Definition 1.3 (Complete space) *[19] The normed space E is said to be complete if every Cauchy sequence in E converges to an element of E.*

1.1.2 Banach space

Definition 1.4 (Banach space) *[19] If a normed linear space E is complete, it is called a Banach space.*

Theorem 1.1 *[19] Let E be a normed linear space. Then there exists a normed linear space F such that F is complete and E is a dense subset of F . Up to isometry, the space F is unique.*

1.1.3 Hilbert space

Definition 1.5 (Inner-product space) *[16] A complex linear space E is called an inner-product space if to each pair of elements x, y of E there is associated a complex number (x,y) (called the inner product of x and y) with the following properties:*

1. $(x+y,z) = (x,z) + (y,z),$ *for* $x, y, z \in E$,
2. $(x,y) = \overline{(y,x)}$ *[the bar denotes complex conjugate],*
3. $(\alpha x, y) = \alpha(x,y)$, *for all scalars,*
4. $(x,x) \geq 0$ *[it must be real by 2], and* $(x,x) \neq 0$ *if* $x \neq 0$.

Inner-product spaces are special cases of normed linear spaces. This is expressed by the following lemma.

Lemma 1.1 *[16] Let E be a linear space with inner product (.,.). Then the expression*

$$\| x \| = \sqrt{(x,x)}, \quad x \in E, \tag{1.1}$$

defines a norm on E.

Definition 1.6 (Hilbert space) *[16] A Hilbert space is an inner product space which (as a normed linear space) is complete.*

1.1.4 Lebesgue space

Definition 1.7 *Let (X, ν) be a measure space. Given a complex function f, we say $f \in L^2(X)$ on X if f is (Lebesgue) measurable and if*

$$\int_X |\, f \,|^2 \, d\nu < +\infty$$

Then the function f is also said to be square-integrable. In other words, is the set of square-integrable functions. For $f \in L^2(X)$ define

$$\| \, f \, \| = \left(\int_X |\, f \,|^2 \right)^{\frac{1}{2}}$$

We call $\| \, f \, \|$ the $L^2(X)$ norm of f.

In addition, by defining the inner product for $L^2(X)$ of two functions f and g on a measure space X with

$$(f, g) = \int_X f\overline{g} d\nu$$

$L^2(X)$ becomes a Hilbert space.

1.1.5 Sobolev space of first order

Let $\Omega \subset \mathbb{R}^d$ and let v be a function of $L^2(\Omega)$.

Definition 1.8 *We call Sobolev space of first order on Ω the space*

$$H^1(\Omega) = \left\{ v \in L^2(\Omega), \; \partial_{x_i} v \in L^2(\Omega), 1 \leq i \leq d \right\}.$$

The space $H^1(\Omega)$ is endowed with the norm associated to the inner product:

$$(u, v)_{H^1(\Omega)} = \left(\int_\Omega uv + \int_\Omega \sum_{i=1}^{d} \partial_{x_i} u \partial_{x_i} v \right) dx,$$

and we note the corresponding norm:

$$\| v \|_{H^1(\Omega)} = \left(\int_\Omega | u |^2 + \int_\Omega | \nabla u |^2 \right)^{\frac{1}{2}}.$$

Theorem 1.2 *$H^1(\Omega)$ is a Hilbert space.*

1.2 Unbounded linear operators

Definition 1.9 (Linear Operator) *[13] Let E and F be two linear spaces (both real or both complex). Let L be a function with domain D(L) in E and range R(L) in F: Then L is called a linear operator if D(L) is a subspace of E and if :*

1. $L(x_1 + x_2) = Lx_1 + Lx_2$ *and*
2. $L(\alpha x) = \alpha Lx$,

whenever α is a scalar and x_1, x_2 and x are vectors in D(L). If D(L) = X, we often say that L is a linear operator from E into F.

Definition 1.10 *[13] The graph of a linear operator $L : E \longrightarrow F$ is the set of ordered pairs*

$$G(L) = \{(v, Lv), \quad v \in D(L)\} \subset E \times F.$$

Note that the graph is a subspace of $E \times F$.

Lemma 1.2 *[13] The operator $\overline{L}$ is an extension of L if and only if*

$$G(L) \subset G(\overline{L}).$$

Definition 1.11 *[13] We say that an operator L is closed if its graph is closed as a subset of $E \times F$. We call L closable if it has a closed extension. Every closable operator has a smallest closed extension which we call its closure and denoted by L.*

Lemma 1.3 *[13] An operator L is closed if and only if it has the following property. Whenever there is a sequence $x_n \in D(L)$ such that*

1. $x_n \longrightarrow x$ *and*

2. $Lx_n \longrightarrow f$*,*

then

1. $x \in D(L)$ *and*

2. $Lx = f$*.*

We have a similar characterization of a closable operator.

Lemma 1.4 *[13] An operator L is closable if for every sequence* $x_n \in D(L)$ *such that* $x_n \longrightarrow 0$ *we have either*

1. $Lx_n \longrightarrow 0$ *or*

2. $\lim Lx_n$ *does not exist.*

Corollary 1.1 *[13] If L is closable, then* $G(\overline{L}) = \overline{G(L)}$*.*

1.3 Orthogonality and density in Hilbert spaces

Definition 1.12 *[13] Let F be a Hilbert space. We say that two elements of F, x and y are orthogonal if (x, y) = 0. For any subspace M of F, we define the orthogonal complement by*

$$M^{\perp} = \{x \in F/(x, y) = 0, \quad \forall y \in M\}$$

It is clear that M is a closed subspace. If $M^{\perp}$ is also closed, then F is the direct sum of M and $M^{\perp}$: $F = M \oplus M^{\perp}$.

Theorem 1.3 *Let F be a Hilbert space. A subspace M of F is dense if and only if* $M^{\perp} = \{0\}$*.*

Proof First, let's assume that M is dense in F. Let be $f \in M^{\perp} \subset F$, and let $\{f_n\}_{n\in\mathbb{N}}$ sequence of elements of M which converges to f. Since $\langle f,\ f_n\rangle_F = 0$ for every $n \in \mathbb{N}$. By passing to the limit, we conclude that $\|f\|_F = 0$. Then, $f = 0$, which gives $M^{\perp} = \{0\}$.

Contrariwise, suppose that $M^{\perp} = \{0\}$. Then, we have $\left(M^{\perp}\right)^{\perp} = \{0\}^{\perp} = F$, and since $M \subset \overline{M}$, it follows that $\left(\overline{M}\right)^{\perp} \subset M^{\perp}$, and then $\left(M^{\perp}\right)^{\perp} \subset \left(\left(\overline{M}\right)^{\perp}\right)^{\perp}$, but $\overline{M}$ is close, then $\left(\left(\overline{M}\right)^{\perp}\right)^{\perp} = \overline{M}$, it comes $\left(M^{\perp}\right)^{\perp} \subset \overline{M} \Rightarrow F \subset \overline{M}$. Finally, $F = \overline{M}$. ■

1.4 Fractional calculs

Riemann-Liouville type fractional derivation has played an essential role in developing the theory of derivatives and fractional integrals with a view of their applications in pure mathematics (solutions of differential equations of integer order, the definition of new classes of functions, summation of series). However, modern technology requires some revision of the well-known pure mathematics' approach. As a result, many works have appeared, especially in the viscoelastic theory of solid mechanics, where the fractional derivatives are used for a good description of the properties of materials. Mathematical modelling based on rheological models leads naturally to differential equations of fractional order, hence the need to formulate initial conditions of such equations. Applied problems require definitions of fractional derivatives allowing the use of initial conditions physically interpretable, which contain $f(\alpha), f^{(1)}(\alpha)$,...etc. Although initial value problems with initial conditions can be solved mathematically. Their solutions were proposed by M. Caputo (in the sixties) in its definition. He adapted with Mainardi the structure of the theory of viscoelasticity, so we introduce a fractional derivative that is more restrictive than Riemann-Liouville.

This section will present the fractional integration operators and the most used definitions of fractional derivatives and give the most important properties of these notions. We will begin by providing some special functions and functional spaces.

1.4.1 Special functions of the fractional calculus

One of the essential functions of the fractional calculus is Euler gamma function $\Gamma(z)$, which generalizes the factorial $n!$ and allows n also to take non-integer values.

Definition 1.13 *[14]. The Gamma function $\Gamma(z)$ is defined by the integral*

$$\Gamma\left(z\right) = \int_{0}^{+\infty} e^{-t} t^{z-1} dt,$$

where z is an arbitrarily complex number such that $Re(z) > 0$.

We also indicate some properties of the Euler Gamma function:

1. We have $\Gamma(1) = 1$, $\Gamma(0^{+}) = +\infty$. The Euler Gamma function $\Gamma(\alpha)$ is a monotonous and strictly decreasing for $0 < \alpha \leq 1$.
2. One of the basic properties of the Gamma function is that it satisfies the following function equation $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, $Re(\alpha) > 0$.

Definition 1.14 (Mittag-Leffler function) *[37] A function of the Mittag-Leffler type is defined by the series expansion*

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(1 + \alpha k)}, \qquad (\alpha \in \mathbb{C},\ R(\alpha) > 0,\ z \in \mathbb{C}).$$

and its general form is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\beta + \alpha k)}, \qquad (\alpha, \beta \in \mathbb{C},\ R(\alpha) > 0, R(\beta) > 0,\ z \in \mathbb{C}).$$

1.4.2 Caputo's fractional derivative

Definition 1.15 *[10] Let $\alpha > 0$, and let the function S, which is integrable on $(0,\ T)$, we have*

1. *The left Caputo derivative is given by*

$$ {}_{0}^{C}\partial_{t}^{-\alpha} S(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} \frac{S^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau$$

2. *The right Caputo derivative is defined as*

$$ {}_{T}^{C}\partial_{t}^{-\alpha} S(t) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{t}^{T} \frac{S^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau$$

with n is a positive integer satisfying the inequality: $n-1<\alpha<n$.

1.4.3 The Riemann–Liouville integral

Definition 1.16 *[14] The Riemann-Liouville integral of order $\alpha>0$, for an integrable function S, is defined by*

$$D_t^{-\alpha}S(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\frac{S(\tau)}{(t-\tau)^{1-\alpha}}d\tau.$$

1.4.4 Fractional integration by parts

If $u\in L^p(\Omega)$ and $v\in L^q(\Omega)$ either with $\alpha\geq 1$, $p=q=1$, or with $0<\alpha<1$, $\frac{1}{p}+\frac{1}{q}\leq 1+\alpha$, $p,q>1$. Then the fractional integration by parts by defined by the formula [10] :

$$\int_0^T u(t)(D_t^{-\alpha}v)(t)dt=\int_0^T v(t)(D_t^{-\alpha}u)(t)dt. \tag{1.2}$$

1.5 Important inequalities

We will give here some important integral inequalities. These inequalities play an important role in applied mathematics, and also, they are very useful in our next chapters.

1.5.1 Fractional inequalities

Lemma 1.5 *[2] Let $S(t)$ a nonnegative absolutely continuous function verify the inequality*

$${}_0^C\partial_t^\alpha S(t)\leq c_1S(t)+c_2(t),\quad 0<\alpha<1, \tag{1.3}$$

for almost all $t\in[0,T]$, where c_1 is a positive constant and $c_2(t)$ is an integrable nonnegative function on $[0,T]$. Then,

$$S(t)\leq S(0)E_\alpha(c_1t^\alpha)+\Gamma(\alpha)E_{\alpha,\alpha}(c_1t^\alpha)D_t^{-\alpha}c_2(t), \tag{1.4}$$

where

$$E_{\alpha}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma\left(\alpha n+1\right)}\quad \textit{and}\quad E_{\alpha,\nu}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma\left(\alpha n+\nu\right)},$$

are Mittag-Leffler functions.

Proof Let ${}_0^C\partial_t^{\alpha}S(t)-c_1S(t)=g(t)$; then

$$y(t)=y(0)E_{\alpha}(c_1t^{\alpha})+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(c_1(t-\tau)t^{\alpha})g(\tau)d\tau. \tag{1.5}$$

By virtue of the inequality $g(t)\leq c_2(t)$, the positivity of the Mittag-Leffler function $E_{\alpha,\alpha}(c_1(t-\tau)t^{\alpha})$ for given parameters, and the growth of the function $E_{\alpha,\alpha}(t)$, from (1.5), we obtain the inequality

$$y(t)\leq y(0)E_{\alpha}(c_1t^{\alpha})+\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(c_1(t-\tau)t^{\alpha})c_2(\tau)d\tau.$$

$$\leq y(0)E_{\alpha}(c_1t^{\alpha})+\Gamma(\alpha)^{\alpha-1}E_{\alpha,\alpha}(c_1t^{\alpha})D_t^{-\alpha}c_2(t).$$

The proof of the lemma is complete. ■

Lemma 1.6 *[2] On the interval* $[0,T]$*, any absolutely continuous function* $S(t)$ *verifies the following estimate:*

$$S\left(t\right){}^{C}\partial_t^{\beta}S(t)\geq\frac{1}{2}{}^{C}\partial_t^{\beta}S^2\left(t\right),\quad 0<\beta<1, \tag{1.6}$$

Proof Let us rewrite inequality (1.6) in the form

$$\begin{aligned}S\left(t\right){}^{C}\partial_t^{\beta}S(t)-\frac{1}{2}{}^{C}\partial_t^{\beta}S^2\left(t\right) &= \frac{1}{\Gamma(1-\alpha)}S\left(t\right)\int_0^t\frac{S_{\tau}(\tau)}{(t-\tau)^{\alpha}}d\tau-\frac{1}{2\Gamma(1-\alpha)}\int_0^t\frac{2S(\tau)S_{\tau}(\tau)}{(t-\tau)^{\alpha}}d\tau\\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{S_{\tau}(\tau)\left(S\left(t\right)-S\left(\tau\right)\right)}{(t-\tau)^{\alpha}}d\tau\\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{S_{\tau}(\tau)}{(t-\tau)^{\alpha}}\int_{\tau}^tS(\nu)d\nu\\ &= \frac{1}{\Gamma(1-\alpha)}\int_{\tau}^tS(\nu)d\nu\int_0^{\nu}\frac{S_{\tau}(\tau)}{(t-\tau)^{\alpha}}=I\geq 0\end{aligned}$$

Therefore, to prove the lemma, it suffices to show that the integral I is nonnegative. The integral

I takes nonnegative values, since

$$I = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\nu)^{\alpha}\frac{S_\nu(\nu)}{(t-\nu)^{\alpha}}d\nu\int_0^{\nu}\frac{S_\tau(\tau)}{(t-\tau)^{\alpha}}d\tau = \frac{1}{2\Gamma(1-\alpha)}\int_0^t (t-\nu)^{\alpha}\frac{\partial}{\partial t}\left(\int_0^{\nu}\frac{S_\tau(\tau)}{(t-\tau)^{\alpha}}d\tau\right)^2$$
$$= \frac{\alpha}{2\Gamma(1-\alpha)}\int_0^t (t-\nu)^{\alpha-1}\left(\int_0^{\nu}\frac{S_\tau(\tau)}{(t-\tau)^{\alpha}}d\tau\right)^2 \geq 0$$

The proof of the lemma is complete. ■

1.5.2 Poincaré type inequality

Lemma 1.7 ([35]) *For any $n \in \mathbb{N}$, we have*

$$\| \Im_x^{2n} v \|^2_{L^2(0,l)} \leq \left(\frac{l}{2}\right)^{2n} \| v \|^2_{L^2(0,l)}, \tag{1.7}$$

where

$$\Im_x^{2n} v = \int_0^x \int_0^{\xi_1} ... \int_0^{\xi_{2n-1}} v(\eta,t)\, d\eta d\xi_{2n-1}...d\xi_1 = \int_0^x \frac{(x-\xi)^{2n-1}}{(2n-1)!} v(\xi,t)\, d\xi.$$

1.5.3 Gronwall lemma

Gronwall's lemma plays a significant role in estimates containing integrodifferential terms and is frequently used to obtain the a priori estimates in used spaces.

Lemma 1.8 (Gronwall Lemma) *[3] If $f_1(\tau)$, $f_2(\tau)$ and $f_3(\tau)$ are nonnegative functions on $(0,T)$, with $f_1(\tau)$ and $f_2(\tau)$ integrable, and $f_3(\tau)$ is nondecreasing on $(0,T)$ then if*

$$\int_0^{\tau} f_1(t)dt + f_2(\tau) \leq c\int_0^{\tau} f_2(t)dt + f_3(\tau), \tag{1.8}$$

then

$$\int_0^{\tau} f_1(t)dt + f_2(\tau) \leq f_3(\tau).exp(c\tau). \tag{1.9}$$

Proof *We write* (1.8) *in the form*

$$Tf_1 + f_2 \leq Kf_2 + f_3, \tag{1.10}$$

where

$$Tf_1 + f_2 = \int_0^\tau f_1(t)dt$$

and

$$Kf_2 = \int_0^\tau f_2(t)dt. \tag{1.11}$$

Since f_2 is nonnegative function, (1.8) *gives rise to :*

$$f_2 \leq cKf_2 + f_3. \tag{1.12}$$

Obviously the operator K preserves the inequality. If we apply it to (1.11) *and multiply the result by c, we obtain*

$$cKf_2 \leq c^2K^2f_2 + cKf_3. \tag{1.13}$$

Hence

$$Tf_1 + f_2 \leq c^2K^2f_2 + cKf_3 + f_3. \tag{1.14}$$

Continuing this process, we obtain

$$Tf_1 + f_2 \leq c^{n+1}K^{n+1}f_2 + \sum_{m=0}^{n} c^m K^m f. \tag{1.15}$$

It is easy to see that

$$c^{n+1}K^{n+1}f_2 \leq c^{n+1}2^{(n+1)}/(n+1)!.\tau^{n+1} \sup f_2. \tag{1.16}$$

which implies that the first term tends to zero as $n \to \infty$, while the second term on the right-hand side is majored by the function exp$(2c\tau)f_3(\tau)$.

The proof of Lemma 1.8 *is complete.* ■

1.5.4 Cauchy epsilon inequality

$$\mid ab \mid \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \tag{1.17}$$

wich holds for arbitary a and b, and all $\varepsilon > 0$.

1.5.5 Cauchy-Schwartz inequality

Let be Ω an open set of $\mathbb{R}$ ($\Omega \subset \mathbb{R}$)

$$\forall\ u, v \in L^2(\Omega) : \int_\Omega |uv| dx \leq \left(\int_\Omega |u| dx \right)^{\frac{1}{2}} \left(\int_\Omega |v| dx \right)^{\frac{1}{2}},$$

$$(ie) : \| uv \|_{L^1(\Omega)} \leq \| v \|_{L^2(\Omega)} \| u \|_{L^2(\Omega)} .$$

Chapter 2

Existence and uniqueness for a linear fractional integro-differential Equation with an integral and a Neumann conditions

This chapter studies a linear fractional differential equation for which the strong solution's existence and uniqueness are proved. First, we derive a priori bound based on constructing a suitable multiplicator. Then, from the resulting energy estimate, it is possible to establish the solvability of the problem. Then, we demonstrate the generated operator range density.

2.1 Statement of problem

In the region $D = \{(x,t) \in \mathbb{R}^2 : x \in \Omega = [0,1],\ 0 < t < T\}$ such that $T < \infty$, we consider the nonlinear fractional equation

$$\mathcal{L}v = {}^{C}\partial_t^{\beta+1} v - \frac{\partial}{\partial x}\left(\gamma(x,t)\frac{\partial v}{\partial x}\right) - \frac{\partial^2}{\partial x \partial t}\left(\eta(x,t)\frac{\partial v}{\partial x}\right) - \int_0^t \xi(t-z)\, v(x,z)\, dz = f\left(x,t,v,\frac{\partial v}{\partial x}\right). \tag{2.1}$$

with $0 < \beta < 1$.

Associated with initial condition

$$\ell_1 v = v(x,0) = \varphi(x), \quad \ell_2 v = \frac{\partial v(x,0)}{\partial t} = \psi(x), \quad x \in \Omega. \tag{2.2}$$

Neumann type condition

$$\frac{\partial v}{\partial x}(1,t) = 0, \quad t \in (0,T), \tag{2.3}$$

and the boundary condition

$$\int_0^1 v(x,t)\,dx = 0. \tag{2.4}$$

Such that the known functions γ, η, and ξ verify Assumptions 2.1, and data functions f, φ, ψ belong to a suitable function spaces as mentioned in Section **??**.

We define some function spaces and tools required to investigate the following linear problem associated to problem (2.1)- (2.4)

$$\mathcal{L}v = {}^{C}\partial_t^{\beta+1} v - \frac{\partial}{\partial x}\left(\gamma(x,t)\frac{\partial v}{\partial x}\right) - \frac{\partial^2}{\partial x \partial t}\left(\eta(x,t)\frac{\partial v}{\partial x}\right) - \int_0^t \xi(t-z)\, v(x,z)\,dz = f(x,t). \tag{2.5}$$

$$\ell_1 v = v(x,0) = \varphi(x), \quad \ell_2 v = \frac{\partial v(x,0)}{\partial t} = \psi(x), \quad x \in \Omega. \tag{2.6}$$

$$\int_0^1 v(x,t)\,dx = 0, \quad v_x(1,t) = 0, \quad t \in (0,T). \tag{2.7}$$

First, we convert problem $(2.1) - (2.4)$ into an equivalent operator form

$$Lv = \mathcal{F} = (f, \varphi, \psi), \tag{2.8}$$

where the unbounded operator $L = (\mathcal{L}, \ell_1, \ell_2)$ with $L : E \to F$ is defined in $D(L)$ such that

$$D(L) = \begin{cases} v \in L^2(D), {}^{C}\partial_t^{\beta+1} v, \partial v/\partial t, \partial v/\partial x, \partial^2 v/\partial x^2, \partial^3 v/\partial x^2 \partial t \in L^2(D) \\ \int_0^1 v(x,t)\,dx = 0, \quad v_x(1,t) = 0, t \in (0,T), \end{cases} \tag{2.9}$$

and v verify also the initial condition. Here E is Banach space containing elements having the finite norm

$$\| v \|_E^2 = D_t^{\beta-1} \|\Im_x \frac{\partial v}{\partial t}\|_{L^2(\Omega)}^2 + \| \frac{\partial v}{\partial t} \|_{L^2(D)}^2 + \| v \|_{C((0,T),L^2(\Omega))}^2, \tag{2.10}$$

and F is Hilbert space composed of functions normed with

$$\|\mathcal{F}\|_F^2 = \|\Im_x f\|_{L^2(D)}^2 + \|\varphi\|_{L^2(\Omega)}^2 + \|\Im_x \psi\|_{L^2(\Omega)}^2. \tag{2.11}$$

2.2 A priori estimate

Assumption 2.1 *For any $(x,t) \in D$, we suppose that*

$$\begin{aligned} c_0 \leq \gamma(x,t) \leq c_1, \tfrac{\partial\gamma(x,t)}{\partial t} \leq c_2, \tfrac{\partial\gamma(x,t)}{\partial x} \leq c_3 \\ c_4 \leq \eta(x,t), c_5 \leq \tfrac{\partial\eta(x,t)}{\partial t} \leq c_6, \tfrac{\partial\eta(x,t)}{\partial x} \leq c_7 \\ \tfrac{\partial^2\eta(x,t)}{\partial t^2} \leq c_8, \tfrac{\partial^2\eta(x,t)}{\partial t\partial x} \leq c_9, \xi(x,t) \leq c_{10} \end{aligned} \tag{2.12}$$

such that $c_i (i = 0, ..., 10)$ are positive constants.

Theorem 2.1 *Let Assumptions 2.1 be fulfilled. Then, any function $v \in D(L)$ verify the following estimate*

$$\| v \|_E \leq C \| Lv \|_F . \tag{2.13}$$

where $C > 0$ constant independent of v.

Proof We Take the scalar product $L^2(D^\tau)$, of equality (2.5) and the itegro-differential operator $Mv = -2\Im_x^2 \frac{\partial v}{\partial t} = -2\int_0^x \int_0^\xi \frac{\partial v}{\partial t} d\xi dt$, such that $D^\tau = (0,1) \times (0,\tau), \tau \in [0,T]$, we have

$$\begin{aligned} &2\left({}^C\partial_t^{\beta+1} v, -\Im_x^2 \frac{\partial v}{\partial t}\right)_{L^2(D^\tau)} - 2\left(\frac{\partial}{\partial x}\left(\gamma\frac{\partial v}{\partial x}\right), -\Im_x^2\frac{\partial v}{\partial t}\right)_{L^2(D^\tau)} \\ &-2\left(\frac{\partial^2}{\partial x\partial t}\left(\eta\frac{\partial v}{\partial x}\right), -\Im_x^2\frac{\partial v}{\partial t}\right)_{L^2(D^\tau)} - 2\left(\int_0^t \xi(t-z)\, v(x,z)\, dz, -\Im_x^2\frac{\partial v}{\partial t}\right)_{L^2(D^\tau)} \\ &= 2\left(f(x,t), -\Im_x^2\frac{\partial v}{\partial t}\right)_{L^2(D^\tau)}. \end{aligned} \tag{2.14}$$

The integration of the first three terms on the left-hand side (LHS) of equation (2.14), taking into consideration initial and boundary conditions (2.2) – (2.4), gives

$$\begin{aligned}\left({}^{C}\partial_t^{\beta+1}v,-\Im_x^2\frac{\partial v}{\partial t}\right)_{L^2(D^\tau)} &= -\int_0^\tau\left[{}^{C}\partial_t^{\beta}\left(\Im_x\frac{\partial v}{\partial t}\right)\Im_x^2\frac{\partial v}{\partial t}\right]_0^1 dt\\ +\int_0^\tau {}^{C}\partial_t^{\beta}\left(\Im_x\frac{\partial v}{\partial t}\right)\Im_x\frac{\partial v}{\partial t}dxdt &= \int_0^\tau {}^{C}\partial_t^{\beta}\left(\Im_x\frac{\partial v}{\partial t}\right)\Im_x\frac{\partial v}{\partial t}dxdt,\end{aligned} \tag{2.15}$$

$$\begin{aligned}-2\left(\frac{\partial}{\partial x}\left(\gamma(x,t)\frac{\partial v}{\partial x}\right),-\Im_x^2 v_t\right)_{L^2(D^\tau)} &= -\int_{D^\tau}\frac{\partial\gamma}{\partial t}v^2dxdt\\ +2\int_{D^\tau}\frac{\partial\gamma}{\partial x}v\Im_x\frac{\partial v}{\partial t}dxdt+\int_0^1\gamma(x,\tau)v^2(x,\tau)dx&-\int_0^1\gamma(x,0)\varphi^2(x)dx,\end{aligned} \tag{2.16}$$

$$\begin{aligned}-2\left(\frac{\partial^2}{\partial x\partial t}\left(\eta(x,t)\frac{\partial v}{\partial x}\right),-\Im_x^2\frac{\partial v}{\partial t}\right)_{L^2(D^\tau)} &= 2\int_{D^\tau}\eta\left(\frac{\partial v}{\partial t}\right)^2dxdt\\ +\int_0^1\frac{\partial\eta(x,\tau)}{\partial t}v^2(x,\tau)dx-\int_{D^\tau}\frac{\partial^2\eta}{\partial t^2}v^2dxdt&+2\int_{D^\tau}\frac{\partial\eta}{\partial x}\frac{\partial v}{\partial t}\Im_x\frac{\partial v}{\partial t}dxdt\\ +2\int_{D^\tau}\frac{\partial^2\eta}{\partial x\partial t}v\Im_x\frac{\partial v}{\partial t}dxdt-\int_0^1\frac{\partial\eta(x,0)}{\partial t}\varphi^2(x)dx.\end{aligned} \tag{2.17}$$

Substituting (2.15) – (2.17) into (2.14) yields

$$\begin{aligned}&2\left({}^{C}\partial_t^{\beta}\left(\Im_x\frac{\partial v}{\partial t}\right),\Im_x\frac{\partial v}{\partial t}\right)_{L^2(D^\tau)}+2\int_{D^\tau}\eta\left(\frac{\partial v}{\partial t}\right)^2dxdt\\ &+\int_0^1\left(\gamma(x,\tau)+\frac{\partial\eta(x,\tau)}{\partial t}\right)v^2(x,\tau)dx=2\left(f(x,t),-\Im_x^2\frac{\partial v}{\partial t}\right)_{L^2(D^\tau)}\\ &+2\left(\int_0^t\xi(t-z)v(x,z)dz,-\Im_x^2\frac{\partial v}{\partial t}\right)_{L^2(D^\tau)}+\int_{D^\tau}\left(\frac{\partial\gamma}{\partial t}+\frac{\partial^2\eta}{\partial t^2}\right)v^2dxdt\\ &+\int_0^1\left(\gamma(x,0)+\frac{\partial\eta(x,0)}{\partial t}\right)\varphi^2(x)dx-2\int_{D^\tau}\frac{\partial\eta}{\partial x}\frac{\partial v}{\partial t}\Im_x\frac{\partial v}{\partial t}dxdt\\ &-2\int_{D^\tau}\left(\frac{\partial\gamma}{\partial x}+\frac{\partial^2\eta}{\partial x\partial t}\right)v\Im_x\frac{\partial v}{\partial t}dxdt.\end{aligned} \tag{2.18}$$

By applying inequality (1.7), we estimate the first and the last two terms on the right-hand side (RHS) of (2.18); it follows that

$$-2\int_{D^{\tau}} f\Im_x^2\frac{\partial v}{\partial t}dxdt \leq \int_{D^{\tau}} (\Im_x f)^2\, dxdt + \int_{D^{\tau}} \left(\Im_x\frac{\partial v}{\partial t}\right)^2 dxdt, \tag{2.19}$$

$$2\int_{D^{\tau}} \left(\int_0^t \xi\,(t-z)\, v\,(x,z)\, dz\right) \Im_x^2\frac{\partial v}{\partial t}dxdt \leq c_{10}T^2 \parallel v \parallel^2_{L(D^{\tau})} + \frac{1}{2}\int_{D^{\tau}} \left(\Im_x\frac{\partial v}{\partial t}\right)^2 dxdt, \tag{2.20}$$

$$-2\int_{D^{\tau}} \left(\frac{\partial \gamma}{\partial x} + \frac{\partial^2 \eta}{\partial x \partial t}\right) v\Im_x\frac{\partial v}{\partial t}dxdt \leq 2\int_{D^{\tau}} \left\{\left(\frac{\partial \gamma}{\partial x}\right)^2 + \left(\frac{\partial^2 \eta}{\partial x \partial t}\right)^2\right\} v^2 dxdt + \int_{D^{\tau}} \left(\Im_x\frac{\partial v}{\partial t}\right)^2 dxdt, \tag{2.21}$$

$$-2\int_{D^{\tau}} \frac{\partial \eta}{\partial x}\frac{\partial v}{\partial t}\Im_x\frac{\partial v}{\partial t}dxdt \leq c_4\int_{D^{\tau}} \left(\frac{\partial v}{\partial t}\right)^2 dxdt + \frac{1}{c_4}\int_{D^{\tau}} \left(\frac{\partial \eta}{\partial x}\right)^2 \left(\Im_x\frac{\partial v}{\partial t}\right)^2 dxdt. \tag{2.22}$$

By Lemma 1.6, the first term on the LHS of (2.14) becomes

$$2\left({}^C\partial_t^{\beta}\left(\Im_x\frac{\partial v}{\partial t}\right), \Im_x\frac{\partial v}{\partial t}\right)_{L^2(D^{\tau})} \geq \int_{D^{\tau}} {}^C\partial_t^{\beta}\left(\Im_x\frac{\partial v}{\partial t}\right)^2 dxdt. \tag{2.23}$$

Hence, by formulas (2.19) – (2.23) and Assumption (2.12), we obtain

$$\begin{aligned}
&\int_{D^{\tau}} \left({}^C\partial_t^{\beta}\left(\Im_x\frac{\partial v}{\partial t}\right)^2\right) dxdt + \int_0^{\tau} \parallel\frac{\partial v}{\partial t}(.,t)\parallel^2_{L^2(\Omega)}dt + \|v\,(.,\tau)\parallel^2_{L^2(\Omega)} \\
\leq\ & \delta_1\left\{\int_0^{\tau} \|\Im_x f\,(.,t)\parallel^2_{L(\Omega)}dt + \|\varphi\|^2_{L^2(\Omega)} + \int_0^{\tau} \|\Im_x\frac{\partial v}{\partial t}(.,t)\parallel^2_{L(\Omega)}dt\right\} \\
&+\delta_2\left\{\int_0^{\tau} \|v\,(.,t)\parallel^2_{L^2(\Omega)}dt\right\},
\end{aligned}$$

where

$$\delta_1 = \frac{\max\left(1, c_1 + c_6, \frac{5}{2} + \frac{c_7^2}{c_4}\right)}{\min\left(c_4, c_0 + c_5, 1\right)},$$

$$\delta_2 = \frac{c_2 + c_8 + c_3^2 + c_9^2 + c_{10}T^2}{\min\left(c_4, c_6 + c_5, 1\right)}.$$

Now, since

$$\int_0^\tau {}^C\partial_t^\beta \parallel \Im_x \frac{\partial v}{\partial t}^2 \parallel_{L^2(\Omega)} dt = D_\tau^{\beta-1}\|\Im_x \frac{\partial v}{\partial t}\|^2_{L^2(\Omega)} - \frac{\tau^{1-\beta}}{(1-\beta)\Gamma(1-\beta)}\|\Im_x\psi\|^2_{L^2(\Omega)}, \quad (2.24)$$

then

$$\begin{aligned} & D_\tau^{\beta-1}\|\Im_x \frac{\partial v}{\partial t}\|^2_{L^2\Omega)} + \int_0^\tau \parallel \frac{\partial v}{\partial t}(.,t) \parallel^2_{L^2(\Omega)} dt + \parallel v(.,\tau) \parallel^2_{L^2(\Omega)} \\ \leq \; & \delta_3\left\{\int_0^\tau \parallel \Im_x f(.,t) \parallel^2_{L(\Omega)} dt + \parallel \varphi \parallel^2_{L^2(\Omega)} + \|\Im_x\psi\|^2_{L^2(\Omega)} + \int_0^\tau \parallel \Im_x \frac{\partial v}{\partial t}(.,t) \parallel^2_{L(\Omega)} dt\right\} \\ & +\delta_2\left\{\int_0^\tau \parallel v(.,t) \parallel^2_{L^2(\Omega)} dt\right\}, \end{aligned} \quad (2.25)$$

where

$$\delta_3 = \max\left(\delta_1, \frac{T^{1-\beta}}{(1-\beta)\,\Gamma(1-\beta)}\right).$$

We need to drop the last term on the RHS of (2.25). Therefore, we use Gronwall's Lemma, yields

$$\begin{aligned} & D_\tau^{\beta-1}\|\Im_x \tfrac{\partial v}{\partial t}\|^2_{L^2(\Omega)} + \textstyle\int_0^\tau \parallel \tfrac{\partial v}{\partial t}(.,t) \parallel^2_{L^2(\Omega)} dt + \parallel v(.,\tau) \parallel^2_{L^2(\Omega)} \\ \leq \; & \delta_4\left\{\textstyle\int_0^\tau \parallel \Im_x f(.,t) \parallel^2_{L(\Omega)} d + \parallel \varphi \parallel^2_{L^2(\Omega)} t + \|\Im_x\psi\|^2_{L^2(\Omega)} + \int_0^\tau \|\Im_x \tfrac{\partial v}{\partial t}(.,t) \parallel^2_{L(\Omega)} dt\right\}, \end{aligned} \quad (2.26)$$

where

$$\delta_4 = \exp(\delta_2 T)\,\delta_3,$$

Now, by discarding the last two terms on the LHS of (2.26) then posing $S(\tau) = \int_0^\tau \|\Im_x \frac{\partial v}{\partial t}(.,t)\|^2_{L^2(\Omega)}dt$, ${}^C\partial_\tau^\alpha S(\tau) = D_\tau^{\beta-1}\|\Im_x \frac{\partial v}{\partial t}\|^2_{L^2(\Omega)}$,with $S(0) = 0$, in Lemma (1.5), we obtain

$$\int_0^\tau \|\Im_x \frac{\partial v}{\partial t}(.,t)\|^2_{L^2(\Omega)}dt \leq \delta_5\left\{D_\tau^{-1-\beta} \parallel \Im_x f \parallel^2_{L(\Omega)} + \parallel \varphi \parallel^2_{L^2(\Omega)} + \|\Im_x\psi\|^2_{L^2(\Omega)}\right\}, \quad (2.27)$$

where

$$\delta_5 = \Gamma(\beta)E_{\beta,\beta}(c_{17}T^\beta)\max\left(1, \frac{T^{(\beta+1)}}{(1+\beta)\,\Gamma(1+\beta)}\right).$$

Combining $(2.26)-(2.27)$ yields

$$\begin{aligned} & D_{\tau}^{\beta-1}\|\Im_x \frac{\partial v}{\partial t}\|_{L^2(\Omega)}^2+\int_0^{\tau}\|\frac{\partial v}{\partial t}(.,t)\|_{L^2(\Omega)}^2 dt+\|v(.,\tau)\|_{L^2(\Omega)}^2 \\ \leq\ & \delta_6\left\{D_{\tau}^{-\beta-1}\|\Im_x f\|_{L^2(\Omega)}^2+\int_0^{\tau}\|\Im_x f(.,t)\|_{L(\Omega)}^2 dt+\|\varphi\|_{L^2(\Omega)}^2+\|\Im_x \psi\|_{L^2(\Omega)}^2\right\}, \end{aligned} \tag{2.28}$$

where

$$\delta_6=\max\left(\delta_4 \delta_5, \delta_5\right).$$

From given inequality

$$D_{\tau}^{-1-\beta}\|\Im_x f\|_{L(\Omega)}^2 \leq \frac{T^{\beta}}{\Gamma(1+\beta)} \int_0^{\tau}\|\Im_x f\|_{L(\Omega)}^2 dt, \tag{2.29}$$

we can reduce inequality (2.28) to another one as follows

$$\begin{aligned} & D_{\tau}^{\beta-1}\|\Im_x \frac{\partial v}{\partial t}\|_{L^2(\Omega)}^2+\int_0^{\tau}\|\frac{\partial v}{\partial t}(.,t)\|_{L^2(\Omega)}^2 dt+\|v(.,\tau)\|_{L^2(\Omega)}^2 \\ \leq\ & \delta_7\left\{\int_0^{T}\|\Im_x f(.,t)\|_{L(\Omega)}^2 dt+\|\varphi\|_{L^2(\Omega)}^2+\|\Im_x \psi\|_{L^2(\Omega)}^2\right\}, \end{aligned} \tag{2.30}$$

such that

$$\delta_7=\delta_6\left(1+\frac{T^{\beta}}{\Gamma(1+\beta)}\right).$$

Since the right-hand side of estimate (2.30) is independent of τ, we can take the supremum on the left-hand side with respect to τ over $[0,T]$. Thus, we get the desired inequality (2.13). The proof of Theorem is complete. ■

2.3 Existence of solution

The current section's aim is to prove the existence of the strong solution of problem $(2.5)-(2.7)$. It remains to demonstrate the density of the range $R(L)$.

Proposition 2.1 *[33] The operator L engendered by problem* $(2.1)-(2.4)$ *has a closure.*

Proof Assuming that $u_n \in D(L)$ is a sequence such that:

$$u_n \underset{n\to\infty}{\to} 0 \qquad \text{in } E \tag{2.31}$$

and

$$Lu_n \underset{n\to\infty}{\to} (f, \varphi, \psi) \qquad \text{in } F, \tag{2.32}$$

we must show that $f \equiv 0$, $\varphi \equiv 0$ and $\psi \equiv 0$.

From (2.31), we get:

$$u_n \underset{n\to\infty}{\to} 0 \qquad \text{in } \mathcal{D}'(D).$$

According to the continuity of the derivation defining from $\mathcal{D}'(D)$ to $\mathcal{D}'(D)$; we obtain:

$$\mathcal{L}u_n \underset{n\to\infty}{\to} 0 \qquad \text{in } \mathcal{D}'(D). \tag{2.33}$$

Moreover, it comes from (2.32) that

$$\Im_x \mathcal{L}u_n \underset{n\to\infty}{\to} \Im_x f \qquad \text{in } L^2(D),$$

consequently

$$\mathcal{L}u_n \underset{n\to\infty}{\to} f \qquad \text{in } \mathcal{D}'(D). \tag{2.34}$$

By vertu of the uniqueness of limit in $\mathcal{D}'(D)$, we can conclude from (2.33), and (5.25) that $f \equiv 0$. From (2.32), we have

$$\ell_1 u_n \underset{n\to\infty}{\to} \varphi \qquad \text{in } L^2(0,1). \tag{2.35}$$

According to the norm of the space F, we obtain

$$\|\ell_1 u_n\|_{L^2(0,1)} \leq \|u_n\|_F, \qquad \forall n \in \mathbb{N},$$

then

$$\ell_1 u_n \underset{n\to\infty}{\to} 0 \qquad \text{in } L^2(0,1). \tag{2.36}$$

From (2.35), (2.36), By vertu of the uniqueness of limit in $L^2(0,1)$, we conclude that $\varphi \equiv 0$. Similarly, we show thar $\psi \equiv 0$. ■

Defining the operator equation solution

$$\overline{L}v = \mathcal{F} = (f, \varphi, \psi),$$

as a strong solution of problem (2.5) – (2.7). The inequality (2.13) can be extended into

$$\| v \|_E \leq \| \overline{L}v \|_F, \qquad \forall v \in D\left(\overline{L}\right). \tag{2.37}$$

the above inequality assures the strong solution uniqueness, moreover:

Corollary 2.1 *The range of the operator $\overline{L}$ is a closed in F and $R(\overline{L}) = \overline{R(L)}$ and $\overline{L}^{-1} = \overline{L^{-1}}$.*

Proof It follows from the definition of $R(\overline{L})$ that $R(\overline{L}) \subseteq \overline{R(L)}$. It remains to prove the opposite inclusion. Suppose that $w \in R(L)$, then there exists a sequence $\{w_n\}_{n=1}^{\infty}$ of elements in $R(L)$ such that $\lim_{n\to\infty} w_n = w$. Consequently, there exists a corresponding sequence $u_n \in D(L)$ such that $Lu_n = w_n$. According to Theorem 2.1, we have

$$\| u_m - u_n \|_E \leq C \| Lu_m - Lu_n \|_F, \tag{2.38}$$

when n and $m \to \infty$. Thus $\{u_n\}$ is a fundamental sequence in which converges to an element $u \in F$ and $\overline{L}u = w$, then $w \in R(L)$. This proves Corollary (2.1). ■

Theorem 2.2 *Let the Theorem 2.1 conditions be verified. Then, for any $\mathcal{F} = (f, g, h) \in F$, the problem (2.5) – (2.7) has a unique solution v such that $v = \overline{L}^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$.*

Proposition 2.2 *Let Assumption 2.1 be fulfilled. If for a certain function $g \in L^2(D)$, and every $v \in D(L)$ verifying homogeneous initial conditions, we have*

$$(\mathcal{L}v, g)_{L^2(D)} = 0, \tag{2.39}$$

then g vanishes a.e in D.

Proof Introducing a new function $\sigma(x,t)$ verifies conditions (2.2) and (2.4), and σ, σ_x ,$\Im_t\sigma_x$, $\Im_t\sigma$ and ${}^C\partial_t^{\beta+1}\sigma \in L^2(D)$, then we pose

$$v(x,t) = \Im_t^2\sigma,$$

Where

$$\Im_t\sigma = \int_0^t \sigma(x,s)ds, \quad \Im_t^2\sigma = \int_0^t\int_0^s \sigma(x,s)dzds.$$

Equation (2.39) then becomes

$$\left({}^C\partial_t^{\beta+1}\left(\Im_t^2\sigma\right) - \frac{\partial}{\partial x}\left(\gamma(x,t)\Im_t^2\left(\frac{\partial\sigma}{\partial x}\right)\right) - \frac{\partial^2}{\partial x\partial t}\left(\eta(x,t)\Im_t^2\left(\frac{\partial\sigma}{\partial x}\right)\right) - \int\limits_0^t \xi(t-z)\Im_t^2\sigma(x,z)\,dz, g\right)_{L^2(D)} = 0. \tag{2.40}$$

Now, we consider the function

$$g(x,t) = -\Im_t\Im_x^2\sigma. \tag{2.41}$$

Obviously, the function g included in $L^2(D)$. Equations (2.40) – (2.41) lead to

$$\begin{aligned} &-\left({}^C\partial_t^{\beta+1}\Im_t^2\sigma, \Im_t\Im_x^2\sigma\right)_{L^2(D)} + \left(\frac{\partial}{\partial x}\left(\gamma(x,t)\Im_t^2\left(\frac{\partial\sigma}{\partial x}\right)\right), \Im_t\Im_x^2\sigma\right)_{L^2(D)} \\ &+\left(\frac{\partial^2}{\partial x\partial t}\left(\eta(x,t)\Im_t^2\left(\frac{\partial\sigma}{\partial x}\right)\right), \Im_t\Im_x^2\sigma\right)_{L^2(D)} + \left(\int\limits_0^t \xi(t-z)\Im_t^2\sigma(x,z)\,dz, \Im_t\Im_x^2\sigma\right)_{L^2(D)} \\ = &\ 0. \end{aligned} \tag{2.42}$$

Note that the function σ verifies conditions (2.2) – (2.4) , then we have

$$-\left({}^C\partial_t^{\beta+1}\Im_t^2\sigma, \Im_t\Im_x^2\sigma\right)_{L^2(D)} = -\left({}^C\partial_t^{\beta}\Im_x\Im_t\sigma, \Im_x\Im_t\sigma\right)_{L^2(D)}, \tag{2.43}$$

$$\left(\frac{\partial}{\partial x}\left(\gamma(x,t)\Im_t^2\left(\frac{\partial\sigma}{\partial x}\right)\right),\Im_t\Im_x^2\sigma\right)_{L^2(D)} = \frac{1}{2}\int_0^1\gamma\left(\Im_t^2\sigma\right)^2dx-\frac{1}{2}\int_D\frac{\partial\gamma}{\partial t}\left(\Im_t^2\sigma\right)^2dxdt + \left(\frac{\partial\gamma(x,t)}{\partial x}\left(\Im_t^2\sigma\right),\Im_t\Im_x\sigma\right)_{L^2(D)}, \tag{2.44}$$

$$\left(\frac{\partial^2}{\partial x\partial t}\left(\eta(x,t)\Im_t^2\left(\frac{\partial\sigma}{\partial x}\right)\right),\Im_t\Im_x^2\sigma\right)_{L^2(D)} = \left(\frac{\partial^2\eta}{\partial x\partial t}\Im_t^2\sigma,\Im_t\Im_x\sigma\right)_{L^2(D)} + \left(\frac{\partial\eta}{\partial t}\Im_t^2\sigma,\Im_t\sigma\right)_{L^2(D)}+\int_D\frac{\partial^2\eta}{\partial x^2}\left(\Im_t\Im_x\sigma\right)^2dxdt+\int_D\eta\left(\Im_t\sigma\right)^2dxdt. \tag{2.45}$$

Insertion of equations (2.43) – (2.45) into (2.42), yields

$$\begin{aligned}&-2\left({}^C\partial_t^\beta\Im_x\Im_t\sigma,\Im_x\Im_t\sigma\right)_{L^2(D)}+\int_0^1\gamma\left(\Im_t^2\sigma\right)^2dx=\int_D\frac{\partial\gamma}{\partial t}\left(\Im_t^2\sigma\right)^2dxdt\\&-2\int_D\frac{\partial^2\eta}{\partial x^2}\left(\Im_t\Im_x\sigma\right)^2dxdt-2\int_D\eta\left(\Im_t\sigma\right)^2dxdt-2\left(\left(\frac{\partial^2\eta}{\partial x\partial t}+\frac{\partial\gamma(x,t)}{\partial x}\right)\Im_t^2\sigma,\Im_t\Im_x\sigma\right)_{L^2(D)}\\&-2\left(\frac{\partial\eta}{\partial t}\Im_t^2\sigma,\Im_t\sigma\right)_{L^2(D)}-2\left(\int_0^t\xi(t-z)\Im_s^2\sigma(x,z)\,dz,\Im_t\Im_x^2\sigma\right)_{L^2(D)}.\end{aligned} \tag{2.46}$$

According to Lemma 1.5, we bound the first term on the left-hand side of (2.46); we have

$$2\left({}^C\partial_t^\beta\Im_x\Im_t\sigma,\Im_x\Im_t\sigma\right)_{L^2(D)}\geq{}^C\partial_t^\beta\parallel\Im_x\Im_t\sigma\parallel_{L^2(D)}^2. \tag{2.47}$$

Also, we bound the last three terms on the right-hand side of (2.46) utilizing inequality 1.17, and we get then

$$2\left(\left(\frac{\partial\gamma(x,t)}{\partial x}+\frac{\partial^2\eta}{\partial x\partial t}\right)\Im_t^2\sigma,\Im_t\Im_x\sigma\right)_{L^2(D)}\leq 2\left(c_3^2+c_9^2\right)\parallel\Im_t^2\sigma\parallel_{L^2(D)}^2+\parallel\Im_x\Im_t\sigma\parallel_{L^2(D)}^2, \tag{2.48}$$

$$2\left(\frac{\partial\eta}{\partial t}\left(\Im_t^2\sigma\right),\Im_t\sigma\right)_{L^2(D)}\leq 2c_4\parallel\Im_t^2\sigma\parallel_{L^2(D)}^2+\frac{c_6^2}{2c_4^2}\parallel\Im_t\sigma\parallel_{L^2(D)}^2, \tag{2.49}$$

$$\left(\int_0^t \xi(t-z)\Im_z^2\sigma(x,z)\,dz, \Im_x^2\Im_t\sigma\right)_{L^2(D)} \leq c_{10}T^2 \parallel \Im_t^2\sigma \parallel_{L(D)}^2 + \frac{1}{2} \parallel \Im_x\Im_t\sigma \parallel_{L(D)}^2 . \quad (2.50)$$

The insertion of estimates (2.47) – (2.49) in equation (2.46) gives

$$D_\tau^{\beta-1}\|\Im_x\Im_t\sigma\|_{L^2(D)}^2 + \int_0^1 \left(\Im_t^2\sigma\right)^2 dx \leq \delta_8\left(\int_D \left(\Im_t^2\sigma\right)^2 dxdt + \|\Im_x\Im_t\sigma\|_{L^2(D)}^2\right), \quad (2.51)$$

with

$$\delta_8 = \frac{\max\left(c_2 + 2\left(c_3^2 + c_9^2\right) + c_{10}T^2 + \frac{c_6^2}{2c_4}, 2c_{10} + \frac{3}{2}\right)}{\min(1, c_0)}. \quad (2.52)$$

Eliminating the first term on the left-hand side of (2.51), using Lemma 1.6, with

$$S(\tau) = \int_0^\tau \int_0^1 \left(\Im_t^2\sigma\right)^2 dxdt. \quad (2.53)$$

Observe that $S(0) = 0$, then we get

$$S(\tau) \leq T\exp(T\delta_8)\|\Im_x\Im_t\sigma\|_{L^2(D^\tau)}^2, \quad (2.54)$$

Similarly, by discarding the second integral on the left-hand side of (2.51) and applying (2.54), we obtain

$$D_\tau^{\beta-1} \parallel \Im_x\Im_t\sigma \parallel_{L^2(D)}^2 \leq \delta_8(T\exp(T\delta_8) + 1) \parallel \Im_x\Im_t\sigma \parallel_{L^2(D)}^2 . \quad (2.55)$$

By Lemma 1.6, with

$$S(\tau) = \int_0^\tau \int_0^1 (\Im_x\Im_t\sigma)^2 dxdt,$$

and

$${}^C\partial_\tau^\beta S(\tau) = D_\tau^{\beta-1}\|\Im_x\Im_t\sigma\|_{L^2(D)}^2, S(0) = 0,$$

it follows that

$$\begin{aligned}\|\Im_x\Im_t\sigma\|_{L^2(D^\tau)}^2 &\leq S(0)E_\beta(\delta_8(T\exp(T\delta_8)+1)\tau^\beta) + \Gamma(\beta)E_{\beta,\beta}(\delta_8(T\exp(T\delta_8)+1)\tau^\beta)D_\tau^{-\beta}(0)\\ &= 0, \end{aligned} \quad (2.56)$$

for any $\tau \in [0, T]$. Hence inequality (2.56) shows that $g = 0$ ae in D. Continuing the Theorem 2.2 proof, we assume that for a certain function $G = (g, g_0, g_1) \in R(L)^{\perp}$, we have

$$(\mathcal{L}v, g)_{L^2(D)} + (l_1 v, g_0)_{L^2(\Omega)} + (l_2 v, g_1)_{L^2(\Omega)} = 0, \tag{2.57}$$

then we should show that $g_0 = 0, g_1 = 0$. Putting $v \in D(L)$, verifying homogeneous initial conditions into (2.57), yields

$$(\mathcal{L}v, g)_{L^2(D)} = 0, \quad \forall v \in D(L). \tag{2.58}$$

By applying Proposition 2.2 to (2.58), we see that $g = 0$. Consequently, (2.57) becomes

$$(l_1 v, g_0)_{L^2(\Omega)} + (l_2 v, g_1)_{L^2(\Omega)} = 0 \quad \forall v \in D(L), \tag{2.59}$$

Since $l_1 v$ and $l_2 v$ are independent and their ranges l_1 and l_2 are everywhere dense in $L^2(\Omega)$, we conclude that $g_0 = g_1 = 0$, this complete the proof of Theorem 2.2. ■

Chapter 3

The study of nonlinear nonlocal fractional Integro-Differential Equation

In the current chapter, we solve the nonlinear problem by introducing an iterative process depending on the initial results.

3.1 Position of the auxiliary problem

This section is devoted to solving the main problem (2.1) − (2.4). Consider now the auxiliary problem with the homogeneous equation:

$$\mathcal{L}V = {}^{C}\partial_t^{\beta+1}V - \frac{\partial}{\partial x}\left(\gamma(x,t)\frac{\partial V}{\partial x}\right) - \frac{\partial^2}{\partial x \partial t}\left(\eta(x,t)\frac{\partial V}{\partial x}\right) - \int_0^t \xi(t-z)\,V(x,z)\,dz = 0, \quad (3.1)$$

$$\ell_1 V = V(x,0) = 0, \quad \ell_2 V = \frac{\partial V(x,0)}{\partial t} = 0, \quad x \in \Omega, \quad (3.2)$$

$$\int_0^1 V(x,t)\,dx = 0, \quad V_x(1,t) = 0, \quad t \in (0,T). \quad (3.3)$$

If V and v are solutions of problems respectively $(2.7)-(2.5)$,$(2.1)-(2.4)$, then $h=v-V$ satisfies

$$\begin{aligned}\mathcal{L}h &= {}^{C}\partial_t^{\beta+1}h-\frac{\partial}{\partial x}\left(\gamma(x,t)\frac{\partial h}{\partial x}\right)-\frac{\partial^2}{\partial x\partial t}\left(\eta(x,t)\frac{\partial h}{\partial x}\right)-\int\limits_0^t\xi(t-z)\,h(x,z)\,dz\\ &= G\left(x,t,h,\frac{\partial h}{\partial x}\right)\end{aligned}\tag{3.4}$$

$$\ell_1 h=h(x,0)=0,\quad \ell_2 h=\frac{\partial h}{\partial t}(x,0)=0,\quad x\in\Omega,\tag{3.5}$$

$$\int\limits_0^1 h(x,t)\,dx=0,\quad h_x(1,t)=0,\quad t\in(0,T),\tag{3.6}$$

such that the function $G(x,t,h,\frac{\partial h}{\partial x})=f(x,t,h+V,\frac{\partial h}{\partial x}+\frac{\partial V}{\partial x})$, verifies the following condition

$$|G(x,t,w_1,y_1)-G(x,t,w_2,y_2)|\le L\left(|w_1-w_2|+|y_1-y_2|\right)\quad \text{for all } (x,t)\in D.\tag{3.7}$$

Theorem 2.2 shows that the solution of problem $(2.5)-(2.7)$ is unique and depends continuously on the initial data. It remains to establish a similar proof for problem $(3.4)-(3.6)$. We introduce the space

$$\widetilde{C}^1(D)=\left\{w\in C^1(D)\quad\text{such that,}\quad\frac{\partial w^2}{\partial t\partial x}\in C(D)\right\}.\tag{3.8}$$

Suppose that h and $u\in\widetilde{C}^1(D)$ verify homogenous initial and boundary condtions $h(x,T)=0, h(x,0)=0, \int_0^1 h(x,t)dx=0$. For $u\in\widetilde{C}^1(D)$, we have

$$\begin{aligned}(\mathcal{L}h,\Im_x u)_{L^2(D)} &= \left({}^{C}\partial_t^{\beta+1}h,\Im_x u\right)_{L^2(D)}-\left(\frac{\partial}{\partial x}\left(\gamma(x,t)\frac{\partial h}{\partial x}\right),\Im_x u\right)_{L^2(D)}\\ &-\left(\frac{\partial^2}{\partial x\partial t}\left(\eta(x,t)\frac{\partial h}{\partial x}\right),\Im_x u\right)_{L^2(D)}-\left(\int\limits_0^t\xi(t-z)\,h(x,z)\,dz,\Im_x u\right)_{L^2(D)}.\end{aligned}\tag{3.9}$$

Computation of all terms of equation (3.9), using conditions on h and u, gives

$$\left({}^{C}\partial_t^{\beta+1}h, \Im_x u\right)_{L^2(D)} = -\left({}^{C}\partial_t^{\beta+1}\Im_x h, u\right)_{L^2(D)}, \tag{3.10}$$

$$-\left(\frac{\partial}{\partial x}\left(\gamma\frac{\partial h}{\partial x}\right), \Im_x u\right)_{L^2(D)} = \left(\gamma\frac{\partial h}{\partial x}, u\right)_{L^2(D)}, \tag{3.11}$$

$$-\left(\frac{\partial^2}{\partial x\partial t}\left(\eta\frac{\partial h}{\partial x}\right), \Im_x u\right)_{L^2(D)} = \left(\frac{\partial}{\partial t}\left(\eta\frac{\partial h}{\partial x}\right), u\right)_{L^2(D)}, \tag{3.12}$$

$$-\left(\int_0^t \xi\left(t-z\right)h\left(x,z\right)dz, \Im_x u\right)_{L^2(D)} = \left(\int_0^t \xi\left(t-z\right)\Im_x h\left(x,z\right)dz, u\right)_{L^2(D)}. \tag{3.13}$$

Insertion of (3.10) – (3.13) into (3.9) yields

$$R(h,u) = (u, \Im_x G)_{L^2(D)}. \tag{3.14}$$

Such that

$$\begin{aligned} R(h,u) = &-\left({}^{C}\partial_t^{\beta+1}\Im_x h, u\right)_{L^2(D)} + \left(\gamma\frac{\partial h}{\partial x}, u\right)_{L^2(D)} \\ &+\left(\frac{\partial}{\partial t}\left(\eta\frac{\partial h}{\partial x}\right), u\right)_{L^2(D)} + \left(\int_0^t \xi\left(t-z\right)\Im_x h\left(x,z\right)dz, u\right)_{L^2(D)}. \end{aligned} \tag{3.15}$$

Definition 3.1 *A function $h \in L^2(0,T,H^1(\Omega))$ is considered as the problem (3.4) – (3.6) weak solution if it satisfies (3.6) and (3.14) holds.*

Constructing an iteration sequence as follows: Let $h(0) = 0$, and let defining the sequence $(h^{(n)})_n \in \mathbb{N}$ as follows: If $h^{(n-1)}$ is given, then for $n \in \mathbb{N}$ solve the following problem:

$$\begin{aligned} \mathcal{L}h &= {}^{C}\partial_t^{\beta+1}h^{(n)} - \frac{\partial}{\partial x}\left(\gamma\frac{\partial h^{(n)}}{\partial x}\right) - \frac{\partial^2}{\partial x\partial t}\left(\eta\frac{\partial h^{(n)}}{\partial x}\right) - \int_0^t \xi\left(t-z\right)h^{(n)}\left(x,z\right)dz \\ &= G\left(x,t,h^{(n-1)}, \frac{\partial h^{(n-1)}}{\partial x}\right), \end{aligned} \tag{3.16}$$

$$\ell_1 h^{(n)} = h^{(n)}(x,0) = 0, \quad \ell_2 h^{(n)} = \frac{\partial h^{(n)}}{\partial t}(x,0) = 0, \quad x \in \Omega, \tag{3.17}$$

$$\int_0^1 h^{(n)}(x,t)\,dx = 0, \quad h_x^{(n)}(1,t) = 0, \quad t \in (0,T). \tag{3.18}$$

3.2 The existence of the solution

Theorem 3.1 *For each fixed n, assume that the solution of each problem* (3.16)−(3.18) $h^{(n)}(x,t)$ *is unique. If we put* $H^{(n)}(x,t) = h^{(n+1)}(x,t) - h^{(n)}(x,t)$, *then we obtain*

$$\mathcal{L}H^{(n)} = {}^{C}\partial_t^{\beta+1} H^{(n)} - \frac{\partial}{\partial x}\left(\gamma \frac{\partial H^{(n)}}{\partial x}\right) - \frac{\partial^2}{\partial x \partial t}\left(\eta \frac{\partial H^{(n)}}{\partial x}\right) - \int_0^t \xi(t-z)\, H^{(n)}(x,z)\,dz = \Psi^{(n-1)}(x,t) \tag{3.19}$$

$$\ell_1 H^{(n)} = H^{(n)}(x,0) = 0, \quad \ell_2 H^{(n)} = \frac{\partial H^{(n)}(x,0)}{\partial t} = 0, \quad x \in \Omega, \tag{3.20}$$

$$\int_0^1 H^{(n)}(x,t)\,dx = 0, \quad H_x^{(n)}(1,t) = 0, \quad t \in (0,T), \tag{3.21}$$

with

$$\Psi^{(n-1)}(x,t) = G\left(x,t,h^{(n)},\frac{\partial h^{(n)}}{\partial x}\right) - G\left(x,t,h^{(n-1)},\frac{\partial h^{(n-1)}}{\partial x}\right).$$

Lemma 3.1 *Under Assumptions* 2.1, *and supposing that the condition* (3.7) *holds, then for the linearized problem* (3.19) − (3.21), *the following estimate holds*

$$\|H^{(n)}\|_{L^2(0,T,H^1(\Omega))} \leq K \|H^{(n-1)}\|_{L^2(0,T,H^1(\Omega))}, \tag{3.22}$$

where $K > 0$ *is constant given by*

$$K = \exp(\delta_{10} T)\left(1 + \Gamma(\beta) E_{\beta,\beta}(\delta_9 \exp(\delta_{10} T)\, t^{\beta}) \frac{T^{\beta}}{\Gamma(1+\beta)}\right).$$

Proof We take the scalar product in $L^2(D^\tau)$, $\tau \in [0,T]$ of (3.19) and the integrodifferential operator $MH^{(n)} = -\Im_x^2 H^{(n)}$, we get

$$
\begin{aligned}
&2\left({}^{C}\partial_t^{\beta+1}H^{(n)},-\Im_x^2\frac{\partial H^{(n)}}{\partial t}\right)_{L^2(D^\tau)}-2\left(\frac{\partial}{\partial x}\left(\gamma\left(x,t\right)\frac{\partial H^{(n)}}{\partial x}\right),-\Im_x^2\frac{\partial H^{(n)}}{\partial t}\right)_{L^2(D^\tau)}\\
&-2\left(\frac{\partial^2}{\partial x\partial t}\left(\eta\left(x,t\right)\frac{\partial H^{(n)}}{\partial x}\right),-\Im_x^2\frac{\partial H^{(n)}}{\partial t}\right)_{L^2(D^\tau)}-2\left(\int\limits_0^t\xi\left(t-z\right)H^{(n)}\left(x,z\right)dz,-\Im_x^2\frac{\partial H^{(n)}}{\partial t}\right)_{L^2(D^\tau)}\\
&=2\left(\Psi^{(n-1)}\left(x,t\right),-\Im_x^2\frac{\partial H^{(n)}}{\partial t}\right)_{L^2(D^\tau)}. \qquad (3.23)
\end{aligned}
$$

Integrations by parts all terms of (3.23), by using conditions (3.20) – (3.21), yields

$$
\begin{aligned}
&\left({}^{C}\partial_t^{\beta+1}H^{(n)},-\Im_x^2\frac{\partial H^{(n)}}{\partial t}\right)_{L^2(D^\tau)}=-\int_0^\tau\left[{}^{C}\partial_t^{\beta}\left(\Im_x\frac{\partial H^{(n)}}{\partial t}\right)\Im_x^2\frac{\partial H^{(n)}}{\partial t}\right]_0^1dt\\
&+\int_0^\tau{}^{C}\partial_t^{\beta}\left(\Im_x\frac{\partial H^{(n)}}{\partial t}\right)\Im_x\frac{\partial H^{(n)}}{\partial t}dxdt=\int_0^\tau{}^{C}\partial_t^{\beta}\left(\Im_x\frac{\partial H^{(n)}}{\partial t}\right)\Im_x\frac{\partial H^{(n)}}{\partial t}dxdt, \qquad (3.24)
\end{aligned}
$$

$$
\begin{aligned}
&-2\left(\frac{\partial}{\partial x}\left(\gamma\left(x,t\right)\frac{\partial H^{(n)}}{\partial x}\right),-\Im_x^2H_t^{(n)}\right)_{L^2(D^\tau)}=-\int_{D^\tau}\frac{\partial\gamma}{\partial t}(H^{(n)})^2dxdt\\
&+2\int_{D^\tau}\frac{\partial\gamma}{\partial x}H^{(n)}\Im_x\frac{\partial H^{(n)}}{\partial t}dxdt+\int_0^1\gamma\left(x,\tau\right)(H^{(n)})^2\left(x,\tau\right)dx, \qquad (3.25)
\end{aligned}
$$

$$
\begin{aligned}
&-2\left(\frac{\partial^2}{\partial x\partial t}\left(\eta\left(x,t\right)\frac{\partial H^{(n)}}{\partial x}\right),-\Im_x^2\frac{\partial H^{(n)}}{\partial t}\right)_{L^2(D^\tau)}=2\int_{D^\tau}\eta\left(\frac{\partial H^{(n)}}{\partial t}\right)^2dxdt\\
&+\int_0^1\frac{\partial\eta\left(x,\tau\right)}{\partial t}(H^{(n)})^2\left(x,\tau\right)dx-\int_{D^\tau}\frac{\partial^2\eta}{\partial t^2}(H^{(n)})^2dxdt+2\int_{D^\tau}\frac{\partial\eta}{\partial x}\frac{\partial H^{(n)}}{\partial t}\Im_x\frac{\partial H^{(n)}}{\partial t}dxdt\\
&+2\int_{D^\tau}\frac{\partial^2\eta}{\partial x\partial t}H^{(n)}\Im_x\frac{\partial H^{(n)}}{\partial t}dxdt. \qquad (3.26)
\end{aligned}
$$

Substituting (3.24) – (3.26) into (2.14), yields

$$\begin{aligned}
&2\left({}^{C}\partial_t^{\beta}\left(\Im_x\frac{\partial H^{(n)}}{\partial t}\right),\Im_x\frac{\partial H^{(n)}}{\partial t}\right)_{L^2(D^\tau)}+2\int_{D^\tau}\eta\left(\frac{\partial H^{(n)}}{\partial t}\right)^2 dxdt\\
&+\int_0^1\left(\gamma(x,\tau)+\frac{\partial\eta(x,\tau)}{\partial t}\right)(H^{(n)})^2(x,\tau)\,dx=2\left(\Psi^{(n-1)}(x,t),-\Im_x^2\frac{\partial H^{(n)}}{\partial t}\right)_{L^2(D^\tau)}\\
&+2\left(\int_0^t\xi(t-z)H^{(n)}(x,z)\,dz,-\Im_x^2\frac{\partial H^{(n)}}{\partial t}\right)_{L^2(D^\tau)}+\int_{D^\tau}\left(\frac{\partial\gamma}{\partial t}+\frac{\partial^2\eta}{\partial t^2}\right)(H^{(n)})^2dxdt\\
&-2\int_{D^\tau}\frac{\partial\eta}{\partial x}\frac{\partial H^{(n)}}{\partial t}\Im_x\frac{\partial H^{(n)}}{\partial t}dxdt-2\int_{D^\tau}\left(\frac{\partial\gamma}{\partial x}+\frac{\partial^2\eta}{\partial x\partial t}\right)H^{(n)}\Im_x\frac{\partial H^{(n)}}{\partial t}dxdt. \qquad (3.27)
\end{aligned}$$

Estimating the first and the last two terms on the right-hand side (RHS) of (3.27) by applying inequality (1.7); it follows that

$$-2\int_{D^\tau}\Psi^{(n-1)}(x,t)\,\Im_x^2\frac{\partial H^{(n)}}{\partial t}dxdt\le\int_{D^\tau}\left(\Psi^{(n-1)}(x,t)\right)^2dxdt+\int_{D^\tau}\left(\Im_x\frac{\partial H^{(n)}}{\partial t}\right)^2dxdt, \quad (3.28)$$

$$2\int_{D^\tau}\left(\int_0^t\xi(t-z)H^{(n)}(x,z)\,dz\right)\Im_x^2\frac{\partial H^{(n)}}{\partial t}dxdt\le c_{10}T^2\parallel H^{(n)}\parallel^2_{L(D^\tau)}+\frac{1}{2}\int_{D^\tau}\left(\Im_x\frac{\partial H^{(n)}}{\partial t}\right)^2dxdt, \quad (3.29)$$

$$\begin{aligned}
-2\int_{D^\tau}\left(\frac{\partial\gamma}{\partial x}+\frac{\partial^2\eta}{\partial x\partial t}\right)H^{(n)}\Im_x\frac{\partial v}{\partial t}dxdt\quad&\le 2\int_{D^\tau}\left\{\left(\frac{\partial\gamma}{\partial x}\right)^2+\left(\frac{\partial^2\eta}{\partial x\partial t}\right)^2\right\}(H^{(n)})^2dxdt\\
&+\int_{D^\tau}\left(\Im_x\frac{\partial H^{(n)}}{\partial t}\right)^2dxdt, \qquad (3.30)
\end{aligned}$$

$$-2\int_{D^\tau}\frac{\partial\eta}{\partial x}\frac{\partial H^{(n)}}{\partial t}\Im_x\frac{\partial H^{(n)}}{\partial t}dxdt\le c_4\int_{D^\tau}\left(\frac{\partial H^{(n)}}{\partial t}\right)^2dxdt+\frac{1}{c_4}\int_{D^\tau}\left(\frac{\partial\eta}{\partial x}\right)^2\left(\Im_x\frac{\partial H^{(n)}}{\partial t}\right)^2dxdt. \quad (3.31)$$

By Lemma 1.6, the first term on the LHS of (3.27) becomes

$$2\left({}^{C}\partial_t^{\beta}\left(\Im_x\frac{\partial H^{(n)}}{\partial t}\right),\Im_x\frac{\partial H^{(n)}}{\partial t}\right)_{L^2(D^\tau)}\ge\int_{D^\tau}{}^{C}\partial_t^{\beta}\left(\Im_x\frac{\partial H^{(n)}}{\partial t}\right)^2dxdt. \quad (3.32)$$

Hence, by formulas (3.28) – (3.32) and Assumption (2.1), we obtain

$$D^{\beta-1}\|\Im_x \frac{\partial H^{(n)}}{\partial t}\|^2_{L^2(\Omega)} + (c_0+c_5) \parallel H^{(n)}(.,\tau) \parallel^2_{L^2(\Omega)} \leq \int_0^\tau \parallel \Im_x \Psi^{(n-1)}(x,t) \parallel^2_{L(\Omega)} dt$$
$$+\left(\frac{5}{2}+\frac{c_7^2}{2c_4}\right)\int_0^\tau \parallel \Im_x \frac{\partial H^{(n)}}{\partial t}(.,t) \parallel^2_{L(\Omega)} dt + \left(2\left(c_3^2+c_9^2\right)+c_2+c_8+c_{10}T^2\right)$$
$$\times \int_0^\tau \parallel H^{(n)}(.,t) \parallel^2_{L^2(\Omega)} dt. \tag{3.33}$$

On the other hand, applying to equation (3.19) the operator $\Im_x$, and taking into consederation condition (3.21), multiplying the resulted equation with $\frac{\partial T^{(n)}}{\partial x}$ and integrating over D^τ, we get

$$\int_{D^\tau} {}^C\partial_t^{\beta+1}\Im_x H^{(n)}\frac{\partial H^{(n)}}{\partial x}dxdt - \int_{D^\tau}\gamma(x,t)\left(\frac{\partial H^{(n)}}{\partial x}\right)^2 dxdt$$
$$-\int_{D^\tau}\frac{\partial}{\partial t}\left(\eta(x,t)\frac{\partial H^{(n)}}{\partial x}\right)\frac{\partial H^{(n)}}{\partial x}dxdt - \int_{D^\tau}\int_0^t \xi(t-z)\Im_x H^{(n)}(x,z)\frac{\partial H^{(n)}}{\partial x}dzdxdt$$
$$= \int_{D^\tau}\Im_x\Psi^{(n-1)}(x,t)\frac{\partial H^{(n)}}{\partial x}dxdt. \tag{3.34}$$

After integartion by parts of all terms of (3.34) and taking into consideration conditions (3.20) ,(3.21) and using inequality (1.17), we have

$$\int_{D^\tau} {}^C\partial_t^{\beta+1}H^{(n)}H^{(n)}dxdt + c_0\int_0^\tau \parallel \frac{\partial H^{(n)}}{\partial x}(.,t) \parallel_{L^2(\Omega)} dt + \frac{1}{2}c_4 \parallel \frac{\partial H^{(n)}}{\partial x}(.,\tau) \parallel^2_{L^2(\Omega)}$$
$$\leq \frac{1}{2}\int_0^\tau \parallel \Psi^{(n-1)} \parallel_{L^2(\Omega)} dt + \left(c_{10}T^2+1\right)\int_0^\tau \parallel H^{(n)}(.,t) \parallel_{L^2(\Omega)} dt. \tag{3.35}$$

Combination of inequalities (3.34) – (3.35) gives

$$D_\tau^{\beta-1}\|\Im_x \frac{\partial H^{(n)}}{\partial t}\|_{L^2(\Omega)} + \int_0^\tau \partial_t^{\beta+1}H^{(n)}H^{(n)}dxdt + c_0\int_0^\tau \parallel \frac{\partial H^{(n)}}{\partial x}(.,t) \parallel_{L^2(\Omega)} dt$$
$$+\frac{1}{2}c_4 \parallel \frac{\partial H^{(n)}}{\partial x}(.,\tau) \parallel^2_{L^2(\Omega)} + (c_0+c_5) \parallel H^{(n)}(.,\tau) \parallel_{L^2(\Omega)} \leq \int_0^\tau \parallel \Psi^{(n-1)} \parallel_{L^2(\Omega)} dt$$
$$+\left(\frac{5}{2}+\frac{C_7^2}{2C_4}\right)\int_0^\tau \parallel \Im_x \frac{\partial H^{(n)}}{\partial t}(.,t) \parallel_{L^2(\Omega)} dt + \left(\left(2(c_3^2+c_9^2\right)+c_2+c_8+c_{10}T^2+1\right)$$
$$\times \int_0^\tau \parallel H^{(n)}(.,t) \parallel^2_{L^2(\Omega)} dt. \tag{3.36}$$

Eliminating the last term on the RHS of (3.36), by using Gronwell's Lemma , it comes

$$\begin{aligned} & D_\tau^{\beta-1}\|\Im_x\frac{\partial H^{(n)}}{\partial t}\|_{L^2(\Omega)}+\int_0^\tau \partial_t^{\beta+1}H^{(n)}H^{(n)}dxdt+(c_0+c_5)\parallel H^{(n)}(.,\tau)\parallel_{L^2(\Omega)} \\ & +\frac{1}{2}c_4\parallel\frac{\partial H^{(n)}}{\partial x}(.,\tau)\parallel^2_{L^2(\Omega)}+c_0\int_0^\tau\parallel\frac{\partial H^{(n)}}{\partial x}(.,t)\parallel_{L^2(\Omega)}dt \\ \leq\ & \exp(\delta_{10})\left\{\int_0^\tau\parallel\Psi^{(n-1)}\parallel^2_{L(\Omega)}dt+\delta_9\int_0^\tau\parallel\Im_x\frac{\partial H^{(n)}}{\partial t}(.,t)\parallel^2_{L(\Omega)}dt\right\}. \end{aligned} \tag{3.37}$$

Where

$$\delta_9=\frac{5}{2}+\frac{C_7^2}{2C_4}.$$
$$\delta_{10}=2\left(c_3^2+c_9^2\right)+c_2+c_8+c_{10}T^2+1.$$

To discard the last integral on the RHS of inequality (3.37), let's drop the three first elements then use the Growall's Lemma, it follows

$$\int_0^\tau\|\Im_x\frac{\partial H^{(n)}}{\partial t}\|^2_{L^2(\Omega)}dt\leq\Gamma(\beta)E_{\beta,\beta}(\delta_9\exp(\delta_{10}T)\,t^\beta)\exp(\delta_9 t)\,D_t^{-\beta}\parallel\Psi^{(n-1)}\parallel^2_{L(\Omega)}. \tag{3.38}$$

On the other hand, via the condition (3.7), we get

$$\int_0^\tau\parallel\Psi^{(n-1)}\parallel^2_{L(\Omega)}dt\leq 2L^2\int_0^\tau\left(\parallel H^{(n-1)}(.,t)\parallel^2_{L^2(\Omega)}+\parallel\frac{\partial H^{(n-1)}(.,t)}{\partial x}\parallel^2_{L^2(\Omega)}\right)dt. \tag{3.39}$$

Combination of (3.37) – (3.39) and by using (2.29), we get

$$\begin{aligned} & D^{\beta-1}\|\Im_x\frac{\partial H^{(n)}}{\partial t}\|^2_{L^2(\Omega)}+\int_0^\tau \partial_t^{\beta+1}H^{(n)}H^{(n)}dxdt \\ & +\int_0^\tau\parallel\frac{\partial H^{(n)}}{\partial x}(.,t)\parallel^2_{L^2(\Omega)}dt+\parallel\frac{\partial H^{(n)}}{\partial x}(.,\tau)\parallel^2_{L^2(\Omega)}+\parallel H^{(n)}(.,\tau)\parallel^2_{L^2(\Omega)}\leq \\ & \delta_{11}L^2\int_0^T\left(\|H^{(n-1)}(.,t)\|^2_{L^2(\Omega)}+\|\frac{\partial H^{(n-1)}}{\partial x}(.,t)\|^2_{L^2(\Omega)}\right)dt. \end{aligned} \tag{3.40}$$

where

$$\delta_{11}=\exp(\delta_{10}T)\left(1+\Gamma(\beta)E_{\beta,\beta}(\delta_9\exp(\delta_{10}T)\,t^\beta)\frac{T^\beta}{\Gamma(1+\beta)}\right).$$

After discaring the first two terms on the LHS of inequality (3.40), we get

$$\begin{aligned}&\int_0^{\tau} \parallel \frac{\partial H^{(n)}}{\partial x}(.,t) \parallel_{L^2(\Omega)}^2 dt + \parallel H^{(n)}(.,\tau) \parallel_{L^2(\Omega)}^2 + \parallel \frac{\partial H^{(n)}}{\partial x}(.,\tau) \parallel_{L^2(\Omega)}^2 \\ \leq \ & \delta_{11} L^2 \int_0^T \left(\| H^{(n-1)}(.,t) \|_{L^2(\Omega)}^2 + \| \frac{\partial H^{(n-1)}}{\partial x}(.,t) \|_{L^2(\Omega)}^2 \right) dt. \end{aligned} \tag{3.41}$$

Here, the RHS doesn't depend on τ so, we can replace the LHS by upper bounds with respect to τ, we obtain

$$\begin{aligned}&\int_0^{T} \parallel \frac{\partial H^{(n)}}{\partial x}(.,t) \parallel_{L^2(\Omega)}^2 dt + \parallel H^{(n)}(.,\tau) \parallel_{L^2(\Omega)}^2 + \parallel \frac{\partial H^{(n)}}{\partial x}(.,\tau) \parallel_{L^2(\Omega)}^2 \\ \leq \ & \delta_{11} L^2 \int_0^T \left(\| H^{(n-1)}(.,t) \|_{L^2(\Omega)}^2 + \| \frac{\partial H^{(n-1)}}{\partial x}(.,t) \|_{L^2(\Omega)}^2 \right) dt \end{aligned} \tag{3.42}$$

Now, let's integrate over $(0,T)$, we get

$$\begin{aligned}&\int_0^{T} \parallel H^{(n)}(.,\tau) \parallel_{L^2(\Omega)}^2 dt + \int_0^T \parallel \frac{\partial H^{(n)}}{\partial x}(.,t) \parallel_{L^2(\Omega)}^2 dt \\ \leq \ & \delta_{12} L^2 \int_0^T \left(\| H^{(n-1)}(.,t) \|_{L^2(\Omega)}^2 + \| \frac{\partial H^{(n-1)}}{\partial x}(.,t) \|_{L^2(\Omega)}^2 \right) dt. \end{aligned} \tag{3.43}$$

$$\delta_{12} = \frac{\delta_{11} L^2 T}{\min(1,T)}.$$

We get then the desired inequality (3.22)

$$\|H^{(n)}\|_{L^2(0,T,H^1(\Omega))} \leq \delta_{12} \|H^{(n-1)}\|_{L(0,T,H^1(\Omega))}. \tag{3.44}$$

This complete the proof of Lemma 3.1. ■

Using the convergence of serie criteria we conclude that $\sum_{n=1}^{\infty} H^{(n)}$ converges if $\frac{\delta_{11}L^2T}{\min(1,T)} < 1$, namely if $L < \sqrt{\frac{\min(1,T)}{\delta_{11}T}}$. Since $H^{(n)}(x,t) = h^{(n+1)}(x,t) - h^{(n)}(x,t)$, then the sequence $\left(h^{(n)}\right)_{n\in\mathbb{N}}$ given by $h^{(n)}(x,t) = \sum_{i=0}^{n-1} H^{(i)} + h^{(0)}(x,t)$, $i \in \mathbb{N}$ converges to a function $h \in L^2((0,T), H^1(0,1))$.

In order to show that this limit is the solution of problem (3.19) – (3.21), it is sufficient to demonstrate that h verifies (3.6) and (3.14).

We have from problem (3.16) $-$ (3.18) that

$$R(h^{(n)},u)=\left(u,\Im_x G\left(x,t,h^{(n-1)},\frac{\partial h^{(n-1)}}{\partial x}\right)\right)_{L^2(D)} \tag{3.45}$$

Precisely

$$\begin{aligned} R(h^{(n)}-h,u)+R(h,u) \quad &= \left(u,\Im_x G\left(x,t,h^{(n-1)},\frac{\partial h^{(n-1)}}{\partial x}\right)-\Im_x G\left(x,t,h,\frac{\partial h}{\partial x}\right)\right)_{L^2(D)} \\ &+\left(u,\Im_x G\left(x,t,h,\frac{\partial h}{\partial x}\right)\right)_{L^2(D)} \end{aligned} \tag{3.46}$$

using equation (3.16), then (3.46) becomes

$$\begin{aligned} R(h^{(n)}-h,u) &= -\left({}^C\partial_t^{\beta+1}\Im_x\left(h^{(n)}-h\right),u\right)_{L^2(D)}+\left(\gamma\frac{\partial\left(h^{(n)}-h\right)}{\partial x},u\right)_{L^2(D)} \\ &+\left(\frac{\partial}{\partial t}\left(\eta\frac{\partial\left(h^{(n)}-h\right)}{\partial x}\right),u\right)_{L^2(D)}+\left(\int_0^t \xi\left(t-z\right)\Im_x\left(h^{(n)}-h\right)(x,z)\,dz,u\right)_{L^2(D)} \end{aligned} \tag{3.47}$$

Integrating by parts all terms on the LHS, taking into consideration conditions on v and w, this transform (3.47) into

$$\begin{aligned} R(h^{(n)}-h,u) &= -\left({}^C\partial_t^{\beta+1}\left(h^{(n)}-h\right),\Im_x u\right)_{L^2(D)}+\left(\gamma\frac{\partial\left(h^{(n)}-h\right)}{\partial x},u\right)_{L^2(D)} \\ &+\left(\eta\frac{\partial\left(h^{(n)}-h\right)}{\partial x},\frac{\partial u}{\partial t}\right)_{L^2(D)}+\left(\int_0^t \xi\left(t-z\right)\Im_x\left(h^{(n)}-h\right)(x,z)\,dz,u\right)_{L^2(D)} \end{aligned} \tag{3.48}$$

Applying Cauchy-Schwartz inequality, yields

$$R(h^{(n)}-h,u)\leq \delta_{13}\parallel h^{(n)}-h\parallel_{L^2(0,T,H^1(\Omega))}\left(\parallel u\parallel_{L^2((D)}+\parallel\frac{\partial u}{\partial t}\parallel_{L^2((D)}\right) \tag{3.49}$$

where

$$\delta_{13} = \max\left(c_1 + T\frac{c_0}{2}, c_6\right)$$

and we have from (3.46) the following estimate

$$\left(u, \Im_x G\left(x,t,h^{(n-1)}, \frac{\partial h^{(n-1)}}{\partial x}\right) - \Im_x G\left(x,t,h,\frac{\partial h}{\partial x}\right)\right)_{L^2(D^T)} \leq \frac{L}{\sqrt{2}} \parallel h^{(n)} - h \parallel_{L^2((0,T,H^1(\Omega))} \parallel u \parallel_{L^2((D)} \tag{3.50}$$

Passing to the limit $n \to \infty$ in (3.48), and taking into consideration (3.49)-(3.50) ,we obtain

$$R(h,u) = \left(u, \Im_x G\left(x,t,h,\frac{\partial h}{\partial x}\right)\right)_{L^2(D)} \tag{3.51}$$

To conclude that problem $(3.19)-(3.21)$ admits a weak solution, we prove that (3.6) holds. Since $lim_{n\to\infty} \parallel h^{(n)} - h \parallel_{L^2((0,T;H^1(\Omega))}= 0$ then, we deduce that $\int_0^1 hdx = 0$ and $\frac{\partial h}{\partial x}(1,t) = 0$.

Therefore, we have estabilished this result:

Theorem 3.2 *Suppose that conditions of Lemma* 3.1 *hold, and that* $L < \sqrt{\frac{\min(1,T)}{\delta_{11}T}}$*, then the problem* (3.4)-(3.6) *admits a weak solution in* $L^2(0,T,H^1(\Omega))$.

3.3 The uniqueness of the solution

Now, we prove the uniqueness of solution of problem $(3.4) - (3.6)$.

Theorem 3.3 *Under conditions of Lemma* (3.1)*, the problem* $(3.4) - (3.6)$ *admits a unique.*

Proof Suppose that the problem $(3.4) - (3.6)$ admits v_1 and v_2 as solutions in $L^2(0,T,H^1(\Omega))$, then $H = v_1 - v_2$ belongs to $L^2(0,T,H^1(\Omega))$ and verifies

$$\mathcal{L}H = {}^C\partial_t^{\beta+1}H - \frac{\partial}{\partial x}\left(\gamma(x,t)\frac{\partial H}{\partial x}\right) - \frac{\partial^2}{\partial x \partial t}\left(\eta(x,t)\frac{\partial H}{\partial x}\right) - \int\limits_0^t \xi(t-z)\,H(x,z)\,dz = \Psi(x,t) \tag{3.52}$$

$$\ell_1 H = H(x,0) = 0, \quad \ell_2 H = H_t(x,0) = 0, \quad x \in \Omega, \tag{3.53}$$

$$\int_0^1 H(x,t)\,dx = 0, \quad H_x(1,t) = 0, \quad t \in (0,T), \tag{3.54}$$

where $\Psi(x,t) = G\left(x,t,v_1,\frac{\partial v_1}{\partial x}\right) - G\left(x,t,v_2,\frac{\partial v_2}{\partial x}\right)$.

This will be done by establishing the same proof of Lemma 3.1; we obtain

$$\|H\|_{L^2(0,T,H^1(\Omega))} \leq K\|H\|_{L^2(0,T,H^1(\Omega))} \tag{3.55}$$

Since $K < 1$, then from (3.22) we have $(1-K)\|H\|_{L^2(0,T,H^1(\Omega))} \leq 0$, from which we deduce that $v_1 = v_2$ in $L^2((0,T), H^1(\Omega))$. ■

Conclusion

This thesis studied the existence, uniqueness, and continuous dependence of a weak solution for some nonlinear fractional mixed problems classes with nonlocal conditions (nonlocal boundary conditions). The used method is one of the most efficient functional analysis methods for solving partial differential equations with nonlocal conditions, the so-called energy-integral method or a priori estimates method. We constructed a suitable multiplicator, which provided a priori estimate, from which it was possible to establish the solvability of the problem. However, the great flexibility of the method has its disadvantage. The significant difficulty of choosing an adequate multiplicator to the considered problem is the crucial step of establishing the a priori estimate. We first found a priori estimate for an associated linear problem and the range density, ensuring the solvability of this problem. Then, by applying an iterative process based on the obtained results for the linear problem, we proved the existence, uniqueness, and continuous dependence of a weak solution of the main problem.

Perspectives

Questions that can be posed here are

- There are many fractional derivatives approaches. What difference can make the change of the derivative Caputo to the present study?

- The present work consists of a time fractional nonlocal nonlinear problem, is it possible to obtain the existence and the uniqueness of the solutions when we add the spacial fractional derivative?

- Or if we add mixed fractional derivative (Riemann-Liouville with Caputo ...)

We plan in the future to study the existence and uniqueness of solutions of time and space fractional nonlinear nonlocal problems, by the method of energy inequally method.

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Abstract

This thesis studies the existence and uniqueness of a weak solution for fractional nonlinear mixed problem classes with a nonlocal condition. The used method is one of the sufficient functional analysis methods for solving partial differential equations with integral boundary conditions, the so-called energy-integral method or a priori estimates method. First, we establish a priori estimate for the associated linear problem and demonstrate the generated operator range density. Then, by applying an iterative process based on the obtained results for the linear problem, we prove the existence, uniqueness and continuous dependence of the weak solution of the nonlinear problem.

Keywords: Existence and uniqueness of solution, A priori estimate, Fractional derivatives and integrals, Integral condition.

Résumé

Dans cette thèse, on étudie l'existence et l'unicité d'une solution faible pour une classe de problèmes mixtes non-linéaires fractionnaires avec une condition non-locale. La méthode utilisée est l'une des méthodes d'analyse fonctionnelle pour résoudre des équations aux dérivées partielles avec des conditions aux limites intégrales, dite ; la méthode des estimations a priori. Tout d'abord, on établit une estimation a priori pour le problème linéaire associé et on démontre la densité de l'image d'opérateurs engendré par le problème considéré. Ensuite, en appliquant un processus itératif basé sur les résultats obtenus pour le problème linéaire, on prouve l'existence, l'unicité et la dépendance continue de la solution faible du problème non-linéaire.

Mots clés: Existence et unicité de la solution, Estimation a priori, Dérivées et intégrales fractionnaire, Condition intégrale.

الملخص

تدرس هذه الأطروحة وجود و وحدانية الحل الضعيف لمعادلة غير خطية كسرية. الطريقة المستخدمة هي إحدى طرق التحليل الدالي الكافية لحل المعادلات التفاضلية الجزئية بشروط حدية متكاملة ، والتي تسمى طريقة التقديرات المسبقة. أولاً، قمنا بإنشاء تقدير مسبق للمشكلة الخطية المصاحبة ثم أثبتنا كثافة صورة المؤثر المولد بالمسألة المعطاة. بعد ذلك، من خلال تطبيق عملية تكرارية بناءً على النتائج التي تم الحصول عليها للمسألة الخطية، نثبت وجود الحل الضعيف للمشكلة غير الخطية.

كلمات مفتاحية : وجود ووحدانية الحل، تقديرات مسبقة، مشتقة وتكامل كسريان ، شرط تكاملي.