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# On the distribution of certain Pisot numbers

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**Abstract**

A Pisot number $\theta$ is said to be simple if the beta-expansion of its fractional part, in base $\theta$, is finite. Let $\mathbb{P}$ be the set of such numbers, and $\mathbb{S} \setminus \mathbb{P}$ be the complement of $\mathbb{P}$ in the set $\mathbb{S}$ of Pisot numbers. We show several results about the derived sets of $\mathbb{P}$ and of $\mathbb{S} \setminus \mathbb{P}$. A Pisot number $\theta$, with degree greater than 1, is said to be strong, if it has a proper real positive conjugate which is greater than the modulus of the remaining conjugates of $\theta$. The set, say $\mathbb{X}$, of such numbers has been defined by Boyd (1993) [5], and is contained in $\mathbb{S} \setminus \mathbb{P}$. We also prove that the infimum of the $j$-th derived set of $\mathbb{X}$, where $j$ runs through the set of positive rational integers, is at most $j + 2$.


*Keywords:* Beta-expansion; Pisot numbers; Beta-numbers

## 1. Introduction

For a real number $\beta > 1$, let $(r_n(\beta), \varepsilon_n(\beta))_{n \geq 1} = (r_n, \varepsilon_n)_{n \geq 1}$ be the sequence defined by the relations

$$(r_n, \varepsilon_n) := (\{\beta r_{n-1}\}, [\beta r_{n-1}]),$$

where { } and [ ] are, respectively, the fractional and integer part functions, $r_0 = r_0(\beta) := \{\beta\}$, and $n$ runs through the set $\mathbb{N}$ of positive rational integers. Then, $r_n \in [0, 1[$, $\varepsilon_n \in$

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$\{0, 1, \ldots, \max_{k\in\mathbb{N}\cap[1,\beta[} k\}$, $\{\beta\} = \sum_{n\geq 1} \varepsilon_n/\beta^n$, and we write

$$\{\beta\} \equiv \sum_{n\geq 1} \frac{\varepsilon_n}{\beta^n}.$$

The sequence $(\varepsilon_n)_{n\geq 1}$, called beta-expansion of $\{\beta\}$, was defined, in a more general context, by Rényi [17]. If this sequence is periodic, then $\beta$ is said to be a beta-number; in particular, when $(\varepsilon_n)_{n\geq 1}$ terminates only by zeros, the number $\beta$ is called a simple beta-number. Beta-numbers have been defined and studied by Parry [15]. One of the results of [15], asserts that beta-numbers are algebraic integers, the conjugates of a beta-number $\beta$, are of modulus at most $\min(2, \beta)$, and simple beta-numbers are dense in the interval $[1, \infty[$.

An easy argument shows that Pisot numbers are beta-numbers [19]. A Pisot number is a real algebraic integer greater than 1, whose other conjugates are of modulus less than 1. Some results related to the beta-expansion of real numbers, in a base which is a Pisot number, may be found in [1,6,10,24]. A large amount of the structure of the set, usually denoted as $\mathbb{S}$, of Pisot numbers is understood. For example, $\mathbb{S}$ is closed in the real line $\mathbb{R}$ [18], $\min \mathbb{S} = \theta_0 = 1.3247\ldots$, where $\theta_0^3 - \theta_0 - 1 = 0$ [20], the elements of $\mathbb{S}\cap]1, 1.6183\ldots[$ are known [8] (see also Theorem 7.2.1 of [4], or the proof of Theorem 1 below), $\min \mathbb{S}^{(1)} = \frac{1+\sqrt{5}}{2} = 1.6180\ldots$ [8], a complete list of the elements of $\mathbb{S}^{(1)}\cap]1, 2[$ is exhibited in [2] (see also Theorem 1.6 of [14]), and $\min \mathbb{S}^{(2)} = 2$ [11]. Throughout, the notation $I^{(j)}$, where $I \subset \mathbb{R}$ and $j \in \mathbb{N}$, means the $j$-th derived set of $I$.

The distribution of the elements of the set, say $\mathbb{P}$, of Pisot numbers which are simple beta-numbers, is not completely known. In a very recent paper, Panju [14] determined the beta-expansion of $\{\beta\}$, when $\beta$ runs through the set of regular Pisot numbers in the interval $]1, 2[$. A regular Pisot number less than 2, is an element of $\mathbb{S}^{(1)}\cap]1, 2[$, or is a term of a sequence that tend to an element of $\mathbb{S}^{(1)}\cap]1, 2[$. Hence, for any $\varepsilon > 0$, there are at most finitely many non-regular Pisot numbers in $[1, 2 - \varepsilon]$. The sequences of Pisot numbers that approach any element of $\mathbb{S}^{(1)}\cap]1, 2[$ are completely understood [23].

The following theorem collects some results about the structures of the set $\mathbb{P}$ and of its complement $\mathbb{S}\backslash\mathbb{P}$ in $\mathbb{S}$. The first and second parts are essentially consequences of the computation done in [14].

**Theorem 1.** *With the notation above, we have:*

(i) $\mathbb{P}\cap]1, \left(1+\sqrt{5}\right)/2] = \mathbb{S}\cap]1, \left(1+\sqrt{5}\right)/2]$;
(ii) $\inf \mathbb{P}^{(2)} = \inf(\mathbb{S} \setminus \mathbb{P})^{(2)} = 2$, *and the two sets* $\mathbb{P}$ *and* $\mathbb{S} \setminus \mathbb{P}$ *are not closed;*
(iii) *if* $a \in \mathbb{N} \cap [2, \infty[$, *then* $a \in \mathbb{P}^{(2)}$.

It is clear that beta-numbers of degree 1, belong to $\mathbb{P}$, and a short computation shows that quadratic beta-numbers, are also Pisot numbers (see the proof of Theorem 2(i) below). In [3], Bassino has proved that simple cubic beta-numbers are Pisot numbers too. Next, we complete these results by proving the following theorem.

**Theorem 2.** *The following propositions are true.*

(i) *Let* $\beta$ *be a beta-number, with degree* $d$. *Then,* $\beta$ *is a Pisot number, when* $d \leq 2$, *or when* $\beta$ *is simple and* $d = 3$.
(ii) *There are infinitely many cubic beta-numbers which are not Pisot numbers.*
(iii) *For any* $d \in \mathbb{N} \cap [4, \infty[$, *there are infinitely many simple beta-numbers, with degree* $d$, *which are not Pisot numbers.*

A Pisot number $\theta$ with degree $d$ is said to be strong, if $d = 1$, or if $d \geq 2$ and $\theta$ has a proper real positive conjugate, which is greater than the absolute value of the $d-2$ remaining conjugates of $\theta$. The set, say $\mathbb{X}$, of strong Pisot numbers has been defined by Boyd [5], and its elements appear in a paper of Dubickas [7] as examples of $Z$-numbers. A theorem of Pisot [16], asserts that a real number $\beta > 1$, is a Pisot number, if and only if, there is $\lambda \in \mathbb{R} \setminus \{0\}$, such that $\sum_{n\in\mathbb{N}} \|\lambda\beta^n\|^2 < \infty$, where $\|t\| = \min\{\{t\}, 1-\{t\}\}$ is the usual distance from $t$ to the ring of rational integers $\mathbb{Z}$. Using this result of Pisot, the author obtained, recently [27], a characterization of the elements of $\mathbb{X}$, among real numbers greater than 1:

$$\beta \in \mathbb{X} \Longleftrightarrow \forall \varepsilon > 0, \quad \exists \lambda \in \mathbb{R} \setminus \{0\} \mid \sum_{n\geq 0} \{\lambda\beta^n\}^2 < \varepsilon.$$

It is worth noting, that the relation above, does not allow to prove that $\mathbb{X}$ is a closed subset of $\mathbb{R}$, in a similar way that it has been done by Salem, for the set $\mathbb{S}$ [18]. I am not able to prove (or disprove) that $\mathbb{X}$ is closed. The following theorem gives explicitly some elements of $\mathbb{X}^{(j)}$, where $j$ runs through $\mathbb{N}$.

**Theorem 3.** *For $a \in \mathbb{N} \cap [3, \infty[$, $k \in \{1, \ldots, a-2\}$, and $\{n_1, \ldots, n_k\} \subset \mathbb{N}$, let $P_{(n_k,\ldots,n_0;a)}$ be the polynomial defined by the relation*

$$P_{(n_k,\ldots,n_0;a)}(x) = x^{n_k} P_{(n_{k-1},\ldots,n_0;a)}(x) + 1, \tag{1}$$

*where $P_{(n_0;a)}(x) := x - a$. Then, $P_{(n_k,\ldots,n_0;a)}$ is the minimal polynomial of a strong Pisot number $\theta_{(n_k,\ldots,n_0;a)} \in ]a-1, a]$, and $\lim_{n_k\to\infty} \theta_{(n_k,\ldots,n_0;a)} = \theta_{(n_{k-1},\ldots,n_0;a)}$.*

A proposition in [27], asserts that $\mathbb{N} \cap [3, \infty[ \subset \mathbb{X}^{(1)}$. Theorem 3 improves this proposition.

**Corollary.** *With the notation of Theorem 3, $\theta_{(n_k,\ldots,n_0;a)} \in \mathbb{X}^{(a-2-k)}$.*

It follows immediately that $\theta_{(n_0;a)} = a \in \mathbb{X}^{(a-2)}$, and so $\inf \mathbb{X}^{(j)} \leq j+2$, for all $j \in \mathbb{N}$. Notice also that the corollary above completes Theorem 1(iii), since a strong Pisot number $\theta$, with degree greater than 1, has a real positive conjugate, and so $\theta \in \mathbb{S} \setminus \mathbb{P}$ (it is easy to see from the proof of Theorem 3, when $k \geq 1$, that $\theta_{(n_k,\ldots,n_0;a)} \in ]a-1, a[$, and so $\theta_{(n_k,\ldots,n_0;a)}$ is not of degree 1).

A Pisot number whose other conjugates are of modulus less than $\varepsilon$, where $\varepsilon$ is fixed in the interval $]0, 1]$, is called an $\varepsilon$-Pisot number [9]. Let $K$ be a real algebraic number field. Then, a result of Fan and Schmeling [9] asserts that the set of $\varepsilon$-Pisot numbers generating $K$, is relatively dense in $[1, \infty[$, that is there is a positive constant $\rho$, depending only on $\varepsilon$ and $K$, such that each subinterval of $[1, \infty[$, of length $\rho$, contains an $\varepsilon$-Pisot numbers generating $K$. We may ask the same question for strong Pisot numbers, when the field $K$ has a proper real conjugate, and in particular when $K$ is generated by a Salem number. Recall that a Salem number is a real algebraic integer greater than 1 whose other conjugates are of modulus at most 1 and with a conjugate of modulus 1. Using some results of Meyer on harmonious sets [12], we obtain the following theorem.

**Theorem 4.** *Let $K$ be a real algebraic number field, with degree $d$, having at least one proper real conjugate when $d \geq 2$, and let $\varepsilon \in ]0, 1]$. Then, the following assertions are true.*

(i) *The set of $\varepsilon$-strong Pisot numbers generating $K$, is relatively dense in $[1, \infty[$.*
(ii) *If $K$ is generated by a Salem number, then there is a finite subset, say $F = F(K, \varepsilon)$, of the integers of $K$ such that each Salem number generating $K$ can be written as a sum of an element of $F$ and an $\varepsilon$-strong Pisot number with degree $d$.*

In this paper, when we speak about a conjugate, the degree or the minimal polynomial of an algebraic integer, we mean over the field of the rationals $\mathbb{Q}$. Irreducible polynomials or the degree of a number field are considered over $\mathbb{Q}$ too. The proof of Theorem 2 uses elementary statements, and the one of Theorem 1 is mainly based on some results of [14]. To simplify the proof of Theorem 3, we shall use a theorem of Smyth [21], which says that two conjugates of a Pisot number having the same modulus are necessary complex conjugates. An easy application of an argument, due to Meyer, gives Theorem 4(i). Finally, notice that the proof of Theorem 4(ii) is similar to the one of Theorem 2 of [25], with minor modifications; for convenience of the reader, we give some details of this proof.

## 2. The proofs

**Proof of Theorem 1.** (i) In [8], Dufresnoy and Pisot have shown that the minimal polynomial of a Pisot number, say $\theta$, less than $\theta_\infty := \left(1+\sqrt{5}\right)/2$, is one of the following polynomials: $A(x) = x^6 - 2x^5 + x^4 - x^2 + x - 1$, $B_{2k}(x) = (x^{2k}(x^2 - x - 1) + 1)/(x - 1)$, $B_{2k+1}(x) = (x^{2k+1}(x^2 - x - 1) + 1)/(x^2 - 1)$, or $C_k(x) = x^k(x^2 - x - 1) + (x^2 - 1)$, where $k$ runs through $\mathbb{N}$. Hence, the root $\theta = 1.5617\ldots$ of the polynomial $A$ is the only non-regular Pisot number in $[1, \theta_\infty]$. A direct calculation gives for this last case, $r_0 = \theta - 1$, $(r_1, \varepsilon_1) = (\theta^2 - \theta, 0)$, $(r_2, \varepsilon_2) = (\theta^3 - \theta^2 - 1, 1)$, $(r_3, \varepsilon_3) = (\theta^4 - \theta^3 - \theta, 0)$, $(r_4, \varepsilon_4) = (\theta^5 - \theta^4 - \theta^2, 0)$, $(r_5, \varepsilon_5) = (\theta^6 - \theta^5 - \theta^3 - 1, 1)$, $(r_6, \varepsilon_6) = (\theta^7 - \theta^6 - \theta^4 - \theta, 0)$ and $r_7 = \theta^8 - \theta^7 - \theta^5 - \theta^2 - 1 = (\theta^2 + \theta + 1)A(\theta) = 0$; thus

$$\{\theta\} \equiv \frac{1}{\theta^2} + \frac{1}{\theta^5} + \frac{1}{\theta^7}$$

and $\theta \in \mathbb{P}$. If $B_{2k}(\theta) = 0$ (resp., if $B_{2k+1}(\theta) = 0, C_{2k}(\theta) = 0, C_{2k+1}(\theta) = 0$), then the results of [14] yield $\{\theta\} \equiv \sum_{j=1}^{k-1} \frac{1}{\theta^{2j}} + \sum_{j=k+1}^{2k} \frac{1}{\theta^{2j}}$ (resp., $\{\theta\} \equiv \sum_{j=0}^{k} \frac{1}{\theta^{2j}}$, $\{\theta\} \equiv \left(\sum_{j=1}^{k-1} \frac{1}{\theta^{2j}}\right) + \frac{1}{\theta^{4k-1}}$, $\{\theta\} \equiv \left(\sum_{j=1}^{k-1} \frac{1}{\theta^{2j}}\right) + \frac{1}{\theta^{2k+1}} + \frac{1}{\theta^{4k+2}}$), and so $\theta \in \mathbb{P}$. Theorem 1(i) follows immediately, since $\{\theta_\infty\} = \theta_\infty - 1 \equiv 1/\theta_\infty$.

(ii) From the result above we have $\inf \mathbb{P}^{(1)} = \left(1+\sqrt{5}\right)/2$. Let $\theta_k > 1$ be a root of the polynomial $x^k(x^2 - x - 1) - (x^2 - 1)$, where $k \in \mathbb{N}$. Then, $\theta_k \in \mathbb{S} \cap ]\theta_\infty, \infty[$, $\lim_{k\to\infty} \theta_k = \theta_\infty$ [8], and $\theta_k \in \mathbb{S} \setminus \mathbb{P}$ [14]; thus

$$\inf \mathbb{S} \setminus \mathbb{P} = \inf(\mathbb{S} \setminus \mathbb{P})^{(1)} = \left(1+\sqrt{5}\right)/2.$$

In particular, we see that $\mathbb{S} \setminus \mathbb{P}$ is not closed. From the proof of Theorem 1(iii), we shall easily deduce that each interval of the form $[a, a+1]$, where $a \in \mathbb{N} \cap [2, \infty[$, contains infinitely many elements of the set $\mathbb{P}^{(1)} \cap \mathbb{S} \setminus \mathbb{P}$, and from this we obtain the non-closure of $\mathbb{P}$. Notice also that the equality $\inf(\mathbb{S})^{(2)} = 2$, implies $\inf(\mathbb{S} \setminus \mathbb{P})^{(2)} \geq 2$ and $\inf \mathbb{P}^{(2)} \geq 2$. Now, let $L_k(x) := x^k - \sum_{j=1}^{k} x^{k-j}$, where $k \in \mathbb{N} \cap [2, \infty[$. It is well known that $L_k$ is the minimal polynomial of a Pisot number, say $l_k$, satisfying $\lim_{k\to\infty} l_k = 2$ and $l_k \in \mathbb{S}^{(1)} \cap ]1, 2[$ (see [2] or [23]). If we consider the sequence of Pisot numbers $(\theta_{j,k})_{j \geq k+1}$, where $\theta_{j,k}$ is a root of the polynomial

$$E_{j,k}(x) := x^j L_k(x) - (x^{k+1} - 1),$$

then $\theta_{j,k} \in \mathbb{S} \setminus \mathbb{P}$ [14]. It follows by the equality $\lim_{j\to\infty} \theta_{j,k} = l_k$, that $\inf(S \setminus P)^{(2)} \leq 2$ and so $\inf(S \setminus P)^{(2)} = 2$ (using the relation $l_k^k - \sum_{j=1}^{k} l_k^{k-j} = 0$, a short computation shows that

$\{l_k\} = l_k - 1 \equiv \sum_{j=1}^{k-1} 1/l_k^j$, and so $l_k \in \mathbb{P}$; this shows again that $\mathbb{S} \setminus \mathbb{P}$ is not closed). Similarly as for $E_{j,k}$, we obtain that each polynomial of the form

$$x^j L_k(x) - (x^k - 1)/(x - 1),$$

where $j \geq k + 1$, has a root, say again $\theta_{j,k}$, which is a Pisot number. Moreover, we have $\theta_{j,k} \in \mathbb{P}$, $\lim_{j\to\infty} \theta_{j,k} = l_k$ and so $\inf P^{(2)} = 2$.

(iii) Let $a \in \mathbb{N} \cap [2, \infty[$. Then, the polynomial

$$P(x) = P_{(k,a)}(x) := x^k(x - (a + 1)) + (a - 1),$$

where $k \in \mathbb{N}$, is the minimal polynomial of a Pisot number $\theta_k = \theta_{(k,a)}$. Indeed, we have for $|z| = 1$, $\left|z^k(z - (a + 1))\right| = |z - (a + 1)| \geq a > a - 1$, and so by Rouché's theorem, the polynomial $P$ has $k$ roots inside the unit circle, and one root $\theta_k$ with modulus greater than 1 (the modulus of the product of the roots of $P$ is $a - 1 \geq 1$). Moreover, the relations $P(a+1) = a-1$, $P(a) = a - a^k - 1$ and $|\theta_k - (a + 1)| = (a - 1)/\theta_k^k$, yield $\theta_k \in ]a, a+1[\cap \mathbb{S}$, and $\lim_{k\to\infty} \theta_k = a + 1$. Now, consider the polynomial

$$M(x) = M_{(n,k,a)}(x) := x^{n-k} P_{(k,a)}(x) + 1,$$

where $n \in [k + 1, \infty[$. Then,

$$M(x) = (x - 1)\left(x^n - a\sum_{j=1}^{k} x^{n-j} - \sum_{j=k+1}^{n} x^{n-j}\right),$$

and by a result of Perron, cited in [10], the factor $x^n - a\sum_{j=1}^{k} x^{n-j} - \sum_{j=k+1}^{n} x^{n-j}$ of $M(x)$, is the minimal polynomial of a Pisot number $\beta_n = \beta_{(n,k,a)}$. It is easy to check that $\beta_n \in \mathbb{P}$ (this result is also a corollary of Theorem 2 of [10]), and the equality $|P(\beta_n)| = 1/\beta_n^{n-k}$ gives $\lim_{n\to\infty} \beta_n = \theta_k$. It follows that each $\theta_k$ is a limit of a sequence of elements of $\mathbb{P}$, and so $a + 1 \in \mathbb{P}^{(2)}$. This ends the proof of Theorem 1(iii), because the relation $2 \in \mathbb{P}^{(2)}$, follows from Theorem 1(ii). Finally, notice that $\theta_k \in \mathbb{S} \setminus \mathbb{P}$, since $P(0) = a - 1$ and $P(1) = -1$; thus $a+1 \in (\mathbb{S}\setminus\mathbb{P})^{(1)}$, and the sets $\mathbb{P}$ and (again) $\mathbb{S}\setminus\mathbb{P}$ are not closed, since $\beta_{(n,k,a)} \in \mathbb{P}$, $\theta_{(k,a)} \in \mathbb{S}\setminus\mathbb{P}$ and $a + 1 \in \mathbb{P}$. □

**Proof of Theorem 2.** (i) Let $\beta$ be a beta-number with degree $d$. If $d = 1$, then $\beta$ is a rational integer, $\{\beta\} = 0$ and $\beta \in \mathbb{P}$. Suppose $d = 2$. Since a beta-number has no conjugate, other than $\beta$ itself, in the interval $]1, \infty[$ [22], the conjugate $\gamma$ of a quadratic beta-number $\beta$, other than $\beta$, belongs to the interval $]-2, 1[$. To show that $\beta \in \mathbb{S}$, it suffices to prove that $\gamma \notin ]-2, -1[$.

A short computation gives that each interval of the form $[n, n + 1]$, where $n \in \mathbb{N}$, contains $2n - 1$ quadratic Pisot numbers $\theta$: the minimal polynomial of $\theta$ is $P(x) = x^2 - (n + 1)x + p$, with $p \in \{1, \ldots, n - 1\}$, or $Q(x) = x^2 - nx - p$, with $p \in \{1, \ldots, n\}$. If $P(\theta) = 0$, then $\theta$ has a conjugate in $]0, 1[$ and so $\theta \in \mathbb{S} \setminus \mathbb{P}$; otherwise $\{\theta\} = \theta - n \equiv \frac{p}{\theta}$ and $\theta \in \mathbb{P}$.

Now, assume on the contrary that $\gamma \in ]-2, -1[$. Then, $\beta + 1$ is a quadratic Pisot number, and so $\beta$ is a root of one of the irreducible polynomials $Q(x + 1) = x^2 - (n - 2)x - (n + p - 1)$, where $p \in \{1, \ldots, n\}$. If $\beta$ is simple, then there is $k \in \mathbb{N}$, such that

$$\{\beta\} = \beta - \varepsilon_0 \equiv \sum_{j=1}^{k} \frac{\varepsilon_j}{\beta^j},$$

and so $Q(x + 1)$ divides, in the ring $\mathbb{Z}[x]$, the polynomial $x^{k+1} - \varepsilon_0 x^k - \cdots - \varepsilon_k$; thus $n + p - 1$ divides $\varepsilon_k$ in $\mathbb{Z}$, and this leads to a contradiction, since $n + p - 1 \geq n$ and $1 \leq \varepsilon_k \leq [\beta] < \beta \leq n$.

Similarly, if $\beta$ is not simple, then we have

$$\{\beta\} = \beta - \varepsilon_0 \equiv \sum_{j=1}^{k-1} \frac{\varepsilon_j}{\beta^j} + \sum_{l=0}^{\infty} \sum_{j=k}^{k+q-1} \frac{\varepsilon_j}{\beta^{j+lq}},$$

for some $q$ and $k \in \mathbb{N}$ (if $k$ exists). By the equation $\beta - \varepsilon_0 = \sum_{j=1}^{k-1} \frac{\varepsilon_j}{\beta^j} + \frac{\beta^q}{\beta^q - 1} \sum_{j=k}^{k+q-1} \frac{\varepsilon_j}{\beta^j}$, we obtain a monic polynomial $R(x)$, which is a multiple of $Q(x+1)$ in $\mathbb{Z}[x]$, and this relation leads also to a contradiction, because $|R(0)| = \left|\varepsilon_{k-1} - \varepsilon_{k+q-1}\right| \leq [\beta] < n$. Finally, notice that the cubic case follows from [3].

(ii) Consider the family of polynomials

$$P(x) = P_k(x) := x^3 - kx^2 - (k+1)x + 1,$$

where $k \in \mathbb{N} \cap [2, \infty[$. Then, the relations $P(-2) = -5 - 2k$, $P(-1) = 1 = P(0)$, $P(1) = 1 - 2k$, $P(k) = 1 - k(k+1)$ and $P(k+1) = 1$, imply that $P$ has a root in each one of the following intervals $]-2, -1[$, $]0, 1[$ and $]k, k+1[$, and so $P$ is irreducible. Let $\beta = \beta_k$ be the root of $P$ which belongs to the interval $]k, k+1[$. Then, $\beta \notin \mathbb{S}$, $[\beta] = k$ and $\{\beta\} = \beta - k = (k+1)/\beta - 1/\beta^2$. Hence, $\beta\{\beta\} = k + 1 - 1/\beta$, $k < \beta\{\beta\} < k+1$, $\varepsilon_1 = k$, $r_1 = 1 - 1/\beta$, $\beta r_1 = \beta - 1$, $\varepsilon_2 = k - 1$ and $r_2 = \{\beta\}$. It follows that $r_{j+2} = r_j$, $\varepsilon_{2j-1} = k$ and $\varepsilon_{2j} = k - 1$, $\forall j \in \mathbb{N}$; thus

$$\{\beta\} \equiv \sum_{j \geq 1} \left( \frac{k}{\beta^{2j-1}} + \frac{k-1}{\beta^{2j}} \right)$$

and $\beta$ is a beta-number.

(iii) Let $d \in \mathbb{N} \cap [4, \infty[$. First, suppose that $d$ is even, and consider the polynomial

$$P(x) = P_d(x) := x^d - px^{d-1} - px^{d-2} - p,$$

where $p$ is prime. Then, $P$ is $p$-Eisenstein, $P(p) = -p^{d-1} - p$, $P(p+1) = (p+1)^{d-2} - p$, and so $P$ is the minimal polynomial of a real number $\beta = \beta_d \in ]p, p+1[$. Furthermore, we have $[\beta] = p$, $\{\beta\} = \beta - p = p/\beta + p/\beta^{d-1}$, $\beta\{\beta\} = p + p/\beta^{d-2}$, $(r_1, \varepsilon_1) = (p/\beta^{d-2}, p)$, $(r_j, \varepsilon_j) = (p/\beta^{d-1-j}, 0)$, $\forall j \in \{2, \ldots, d-2\}$, and $(r_{d-1}, \varepsilon_{d-1}) = (0, p)$. Hence, $\{\beta\} \equiv \frac{p}{\beta} + \frac{p}{\beta^{d-1}}$ and $\beta$ is a simple beta-number which is not a Pisot number, since $\beta$ has a conjugate in $]-2, -1[$ (we have $P(-1) = 1 - p < 0$ and $P(-2) = 2^d + 2^{d-2}p - p > 0$).

Now, assume that $d$ is odd, and consider

$$P(x) = P_d(x) := x^d - 2px^{d-1} - 2px^{d-2} - 2px^{d-4} - p,$$

where $p$ is prime. Similarly as for the case where $d$ is even, we have that $P$ is $p$-Eisenstein, $P(2p) < 0$, $P(2p+1) > 0$, $P(-1) > 0$, $P(-2) < 0$, and so $P$ is the minimal polynomial of an algebraic integer $\beta = \beta_d \in ]2p, 2p+1[$, having a conjugate in the interval $]-2, -1[$. Moreover, $[\beta] = 2p$, $\{\beta\} = \beta - 2p$ and an easy calculation gives $\{\beta\} \equiv \frac{2p}{\beta} + \frac{2p}{\beta^3} + \frac{p}{\beta^{d-1}}$. Hence, $\beta$ is a simple beta-number which does not belong to $\mathbb{S}$. □

**Proof of Theorem 3.** To simplify the notation, set $P_k := P_{(n_k, \ldots, n_0; a)}$, $\forall k \in \{0, \ldots, a-2\}$. It is clear that $P_1(1) = 2 - a$, $P_1(a-1) = 1 - (a-1)^{n_1} \leq -1$ and $P_1(a) = 1$. Assume that we have $P_{k-1}(1) = k - a$, $P_{k-1}(a-1) \leq -1$ and $P_{k-1}(a) \geq 1$, where $k \geq 2$. Then, (1) gives immediately $P_k(1) = k + 1 - a$, $P_k(a-1) \leq -1$ and $P_k(a) \geq 1$. It follows by induction that $P_k(1) < 0$, $P_k(a-1) < 0$ and $P_k(a) > 0$. Hence, if $k \in \{1, \ldots, a-2\}$, then the polynomial

$P_k$ has two real roots, one $\rho_k = \rho_{(n_k,\dots,n_0;a)} \in ]0,1[$ (recall that $P_k(0) = 1$), and the other $\theta_k = \theta_{(n_k,\dots,n_0;a)} \in ]a-1, a[$. Furthermore, the equalities $P_k(\rho_{k-1}) = P_k(\theta_{k-1}) = 1$, where $k \in \{2,\dots,a-2\}$, obtained from (1), yield $\rho_k \in ]\rho_{k-1}, 1[$ and $\theta_k \in ]a-1, \theta_{k-1}[$; thus

$$a - 1 < \theta_{a-2} < \cdots < \theta_0 := \theta_{(n_0;a)} = a. \tag{2}$$

It is clear that $\theta_0 \in \mathbb{X}$. For $k \in \{1,\dots,a-2\}$, we shall prove that $\rho_k$ is a conjugate of $\theta_k$, and the other conjugates of $\theta_k$ are of modulus less than $\rho_k$. Let $z$ be a complex number with modulus 1. Then, $|P_0(z)| \geq a-1$. Moreover, if $|P_{k-1}(z)| \geq a-k$ for some $k \geq 1$, then by (1) we obtain $|P_k(z)| \geq |P_{k-1}(z)| - 1 \geq a-(k+1)$, $|P_k(z)| \geq 1$ for all $k \in \{0,\dots,a-2\}$, and

$$|P_k(z)| > 1, \quad \forall k \leq a-3. \tag{3}$$

Now, a simple induction shows that $\theta_k \in \mathbb{S}$ and $P_k$ is the minimal polynomial of $\theta_k$. Indeed, this last proposition is trivially true for $k = 0$. Assume that $P_{k-1}$ is the minimal polynomial of the Pisot number $\theta_{k-1}$, where $k \in \{1,\dots,a-2\}$. Then, using Rouché's theorem, the relation (1) together with (3) give that $P_k$ has $n_k + \deg(P_{k-1}) - 1 = \deg(P_k) - 1$ roots with modulus less than 1, and the result follows immediately, since $P_k$ has one root with modulus $>1$, namely $\theta_k$.

To complete the proof of the relation: $\theta_k \in \mathbb{X}$, consider a conjugate $\alpha$ of $\theta_k$, with modulus less than 1, where $k \geq 1$. Then, writing (1) explicitly, we obtain

$$\alpha^{n_k+\cdots+n_1+1} - a\alpha^{n_k+\cdots+n_1} + \alpha^{n_k+\cdots+n_2} + \cdots + \alpha^{n_k} + 1 = 0,$$
$$|\alpha|^{n_k+\cdots+n_1+1} + |\alpha|^{n_k+\cdots+n_2} + \cdots + |\alpha|^{n_k} + 1 \geq a\,|\alpha|^{n_k+\cdots+n_1},$$

and so

$$P_k(|\alpha|) \geq 0.$$

Consequently, to show the inequality $|\alpha| \leq \rho_k$, it suffices to verify that $P_k(t) < 0, \forall t \in ]\rho_k, 1[$. In fact, we use again an induction on $k$, to show that $P_k'(t) < 0$ and $P_k(t) < 0, \forall t \in ]\rho_k, 1[$. Clearly, $P_1'(t) = (n+1)t^{n-1}(t - nr/(n+1)) < 0, \forall t \in ]0, nr/(n+1)[$, and so the quantities $P_1'(t)$ and $P_1(t)$ are negative when $t \in ]\rho_1, 1[$, since $]\rho_1, 1[ \subset ]0, nr/(n+1)[$ and $P_1(\rho_1) = 0$. Suppose that $P_{k-1}(t) < 0$ and $P_{k-1}'(t) < 0$, where $t \in ]\rho_{k-1}, 1[$ and $k \in \{2,\dots,a-2\}$. Then, the relation $P_k'(t) = t^{n_k-1}(n_k P_{k-1}(t) + tP_{k-1}'(t))$, gives $P_k'(t) < 0$ for $t \in ]\rho_k, 1[$, since $]\rho_k, 1[ \subset ]\rho_{k-1}, 1[$, and so

$$P_k(t) < P_k(\rho_k) = 0,$$

$\forall t \in ]\rho_k, 1[$. Hence, $|\alpha| \leq \rho_k$, and by the above mentioned result of Smyth [21] we deduce that $|\alpha| < \rho_k$; thus $\theta_k \in \mathbb{X}$.

Now, the relation (1) together with (2) yield

$$\lim_{n_k \to \infty} P_{k-1}(\theta_k) = 0 \quad \text{for } k \geq 2,$$

since $|P_{k-1}(\theta_k)| = 1/\theta_k^{n_k} < 1/(a-1)^{n_k}$; thus

$$\lim_{n_k \to \infty} \theta_k = \theta_{k-1}, \tag{4}$$

because $\theta_{k-1}$ is the unique root with modulus greater than 1 of the polynomial $P_{k-1}$. □

**Proof of the Corollary.** With the same notation as in the proof above, the relation (4) gives immediately $\theta_{a-2-k} \in \mathbb{X}^{(k)}, \forall k \in \{1,\dots,a-3\}$. The inequality $|\theta_1 - a| = \frac{1}{\theta_1^{n_1}} < \frac{1}{(a-1)^{n_1}}$, implies also $\lim_{n_1 \to \infty} \theta_1 = a$, and so $\theta_0 \in \mathbb{X}^{(a-2)}$. □

**Remark.** If we consider the polynomials $x^n(x-r)+k$, where the rational integers $r$ and $k$ satisfy $r \geq k+2 \geq 4$, then by the same way as in the proof of Theorem 3, we obtain families of strong Pisot which are not units.

**Proof of Theorem 4.** To show Theorem 4(i), we shall exhibit a real model set $\Lambda_\varepsilon$, such that each element of $\Lambda_\varepsilon \cap ]1,\infty[$ is an $\varepsilon$-strong Pisot number generating $K$, and after this, we conclude by a result of Meyer [12], which says that a real model set is a relatively dense subset of $\mathbb{R}$ (for more details on model sets, see also [13] or [26]). It is clear that Theorem 4(i) is true for $d=1$, since the set of $\varepsilon$-strong Pisot number generating $\mathbb{Q}$ is $\mathbb{N} \cap [2,\infty[$. Now, suppose that $d \geq 2$, and let $\sigma_1,\ldots,\sigma_d$ be the distinct embeddings of $K$ in $\mathbb{C}$, where the first $r$ ones are real, $\sigma_1$ is the identity of $K, r \geq 2$, and $\sigma_{j+(d-r)/2}(\alpha)$ is the complex conjugate $\overline{\sigma_j(\alpha)}$ of $\sigma_j(\alpha)$, where $j \in \{r+1,\ldots,r+(d-r)/2\}$ and $\alpha \in K$. For a fixed base $\{\omega_1,\ldots,\omega_d\}$ of the ring of the integers of $K$, consider the linear forms $l_1,\ldots,l_d$ defined on $\mathbb{R}^d$ by the relations

$$l_j(x_1,\ldots,x_d)=\sum_{k=1}^{d} x_k\sigma_j(\omega_k)$$

when $j \in \{1,2,\ldots,r\}$, and

$$l_j(x_1,\ldots,x_d)=\sum_{k=1}^{d} x_k(\sigma_j(\omega_k)+\sigma_{j+(d-r)/2}(\omega_k))/2$$

and

$$l_{j+(d-r)/2}(x_1,\ldots,x_d)=\sum_{k=1}^{d} x_k(\sigma_j(\omega_k)-\sigma_{j+(d-r)/2}(\omega_k))/2i,$$

where $i^2=-1$, when $j \in \{r+1,\ldots,r+(d-r)/2\}$. Consider also the subset

$$\Omega_\varepsilon = ]\varepsilon/2,\varepsilon[\times(]0,\varepsilon/2[)^{r-2}\times\bigcap_{j=1}^{(d-r)/2}\{(y_1,\ldots,y_{d-r})\in\mathbb{R}^{d-r} \mid y_j^2+y_{j+(d-r)/2}^2<\varepsilon^2/4\}$$

of the Euclidean space $\mathbb{R}^{d-1}$. Then, the set

$$\Lambda_\varepsilon=\left\{\sum_{k=1}^{d} p_k\omega_k \mid (p_1,\ldots,p_d)\in\mathbb{Z}^d, (l_2(p_1,\ldots,p_d),\ldots,l_d(p_1,\ldots,p_d))\in\Omega_\varepsilon\right\},$$

is a real model set [12]. It is clear that $\Lambda_\varepsilon$ is contained in the ring of the integers of $K$. Let $\alpha \in \Lambda_\varepsilon$. Then, the relation $\prod_{j=1}^{d}|\sigma_j(\alpha)| \geq 1$, gives $|\alpha|>1$, and so the conjugates of $\alpha$ are exactly the numbers $\sigma_1(\alpha),\ldots,\sigma_d(\alpha)$. In particular, if $\alpha>0$, then $\alpha>1, \varepsilon/2<\sigma_2(\alpha)<\varepsilon$, and $|\sigma_j(\alpha)|<\varepsilon/2$ for $j \geq 3$; thus $\alpha$ is an $\varepsilon$-strong Pisot number, and this ends the proof of Theorem 4(i).

Now, suppose that $K$ is generated by a Salem number. Then, $r=2$ and $d \geq 4$. Let $\Lambda$ be a model set defined by

$$\Lambda=\left\{\sum_{k=1}^{d} p_k\omega_k \mid (p_1,\ldots,p_d)\in\mathbb{Z}^d, (l_2(p_1,\ldots,p_d),\ldots,l_d(p_1,\ldots,p_d))\in\Omega\right\},$$

where $\Omega=[-1,1]\times\bigcap_{j=1}^{(d-2)/2}\{(y_1,y_2,\ldots,y_{d-2})\in\mathbb{R}^{d-2}, y_j^2+y_{j+d-2}^2\leq 1\}$. Then, a simple computation shows that $\alpha \in \Lambda$ if and only if $\alpha$ is an integer of $K$ and $|\sigma_j(\alpha)| \leq 1$ for each

$j \in \{2, 3, \ldots, d\}$; thus Salem numbers generating $K$ belong to $\Lambda$. After this we conclude, similarly as in the proof of Theorem 2 of [25], by using Proposition 7.9 of [13], which asserts that there is a finite subset $F_\varepsilon$ of $K$ such that $\Lambda \subset \Lambda_\varepsilon + F$. $\square$

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