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Discrete Picone inequalities and some applications

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ملخص

الهدف الأساسي من هذا المذكرة هو دراسة بعض المسائل المتعلقة بالمؤثرات غير المحلية، إذ قمنا بدراسة مسألتين مختلفتين، الأولى تتعليق بالقيم الذاتية المعممة و الأخرى بطرف شاذ غير خطي، و قد استعملنا لدراسة هذه المسائل طريقة الصيغة التغيرية لإثبات أن المسألتين تتمتعان بحل، و من أجل اثبات وحدانية الحلول لكليهما استعملنا متراجحة **بيكون** ، و لتطبيق هذا الأخير استعملنا بعض النتائج التي تتعلق بالإنتظامية المثبت في دراسة نشرت مؤخرا.
تقع هذه الدراسة في ثلاثة فصول:
خصص الفصل الأول للتذكير ببعض المفاهيم الأساسية المستعملة في هذا العمل.
يتناول الفصل الثاني دراسة لمراحل تطور متراجحة **بيكون** في الحالة المحلية وغير المحلية التي ستكون ذات فائدة بالغة في الفصل القادم.
الفصل الثالث يتناول وجود، عدم وجود، انتظامية و وحدانية الحل الضعيف للمسألتين الذكورتين آنفا.

الكلمات المفتاحية

مؤثرات غير محلية وغير متجانسة ، عدم الوجود، انتظامية النتائج، الحل الموجب، طرف شاذ و غير خطي ، الصيغة التغايرية.

Abstract

The main objective of this thesis is to study some problems related to eigenvalues and singular ones. We employ variational methods in order to show the existence of positive weak solutions in both cases. Thanks to the results obtained recently research together with a new version of the Picone inequality, we also establish the uniqueness results.
We divided this work into three chapters:

In the first chapter, we begin by recalling some of the basic and preliminary concepts used in this work.

The second chapter deals with the definition of Picone inequality in local and non-local cases, which we will need in the next chapter.

The third chapter deals with the presence of existence, non-existence, regularity, and the uniqueness of the weak solution to two problems related to non-local and non-homogeneous operators, the first for the generalized eigenvalues and the second for the singular.

key-words : Fractional p-Laplacian operator, non-existence, regularity results , positive solutions, singular nonlinearity, variatianal methods.

Function spaces

$L^p(\Omega) := \left\{ u : \Omega \to \mathbb{R}^N : u \text{ is measurable and } \int_\Omega |u|^p \, dx < \infty \right\}, 1 \leq p < \infty.$

$L^\infty(\Omega) := \{u : \Omega \to \mathbb{R} : u \text{ is measurable and } |u(x)| \leq C \text{ a.e. in } \Omega \text{ for some constant } C\}.$

$C(\Omega)$ space of continuously fanctions on Ω.

$C(\overline{\Omega})$ functions in $C(\Omega)$ where the function $x \mapsto u(x)$ admits a continuous extension to $\overline{\Omega}$.

$C_c^\infty(\Omega) := \left\{ \varphi : \mathbb{R}^N \to \mathbb{R} : \varphi \in C^\infty(\mathbb{R}^N) \quad \text{and} \quad \operatorname{supp}(\varphi) \Subset \Omega \right\}.$

$C^{0,\alpha}(\overline{\Omega}) := \left\{ u \in C(\overline{\Omega}), \sup_{x,y \in \overline{\Omega},\, x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty \right\}, \text{ with } 0 < \alpha < 1.$

$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dx dy < \infty \right\}, \text{ with } 0 < s < 1 \text{ and } 1 \leq p < \infty.$

$W_0^{s,p}(\Omega) := \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.$

$W := W_0^{s_1,p}(\Omega) \cap W_0^{s_2,q}(\Omega)$

Contents

Introduction

In this work, we study some results of the fractional non-homogeneous problem involving fractional and non-homogeneous operators. These kinds of non-local operators have their applications in the real world such as optimization, finance, phase transitions, soft thin films, and image processing. The fractional Laplacian also provides a model to describe certain jump Lévy processes in probability theory and porous media in physical and among others in various fields, see [2, 4, 15, 27, 41] and the references therein.
We point out that, the paper [40] presented two evolution models of flows in porous media involving fractional operators:

• The first model is based on Darcy's law and is given by

$$\begin{aligned} \partial_t u = \nabla\cdot(u\nabla \mathrm{P}) \quad in \quad (0,\infty)\times \mathrm{R}^{\mathrm{N}}, \\ \mathrm{P} = (-\Delta)^{-s} u \quad \text{in} \quad (0,\infty)\times \mathrm{R}^{\mathrm{N}}, \\ u(0,x) = u_0(x) \qquad \mathrm{R}^{\mathrm{N}} \end{aligned}$$

where u is the particle density of the fluid, P is the pressure and $(-\Delta)^{-s}$ is the inverse of the fractional Laplace operator (i.e. $p=2$). The initial data u_0 is a non-negative, bounded and integrable function in R^{N}.
• The second model in analogy to classical models of transport through porous media is described in the non local case by

$$\partial_t u + (-\Delta)^s (u^m) = 0 \tag{1}$$

For $s \longleftrightarrow 1$ and $m=1$, the limiting model (1) is the well known heat equation. Furthermore if$m>1$, (1) is known as the porous media equation (PME for short) whereas in case $m<1$ it is referred as the fast diffusion equation (FDE for short).

In this thesis, we study two problems:

• Firstly

$$(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \lambda\, a_r(x) u^{r-1}, \quad u>0 \quad \text{in } \Omega; \quad u=0, \quad \text{in } \mathbb{R}^{\mathrm{N}}\setminus\Omega \qquad (\textbf{P-(i)})$$

where $r=p$ or q, as follows.

We say that a non-negative function $u \in \mathbf{W}$ is called a weak solution to (**P-(i)**) if, for any $\varphi \in \mathbf{W}$ we have:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2q}}dxdy = \lambda\int_\Omega a_r(x)u^{r-1}\varphi dx. \quad (2)$$

In addition if u satisfies $u > 0$ in Ω, we call u positive weak solution.

Theorem 0.1. *If $\lambda \le \lambda_{1,s,r}(a_r)$ holds, then (**P-(i)**) has no nontrivial solutions.*

Theorem 0.2. *If $\lambda > \lambda_{1,s_2,q}(a_q)$. Then (**P-(i)**) has at least one positive solution u. In addition, $u \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for any $\sigma \in (0, s_1)$ and $\sigma' > s_1$, there exists a positive constant $c = c(\sigma, \sigma') > 0$, such that:*

$$c^{-1}d^{\sigma'} \le u \le c\, d^{\sigma} \quad in\, \Omega.$$

Moreover, the solution is unique.

Theorem 0.3. *We set the following non-local Rayleigh quotient:*

$$\underline{\lambda}_{s,s*,r,r^*}(a) := \inf_{u\in \boldsymbol{W}}\left\{\frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^r}{|x-y|^{N+sr}}dxdy + \frac{r}{r^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{r*}}{|x-y|^{N+s^*r^*}}dxdy}{\displaystyle\int_\Omega a_r(x)u^r\,dx}\right\}.$$

where $r = p$ (or q), with $s = s_1$ (or s_2) if $r^* = q$ (or p), with $s = s_2$ (or s_1, respectively). Then, $\underline{\lambda}_{s,s^*,r,r^*}(a_r) = \lambda_{1,s,r}(a_r)$. In addition, the infimum is not attained.

Secondly:

$$(-\Delta)_p^{s_1}u + (-\Delta)_q^{s_2}u = u^{-\delta} + b(x,u), \quad u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N\setminus\Omega \quad (\textbf{P-(ii)})$$

We say that $u \in \mathbf{W}\cap L^\infty(\Omega)$ is a positive weak solution to the problem (**P-(ii)**), if

$$\operatorname*{ess\,inf}_K u > 0 \quad \text{over every compact set } K \subset \Omega$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1p}}dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2q}}dxdy = \int_\Omega u^{-\delta}\varphi dx + \int_\Omega b(x,u)\varphi dx \quad (3)$$

for all $\varphi \in C_c^\infty(\Omega)$.

Theorem 0.4. *Let $0 < s_2 \le s_1 < 1$ and $1 < \upsilon < q \le p < \infty$. Assume that b satisfies **(H1)**-**(H2)**. Then, there exists a unique nontrivial weak solution u to the problem (**P-(ii)**). In addition, $u \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for any $\sigma \in (0, s_1)$ and $\sigma' > s_1$, there exists a positive constant $c = c(\sigma, \sigma') > 0$, such that:*

$$c^{-1}d^{\sigma'} \le u \le c\, d^{\sigma} \quad in\, \Omega.$$

CHAPTER 1

Preliminaries and functional setting

This chapter is meant to provide an overview of the functional analysis that will be used afterward. Moreover, we present some basic facts concerning the necessary function of spaces (see [5]).

1 Notation and function spaces

1.1 Fourier transform of tempered distributions

In this section we just recall briefly the notion of Fourier transform of a tempered distribution. First of all, we consider the Schwartz space of rapidly decaying $C^{\infty}(\mathbb{R})$ functions whose topology is generated by the seminorms $\{p_j\}_{j\in\mathbb{N}}$ defined as

$$p_j(\phi) := \sup_{x\in \mathrm{R}^n} (1+|x|)^j \sum_{|\alpha|\leq j} |\mathrm{D}^{\alpha}\phi(x)|$$

where $\phi \in j(\mathrm{R}^n)$. More precisely, j contains the smooth functions ϕ satisfying

$$\sup_{x\in \mathrm{R}^n} |x^{\alpha}\mathrm{D}^{\beta}\phi(x)| < +\infty$$

for all multi indices α and $\beta \in \mathrm{N}_0^n$.
The natural locally convex topology on J can be characterized by the following notion of convergence:
the sequence $\{\phi_j\}_{j\in\mathbb{N}}$ converges to 0 in J if and only if

$$\lim_{j\to+\infty} x^{\alpha}\mathrm{D}^{\beta}\phi_j(x) = 0, \quad \text{for all} \quad \alpha \quad \text{and} \quad \beta \in \mathrm{N}_0^n$$

We denote by

$$j_{\phi}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathrm{R}^{\mathrm{N}}} \exp^{-i\xi\cdot x} \phi(x)\,dx \tag{1.1}$$

The Fourier transform of a function $\phi \in j$. Note that, for every $\phi \in j$, one has that $\mathrm{F}_{\phi} \in j$.
It may be readily verified that the Fourier transform (1.1) and the inverse Fourier transform, given by

$$j^{-1}\varphi(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^{\mathbb{N}}} \exp^{i x.\xi} \varphi(\xi)\,d\xi \tag{1.2}$$

are both continuous on $j(\mathbb{R}^{\mathbb{N}})$ into $j(\mathbb{R}^{\mathbb{N}})$. Moreover, since

$$j^{-1}j\varphi = jj^{-1}\varphi = \varphi$$

each of them is, in fact, an isomorphism and a homeomorphism of $j(\mathbb{R}^{\mathbb{N}})$ onto $j(\mathbb{R}^{\mathbb{N}})$.
Now let j' be the topological dual of j. As usual, a tempered distribution is an element of j. If $t \in j$, the Fourier transform of T can be defined as the tempered distribution given by

$$\langle j\mathrm{T}, \varphi\rangle := \langle \mathrm{T}, j\varphi\rangle \tag{1.3}$$

for every $\varphi \in j$, where $<,>$ denotes the usual duality bracket between j and its dual j'.
By using definition 1.1, one has

$$u \in \mathrm{L}^2(\mathbb{R}^{\mathbb{N}}) \quad \text{if and only if} \quad ju \in \mathrm{L}^2(\mathbb{R}^{\mathbb{N}}) \tag{1.4}$$

and

$$\|u\|_{\mathrm{L}^2(\mathbb{R})} = \|ju\|_{\mathrm{L}^2(\mathbb{R})} \tag{1.5}$$

for every $u \in \mathrm{L}^2(\mathbb{R})$. Formula 1.5 is the so-called Parseval–Plancherel formula, which will be crucial in what follows for proving the equivalence between the fractional spaces $\mathrm{H}^s(\mathbb{R}^n)$ and $\hat{\mathrm{H}}^s(\mathbb{R}^n)$ (see Corollary 1.15). For a detailed introduction to the classical theory of distribution and Fourier transform, we refer to the monograph [36] and the recent book [14] for several applications to elliptic problems of linear and nonlinear functional analysis.

1.2 Fractional Sobolev spaces

Let Ω be a possibly nonsmooth, open set of the Euclidean space $\mathbb{R}$ and $p \in [1, +\infty)$. For any $s > 0$, we would define the fractional Sobolev space $\mathrm{W}^{s,p}(\Omega)$. In the literature, fractional Sobolev-type spaces are also called Aronszajn, Gagliardo, or Slobodeckij spaces, by the names of the ones who introduced them, almost simultaneously see [**?**], [21], [37].
If $s \geq 1$ is a positive integer, we denote by $\mathrm{W}^{s,p}(\Omega)$ the classical Sobolev space equipped with the standard norm

$$\|u\|_{W^{s,p}(\Omega)} := \sum_{0\leq|\alpha|\leq s} \|\mathrm{D}^\alpha u\|_{\mathrm{L}^p(\Omega)}$$

for every $u \in \mathrm{W}^{s,p}(\Omega)$, where here and in what follows $\|.\|_{\mathrm{L}^p(\Omega)}$ denotes the usual norm in $\mathrm{L}^p(\Omega)$, and D^α stands for the α-distributional derivative. This section is devoted to the definition of fractional Sobolev spaces; that is, here we are interested in the case where $s \notin \mathrm{N}$.
For a fixed $s \in (0,1)$, we recall that the Sobolev space$\mathrm{W}^{s,p}(\Omega)$ is defined as follows:

$$\mathrm{W}^{s,p}(\Omega) := \left\{ u \in \mathrm{L}^p(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{n/p+s}} \in \mathrm{L}^p(\Omega\times\Omega) \right\}.$$

It is endowed with the natural norm

$$\|u\|_{\mathrm{W}^{s,p}(\Omega)} := \left(\int_\Omega |u(x)|^p dx + \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}. \tag{1.6}$$

where the term

$$[u]_{\mathrm{W}^{s,p}(\Omega)} := \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}. \tag{1.7}$$

is the Gagliardo seminorm of u.
When $s > 1$ and $s \notin \mathrm{N}$, we can write $s = m + \sigma$, where $m \in \mathrm{N}$ and $\sigma \in (0,1)$. We can define $\mathrm{W}^{s,p}(\Omega)$ as follows:

$$\mathrm{W}^{s,p}(\Omega) := \{ u \in \mathrm{W}^{m,p}(\Omega) : \mathrm{D}^\alpha u \in \mathrm{W}^{\sigma,p}(\Omega) \text{for any} \quad \alpha s.t. |\alpha| = m \}$$

In this case, $W^{s,p}(\Omega)$is endowed with the norm

$$\|u(x)\|_{W^{s,p}(\Omega)} := \left(\|u(x)\|^p_{W^{m,p}(\Omega)} + \sum_{|\alpha|=m} \|D^\alpha u\|^p_{W^{\sigma,p}(\Omega)} \right)^{\frac{1}{p}} \tag{1.8}$$

for every$u \in W^{s,p}(\Omega)$. All in all, the space $W^{s,p}(\Omega)$ is well defined and is a Banach space for every $s > 0$.
As in the clas sical case (i.e., $s\ in N$), any function in the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$ can be approximated by a sequence of smooth functions with compact support. Indeed, for any $s > 0$,

$$\overline{C_0^\infty(R^n)}^{\|.\|_{W^{s,p}(R^n)}} = W^{s,p}(R^n);$$

that is, the space $c_0^\infty(\mathbb{R}^n)$ is dense in $W^{s,p}(R^n)$.
In general, if $\Omega \subset R^n$, the space$c_0^\infty(\mathbb{R})$ is not dense in $c_0^\infty(\mathbb{R})$. Hence, we denote by $c_0^\infty(R^n)$ the closure of $c_0^\infty(\mathbb{R})$ with respect to the norm ,$\|.\|_{W^{s,p}(\Omega)}$;that is,

$$W_0^{s,p}(\Omega) := \overline{C_0^\infty(R^n)}^{\|.\|_{W^{s,p}(\Omega)}}.$$

With this definition, we can also construct $W^{s,p}(\Omega)$ when $s < 0$. Indeed, for $s < 0$ and $p \in (1, +\infty)$, we can define

$$W^{s,p}(\Omega) := (W_0^{-s,q}(\Omega))'$$

that is , $W^{s,p}(\Omega)$ is the dual space of $W_0^{-s,q}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$

Theorem 1.1. *Let $s \in (0,1)$ and $p \in [1, +\infty)$such that $sp < n$. Then there exists a positive constant* $C := C(n, p, s)$ *such that, for any $u \in W^{s,p}(R^n)$*

$$\|u(x)\|_{L^{p^*s}(R^n)} \leq c \int_{R^n \times R^n} \frac{|u(x) - u(y)|}{|x-y|^{n+ps}} dxdy$$

where the constant

$$p_s^* := \frac{pn}{n - sp}$$

is the so–called fractional critical exponent. Consequently, the space is $W^{s,p}(R^n)$ *continuously embedded in* $L^q(R^n)$ *for any $q \in [p, p_s^*]$. Moreover, the embedding* $W^{s,p}(R^n) \hookrightarrow L^q_{loc(R^n)}$ *is compact for every $q \in [p, p_s^*)$. In an extension domain Ω, the following embedding result holds:*

Theorem 1.2. *Let $s \in (0,1)$ and $p \in [1, +\infty)$such that $sp < n$. Let $\Omega \subset R^n$be an extension domain for* $W^{s,p}$. *Then there exists a positive constant*$C := C(n, p, s, \Omega)$ *such that, for any ,$u \in W^{s,p}(\Omega)$,*

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{w^{s,p}(\Omega)},$$

for any $q \in [p, p_s^*]$. Moreover, the embedding $W^{s,p}(R^n) \hookrightarrow L^q_{loc(R^n)}$ *is compact for every $q \in [p, p_s^*)$.that is, the space*$W^{s,p}(\Omega)$ *is continuously embedded in*$L^q(\Omega)$ *for any $q \in [p, p_s^*]$. If, in addition, Ω is bounded, then the space* $W^{s,p}(\Omega)$ *is compactly embedded in*$L^q(\Omega)$*for any $q \in [p, p_s^*)$*

1.3 The fractional Laplacian operator

Nonlocal equations have attracted much attention in recent decades. The basic operator involved in this kind of problems is the so-called fractional Laplacian $(-\Delta^s)$ with $s \in (0,1)$. This operator and its generalization appear in many areas of mathematics, such as, for

instance, harmonic analysis, probability theory, potential theory, quantum mechanics, statistical physics, and cosmology, as well as in many applications, as we highlighted at the beginning of this chapter. This section is devoted to the definition of this operator and to its properties.
Let $s \in (0,1)$, and define the operator$(-\Delta^s) : j : j \longrightarrow L^2(\mathbb{R}^n)$ by

$$(-\Delta^s)u(x) := c(n,s) \lim_{\varepsilon \longrightarrow 0} \int_{\mathbb{R}^n / B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy \qquad x \in \mathbb{R}^n, \tag{1.9}$$

where $B(x,\varepsilon)$ is the ball centered at $x \in \mathbb{R}^n$ with radius ε, and $C(n,s)$ is the following (positive) normalization constant:

$$C(n,s) := \left(\frac{1-\cos(\zeta_1)}{|\zeta_1|^{n+2s}} d\zeta \right)^{-1} \tag{1.10}$$

with$\zeta = (\zeta_1, \zeta'), \zeta' \in \mathbb{R}^{n-1}$. The operator defined in (1.8) is the fractional Laplacian. Commonly, in defining $(-\Delta^s)$, the abbreviation for “in the principal–value sense” is adopted. Precisely, setting

$$\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy \qquad x \in \mathbb{R}^n,$$

we can write

$$(-\Delta^s)u(x) := c(n,s)\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy \qquad x \in \mathbb{R}^n, \qquad x \in \mathbb{R}^n, \tag{1.11}$$

The singular integral given in 1.10 can be written as a weighted second–order differential quotient as follows (see [16], lemma 3.2): Some recent results on fractional Laplacian equations can be found in [10], [11], [12], [13], [16], [17], [18], [33], [34], [32], [7] and references therein. Moreover, very recently, a new nonlocal and nonlinear operator (the fractional p–Laplacian $(-\Delta^s)$ was considered (see, e.g., the papers [30, 31] and references therein). Namely, for $p \in (1,+\infty)$, $s \in (0,1)$, and u smooth enough, it is defined as

$$(-\Delta^s)u(x) = \text{P.V.} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}} dy \qquad x \in \mathbb{R}^n, \qquad x \in \mathbb{R}^n, \tag{1.12}$$

Up to some normalization constant depending on n, p and s, this definition is consistent with the one of the fractional Laplacian $(-\Delta^s)$ in the case $p = 2$.ators, we refer the reader to the seminal paper of Caffarelli [9] (see also the recent papers [6], [35], [35])

1.4 Lebesgue's dominated convergence theorem

Let f_n be a sequence of complex-valued measurable functions on a measure space $(S, \sum, \mu)$. Suppose that the sequence converges pointwise to a function $f$and is dominated by some integrable function g in the sense that

$$|f_n(x)| \leq g(x)$$

for all numbers n in the index set of the sequence and all points $x \in S$. Then f is integrable (in the Lebesgue sense) and

$$\lim_{n \to \infty} \int_S |f_n - f| d\mu = 0$$

which also implies

$$\lim_{n \to \infty} \int_S f_n d\mu = \int_S f d\mu$$

1.5 Fatou's Lemma

Lemma 1.3. *Giver a mesure space* (Ω, F, μ) *and a set* $X \in F$ *,let* $\{f_n\}$ *be a sequence of* $(F, B_{\mathbb{R}\geq})$ *mesurable non negative functions* $f_n : X \to [0, +\infty]$ *. Define the function* $f : X \to [0, +\infty]$ *by setting* $f(x) = \lim_{n\to\infty} \inf f_n(x)$*, for every* $x \in X$
then f *is* $(F, B_{\mathbb{R}\geq})$ *mesurable , and also* $\int_X f d\mu \leq \liminf_{n\longrightarrow\infty} f_n d\mu$*. where the integrals may be infinite*

Now, we recall the embedding of $W_0^{s_1,p}(\Omega)$ in $W_0^{s_2,q}(\Omega)$ in the following Lemma:

Lemma 1.4. *(see [26, Lemma 2.1]) Let* $1 < q \leq p < \infty$ *and* $0 < s_2 < s_1 < 1$*, then there exists a constant* $C = C(|\Omega|, N, p, q, s_1, s_2) > 0$ *such that*

$$\|u\|_{W_0^{s_2,q}(\Omega)} \leq C \|u\|_{W_0^{s_1,p}(\Omega)}$$

for all $u \in W_0^{s_1,p}(\Omega)$.

Remark 1.5. *The embedding in Lemma 1.4 when* $s_1 = s_2$*, with* $p \neq q$ *does not hold, see [28, Theorem 1.1] for the counterexample. For that, we consider the space* $\boldsymbol{W} := W_0^{s_1,p}(\Omega)$*, in the case* $0 < s_2 < s_1 < 1$*, and if* $s = s_1 = s_2$*, we take* $\boldsymbol{W} := W_0^{s,p}(\Omega) \cap W_0^{s,q}(\Omega)$*, equipped with the Cartesian norm* $\|\cdot\|_{\boldsymbol{W}} := \|\cdot\|_{W_0^{s,p}(\Omega)} + \|\cdot\|_{W_0^{s,q}(\Omega)}$*. Notice that* $W_0^{s,r}(\Omega)$ *is a separable reflexive Banach space with* $s \in \{s_1, s_2\}$ *and* $r \in \{p, q\}$*, then* $\boldsymbol{W}$ *is also a separable reflexive Banach space.*

CHAPTER 2

Picone inequality

1 Picone inequality in the local case

We begin this chapter with the following first version of Picone equality (se [29]):

$$\nabla u \nabla (\frac{v^2}{u}) - |\nabla v|^2 = -|\nabla v - \nabla u (\frac{v}{u})|^2 \tag{2.1}$$

where $u, v \geq 0$ are differentiable functions , with $u > 0$, We refer here, this version was used to prove a comparaison principle for ordinary differentil equations of sturm-liouville type.

1.1 Linear Picone inequalities

After that, the results in [1] extend this inequality to the nonlinear p–Laplace operator as the following Theorem:

Theorem 1.1. *([1]) Let $v \geq 0$, $u \geq 0$ be differentiable. Denote*

$$\mathrm{L}(u,v) = |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v^p| - p\frac{u^{p-1}}{v^{p-1}}\nabla u |\nabla v|^{p-2}\nabla v \tag{2.2}$$

$$\mathrm{R}(u,v) = |\nabla u|^p - \nabla(\frac{u^p}{v^{p-1}})|v^{p-2}|\nabla v \tag{2.3}$$

then

$$\mathrm{L}(u,v) = \mathrm{R}(u,v)$$

Morever, $\mathrm{L}(u,v) \geq 0$, *and* $\mathrm{L}(u,v) = 0$ *a.e* Ω *if and only if* $\nabla(\frac{u}{v}) = 0$ *a.e.*Ω, *i.e.* $u = kv$ *for some constant* k *in each component of* Ω.

Remark 1.2. *In the linear case $p = 2$, inequality is a direct consequence of the simple identity.*

More recently, non-homogeneous Picone inequalities of (2.2), were established. The first contribution is obtained in [8] and states as follows:

$$|\nabla u|^{p-2}\nabla u\nabla(\frac{v^q}{u^{q-1}}) \le |\nabla v|^q|\nabla u|^{p-q} \tag{2.4}$$

For $p \neq q$, as a plain consequence of Young inequality, (2.4) implies

$$\frac{1}{p}\langle\nabla H(\nabla u), \nabla(\frac{v^q}{u^{q-1}})\rangle \le \frac{q}{p} + \frac{p-q}{p}H(\nabla u) \tag{2.5}$$

Observe that for $q = 1$, the previous inequality reduces to

$$H(\nabla u) + \langle\nabla H(\nabla u), \nabla(v) - \nabla(u)\rangle \le H(\nabla v) \tag{2.6}$$

which just follows from the convexity of $z \to H(z)$. On the other hand, by applying (2.3) with the choices(here $\varepsilon > 0$) $U = (\varepsilon + H(\nabla u))^{\frac{-1}{p}} u$ and $V = (\varepsilon + H(\nabla v))^{\frac{-1}{p}} v$,we get

$$\frac{1}{p}(\varepsilon + H(\nabla u))^{\frac{q-p}{p}}(\varepsilon + H(\nabla v))^{\frac{-q}{p}}\langle\nabla H(\nabla u), \nabla(\frac{v^q}{u^{q-1}}\rangle \le 1 \tag{2.7}$$

By multiplying the previous by $(\varepsilon + H(\nabla u))^{\frac{p-q}{p}}$ $(\varepsilon + H(\nabla u))^{\frac{q}{p}}$ and then letting ε goes to 0,we get and a second form of identity is given in as follows:

$$|\nabla u|^{q-2}\nabla u\nabla(\frac{v^p}{u^{p-1}}) \le |\nabla v|^{q-2}\nabla v\nabla(\frac{v^{p-q+1}}{u^{p-q}}) \tag{2.8}$$

where u, v are nonnegative differentiable functions,with $u > 0, 1 < q \le p$

1.2 Non-linear Picone inequalities

First, the nonlinear Picone inequality analogue of (2.1) in connection to the Laplace operator has been obtained in [39], as follows:

$$\nabla u\nabla(\frac{v^2}{f(u)}) \le \alpha|\nabla v|^2 \tag{2.9}$$

for differentiable functions u and v, with $u \neq 0$ and where $f(y) \neq 0$ when $y \neq 0$ together with $f'(y) \ge \frac{1}{\alpha}$.

In [3], the author provides an extension of the above result to the $p-$ Laplace operator (with $\alpha = 1$) as follows:

Theorem 1.3. *Let $v > 0$ and $u \ge 0$ be two non-constant differentiable functions in Ω Also assume that $f'(y) \ge (p-1)[f(y)^{\frac{p-2}{p-1}}]$ for all y. Define*

$$L(u, v) = |\nabla u|^p - \frac{pu^{p-1}\nabla u|\nabla v|^{p-2}\nabla v}{f(v)} + \frac{u^p f'(v)|\nabla v|}{[f(y)]^2}$$

$$R(u, v) = |\nabla u|^p - \nabla\left(\frac{u^p}{f(v)}\right)|\nabla v|^{p-2}\nabla v$$

Then $L(u, v) = R(u, v) \ge 0$. Moreover $L(u, v) = 0$ a.e Ω if and only if $\nabla(\frac{u}{v}) = 0$.

Remark 1.4. *when $p = 2$ and $f(y) = y$ we get the classical Picone's Identity*

$$|\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u\nabla v = |\nabla u|^2 + \nabla\frac{u^2}{v^2}\nabla v \ge 0 \tag{2.10}$$

for Laplacian and when $p = 2$ we get back its nonlinear version

$$|\nabla u|^2 + \frac{|\nabla u|^2}{f'(v)} + (\frac{u\sqrt{f'(v)}\nabla v}{f(v)} - \frac{\nabla u}{\sqrt{f'(v)}})^2 = |\nabla u|^2 - \nabla(\frac{u^2}{f(v)})\cdot\nabla v \ge 0 \tag{2.11}$$

2 Picone inequality in the nonlocal case

We start by [8], the authors proved the following Picone inequality:

Proposition 2.1. *([8]) Let $1 < p < \infty$ and $1 < q \leq p$.*
Let u, v be two measurable functions with $v \geq 0$ and $u > 0$, then

$$|u(x)-u(y)|^{p-2}\left(u(x)-u(y)\right)\left[\frac{v(x)^q}{u(x)^{q-1}}-\frac{v(y)^q}{u(y)^{q-1}}\right] \leq |v(x)-v(y)|^q|u(x)-u(y)|^{p-q} \quad (2.12)$$

Proof. (see [8]) we notice at first that is sufficient to prove

$$|u(x)-u(y)|^{q-2}\left(u(x)-u(y)\right)\left[\frac{v(x)^q}{u(x)^{q-1}}-\frac{v(y)^q}{u(y)^{q-1}}\right] \leq |v(x)-v(y)|^q \quad (2.13)$$

since 2.12 then follows by multiplying the previous inequality by $|u(x)-u(y)|^{p-q}$. At this aim, let us start by observing that if $u(x) = u(y)$, inequality 2.13 is trivially satisfied. We take then $u(x) \neq u(y)$ and we can always suppose that $u(x) < u(y)$, up to exchanging the role of x and y. We further observe that if $v(y) = 0$, inequality 2.13 is again trivially satisfied, since

$$|u(x)-u(y)|^{q-2}\left(u(x)-u(y)\right)\left[\frac{v(x)^q}{u(x)^{q-1}}-\frac{v(y)^q}{u(y)^{q-1}}\right] \leq 0$$

We can thus suppose that $v(y) \neq 0$, then we rewrite the left-hand side of 2.13 as

$$\begin{aligned}&|u(x)-u(y)|^{q-2}\left(u(x)-u(y)\right)\left[\frac{v(x)^q}{u(x)^{q-1}}-\frac{v(y)^q}{u(y)^{q-1}}\right]\\ &= u(x)^q\left(\frac{v(y)}{u(y)}\right)^q\left[\left(1-\frac{u(y)}{u(x)}\right)^{q-1}\left(\left(\frac{v(x)u(y)}{v(y)u(x)}\right)^q-\frac{u(y)}{u(x)}\right)\right]\end{aligned}$$

while the right-hand side of 2.13 rewrites as

$$|v(x)-v(y)|^q = u(x)^q\left(\frac{v(y)^q}{u(y)^q}\right)\left|\left(\frac{v(x)u(y)}{v(y)u(x)}\right)-\frac{u(y)}{u(x)}\right|^q$$

Then if we set

$$\mathrm{A} = \frac{v(x)u(y)}{v(y)u(x)} \text{ and } t = \tfrac{u(y)}{u(x)}$$

the previous manipulations show that 2.12 is equivalent to the following

$$(1-t)^{q-1}\left(\mathrm{A}^q - t\right) \leq |\mathrm{A}-t|^q$$

for $0 \leq t \leq 1$
The previous elementary inequality is true (see [19, Lemma 2.6]), thus we get the desired conclusion. □

In [23], authors extend the results obtained in [19], as follows:

Theorem 2.2. *([23]) Let $1 < p < \infty$ and $1 < q \leq p$: Let $u; v$ be two Lebesgue-measurable functions in Ω ; with $v \geq 0$ and $u > 0$; then*

$$\left[u(x)-u(y)\right]^{q-1}\left[\frac{v(x)^p}{u(x)^{p-1}}-\frac{v(y)^p}{u(y)^{p-1}}\right] \leq \left[v(x)-v(y)\right]^{q-1}\left[\frac{v(x)^{p-q+1}}{u(x)^{p-q}}-\frac{v(y)^{p-q+1}}{u(y)^{p-q}}\right] \quad (2.14)$$

Moreover, the equality in 2.14 holds in if and only if $u = kv$; for some constant $k > 0$:

To proof this Theorem, we need the following technical Lemma:

Lemma 2.3. *([23]) Let* $1 < p < \infty$ *and* $1 < q \leq p$*: Then for all* $0 \leq t \leq 1$ *and* $A \in \mathbb{R}^+$*; we have:*

$$(1-t)^{q-1}\left(A^p - t\right) \leq |A-t|^{q-2}(A-t)\left(A^{p-q+1}-t\right) \tag{2.15}$$

Moreover, (2.15) *is always strict unless* $A = 1$ *or* $t = 0$*:*

Proof. (see [23]) Since the case $p = q$ is covered by [[19], Lemma 2.6], we assume that $1 < q < p$:
First, for $t = 0$; (2.15) is obviously satisfied. Let us assume $t > 0$:
Let us start with the case $A^p < t$; this implies that $A < 1$: We distinguish three cases:
(1) Suppose that $A^{p-q+1} \geq t$; we obtain $A > A^{p-q+1} \geq t > A^p$; then (2.15) follows from $A^p - t < 0$ and $(A-t)\left(A^{p-q+1}-t\right) \geq 0$
(2) If $t \geq A > A^{p-q+1}$; then $t \geq A > A^{p-q+1} > A^p$: Hence, (2.15) again follows.
(3) Finally, if $A > t > A^{p-q+1}$; we observe that $(1-t)^{q-1} \geq (A-t)^{q-1}$ and $A^p - t < A^{p-q+1} - t < 0$:
Then, by multiplying the previous two inequalities, we obtain (2.15).
We now assume $A^p > t$ (note that if $A^p = t$; (2.15) is obvious). Since $t \leq 1$, this implies that $A > t$, we then define g as below :

$$g(A) = \frac{(A-t)^{q-1}(A^{p-q+1}-t)}{A^p - t}$$

After straightforward computations, the derivative of g with respect to A; denoted by $g'(A)$, verifies

$$\begin{aligned}
g'(A) = &\\
&\frac{(q-1)(A-t)^{q-2}\left[\left(A^{p-q+1}-t\right)(A^p-t)-(A-t)(A^{2p-q}-tA^{p-q})\right]+pt(A-t)^{q-1}(A^{p-1}-A^{p-q})}{(A^p-t)^2}\\
&= \frac{t(q-1)(A-t)^{q-2}\left[A^{p-q}(A^p-A^q-t)+t\right]+pt(A-t)^{q-1}(A^{p-1}-A^{p-q})}{(A^p-t)^2}\\
&= \frac{t(A-t)^{q-2}\left[(q-1)\left(\frac{A^p-t}{A^q}\right)(A^p-A^q)+p(A-t)\left(A^{p-1}-A^{p-q}\right)\right]}{(A^p-t)^2}
\end{aligned}$$

Now, we note that $g'(A)$ is positive if $A > 1$ whereas it is negative if $0 < A < 1$. Noting $g'(1) = 0$, we get that $A = 1$ is a global minimum point of the function g: Then $g(A) \geq g(1)$
for all $A > t^{\frac{1}{p}}$. The proof is now complete. □

From Lemma (2.3), we deduce the proof of Theorem (2.2):

Proof of theorem (2.2). (see [23]) First, note that if $p = q$; then (2.14) is obviously satisfied from (1.4). Therefore, since the inequality (2.14) is invariant under the permutation $(x, y) \longrightarrow (y, x)$, we can suppose in the sequel that $u(x) \geq u(y)$ together with $p > q$.
Now, the left-hand side expression of (2.14) can be rephrased as:

$$\begin{aligned}
&|u(x)u(y)|^{q-2}\left(u(x)-u(y)\right)\left[\frac{v(x)^p}{u(x)^{p-1}} - \frac{v(y)^p}{u(y)^{p-1}}\right]\\
&= u(x)^q\left(\frac{v(y)}{u(y)}\right)^p\left[\left(1-\frac{u(y)}{u(x)}\right)^{q-1}\left(\left(\frac{v(x)u(y)}{v(y)u(x)}\right)^p - \frac{u(y)}{u(x)}\right)\right]
\end{aligned}$$

and the right-hand side

$$|v(x)-v(y)|^{q-2}\left(v(x)-v(y)\right)\left[\frac{v(x)^{p-q+1}}{u(x)^{p-q}}-\frac{v(y)^{p-q+1}}{u(y)^{p-q}}\right]$$

$$= u(x)^q\left(\frac{v(y)}{u(y)}\right)^p|\left(\frac{v(x)u(y)}{v(y)u(x)}\right)-\frac{u(y)}{u(x)}|^{q-2}\left(\left(\frac{v(x)u(y)}{v(y)u(x)}\right)-\frac{u(y)}{u(x)}\right)\left(\left(\frac{v(x)u(y)}{v(y)u(x)}\right)^{p-q+1}-\frac{u(y)}{u(x)}\right)$$

Setting $A=\dfrac{v(x)u(y}{v(y)u(x)}, t=\frac{u(y)}{u(x)}$, and applying Lemma (2.3), we obtain the desired conclusion. On the other hand, since $t\neq 0$; we remark that the equality in (2.14) holds if and only $A=1$, i.e.

$$\frac{u(x)}{v(x)}=\frac{u(y)}{v(y)}$$

from which we get $u=kv$ a.e. in $\Omega$for some $k>0$. □

3 Comparison principle

Definition 3.1. *Let* X *be a real vector space. Let* C *be a non empty convex cone in* X*. A functional* $W:C\longrightarrow\mathbb{R}$ *will be called ray–strictly convex (strictly convex,respectively) if it satisfies*

$$w((1-t)v_1+tv_2)\leq(1-t)w(v_1)+tw(v_2),$$

for all $v_1,v_2\in C$ and for all $t\in(0,1)$, where the inequality is always strict unless $\frac{v_1}{v_2}\equiv c>0$ (always strict unless $v_1\equiv v_2$, respectively).

Proposition 3.2. *(Discrete hidden convexity [[8], Proposition 4.1]).Let $1<p<\infty$ and $1<q\leq p$. For every $u_0,u_1\geq 0$, we define*

$$\sigma_t(x)=[(1-t)u_0^q(x)+tu_1^q(x)]^{\frac{1}{q}}\qquad t\in[0,1]x\in\mathbb{R}^N.$$

Then

$$|\sigma_t(x)\sigma_t(y)|^p\leq|(1-t)u_0(x)+tu_0(y)|^p+|tu_1(x)+tu_1(y)|^p\qquad t\in[0,1]x,y\in\mathbb{R}^N.$$

Proposition 3.3. *[22] Let $1<p<\infty$ and $1<r\leq p$. The functional $W:v_+^r\longrightarrow\mathbb{R}^+$ defined by*

$$W(w):=\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w(x)^{\frac{1}{r}}-w(y)^{\frac{1}{r}}|^p}{|x-y|^{N+sp}}dxdy,\tag{2.16}$$

is ray-strictly convex on V_+^r. Furthermore, if $p\neq r$, then W is even strictly convex on V_+^r.

Proof. (see [22]) According to Definition 3.1, let us consider any $w_1,w_2\in V_+^r$ and $t\in[0,1]$. Let us denote $w=tw_1+(1-t)w_2$, we obtain by Proposition 3.2

$$w(w)\leq tw(w_1)+(1-t)w(w_2)\tag{2.17}$$

If the equality holds, then

$$|w(x)^{\frac{1}{r}}-w(y)^{\frac{1}{r}}|^p=t|w_1(x)^{\frac{1}{r}}-w_1(y)^{\frac{1}{r}}|^p+(1-t)|w_2(x)^{\frac{1}{r}}-w_2(y)^{\frac{1}{r}}|^p$$

a.e $x,y\in\mathbb{R}^n$...If $p=r$, we obtain

$$|\|a\|_{l^r}-\|b\|_{l^r}|^r=\|a-b\|_{l^r}^r\qquad a.ex,y\in\mathbb{R}^n$$

where $\|.\|_{l^r}$ denotes the l^r norm in $\mathbb{R}^2$ and

$$a=((tw_1(x))^{\frac{1}{r}},(1-t)w_2(x))^{\frac{1}{r}}),\quad b=((tw_1(y))^{\frac{1}{r}},(1-t)w_2(y))^{\frac{1}{r}})$$

Since $r>1$, there exists a constant $c>0$ such that $w_1=cw_2$ a.e. $x\in\mathbb{R}^N$ Then, W is ray –strictly convex on v_+^r . On the other hand, if $p\neq r$ thanks to the stric convexity of $\tau\longrightarrow\tau^{\frac{p}{r}}$ on $\mathbb{R}^+$, we obtain $w_1=w_2$ a.e $x,y\in\mathbb{R}^n$ and Wis strictly convex on v_+^r □

Proposition 3.4. *(Discrete Picone inequality [[8], Proposition 4.2]) Let* $1<p<\infty$ *and* $1<r\leq p$. *Let* u,v *be two Lebesgue-measurable functions with* $v\geq 0$ *and* $u>0$. *Then*

$$\begin{aligned}&\left|u(x)-u(y)\right|^{p-2}\left(u(x)-u(y)\right)\left[\frac{v(x)^r}{u(x)^{r-1}}-\frac{v(y)^r}{u(y)^{r-1}}\right]\\&\leq\left|v(x)-v(y)\right|^r\left|u(x)-u(y)\right|^{p-r}\end{aligned}$$

Lemma 3.5. *(see [22]) Let* $1<p<\infty$. *Then, for* $1<r\leq p$ *and for any* u,v *two measurable and positive functions in* Ω.

$$\begin{aligned}&|u(x)-u(y)|^{p-2}(u(x)-u(y))\left[\frac{u(x)^r-v(x)^r}{u(x)^{r-1}}-\frac{u(y)^r-v(y)^r}{u(y)^{r-1}}\right]+\\&|v(x)-v(y)|^{p-2}(v(x)-v(y))\left[\frac{v(x)^r-u(x)^r}{v(x)^{r-1}}-\frac{v(y)^r-u(y)^r}{v(y)^{r-1}}\right]\geq 0\end{aligned}\tag{2.18}$$

for a.e. $x,y\in\Omega$. *Moreover, if* $u,v\in W_0^{s,p}(\Omega)$ *and if the equality occurs in (2.18) for a.e.*$x,y\in\Omega$, *then we have the following two statements:*
(1) $u/v\equiv const>0$ *a.e in* Ω.
(2) If also $p\neq r$, *then* $u\equiv v$ *a.e. in* Ω.

Proof. (see [22]) Let u,v be two measurable functions such that $u,v>0$ in Ω and $1<r\leq p$. Then by using Proposition 3.4, we obtain for $x,y\in\Omega$,

$$|u(x)-u(y)|^{p-2}(u(x)-u(y))\left[\frac{v(x)^r}{u(x)^{r-1}}-\frac{v(y)^r}{u(y)^{r-1}}\right]\leq|v(x)-v(y)|^r|u(x)-u(y)|^{p-r}\tag{2.19}$$

Let us start with the case $r=p$. By using the above inequality,in this case, we obtain

$$|u(x)-u(y)|^{p-2}(u(x)-u(y))\left[\frac{u(x)^p-v(x)^p}{u(x)^{p-1}}-\frac{u(y)^p-v(y)^p}{u(y)^{p-1}}\right]\geq|u(x)-u(y)|^p-|(v(x)-v(y))|^p\tag{2.20}$$

By exchanging the roles of u and v, we obtain

$$|v(x)-v(y)|^{p-2}(v(x)-v(y))\left[\frac{v(x)^p-u(x)^p}{v(x)^{p-1}}-\frac{v(y)^p-u(y)^p}{v(y)^{p-1}}\right]\geq|v(x)-v(y)|^p-|(u(x)-u(y))|^p\tag{2.21}$$

Combining (2.20) and (2.21), we obtain

$$\begin{aligned}&|u(x)-u(y)|^{p-2}(u(x)-u(y))\left[\frac{u(x)^p-v(x)^p}{u(x)^{p-1}}-\frac{u(y)^p-v(y)^p}{u(y)^{p-1}}\right]+\\&|v(x)-v(y)|^{p-2}(v(x)-v(y))\left[\frac{v(x)^p-u(x)^p}{v(x)^{p-1}}-\frac{v(y)^p-u(y)^p}{v(y)^{p-1}}\right]\geq 0\end{aligned}$$

which concludes the proof of (2.18) for $r = p$. We deal finally with the case $1 < r < p$. By using Young's inequality, (2.19) implies

$$|u(x)-u(y)|^{p-2}(u(x)-u(y))\left[\frac{u(x)^r - v(x)^r}{u(x)^{r-1}} - \frac{u(y)^r - v(y)^r}{u(y)^{r-1}}\right] \geq \frac{r}{p}\left[|u(x)-u(y)|^p - |v(x)-v(y)|^p\right] \tag{2.22}$$

Reversing the role of u and v:

$$|v(x)-v(y)|^{p-2}(v(x)-v(y))\left[\frac{v(x)^r - u(x)^r}{v(x)^{r-1}} - \frac{v(y)^r - u(y)^r}{v(y)^{r-1}}\right] \geq \frac{r}{p}\left[|v(x)-v(y)|^p - |u(x)-u(y)|^p\right] \tag{2.23}$$

Adding the above inequalities, we obtain (2.18) Now, let us consider $u, v \in W_0^{s,p}(\Omega)$, such that $u > 0$, $v > 0$ a.e. inΩ and $\theta \in (0,1)$. Setting $w := (1-\theta)u^r + \theta v^r$, one can easily check that $w \in V_+^r$. Thus, by Proposition 3.3, it is easy to prove that the function, defined in $[0.1]$,

$$\theta \longrightarrow \phi(\theta) := W(w) = W((1-\theta)u^r + \theta v^r)$$

is convex, differentiable and for $\theta \in (0,1)$:

$$\phi'(\theta) = \int_{R^{2N}\setminus(\Omega^c\times\Omega^c)} \frac{|w(x)^{\frac{1}{r}} - w(y)^{\frac{1}{r}}|^{p-2}(w(x)^{\frac{1}{r}} - w(y)}{|x-y|^{N+sp}} \tag{2.24}$$

Finally, let us assume that the equality in (2.18) holds. By the monotonicity of $\phi' : (0,1) \longrightarrow R$, we deduce that $\phi'(\theta) = const$in $(0,1)$. It follows that $\phi : [0,1] \longrightarrow R$ must be linear, i.e.

$$\phi(\theta) = W(w) = (1-\theta)\phi(0) + \theta\phi(1) = (1-\theta)W(u^r) + \theta W(v^r),$$

for all $\theta \in [0,1]$. We conclude that $u \equiv const.v$ with $const > 0$ and if $p \neq r$, then $u \equiv v$, thanks to Proposition 3.3 □

From the result in Lemma 3.5, we can show an extended version of the famous Diaz-Saa inequality to the fractional and non-homogeneous operator, as follows:

Lemma 3.6. *(Diaz-Saa inequality) Let $1 < p < \infty$ and $0 < s_1, s_2 < 1$. Then, for $1 < q \leq p$, we have in the sense of distributions:*

$$\int_\Omega \left(\frac{(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u}{u^{q-1}} - \frac{(-\Delta)_p^{s_1} v + (-\Delta)_q^{s_2} v}{v^{q-1}}\right)(u^q - v^q)dx \geq 0 \tag{2.25}$$

for any $u, v \in \mathbf{W}$ positive in Ω such that $\frac{u}{v}, \frac{v}{u} \in L^\infty(\Omega)$. Moreover, if the equality occurs in (2.25), then we have the following two statements:

- $\frac{u}{v} \equiv const > 0$ *a.e. in* Ω.
- *If also $p \neq q$ then $u \equiv v$ a.e. in Ω.*

Remark 3.7. *It is easy to see that (2.25) is a distributional interpretation of the following inequality:*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)}{|x-y|^{N+s_1 p}}\left[\frac{u(x)^q-v(x)^q}{u(x)^{q-1}}-\frac{u(y)^q-v(y)^q}{u(y)^{q-1}}\right]dxdy$$
$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p-2}\big(v(x)-v(y)\big)}{|x-y|^{N+s_1 p}}\left[\frac{v(x)^q-u(x)^q}{v(x)^{q-1}}-\frac{v(y)^q-u(y)^q}{v(y)^{q-1}}\right]dxdy$$
$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{q-2}\big(u(x)-u(y)\big)}{|x-y|^{N+s_2 q}}\left[\frac{u(x)^q-v(x)^q}{u(x)^{q-1}}-\frac{u(y)^q-v(y)^q}{u(y)^{q-1}}\right]dxdy$$
$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{q-2}\big(v(x)-v(y)\big)}{|x-y|^{N+s_2 q}}\left[\frac{v(x)^q-u(x)^q}{v(x)^{q-1}}-\frac{v(y)^q-u(y)^q}{v(y)^{q-1}}\right]dxdy\geq 0$$

for any $u,v\in \boldsymbol{W}$ *positive in* Ω *such that* $u/v,\ v/u\in L^\infty(\Omega)$*. These integrals are defined as Lebesgue integrals. Indeed, for any* $x,y\in\mathbb{R}^N$:

• *There exists* $M>0$ *such that:*

$$0\leq \frac{u^q}{v^{q-1}}=\left(\frac{u}{v}\right)^{q-1}u\leq M\,u \quad \textit{and} \quad 0\leq \frac{v^q}{u^{q-1}}=\left(\frac{v}{u}\right)^{q-1}v\leq M\,v.$$

Then $\dfrac{u^q}{v^{q-1}}\in L^p(\mathbb{R}^N)$ *and* $\dfrac{v^q}{u^{q-1}}\in L^p(\mathbb{R}^N)$*, and vanish in* $\mathbb{R}^N\setminus\Omega$*.*

• *Having in mind Lagrange's Mean Value Theorem, we obtain:*

$$\begin{aligned}\left|\frac{u^q(x)}{v^{q-1}(x)}-\frac{u^q(y)}{v^{q-1}(y)}\right| &\leq \left|\frac{u^q(x)}{v^{q-1}(x)}-\frac{u^q(y)}{v^{q-1}(x)}\right|+\left|\frac{u^q(y)}{v^{q-1}(x)}-\frac{u^q(y)}{v^{q-1}(y)}\right|\\
&=\frac{|u^q(x)-u^q(y)|}{v^{q-1}(x)}+u^q(y)\,\frac{|v^{q-1}(x)-v^{q-1}(y)|}{v^{q-1}(x)\,v^{q-1}(y)}\\
&\leq \frac{q\max\{u^{q-1}(x),u^{q-1}(y)\}}{v^{q-1}(x)}\,|u(x)-u(y)|\\
&\qquad+\frac{u^q(y)\,(q-1)\max\{v^{q-2}(x),v^{q-2}(y)\}}{v^{q-1}(x)\,v^{q-1}(y)}\,|v(x)-v(y)|\\
&\leq C\left(q,\left\|\frac{u}{v}\right\|_{L^\infty(\Omega)}\right)\big(|u(x)-u(y)|+|v(x)-v(y)|\big).\end{aligned}$$

Hence, $u^q/v^{q-1},\ v^q/u^{q-1}\in \boldsymbol{W}$.

CHAPTER 3

Applications to Differential Equations

The purpose of this chapter is to study the existence, non-existence, uniqueness, and regularity of the weak solutions to the following non-linear problem involving fractional operators:

$$(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = g(x,u), \quad u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \tag{\textbf{P}}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with boundary of class $C^{1,1}$, $N > s_1 p$, $0 < s_2 \le s_1 < 1$ and $1 < q \le p < \infty$. Here $(-\Delta)_r^s$ is the fractional $r-$Laplace operator, defined for $s \in \{s_1, s_2\}$ and $r \in \{p, q\}$, as

$$(-\Delta)_r^s u(x) := 2\,\textbf{P.V.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{r-2}(u(x)-u(y))}{|x-y|^{N+sr}}\, dy$$

where **P.V.** denotes the Cauchy principal value. For problem (**P**), we consider the following two types of nonlinearities to the function g:

• **Case (i)** (Generalized eigenvalue problems):

$$g(x,u) = \lambda\, a_q(x) u^{q-1} \quad \text{such that} \quad r = q$$

where $a_q \in (L^\infty(\Omega))^+ \setminus \{0\}$ and λ is a positive real number.

• **Case (ii)** (Singular nonlinearity problems):

$$g(x,u) = \frac{1}{u^\delta} + b(x,u) \quad \text{with} \quad 0 < \delta < 1.$$

Our Hypotheses on the function $b : \Omega \times \mathbb{R} \to \mathbb{R}$ are the following:

(H1) $b : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, such that $b(x,0) \equiv 0$ and b is positive on $\Omega \times \mathbb{R}^+ \setminus \{0\}$.

(H2) Let $1 < \upsilon < q$, for a.e. $s \mapsto \dfrac{b(x,s)}{s^{\upsilon-1}}$ is decreasing in $\mathbb{R}^+ \setminus \{0\}$.

1 Fractional and non-homogeneous eigenvalue problems

We define the notion of weak solutions to the problem (**P**) in **case (i)**, i.e.

$$(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \lambda\, a_r(x) u^{r-1}, \quad u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \tag{\textbf{P-(i)}}$$

where $r = p$ or q, as follows:

Definition 1.1. *A non-negative function $u \in \boldsymbol{W}$ is called a weak solution to* (**P-(i)**) *if, for any $\varphi \in \boldsymbol{W}$ we have:*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x)-u(y)\right|^{p-2}\left(u(x)-u(y)\right)\left(\varphi(x)-\varphi(y)\right)}{|x-y|^{N+s_1 p}}dxdy$$
$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x)-u(y)\right|^{q-2}\left(u(x)-u(y)\right)\left(\varphi(x)-\varphi(y)\right)}{|x-y|^{N+s_2 q}}dxdy$$
$$=\lambda\int_\Omega a_r(x)u^{r-1}\varphi dx. \tag{3.1}$$

In addition if u satisfies $u>0$ in Ω, we call u positive weak solution.

In the following, we present an important remark about the regularity of weak solutions to fractional and non-homogeneous equations that we will use several times in the sequel:

Remark 1.2. *Let $u_0 \in \boldsymbol{W} \cap L^\infty(\Omega)$ be a nontrivial weak solution to the problem* (**P-(i)**). *Then, Theorem 2.3 in [24], Corollary 2.4 and Remark 2.3 in [25] provide the $C^{0,\alpha}(\overline{\Omega})$-regularity of u_0, for some $\alpha \in (0, s_1)$. By [24, Theorem 2.5], we infer that $u_0 > 0$ in Ω. Finally, by the Hopf's Lemma [24, Proposition 2.6] implies that $u_0 \geq k\, d^{s_1+\epsilon_0}(x)$ for some $k = k(\epsilon_0) > 0$ and for any $\epsilon_0 > 0$. Again by using [24, Proposition 3.11], we get that, for all $\sigma \in (0, s_1)$ there exists a constant $K = K(\sigma) > 0$ such that $u_0 \leq K\, d^\sigma(x)$ in Ω.*

We first investigate the non-existence of positive weak solutions for the problem (**P-(i)**):

Theorem 1.3. *Let $r = p$ or q, with $s = s_1$ or s_2, respectively. If $\lambda \leq \lambda_{1,s,r}(a_r)$ holds, then* (**P-(i)**) *has no nontrivial solutions.*

Next, we state the results of the existence, uniqueness, and regularity:

Theorem 1.4. *Let $0 < s_2 \leq s_1 < 1$ and $1 < q < p < \infty$. Then, we have:*
If $\lambda > \lambda_{1,s_2,q}(a_q)$. Then (**P-(i)**) *has at least one positive solution u. In addition, $u \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for any $\sigma \in (0, s_1)$ and $\sigma' > s_1$, there exists a positive constant $c = c(\sigma, \sigma') > 0$, such that:*

$$c^{-1}d^{\sigma'} \leq u \leq c\, d^\sigma \quad \text{in}\,\Omega.$$

Moreover, the solution is unique.

Theorem 1.5. *We set the following non-local Rayleigh quotient:*

$$\underline{\lambda}_{s,s*,r,r^*}(a) := \inf_{u\in \boldsymbol{W}}\left\{\frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u(x)-u(y)\right|^r}{|x-y|^{N+sr}}dxdy+\frac{r}{r^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u(x)-u(y)\right|^{r*}}{|x-y|^{N+s^*r^*}}dxdy}{\displaystyle\int_\Omega a_r(x)u^r\,dx}\right\}.$$

where $r = p$ (or q), with $s = s_1$ (or s_2) if $r^* = q$ (or p), with $s = s_2$ (or s_1, respectively). Then, $\underline{\lambda}_{s,s^*,r,r^*}(a_r) = \lambda_{1,s,r}(a_r)$. In addition, the infimum is not attained.

In this section, we first prove the non-existence of positive weak solutions for the problem (**P-(i)**).

Proof of Theorem 1.3. Assume by contradiction that $u \in \mathbf{W}$ is a nontrivial solution of (**P-(i)**) and $\lambda \le \lambda_{1,s,r}(a_r)$. Taking u as a test function in (3.1) and by the definition of $\lambda_{1,s,r}(a_r)$, we have that:

$$\lambda_{1,s,r}(a_r) \le \frac{\|u\|^r_{W_0^{s,r}(\Omega)}}{\int_\Omega a_r(x)u^r\,dx} < \lambda = \frac{\|u\|^p_{W_0^{s_1,p}(\Omega)} + \|u\|^q_{W_0^{s_2,q}(\Omega)}}{\int_\Omega a_r(x)u^r\,dx}.$$

This contradicts our hypothesis. □

Next, we establish the existence results and other qualitative properties for the problem (**P-(i)**).

Proof of Theorem 1.4. We distinguish two cases:

- **Case 1:** $s = s_2$ and $r = q$. We divided the proof into 3 steps.

Step 1: Existence of a weak solution.

Consider the energy functional $\mathscr{J}$ corresponding to (**P-(i)**), defined on $\mathbf{W}$ by (with $u^+ = \max\{u,0\}$):

$$\mathscr{J}(u) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+s_1p}}dxdy + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^q}{|x-y|^{N+s_2q}}dxdy - \frac{\lambda}{q}\int_\Omega a_q(x)(u^+)^q\,dx.$$

It is easy to see that $\mathscr{J}$ is well-defined on $\mathbf{W}$. Moreover, $\mathscr{J}$ is weakly lower semi-continuous on $\mathbf{W}$. Indeed, let $(u_n) \subset \mathbf{W}$ converges weakly to some u in $\mathbf{W}$, as $n \to \infty$, we have:

$$\|u\|_{\mathbf{W}} \le \lim_{n\to\infty}\inf\|u_n\|_{\mathbf{W}}.$$

On the other hand, fractional Sobolev embedding [16, Theorem 6.5], implies that, up to a sub-sequence, such that $u_n \to u$ in $L^t(\Omega)$, for every $1 \le t < p^*_{s_1}$ and $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}^N$. By the Dominated Convergence theorem we get:

$$\lim_{n\to\infty}\int_\Omega a(x)(u_n^+)^q\,dx = \int_\Omega a(x)(u^+)^q\,dx.$$

This concludes the weakly lower semi-continuously of $\mathbf{W}$. Finally, $\mathscr{J}$ is also coercive on $\mathbf{W}$. Indeed, for every $u \in \mathbf{W}$, using Hölder inequality and the Sobolev embedding, we obtain:

$$\begin{aligned}
\mathscr{J}(u) &\ge \frac{1}{p}\|u\|^p_{W_0^{s_1,p}(\Omega)} + \frac{1}{q}\|u\|^q_{W_0^{s_2,q}(\Omega)} - \frac{\lambda}{q}\|a_q\|_{L^\infty(\Omega)}\|u\|^q_{L^q(\Omega)}\\
&\ge \frac{1}{p}\|u\|^p_{W_0^{s_1,p}(\Omega)} - |\Omega|^{1-\frac{q}{p}}\frac{\lambda}{q}\|a_q\|_{L^\infty(\Omega)}\|u\|^q_{L^p(\Omega)}\\
&\ge \frac{1}{p}\|u\|^p_{W_0^{s_1,p}(\Omega)} - |\Omega|^{1-\frac{q}{p}}\frac{\lambda}{q\lambda^{\frac{q}{p}}_{1,s_1,p}(1)}\|a_q\|_{L^\infty(\Omega)}\|u\|^q_{W_0^{s_1,p}(\Omega)}\\
&\ge \|u\|^p_{W_0^{s_1,p}(\Omega)}\left[\frac{1}{p} - |\Omega|^{1-\frac{q}{p}}\frac{\lambda}{q\lambda^{\frac{q}{p}}_{1,s_1,p}(1)}\|a_q\|_{L^\infty(\Omega)}\|u\|^{q-p}_{W_0^{s_1,p}(\Omega)}\right].
\end{aligned}$$

Since $1 < q < p$, we conclude that $\mathscr{J}(u) \to +\infty$ when $\|u\|_{\mathbf{W}} \to +\infty$. Then, from above properties $\mathscr{J}$ admits a global minimizer, denoted by u_0. Noting that, with the notation $t = t^+ - t^-$, we have:

$$\begin{aligned}\mathcal{J}(u_0) &= \mathcal{J}(u_0^+)\\ &\quad+\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|(u_0^-)(x)-(u_0^-)(y)\right|^p}{|x-y|^{N+s_1p}}dxdy+\frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|(u_0^-)(x)-(u_0^-)(y)\right|^q}{|x-y|^{N+s_2q}}dxdy\\ &\quad+\frac{2}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|(u_0^+)(x)-(u_0^-)(y)\right|^p}{|x-y|^{N+s_1p}}dxdy+\frac{2}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|(u_0^+)(x)-(u_0^-)(y)\right|^q}{|x-y|^{N+s_2q}}dxdy\\ &\geq \mathcal{J}(u_0^+).\end{aligned}$$

Therefore $u_0\geq 0$. In order to show that $u_0\not\equiv 0$ in Ω, we find a suitable function $u\in\mathbf{W}$ such that $\mathcal{J}(u)<0=\mathcal{J}(0)$. To this aim, for any $t>0$:

$$\begin{aligned}\mathcal{J}(t\phi_{1,s_2,q}(a)) &= \frac{t^p}{p}\|\phi_{1,s_2,q}(a_q)\|^p_{W_0^{s_1,p}(\Omega)}+\frac{t^q}{q}\|\phi_{1,s_2,q}(a_q)\|^q_{W_0^{s_2,q}(\Omega)}-\frac{\lambda t^q}{q}\int_\Omega a(x)\phi_{1,s_2,q}(a_q)^q dx\\ &\leq t^q\left[\frac{t^{p-q}}{p}\|\phi_{1,s_2,q}(a_q)\|^p_{W_0^{s_1,p}(\Omega)}+\frac{\lambda_{1,s_2,q}(a_q)-\lambda}{q\lambda_{1,s_2,q}(a_q)}\|\phi_{1,s_2,q}(a_q)\|^q_{W_0^{s_2,q}(\Omega)}\right]\end{aligned}$$

since $\lambda>\lambda_{1,s_2,q}(a_q)$ and for $t>0$ small enough, we obtain that $\mathcal{J}(t\phi_{1,s_2,q}(a_q))<0$. Since $\mathcal{J}(0)=0$, we deduce that $u_0\not\equiv 0$. From the Gateaux differentiability of $\mathcal{J}$, we have that u_0 satisfies (3.1) i.e. u_0 is a weak solution to (**P-(i)**).

Step 2: Regularity and positivity of weak solutions.

Firstly, we claim that all weak solutions to the problem (**P-(i)**) belongs to $L^\infty(\Omega)$. To this aim, we follow the approach of [20, Theorem 3.2]. Precisely, let $u_0\in\mathbf{W}$ be a weak solution to (**P-(i)**). Setting

$$v_0=\frac{u_0}{\rho\|u_0\|_{L^q(\Omega)}}\quad\text{where}\quad \rho=\max\left\{1,\|u_0\|^{-1}_{L^q(\Omega)}\right\}.$$

Noting that $v_0\in\mathbf{W}$ and $\|v_0\|_{L^q(\Omega)}=\rho^{-1}$. Now, we consider the function w_k defined as follows

$$\begin{cases} w_k(x) & := (v_0(x)-(1-2^{-k}))^+ \quad \text{for } k\in\mathbb{N}\\ w_0(x) & = (v_0(x))^+.\end{cases}$$

We first state the following straightforward observations about $w_k(x)$:

$$w_k\in\mathbf{W}\quad\text{and}\quad w_k=0 \text{ a.e. in } \mathbb{R}^N\setminus\Omega,$$

and

$$\begin{cases} 0\leq w_{k+1}(x)\leq w_k(x) \quad \text{a.e. in } \mathbb{R}^N,\\ v_0(x)<(2^{k+1}+1)w_k(x) \quad \text{for } x\in\{w_{k+1}>0\}.\end{cases} \tag{3.2}$$

Also the inclusion

$$\{w_{k+1}>0\}\subseteq\{w_k>2^{-(k+1)}\}\quad\text{holds for all } k\in\mathbb{N}. \tag{3.3}$$

Now, we set $V_k:=\|w_k\|^q_{L^q(\Omega)}$.

Claim 1. $V_k\to 0$ *as* $k\to\infty$.

Indeed, since $1<q<p$, $\rho\|u_0\|_{L^q(\Omega)}\geq 1$ and by using the following inequality

$$\left|x^+-y^+\right|^l\leq|x-y|^{l-2}(x^+-y^+)(x-y),\quad \text{for } x,y\in\mathbb{R}\quad\text{and}\quad l>1$$

we obtain

$$\begin{aligned}\|w_{k+1}\|^q_{W_0^{s_2,q}(\Omega)} &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_{k+1}(x)-w_{k+1}(y)|^q}{|x-y|^{N+s_2q}}\,dx\,dy\\ &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_{k+1}(x)-w_{k+1}(y)|^p}{|x-y|^{N+s_1p}}\,dx\,dy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_{k+1}(x)-w_{k+1}(y)|^q}{|x-y|^{N+s_2q}}\,dx\,dy\\ &\leq (\rho\,\|u_0\|_{L^q(\Omega)})^{1-p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_0(x)-u_0(y)|^{p-2}(w_{k+1}(x)-w_{k+1}(y))(u_0(x)-u_0(y))}{|x-y|^{N+s_1p}}\,dx\,dy\\ &\quad+(\rho\,\|u_0\|_{L^q(\Omega)})^{1-q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_0(x)-u_0(y)|^{q-2}(w_{k+1}(x)-w_{k+1}(y))(u_0(x)-u_0(y))}{|x-y|^{N+s_2q}}\,dx\,dy\\ &\leq (\rho\,\|u_0\|_{L^q(\Omega)})^{1-q}\left\|a_q\right\|_{L^\infty(\Omega)}\int_{\{w_{k+1}>0\}} u_0^{q-1}w_{k+1}\,dx\\ &\quad=\left\|a_q\right\|_{L^\infty(\Omega)}\int_{\{w_{k+1}>0\}} v_0^{q-1}w_{k+1}\,dx.\end{aligned}$$

This fact combined with (3.2), we obtain

$$\|w_{k+1}\|^q_{W_0^{s_2,q}(\Omega)} \leq C_1(2^{k+1}+1)^{q-1}V_k$$

where $C_1>0$ is a constant. On the other hand, by the Hölder's inequality and fractional Sobolev imbeddings [16, Theorem 6.5], we obtain

$$V_{k+1}=\int_{\{w_{k+1}>0\}} w_{k+1}^q\,dx \leq C_2\|w_{k+1}\|^q_{W_0^{s_2,q}(\Omega)}|\{w_{k+1}>0\}|^{1-\frac{q}{q^*_{s_2}}} \tag{3.4}$$

where $C_2>0$ is a constant. Now, from (3.3) we have

$$V_k=\int_\Omega w_k^q\,dx \geq \int_{\{w_{k+1}>0\}} w_k^q\,dx \geq 2^{-(k+1)q}|\{w_{k+1}>0\}|.$$

Hence, we can write the inequality (3.4) as follows:

$$V_{k+1}\leq C^kV_k^{1+\alpha},\quad \text{for all } k\in\mathbb{N} \tag{3.5}$$

for a suitable constant $C>1$ and $\alpha=\dfrac{s_2q}{N}$. This implies that

$$V_k\leq \frac{\eta^k}{\rho^q},\quad \text{for all } n\in\mathbb{N} \tag{3.6}$$

where $\eta = C^{-\frac{1}{\alpha}}$ and $\rho=\max\left\{1,\|u_0\|^{-1}_{L^q(\Omega)},C^{\frac{1}{q\alpha^2}}\right\}$.

Indeed, by induction arguments, we have:

- Clearly $V_0=\left\|v_0^+\right\|^q_{L^q(\Omega)}\leq \|v_0\|^q_{L^q(\Omega)}=\dfrac{1}{\rho^q}$.
- Now we can assume that (3.6) holds for some $k\in\mathbb{N}$, and by use (3.5) we get

$$V_{k+1}\leq C^kV_k^{1+\alpha}\leq\frac{\eta^{k+1}}{\rho^q}.$$

Since $\eta\in(0,1)$, we deduce that

$$\lim_{k\to\infty}V_k=0 \tag{3.7}$$

Since w_k converges to $(v_0-1)^+$ a.e. in $\mathbb{R}^N$, from (3.7) we infer that $w_k \to 0$ a.e. in Ω. Hence $v_0 \le 1$ a.e. in Ω, which implies $\|u_0\|_{L^\infty(\Omega)} \le \rho \|u_0\|_{L^q(\Omega)}$. Then, we deduce that $u_0 \in L^\infty(\Omega)$. On the other hand, from Remark 1.2, we infer that $u_0 \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ and for any $\epsilon_0 > 0$ there exists a constant $K = K(\epsilon_0) > 0$ such that

$$K^{-1} d^{s_1+\epsilon_0} \le u_0 \le K d^{s_1-\epsilon_0} \text{ in } \Omega.$$

Step 3: Uniqueness of a weak solution.

We apply Lemma 3.6 (The Diaz and Saa inequality) to arrive at the uniqueness of the positive weak solution to the problem (**P-(i)**). For that, we consider $v \in \mathbf{W}$ a weak positive solution of (**P-(i)**). Now, let $\epsilon > 0$, $u_\epsilon = u_0 + \epsilon$, $v_\epsilon = v + \epsilon$ and set

$$\Phi = \frac{u_\epsilon^q - v_\epsilon^q}{u_\epsilon^{q-1}} \quad \text{and} \quad \Psi = \frac{v_\epsilon^q - u_\epsilon^q}{v_\epsilon^{q-1}}.$$

From the inequalities in Remark 3.7, it is easy to see that Φ and Ψ belong to $\mathbf{W}$. Then, from (3.1) we have:

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u_0(x)-u_0(y)\right|^{p-2}\left(u_0(x)-u_0(y)\right)\left(\Phi(x)-\Phi(y)\right)}{|x-y|^{N+s_1p}}dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u_0(x)-u_0(y)\right|^{q-2}\left(u_0(x)-u_0(y)\right)\left(\Phi(x)-\Phi(y)\right)}{|x-y|^{N+s_2q}}dxdy \\
&\quad = \lambda \int_\Omega a_q(x) u_0^{q-1} \Phi dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|v(x)-v(y)\right|^{p-2}\left(v(x)-v(y)\right)\left(\Psi(x)-\Psi(y)\right)}{|x-y|^{N+s_1p}}dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|v(x)-v(y)\right|^{q-2}\left(v(x)-v(y)\right)\left(\Psi(x)-\Psi(y)\right)}{|x-y|^{N+s_2q}}dxdy \\
&\quad = \lambda \int_\Omega a_q(x) v^{q-1} \Psi dx.
\end{aligned}$$

Then adding the above inequalities, we obtain:

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u_0(x)-u_0(y)\right|^{p-2}(u_0(x)-u_0(y))\left(\Phi(x)-\Phi(y)\right)}{|x-y|^{N+s_1p}}dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u_0(x)-u_0(y)\right|^{q-2}(u_0(x)-u_0(y))\left(\Phi(x)-\Phi(y)\right)}{|x-y|^{N+s_2q}}dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|v(x)-v(y)\right|^{p-2}(v(x)-v(y))\left(\Psi(x)-\Psi(y)\right)}{|x-y|^{N+s_1p}}dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|v(x)-v(y)\right|^{q-2}(v(x)-v(y))\left(\Psi(x)-\Psi(y)\right)}{|x-y|^{N+s_2q}}dxdy \\
&\quad = \lambda \int_\Omega a_q(x)\left(\frac{u_0^{q-1}}{u_\epsilon^{q-1}} - \frac{v^{q-1}}{v_\epsilon^{q-1}}\right)(u_\epsilon^q - v_\epsilon^q)dx. \qquad (3.8)
\end{aligned}$$

Now, we follow the proof of [23, Theorem 2.10] in order to pass to the limit in the right-hand side of (3.8). Precisely, we have

$$\left(\frac{u_\epsilon}{v_\epsilon}\right)^q \leq 2^{q-1}\left[\left(\frac{u_0}{v}\right)^q + 1\right] \in L^1(\Omega). \tag{3.9}$$

Indeed, from the Hölder inequality, the fractional Hardy inequality and boundary behaviour of u_0, v (see Remark 1.2), we obtain:

$$\int_\Omega \left(\frac{u_0}{v}\right)^q dx \leq C\int_\Omega \left(\frac{u_0}{d^{s_1+\epsilon_0}(x)}\right)^q dx \leq C\left(\int_\Omega \frac{1}{d^{\frac{pq}{p-q}\epsilon_0}(x)}\right)^{\frac{p-q}{p}} \|u_0\|_{\mathbf{W}}^q < \infty$$

for ϵ_0 small enough and $C = C(\epsilon_0) > 0$. Similarly, we have for ϵ_0 small enough

$$\left(\frac{v_\epsilon}{u_\epsilon}\right)^q \leq 2^{q-1}\left[\left(\frac{v}{u_0}\right)^q + 1\right] \in L^1(\Omega).$$

Now passing to the limit as $\epsilon \to 0$ in (3.8), using Fatou's lemma and the dominated convergence theorem, we obtain that

$$\int_\Omega \left(\frac{(-\Delta)_p^{s_1} u_0 + (-\Delta)_q^{s_2} u_0}{u_0^{q-1}} - \frac{(-\Delta)_p^{s_1} v + (-\Delta)_q^{s_2} v}{v^{q-1}}\right)(u^q - v^q)\,dx = 0.$$

Using Lemma 3.6, we infer that $u_0 = v$, since $1 < q < p$. □

***Proof of Theorem** 1.5.* First, we have

$$\underline{\lambda}_{s,s*,r,r^*}(a_r) = \inf_{u\in\mathbf{W}}\left\{\frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^r}{|x-y|^{N+sr}}dxdy + \frac{r}{r^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{r*}}{|x-y|^{N+s^*r^*}}dxdy}{\displaystyle\int_\Omega a_r(x)u^r\,dx}\right\}$$
$$\geq \lambda_{1,s,r}(a_r) > -\infty. \tag{3.10}$$

Hence $\underline{\lambda}_{s,s*,r,r^*}(a_r)$ exists.

Next, we will follow the same idea in [38, Proposition 4]. For that, let $t > 0$, $v = t\phi_{1,s,r}(a_r)$ and by the definition of $\underline{\lambda}_{s,s*,r,r^*}(a_r)$ and $\lambda_{1,s,r}(a_r)$, we have that

$$\underline{\lambda}_{s,s*,r,r^*}(a_r) \leq \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^r}{|x-y|^{N+sr}}dxdy + \frac{r}{r^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^{r*}}{|x-y|^{N+s^*r^*}}dxdy}{\displaystyle\int_\Omega a_r(x)v^r(x)\,dx}$$
$$= \lambda_{1,s,r}(a_r) + \frac{r\,t^{r^*-r}}{r^*}\frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\phi_{1,s,r}(a_r)(x)-\phi_{1,s,r}(a_r)(y)|^{r*}}{|x-y|^{N+s^*r^*}}dxdy}{\displaystyle\int_\Omega a_r(x)\phi_{1,s,r}(a_r)^r\,dx} \tag{3.11}$$

Now, we distinguish two cases:

Case 1: $r = p$ with $s = s_1$ and $r^* = q$ with $s^* = s_2$.

Since $1 < q < p$ and passing to the limit as $t \to +\infty$ in (3.11), we deduce that

$$\underline{\lambda}_{s_1,s_2,p,q}(a_p) \leq \lambda_{1,s_1,p}(a_p).$$

Case 2: $r = q$ with $s = s_2$ and $r^* = p$ with $s^* = s_1$.

Passing to the limit as $t \to 0$ in (3.11), and using again the fact that $1 < q < p$, we get

$$\underline{\lambda}_{s_2,s_1,q,p}(a_q) \leq \lambda_{1,s_2,q}(a_q).$$

Then, from (3.10), we can see that

$$\underline{\lambda}_{s,s^*,r,r^*}(a_r) = \lambda_{1,s,r}(a_r).$$

On the other hand, we suppose by contradiction that there exists a function $u_0 \in \mathbf{W}$, such that

$$\underline{\lambda}_{s,s*,r,r^*}(a_r) = \frac{\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^r}{|x-y|^{N+sr}}dxdy + \frac{r}{r^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^{r*}}{|x-y|^{N+s^*r^*}}dxdy}{\displaystyle\int_\Omega a_r(x)u_0^r\,dx}$$

$$\geq \lambda_{1,s,r}(a_r) + \frac{\displaystyle\frac{r}{r^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^{r*}}{|x-y|^{N+s^*r^*}}dxdy}{\displaystyle\int_\Omega a_r(x)u_0^r\,dx} > \lambda_{1,s,r}(a_r) = \underline{\lambda}_{s,s*,r,r^*}(a_r)$$

which is a contradiction. □

2 Non-local and non-homogeneous problem with singular

We introduce the notion of the weak solution to the problem (**P**) in **case (ii)**, i.e.

$$(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = u^{-\delta} + b(x,u), \quad u > 0 \quad \text{in } \Omega; \quad u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega \qquad (\textbf{P-(ii)})$$

as follows:

Definition 2.1. *We say that $u \in \mathbf{W} \cap L^\infty(\Omega)$ is a positive weak solution to the problem* (**P-(ii)**)*, if*

$$\operatorname*{ess\,inf}_K u > 0 \quad \textit{over every compact set } K \subset \Omega$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s_1p}}dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{q-2}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s_2q}}dxdy \\
&\quad = \int_\Omega u^{-\delta}\varphi dx + \int_\Omega b(x,u)\varphi dx
\end{aligned} \qquad (3.12)$$

for all $\varphi \in C_c^\infty(\Omega)$.

Our results about the existence and properties of solutions to the problem (**P-(ii)**) are as follows.

Theorem 2.2. *Let $0 < s_2 \leq s_1 < 1$ and $1 < \upsilon < q \leq p < \infty$. Assume that b satisfies **(H1)-(H2)**. Then, there exists a unique nontrivial weak solution u to the problem* (**P-(ii)**)*. In addition, $u \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,s_1)$ and for any $\sigma \in (0,s_1)$ and $\sigma' > s_1$, there exists a positive constant $c = c(\sigma,\sigma') > 0$, such that:*

$$c^{-1}d^{\sigma'} \leq u \leq c\,d^{\sigma} \quad in\,\Omega.$$

First, let $n \in \mathbb{N}^*$, we consider the following auxiliary problem:

$$\begin{gathered} (-\Delta)_p^{s_1} u_n + (-\Delta)_q^{s_2} u_n = (u_n + 1/n)^{-\delta} + b(x, u_n), \\ u_n > 0 \quad \text{in } \Omega; \quad u_n = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{gathered} \tag{$\mathbf{P}_n$}$$

We have the following notion of weak solutions.

Definition 2.3. *We say that $u_n \in \boldsymbol{W}$ is a weak solution of the problem ($\mathbf{P}_n$), if $u_n > 0$ a.e. in Ω and:*

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}\big(u_n(x)-u_n(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s_1 p}} dxdy \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{q-2}\big(u_n(x)-u_n(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s_2 q}} dxdy \\ &\quad = \int_\Omega \left(u_n + \frac{1}{n}\right)^{-\delta} \varphi dx + \int_\Omega b(x,u_n)\varphi dx \end{aligned} \tag{3.13}$$

for all $\varphi \in \boldsymbol{W}$.

Now, we can prove the following results.

Proposition 2.4. *For any $n \in \mathbb{N}^*$, there exists a unique weak solution $u_n \in \boldsymbol{W} \cap C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$ of the problem ($\mathbf{P}_n$). Moreover, the sequence $(u_n)_n$ satisfies:*

$$u_n \le u_{n+1}, \quad \textit{for any } x \in \Omega$$

and for every $\omega \Subset \Omega$ there exists $\sigma = \sigma(\omega) > 0$ (independent on n) such that: for any $n \in \mathbb{N}^*$

$$u_n(x) \ge \sigma \quad \textit{for any } x \in \omega. \tag{3.14}$$

Proof. Define the energy functional $\mathcal{L} : \mathbf{W} \to \mathbb{R}$ by:

$$\begin{aligned} \mathcal{L}(u_n) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+s_1 p}} dxdy + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+s_2 q}} dxdy \\ - \frac{1}{1-\delta}\int_\Omega \left(u_n + \frac{1}{n}\right)^{1-\delta} dx - \int_\Omega B(x,u_n)dx \end{aligned}$$

where

$$B(x,t) = \begin{cases} \displaystyle\int_0^t b(x,s)ds & \text{if } 0 \le t < +\infty, \\ 0 & \text{if } -\infty < t < 0. \end{cases}$$

We extend accordingly the domain of b to all of $\Omega \times \mathbb{R}$ by setting

$$b(x,t) = \frac{\partial B}{\partial t}(x,t) = 0 \quad \text{for } (x,t) \in \Omega \times (-\infty, 0).$$

We note that, $\mathcal{L}$ is well-defined on $\mathbf{W}$. On the other hand, we have that $\mathbf{W}$ is continuously embedded in $W_0^{s_1,p}(\Omega)$, $W_0^{s_2,q}(\Omega)$ and compactly embedded in $L^\upsilon(\Omega)$ (see **(H2)**) together with **(H1)–(H2)**, it is easy to show that $\mathcal{L}$ is weakly lower semi-continuous and coercive on $\mathbf{W}$.

Then, $\mathcal{L}$ admits a global minimizer in $\mathbf{W}$, denoted by u_n. One has $u_n \not\equiv 0$. Indeed, let us consider $\phi \in C_c^1(\Omega)$ non-negative and nontrivial function with compact support in Ω. Then, we obtain

$$\mathscr{L}(t\phi) \leq t^{1-\delta}\left[c_1 t^{p-1+\delta} + c_2 t^{q-1+\delta} - c_3\right], \quad \text{for all } t > 0,$$

where c_1, c_2 and c_3 are independent of t and $c_3 > 0$. Hence, for $t > 0$ small enough, we deduce that $\mathscr{L}(t\phi) < 0 = \mathscr{L}(0)$. Since u_n is a global minimiser for $\mathscr{L}$ and $\mathscr{L}(u_n) \geq \mathscr{L}(u_n^+)$, we deduce that $u_n \geq 0$ and $u_n \not\equiv 0$ a.e. in Ω. From the Gateaux differentiability of $\mathscr{L}$, we have u_n satisfies (3.13) i.e., u_n is a weak solution to ($\mathbf{P}_n$). Moreover, we can see that $u_n \in L^\infty(\Omega)$, for any $n \geq 1$. Indeed, for $n \in \mathbb{N}^*$ be fixed, we define $w_k(x) := (u_n(x) - (1-2^{-k}))^+$ for $k \in \mathbb{N}$, and from **(H1)-(H2)**, we have

$$\begin{aligned}
&\|w_{k+1}\|^q_{W_0^{s_2,q}(\Omega)} \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_{k+1}(x) - w_{k+1}(y)|^q}{|x-y|^{N+s_2 q}}\,dx\,dy \\
&\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_{k+1}(x) - w_{k+1}(y)|^p}{|x-y|^{N+s_1 p}}\,dx\,dy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_{k+1}(x) - w_{k+1}(y)|^q}{|x-y|^{N+s_2 q}}\,dx\,dy \\
&\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p-2}(w_{k+1}(x) - w_{k+1}(y))(u_n(x) - u_n(y))}{|x-y|^{N+s_1 p}}\,dx\,dy \\
&+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{q-2}(w_{k+1}(x) - w_{k+1}(y))(u_n(x) - u_n(y))}{|x-y|^{N+s_2 q}}\,dx\,dy \\
&\leq C(n)\left[\int_{\{w_{k+1}>0\}} w_{k+1}\,dx + \int_{\{w_{k+1}>0\}} u_n^{\upsilon-1} w_{k+1}\,dx\right] \\
&\leq C(n)\left[|w_{k+1} > 0|^{1-\frac{1}{q}}\|w_{k+1}\|_{L^q(\Omega)} + (2^{k+1}+1)^{\upsilon-1}|w_{k+1} > 0|^{1-\frac{\upsilon}{q}}\|w_{k+1}\|^{\upsilon}_{L^q(\Omega)}\right].
\end{aligned}$$

Then, the rest of the proof follows exactly on the same lines of Theorem 1.4 (**Step 2** in **Case 1**). Furthermore, from Remark 1.2, we deduce $u_n \in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0, s_1)$, and $u_n > 0$ in Ω. In particular, by the strong maximum principle, for every $\omega \Subset \Omega$ there exists $\sigma = \sigma(\omega)$, such that

$$u_1(x) \geq \sigma > 0 \quad \text{for any } x \in \omega \Subset \Omega. \tag{3.15}$$

• Now, let $n \in \mathbb{N}^*$, $0 < \epsilon < 1$, $u_{n,\epsilon} = u_n + \epsilon$ and $u_{n+1,\epsilon} = u_{n+1} + \epsilon$. Choosing the test functions:

$$\Phi = \frac{(u^{\upsilon}_{n,\epsilon} - u^{\upsilon}_{n+1,\epsilon})^+}{u^{\upsilon-1}_{n,\epsilon}}, \Psi = \frac{(u^{\upsilon}_{n+1,\epsilon} - u^{\upsilon}_{n,\epsilon})^-}{u^{\upsilon-1}_{n+1,\epsilon}} \in \mathbf{W}$$

in (3.13), we deduce that:

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u_n(x) - u_n(y)\right|^{p-2}(u_n(x) - u_n(y))\left(\Phi(x) - \Phi(y)\right)}{|x-y|^{N+s_1 p}}\,dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u_n(x) - u_n(y)\right|^{q-2}(u_n(x) - u_n(y))\left(\Phi(x) - \Phi(y)\right)}{|x-y|^{N+s_2 q}}\,dxdy \\
&\quad\quad = \int_\Omega \left(u_n + \frac{1}{n}\right)^{-\delta}\Phi\,dx + \int_\Omega b(x,u_n)\Phi\,dx
\end{aligned}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u_{n+1}(x)-u_{n+1}(y)\right|^{p-2}(u_{n+1}(x)-u_{n+1}(y))\left(\Psi(x)-\Psi(y)\right)}{|x-y|^{N+s_1p}}dxdy$$
$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u_{n+1}(x)-u_{n+1}(y)\right|^{q-2}(u_{n+1}(x)-u_{n+1}(y))\left(\Psi(x)-\Psi(y)\right)}{|x-y|^{N+s_2q}}dxdy$$
$$=\int_\Omega\left(u_{n+1}+\frac{1}{n+1}\right)^{-\delta}\Psi dx+\int_\Omega b(x,u_{n+1})\Psi dx.$$

Subtracting the above inequalities, we have

$$\int_{\{u_n>u_{n+1}\}}\int_{\{u_n>u_{n+1}\}}\frac{\left|u_n(x)-u_n(y)\right|^{p-2}\left(u_n(x)-u_n(y)\right)}{\left|x-y\right|^{N+s_1p}}$$
$$\times\left[\frac{u_{n,\epsilon}(x)^\upsilon-u_{n+1,\epsilon}(x)^\upsilon}{u_{n,\epsilon}(x)^{\upsilon-1}}-\frac{u_{n,\epsilon}(y)^\upsilon-u_{n+1,\epsilon}(y)^\upsilon}{u_{n,\epsilon}(y)^{\upsilon-1}}\right]dxdy$$
$$+\int_{\{u_n>u_{n+1}\}}\int_{\{u_n>u_{n+1}\}}\frac{\left|u_n(x)-u_n(y)\right|^{q-2}\left(u_n(x)-u_n(y)\right)}{\left|x-y\right|^{N+s_2q}}$$
$$\times\left[\frac{u_{n,\epsilon}(x)^\upsilon-u_{n+1,\epsilon}(x)^\upsilon}{u_{n,\epsilon}(x)^{\upsilon-1}}-\frac{u_{n,\epsilon}(y)^\upsilon-u_{n+1,\epsilon}(y)^\upsilon}{u_{n,\epsilon}(y)^{\upsilon-1}}\right]dxdy$$
$$+\int_{\{u_n>u_{n+1}\}}\int_{\{u_n>u_{n+1}\}}\frac{\left|u_{n+1}(x)-u_{n+1}(y)\right|^{p-2}\left(u_{n+1}(x)-u_{n+1}(y)\right)}{\left|x-y\right|^{N+s_1p}}$$
$$\times\left[\frac{u_{n+1,\epsilon}(x)^\upsilon-u_{n,\epsilon}(x)^\upsilon}{u_{n+1,\epsilon}(x)^{\upsilon-1}}-\frac{u_{n+1,\epsilon}(y)^\upsilon-u_{n,\epsilon}(y)^\upsilon}{u_{n+1,\epsilon}(y)^{\upsilon-1}}\right]dxdy$$
$$+\int_{\{u_n>u_{n+1}\}}\int_{\{u_n>u_{n+1}\}}\frac{\left|u_{n+1}(x)-u_{n+1}(y)\right|^{q-2}\left(u_{n+1}(x)-u_{n+1}(y)\right)}{\left|x-y\right|^{N+s_2q}}$$
$$\times\left[\frac{u_{n+1,\epsilon}(x)^\upsilon-u_{n,\epsilon}(x)^\upsilon}{u_{n+1,\epsilon}(x)^{\upsilon-1}}-\frac{u_{n+1,\epsilon}(y)^\upsilon-u_{n,\epsilon}(y)^\upsilon}{u_{n+1,\epsilon}(y)^{\upsilon-1}}\right]dxdy$$
$$=\int_{\{u_n>u_{n+1}\}}\left(\frac{\left(u_n+\frac{1}{n}\right)^{-\delta}}{u_{n,\epsilon}^{\upsilon-1}}-\frac{\left(u_{n+1}+\frac{1}{n+1}\right)^{-\delta}}{u_{n+1,\epsilon}^{\upsilon-1}}\right)\left(u_{n,\epsilon}^\upsilon-u_{n+1,\epsilon}^\upsilon\right)dx$$
$$+\int_{\{u_n>u_{n+1}\}}\left(\frac{b(x,u_n)}{u_{n,\epsilon}^{\upsilon-1}}-\frac{b(x,u_{n+1})}{u_{n+1,\epsilon}^{\upsilon-1}}\right)\left(u_{n,\epsilon}^\upsilon-u_{n+1,\epsilon}^\upsilon\right)dx. \tag{3.16}$$

In order to pass to the limit in (3.16), we need to:

First,

$$\left|\left(\frac{\left(u_n+\frac{1}{n}\right)^{-\delta}}{u_{n,\epsilon}^{\upsilon-1}}-\frac{\left(u_{n+1}+\frac{1}{n+1}\right)^{-\delta}}{u_{n+1,\epsilon}^{\upsilon-1}}\right)\left(u_{n,\epsilon}^\upsilon-u_{n+1,\epsilon}^\upsilon\right)\right|\leq n^\delta(u_n+1)^\upsilon$$

Then, by the dominated convergence Theorem, we have

$$\lim_{\epsilon\to 0}\int_{\{u_n>u_{n+1}\}}\left(\frac{\left(u_n+\frac{1}{n}\right)^{-\delta}}{u_{n,\epsilon}^{\upsilon-1}}-\frac{\left(u_{n+1}+\frac{1}{n+1}\right)^{-\delta}}{u_{n+1,\epsilon}^{\upsilon-1}}\right)(u_{n,\epsilon}^{\upsilon}-u_{n+1,\epsilon}^{\upsilon})\,dx$$

$$=\int_{\{u_n>u_{n+1}\}}\left(\frac{\left(u_n+\frac{1}{n}\right)^{-\delta}}{u_{n}^{\upsilon-1}}-\frac{\left(u_{n+1}+\frac{1}{n+1}\right)^{-\delta}}{u_{n+1}^{\upsilon-1}}\right)(u_{n}^{\upsilon}-u_{n+1}^{\upsilon})\,dx\le 0. \qquad (3.17)$$

Second, by Fatou's Lemma and **(H1)**, we have

$$-\liminf_{\epsilon\to 0}\int_{\{u_n>u_{n+1}\}}\frac{b(x,u_n)}{u_{n,\epsilon}^{\upsilon-1}}u_{n+1,\epsilon}^{\upsilon}\,dx\le-\int_{\{u_n>u_{n+1}\}}\frac{b(x,u_n)}{u_{n}^{\upsilon-1}}u_{n+1}^{\upsilon}\,dx, \qquad (3.18)$$

$$-\liminf_{\epsilon\to 0}\int_{\{u_n>u_{n+1}\}}\frac{b(x,u_{n+1})}{u_{n+1,\epsilon}^{\upsilon-1}}u_{n,\epsilon}^{\upsilon}\,dx\le-\int_{\{u_n>u_{n+1}\}}\frac{b(x,u_{n+1})}{u_{n+1}^{\upsilon-1}}u_{n}^{\upsilon}\,dx, \qquad (3.19)$$

$$\lim_{\epsilon\to 0}\int_{\{u_n>u_{n+1}\}}b(x,u_n)u_{n,\epsilon}\,dx=\int_{\{u_n>u_{n+1}\}}b(x,u_n)u_n\,dx, \qquad (3.20)$$

$$\lim_{\epsilon\to 0}\int_{\{u_n>u_{n+1}\}}b(x,u_{n+1})u_{n+1,\epsilon}\,dx=\int_{\{u_n>u_{n+1}\}}b(x,u_{n+1})u_{n+1}\,dx. \qquad (3.21)$$

So, gathering (3.17)-(3.21), passing to the limit sup in (3.16) as $\epsilon\to 0$ and using **(H2)** and Lemma 3.5 (see remark 3.7), we obtain

$$\begin{aligned}
&\int_{\{u_n>u_{n+1}\}}\int_{\{u_n>u_{n+1}\}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{N+s_1p}}\\
&\qquad\times\left[\frac{u_n(x)^{\upsilon}-u_{n+1}(x)^{\upsilon}}{u_n(x)^{\upsilon-1}}-\frac{u_n(y)^{\upsilon}-u_{n+1}(y)^{\upsilon}}{u_n(y)^{\upsilon-1}}\right]dxdy\\
&+\int_{\{u_n>u_{n+1}\}}\int_{\{u_n>u_{n+1}\}}\frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))}{|x-y|^{N+s_2q}}\\
&\qquad\times\left[\frac{u_n(x)^{\upsilon}-u_{n+1}(x)^{\upsilon}}{u_n(x)^{\upsilon-1}}-\frac{u_n(y)^{\upsilon}-u_{n+1}(y)^{\upsilon}}{u_n(y)^{\upsilon-1}}\right]dxdy\\
&+\int_{\{u_n>u_{n+1}\}}\int_{\{u_n>u_{n+1}\}}\frac{|u_{n+1}(x)-u_{n+1}(y)|^{p-2}(u_{n+1}(x)-u_{n+1}(y))}{|x-y|^{N+s_1p}}\\
&\qquad\times\left[\frac{u_{n+1}(x)^{\upsilon}-u_n(x)^{\upsilon}}{u_{n+1}(x)^{\upsilon-1}}-\frac{u_{n+1}(y)^{\upsilon}-u_n(y)^{\upsilon}}{u_{n+1}(y)^{\upsilon-1}}\right]dxdy\\
&+\int_{\{u_n>u_{n+1}\}}\int_{\{u_n>u_{n+1}\}}\frac{|u_{n+1}(x)-u_{n+1}(y)|^{q-2}(u_{n+1}(x)-u_{n+1}(y))}{|x-y|^{N+s_2q}}\\
&\qquad\times\left[\frac{u_{n+1}(x)^{\upsilon}-u_n(x)^{\upsilon}}{u_{n+1}(x)^{\upsilon-1}}-\frac{u_{n+1,\epsilon}(y)^{\upsilon}-u_n(y)^{\upsilon}}{u_{n+1}(y)^{\upsilon-1}}\right]dxdy=0.
\end{aligned}$$

Hence $(u_n(x)-u_{n+1}(x))^+=0$ for any $x\in\mathbb{R}^N$ which implies that $u_n\le u_{n+1}$ for any $x\in\Omega$. This fact combined with (3.15) implies that (3.14) holds true. □

Remark 2.5. *If u_n and v_n are two solutions of the problem* ($\mathbf{P}_n$). *Choose*

$$\Phi = \frac{((u_n+\epsilon)^{\upsilon} - (v_n+\epsilon)^{\upsilon})^+}{(u_n+\epsilon)^{\upsilon-1}} \quad \textit{and} \quad \Psi = \frac{((v_n+\epsilon)^{\upsilon} - (u_n+\epsilon)^{\upsilon})^-}{(v_n+\epsilon)^{\upsilon-1}}$$

as a test function in (3.13)*, and using again the same argument of the monotonicity of $(u_n)_n$ in the proof of Proposition 2.4, it is easy to prove that the weak solution to* ($\mathbf{P}_n$) *is unique.*

Proof of Theorem 2.2. We divided the proof into 3 steps.

Step 1: The sequence $(u_n)_n$ found in Proposition 2.4 is bounded in W. Indeed , we take u_n as test function in (3.13), using **(H1)-(H2)**, Hölder inequality and Sobolev embeddings, we obtain

$$\begin{aligned}
\|u_n\|^p_{W_0^{s_1,p}(\Omega)} + \|u_n\|^q_{W_0^{s_2,q}(\Omega)} &= \int_\Omega \frac{u_n}{\left(u_n+\frac{1}{n}\right)^\delta}dx + \int_\Omega b(x,u_n)u_n dx \\
&\le \int_\Omega u_n^{1-\delta}dx + C\left[\int_\Omega (u_n+u_n^{\upsilon})dx\right] \\
&\le C\left[\|u_n\|^{1-\delta}_{W_0^{s_2,q}(\Omega)} + \|u_n\|_{W_0^{s_2,q}(\Omega)} + \|u_n\|^{\upsilon}_{W_0^{s_2,q}(\Omega)}\right]
\end{aligned}$$

where $C>0$ is a constant.

Assume by contradiction that $\|u_n\|_{\mathbf{W}} \to \infty$. Then we have the following Alternatives:

Alternative 1: $\|u_n\|_{W_0^{s_1,p}(\Omega)} \to +\infty$ and $\|u_n\|_{W_0^{s_2,q}(\Omega)} \to +\infty$.

Then, for n large enough, we have $\|u_n\|_{W_0^{s_1,p}(\Omega)} > 1$ which implies that

$$C_q \le \left[\|u_n\|_{\mathbf{W}}^{1-\delta-q} + \|u_n\|_{\mathbf{W}}^{1-q} + \|u_n\|_{\mathbf{W}}^{\upsilon-q}\right]$$

Since $1<\upsilon<q$ and passing to the limit as $n\to\infty$, we get a contradiction.

Alternative 2: $\|u_n\|_{W_0^{s_1,p}(\Omega)} \to +\infty$ and $\|u_n\|_{W_0^{s_2,q}(\Omega)}$ is bounded.

Symmetrically to Alternative 1.

Alternative 3: $\|u_n\|_{W_0^{s_1,p}(\Omega)}$ is bounded and $\|u_n\|_{W_0^{s_2,q}(\Omega)} \to +\infty$.

Therefore,

$$\begin{aligned}
\|u_n\|^q_{W_0^{s_2,q}(\Omega)} &\le \|u_n\|^p_{W_0^{s_1,p}(\Omega)} + \|u_n\|^q_{W_0^{s_2,q}(\Omega)} \\
&\le C\left[\|u_n\|^{1-\delta}_{W_0^{s_2,q}(\Omega)} + \|u_n\|_{W_0^{s_2,q}(\Omega)} + \|u_n\|^{\upsilon}_{W_0^{s_2,q}(\Omega)}\right]
\end{aligned}$$

and thus

$$1 \le C\left[\|u_n\|^{1-\delta-q}_{W_0^{s_2,q}(\Omega)} + \|u_n\|^{1-q}_{W_0^{s_2,q}(\Omega)} + \|u_n\|^{\upsilon-q}_{W_0^{s_2,q}(\Omega)}\right]$$

$n\to\infty$ leads to a contradiction.

Hence, we deduce that $(u_n)_n$ is uniformly bounded in **W**. Then, we have two points:

- u_n weakly converges to u in **W**. Consequently,

$$\left\{\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{\frac{N+s_1p}{p'}}}\right\} \quad \text{is bounded in } L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)$$

and

$$\left\{ \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))}{|x-y|^{\frac{N+s_2 q}{q'}}} \right\} \quad \text{is bounded in } L^{q'}(\mathbb{R}^N \times \mathbb{R}^N)$$

where $p' = \dfrac{p}{p-1}$ and $q' = \dfrac{q}{q-1}$.

• $u_n \to u$ in $L^t(\Omega)$ for $1 \le t < p^*_{s_1}$ and $u_n \to u$ a.e. in Ω , we obtain

$$\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{\frac{N+s_1 p}{p'}}} \to \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{\frac{N+s_1 p}{p'}}}$$

and

$$\frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))}{|x-y|^{\frac{N+s_2 q}{q'}}} \to \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))}{|x-y|^{\frac{N+s_2 q}{q'}}}$$

a.e. in $\mathbb{R}^N \times \mathbb{R}^N$. It follows that

$$\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{\frac{N+s_1 p}{p'}}} \rightharpoonup \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{\frac{N+s_1 p}{p'}}}$$

and

$$\frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))}{|x-y|^{\frac{N+s_2 q}{q'}}} \rightharpoonup \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))}{|x-y|^{\frac{N+s_2 q}{q'}}}$$

weakly in $L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ and $L^{q'}(\mathbb{R}^N \times \mathbb{R}^N)$, respectively. Then, since $\varphi \in C_c^\infty(\Omega)$, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \Bigg\{ & \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1 p}}\,dx\,dy \\
& + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2 q}}\,dx\,dy \Bigg\} \\
& = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_1 p}}\,dx\,dy \\
& \quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+s_2 q}}\,dx\,dy.
\end{aligned}$$

On the other hand, from Proposition 2.4 for any $\varphi \in C_c^\infty(\Omega)$ with $\operatorname{supp}\varphi = \omega$ there exists $\sigma = \sigma(\omega) > 0$ (independent on n) such that: for any $n \in \mathbb{N}^*$

$$\frac{\varphi}{\left(u_n + \frac{1}{n}\right)^{\delta}} \le \sigma^{-\delta}\varphi \in L^1(\Omega)$$

and by **(H1)-(H2)**, we infer that $b(x,u_n)$ is bounded in $L^{\frac{q}{\upsilon-1}}(\Omega)$ and $b(x,u_n) \to b(x,u)$ a.e. in Ω. Then, by the dominated convergence Theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega \left(u_n + \frac{1}{n}\right)^{-\delta} \varphi\,dx = \int_\Omega u^{-\delta}\varphi\,dx$$

and

$$\lim_{n\to\infty} \int_\Omega b(x,u_n)\varphi\,dx = \int_\Omega b(x,u)\varphi\,dx.$$

By passing to the limit in (3.13),we conclude that u satisfies (**P-(ii)**), i.e.

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s_1p}}dxdy\\&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{q-2}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+s_2q}}dxdy\\&\quad=\int_\Omega u^{-\delta}\varphi dx+\int_\Omega b(x,u)\varphi dx\end{aligned}\tag{3.22}$$

for any $\varphi\in C^\infty(\Omega)$.

Step 2: Regularity and positivity of weak solutions.

We follow approach of the proof [24, Corollary 2.9] to prove $u\in C^{0,\alpha}(\overline{\Omega})$, for some $\alpha\in(0,s_1)$. Therefore, (3.14) which implies that for all $\omega\Subset\Omega$ there exists $\sigma=\sigma(\omega)>0$ such that

$$u\geq\sigma>0\quad\text{for any } x\in\omega.\tag{3.23}$$

Moreover, from **(H1)**-**(H2)**, we have:

$$\begin{aligned}(-\Delta)_p^{s_1}u+(-\Delta)_q^{s_2}u&=u^{-\delta}+b(x,u)\\&\leq\frac{C\left(1+\|u\|^{\delta}_{L^\infty(\Omega)}+\|u\|^{\upsilon+\delta-1}_{L^\infty(\Omega)}\right)}{u^\delta}\end{aligned}$$

for $C>0$ is a constant. On the other hand, from **(H1)**, we have

$$\frac{1}{u^\delta}\leq(-\Delta)_p^{s_1}u+(-\Delta)_q^{s_2}u.$$

Then, by the weak comparison principle [24, Proposition 4.3], for any $\epsilon_0>0$ there exists a constant $K=K(\epsilon_0)>0$ such that

$$K^{-1}d^{s_1+\epsilon_0}\leq u\leq Kd^{s_1-\epsilon_0}\text{ in }\Omega.$$

This fact combined with the fractional Hardy inequality , we obtain

$$\begin{aligned}\int_\Omega u^{-\delta}\varphi dx&\leq C(\epsilon)\int_\Omega\frac{\varphi}{d^{(s_1+\epsilon_0)\delta}}dx\\&\leq C(\epsilon)\left(\int_\Omega\frac{1}{d^{\frac{p((s_1+\epsilon)\delta-s_1)}{p-1}}(x)}\right)^{\frac{p-1}{p}}\|\varphi\|_{\mathbf{W}}<\infty\end{aligned}$$

since $0<\delta<1$ for ϵ small enough and $C(\epsilon)>0$ is a constant. By density arguments, we conclude that (3.22) is satisfied for any $\varphi\in\mathbf{W}$.

Step 3: Uniqueness of a weak solution.

Let u_1 and u_2 be two weak solutions of (**P-(ii)**). Let $\epsilon\in(0,1)$, we choose

$$\Phi=\frac{(u_1+\epsilon)^\upsilon-(u_2+\epsilon)^\upsilon}{(u_1+\epsilon)^{\upsilon-1}}\quad\text{and}\quad\Psi=\frac{(u_2+\epsilon)^\upsilon-(u_1+\epsilon)^\upsilon}{(u_2+\epsilon)^{\upsilon-1}}$$

as test functions in (3.12). Then adding the equations, we deduce

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_1(x)-u_1(y)|^{p-2}(u_1(x)-u_1(y))\left(\Phi(x)-\Phi(y)\right)}{|x-y|^{N+s_1p}}dxdy$$
$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_1(x)-u_1(y)|^{q-2}(u_1(x)-u_1(y))\left(\Phi(x)-\Phi(y)\right)}{|x-y|^{N+s_2q}}dxdy$$
$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_2(x)-u_2(y)|^{p-2}(u_2(x)-u_2(y))\left(\Psi(x)-\Psi(y)\right)}{|x-y|^{N+s_1p}}dxdy$$
$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_2(x)-u_2(y)|^{q-2}(u_2(x)-u_2(y))\left(\Psi(x)-\Psi(y)\right)}{|x-y|^{N+s_2q}}dxdy$$
$$=\int_\Omega \left[\frac{u_1^{-\delta}}{(u_1+\epsilon)^{\upsilon-1}}-\frac{u_2^{-\delta}}{(u_2+\epsilon)^{\upsilon-1}}\right]((u_1+\epsilon)^\upsilon-(u_2+\epsilon)^\upsilon)dx$$
$$+\int_\Omega\left(\frac{b(x,u_1)}{(u_1+\epsilon)^{\upsilon-1}}-\frac{b(x,u_2)}{(u_2+\epsilon)^{\upsilon-1}}\right)((u_1+\epsilon)^\upsilon-(u_2+\epsilon)^\upsilon)dx. \tag{3.24}$$

On the other hand, according to boundary behaviour of u_1 and u_2, we have

$$\frac{1}{u_1^\delta}\left(\frac{u_2+\epsilon}{u_1+\epsilon}\right)^\upsilon \leq 2^{\upsilon-1}\left[\frac{1}{u_1^\delta}\left(\frac{u_2}{u_1}\right)^\upsilon+\frac{1}{u_1^\delta}\right]\in L^1(\Omega). \tag{3.25}$$

Indeed, since $N > s_1\upsilon$ and $\upsilon > \delta$, we have

$$\int_\Omega \frac{1}{u_1^\delta}dx \leq C(\epsilon)\int_\Omega \frac{1}{d^{\delta(s_1+\epsilon_0)}(x)}dx<\infty$$

and

$$\int_\Omega \frac{1}{u_1^\delta}\left(\frac{u_2}{u_1}\right)^\upsilon dx \leq C(\epsilon)\int_\Omega \frac{1}{d(x)^{\delta(s_1+\epsilon_0)}}\left(\frac{u_2}{d^{s_1+\epsilon_0}(x)}\right)^\upsilon dx\leq C(\epsilon)\int_\Omega \frac{u_2^{\upsilon-\delta}}{d^{(2\delta+\upsilon)\epsilon_0+s_1\upsilon}(x)}dx<\infty$$

for ϵ_0 small enough and $C(\epsilon_0) > 0$. Similarly, we have for ϵ_0 small enough

$$\frac{1}{u_2^\delta}\left(\frac{u_1+\epsilon}{u_2+\epsilon}\right)^\upsilon \leq 2^{\upsilon-1}\left[\frac{1}{u_2^\delta}\left(\frac{u_1}{u_2}\right)^\upsilon+\frac{1}{u_2^\delta}\right]\in L^1(\Omega). \tag{3.26}$$

Finally, gathering (3.9), (3.25), (3.26), using $u_1, u_2 \in L^\infty(\Omega)$ and $b(x,u_1), b(x,u_2) \in L^\infty(\Omega)$ which combined with Fatou's lemma, the dominated convergence theorem, **(H2)** and passing to the limit in (3.24) as $\epsilon \to 0$ yield

$$\int_\Omega\left(\frac{(-\Delta)_p^{s_1}u_1+(-\Delta)_q^{s_2}u_1}{u_1^{\upsilon-1}}-\frac{(-\Delta)_p^{s_1}u_2+(-\Delta)_q^{s_2}u_2}{u_2^{\upsilon-1}}\right)(u_2^\upsilon-u_2^\upsilon)dx=0.$$

From Lemma 3.6, we infer that $u_1 = u_2$ since $1 < \upsilon < q \leq p$. □

Conclusion

he main goals of the present work are to discuss non-existence, existence, uniqueness, and Hölder regularity results, in two problem. More precisely, by using the variational methods, we show the existence of a positive weak solution. On the other hand, the results obtained in [24] and [25] combined with the boundedness of weak solution provide the regularity results. This fact together with a new version of the Diaz-Saa inequality yields the uniqueness of weak solutions to the problem.

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