

People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
University of Oum El Bouaghi
Faculty of: Exact Sciences and Sciences of Nature and Life



Thesis

Presented to obtain

3rd Cycle Doctorate

Branch: Mathematics

Specialty: Applied Mathematics

Title :

On solvability and dynamic of partial differential and difference equations with integer and fractional order

Presented by :

CHEBANA Zainouba

Publicly defended on : 11/02/2025

In front of the following committee members :

N°	First and last name	Grade	University	Quality
01	Nasserdine Kechkar	Prof.	University of OEB	President
02	Taki-eddine Oussaeif	Prof.	University of OEB	Supervisor
03	Adel Ouannas	Prof.	University of OEB	Co-reporter
04	Sofiane Dehilis	MCA	University of OEB	Examiner
05	Imad Rezzoug	Prof.	University of OEB	Examiner
06	Khaled Saoudi	Prof.	University of Khenchla	Examiner
07	Abdelfatteh Bouziani	Prof.	University of OEB	Invited
08	Abdelhamid Ayadi	Prof.	University of OEB	Invited

Dedications

I extend my deepest gratitude to my parents, whose unwavering support and love have been my guiding light in life.

To my brothers, Fouaz, Ammar, and Abdourahmane, for always standing by me and supporting me throughout this journey.

To my sisters, Assia and Besma, for their constant encouragement and love.

To our beloved children, Meriem, Abderrahim, Layla, Salah Eddine, Nedjma, Wail, and Assad, who bring joy and meaning to every day.

A heartfelt thank you to all of my professors, especially Dr. Kachkar Nasser Eddine, Dr. Taki-eddine Oussaef, Dr. Ouanas Adel, Dr. Rezzoug Imad, Dr. Dehilis Soufiane, Dr. Ben Brahim Abdul Waheb, and Dr. Makhlouf Sadiq, Dr. Saadi Mohammed for their invaluable guidance and support.

Finally, to the pure soul of Ameline and my grandmother, whom I wish could have been here to share in this significant moment of my life. May God bless their souls and grant them a place in Paradise.

Thank you!

Chebana & Fainouba ★

Acknowledgments

First and foremost, profound gratitude is extended to Almighty Allah for bestowing upon me the fortitude and perseverance to complete this intellectual endeavor. His divine guidance has been the compass throughout this scholarly journey.

Sincere Appreciation is humbly offered to my esteemed supervisor, Professor Takieddine Oussaeif, and co-supervisor, Mr. Adel Ounass, whose erudite counsel and humane approach have profoundly shaped this work. The privilege of their mentorship remains an enduring professional honor.

Respectful Thanks are tendered to Professor Kachkar Nassereddine for graciously accepting to chair my defense committee. His scholarly presence elevates this academic exercise.

To the distinguished members of the examination committee - Professor Abdelfattah Bouziani, Professor Dehilis Sofiane, Professor Rezzoug Imad, and Professor Saoudi Khalel - I express Grateful Recognition for their meticulous evaluation and insightful critique of this research.

Special Commendation is reserved for Professor Benbrahim Abdelwaheb, whose unwavering support and pedagogical encouragement transformed challenges into opportunities for intellectual growth.

The medical fraternity receives Heartfelt Appreciation, particularly Dr. Saadi and Dr. Dehilis Sofiane, whose altruistic support and clinical insights enriched this multidisciplinary endeavor.

To my fellow researchers and doctoral candidates - your camaraderie and intellectual discourse have been both beacon and bastion throughout this odyssey.

Ultimate Gratitude is reserved for my family pillars, Chebana & Zainouba, whose silent sacrifices and emotional sustenance formed the bedrock of this achievement.

Chebana & Zainouba ★

Abstract

This thesis delves into the enigmatic mathematical realms of solutions associated with nonlinear partial differential equations (PDEs), probing their existence, uniqueness, and explosive behaviors under nonlocal integral constraints. Through a methodological tapestry weaving rigorous theoretical analysis with practical insights, the study embarks on an intellectual journey across layered complexities—from foundational analytical frameworks to unbounded temporal phenomena.

The introductory chapter constructs a robust analytical scaffold via a synthesis of advanced function spaces and cornerstone theorems, bolstered by innovative techniques such as Faedo-Galerkin, energy methodologies, to establish existence under critical conditions. This chapter lays a methodological cornerstone for subsequent explorations of solutions and their behavioral unraveling.

The second chapter immerses into the interplay of nonlinear parabolic equation with a generalized nonlinear integral condition of second type, unveiling mathematical thresholds that demarcate stable solutions from finite-time blow-ups. Qualitative insights emerge on how behaviors morph under initial condition and parameter perturbations.

Chapter 3 transitions to the singular and degenerate quasi-linear equations, dissecting latent connections between linear and nonlinear architectures. Here, precise conditions governing solution emergence and uniqueness are illuminated, supported by numerical analyses that juxtapose theoretical predictions against empirical manifestations.

Subsequent chapters extend the analytical paradigm to explore the solvability of nonlinear hyperbolic equations with mixed nonlinear and linear integral Neumann boundary conditions and the solvability of nonlinear fractional problems with second-kind integral conditions and boundary specifications, invoking energy inequality techniques and fractional calculus to explore mathematical memory effects and nonlocal interactions.

In conclusion, this work transcends mere theoretical expansion of nonlinear PDE analysis, illuminating the practical ramifications of blow-up phenomena across diverse scientific and engineering domains. The findings underscore the necessity of integrating analytical and numerical methodologies to decode latent complexities in these systems, offering a referential framework for pioneering future explorations in nonlinear mathematics and its transformative applications.

keywords: Existence, uniqueness, blow up, decay, numerical solution, nonlinear equation, fractional partial differential equation, difference equation.

ملخص الأطروحة

تعمق هذه الأطروحة في العوالم الرياضية الغامضة لحلول المعادلات التفاضلية الجزئية غير الخطية ، مستكشفة وجودها، ووحدانيتها، وسلوكياتها الانفجارية تحت قيود تكاملية غير موضعية. من خلال نسج منهجي يدمج التحليل النظري الصارم مع الرؤى التطبيقية، تنطلق الدراسة في رحلة فكرية عبر تعقيدات متعددة الطبقات—من الأطر التحليلية الأساسية إلى الظواهر الزمنية غير المحدودة.

يُشيد الفصل التمهيدي سقالة تحليلية متينة عبر دمج فضاءات الدوال المتقدمة ونظريات أساسية، مدعومة بتقنيات مبتكرة مثل طريقة فايدو-جاليركين والمنهجيات الطاقوية، لإثبات وجود الحلول تحت شروط حرجة. يُرسي هذا الفصل حجراً منهجياً لاستكشافات لاحقة لسلوكيات الحلول وتفكيكها.

يغوص الفصل الثاني في التفاعل بين المعادلة قطع مكافئ غير الخطية مع شرط تكاملي غير خطي معمم من النوع الثاني، كاشفاً عن عتبات رياضية تفصل بين الحلول المستقرة وتلك المنفجرة في زمن محدود. تبرز رؤى نوعية حول كيفية تحول السلوكيات تحت اضطرابات الشروط الأولية والمعلمات.

ينتقل الفصل الثالث إلى المعادلات شبه الخطية المتدهورة والمتفردة، مُحللاً الروابط الخفية بين الهياكل الخطية وغير الخطية. هنا تُضاء الشروط الدقيقة التي تحكم ظهور الحلول ووحدانيتها، مدعومة بتحليلات عديدة تُقارن التوقعات النظرية مع المظاهر التجريبية.

تمتد الفصول اللاحقة إلى استكشاف قابلية حل المعادلات الزائدية غير الخطية بشروط نيومان التكاملية المختلطة، وقابلية حل المسائل الكسرية غير الخطية بشروط تكاملية من النوع الثاني ومواصفات حدية، مستخدمة تقنيات عدم المساواة الطاقوية وحساب الكسور لاستكشاف تأثيرات الذاكرة الرياضية والتفاعلات غير الموضعية.

في الختام، يتجاوز هذا العمل التوسع النظري في تحليل المعادلات التفاضلية الجزئية غير الخطية، مُسلطاً الضوء على التداعيات العملية لظواهر الانفجار عبر مجالات علمية وهندسية متنوعة. تؤكد النتائج على ضرورة دمج المنهجيات التحليلية والعديدية لفك تعقيدات هذه الأنظمة، مقدماً إطاراً مرجعياً لاستكشافات مستقبلية رائدة في الرياضيات غير الخطية وتطبيقاتها التحويلية.

الكلمات المفتاحية: الوجود، الوحدانية، الانفجار، التلاشي، الحل العددي، المعادلة غير الخطية، المعادلة التفاضلية الجزئية الكسرية، معادلة الفروق.

Contents

1 Preliminaries	2
1.1 Functional spaces	2
1.1.1 Spaces involving time	2
1.2 Some essential theorems and corollaries	4
1.2.1 Theorem of convergence and compactness	5
1.3 Global and local solutions	8
1.4 Inequalities	9
1.5 Techniques for establishing existence and uniqueness of solutions	11
1.5.1 Faedo-Galerkin method	12
1.5.2 The energy method	13
1.5.3 Fixed-point methods	18
1.5.4 Continuation principle	18
1.5.5 Nonextendable solution	19
1.6 Blow-up methods	20
1.6.1 Kaplan's first eigenvalue method	20
1.6.2 Energy methods	22
1.7 Decay analysis for nonlinear parabolic and hyperbolic equations	30
1.7.1 Energy methods	31
1.8 Fractional calculus preliminaries	38
1.9 Numerical simulation tools	41
1.9.1 Interpolation approximations of the fractional Caputo derivatives	41
2 Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type	47
2.1 Solvability of the problem for $b = c = 0$	47
2.2 Finite-time blow-up solutions to a super-linear problem	73
2.2.1 Numerical simulation of blow-up	75

Contents

2.2.2	Numerical examples	78
2.3	Solvability of the problem for $b \neq 0, c = 0$	80
2.4	Solvability of a semi-linear parabolic equation with second integral conditions	91
2.4.1	Existence of the solution to the semi-linear problem (PI_1)	93
3	Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition	106
3.1	Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with nonlinear integral condition	107
3.1.1	Study of the linear problem	107
3.2	Investigation of the existence and uniqueness of weak solutions to the non-linear problem	117
3.3	Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a linear integral condition	121
3.3.1	Solvability of the linear problem	122
3.3.2	Solvability of the nonlinear problem	128
3.3.3	Construction of a blow-up solution of Fujita type	133
4	Exploring Solvability of Nonlinear Hyperbolic Equations with Mixed Nonlinear and Linear Integral Neumann Boundary Conditions	140
4.1	Analysis of the solvability of the solution of linear hyperbolic problem with integral condition of second type by the method of Fadeo-galarkin	141
4.1.1	Position of the linear problem (P_1)	141
4.1.2	Variational formulation	142
4.1.3	Study of the existence of weak solution of the problem (P_2)	142
4.2	Exploration of weak solutions in the context of nonlinear source with $b=0$ and homogeneous boundary conditions	152
5	Solvability of Nonlinear Fractional Problems with Second Kind Integral Conditions and Boundary Specifications	158
5.1	A nonlinear fractional problem with a second kind integral condition for time-fractional partial differential equation	158

Contents

5.1.1	Formulation of the non-linear problem (P_1)	158
5.1.2	Study of the associated linear problem	159
5.2	Solvability of fractional Fujita problems with specified boundary conditions using the energy inequality method	170
5.2.1	Formulation of the problem	170
3.2	A priori bound	172
4.3	Existence and uniqueness of solution	177

Introduction

During the second half of the nineteenth century, many mathematicians focused on exploring the complexities of partial differential equations (PDEs). The surge in interest surrounding partial differential equations (PDEs) largely stems from their ability to encapsulate many of the core laws governing nature. These equations frequently occur in the mathematical analysis of numerous scientific and engineering problems. As the field evolved, the next significant step involved establishing a comprehensive theory and developing various techniques to solve PDEs. The introduction of key concepts like distributions has brought the modern theory of linear PDEs to a more mature phase. Today, PDEs hold a prominent position in contemporary mathematics, playing a pivotal role in areas such as physics, geometry, and analysis[40].

Although nonlinear PDEs have ancient origins, they have experienced remarkable advances during the second half of the twentieth century. The primary catalyst for this development has been the study of nonlinear wave propagation problems, which arise in various fields, including fluid dynamics, nonlinear optics, solid mechanics, plasma physics, quantum field theory, and condensed-matter physics.

Numerous studies have focused on the solutions of nonlinear fractional [51] and ordinary differential equations[27], particularly in the context of various boundary conditions. These investigations are of significant interest because such equations provide precise and realistic models to describe complex phenomena in the natural and applied sciences. Among these boundary conditions, integral boundary conditions have attracted attention due to their applicability in a wide range of disciplines. For instance, they are frequently employed in the modeling of population dynamics, blood flow within biological systems, chemical engineering processes, cellular interactions, underground water flow, heat transmission, plasma physics, thermoelasticity, and other fields. The versatility of integral boundary conditions makes them a critical tool for solving real-world problems. A substantial body of research and results on these applications and their theoretical underpinnings can be found extensively throughout the existing literature, highlighting the growing importance of these mathematical techniques in advancing both scientific understanding and technological innovation.

Contents

The complexity of nonlinear PDEs posed significant challenges, requiring sophisticated mathematical tools such as perturbation techniques and advanced numerical methods to solve and analyze them. These equations often exhibit chaotic behavior, blow-up phenomena, and the formation of singularities, making them central to ongoing research. Advancements in fractional differential equations also paralleled the rise of nonlinear PDEs, offering enhanced frameworks for modeling systems with memory effects and anomalous diffusion. Including integral boundary conditions further broadened the scope of differential equations, allowing for more comprehensive models in areas such as population dynamics and heat transfer [24].

Blow-up and decay phenomena are fundamental aspects in the study of nonlinear partial differential equations, particularly within the contexts of parabolic and hyperbolic problems. These phenomena are of significant academic interest due to their implications for the long-term behavior of solutions. Specifically, finite-time blow-up describes a scenario where the solution of an equation becomes singular or unbounded within a finite time interval, a behavior often driven by the intensity of the nonlinearity and the structure of the initial and boundary conditions. The profiles of such blow-ups can be highly varied, manifesting as pointwise blow-up at specific spatial locations, global blow-up affecting the entire domain, or self-similar blow-up that exhibits a predictable form as the singularity is approached. Conversely, decay phenomena characterize the attenuation of solutions over time, with behaviors ranging from exponential decay—commonly associated with strong dissipative mechanisms—to polynomial decay, which occurs in the presence of weaker dissipation. The interplay between these outcomes is intricately linked to the energy distribution and dissipation mechanisms inherent in the system, making the analysis of blow-up and decay critical in applications such as heat conduction, wave propagation, and stress analysis. Theoretical insights into these phenomena provide a deeper understanding of the stability and instability characteristics of various physical and engineering models, where accurately predicting finite-time singularities or long-term stabilization is essential.

An integral boundary condition is a type of boundary condition used in PDEs where the value of the solution at the boundary is defined by an integral involving the solution itself, often across the entire spatial domain or a subset of it. Unlike traditional boundary conditions like Dirichlet or Neumann conditions, which specify the value or derivative of the solution at the boundary, integral boundary conditions introduce a nonlocal constraint, linking the solution at different points through integration [21], [50], [8] and [7].

Contents

Integral boundary conditions (IBCs) can have the first type:

$$\int_{\Omega} u(x, t) = E(t), \quad \int_{\Omega} k(x, t)u(x, t)dx = 0,$$

where $t \in (0, T)$, $\Omega \subset \mathbb{R}^n$ and k is a given function. The second type occurs when the Dirichlet or Neumann condition is modeled by an integral involving the solution. For example, we can have:

$$u(x, t)|_{\partial\Omega} = \int k(x, t)u(x, t)dx,$$

This is a versatile mathematical tool used to model systems where the boundary's influence depends on the overall solution behavior. These conditions are particularly relevant in fields such as conservation laws, population dynamics, heat transfer, and fluid dynamics, where the conservation of energy or mass across boundaries is crucial. IBCs offer advantages like enhanced modeling precision, the ability to capture complex system interactions, and the representation of intricate boundary behaviors. Researchers have made significant contributions to the understanding and application of IBCs, including Batten's early work[3], Cannon's theoretical foundations[16], [17], [18], Ionkin's analytical techniques[37], Kamynin's innovative solution strategies, and contemporary advancements in methodology and computational methods[39].

Integral boundary conditions significantly influence the dynamics of blow-up and decay phenomena in parabolic and hyperbolic equations[21] [47] [41] [54] [6] [56] [57], introducing complex behaviors that differ markedly from classical boundary conditions. These conditions can fundamentally alter both the occurrence and nature of finite-time blow-up, potentially creating multiple blow-up points through non-local effects and modifying blow-up rates through integral feedback mechanisms. The decay properties are similarly affected, often exhibiting slower-than-exponential rates due to memory effects, with patterns ranging from polynomial to logarithmic decay[60]. The interplay between blow-up and decay mechanisms is characterized by a delicate balance between destabilizing nonlinear terms and stabilizing integral effects, where the integral terms can either accelerate or delay blow-up time while simultaneously introducing non-local memory effects that modify decay patterns. This complexity necessitates sophisticated analytical tools, including modified maximum principles, integral inequalities, and adapted energy methods, to fully understand the system's behavior. From an applications perspective, these dynamics are crucial in various physical and biological systems, such as heat propagation

Contents

with memory, reaction-diffusion with integral feedback, and population dynamics with age structure. The ongoing research in this field focuses on both theoretical aspects - including the classification of blow-up scenarios and sharp decay estimates - and practical considerations such as developing robust numerical methods for blow-up detection and control strategies for system stabilization. Understanding these dynamics is essential for predicting critical phenomena, controlling system behavior, and optimizing practical applications across various scientific domains[26].

The organization of thesis of the main work

The primary aim of this thesis is to investigate the solvability, blow-up behavior, and decay properties of nonlinear evolution problems, with a specific focus on those subject to nonlinear integral conditions. To accomplish this objective, the thesis is structured into the following main chapters:

Introduction and Background: This chapter provides an overview of nonlinear evolution problems, outlining their significance in various scientific fields and the motivation behind studying their solvability and behavior. It also introduces key concepts and terminology essential for understanding the subsequent chapters.

Functional Spaces and Preliminaries: Here, we establish the necessary functional spaces and foundational theorems that underpin the analysis of nonlinear evolution problems. This chapter emphasizes the mathematical tools required for addressing the challenges posed by these complex equations.

In the other chapters, we explore the solvability, blow-up behavior, and stability of solutions for various classes of partial differential equations (PDEs), focusing on both classical and fractional order problems. These investigations are motivated by the increasing importance of nonlinear PDEs in modeling complex physical, biological, and engineering systems, where nonlinearities and fractional dynamics play pivotal roles.

The study begins by addressing the solvability and blow-up phenomena for a nonlinear parabolic equation subject to a generalized nonlinear integral condition of the second type. This sets the foundation for understanding how integral conditions influence solution behavior, particularly in contexts where blow-up or decay is expected.

The second chapter extends this analysis to singular and degenerate semilinear parabolic equations, incorporating both linear and nonlinear integral conditions. This chapter emphasizes the interplay between degeneracy, nonlinearity, and bound-

Contents

ary conditions, leading to a deeper understanding of solution regularity and blow-up dynamics.

Building on this, the third chapter focuses on linear hyperbolic problems with second-type integral conditions, employing the Faedo-Galerkin method to establish solvability. This chapter illustrates how classical techniques can be adapted to handle complex boundary conditions.

In the fourth chapter, we turn our attention to fractional calculus, investigating nonlinear fractional PDEs with second-kind integral conditions. The fractional framework provides a natural extension of classical problems, allowing for the modeling of memory effects and anomalous diffusion.

The fifth chapter continues the exploration of fractional problems, specifically examining semi-linear fractional parabolic equations with Dirichlet boundary conditions. This analysis reveals the unique characteristics of fractional diffusion and its impact on solution dynamics.

Finally, the thesis concludes with an investigation of fractional Fujita-type problems, focusing on their solvability under specified boundary conditions using the energy inequality method. This chapter highlights the intricate relationship between fractional order, boundary conditions, and solution behavior, providing new insights into the blow-up and global existence of solutions.

Conclusion and Future Directions

The thesis is concluded by summarizing the main findings related to solvability, blow-up, and decay of nonlinear evolution problems. The broader implications of the results are discussed, particularly in applied fields like physics, engineering, and biology. Suggestions for future research, including the extension of these methods to more complex systems or higher-dimensional problems, are provided.

Preliminaries

This chapter lays the groundwork for the thesis by stating fundamental mathematical concepts, theorems, and analytical techniques.

It begins with a discussion of functional spaces, particularly those involving time-dependent functions, and introduces key convergence and compactness theorems that are essential for understanding the solution behavior. The chapter delves into the concepts of global and local solutions, along with fundamental inequalities used to bound solution properties.

A variety of methods for proving existence and uniqueness of solutions are presented, including the Faedo-Galerkin method, energy method, fixed-point techniques, and continuation principles. These methods form the core analytical toolkit for investigating solutions. The chapter also explores techniques for analyzing blow-up phenomena, such as Kaplan's first eigenvalue method, and discusses decay properties of nonlinear equations.

To support the modeling of fractional and complex dynamics, the chapter introduces preliminary concepts in fractional calculus and relevant simulation tools. Collectively, these sections provide a solid foundation for the in-depth analysis presented in subsequent chapters.

1.1 Functional spaces

Functional spaces are fundamental in the study of partial differential equations (PDEs), and functional analysis. In the context of PDEs and mathematical analysis, the most commonly used functional spaces include Lebesgue spaces, Sobolev spaces, and Hilbert spaces[?].

1.1.1 Spaces involving time

The section explores the idea of function spaces where each point in time is mapped to an element of a Banach space. These Banach-valued functions require careful consideration of their properties, including continuity, measurability, and integrability with respect to time. The treatment of these properties involves tools from both functional analysis and real analysis, such as Bochner integrals, which

Chapter 1. Preliminaries

extend the concept of integration to Banach-valued functions, and Lebesgue spaces, and describe the integrability conditions these functions must satisfy.

The space $L^p(0, T; X)$

The space $L^p(0, T; X)$ consists of Banach-valued measurable functions $f : [0, T] \rightarrow X$. In these spaces, functions must not only be measurable but must also satisfy integrability conditions with respect to the time variable. Specifically, for a given p -norm, the p -th power of the function's norm (in the Banach space) must be integrable over the domain. These L^p -spaces extend classical spaces of integrable functions by accommodating functions that take values in Banach spaces rather than simple real or complex numbers. Such spaces are critical for the mathematical treatment of evolution equations, where time-dependent solutions often reside in these types of functional spaces. These spaces are equipped with the following norms:

$$\begin{cases} \text{for } 1 \leq p < \infty & \|f\|_{L^p(0,T;X)} := \left(\int_0^T \|f(t)\|^p dt \right)^{\frac{1}{p}} < \infty, \\ p = \infty & \|f\|_{L^p(0,T;X)} := \operatorname{ess\,sup}_{0 \leq t \leq T} \|f(t)\| < \infty. \end{cases}$$

The space $C([0, T]; X)$

The space $C([0, T]; X)$ includes all continuous functions $f : [0, T] \rightarrow X$, defined by the norm:

$$\|f\|_{C([0,T];X)} := \max_{0 \leq t \leq T} \|f(t)\| < \infty.$$

The space $W^{1,p}([0, T]; X)$

The Sobolev space $W^{1,p}(0, T; X)$ meaning these functions are measurable and their p -th power is integrable with respect to time over the interval $(0, T)$. Additionally, a crucial requirement for belonging to this space is that the weak derivative of the function f with respect to time is in $L^p(0, T; X)$. The norm on $W^{1,p}(0, T; X)$ is given by:

$$\begin{cases} \text{for } 1 \leq p < \infty & \|f\|_{W^{1,p}(0,T;X)} := \left(\int_0^T (\|f(t)\|^p + \|f'(t)\|^p) dt \right)^{\frac{1}{p}} < \infty, \\ p = \infty & \|f\|_{W^{1,p}(0,T;X)} := \operatorname{ess\,sup}_{0 \leq t \leq T} (\|f(t)\| + \|f'(t)\|) < \infty. \end{cases}$$

Remark 1.1

We set: $H^1([0, T]; X) = W^{1,2}([0, T]; X)$.

Theorem 1.1

Let $f \in W^{1,p}(0, T; X)$ for some $1 \leq p \leq \infty$. Then

- $f \in C([0, T]; X)$,
- $f(t) = f(s) + \int_s^t f'(\tau) d\tau$, for all $0 \leq s \leq t \leq T$,
- $\max_{0 \leq t \leq T} \|f(t)\| \leq C \|f\|_{W^{1,p}(0,T;X)}$ with the constant C depending only on T .

Proof See the book [30]. ■

Remark 1.2

Given a bounded open set $\Omega \subset\subset V$, and $\tilde{u} = Lu$ (L is a bounded linear operator), we get that $\tilde{u} \in L^2(0, T; H^2(V))$ and

$$\|\tilde{u}\|_{L^2(0,T;H^2(V))} \leq C \|u\|_{L^2(0,T;H^2(\Omega))},$$

an appropriate constant C . In addition, $\tilde{u}' \in L^2(0, T; H^2(V))$, with the estimate

$$\|\tilde{u}'\|_{L^2(0,T;H^2(V))} \leq C \|\tilde{u}\|_{L^2(0,T;H^2(\Omega))}.$$

1.2 Some essential theorems and corollaries

In mathematical analysis, pointwise convergence is a critical concept that describes how a sequence of functions behaves as it approaches a limiting function at each individual point in the domain.

Theorem 1.2: (Dominated convergence theorem)

Let $A \subset \mathbb{R}^n$ be measurable and $\{f_j\}$ a sequence of measurable functions that converge pointwise to a limit on a set A . Let be given a sequence of measurable functions f_j that converges to a function f at every point $x \in A$, meaning that for each x , the value of $f_j(x)$ gets arbitrarily close to $f(x)$ as j increases. If there exists a function $g \in L^1(A)$ such that $|f_j(x)| < g(x)$ for every index j and for all $x \in A$, then

$$\lim_{j \rightarrow \infty} \int_A f_j(x) dx = \int_A (\lim_{j \rightarrow \infty} f_j(x)) dx.$$

Theorem 1.3: (An Imbedding Theorem for L^p Spaces)

Suppose that $mes(\Omega) = \int_{\Omega} 1 dx < \infty$ and $1 \leq p \leq q \leq \infty$. If $u \in L^q(\Omega)$, then $u \in L^p(\Omega)$ and

$$\|u\|_p \leq (mes(\Omega))^{\frac{1}{p} - \frac{1}{q}} \|u\|_q.$$

Hence

$$L^q(\Omega) \rightarrow L^p(\Omega).$$

If $u \in L^\infty(\Omega)$, then

$$\lim_{p \rightarrow \infty} \|u\|_p = \|u\|_\infty.$$

Finally, if $u \in L^p(\Omega)$ when p varies across the range $1 \leq p < \infty$, and if there exists a constant K such that for all such p

$$\|u\|_p \leq K.$$

then $u \in L^\infty(\Omega)$ and

$$\|u\|_\infty \leq K.$$

1.2.1 Theorem of convergence and compactness

This section presents the essential theorems and lemmas of the main techniques of establishing existence and uniqueness of solutions used in the thesis

Lemma 1.1

In a reflexive Banach space, every bounded set is weakly compact. In particular, every sequence from a bounded set contains a weakly convergent subsequence.

Lemma 1.2

Consider a Banach space V , defined by $V = (V^*)'$ where V^* denotes a reflexive Banach space. Then, any bounded subset of V is weakly star compact, i.e. every sequence in a bounded subset of V contains a subsequence that converges weakly star.

Theorem 1.4

Suppose that V_0, V, V_1 are three Banach spaces with V_0, V_1 reflexive. Assume that V_0 be continuously imbedded into V , which in turn is continuously embedded in V_1 , and the imbedding from V_0 into V is compact. For any given p_0, p_1 with $1 < p_0, p_1 < \infty$, set:

$$W = \{v \mid v \in L^{p_0}([0, T], V_0), v_t \in L^{p_1}([0, T], V_1)\}$$

Then the imbedding from W into $L^{p_0}([0, T], V)$ is compact.

Lemma 1.3

Let Ω be an arbitrary domain in \mathbb{R}^n . Let $u_n(x), u(x)$ be real functions in $L^p(\Omega)$, ($1 \leq p \leq \infty$) such that u_n strongly converges to u in $L^p(\Omega)$. Then if $1 \leq p < \infty$, u_n has a subsequence almost everywhere converging to u and if $p = \infty$, then u_n itself converges almost everywhere to u .

Lemma 1.4

Let V be a Banach space, and $V = (V^*)'$ with V^* being another Banach space. Suppose that, for $1 < p \leq \infty$,

$$\begin{cases} u_n \rightarrow u & \text{weakly star in } L^p([0, T], V), \\ u_n \rightarrow u & \text{weakly star in } L^p([0, T], V). \end{cases}$$

Then

$$u_n(0) \rightarrow u(0) \text{ weakly star in } B.$$

Remark 1.3

Suppose that $1 < p < \infty$, and V is a reflexive Banach space. Then the weakly star convergence in the above lemma is equivalent to the weak convergence.

Lemma 1.5

Let V be a reflexive Banach space and ϱ an abstract Lipschitz continuous function defined in $[0, T]$ and valued in V . Then, $\varrho \in L^1([0, T], B)$ and

$$\varrho(t) = \varrho(0) + \int_0^t \varrho'(\tau) d\tau.$$

The Schauder theorem

In our work, we have used the following version of the Schauder theorem.

Theorem 1.5: (Schauder-Tychonoff theorem)

Let E be a separated LCTVS (locally convex topological vector space), K a nonvoid compact convex subset of E , and f a continuous map from K into itself. Then f admits at least one fixed point.

Corollary 1.1

Let S be a separated locally convex topological vector space (LCTVS) with the property that the closed convex envelope of any compact subset is also compact. Let Υ be a nonempty, closed, and convex subset of S , and v be a continuous function mapping from Υ to itself, with $v(\Upsilon)$ being relatively compact within S . Then, v has at least one fixed point”.

Proof Let B be the convex envelop of $f(A)$ (B the smallest convex set containing $f(A)$) and K the closure of B in E . Then, $K \subset A$. Since $f(A)$ is relatively compact, K is also nonvoid and convex. As f is continuous on A , we have

$$f(K) = f(\overline{B}) = f(B \cap A) \subset \overline{f(B)} \subset \overline{f(A)} = \overline{B} = K.$$

It only remains to apply the Schauder-Tychonoff theorem to $f|_K$. ■

Theorems of existence for ODEs

The existence of solutions for ODEs is a fundamental aspect of the theory of differential equations. Several key theorems provide conditions under which solutions exist. Below are some of the most important theorems.

Theorem 1.6: (Picard–Lindelöf theorem)

Let $Q \subseteq \mathbb{R} \times \mathbb{R}^n$ be a closed rectangle with $(t_0, y_0) \in \text{int}(D)$, the interior of Q . Let $f : Q \rightarrow \mathbb{R}^n$ be a function that is continuous in t and Lipschitz continuous in y (with Lipschitz constant independent from t). Then, there exists some $\epsilon > 0$ such that the initial-value problem

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = y_0. \end{cases}$$

has a unique solution $y(t)$ on the interval $[t_0 - \epsilon, t_0 + \epsilon]$.

Theorem 1.7: (Carathéodory theorem)

Let f be a function defined on a rectangle R , such that for each fixed x , it is measurable in t , and for each fixed t , it is continuous in x . Assume that there exists a Lebesgue-integrable function m on the interval $|t - \tau| \leq a$, satisfying the condition:

$$|f(t, x)| \leq m(t) \quad ((t, x) \in R).$$

Then, a solution φ of (E) exists in the extended sense over some interval $|t - \tau| \leq \beta$, $(\beta > 0)$, and satisfies the initial condition $y_0(\tau) = \xi$.

1.3 Global and local solutions

In the realm of nonlinear evolution equations, the exploration of global and local solutions remains a vibrant area of research to this day. The investigation into these solutions is crucial because it directly impacts the understanding of the behavior and dynamics of the equations involved. Researchers are particularly focused on several key aspects:

- **Local solution:** A local solution of a PDE means that the solution exists only on a finite interval or neighborhood. Often the focus is on proving the existence and properties of local solutions first, and this encourages the

researchers to build and develop important methods and theories in the generalized sense or weak sense like the Faedo-Galerkin method, a prior estimate, fixed point.

- **Global solution:** The solution that exists globally in time is necessary to comprehend how the evolution problem behaves over the whole domain or time range of interest and to ascertain whether a continuous dependence on the initial or boundary conditions exists. This issue is still the main concern in the research of PDEs. Since the 1960s the exploration of global solutions in differential equations encompasses an examination of how solutions behave across their entire domain, particularly as time progresses towards infinity.

1.4 Inequalities

In this section, we focus on the application of key inequalities to estimate the energy and other critical terms associated with the solutions of the given PDEs. These inequalities allow us to handle the nonlinearities present in the system and provide a framework for proving the boundedness of the solution, preventing blow-up phenomena, and ensuring global solvability.

Theorem 1.8: (Gronwall inequality [Gronwall (1919)]).

Let $u(t)$ be a continuous function defined on the interval $[t_0, t_1]$ and

$$u(t) \leq \alpha + \beta \int_{t_0}^t u(s) ds,$$

where α and β are nonnegative constants. Then,

$$u(t) \leq \alpha \exp(\beta(t - t_0)), t \in [t_0, t_1].$$

Theorem 1.9: (Bellman inequality [Bellman (1943)]).

Let α be a positive constant, $u(t)$ and $\beta(t)$, $t \in [t_0, t_1]$ real-valued continuous functions, with $\beta(t) \geq 0$, satisfying

$$u(t) \leq \alpha + \int_{t_0}^t \beta(s)u(s)ds,$$

Then ,

$$u(t) \leq \alpha \exp\left(\int_{t_0}^t \beta(s)ds\right), t \in [t_0, t_1].$$

Bellman also proved that if $u(t)$ and $\beta(t)$ are continuous functions, $\beta(t)$ is a nonnegative function, and

$$u(t) \leq u(t_0) + \alpha \int_{t_0}^t \beta(s)u(s)ds, t \in [t_0, t_1]$$

then

$$u(t_0) \exp\left(-\int_{t_0}^t \beta(s)ds\right) \leq u(t) \leq u(t_0) \exp\left(\int_{t_0}^t \beta(s)ds\right), t \in [t_0, t_1].$$

- **An Interpolation inequality:** Let $1 < p < q < r$, so that $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}$, for some θ satisfying $0 < \theta < 1$. If $u \in L^p(\Omega) \cap L^r(\Omega)$, then $u \in L^q(\Omega)$ and

$$\|u\|_q \leq \|u\|_p^\theta \|u\|_r^{1-\theta}.$$

- **A Reverse Minkowski inequality:** Let $0 < p < 1$. If $u, v \in L^p(\Omega)$

$$\| |u| + |v| \|_p \geq \|u\|_p + \|v\|_p.$$

- **A Reverse Holder inequality:** Let $0 < p < 1$ such that $p' = \frac{p}{p-1} < 0$. If $f \in L^p(\Omega)$ and $\int_{\Omega} |g(x)|^{p'} dx < \infty$, then

$$\int_{\Omega} |f(x) + g(x)| dx \geq \left(\int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |g(x)|^{p'} dx \right)^{\frac{1}{p'}}.$$

1.5 Techniques for establishing existence and uniqueness of solutions

In mathematical analysis, particularly in the study of differential and functional equations, establishing the existence and uniqueness of solutions is crucial. Several techniques and theorems are employed to guarantee that a solution exists and that it is the only one under specified conditions. In summary, establishing the existence and uniqueness of solutions to differential equations is a critical aspect of mathematical analysis[?]. The concept of well-posedness introduced by Hadamard, along with various techniques such as fixed-point theorems, energy methods, variational methods, and the use of Sobolev spaces, provides a robust framework for rigorously analyzing boundary-value problems. These methods not only enhance our theoretical understanding but also have significant implications for practical applications across various fields. Below are some key methods used to achieve this. The concept of well-posedness for boundary-value problems of partial differential equations was introduced by the French mathematician Jacques Salomon Hadamard around 1932.

Definition 1.1

(Hadamard's well-posedness) A problem is said to be well-posed. If

1. it has a unique solution,
2. the solution depends continuously on the given data

Otherwise, the problem is said to be ill-posed.

A comprehensive study of the significance of specific mathematical methods is essential for two primary reasons. First, it is crucial to recognize the appropriate contexts in which these methods should be employed. This understanding ensures that mathematicians and researchers can effectively apply the right approach to solve various problems. Secondly, gaining insight into the limitations and breakdowns of these methods enables a deeper comprehension of the nature of the problems being addressed. By identifying the scenarios in which a method may fail, practitioners can better navigate complex challenges and refine their analytical strategies.

In light of these considerations, this section will detail the various methods we have utilized to tackle the problems at hand. Each method will be examined not only in terms of its effectiveness but also regarding its boundaries and limitations. This dual focus will provide a well-rounded perspective on the methods and enhance

our understanding of their applicability in different mathematical contexts..

1.5.1 Faedo-Galerkin method

It originated from the idea of using an orthogonality condition, commonly attributed to Galerkin in 1915. Referred to as the Faedo-Galerkin method when applied to time-dependent problems, this emerged later within the framework of functional analysis in the hands of the Italian mathematician Gaetano Fichera, who made significant contributions to its development that has undergone refinement over time.[?]

The Faedo-Galerkin method has a senseibility of application on the three steps:

- The first step consists in constructing solutions of certain finite-dimensional approximations of the main problem, by selecting a set of basis functions that form an orthogonal basis in the function space (F). A sequence (ω_n) of functions having the following properties is introduced:

* $\forall j = 1, \dots, m : \omega_j \in V \cap H$

* The family $\{\omega_1, \omega_2, \dots, \omega_m\}$ is linearly independent;

* The space $B_m = [\omega_1, \omega_2, \dots, \omega_m]$ generated by the family, $\{\omega_1, \omega_2, \dots, \omega_m\}$, is dense in $B \cap F$.

Let $u_m = u_m(t)$ be an approximate solution given by

$$u_m(t) = \sum_{i=1}^m \alpha_{ik} w_i,$$

and completed by the following initial conditions:

$$\left\{ \begin{array}{l} u_m(t) = u_{0m}, \\ u_{0m} = \sum_{i=1}^m \beta_{im} w_i \xrightarrow{m \rightarrow \infty} u_0. \end{array} \right.$$

Remark 1.4

We could take $\{\omega_1, \omega_2, \dots\}$ to be the complete set appropriately normalized eigenfunctions for the diffusion operator.

- The second step involves deriving a priori estimates for the approximate solution of the PDE problem. This process is crucial as it provides bounds on the solution's behavior before obtaining the actual solution. By establishing these estimates, we can ensure that the approximate solutions not only converge but also remain stable within the defined parameters of the problem.

This step often includes analyzing the dependencies of the solution on initial and boundary conditions, as well as the properties of the differential operator involved. A priori estimates serve as a foundation for proving the existence and uniqueness of solutions, and play a vital role in assessing the accuracy of numerical methods used in solving the PDE.

- The third step concerns the process of taking limits within the Faedo-Galerkin method and is intended for evaluating convergence and understanding the accuracy of solutions. Its contributions to approximation theories and the establishment of independent error estimates have had a lasting impact on the field, enhancing both theoretical and practical approaches to solving differential equations of both the Galerkin subspace and time discretization, establishing it as a significant tool for the analysis of mathematical systems. Thus, the continuous operator problem is converted to a discrete problem by applying linear constraints determined by finite sets of basis functions. This allows for a more straightforward analysis and solution of problems through weak formulations providing a comprehensive and robust approach for achieving convergence to the true solution. Its adaptability allows it to be applied to various types of problems, encompassing both linear and nonlinear systems. Moreover, its ability to be combined with other methodologies enhances its effectiveness and establishes it as a fundamental component in contemporary numerical analysis and computational mathematics. This framework is especially valuable for addressing complex problems where traditional methods may struggle, ultimately leading to accurate and dependable solutions. This enables its applicability to demonstrating its versatility and usefulness in solving a wide range of mathematical problems. Despite its advantages, the method faces numerous challenges, particularly the difficulty of guaranteeing convergence to the true solution in high-dimensional evolution problems, essentially, because of the complexity in selecting the basis functions and the inadequacy of the choice of function sets that may lead to inaccuracies in the solutions. Additionally, the complexity involved in computing a priori estimates poses challenges when applying the Faedo-Galerkin method to specific types of boundary conditions and handling them.

1.5.2 The energy method

In the mid-20th century, researchers began to explore energy estimates as a powerful tool in the analysis of PDEs, significantly advancing the field of mathematical

Chapter 1. Preliminaries

analysis. The method of energy inequalities emerged as a critical framework for proving the existence of solutions to various classes of PDEs. By establishing these energy inequalities, mathematicians were able to derive crucial information regarding the behavior, stability, and convergence of solutions.

The core idea behind the energy method involves constructing energy estimates that provide bounds on the energy of the system described by the differential equations. This typically entails defining a suitable energy functional that encapsulates the essential characteristics of the system. By demonstrating that this energy functional remains bounded over time, researchers can ensure that the energy of the system is controlled and finite. In essence, this boundedness implies that the solutions do not exhibit unbounded growth, which is critical for the existence of well-behaved solutions.

By proving that the energy remains finite, the energy inequality method establishes a direct link between the mathematical properties of the PDEs and the physical interpretations of the solutions. This approach allows researchers to make significant progress in various fields, including physics, where it helps in understanding phenomena such as heat conduction, wave propagation, and fluid dynamics. In engineering, energy estimates are fundamental in analyzing systems subjected to dynamic loads, ensuring the stability and safety of structures and materials.

Furthermore, the energy method has proven invaluable in the context of nonlinear PDEs, where traditional methods may falter. In these cases, the method can reveal the existence of solutions under conditions that would otherwise lead to blow-up or other pathological behaviors. By carefully analyzing the energy flow and dissipation, researchers can gain insights into the qualitative behavior of solutions, including their long-term stability and convergence to equilibrium states.

Overall, the development and application of energy inequalities have laid a robust foundation for the analysis of PDEs, facilitating advancements in both theoretical mathematics and practical applications. As this method continues to evolve, it remains a cornerstone in the study of nonlinear dynamics, equipping researchers with the tools needed to tackle increasingly complex problems across diverse scientific and engineering domains.

A priori estimate

The a priori estimates, totally dependent to a bound on the solution of a mathematical problem, are obtained before the actual solution is known to exist and guesses the functional space of the solution by providing bounds on solution norms in

appropriate function spaces. They are is important for proving the well-posedness of the problems and suggests the corresponding analysis of PDEs. Also, a priori estimates can be of great assistance in works where the authors (articles of taki and his superviosor) are dealing with delay and functional-differential equations on the basis of an analogous abstract theorem, and have derived very precise results in the case of ODEs, up to the point of postulating what now emerges as a special case. The advantage of the present approach lies in its broad scope of application to various classes of evolution equations and its provision of a unified treatment for different equations. To get an a priori estimate of the problem, the proof must be based on one of based our proof on three techniques depending on the nature of the considered problem

- Pointwise estimates: These provide bounds on the solution or its derivatives at individual points in the domain and often use the maximum principle, comparison principles, or other inequalities (Sobolev space, Poincare inequalities,...). They are handy for establishing regularity properties of solutions, such as Hölder or Lipschitz continuity (uniqueness and stability of solution).
- Integral estimates: These provide bounds on the solution or its derivatives in integral norms, such as L^p or Sobolev norms by multiplying the PDE by a test function, integrating over the domain, and then applying the functional analysis techniques to get crucial for proving the existence and uniqueness of solutions.
- Energy estimates: These from a specific type of integral estimates that exploit the energy of the PDE to provide bounds on the energy of the solution. They are generally used in the analysis evolution of PDEs.

Energy inequalities method (a priori estimation method)

The energy inequalities method for solving PDEs is a powerful tool, particularly in the context of the existence and uniqueness of solutions. It is widely used in various branches of mathematics and physics, including heat transfer, electromagnetism, quantum mechanics, elasticity, fluid mechanics, cosmology, and sonic boom equations. The energy inequalities method is particularly effective for linear PDEs and nonlinear under conditions. It is useful in the same way as their precursors, for studying the well-posedness, existence, uniqueness, and stability of solutions to partial differential equations.

In the a priori estimation method, also called the energy inequality method, multipliers are constructed according to the specific problem at hand. These

Chapter 1. Preliminaries

multipliers yield a priori estimates that are crucial for proving the existence of solutions.

The energy inequality technique generally follows these steps:

- Formulation of the problem : The problem is formulated as an operator equation

$$Lu = f,$$

Here, L represents a linear operator, and f is a known function. The operator L is applied to a domain E , which includes functions u that meet specific criteria, such as boundedness and smoothness.

- Choosing and constructing the multipliers : The construction of multipliers is a crucial step in the energy inequality method. Generally, we denote by Mu the effect of any multiplier M on u . These multipliers are used to transform the operator equation into an equivalent form that can be used to establish the prior estimates. The multipliers are chosen such that they satisfy certain conditions, like being smooth and having a finite norm.
- A priori estimates: The multipliers are used to derive a priori estimates for the solution u . These estimates are typically in the form of an energy inequality:

$$\|u\|_E \leq C \|Lu\|_F \quad \forall u \in D(L), \quad (1.1)$$

often involving function spaces such as Banach spaces and Hilbert spaces. This is an inequality involving the square integral of the solution and its derivatives.

Consequences:

- The solution u is both unique and exhibits continuous dependence on the initial and boundary conditions.

Proof For u_1 and u_2 being solutions of the problem, we can leverage the linearity of the operator L . This property allows us to combine these solutions to analyze the behavior of the system more effectively. This we get

$$L(u_1 - u_2) = 0,$$

Which gives

$$\|u_1 - u_2\|_E \leq 0 \quad \forall u \in D(L),$$

Then,

$$\|u_1 - u_2\|_E = 0 \implies u_1 = u_2.$$

- The operator L from E to F having a closure \bar{L} , there is a constant C independent of u such that

$$\|u\|_E \leq C \|\bar{L}u\|_F \quad \forall u \in D(\bar{L}).$$

- **Existence (Solvability):** The proof of the existence of a solution to the problem is established by leveraging the a priori estimates derived in the preceding step, in conjunction with the application of the subsequent lemma. By utilizing the a priori estimates, we can demonstrate that the approximate solutions remain bounded and converge towards a potential solution. This convergence is essential in ensuring that the solutions do not diverge and that they satisfy the necessary conditions of the problem. In the subsequent lemma, we apply specific criteria that provide a framework for establishing the existence of a limit point within the solution space.

Lemma 1.6:

$$R(L) = R(\bar{L}).$$

. The fact that the range of L is dense in F guarantees that for any element $(f, \phi) \in F$

1- L is injective and has a closed range, and we now need to show that every element f in E can be written as Lv for some $v \in D(L)$.

2- By the Banach open mapping theorem, since L is a closed operator with a closed range.

3- Operator L is surjective in its range. For any $f \in R(L)$, there exists $v \in D(L)$ such that $Lv = f$. Existence is usually established by demonstrating that the range of the operator L is dense in the functional space F (which consists of pairs (f, ϕ)), ensuring that the operator L maps onto a dense subset of \mathcal{F} . This characteristic is utilized to establish the existence of solutions. Suppose that a sequence of functions in the L domain converges to zero while their images under L converge to a certain limit. In that case, that limit must be zero, ensuring the closure of the operator.

If L is surjective (its range is the entire space), it must also be injective for L to be closed and have a dense range. This follows from the open mapping theorem, which states that a bounded linear operator between Banach spaces that is surjective is also boundedly invertible, implying injectivity.

$$\forall v \in D(L) \quad (Lu, v) = 0 \implies u = 0,$$

$$\forall v \in D(l) \quad (lu, v) = 0 \implies lu = 0.$$

1.5.3 Fixed-point methods

Since 1930, the influential Polish mathematician Juliusz Schauder was introduced by the paper "*The Fixed Point Theorem in Function Spaces*", which gives future developments in the theory of nonlinear functional analysis. Before Schauder, the Banach fixed-point theorem (the contraction mapping theorem) was well-known. However, the Banach theorem applied only to contractions, limiting its applicability. Schauder's theorem extended the scope by applying to continuous, compact operators, thus broadening the range of problems that could be addressed. The development of Schauder's fixed-point theorems highlights their profound impact on the study of evolution problems.

In order to apply the Schauder's theorem on a parabolic (hyperbolic) problem we first need to define a closed convex set. Let it be:

$$X = \{u \in Y; u(0) = u_0\};$$

where $(Y, \|\cdot\|_Y)$ is a Banach space, supposedly containing the solution, and u_0 the initial boundary condition. By letting $v = \vartheta(u)$, the solution v of the main problem corresponds to u , where $\vartheta : X \rightarrow X$ (X being a separated locally convex topological vector space satisfying that the closed convex envelope of any compact set is compact), to apply the theorem of Schauder, one must follow this plan.

- 1- Begin by considering the mapping $\vartheta : B_R \rightarrow B_R$, where $B_R = \{u \in X; \|u\|_Y \leq R\}$, and B_R is defined as a closed, convex, and non-empty subset of the space X .
- 2- Show that ϑ is mapped in itself (for the condition on T and R).
- 3- Prove the continuity of ϑ .
- 4- Show that $\phi(B_R)$ is relatively compact in Y .

1.5.4 Continuation principle

The continuation principle is a key tool in proving the global existence of solutions for evolution problems. It is based on the idea that if a solution can be shown to exist locally in time and remains bounded, then this solution can be extended to exist for all time. Show that if the solution remains bounded on $[0, T_{max}]$, then it can be extended beyond T_{max} or extend the solution using the boundedness of u , implying $T_{max} = \infty$. Generally, after proving the local existence, the proof is then based on a contradiction argument. However, the technique needs more attention and focus because this is not always available for all the research and problems.

1.5.5 Nonextendable solution

Definition 1.2: nonextendable solution

Every solution can be expanded to meet this condition. In other words, any given solution is a limitation of a specific nonextendable solution within a shorter time interval.

In another way, the nonextendability of a solution is related to the concept of blow-up, where the solution becomes unbounded or singular within a finite time interval. Also, the concept of nonextendable solutions is closely tied to blow-up phenomena in differential equations. A nonextendable or maximal solution denotes the largest interval where a solution exists uniquely. If a solution cannot be extended past a certain point, it is labeled nonextendable. This moment is typically linked with the solution becoming unbounded or singular within a finite time period, a situation known as blow-up.

Consider the abstract Cauchy problem:

$$\begin{cases} \frac{du}{dt} = A(u), \\ u(0) = u_0. \end{cases} \quad (1.2)$$

Let B be a Banach space equipped with the norm $\|\cdot\|$ and $u_0 \in B$. A solution of the Cauchy problem (1.2) is a function $u \in C^1([0, T_0], B)$, $T_0 > 0$. We analyze the equation using strong differentiation within the Banach space B for all $t \in [0, T_0]$. The following theorem is significant.

Theorem 1.10

Let $A(u)$ be a locally Lipschitz-continuous operator, that is, there exists a function $\chi : [0, +\infty) \rightarrow [0, +\infty)$, which is bounded on any bounded subset of the ray $[0, +\infty)$ and is such that, for any $u_1, u_2 \in B$,

$$\|A(u_1) - A(u_2)\| \leq \chi(\max(\|u_1\|, \|u_2\|)) \|u_1 - u_2\|.$$

Then, there exists a unique solution such that either $T_0 = +\infty$ or $T_0 < +\infty$, and in the latter case we have

$$\lim_{t \rightarrow T_0} \|u(t)\| = +\infty.$$

1.6 Blow-up methods

Blow-up methods are powerful analytical techniques used to study the behavior of solutions to PDEs and other mathematical problems, particularly in the context of singularities. These methods involve rescaling or transforming the problem to gain insights into the nature of solutions as they approach critical points or boundaries. Here as an overview of blow-up methods and their applications: The earliest significant contributions to the study of blow-up phenomena in parabolic equations are attributed to Samuel Kaplan and Hiroshi Fujita. Their pioneering works laid the foundation for understanding the conditions under which solutions to certain parabolic PDEs exhibit blow-up behavior, where the solution becomes unbounded in finite time.[?],[?],[?],[?]

The term 'blow-up' describes the situation where the solution to a differential equation grows without bounds in a finite amount of time. This is important for analyzing several physical and mathematical models.

The present section is a result of an important question: after studying the local existence the intention of researchers directly goes to the question what happen after T ? To give an answer to this simple question the researchers shall develop techniques based on it.

Definition 1.3

Blow-up signifies a sudden infinite growth in a finite time span. Blow-up is often caused by an imbalance between diffusion and reaction terms.

1.6.1 Kaplan's first eigenvalue method

In 1930, Kaplan introduced the first eigenvalue method which a technique used to prove the finite time blow-up. The method uses the first eigenvalue λ_1 to establish an upper bound for the blow-up time. This involves deriving a differential inequality to obtain a lower bound for the blow-up time. The advantage of using this method resides in its flexibility to be used in various applications.

Kaplan's first eigenvalue method is a tool for analyzing finite time blow-up in differential equations due to its ability to relate blow-up to spectral properties. However, based on the information gathered in few references, because of the lack of dedicated monographs suggests that while finite-time blow-up and Kaplan's first eigenvalue method are active areas of research, there does not appear to be a

comprehensive book solely focused on these topics.

The research papers and online resources indicate that Kaplan’s first eigenvalue method, that more deep research maybe needed to cover this subject.

Eigenvalues and eigenfunctions

Let us consider the following boundary-value problem:

$$\begin{cases} Lu = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

where Ω is an open bounded domain, and λ represents the eigenvalue of L provided there exists a nontrivial solution u of (1.3).

Eigenvalues of symmetric elliptic operators

For L is the symmetric elliptic operator, by assuming that Ω is connected, the associated bilinear form $B [,]$ satisfies

$$\forall u, v \in H_0^1(\Omega) : B [u , v] = B [v , u] .$$

Theorem 1.11

1. Each eigenvalue of L is real.
2. If we repeat each eigenvalue according to its (finite) multiplicity, we have $\{\lambda_k\}_{k=1}^\infty$, where $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$, and $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$.
3. There exists an orthonormal basis $\{u_k\}_{k=1}^\infty$ of $L^2(\Omega)$, with each $u_k \in H_0^1(\Omega)$ being an eigenfunction corresponding to λ_k for $k = 1, 2, 3, \dots$:

$$\begin{cases} Lu_k = \lambda_k u_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Remark 1.5

- 1- A symmetric operator implies that all the eigenvalues are real, positive, and the corresponding eigenfunctions form an orthonormal basis of $L^2(\Omega)$.
- 2- $\lambda_1 > 0$ is called the principal eigenvalue of L .

Theorem 1.12: Variational principle for the principal eigenvalue

We have $\lambda_1 = \min \{ B[u, u] \mid u \in H_0^1(\Omega), \|u\|_{L^2(\Omega)} = 1 \}$. Furthermore, the above minimum is attained for a function u_1 , positive in Ω , which solves

$$\begin{cases} Lu_1 = \lambda_1 u_1 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.5)$$

Finally, if u is any weak solution of

$$\begin{cases} Lu = \lambda_1 u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.6)$$

then u is a multiple of u_1 .

The technique steps of the Kaplan's first eigenvalue method are as follows.

Step 01: Determine the first eigenvalue λ_1 ($\lambda_1 \neq 0$ and nonnegative eigenfunction) along with the associated eigenfunction for the relevant eigenvalue problem defined on the domain Ω with the specified boundary conditions.

Step 02: Test the nonlinear problem with the eigenfunction and simplifying the expression to get an ODE.

Step 03: Analyze and solve the ODE to find the conditions under which a solution to a PDE experiences finite-time blow-up.

1.6.2 Energy methods

Energy methods are a powerful analytical approach in the study of PDEs and dynamical systems, particularly when investigating blow-up phenomena. These methods involve analyzing energy-like quantities associated with the solutions to assess their behavior as they approach singularities. Below is an overview of how energy methods can be applied to study blow-up.

1. Basic concept of energy methods typically involve defining an energy functional associated with the system. This energy functional often represents a physical quantity such as kinetic energy, potential energy, or a combination of both. By analyzing the properties of this functional, one can infer the behavior of solutions over time.

Energy method of H. A. Levine

The energy method, developed by H. A. Levine, was originally designed specifically for investigating nonlinear equations. However, because of its succinctness, it has become essential for analyzing the blow-up behavior in a broad range of mathematical physics problems. There are numerous works regarding blow-up phenomena by using the method of H. Levine. Levine’s method relies on the construction of a suitable Lyapunov functional, which serves as a tool to establish the existence and characterization of blow-up solutions. In particular, the method involves identifying an appropriate energy functional and demonstrating that its time derivative Alongside the solutions of the evolution equation, specific conditions must be satisfied that can lead to the phenomenon known as blow-up of the solutions. Essentially, Levine’s method utilizes the inherent properties of the solutions to derive insights into their behavior under certain circumstances of an energy functional associated with the PDE to demonstrate that the solution cannot remain bounded for all time and must therefore blow up.

Application of H. Levine’s method to nonlinear parabolic equations :

Consider the abstract parabolic problem:

$$\begin{cases} \frac{du}{dt} = -\Lambda u + \Gamma(u(t)), & t \in [0, T), \\ u(0) = u_0, \end{cases} \tag{1.7}$$

where Λ represents a positive linear operator defined within a subdomain D of a real or Hilbert space, and $\Gamma : D \rightarrow H$ (H is a real Hilbert space), and the nonlinear term $\Gamma(u)$ complies with the following conditions:

- for every $x \in D$, the function $\Gamma(x)$ possesses a symmetric Fréchet derivative denoted by Γ_x , and the mapping $x \rightarrow \Gamma_x$ is strongly continuous;
- the scalar $\Psi : D \rightarrow \mathbb{R}$ defined by the formula:

$$\Psi(x) = \int_0^1 (\Gamma(\rho x), x) d\rho, \tag{1.8}$$

for $x \in D$ satisfies the subsequent inequalities for a given constant $\alpha > 0$:

$$\begin{aligned} 2(\alpha + 1)\Psi(x) &\leq (x, \Gamma(x)), \\ \Psi(u_0) &> \frac{1}{2}(u_0, \Lambda(0)u_0). \end{aligned} \tag{1.9}$$

Theorem 1.13

Under these assumptions and that $u : [0, T) \rightarrow D$ is a strictly continuous and differentiable function with respect to the norm of D , and is a nonextendable solution of problem (1.3). For any t , let the above conditions for $\Gamma : D \rightarrow H$ hold, and let the symmetric operator Λ fulfill the inequalities:

1. $(x, \Lambda(t)x) \geq 0$ for all $x \in D$.
2. $(x, \Lambda(t)x) \leq 0$ for all $x \in D$.

Then, the lifetime T of the solution is bounded:

$$\lim_{t \rightarrow T^-} \int_0^t (u(\tau), u(\tau)) d\tau = +\infty, \quad \limsup_{t \rightarrow T^-} (u(t), u(t)) = +\infty,$$

As a result of our analysis, we derive the following upper bound:

$$T \leq \frac{(2\alpha + 1)(u_0, u_0)}{2\alpha^2(\alpha + 1)} \left[\left((\Gamma u_0, u_0) - \frac{1}{2}(u_0, \Lambda(0)u_0) \right) \right]^{-1}. \quad (1.10)$$

Remark 1.6

The straightforward equation of the type $u_t = \Delta u + u_x + u^2$, is well-suited for the application of Levine's energy method. This approach facilitates the analysis of the energy dynamics of solutions, helping to uncover stability properties and conditions leading to blow-up, there by enriching our understanding of nonlinear evolution equations. However, the blow-up result can also be easily derived using alternative methods. This is due to the fact that the operator $\Lambda = -\frac{d^2}{dx^2} - \frac{d}{dx}$ is not symmetric.

In the subsequent sections, we will detail the mathematical formulation of the problem, the assumptions made, and the key results obtained from our analysis. This focused examination will serve as a foundation for exploring more complex systems in future research, ultimately enhancing our understanding of ignition dynamics in solid fuels:

$$\begin{cases} u_t - \Delta u = |u|^{p-1} u & \text{in } \Omega_T = \Omega \times (0, T), \\ u(x, 0) = \varphi(x) & \text{for } x \in \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0, T). \end{cases} \quad (\text{P})$$

The domain Ω can be either bounded or unbounded, $p > 1$, and φ is a known function.

Theorem 1.14

Let Ω be a smooth domain, φ be the initial condition of equation (P). Assume that the following conditions hold:

1. **Finite-Time Blow-Up:** If, for all $t > 0$,

$$-\frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 dx + \frac{1}{p+1} \int_{\Omega} |\varphi|^{p+1} dx > 0,$$

then the solution of equation (P) must blow up in finite time. This result highlights a key aspect of the system's dynamics, with implications for the underlying mathematical model and its real-world applications.

2. **Global Existence:** If the solution φ of equation (P) is global and satisfies

$$-\frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 dx + \frac{1}{p+1} \int_{\Omega} |\varphi|^{p+1} dx < 0, \quad \text{for all } t > 0,$$

then the solution remains globally bounded in time.

Proof Initially, we construct the energy functional by multiplying the equation by u_t and subsequently integrating over the domain Ω . This leads to the following equation:

$$\int_{\Omega} u_t^2 dx + \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} |\nabla u|^2 dx \right) = \frac{1}{p+1} \frac{d}{dt} \left(\int_{\Omega} |u|^{p+1} dx \right). \quad (1.11)$$

Then, the appropriate energy functional $J(t)$ is a typical choice for analyzing the comprehensive analysis of the behavior of solutions to the evolution equation

$$J(t) = -\frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx.$$

From equation (1.11), we get:

$$\frac{d}{dt} J(t) = \int_{\Omega} u_t^2 dx \geq 0.$$

Integrating the latter $(0, T)$ implies

$$J(t) = J(0) + \int_0^t \int_{\Omega} u_t^2 dx dt. \quad (1.12)$$

Chapter 1. Preliminaries

Now, construct a concave functional that is related to the solution of the equation in (P) :

$$\Upsilon(t) = \int_0^t \int_{\Omega} u^2 dx dt + A, \quad (1.13)$$

Where A is a positive integer to be determined. Next, by differentiating (1.13) , we get

$$\Upsilon'(t) = \int_{\Omega} u^2 dx. \quad (1.14)$$

Therefore,

$$\Upsilon''(t) = \frac{d}{dt} \int_{\Omega} u^2 dx.$$

From the procedure of multiplying the equation in (P) by u , and then integrating the resulting expression on the domain Ω , we can derive important energy estimates that provide insights into the behavior of the solution. Here is how this process typically unfolds:

$$\Upsilon''(t) = \frac{d}{dt} \int_{\Omega} u^2 dx = -2 \int_{\Omega} |\nabla u|^2 dx + 2 \int_{\Omega} |u|^{p+1} dx. \quad (1.15)$$

Upon examining the terms present in $J(t)$ and $\Upsilon''(t)$, we can deduce the following relationships from the procedure outlined above:

$$\begin{aligned} \Upsilon''(t) &\geq 2 \left(-(1+p) \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |u|^{p+1} dx \right), \\ &\geq 4(1+\beta) \left(J(0) + \int_0^t \int_{\Omega} u_t^2 dx dt \right). \end{aligned}$$

where the last equality is from (1.12) . Clearly,

$$\Upsilon'(t) = \int_{\Omega} u^2 dx = 2 \int_0^t \int_{\Omega} u_t u dx dt + \int_{\Omega} |\varphi|^2 dx. \quad (1.16)$$

It follows that, for any $\delta > 0$, we can establish the following results regarding the

Chapter 1. Preliminaries

behavior of the solution u :

$$\begin{aligned}
 \Upsilon'(t)^2 &= \left(2 \int_0^t \int_{\Omega} u_t u dx dt + \int_{\Omega} |\varphi|^2 dx \right)^2, \\
 &\leq 4 \left(\frac{\delta}{2} \left(\int_0^t \int_{\Omega} u_t u dx dt \right)^2 + \frac{1}{2\delta} \left(\int_{\Omega} |\varphi|^2 dx \right)^2 \right), \\
 &\leq 2\delta \left(\int_0^t \int_{\Omega} u_t^2 dx dt \int_0^t \int_{\Omega} u^2 dx dt \right) + \frac{2}{\delta} \left(\int_{\Omega} |\varphi|^2 dx \right)^2 \quad (1.17)
 \end{aligned}$$

Combining the above estimates, we find that for $\delta > 0$,

$$\begin{aligned}
 \Upsilon''(t)\Upsilon(t) - \alpha\Upsilon'(t)^2 &\geq 2(1+p) \left(J(0) + \int_0^t \int_{\Omega} u_t^2 dx dt \right) \left(\int_0^t \int_{\Omega} u^2 dx dt + A \right) \\
 &\quad - \alpha \left[2\delta \left(\int_0^t \int_{\Omega} u_t^2 dx dt \int_0^t \int_{\Omega} u^2 dx dt \right) + \frac{2}{\delta} \left(\int_{\Omega} |\varphi|^2 dx \right)^2 \right].
 \end{aligned}$$

we have $2(1+p) - 2\alpha\delta > 0$, $J(0) > 0$. and we can choose A to be large enough so that

$$\Upsilon''(t)\Upsilon(t) - \alpha\Upsilon'(t)^2 > 0. \quad (1.18)$$

From (1.18), it follows

$$\frac{d}{dt} \left(\frac{\Upsilon'(t)}{\Upsilon^\alpha(t)} \right) > 0,$$

so that

$$\forall t > 0, \quad \frac{\Upsilon'(t)}{\Upsilon^\alpha(t)} > \frac{\Upsilon'(0)}{\Upsilon^\alpha(0)}.$$

It can be concluded that the expression $\Upsilon(t) = \int_0^t \int_{\Omega} u^2 dx dt + A$ cannot remain finite for all values of t . ■

Application of H. Levine's method to nonlinear hyperbolic equations

Todorova's energy method

The energy method proposed by G. Todorova in 1994 contributes to the field of mathematical physics by providing a specialized technique for analyzing nonlinear PDEs that contain the term $|u_t|$, in addition to a nonlinear source term.

Chapter 1. Preliminaries

Consider the following a nonlinear hyperbolic problem with linear damping u_t :

$$\begin{cases} u_{tt} + au_t = Pu + F(u), \\ u(0) = \varphi(x), \\ u_t(0) = \psi(x). \end{cases} \quad (1.19)$$

The symmetry of the operator $P(t) : D \rightarrow H$ plays a crucial role in ensuring that $(x, P(t)x) \geq 0$ for all $x \in D$. If $v : [0, \infty) \rightarrow H$ is a strictly continuous differentiable mapping and both $v(t)$ and $\frac{dv}{dt}$ belong to D for all $t \geq 0$, then the function $(v(t), P(t)v(t))$ is continuously differentiable, and the following inequality holds:

$$Q(v, v) \equiv \frac{d}{dt}(v, Pv) - 2\left(\frac{dv}{dt}, Pv\right) \leq 0.$$

Assume that the operator $P(t) = P \geq 0$ is time-independent allows for a clearer analysis of the solution behavior.

Applying Levine's method, we derive sufficient conditions for blow-up in the initial-boundary-value problem.

Theorem 1.15

Assume that F satisfies all the conditions in the last section. Let us treat the nonextendable solutions in time $u : [0, T) \rightarrow H$ of the problem (1.19). Moreover, assume that the function u_0 satisfies the inequality:

$$G(\varphi) > \frac{1}{2}\left((\varphi, P\varphi) + \left(\frac{a}{2\alpha}\right)^2(\varphi, \varphi)\right). \quad (1.20)$$

Then, for all initial functions $\psi \in D$ satisfying the inequalities:

$$\|\psi\| \leq r(\varphi), \|\psi\| \cos \theta > \frac{a\|\varphi\|}{2\theta},$$

where

$$\cos \theta = \frac{(\varphi, \psi)}{\|\varphi\| \|\psi\|}, r(\varphi) = \sqrt{2G(\varphi) - (\varphi, A\varphi)},$$

the associated solution experiences blow-up in a limited time frame:

$$T \leq \frac{1}{a} \ln \left(\frac{2\alpha\|\psi\| \cos \theta}{2\alpha\|\psi\| \cos \theta - a\|\varphi\|} \right).$$

Proof Let $\phi(t) = (u, u)$. By differentiation this function twice by time, we obtain

Chapter 1. Preliminaries

the:

first derivative:

$$\phi'(t) = 2(u, u_t), \quad (1.21)$$

and the second derivative:

$$\phi''(t) = 2(u, u_{tt}) + 2(u_t, u_t). \quad (1.22)$$

The application of the Cauchy–Bunyakovsky inequality yields:

$$(u, u_t)^2 \leq (u_t, u_t)(u, u),$$

i.e.

$$0 \leq (u_t, u_t)(u, u) - (u, u_t)^2,$$

This can be represented in the following form:

$$G = (u_t, u_t)(u, u) - (u, u_t)^2,$$

$$4(\alpha + 1)G = 2\phi(2\alpha + 2)(u_t, u_t) - (\alpha + 1)(\phi')^2 \geq 0.$$

It is now easy to prove the estimate

$$\phi\phi'' - (\alpha + 1)(\phi')^2 \geq 4(\alpha + 1)G + 2\phi((u, u_{tt}) - (2\alpha + 1)(u_t, u_t)). \quad (1.23)$$

In the next step, let us introduce the function:

$$X(t) \equiv (u, u_{tt}) - (2\alpha + 1)(u_t, u_t) \quad (1.24)$$

and examine the last term of (1.23) separately. Differentiating $X(t)$ with respect to time gives

$$\begin{aligned} X'(t) &= a(4\alpha + 2)(u_t, u_t) - (4\alpha + 2)(u_t, F(u)) \\ &\quad + \frac{d}{dt}[2\alpha(u, Au) - a(u, u_t) + (u, F(u))]. \end{aligned} \quad (1.25)$$

By integrating (1.25) from 0 to t and applying the properties of P and C along

with the definition of X , it follows:

$$\begin{aligned} X(t) &\geq -a(u, u_t) + 2(2\alpha + 1)C(\varphi) - (2\alpha + 1)((\varphi, P\varphi) + (\psi, \psi)), \\ &\geq -a\frac{\phi'(t)}{2}. \end{aligned} \tag{1.26}$$

In the previous inequality, we have applied condition (1.20). Substituting (1.26) into (1.23), we derive the ordinary differential inequality

$$\phi\phi'' - (\alpha + 1)(\phi')^2 \geq -a\phi\phi' \Rightarrow (\phi^{-\alpha})'' + a(\phi^{-\alpha})' \leq 0.$$

Integrating this inequality twice by time, we arrive at the lower estimate:

$$\phi^\alpha \geq \frac{\phi^\alpha(0)}{1 - \alpha(1 - e^{-at})\frac{\phi'(0)}{\phi(0)}}. \tag{1.27}$$

Under the condition

$$\alpha\frac{\phi'(0)}{\phi(0)} > 1.$$

the denominator in (1.27) becomes zero at a specific moment in time; say $t = t_0$. Consequently, we have established sufficient conditions for the blow-up of a solution within a finite time, $\lim_{t \rightarrow T} \|u(t)\| = +\infty$, and the upper estimate of the lifetime:

$$T \leq t_0 = \frac{1}{a} \ln \left(\frac{2\alpha(\varphi, \psi)}{2\alpha(\varphi, \psi) - a\|\varphi\|} \right).$$

The theorem is fully proven. ■

1.7 Decay analysis for nonlinear parabolic and hyperbolic equations

Decay analysis is an important topic in the study of PDEs, particularly for nonlinear parabolic and hyperbolic equations. This behavior is often characterized by the concept of asymptotic stability, where the solution converges to a stable steady state as time goes to infinity. The decay rate and the long-time behavior of solutions are influenced by factors such as the nonlinearity, the dissipation mechanisms, and the initial and boundary conditions.

The mathematical analysis of decay for nonlinear parabolic and hyperbolic equations

involves a wide range of techniques, including: energy methods, functional analytic techniques, spectral theory, and others.

Definition 1.4

Decay means that the solution $u(x, t)$ approaches the zero function as $t \rightarrow \infty$.

Definition 1.5

Exponential decay dominates the long-term behavior whose property is a strong stability and a convergence result for the nonlinear evolution equation, and it has important implications for the long-term behavior of the solution.

1.7.1 Energy methods

The method of energy is a powerful technique used to establish the exponential decay. The core strategy for achieving exponential decay estimates in the context of nonlinear parabolic equations consists in demonstrating that the problem is dissipative. This is primarily done by confirming that the diffusion term is coercive and that the reaction term contributes through a significant dissipation. This fact, combined with Poincaré’s inequality, allows to control the functional energy $E(t)$ and derive the exponential decay estimate.

The exponential decay for an parabolic equation with Dirichlet conditions and parabolic equation with an integral condition.

Consider the nonlinear heat equation in the spatial domain $\Omega = (0, 1)$:

$$\begin{cases} u_t - \Delta u = u^p & \forall x, t \in Q_T, \\ u(x, 0) = \varphi(x) & \forall x \in \Omega, \\ u(0, t) = u(1, t) = 0 & \forall t \in [0, T]. \end{cases} \quad (1.28)$$

Theorem 1.16

Let $T > 0$, $\Omega = (0, 1)$ and $E(0)$, λ positive constants. Then, the problem (P_{m1}) admits a solution $u(x, t)$ satisfying the following exponential decay estimate:

$$E(t) = E(0)e^{-\lambda t}.$$

Proof Step 01: Construct the energy functional:

Chapter 1. Preliminaries

$$\begin{aligned} \int_{\Omega} u_t u - \int_{\Omega} \Delta u u &= \int_{\Omega} u^{p+1}, \\ \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega} u^2 \right) &= - \int_{\Omega} \nabla u^2 + \int_{\Omega} u^{p+1}, \end{aligned} \quad (1.29)$$

By using the Gagliardo-Nirenberg interpolation inequality, for any $\theta \in (0, 1)$, we have

$$\begin{aligned} \|u\|_{p+1}^{p+1} &\leq C \|\nabla u\|_2^{(p+1)\theta} \|u\|_2^{(p+1)(1-\theta)}, \\ &\leq 2^{\frac{p-1}{2}} C (M(t))^{\frac{p-1}{2}} \|\nabla u\|_2^2. \end{aligned} \quad (1.30)$$

Step 02: On the other hand, by using the poincaré inequality on (1.29), we get

$$\begin{aligned} \frac{d}{dt} M(t) &= - \int_{\Omega} \nabla u^2 + \int_{\Omega} u^{p+1}, \\ &\leq -M(t) \left(C \frac{2}{c_{\Omega}} - (M(t))^{\frac{p-1}{2}} \right). \end{aligned}$$

This integral seems to be challenging to solve explicitly for a general p , but we can analyze the behavior near $M(t) = 0$ to understand the exponential decay.

Step 03: By assuming that $M(t)$ is small, $(M(t))^{\frac{p-1}{2}}$ will be much smaller than $C \frac{2}{c_{\Omega}}$, so that we can use the approximation:

$$\frac{d}{dt} M(t) \approx -C \frac{2}{c_{\Omega}} M(t),$$

By setting $\lambda = C \frac{2}{c_{\Omega}}$, we can solve the simplified differential equation:

$$\frac{d}{dt} M(t) \approx -\lambda M(t).$$

Thus, we conclude that the solution to this equation is given by

$$M(t) = M(0)e^{-\lambda t}.$$

where $M(0)$ is the initial energy. ■

Now, we present another study of the same problem but with a second-kind integral condition.

Let $Q_T = \{(x, t) \in \Omega \times (0, T) \mid \Omega = (0, l)\}$. Consider the nonlinear heat problem

with an integral boundary condition:

$$\begin{cases} u_t - \Delta u = u^p & \forall x, t \in Q_T, \\ u(x, 0) = \varphi(x) & \forall x \in \Omega, \\ u(0, t) = 0 & \forall t \in [0, T], \\ u(l, t) = \int_{\Omega} k(x, t)u(x, t)dx & \forall t \in [0, T]. \end{cases} \quad (1.31)$$

where k is a bounded function, φ a known function, and $p \geq 1$.

Theorem 1.17

Let $\Omega = (0, l)$ and $E(0)$, λ a positive constants. Then, the problem (1.31) admits a solution $u(x, t)$ satisfying the following exponential decay estimate:

$$E(t) = E(0)e^{-\lambda t}.$$

Proof First, let us define

$$E(t) = \frac{1}{2} \int_{\Omega} u^2,$$

by constructing the energy functional with the use of

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \int_{\Omega} u^2 \right) &= - \int_{\Omega} \nabla u^2 + \int_{\Omega} u^{p+1} + \left(\int_{\Omega} k u dx \right) \left(\int_{\Omega} u \right), \\ &\leq - \int_{\Omega} \nabla u^2 + C (E(t))^{\frac{p-1}{2}} \|\nabla u\|_2^2 + \frac{\varepsilon}{2} k_{\max} \left(\int_{\Omega} u \right)^2 + \frac{1}{2\varepsilon} \left(\int_{\Omega} u \right)^2, \\ &\leq - \int_{\Omega} \nabla u^2 + C (E(t))^{\frac{p-1}{2}} \|\nabla u\|_2^2 + l^2 \left(\frac{\varepsilon}{2} k_{\max} + \frac{1}{2\varepsilon} \right) \int_{\Omega} \nabla u^2. \end{aligned} \quad (1.32)$$

So, from (1.32), we conclude that

$$\begin{aligned} \frac{dE(t)}{dt} &\leq \left(-1 + l^2 \left(\frac{\varepsilon}{2} k_{\max} + \frac{1}{2\varepsilon} \right) \right) \int_{\Omega} \nabla u^2 + C (E(t))^{\frac{p-1}{2}} \|\nabla u\|_2^2 \\ &\leq - \|\nabla u\|_2^2 \left(\left(1 - \left(\frac{\varepsilon}{2} k_{\max} + \frac{1}{2\varepsilon} \right) \right) - C (E(t))^{\frac{p-1}{2}} \right), \\ &\leq \frac{-1}{l} \|u\|_2^2 \left(\left(1 - \left(\frac{\varepsilon}{2} k_{\max} + \frac{1}{2\varepsilon} \right) \right) - C (E(t))^{\frac{p-1}{2}} \right), \\ &\leq \frac{-2}{l} E(t) \left(\left(1 - \left(\frac{\varepsilon}{2} k_{\max} + \frac{1}{2\varepsilon} \right) \right) - C (E(t))^{\frac{p-1}{2}} \right). \end{aligned}$$

This integral can be challenging to solve explicitly for a general p , but we can analyze the behavior near $E(t) = 0$ to understand the exponential decay. By assuming

Chapter 1. Preliminaries

that $E(t)$ is small, $(E(t))^{\frac{p-1}{2}}$ will be much smaller than $\left(\frac{-2}{l} \left(1 - \left(\frac{\varepsilon}{2}k_{\max} + \frac{1}{2\varepsilon}\right)\right)\right)$, so that we can employ the approximation:

$$\frac{d}{dt}E(t) \approx -\lambda E(t),$$

where $\lambda = \frac{2}{l} \left(1 - \left(\frac{\varepsilon}{2}k_{\max} + \frac{1}{2\varepsilon}\right)\right)$. Therefore, we can solve the simplified differential equation:

$$\frac{d}{dt}E(t) \approx -\lambda E(t),$$

Thus, the solution to this equation is

$$E(t) = E(0)e^{-\lambda t}.$$

where $E(0)$ is the initial energy. ■

Exponential decay for a hyperbolic equation with Dirichlet conditions and a hyperbolic equation with an integral condition

This section comprises two parts. First, we present the proof of the exponential decay for a hyperbolic equation with a Dirichlet boundary conditions. Then, we consider a more complex hyperbolic problem with an integral boundary condition to study the exponential decay results under different assumptions in combination with the above method. We start by considering the hyperbolic equation with a Dirichlet boundary condition in a bounded domain $\Omega = (0, l)$, $l > 0$:

$$\begin{cases} u_{tt} - a\Delta u + bu_t = u^p & \forall x, t \in Q_T, \\ u(x, 0) = \varphi(x) & \forall x \in \Omega, \\ u_t(x, 0) = \psi(x) & \forall x \in \Omega, \\ u(0, t) = u(l, t) = 0 & \forall t \in [0, T]. \end{cases} \quad (1.33)$$

Where the parameters $a > 0$ and $p > 1$, are given with the functions

φ and ψ , we note that the term involving b signifies the damping effect present in the system.

Parameter Significance: The condition $a > 0$ typically indicates a stabilizing influence within the system, ensuring that any growth or oscillatory behavior is moderated.

The parameter $p > 1$ suggests a nonlinear response, which can lead to complex dynamics that depend on the magnitude of the state variables.

Chapter 1. Preliminaries

Functions φ and ψ are predefined and can represent various aspects of the system, such as initial conditions, external forces, or specific characteristics that influence the behavior of the solutions. Their forms are critical, as they can introduce nonlinearity or time dependency, affecting the overall dynamics.

Damping Effect: The damping term, represented by b , plays a vital role in the system's stability. It works to counteract growth and oscillations, providing a corrective mechanism that leads to a more stable and controlled evolution of the system. This effect is particularly important in physical systems where resistance or friction acts to bring motion to equilibrium.

Theorem 1.18

Let $\Omega = (0, l)$, $a > 0$, $p > 1$, $\frac{\varepsilon}{2} > b$, $\forall \varepsilon > 0$, and φ, ψ are given functions. Then, the energy functional is given by

$$E(t) = \frac{1}{2} \|u_t\|_2^2 + \frac{a}{2} \|\nabla u\|_2^2.$$

Moreover, there exist positive constants $E(0)$ and λ (depending on the problem parameters) such that the solution $u(x, t)$ satisfies the following exponential decay estimate:

$$E(t) = E(0)e^{-(2b + \max(2b, \varepsilon))t}$$

Proof As usual, we first start by the energy functional defined by multiplying the main equation of the problem (1.33) on u_t , and then integrating the obtained equation over Ω . We get

$$\begin{aligned} \int_{\Omega} u_{tt} u_t dx - a \int_{\Omega} \Delta u u_t dx &= \int_{\Omega} u^p u_t dx, \\ &\leq \frac{1}{2} \int_{\Omega} u^{2p} dx + \left(-b + \frac{1}{2}\right) \int_{\Omega} u_t^2 dx. \end{aligned} \quad (1.34)$$

Next, by using the inequality:

$$\|u\|_{2p}^{2p} \leq C \|\nabla u\|_2^{2p\theta} \|u\|_2^{2p(1-\theta)}.$$

for $\theta = \frac{1}{p}$, we obtain

$$\|u\|_{2p}^{2p} \leq C \|\nabla u\|_2^2 \|u\|_2^{2(p-1)}.$$

Chapter 1. Preliminaries

Applying the Poincaré inequality gives

$$\begin{aligned} \|u\|_{2p}^{2p} &\leq CC_\Omega \|\nabla u\|_2^2 \|\nabla u\|_2^{2(p-1)}, \\ &\leq C' \|\nabla u\|_2^{2p} \leq \left(\frac{2}{a}\right)^p C'(M(t))^p \end{aligned} \quad (1.35)$$

where $C' = \frac{C'a}{2}$, and $M(t) = \frac{1}{2} \|u_t\|_2^2 + \frac{a}{2} \|\nabla u\|_2^2$. By combining (1.34) and (1.35), it follows that

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} \|u_t\|_2^2 + \frac{a}{2} \|\nabla u\|_2^2 \right] &\leq \left(\frac{2}{a}\right)^p \frac{C'}{\varepsilon} (M(t))^p + \left(-b + \frac{\varepsilon}{2}\right) \int_\Omega u_t^2 dx, \\ \frac{d}{dt} M(t) &\leq \left(\frac{2}{a}\right)^p \frac{C'}{\varepsilon} (M(t))^p + \left(-b + \frac{1}{2}\right) M(t) + \frac{2}{a} \left(-b + \frac{1}{2}\right) M(t), \\ &\leq \left(\frac{2}{a}\right)^p \frac{C'}{\varepsilon} E(t)^p + (-2b + \max(2b, \varepsilon)) E(t), \\ &\leq -M(t) \left((-2b + \max(2b, \varepsilon)) - \left(\frac{2}{a}\right)^p \frac{C'}{\varepsilon} M(t)^{p-1} \right) \end{aligned}$$

For $\frac{\varepsilon}{2} > b, \forall \varepsilon > 0$ assuming that $(-2b + \max(2b, \varepsilon)) - \left(\frac{2}{a}\right)^p \frac{C'}{\varepsilon} M(t)^{p-1} > 0$, we have

$$\left(\frac{-2b + \max(2b, \varepsilon)}{\left(\frac{2}{a}\right)^p \frac{C'}{\varepsilon}} \right)^{\frac{1}{p-1}} > M(t).$$

We observe that if $M(t)$ is small, then $M(t)^{p-1}$ also small since

$$\frac{C'}{2} M(t)^p + \left(1 + \frac{2}{a}\right) \left(-b + \frac{1}{2}\right) M(t)$$

To ensure the exponential decay, we need the term

$$\frac{dM(t)}{dt} \approx -(-2b + \max(2b, \varepsilon)) M(t).$$

So, we get the solution to this equation as follows:

$$E(t) = E(0)e^{-(2b + \max(2b, \varepsilon))t},$$

where $E(0)$ is the initial energy. ■

Now, we must highlight a special problem, Consider the following non-linear hyperbolic initial-boundary value problem. For any $T > 0$ and $\Omega_T = \Omega \times (0, T)$,

Chapter 1. Preliminaries

find $u = u(x, t)$ satisfying the PDE:

$$u_{tt} - a\Delta u + bu_t = u^p, \quad (1.36)$$

with the prescribed initial condition:

$$\begin{cases} u(x, 0) = \varphi(x) & \forall x \in \Omega, \\ u_t(x, 0) = \psi(x) & \forall x \in \Omega. \end{cases} \quad (1.37)$$

This problem involves both a homogeneous Dirichlet condition and an integral boundary condition:

$$\begin{aligned} u(0, t) &= 0 & \forall t > 0, \\ u_x(1, t) &= \int_{\Omega} k(x, t)u(x, t)dx & \forall t > 0. \end{aligned} \quad (1.38)$$

Let us show that the energy of this problem decays exponentially over time. This approach can be used to establish the existence and uniqueness of a solution, as well as to study the long-time behavior of the solution.

Theorem 1.19

Let $\Omega = (0, l)$ be the domain under consideration. Assume that the following conditions hold: $\theta \max(\varepsilon, 1) < \frac{b}{2} < k, \forall \varepsilon, \theta > 0$, and φ, ψ are given function, Then, the energy functional associated can be expressed as:

$$E(t) = \frac{1}{2} \|u_t\|_2^2 + \frac{a}{2} \|\nabla u\|_2^2.$$

Furthermore, there exist positive constants $E(0)$ and λ , which depend on the problem parameters, such that the solution $u(x, t)$ obeys the following exponential decay estimate:

$$E(t) = E(0)e^{-(b-\theta \max(\varepsilon, 1))t}.$$

Proof By multiplying of equation (1.36) with u_t , we arrive at the following outcome:

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} \int_{\Omega} u_t^2 dx + \frac{a}{2} \int_{\Omega} u_x^2 dx \right] &= -b \int_{\Omega} u_t^2 dx + a \int_{\Omega} k u dx u(1, t) + \int_{\Omega} u^p u_t dx. \\ &\leq -b \int_{\Omega} u_t^2 dx + \frac{ak^2}{2\varepsilon} \int_{\Omega} u^2 dx + \frac{a\varepsilon}{2} \int_{\Omega} u_x^2 dx \\ &\quad + \frac{1}{2\theta} \int_{\Omega} u^{2p} dx + \frac{\theta}{2} \int_{\Omega} u_t^2 dx. \end{aligned}$$

Using the Gagliardo-Nirenberg interpolation inequality implies

$$\begin{aligned} & \frac{d}{dt} \left[\frac{1}{2} \int_{\Omega} u_t^2 dx + \frac{a}{2} \int_{\Omega} u_x^2 dx \right] \\ & \leq \frac{C}{2b} E(t)^p - bE(t) + \left(\frac{k^2}{\varepsilon} + \varepsilon + b \right) \frac{a}{2} \int_{\Omega} u_x^2 dx + \frac{\theta}{2} \int_{\Omega} u_t^2 dx. \end{aligned}$$

By assuming that $\theta = \frac{k^2}{\varepsilon} + \varepsilon + b$, we get, $\theta\varepsilon = \varepsilon^2 + \varepsilon b + k^2$ for $\frac{b}{2} < k$. Therefore,

$$\frac{dE(t)}{dt} \leq \frac{C}{2b} E(t)^p + (-b + \theta \max(\varepsilon, 1)) E(t),$$

and for $\theta \max(\varepsilon, 1) < b$, we obtain

$$\frac{dE(t)}{dt} \leq -E(t) \left((b - \theta \max(\varepsilon, 1)) - \frac{C}{2b} E(t)^{p-1} \right),$$

Subsequently, by assuming that $E(t)$ is small, $(E(t))^{p-1}$ will be much smaller than $(b - \theta \max(\varepsilon, 1))$, and we can use the approximation:

$$\frac{d}{dt} E(t) \approx -\lambda E(t),$$

where $\lambda = b - \theta \max(\varepsilon, 1)$. Then, we can solve the simplified differential equation:

$$\frac{d}{dt} E(t) \approx -\lambda E(t),$$

whose solution is:

$$E(t) = E(0)e^{-\lambda t}.$$

where $E(0)$ is the initial energy. ■

1.8 Fractional calculus preliminaries

Fractional calculus provides powerful tools for modeling and analyzing complex systems that cannot be adequately described using traditional integer-order approaches. By extending the concepts of derivatives and integrals to fractional orders, it opens new avenues for research and application across a range of scientific and engineering disciplines. Fractional calculus is a branch of mathematical analysis that extends the concept of derivatives and integrals to non-integer (fractional) orders. It provides a framework for modeling various phenomena in nature that

Chapter 1. Preliminaries

are better described by fractional rather than classical derivatives. In the history of fractional calculus, as early as 1695, L'Hôpital posed the question regarding the interpretation of $\frac{d^n}{dx^n}$ if $n = \frac{1}{2}$, essentially asking, that is "What if n is fractional?". Leibniz responded by stating: "This represents an apparent paradox from which valuable insights will eventually arise," adding that " $d^{\frac{1}{2}}$ will equate to $x\sqrt{x} : x$ ". S.F.Lacroix was the first to mention in some two pages a derivative of arbitrary order in his 700 page textbook of 1819. Thus, for $y = x^\alpha$, $\alpha \in \mathbb{R}_+$, he demonstrated that

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + \frac{1}{2})}x^{\alpha - \frac{1}{2}}.$$

In particular, he had $\left(\frac{d}{dx}\right)^{\frac{1}{2}}x = 2\sqrt{\frac{x}{\pi}}$ (the same results as by the present day Riemann-Liouville definition below). Fractional calculus is a rich field with numerous definitions and properties that extend traditional calculus concepts to non-integer orders. Its applications across various disciplines highlight its importance and versatility in solving complex problems in science and engineering. Understanding the foundational definitions and properties of fractional derivatives is crucial for exploring further advancements in this area. Fractional calculus is a very productive field of mathematics having many applications in various fields of science and engineering. In this section, several common definitions and properties of fractional derivatives will be introduced.

Definition 1.6: (Gamma function) :

The function $\Gamma(z)$, known as the Euler gamma function, is expressed by the Euler integral of the second kind as follows:

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t) dt \quad (R(z) > 0),$$

Remark 1.7

- $\Gamma(z + 1) = z\Gamma(z)$.
- $\Gamma(n + 1) = n!$ ($n \in \mathbb{N}^*$) as usual $0! = 1$.
- $\Gamma(n) = (n - 1)!$.

Definition 1.7

For any $0 < \alpha < 1$, the Caputo and Riemann-Liouville derivatives are defined as follows:

- The left Caputo derivative:

$${}_0^C D_t^\alpha \mu(x, t) := \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{\partial^n \mu(x, \tau)}{\partial \tau^n} (t - \tau)^{-\alpha+n-1} d\tau.$$

- The right Caputo derivative:

$${}_t^C D_T^\alpha \mu(x, t) := \frac{(-1)^n}{\Gamma(n - \alpha)} \int_t^T \frac{\partial^n \mu(x, \tau)}{\partial \tau^n} (\tau - t)^{n-\alpha-1} d\tau.$$

- The left Riemann-Liouville derivative :

$${}_0^R D_t^\alpha \mu(x, t) := \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \mu(x, \tau) (t - \tau)^{n-\alpha-1} d\tau.$$

- The right Riemann-Liouville derivatives :

$${}_t^R D_T^\alpha \mu(x, t) := \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_t^T \mu(x, \tau) (\tau - t)^{n-\alpha-1} d\tau.$$

Here, $n = [\alpha] + 1$ for $\alpha \in \mathbb{N}_+$, $\delta \neq n + \frac{1}{2}$,

Definition 1.8

The semi-norms $|\cdot|_{l_{H^\delta(\Omega)}}$, $|\cdot|_{r_{H^\delta(\Omega)}}$, and $|\cdot|_{c_{H^\delta(\Omega)}}$ are said to be equivalent. This equivalence is denoted by

$$|\cdot|_{l_{H^\delta(\Omega)}} \cong |\cdot|_{r_{H^\delta(\Omega)}} \cong |\cdot|_{c_{H^\delta(\Omega)}}. \quad (1.39)$$

Lemma 1.7

For every real $\delta > 0$, the space ${}^R H_0^\delta(\Omega)$ with respect to the norm (1.39) is complete.

1.9 Numerical simulation tools

Simulation tools are software applications or platforms designed to model, analyze, and visualize complex systems across various fields, including engineering, physics, finance, biology, and social sciences. These tools enable users to create simulations that replicate real-world processes, allowing for experimentation and analysis without the constraints of physical setups.

1.9.1 Interpolation approximations of the fractional Caputo derivatives

Interpolation approximations of fractional Caputo derivatives are crucial for numerically solving fractional differential equations (FDEs). Given the complexities associated with fractional calculus, these methods help in estimating the fractional derivatives when analytical solutions are difficult or impossible to obtain.

Theorem 1.20

The L^1 -method for the Caputo derivative is given by

$${}_0^C D_t^\alpha f(t_n) = C \sum_{k=0}^{n-1} (f(t_{k+1}) - f(t_k)) \left[(n-k-1)^{1-\alpha} - (n-k)^{1-\alpha} \right] + O(\alpha t^{2-\alpha}).$$

where $C = \frac{\Delta t^{-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}$.

The L^2 -method for the Caputo derivative is given by:

$${}_0^C D_t^\alpha f(t_n) = \sum_{k=-1}^n C_k f(t_{n-k}) + O(\alpha t^{3-\alpha}),$$

where

$$C_k = A \begin{cases} 1, & k = -1 \\ 2^{2-\alpha} - 3, & k = 0 \\ (k+2)^{-\alpha+2} - 3(k+1)^{-\alpha+2} \\ \quad + 3k^{-\alpha+2} - (k-1)^{-\alpha+2}, & 1 \leq k \leq n-2 \\ -(n-2)^{-\alpha+2} + 3(n-1)^{-\alpha+2} - 2n^{-\alpha+2}, & k = n-1 \\ n^{-\alpha+2} - (n-1)^{-\alpha+2}, & k = n. \end{cases}$$

where $A = \frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)}$.

L^1 -approximation

The analysis of numerical schemes based on the L^1 -approximation has been an active area of research in the field of fractional calculus. The L^1 -approximation is one of the popular ways to discretize the Caputo fractional derivative of order α ($0 < \alpha < 1$).

For any positive integer n , let us set: $\Delta t = \frac{T}{n}$, $t_k = k\Delta t$, $0 \leq k \leq n$ and $a_l^{(\alpha)} = (l+1)^{1-\alpha} - l^{1-\alpha}$, $l \geq 0$.

The Caputo fractional derivative at $t = t_n$ can be written as

$$\begin{aligned} {}_0^C D_t^\alpha f(t_n) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{f'(t)}{(t_n-t)^\alpha} dt, \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{f'(t)}{(t_n-t)^\alpha} dt. \end{aligned}$$

with $[t_k, t_{k+1}] \subset [0, t_n]$, $\forall k \in \overline{0, n}$. We have the following approximation of $f'(t)$

$$\begin{aligned} f'(t) &= \frac{f(t_k + \Delta t) - f(t_k)}{\Delta t} + o(\Delta t), \\ &\simeq \frac{f(t_{k+1}) - f(t_k)}{\Delta t}. \end{aligned}$$

Therefore,

$$\begin{aligned} {}_0^C D_t^\alpha f(t_n) &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \frac{f(t_{k+1}) - f(t_k)}{\Delta t} \int_{t_k}^{t_{k+1}} \frac{1}{(t_n-t)^\alpha} dt, \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \frac{f(t_{k+1}) - f(t_k)}{\Delta t} \int_{t_k}^{t_{k+1}} (t_n-t)^{-\alpha} dt, \\ {}_0^C D_t^\alpha f(t_n) &= \frac{\Delta t^{-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \sum_{k=0}^{n-1} (f(t_{k+1}) - f(t_k)) [(n-k-1)^{1-\alpha} - (n-k)^{1-\alpha}]. \end{aligned} \tag{1.40}$$

Setting:

$$b_k = \frac{\Delta t^{-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} [(k+1)^{1-\alpha} - k^{1-\alpha}].$$

Chapter 1. Preliminaries

produces

$${}_0^C D_t^\alpha f(t_n) = \sum_{k=0}^{n-1} b_k (f(t_{k+1}) - f(t_k)). \quad (1.41)$$

the L^1 -approximation for the Caputo fractional derivative is thus determined by (1.40).

For the Riemann–Liouville fractional derivative:

$${}_0^R D_t^\alpha f(t_n) = {}_0^C D_t^\alpha f(t_n) + \frac{f'(0)}{\Gamma(1-\alpha)} t_n^{-\alpha}. \quad (1.42)$$

we can obtain

$${}_0^R D_t^\alpha f(t_n) = \sum_{k=0}^{n-1} b_k (f(t_{k+1}) - f(t_k)) + \frac{f'(0)}{\Gamma(1-\alpha)} t_n^{-\alpha}.$$

The error estimate of the L^1 -method is easily expressed as follows:

$$\left| \sum_{k=0}^{n-1} b_k (f(t_{k+1}) - f(t_k)) + \frac{f'(0)}{\Gamma(1-\alpha)} t_n^{-\alpha} - {}_0^R D_t^\alpha f(t_n) \right| \leq C \Delta t^{2-\alpha},$$

where C is a positive constant that depends onely on the order α and and the function f .

L^2 - and L_C^2 -approximations

The L^2 -method and its variant L_C^2 -method are used to discretize the Riemann–Liouville derivative of order α ($1 < \alpha < 2$), which can be obtained in a similar way to that of the $L1$ method. For the Caputo derivative with order $1 < \alpha < 2$, we have

$${}_0^C D_t^\alpha f(t_n) = \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{f''(t)}{(t_n-t)^{\alpha-1}} dt.$$

By using the property of integral: $\int_0^a g(t) dt = \int_0^a g(a-t) dt$, we get

$$\frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{f''(t_n-t)}{t^{\alpha-1}} dt.$$

Chapter 1. Preliminaries

$$\begin{aligned} {}_0^C D_t^\alpha f(t_n) &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{f''(t_n-t)}{t^{\alpha-1}} dt. \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{f''(t_n-t)}{t^{\alpha-1}} dt. \end{aligned}$$

On each subinterval $[t_k, t_{k+1}]$, we have

$$f''(t) \simeq \frac{f(t_{k+1}) - 2f(t_k) + f(t_{k-1}))}{\Delta t^2}.$$

Hence,

$$\begin{aligned} {}_0^C D_t^\alpha f(t_n) &\approx \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^n \frac{f(t_n-t_{k+1}) - 2f(t_n-t_k) + f(t_n-t_{k-1}))}{\Delta t^2} \int_{t_k}^{t_{k+1}} \frac{1}{t^{\alpha-1}} dt \\ &\approx \frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)} \sum_{k=0}^n \left(f(t_n-t_{k+1}) - 2f(t_n-t_k) \right. \\ &\quad \left. + f(t_n-t_{k-1}) \right) \left((k+1)^{-\alpha+2} - k^{-\alpha+2} \right) \\ &\approx \sum_{k=0}^n (f(t_n-t_{k+1}) - 2f(t_n-t_k) + f(t_n-t_{k-1})) b_k. \end{aligned} \quad (1.43)$$

where

$$b_k = \frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)} \left((k+1)^{-\alpha+2} - k^{-\alpha+2} \right).$$

The scheme can be rewritten as follows:

$${}_0^C D_t^\alpha f(t_n) = \sum_{k=-1}^n C_k f(t_{n-k}),$$

where

$$C_k = C \begin{cases} 1, & k = -1 \\ 2^{2-\alpha} - 3, & k = 0 \\ (k+2)^{-\alpha+2} - 3(k+1)^{-\alpha+2} + 3k^{-\alpha+2} - (k-1)^{-\alpha+2}, & 1 \leq k \leq n-2 \\ -(n-2)^{-\alpha+2} + 3(n-1)^{-\alpha+2} - 2n^{-\alpha+2}, & k = n-1 \\ n^{-\alpha+2} - (n-1)^{-\alpha+2}, & k = n \end{cases} \quad (1.44)$$

For $C = \frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)}$. This leads to the following L^2 -approximation for the Riemann-Liouville derivative:

$$\begin{aligned} {}_0^R D_t^\alpha f(t_n) &\approx {}_0^C D_t^\alpha f(t_n) + \frac{f(0)t_n^{-\alpha}}{\alpha(1-\alpha)} + \frac{f'(0)t_n^{1-\alpha}}{\alpha(1-\alpha)}, \\ &\approx \sum_{k=-1}^n C_k f(t_{n-k}) + \frac{f(0)t_n^{-\alpha}}{\alpha(1-\alpha)} + \frac{f'(0)t_n^{1-\alpha}}{\alpha(1-\alpha)}. \end{aligned}$$

Let us pass to the L_C^2 -method. For the Caputo derivative with order $1 < \alpha < 2$, we have

$$\begin{aligned} {}_0^C D_t^\alpha f(t_n) &= \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{f''(t)}{(t_n-t)^{\alpha-1}} dt, \\ &= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{f''(t)}{(t_n-t)^{\alpha-1}} dt. \end{aligned} \quad (1.45)$$

An approximation of the second-order derivative f'' can be performed by using the four-point discretization. Indeed, starting from the formula:

$$f''(t_k) = \frac{f'(t_{k+1}) + f'(t_{k-1}))}{2\Delta t},$$

and using $f'(t_{k+1}) = \frac{f(t_{k+2}) - f(t_{k+1})}{\Delta t}$, and $f'(t_{k-1}) = \frac{f(t_k) - f(t_{k-1})}{\Delta t}$, yield the four-point formula:

$$f''(t_k) = \frac{f(t_{k+2}) - f(t_{k+1}) + f(t_k) - f(t_{k-1}))}{2\Delta t^2}. \quad (1.46)$$

Finally, combining (1.45) and (1.46) leads to

$${}_0^C D_t^\alpha f(t_n) \approx \frac{1}{\Gamma(3-\alpha)} \sum_{k=0}^{n-1} (f(t_{k+2}) - f(t_{k+1}) + f(t_k) - f(t_{k-1})) [t_{n-k-1}^{-2+\alpha} - t_{n-k}^{-2+\alpha}].$$

By a similar reasoning, we can get

$$\begin{aligned} {}_0^R D_t^\alpha f(t_n) &\approx {}_0^C D_t^\alpha f(t_n) + \frac{f(0)t_n^{-\alpha}}{\alpha(1-\alpha)} + \frac{f'(0)t_n^{1-\alpha}}{\alpha(1-\alpha)}, \\ &\approx \frac{1}{\Gamma(3-\alpha)} \sum_{k=0}^{n-1} (f(t_{k+2}) - f(t_{k+1}) + f(t_k) - f(t_{k-1})) [t_{n-k-1}^{-2+\alpha} - t_{n-k}^{-2+\alpha}] \\ &\quad + \frac{f(0)t_n^{-\alpha}}{\alpha(1-\alpha)} + \frac{f'(0)t_n^{1-\alpha}}{\alpha(1-\alpha)}. \end{aligned}$$

Chapter 1. Preliminaries

The L^2 and L^2_C methods provide robust frameworks for approximating the fractional Caputo derivatives. By focusing on error minimization in the L^2 -norm and incorporating corrective terms, these methods enhance the accuracy and reliability of numerical solutions to fractional differential equations. Their versatility makes them valuable tools for researchers and practitioners across various disciplines.

Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

This chapter investigates the solvability and blow-up behavior of weak solutions for a specific class of nonlinear parabolic equations characterized by generalized nonlinear integral conditions of the second kind.

The chapter is structured into sections, each dedicated to developing the analytical framework required to study the existence and uniqueness of weak solutions to the nonlinear parabolic equation under consideration. These models are directly relevant to the main problem of interest (2.1). Subsequently, we will explore the blow-up phenomena associated with these weak solutions. [?],[?], [7],[21].

Let $(0, T)$ and $\Omega = (0, 1)$ be open bounded intervals in \mathbb{R} . Consider the nonlinear parabolic equation in the domain $Q_T = \Omega \times (0, T)$:

$$\begin{cases} u_t - a\Delta u + bu + cu^p = f(x, t, u) & \text{in } Q_T, \\ u(x, 0) = \varphi(x) & \forall x \in \Omega, \\ u_x(0, t) = \int_0^1 k_1(x, t)g(u(x, t))dx & \forall t \in (0, T), \\ u_x(1, t) = \int_0^1 k_2(x, t)h(u(x, t))dx & \forall t \in (0, T). \end{cases} \quad (2.1)$$

where φ is a given function, a, b and c positive integers, $p \geq 1$, and k_1, k_2 bounded functions.

2.1 Solvability of the problem for $b = c = 0$

Consider the problem (2.1) with appropriate boundary and initial conditions. We will analyze this problem under the specific parameters $b = 0, c = 0$ and g, h are

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

Lipschitzian functions satisfying:

$$\|g(u)\| \leq c_1 \|u\|, \quad (2.2)$$

$$\|h(u)\| \leq c_2 \|u\|, \quad (2.3)$$

c_1, c_2 being positive constants.

Let us study the problem:

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} - a \frac{\partial^2 u}{\partial x^2} = f(u), & x, t \in Q \\ u(x, 0) = \varphi(x), & 0 \leq x \leq 1 \\ u_x(0, t) = \int_0^1 k_1(x, t) g(u(x, t)) dx, & 0 \leq t \leq T \\ u_x(1, t) = \int_0^1 k_2(x, t) h(u(x, t)) dx & 0 \leq t \leq T. \end{array} \right. \quad (\text{P2})$$

In order to solve this problem, we focus on the corresponding linear equation to demonstrate the existence of a solution:

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} - a \frac{\partial^2 u}{\partial x^2} = f(x, t), & (x, t) \in Q \\ u(x, 0) = \varphi(x), & 0 \leq x \leq 1 \\ u_x(0, t) = \int_0^1 k_1(x, t) g(u(x, t)) dx, & 0 \leq t \leq T \\ u_x(1, t) = \int_0^1 k_2(x, t) h(u(x, t)) dx & 0 \leq t \leq T. \end{array} \right. \quad (\text{P3})$$

Let us take into account the following auxiliary problem that involves a homogeneous equation:

$$\mathcal{L}w = \frac{\partial w}{\partial t} - a \frac{\partial^2 w}{\partial x^2} = 0,$$

with the initial condition:

$$\ell w = w(x, 0) = \varphi(x),$$

and the nonlinear integral conditions of the second kind:

$$w_x(0, t) = \int_0^1 k_1(x, t) g(w(x, t) + y(x, t)) dx,$$

$$w_x(1, t) = \int_0^1 k_2(x, t) h(w(x, t) + y(x, t)) dx.$$

Let g^* and h^* be two functions defined by

$$g^*(w) = g(w, y) \text{ and } h^*(w) = h(w, y),$$

and verifying the following inequality:

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

$$\|g^*(w)\|_{L^2(Q)} \leq b_1 \|w\|_{L^2(Q)} + b_2, \text{ and } \|h^*(w)\|_{L^2(Q)} \leq b_3 \|w\|_{L^2(Q)} + b_4,$$

b_1, b_2, b_3 and b_4 are positive constants. As a result, the corresponding auxiliary

problem, when considering the homogeneous version of the equation, can be expressed in the following form:

$$\begin{cases} \mathcal{L}w = \frac{\partial w}{\partial t} - a \frac{\partial^2 w}{\partial x^2} = 0, & (x, t) \in Q \\ \ell w = w(x, 0) = \varphi(x), & x \in (0, 1) \\ w_x(0, t) = \int_0^1 \eta_0(x, t) g^*(w(x, t)) dx, & t \in (0, T) \\ w_x(1, t) = \int_0^1 \eta_1(x, t) h^*(w(x, t)) dx. & t \in (0, T). \end{cases} \quad (\text{P2})$$

Assuming that u is a solution to problem (P1) and w is a solution to problem (P2),

it follows that $y = u - w$ is a solution of the following problem:

$$\begin{cases} \mathcal{L}y = \frac{\partial y}{\partial t} - a \frac{\partial^2 y}{\partial x^2} = f(x, t), & (x, t) \in Q \\ \ell y = y(x, 0) = 0, & x \in (0, 1) \\ y_x(0, t) = 0, & t \in (0, T) \\ y_x(1, t) = 0, & t \in (0, T). \end{cases} \quad (\text{P3})$$

To establish the existence of solutions for the problem (P2), we only need to convert it into a nonlinear ODE. We do this by integrating the equation in (P2) over the domain Ω , we have:

$$\int_0^1 w_t dx - a \int_0^1 w_{xx} dx = 0, \quad \forall x \in \Omega;$$

But,

$$\begin{aligned} \int_0^1 w_{xx} dx &= w_x(1, t) - w_x(0, t) \\ &= \int_0^1 \eta_1(x, t) h^*(w(x, t)) dx - \int_0^1 \eta_0(x, t) g^*(w(x, t)) dx, \end{aligned}$$

then, we get

$$\int_0^1 w_t dx - a \int_0^1 \eta_1(x, t) h^*(w(x, t)) dx + a \int_0^1 \eta_0(x, t) g^*(w(x, t)) dx = 0.$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

i.e.

$$\int_0^1 [w_t - a(\eta_1(x, t) h^*(w(x, t)) dx + \eta_0(x, t) g^*(w(x, t)))] dx = 0,$$

Thus,

$$\int_0^1 (w_t - F(t, w(x, t))) dx = 0, \quad (2.4)$$

where

$$F(t, w(x, t)) = a\eta_1(x, t) h^*(w(x, t)) - a\eta_0(x, t) g^*(w(x, t)).$$

Therefore, it is clear that a function ψ exists such that:

$$w_t - F(t, w(x, t)) = \psi(x, t), \quad \text{with} \quad \int_0^1 \psi(x, t) dx = 0.$$

Consequently, we get

$$w_t = G(t, w(t)),$$

where

$$G(t, w(t)) = F(t, w(x, t)) + \psi(x, t).$$

G , known as a Carathéodory mapping, is applicable due to the following:

- * The continuity of the function F with respect to w implies G for w as well,
- * Given that F and ψ are measurable with respect to t , we can conclude that G is measurable for t as well. Moreover,

$$\begin{aligned} \|G(t, w)\| &\leq a \|\eta\|_\infty (\|h^*(w)\| + \|g^*(w)\|) + \|\psi(x, t)\|_{L^2} + b_2 + b_4 \\ &\leq b_2 + b_4 + \|\psi(x, t)\|_{L^2} + 2a \|\eta\|_\infty \|w\|, \end{aligned}$$

Thus, through the application of the existence and uniqueness theorem, we derive $w \in W^{1,1}$ and, using Nemytskii mappings in Lebesgue spaces, we can verify that w_t is in $L^2[0, T]$

Based on these results, we can conclude that the problem (P2) has a unique solution. Consequently, we need to address and demonstrate that problem (P3) has a unique strong solution.

Let's explore the following auxiliary problem with homogeneous conditions:

$$\left\{ \begin{array}{ll} \mathcal{L}v = \frac{\partial v}{\partial t} - a \frac{\partial^2 v}{\partial x^2} = f(x, t) & (x, t) \in Q_\tau, \\ \ell v = v(x, 0) = 0, & x \in (0, 1), \\ v_x(0, t) = 0 & t \in (0, T), \\ v_x(1, t) = 0 & t \in (0, T). \end{array} \right. \quad (\text{P4})$$

Evaluation of the uniqueness of solution for the problem (P₄)

To evaluate the uniqueness of the solution for the problem (P₄), we need to establish conditions under which the weak solution to the nonlinear parabolic equation is unique.

Theorem 2.1

For every solution v to (P4) belonging to E , the following inequality holds:

$$\|v\|_E \leq c \|Lv\|_F, \quad (2.5)$$

where c represents a positive constant that is independent of v .

Proof Let us assume that a solution v to the problem (P4) exists. By multiplying the equation in (P4) by v and integrating over Q , where $Q = \Omega \times (0, \tau)$, we get

$$\int_{Q^\tau} v_t \cdot v - a \int_{Q^\tau} v_{xx} \cdot v = \int_{Q^\tau} f(x, v) \cdot v. \quad (2.6)$$

Carrying out integration by parts for each term in the left-hand expression of (2.6) over Q , $0 < \tau < T$, we obtain

$$\frac{1}{2} \int_0^1 v(x, \tau)^2 dx + a \int_{Q^\tau} v_x^2 dt = \int_{Q^\tau} f \cdot v dx dt,$$

By multiplying the equation by 2, and using the Cauchy inequality, we have

$$\int_{Q^\tau} f \cdot v dx dt \leq \frac{1}{2} \|f\|_{L^2(Q)}^2 + \frac{1}{2} \|v\|_{L^2(Q)}^2.$$

Then, the (4.11) becomes

$$\int_0^1 v(x, \tau)^2 dx + 2a \|v_x\|_{L^2(Q^\tau)}^2 \leq \|f\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2, \quad (2.7)$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

Using Gronwall's lemma in (3.7), we obtain

$$\int_0^1 v(x, \tau)^2 dx + \|v_x\|_{L^2(Q^\tau)}^2 \leq \frac{\exp(T)}{\min\{1, 2a\}} \|f\|_{L^2(Q)}^2.$$

By passing to the maximum over $(0, T)$, we get

$$\|v\|_{L^\infty(0, T; L^2(\Omega))}^2 + \|v_x\|_{L^2(Q)}^2 \leq c^2 \|f\|_{L^2(Q)}^2.$$

Finally, we obtain the desired inequality, with $c = \sqrt{\frac{\exp(T)}{\min\{1, 2a\}}}$. ■

Corollary 2.1

If, for any $s \in D(L)$, the estimate:

$$\|s\|_E \leq C \|\mathcal{F}\|_F,$$

is valid, then the solution v to problem (P3) if it exists, is unique.

Proof Consider s_1 and s_2 as two distinct solutions to the given problem (P3):

$$\begin{cases} Ls_1 = \mathcal{F} \\ Ls_2 = \mathcal{F} \end{cases} \implies Ls_1 - Ls_2 = 0,$$

and since L is linear we obtain

$$L(s_1 - s_2) = 0,$$

Next, according to (2.5) we get

$$\|s_1 - s_2\|_E^2 \leq c \|0\|_F^2 = 0,$$

which gives

$$s_1 = s_2.$$

■

Study of the existence of the solution of problem (P4)

In this section, an analysis of the existence of a solution to the given problem is under taken. This involves exploring the conditions under which a solution can be determined, as well as identifying the mathematical framework that guarantees

this existence.

Proposition 2.1

The operator L , mapping from the space E to the space F possesses a closure.

Proof Let $\{v_n\} \subset D(L)$ be such as:

$$v_n \longrightarrow 0 \text{ in } E,$$

and

$$Lv_n \longrightarrow (f; \varphi) \text{ in } F, \tag{2.8}$$

It must be demonstrated that

$$f \equiv 0 \text{ and } \varphi \equiv 0.$$

The convergence of v_n towards 0 in E implies

$$v_n \longrightarrow 0 \text{ in } D'(Q). \tag{2.9}$$

Considering the continuity of the derivation $D'(Q)$ within the space $D'(Q)$, By analyzing how continuity influences the properties of solutions and the overall behavior of the system, we gain valuable insights into the dynamics governed by the relation. The relation (2.9) encompasses several key aspects that are critical for understanding the involved dynamics:

$$\mathcal{L}v_n \longrightarrow 0 \text{ in } D'(Q), \tag{2.10}$$

Otherwise, the convergence of $\mathcal{L}v_n$ towards f in $L^2(Q)$ leads to

$$\mathcal{L}v_n \longrightarrow f \text{ in } D'(Q). \tag{2.11}$$

Due to the uniqueness of the limit within the space $D'(Q)$, we are able to derive after calculations based on this property from the following expressions or principles (3.8) and (4.15) that:

$$f = 0.$$

Then, (2.8) gives

$$\ell v_n \longrightarrow \varphi \text{ in } L^2(\Omega).$$

On the other hand, it is essential to consider the following aspects or factors that

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

may influence our analysis or understanding of the situation at hand. We have

$$\begin{aligned}\|v_n\|_E^2 &= \|v_n\|_{C(0,T, L^2(\Omega))}^2 + \|\partial_x v_n\|_{L^2(Q)}^2 \\ \|v_n\|_E^2 &\geq \|v_n(x, 0)\|_{L^2(\Omega)}^2.\end{aligned}$$

By passing to the limits, it follows that

$$\lim_{n \rightarrow +\infty} \|v_n\|_E^2 \geq \|\varphi(x)\|_{L^2(\Omega)}^2,$$

Since $v_n \rightarrow 0$ in E and then $\|v_n\|_E^2 \rightarrow 0$ in E , we find

$$\|\varphi(x)\|_{L^2(\Omega)}^2 \leq 0,$$

from where $\varphi = 0$. ■

Definition 2.1

The solution of the equation

$$\bar{L}u = \mathcal{F}$$

is said to be a strong generalized solution to problem (P3).

- The theorem (5.25) applies to a strong generalized solution, meaning we have the inequality:

$$\|u\|_E \leq K \|\bar{L}u\|_F \quad \forall u \in D(\bar{L}). \quad (2.12)$$

As a result, this inequality leads to several corollaries that clarify the implications and applications of the established relationship.

Corollary 2.2

The solution to the problem (P3), if it exists, is unique and exhibits continuous dependence on the parameter $\mathcal{F} \in F$.

This means that small variations in the input F will result in corresponding small changes in the solution, ensuring stability and robustness in the system's response.

Corollary 2.3

The set of values $R(\bar{L})$ associated with the operator \bar{L} is equal to $\overline{R(\bar{L})}$.

Proof Let $z \in \overline{R(\bar{L})}$, which implies that there exists a Cauchy sequence $\{z_n\}_{n \in \mathbb{N}}$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

in F composed of elements from the entire range $R(L)$ such as:

$$\lim_{n \rightarrow +\infty} z_n = z.$$

Consequently, for each element in this sequence, we can define a corresponding $y_n \in D(L)$ such as:

$$Lv_n = z_n.$$

From the estimate (2.12), we obtain:

$$\|v_p - v_q\|_E \leq k \|Lv_p - Lv_q\|_F \rightarrow 0.$$

When p and q tend to infinity, we can conclude that $\{v_n\}_n$ forms a Cauchy sequence in E . Therefore, since E is a Banach space, there exists an $s \in E$ such that:

$$\lim_{n \rightarrow +\infty} v_n = v \text{ in } E.$$

Under the definition of the operator \bar{L} , consider a sequence $(v_n)_{n \in \mathbb{N}}$ in E such that

$$\lim_{n \rightarrow +\infty} v_n = v \text{ in } E.$$

If, additionally,

$$\lim_{n \rightarrow +\infty} Lv_n = \lim_{n \rightarrow +\infty} z_n = z,$$

then it follows that

$$\lim_{n \rightarrow +\infty} \bar{L}v_n = z.$$

This conclusion relies on the fact that \bar{L} is a closed operator. Consequently, we deduce that

$$\bar{L}y = z.$$

The function y satisfies the following condition:

$$v \in D(\bar{L}), \bar{L}v = z.$$

Thus $z \in R(\bar{L})$, and so,

$$\overline{R(L)} \subset R(\bar{L}).$$

Consequently, we can conclude that $R(\bar{L})$ is closed because of its completeness. In metric spaces, any complete subspace is inherently closed, even if the larger space itself lacks this property. This characteristic ensures that all limit points of $R(\bar{L})$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

are contained within $R(L)$. Next, we show the reverse inclusion.

Let $z \in R(\bar{L})$. Hence, there exists a Cauchy sequence

$\{z_n\}_n$ in F that converges to z constituted of the elements of the whole $R(\bar{L})$. This completeness is a vital aspect of understanding the behavior of the operator L and its implications in functional analysis, such as:

$$\lim_{n \rightarrow +\infty} z_n = z.$$

where $z \in R(\bar{L})$. Since $R(\bar{L})$ is a closed subset of a complete space F , we conclude that $R(\bar{L})$ is also complete. Furthermore, for each term of the Cauchy sequence $\{z_n\}$ in $R(L)$, there exists a corresponding term $v_n \in D(\bar{L})$ such as:

$$\bar{L}v_n = z_n.$$

From the estimate (2.12), we obtain

$$\|v_p - v_q\|_E \leq m \|\bar{L}v_p - \bar{L}v_q\|_F \rightarrow 0.$$

When p and q tend towards infinity, we can deduce that $\{v_n\}_n$ is a Cauchy sequence in the space E . Since E is a Banach space, there exists an element $v \in E$ such that

$$\lim_{n \rightarrow +\infty} v_n = v \text{ in } E.$$

Once more, there is then a corresponding sequence $\{L(v_n)\}_n \subset R(L)$ such as:

$$Lv_n = \bar{L}v_n \text{ over } R(L), \forall n \in \mathbb{N}.$$

Thus,

$$\lim_{n \rightarrow +\infty} Lv_n = z,$$

Consequently, $z \in \overline{R(L)}$, and so,

$$R(\bar{L}) \subset \overline{R(L)}.$$

Theorem 2.2

The solution of (P3) exists.

Proof We need to establish that $R(L)$ is dense in F . To demonstrate the existence of a solution, it is sufficient to show that Ly is surjective. According to the density

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

of \bar{L} we have $\overline{R(\bar{L})} = F$. As a result, we conclude that the following statements hold. Highlighting the significant findings of our analysis $R(L)^\perp = \{0\}_F$, we need to follow a structured approach. We have

$$\begin{aligned} R(L)^\perp &= \{w \in F, \langle w, \mathcal{F} \rangle_F = 0, \forall \mathcal{F} \in R(L)\}, \\ &= \{(w, w_0) \in L^2(Q), \langle w, f \rangle_{L^2(Q)} + \langle w_0, \varphi \rangle_{L^2(Q)} = 0, \forall f \in L^2(Q), \forall \varphi \in L^2(Q)\} \end{aligned}$$

and

$$D_0(L) = \{v \in E, v(x, 0) = 0\}.$$

Then,

$$w_0 = 0.$$

It remains to demonstrate that $w = 0$. We have

$$\langle w, Lv \rangle_{L^2(Q)} = \int_0^1 \int_0^T wLv = 0.$$

Setting: $w = v$, we obtain

$$\int_0^1 \int_0^T v(v_t - a\Delta v) = \int_0^1 \int_0^T v \cdot v_t - a \int_0^1 \int_0^T v \cdot v_{xx} = 0.$$

Therefore,

$$\int_0^1 \int_0^T v \cdot v_t = a \int_0^1 \int_0^T v \cdot v_{xx}.$$

By applying integration by parts, we obtain the following results that provide deeper insights into the problem at hand:

$$\frac{1}{2} \int_0^1 v^2(x, T) = -a \int_0^1 v_x^2 \leq 0.$$

Finally, we get

$$v = 0 \implies w = 0.$$

■

Numerical study of the problem (P3) For the numerical solution of the considered problem (P3) we apply the compact finite difference technique. First, we partition the interval $[0, 1]$ into M equal parts by taking the spacestep $\Delta x = \frac{1}{M}$ and the interval $[0, T]$ into N equal parts by using the timestep $\Delta t = \frac{1}{N}$. By u_i^n we use, the grid point (x_i, t_n) are given by : $x_i = i\Delta x, i = 0, 1, \dots, M. t_n = n\Delta t(n =$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

$0, \dots, N$). The notations $u_i^n, f_i^n, g(u_i^n), h(u_i^n), \left(\frac{\partial g(u)}{\partial u}\right)_i^n$ and $\left(\frac{\partial h(u)}{\partial u}\right)_i^n$ are used for the approximations of $u(x_i, t_n), f(x_i, t_n), g(u(x_i, t_n)), h(u(x_i, t_n)), \frac{\partial g(u(x_i, t_n))}{\partial u}$ and $\frac{\partial h(u(x_i, t_n))}{\partial u}$ respectively. By using the finite difference scheme and by multiplying the operator $\left(1 + \frac{(\Delta x)^2}{12} \delta_x^2\right)$, we obtain :

$$\delta_t u_i^n + \frac{(\Delta x)^2}{12} \delta_x^2 (\delta_t u_i^n) - a \delta_x^2 u_i^n = f_i^n + \frac{(\Delta x)^2}{12} \delta_x^2 f_i^n$$

Setting: $r = a \frac{\Delta t}{(\Delta x)^2}$, the latter scheme can be written as follows:

$$\begin{aligned} & \left(\frac{1}{12} - r\right) u_{i+1}^n + \left(\frac{5}{6} + 2r\right) u_i^n + \left(\frac{1}{12} - r\right) u_{i-1}^n \\ &= \frac{1}{12} u_{i+1}^{n-1} + \frac{5}{6} u_i^{n-1} + \frac{1}{12} u_{i-1}^{n-1} + \frac{\Delta t}{12} (f_{i+1}^n + 10f_i^n + f_{i-1}^n). \end{aligned} \quad (2.13)$$

We still have to predetermine the two unknowns u_0^n et u_M^n . To this purpose, we approximate the integral conditions numerically by the composite Simpson rule. We have chosen this approximation because it is of the same order of precision. As it requires an even number of sub-intervals, we set: $M = 2i$. In doing so, we have

$$\begin{aligned} u_x(0, t_n) &= \int_0^1 k_0(x, t_n) g(u(x, t_n)) dx \\ &= \frac{\Delta x}{3} \left(k_0(x_0, t_n) g(u_0^n) + 4 \sum_{i=1}^{\frac{M}{2}} k_0(x_{2i-1}, t_n) g(u_{2i-1}^n) \right. \\ & \quad \left. + 2 \sum_{i=1}^{\frac{M}{2}-1} k_0(x_{2i}, t_n) g(u_{2i}^n) + k_0(x_M, t_n) g(u_M^n) \right) \end{aligned}$$

Thus,

$$\begin{aligned} & 3u_x(0, t_n) - \Delta x k_0(x_0, t_n) g(u_0^n) - \Delta x k_0(x_M, t_n) g(u_M^n) \\ &= \Delta x \left(4 \sum_{i=1}^{\frac{M}{2}} k_0(x_{2i-1}, t_n) g(u_{2i-1}^n) + 2 \sum_{i=1}^{\frac{M}{2}-1} k_0(x_{2i}, t_n) g(u_{2i}^n) \right) \end{aligned} \quad (2.14)$$

So that

$$\begin{aligned} u_x(1, t_n) &= \int_0^1 k_1(x, t_n) h(u(x, t_n)) dx \\ &= \frac{\Delta x}{3} \left(k_1(x_0, t_n) h(u_0^n) + 4 \sum_{i=1}^{\frac{M}{2}} k_1(x_{2i-1}, t_n) h(u_{2i-1}^n) \right. \\ & \quad \left. + 2 \sum_{i=1}^{\frac{M}{2}-1} k_1(x_{2i}, t_n) h(u_{2i}^n) + k_1(x_M, t_n) h(u_M^n) \right) \end{aligned}$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

i.e.

$$\begin{aligned} & 3u_x(1, t_n) - \Delta x k_1(x_0, t_n)h(u_0^n) - \Delta x k_1(x_M, t_n)h(u_M^n) \\ = & \Delta x \left(4 \sum_{i=1}^{\frac{M}{2}} k_1(x_{2i-1}, t_n)h(u_{2i-1}^n) + 2 \sum_{i=1}^{\frac{M}{2}-1} k_1(x_{2i}, t_n)h(u_{2i}^n) \right) \end{aligned} \quad (2.15)$$

Of course, a linearization technique will be developed to overcome the difficulty. Using Taylor's series expansion of the nonlinear terms $g(u_i^n) = g(u(x_i, t_n))$ and $h(u_i^n) = h(u(x_i, t_n))$, we obtain

$$g(u_i^n) = g(u_i^{n-1}) + \left(\frac{\partial g(u)}{\partial u} \right)_i^{n-1} (u_i^n - u_i^{n-1}) + \dots \quad (2.16)$$

$$h(u_i^n) = h(u_i^{n-1}) + \left(\frac{\partial h(u)}{\partial u} \right)_i^{n-1} (u_i^n - u_i^{n-1}) + \dots \quad (2.17)$$

It follows that

$$a_0^n u_0^n + a_1^n u_1^n + a_2^n u_2^n + a_3^n u_3^n + a_4^n u_4^n + \dots + a_{M-1}^n u_{M-1}^n + a_M^n u_M^n = L_M^n, \quad (2.18)$$

where

$$\left\{ \begin{aligned} a_0^n &= -25 - 4(\Delta x)^2 k_0(x_0, t_n) \left(\frac{\partial g(u)}{\partial u} \right)_0^{n-1}, \\ a_1^n &= 48 - 16(\Delta x)^2 k_0(x_1, t_n) \left(\frac{\partial g(u)}{\partial u} \right)_1^{n-1}, \\ a_2^n &= -36 - 8(\Delta x)^2 k_0(x_2, t_n) \left(\frac{\partial g(u)}{\partial u} \right)_2^{n-1}, \\ a_3^n &= 16 - 16(\Delta x)^2 k_0(x_3, t_n) \left(\frac{\partial g(u)}{\partial u} \right)_3^{n-1}, \\ a_4^n &= -3 - 8(\Delta x)^2 k_0(x_4, t_n) \left(\frac{\partial g(u)}{\partial u} \right)_4^{n-1}, \\ a_{2i-1}^n &= -16(\Delta x)^2 k_0(x_{2i-1}, t_n) \left(\frac{\partial g(u)}{\partial u} \right)_{2i-1}^{n-1} u_{2i-1}^n \quad ; i = 3, \dots, \frac{M}{2}, \\ a_{2i}^n &= -8(\Delta x)^2 k_0(x_{2i}, t_n) \left(\frac{\partial g(u)}{\partial u} \right)_{2i}^{n-1} u_{2i}^n \quad ; i = 3, \dots, \frac{M}{2} - 1, \end{aligned} \right. \quad (2.19)$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

and

$$\begin{aligned}
 L_M^n = & 16(\Delta x)^2 \sum_{i=1}^M k_1(x_{2i-1}, t_n) \left[h(u_{2i-1}^{n-1}) - \left(\frac{\partial h(u)}{\partial u} \right)_{2i-1}^{n-1} u_{2i-1}^{n-1} \right] \\
 & + 8(\Delta x)^2 \sum_{i=1}^{M-1} k_1(x_{2i}, t_n) \left[h(u_{2i}^{n-1}) - \left(\frac{\partial h(u)}{\partial u} \right)_{2i}^{n-1} u_{2i}^{n-1} \right] \\
 & + 4(\Delta x)^2 \left[k_1(x_0, t_n) \left[h(u_0^{n-1}) - \left(\frac{\partial h(u)}{\partial u} \right)_0^{n-1} u_0^{n-1} \right] \right] \\
 & + 4(\Delta x)^2 \left[k_1(x_{2M}, t_n) \left[h(u_{2M}^{n-1}) - \left(\frac{\partial h(u)}{\partial u} \right)_{2M}^{n-1} u_{2M}^{n-1} \right] \right]. \quad (2.20)
 \end{aligned}$$

Likewise, we have :

$$b_0^n u_0^n + \dots + b_{2M-4}^n u_{2M-4}^n + b_{2M-3}^n u_{2M-3}^n + b_{2M-2}^n u_{2M-2}^n + b_{2M-1}^n u_{2M-1}^n + b_{2M}^n u_{2M}^n = \gamma_M^n, \quad (2.21)$$

where

$$\left\{ \begin{aligned}
 b_0^n &= -4(\Delta x)^2 k_1(x_0, t_n) \left(\frac{\partial h(u)}{\partial u} \right)_0^{n-1}, \\
 b_{2M-4}^n &= 3 - 8(\Delta x)^2 k_1(x_{2M-4}, t_n) \left(\frac{\partial h(u)}{\partial u} \right)_{2M-4}^{n-1}, \\
 b_{2M-3}^n &= -16 - 16(\Delta x)^2 k_1(x_{2M-3}, t_n) \left(\frac{\partial h(u)}{\partial u} \right)_{2M-3}^{n-1}, \\
 b_{2M-2}^n &= 36 - 8(\Delta x)^2 k_1(x_{2M-2}, t_n) \left(\frac{\partial g(u)}{\partial u} \right)_{2M-2}^{n-1}, \\
 b_{2M-1}^n &= -48 - 16(\Delta x)^2 k_1(x_{2M-1}, t_n) \left(\frac{\partial h(u)}{\partial u} \right)_{2M-1}^{n-1}, \\
 b_{2M}^n &= 25 - 4(\Delta x)^2 k_1(x_{2M}, t_n) \left(\frac{\partial g(u)}{\partial u} \right)_{2M}^{n-1}, \\
 b_{2i-1}^n &= -16(\Delta x)^2 k_1(x_{2i-1}, t_n) \left(\frac{\partial h(u)}{\partial u} \right)_{2i-1}^{n-1}, \quad i = 1, \dots, M-2, \\
 b_{2i}^n &= -8(\Delta x)^2 k_1(x_{2i}, t_n) \left(\frac{\partial h(u)}{\partial u} \right)_{2i}^{n-1}, \quad i = 1, \dots, M-2,
 \end{aligned} \right. \quad (2.22)$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

and

$$\begin{aligned}
 \gamma_M^n = & 16(\Delta x)^2 \sum_{i=1}^M k_1(x_{2i-1}, t_n) \left[h(u_{2i-1}^{n-1}) - \left(\frac{\partial h(u)}{\partial u} \right)_{2i-1}^{n-1} u_{2i-1}^{n-1} \right] \\
 & + 8(\Delta x)^2 \sum_{i=1}^{M-1} k_1(x_{2i}, t_n) \left[h(u_{2i}^{n-1}) - \left(\frac{\partial h(u)}{\partial u} \right)_{2i}^{n-1} u_{2i}^{n-1} \right] \\
 & + 4(\Delta x)^2 \left[k_1(x_0, t_n) \left[h(u_0^{n-1}) - \left(\frac{\partial h(u)}{\partial u} \right)_0^{n-1} u_0^{n-1} \right] \right] \\
 & + 4(\Delta x)^2 \left[k_1(x_{2M}, t_n) \left[h(u_{2M}^{n-1}) - \left(\frac{\partial h(u)}{\partial u} \right)_{2M}^{n-1} u_{2M}^{n-1} \right] \right].
 \end{aligned} \tag{2.23}$$

Combining (2.18), (2.20), with (2.85) yields an $(M + 1) \times (M + 1)$ system of linear equations that can be written in the following matrix form:

$$A^n U^n = B^n,$$

where

$$A^n = \begin{pmatrix} a_0^n & a_1^n & a_2^n & a_3^n & a_4^n & \dots & a_{M-4}^n & a_{M-3}^n & a_{M-2}^n & a_{M-1}^n & a_M^n \\ \frac{1}{12} - r & \frac{5}{6} + 2r & \frac{1}{12} - r & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & & & & & & & \vdots \\ 0 & & & \ddots & & & & 0 & \frac{1}{12} - r & \frac{5}{6} + 2r & \frac{1}{12} - r \\ b_0^n & b_1^n & b_2^n & b_3^n & b_4^n & \dots & b_{2M-4}^n & b_{2M-3}^n & b_{2M-2}^n & b_{2M-1}^n & b_{2M}^n \end{pmatrix},$$

$$U^{n+1} = \begin{pmatrix} u_0^n \\ u_1^n \\ \vdots \\ u_{M-1}^n \\ u_M^n \end{pmatrix},$$

$$B^n = \begin{pmatrix} L_M^n \\ L_1^n \\ \vdots \\ L_{2M-1}^n \\ \gamma_M^n \end{pmatrix},$$

with $a_0^n, \dots, a_M^n, b_0^n, \dots, b_M^n, L_M^n$ and γ_M^n being the coefficients in (2.19), (2.22), and (2.23) respectively.

Numerical experiments To test the above algorithm described in Section 2.3, we use two examples with known analytical solutions.

Example 1: Consider the heat equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = (1 + \pi^2) \exp(t) \cos(\pi x), \quad 0 < x < 1, \quad 0 < t \leq T, \quad (2.24)$$

with the initial condition:

$$u(x, 0) = \cos(\pi x), \quad 0 < x < 1, \quad (2.25)$$

and the nonlinear nonlocal boundary conditions:

$$u_x(0, t) = \int_0^1 \sin(\pi x) u^3(x, t) dx, \quad 0 < t \leq T, \quad (2.26)$$

$$u_x(1, t) = \int_0^1 \sin(\pi x) u^5(x, t) dx, \quad 0 < t \leq T. \quad (2.27)$$

The analytic solution is

$$u(x, t) = \cos(\pi x) \exp(t). \quad (2.28)$$

In Table 2.1 and Table 2.2 we present computed results with $h = 0.05, 0.005$ and $k = 0.4$ using the proposed numerical scheme for $x = 0.1$ and $t = 0.01, 0.02, 0.03, \dots, 0.1$. Table 2.3 gives the maximum errors of the numerical solutions and the experimental order of convergence. The maximum error is defined as follows:

$$Er = \|u - u_{hk}\|_\infty = \max_{0 \leq k \leq N} \{ \max_{0 \leq i \leq M} |u(x_i, t_k) - u_i^k| \},$$

whereas the experimental order of convergence for the scheme is calculated using the formula:

$$order = \frac{\ln(Er(h_{i-1})/Er(h_i))}{\ln(h_{i-1}/h_i)}.$$

From the tables, it is clear that the obtained numerical results are in good agreement with the exact ones. Moreover, the new scheme is fourth-order accurate in

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

Table 2.1: Comparison of exact solutions with the numerical results at $x = 0.1$ for $h = 0.05$ and $h = 0.005$ for Example 1.

t_i	exact	CBES $h = 0.05$	CBES $h = 0.005$
0.01	0.96061479	0.96061503	0.96061479
0.02	0.97026913	0.97026948	0.97026913
0.03	0.98002050	0.98002092	0.98002050
...
0.1	1.05108000	1.05108060	1.05108000

Table 2.2: Maximum errors and the experimental order of convergence for Example 44.

M	N	maximum errors	order
4	40	3.88×10^{-4}	
8	640	$2.50 \cdot 10^{-5}$	3.953
16	10240	$1.57 \cdot 10^{-6}$	3.995
32	163840	$9.83 \cdot 10^{-8}$	3.997

space. Figure 2.1 illustrates the exact solution and an approximate solution of Example 1 by CBES .

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

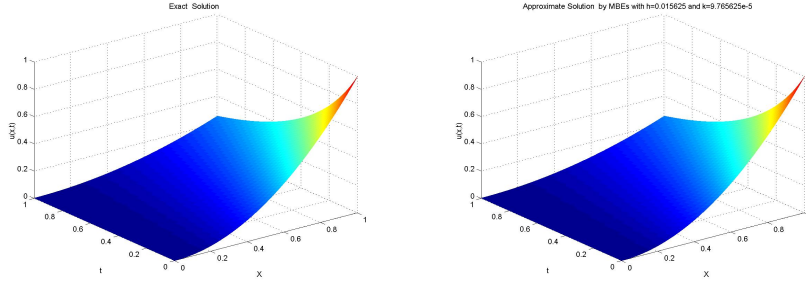


Figure 2.1: (a) Exact and (b) Approximate Solution by CBES for Example 1.

Example 2: The second test problem is that of the PDE:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 2t - 2\pi(3x^2 - 3x) \cos\left(2\pi\left(x^3 - \frac{3}{2}x^2\right)\right) + 4\pi(3x^2 - 3x)^2 \sin\left(2\pi\left(x^3 - \frac{3}{2}x^2\right)\right),$$

$$0 < x < 1, \quad 0 < t \leq T, \quad (2.29)$$

with the initial condition:

$$u(x, 0) = \sin\left(2\pi\left(x^3 - \frac{3}{2}x^2\right)\right), \quad 0 < x < 1, \quad (2.30)$$

and the nonlocal boundary conditions:

$$u_x(0, t) = \int_0^1 2\pi(3x^2 - 3x) \cos\left(2\pi\left(x^3 - \frac{3}{2}x^2\right)\right) e^{u(x,t)} dx, \quad 0 < t \leq T, \quad (2.31)$$

$$u_x(1, t) = \int_0^1 2\pi(3x^2 - 3x) \cos\left(2\pi\left(x^3 - \frac{3}{2}x^2\right)\right) \frac{1}{1 + u(x, t)} dx, \quad 0 < t \leq T. \quad (2.32)$$

The analytic solution is

$$u(x, t) = \sin\left(2\pi\left(x^3 - \frac{3}{2}x^2\right)\right) + t^2. \quad (2.33)$$

In Table 2.3, we present results with for $h = 0.05$ and $h = 0.005$ and $r = 0.4$ using the proposed numerical scheme for $x = 0, 1$ and $t = 0, 01; 0, 02; 0, 03; \dots; 0, 1$. Table 2.4, gives the maximum errors of the numerical solutions.

Figure 2.2 illustrate the exact solution and an approximate solution of Example 2 by CBES .

From the shown tables, it is clear that the numerical are in good agreement with the exact ones.

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

Table 2.3: Comparison of exact solutions with numerical results at $x = 0.1$ for $h = 0.05$ and $h = 0.005$

t_i	exact	CBES $h = 0.05$	CBES $h = 0.005$
0.01	0.30911699	0.30912416	0.30911707
0.02	0.30941699	0.3094286	0.30941711
0.03	0.30991699	0.30993220	0.30991715
...
0.1	0.31901699	0.31904755	0.31901730

Table 2.4: Maximum errors and the experimental order of convergence for Example 2.

M	N	maximum errors	order
4	40	4.749373×10^{-3}	
8	640	$2.949950 \cdot 10^{-4}$	4.008
16	10240	$1.840864 \cdot 10^{-5}$	4.002
32	163840	$1.150093 \cdot 10^{-6}$	4.0005

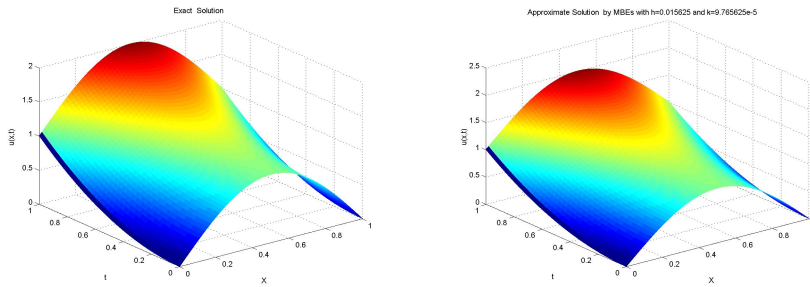


Figure 2.2: (a) Exact and (b) Approximate Solution by CBES for Example 2.

if u represents a solution to the problem denoted as (P3), and w denotes a solution to the problem labeled as (P2). then we can analyze the relationship or interactions between these solutions in the context of their respective frameworks $y = u - w$ satisfies

$$\left\{ \begin{array}{ll} \mathcal{L}y = \frac{\partial y}{\partial t} - a \frac{\partial^2 y}{\partial x^2} = G(x, t, y) & x, t \in Q \\ y(x, 0) = 0 & x \in (0, 1) \\ \frac{\partial y}{\partial x}(x, 0) = 0 & t \in (0, T) \\ \frac{\partial y}{\partial x}(1, t) = 0 & t \in (0, T) \end{array} \right. \quad (P3')$$

where $G(x, t, y) = f(x, t, y)$.

Since the function f is Lipschitzian, the function G is also Lipschitzian, i.e. there

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

exists a positive constant k such that the following condition holds:

$$\|G(x, t, y_1) - G(x, t, y_2)\|_{L^2(Q)} \leq k\|y_1 - y_2\|_{L^2(Q)}, \quad (2.34)$$

for any $(y_1, y_2) \in (L^2(Q))^2$.

Therefore, we have to demonstrate that problem (P3') has a unique weak solution. To start, we will clarify the concept of the solution that we are examining. Let us define the following parameters that are essential for understanding the solution $v = v(x, t)$ belonging to the space $L^2(0, T; H^1(0, 1))$.

By multiplying the equation in (P3') by v and integrating it on Q , we arrive at

$$\int_Q \frac{\partial y}{\partial t} v dx dt - a \int_Q \frac{\partial^2 y}{\partial x^2} v dx dt = \int_Q v G(x, t, y) dx dt.$$

Now, by using integration by parts along with the stipulated condition on y , we obtain the following equation:

$$\int_Q \frac{\partial y}{\partial t} v dx dt + a \int_Q \frac{\partial y}{\partial x} \frac{\partial v}{\partial x} dx dt = \int_Q v G(x, t, y) dx dt, \quad (2.35)$$

From (2.67), it follows that

$$\Lambda(r, v) = \int_Q v G(x, t, r) dx dt, \quad (2.36)$$

where

$$\Lambda(y, v) = \int_Q \frac{\partial y}{\partial t} v dx dt + a \int_Q \frac{\partial y}{\partial x} \frac{\partial v}{\partial x} dx dt.$$

We can present the following definition.

Definition 2.2

A function $y \in L^2(0, T; H^1(0, 1))$ is a weak solution to problem (P3') if it satisfies the following conditions (2.68) and $y_x(0, t) = y_x(1, t) = 0$.

Now, we construct a recursive sequence that begins with $y^{(0)} = 0$, and is such that given the previous element $y^{(n-1)}$, the next element in the sequence is iteratively determined.

We will address the following problem, which consists in analyzing its components

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

and seeking a solution through appropriate methods and techniques:

$$\begin{cases} \frac{\partial y^{(n)}}{\partial t} - a \frac{\partial^2 y^{(n)}}{\partial x^2} = G(x, t, y^{(n-1)}) \\ y^{(n)}(x, 0) = 0 \\ \frac{\partial y^{(n)}}{\partial x}(0, t) = \frac{\partial y^{(n)}}{\partial x}(1, t) = 0. \end{cases} \quad (\text{P5})$$

According to Theorem (5.25), for each fixed n , the problem denoted as (P5) has a unique solution. This establishes a foundational aspect of our analysis, ensuring that the solutions are well-defined for each specific case of n through defining $y^{(n)}(x, t)$.

We will explore the proof, by assuming now that $z^{(n)}(x, t) = y^{(n+1)}(x, t) - y^{(n)}(x, t)$, to get a new problem:

$$\begin{cases} \frac{\partial z^{(n)}}{\partial t} - a \frac{\partial^2 z^{(n)}}{\partial x^2} = p^{(n-1)}(x, t) \\ z^{(n)}(x, 0) = 0 \\ \frac{\partial z^{(n)}}{\partial x}(0, t) = \frac{\partial z^{(n)}}{\partial x}(1, t) = 0, \end{cases} \quad (\text{P(z)})$$

where

$$p^{(n-1)}(x, t) = G\left(x, t, z^{(n)}, \frac{\partial z^{(n)}}{\partial x}\right) - G\left(x, t, z^{(n-1)}, \frac{\partial z^{(n-1)}}{\partial x}\right).$$

Lemma 2.1

Assuming that the condition specified in (2.66) is satisfied, we can derive the following a priori estimate for the problem labeled as (P(z)):

$$\|z^n\|_{L^2(0,T; H^1(0,1))} \leq c \|z^{n-1}\|_{L^2(0,T; H^1(0,1))},$$

where

$$c = \sqrt{\frac{k^2 \exp\left(\frac{T}{2}\right)}{\min\left\{\frac{1}{2}, a\right\}}}.$$

Proof Multiplying the equation

$$\frac{\partial z^{(n)}}{\partial t} - a \frac{\partial^2 z^{(n)}}{\partial x^2} = p^{(n-1)}(x, t), \quad (2.37)$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

by $z^{(n)}(x, t)$, and integrating it over Q_τ , we obtain

$$I_1 = I_2,$$

where

$$\begin{aligned} I_1 &= \int_{Q_\tau} \frac{\partial z^{(n)}}{\partial t}(x, t) \cdot z^{(n)}(x, t) dxdt - a \int_{Q_\tau} \frac{\partial^2 z^{(n)}}{\partial x^2}(x, t) \cdot z^{(n)}(x, t) dxdt, \\ I_2 &= \int_{Q_\tau} p^{(n-1)}(x, t) \cdot z^{(n)}(x, t) dxdt. \end{aligned}$$

To calculate I_1 , we apply integration by parts while taking into account the initial and boundary conditions. This leads to the following expression:

$$I_1 = \frac{1}{2} \int_0^1 (z^{(n)}(x, \tau))^2 dx + a \int_{Q_\tau} \left(\frac{\partial z^{(n)}}{\partial x}(x, t) \right)^2 dxdt.$$

For I_2 , using the Cauchy inequality, we find the inequality:

$$I_2 \leq \frac{1}{2} \int_Q (z^{(n)})^2 + \frac{1}{2} \int_Q (p^{(n-1)})^2$$

On the other hand, we can also express $p^{n-1}(x, t)$ as follows:

$$\left| p^{n-1}(x, t) \right|^2 = \left| G \left(x, t, z^{(n)}, \frac{\partial z^{(n)}}{\partial x} \right) - G \left(x, t, z^{(n-1)}, \frac{\partial z^{(n-1)}}{\partial x} \right) \right|^2.$$

Since G is Lipschitzian, we get

$$\int_Q \left| p^{n-1}(x, t) \right|^2 dxdt \leq 2k^2 \int_Q \left(\left| z^{(n-1)} \right|^2 + \left| z_x^{(n-1)} \right|^2 \right) dxdt,$$

so that

$$\int_Q \left| p^{n-1}(x, t) \right|^2 dxdt \leq 2k^2 \|z^{n-1}\|_{L^2(0,T; H^1(0,1))}^2. \quad (2.38)$$

Finally

$$\begin{aligned} & \frac{1}{2} \|z^n(x, \tau)\|_{L^2(0,1)}^2 + a \int_{Q_\tau} \left(\frac{\partial z^{(n)}}{\partial x}(x, t) \right)^2 dxdt \\ & \leq \frac{1}{2} \|z^{(n)}\|_{L^2(0,T; L^2(0,1))}^2 + k^2 \|z^{n-1}\|_{L^2(0,T; H^1(0,1))}^2, \end{aligned}$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

By applying Gronwall's lemma and then considering the maximum on the interval $(0, T)$, we arrive at the following inequality:

$$\begin{aligned} & \|z^{(n)}\|_{L^\infty(0,T; L^2(0,1))}^2 + \|z^{(n)}\|_{L^2(0,T; H^1(0,1))}^2 \\ & \leq \frac{k^2 \exp\left(\frac{T}{2}\right)}{\min\left\{\frac{1}{2}, a\right\}} \|z^{(n-1)}\|_{L^2(0,T; H^1(0,1))}^2 \end{aligned}$$

Thus,

$$\|z^{(n)}\|_{L^2(0,T; H^1(0,1))} \leq c \|\delta z^{(n-1)}\|_{L^2(0,T; H^1(0,1))}.$$

where

$$c = \sqrt{\frac{k^2 \exp\left(\frac{T}{2}\right)}{\min\left\{\frac{1}{2}, a\right\}}},$$

This completes the proof of Lemma 11.

The established conditions and expressions guarantee the convergence of the series $\sum_{n=1}^{\infty} z^{(n)}$ under the following specified assertion:

$$\sqrt{\frac{k^2 \exp\left(\frac{T}{2}\right)}{\min\left\{\frac{1}{2}, a\right\}}} < 1,$$

that can be expressed in the form:

$$k < \sqrt{\frac{\min\left\{\frac{1}{2}, a\right\}}{\exp\left(\frac{T}{2}\right)}}.$$

Since $z^{(n)}(x, t) = y^{(n+1)}(x, t) - y^{(n)}(x, t)$, and $y^{(0)}(x, t) = 0$, we can now proceed to

$$\begin{aligned} \sum_{i=0}^{n-1} z^{(i)} &= \sum_{i=0}^{n-1} \left(y^{(i+1)}(x, t) - y^{(i)}(x, t) \right) \\ &= y^{(n)}, \end{aligned}$$

and the sequence $\left(y^{(n)} \right)_{n \in \mathbb{N}}$ defined by

$$y^{(n)}(x, t) = \sum_{i=0}^{n-1} z^{(i)},$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

converges to an element $y \in L^2(0, T; H^1(0, 1))$.

By systematically addressing these points, we establish the fact that $\lim_{n \rightarrow \infty} y^{(n)}(x, t) = y(x, t)$ is a solution of the problem (P5) by showing that:

$$\Lambda(y, v) = \int_Q v G \left(x, t, y, \frac{\partial y}{\partial x} \right) dx dt.$$

Therefore, we will examine the weak formulation of problem (P5), by redefining the original problem in a such manner that allows for the inclusion of solutions not necessarily classical. However, these still satisfy the necessary conditions in a weaker sense:

$$\Lambda(y^{(n)}, v) = \int_Q \frac{\partial y^{(n)}}{\partial t} v dx dt + a \int_Q \frac{\partial y^{(n)}}{\partial x} \frac{\partial v}{\partial x} dx dt.$$

To sum up, from the linearity of the operator Λ , we have

$$\begin{aligned} \Lambda(y^{(n)}, v) &= \Lambda(y^{(n)} - y, v) + \Lambda(y, v) \\ &= \int_Q \frac{\partial(y^{(n)} - y)}{\partial t} v dx dt + a \int_Q \frac{\partial(y^{(n)} - y)}{\partial x} \frac{\partial v}{\partial x} dx dt \\ &\quad + \Lambda(y, v) \end{aligned} \tag{2.39}$$

We apply the Cauchy-Schwarz inequality on the term $\Lambda(y^{(n)} - y, v)$, leading us to the following inequality:

$$\begin{aligned} &\int_Q x \frac{\partial(y^{(n)} - y)}{\partial t} v dx dt + a \int_Q x \frac{\partial(y^{(n)} - y)}{\partial x} \frac{\partial v}{\partial x} dx dt + b \int_Q xv(y^{(n)} - y) dx dt \\ &\leq 2 \max\{1, a\} \cdot \|v_x\|_{L^2(0, T; H^1(0, 1))} \left[\begin{array}{l} \| (y^{(n)} - y)_t \|_{L^2(0, T; H^1(0, 1))} \\ + \| (y^{(n)} - y) \|_{L^2(0, T; H^1(0, 1))} \end{array} \right]. \end{aligned}$$

As

$$y^{(n)} \longrightarrow y \quad \text{in } L^2(0, T; H^1(0, 1)),$$

we get

$$\begin{aligned} y^{(n)} &\longrightarrow y \quad \text{in } L^2(0, T; L^2(0, 1)), \\ y_t^{(n)} &\longrightarrow y_t \quad \text{in } L^2(0, T; L^2(0, 1)), \\ y_x^{(n)} &\longrightarrow y_x \quad \text{in } L^2(0, T; L^2(0, 1)). \end{aligned}$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

Taking the limit as n approaches $+\infty$, we obtain the following result:

$$\lim_{n \rightarrow +\infty} \Lambda(y^{(n)} - y, v) = 0. \tag{2.40}$$

This limit enables us to draw significant conclusions about the convergence of the sequence $y^{(n)}$ and its relationship with the function v , ultimately shedding light on the properties of the weak formulation we are considering. According to (2.74), the process allows to pass to the limit in (2.73), and we obtain

$$\lim_{n \rightarrow +\infty} \Lambda(y^{(n)}, v) = \Lambda(y, v).$$

■

Theorem 2.3

If the solution of the problem (P3) satisfies (2.66) and

$$k < \sqrt{\frac{\min\left\{\frac{1}{2}, a\right\}}{\exp\left(\frac{T}{2}\right)}},$$

then the problem (P3) admits a weak solution belonging to $L^2(0, T; L^2(0, 1))$.

Now, we show that the solution to the problem (P3) is unique.

Theorem 2.4

The solution is unique if the condition (2.66) is satisfied.

Proof Assume that y_1 and y_2 in $L^2(0, T; H^1(0, 1))$ are two solutions to (P3), so

$$y = y_1 - y_2,$$

is also a solution in $L^2(0, T; H^1(0, 1))$ to the problem:

$$\frac{\partial y}{\partial t} - a \frac{\partial^2 y}{\partial x^2} = G(x, t, y),$$

$$\begin{aligned} y(x, 0) &= 0, \\ \frac{\partial y}{\partial x}(0, t) &= \frac{\partial y}{\partial x}(1, t) = 0, \end{aligned}$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

$$\begin{aligned}\frac{\partial y}{\partial t} - a \frac{\partial^2 y}{\partial x^2} &= \Psi(x, t), \quad \forall (x, t) \in Q \\ y(x, 0) &= 0, \\ \frac{\partial y}{\partial x}(0, t) &= \frac{\partial y}{\partial x}(1, t) = 0\end{aligned}$$

where

$$\Psi(x, t) = G(x, t, y_1) - G(x, t, y_2).$$

By virtue of lemma (2.1), we obtain

$$\|y\|_{L^2(0,T,H^1(0,1))} \leq c \|y\|_{L^2(0,T,H^1(0,1))}.$$

This yields

$$(1 - c) \|y\|_{L^2(0,T,H^1(0,1))} \leq 0,$$

and since $c \leq 1$, we get

$$\|y\|_{L^2(0,T,H^1(0,1))} = 0,$$

i.e.

$$y_1 = y_2.$$

■

2.2 Finite-time blow-up solutions to a super-linear problem

Here, we analyze the phenomenon of finite-time blow-up of solutions to the semi-linear problem (P2). This investigation concern mainly determining the conditions under which solutions may become unbounded within a finite time interval. We also explore the implications of such a behavior on the mathematical model given by (P2) by taking

$$f(x, t, u) = u^p, g(u(x, t)) = h(u(x, t)) = u^r,$$

where p and r are strictly positive integers. Let $u \in C^{1,2}((0, T) \times (0, 1))$ be a positive function with $u_t \geq 0$, solution of the problem:

$$\begin{cases} \frac{\partial u}{\partial t} - a \frac{\partial^2 u}{\partial x^2} = u^p & \forall (x, t) \in Q_\tau, \\ u(x, 0) = \varphi(x) & \forall x \in (0, 1), \\ u_x(0, t) = \int_0^1 k_1(x, t) u^r dx & \forall t \in (0, T), \\ u_x(1, t) = \int_0^1 k_2(x, t) u^r dx & \forall t \in (0, T). \end{cases} \quad (\text{PB})$$

where k_1 and k_2 are positive and bounded functions, with:

$$\alpha_0 \leq k_0 \leq \beta_0, \quad \alpha_1 \leq k_1 \leq \beta_1 \quad (2.41)$$

Theorem 2.5

Problem (PB) blows up in the finite time:

$$T^* = \frac{[\Upsilon(0)]^{1-s}}{C(s-1)},$$

where $s = \min\{r, p\}$, and $\Upsilon(0) = \int_0^1 \varphi(x) dx$.

Proof Integrating the equation

$$\frac{\partial u}{\partial t} - a \frac{\partial^2 u}{\partial x^2} = u^p$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

on $(0, 1)$, we get

$$\left(\int_0^1 u \, dx\right)_t + a \int_0^1 k_1(x, t) u^r \, dx = \int_0^1 u^p \, dx + a \int_0^1 k_2(x, t) u^r \, dx.$$

Then, the two inequalities (2.41) imply

$$\left(\int_0^1 u \, dx\right)_t + a\beta_0 \int_0^1 u^r \, dx \geq \min\{1, a(\alpha_1 - \beta_0)\} \left[\int_0^1 u^p \, dx + \int_0^1 u^r \, dx \right]$$

Next, by applying the Jensen's inequality, we get:

$$\left(\int_0^1 u \, dx\right)_t \geq \min\{1, a(\alpha_1 - \beta_0)\} \left[\left(\int_0^1 u \, dx\right)^p + \left(\int_0^1 u \, dx\right)^r \right]$$

So that

$$\left(\int_0^1 u \, dx\right)_t \geq 2 \min\{1, a(\alpha_1 - \beta_0)\} \left(\int_0^1 u \, dx\right)^s$$

with

$$\begin{cases} \max\{r, p\} & \text{if } \int_0^1 u \, dx \leq 1, \\ \min\{r, p\} & \text{if } \int_0^1 u \, dx \geq 1. \end{cases}$$

Then

$$\left(\int_0^1 u \, dx\right)_t \geq C \left(\int_0^1 u \, dx\right)^s,$$

where $C = 2 \min\{1, a(\alpha_1 - \beta_0)\}$. Set:

$$\Upsilon(t) = \int_0^1 u(x, t) \, dx$$

to get

$$\Upsilon'(t) \geq C\Upsilon^s(t).$$

Consider the Bernoulli's equation:

$$\Upsilon'(t) - C\Upsilon^s(t) = 0.$$

Its solution is:

$$\Upsilon(t) = \left[\frac{1}{(1-s) \left(Ct + \frac{\Upsilon^{1-s}(0)}{1-s} \right)} \right]^{\frac{1}{s-1}}.$$

Since $\frac{1}{s-1} \geq 1$, it follows that

$$\Upsilon \rightarrow \infty \text{ if } (1-s) \left(Ct + \frac{\Upsilon^{1-s}(0)}{1-s} \right) \rightarrow 0.$$

Finally, we obtain

$$T^* = \frac{\Upsilon^{1-s}(0)}{C(s-1)}.$$

■

2.2.1 Numerical simulation of blow-up

Forward-time centred space (FTCS) We can approximate the time derivative by the forward difference quotient, and use the centered second-order approximation for the spatial derivative in (2.1). Using the notations that were introduced above, the obtained iterative scheme can be written as:

$$u_i^{n+1} = ru_{i+1}^n + (1-2r)u_i^n + ru_{i-1}^n + \Delta t (u_i^n)^p \text{ for } i = 1, \dots, M \text{ and } n = 0, 1, \dots, N, \quad (2.42)$$

where $r = a\Delta t / (\Delta x)^2$.

This procedure is explicit, and we still have to determine the two unknowns u_0^{n+1} and u_M^{n+1} . For this purpose, we need to approximate integrals. Concerning the first derivative at $x = 0$, we use the following forward finite difference scheme of second order:

$$u_x(0, t_{n+1}) \simeq \frac{-u_2^{n+1} + 4u_1^{n+1} - 3u_0^{n+1}}{2\Delta x} \quad (2.43)$$

and for the first derivative at $x = 1$, we rather apply the following backward finite difference scheme of second order:

$$u_x(1, t_{n+1}) \simeq \frac{3u_M^{n+1} - 4u_{M-1}^{n+1} + u_{M-2}^{n+1}}{2\Delta x} \quad (2.44)$$

Combining (2.43) and (2.44) and linearizing the terms $(u_0^{n+1})^r$ and $(u_M^{n+1})^r$, we get

$$\begin{aligned}
& \left[-3 - (\Delta x)^2 r k_0(0, t_{n+1}) (u_0^n)^{r-1} \right] u_0^{n+1} - (\Delta x)^2 r k_0(1, t_{n+1}) (u_M^n)^{r-1} u_M^{n+1} \\
& + 2 (\Delta x)^2 \sum_{i=1}^{M-1} k_0(x_i, t_{n+1}) (u_i^{n+1})^r - (\Delta x)^2 r k_1(0, t_{n+1}) (u_0^n)^{r-1} u_0^{n+1} + \\
& \left[3 - (\Delta x)^2 r k_1(1, t_{n+1}) (u_M^n)^{r-1} \right] u_M^{n+1} \\
= & 4u_1^{n+1} + u_2^{n+1} + (\Delta x)^2 (1-r) k_1(0, t_{n+1}) (u_0^n)^r + (\Delta x)^2 (1-r) k_1(1, t_{n+1}) (u_M^n)^r \\
& + 2 (\Delta x)^2 \sum_{i=1}^{M-1} k_1(x_i, t_{n+1}) (u_i^{n+1})^r.
\end{aligned}$$

By setting:

$$\begin{aligned}
a_1 &= -3 - (\Delta x)^2 r k_0(0, t_{n+1}) (u_0^n)^{r-1}, \\
a_2 &= -(\Delta x)^2 r k_0(1, t_{n+1}) (u_M^n)^{r-1}, \\
b_1 &= -(\Delta x)^2 r k_1(0, t_{n+1}) (u_0^n)^{r-1}, \\
b_2 &= 3 - (\Delta x)^2 r k_1(1, t_{n+1}) (u_M^n)^{r-1}, \\
c_1 &= -4u_1^{n+1} + u_2^{n+1} + (\Delta x)^2 (1-r) k_0(0, t_{n+1}) (u_0^n)^r + (\Delta x)^2 (1-r) k_0(1, t_{n+1}) (u_M^n)^r \\
& + 2 (\Delta x)^2 \sum_{i=1}^{M-1} k_0(x_i, t_{n+1}) (u_i^{n+1})^r, \\
c_2 &= 4u_1^{n+1} + u_2^{n+1} + (\Delta x)^2 (1-r) k_1(0, t_{n+1}) (u_0^n)^r + (\Delta x)^2 (1-r) k_1(1, t_{n+1}) (u_M^n)^r \\
& + 2 (\Delta x)^2 \sum_{i=1}^{M-1} k_1(x_i, t_{n+1}) (u_i^{n+1})^r.
\end{aligned}$$

we get the 2x2 system:

$$\begin{aligned}
a_1 u_0^{n+1} - a_2 u_M^{n+1} &= c_1 \\
b_1 u_0^{n+1} - b_2 u_M^{n+1} &= c_2
\end{aligned}$$

whose solution gives the two unknowns u_0^{n+1} and u_M^{n+1} .

An implicit schema We can approximate the time derivative by the forward difference quotient, and use the centered second-order approximation for the spatial derivative in (2.1) to obtain

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - a \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} = (u_i^n)^p.$$

This scheme can be rewritten in the form:

$$(1-2r)u_i^{n+1} = r u_{i+1}^{n+1} + r u_{i-1}^{n+1} + \Delta t (u_i^n)^p \text{ for } i = 1, \dots, M \text{ and } n = 0, 1, \dots, N, \quad (2.45)$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

where $r = a\Delta t / (\Delta x)^2$. This procedure is explicit, and we still have to determine the two unknowns u_0^{n+1} and u_M^{n+1} . To this end, we approximate integrals numerically by means of the trapezoidal rule. We have

$$\begin{aligned}
 (-1 - r (\Delta x)^2 [k_0(0, t_{n+1})(u_i^n)^{r-1}]) u_0^{n+1} &= (\Delta x)^2 [k_0(0, t_{n+1}) [(1 - r)(u_0^n)^r] \\
 &+ k_0(1, t_{n+1}) [(1 - r)(u_M^n)^r + r(u_M^n)^{r-1}u_M^{n+1}] \\
 &+ 2 \sum_{i=1}^{M-1} k_0(x_i, t_{n+1}) [(1 - r)(u_i^n)^r + r(u_i^n)^{r-1}u_i^{n+1}] \\
 &+ u_2^{n+1}, \tag{2.46}
 \end{aligned}$$

$$\begin{aligned}
 (1 - r (\Delta x)^2 k_1(1, t_{n+1})(u_M^n)^{r-1}) u_M^{n+1} &= (\Delta x)^2 [k_1(0, t_{n+1}) [(1 - r)(u_0^n)^r + r(u_i^n)^{r-1}u_0^{n+1}] \\
 &+ k_1(1, t_{n+1}) [(1 - r)(u_M^n)^r] \\
 &+ 2 \sum_{i=1}^{M-1} k_1(x_i, t_{n+1}) [(1 - r)(u_i^n)^r + r(u_i^n)^{r-1}u_i^{n+1}] \\
 &- u_{M-2}^{n+1}. \tag{2.47}
 \end{aligned}$$

By setting:

$$\begin{aligned}
 a_1 &= (-1 - r (\Delta x)^2 [k_0(0, t_{n+1})(u_i^n)^{r-1}]), \\
 a_2 &= -(\Delta x)^2 k_0(1, t_{n+1})r(u_M^n)^{r-1}, \\
 b_1 &= -(\Delta x)^2 r k_1(0, t_{n+1}) (u_0^n)^{r-1}, \\
 b_2 &= (1 - r (\Delta x)^2 k_1(1, t_{n+1})(u_M^n)^{r-1}), \\
 c_1 &= (\Delta x)^2 [k_1(0, t_{n+1}) [(1 - r)(u_0^n)^r] + k_0(1, t_{n+1}) [(1 - r)(u_M^n)^r] \\
 &+ 2 \sum_{i=1}^{M-1} k_0(x_i, t_{n+1}) [(1 - r)(u_i^n)^r + r(u_i^n)^{r-1}u_i^{n+1}]] + u_2^{n+1}, \\
 c_2 &= (\Delta x)^2 [k_1(0, t_{n+1}) [(1 - r)(u_0^n)^r] + k_1(1, t_{n+1}) [(1 - r)(u_M^n)^r] \\
 &+ 2 \sum_{i=1}^{M-1} k_1(x_i, t_{n+1}) [(1 - r)(u_i^n)^r + r(u_i^n)^{r-1}u_i^{n+1}]] - u_{M-2}^{n+1},
 \end{aligned}$$

we obtain the 2x2 system:

$$\begin{aligned}
 a_1 u_0^{n+1} - a_2 u_M^{n+1} &= c_1 \\
 b_1 u_0^{n+1} - b_2 u_M^{n+1} &= c_2
 \end{aligned}$$

whose solution gives the two unknowns u_0^{n+1} and u_M^{n+1} . Subsequently, considering (1.45), (1.46) and (1.47) together, the global system can be expressed in eth matrix

form:

$$A^{n+1}U^{n+1} = B^{n+1} \quad (2.48)$$

2.2.2 Numerical examples

In this section, we will use the two discrete finite methods derived in the last two sections (explicit and compact finite difference scheme). Three numerical experiments will be considered.

$$\text{Problem 1 : } \left\{ \begin{array}{l} u_t = au_{xx} + u^2 \quad 0 < x < 1, 0 < t < T \\ u_x(0, t) = \int_0^1 k_0(x, t)u^r dx \quad \forall t \in (0, T) \\ u_x(1, t) = \int_0^1 k_1(x, t)u^r dx \quad \forall t \in (0, T) \\ u(x, 0) = A(x+1)^2 \quad x \in (0, 1) \end{array} \right\}$$

$$\text{Problem 2 : } \left\{ \begin{array}{l} u_t = au_{xx} + u^3 \quad 0 < x < 1, 0 < t < T \\ u_x(0, t) = \int_0^1 k_0(x, t)u^r dx \quad \forall t \in (0, T) \\ u_x(1, t) = \int_0^1 k_1(x, t)u^r dx \quad \forall t \in (0, T) \\ u(x, 0) = A(x+1)^2 \quad x \in (0, 1) \end{array} \right\}$$

$$\text{Problem 3 : } \left\{ \begin{array}{l} u_t = au_{xx} + u^4 \quad 0 < x < 1, 0 < t < T \\ u_x(0, t) = \int_0^1 k_0(x, t)u^r dx \quad \forall t \in (0, T) \\ u_x(1, t) = \int_0^1 k_1(x, t)u^r dx \quad \forall t \in (0, T) \\ u(x, 0) = A(x+1)^2 \quad x \in (0, 1) \end{array} \right\}$$

where $k_1(x, t) = \frac{2}{(A^2)*(x+1)^6}$, $k_2(x, t) = \frac{4}{(A^2)*(x+1)^6}$ and $A = 12$.

Since the analytical (exact) solutions to problems 1, 2 and 3 with the associated initial condition are not known, we can only estimate numerically the blow-up times. In the next tables 2.5-2.7, we present blow-up times and the CPU times (CPUT) for different values of the space-step using explicit and compact backward Euler scheme(CBES). The numerical blow-up time is computed as the first time when $\|u^n\|_\infty \geq 10^6$. Our numerical experiments are performed using Matlab on an Intel Core i3 with 2.1 GHz .

h	T^* by explicit scheme	CPUT	T^* by CBES	CPUT
1/40	0.02021781	0.7639	0.02022113	1.316
1/80	0.02021781	4.408	0.02022064	27.229
1/160	0.02021875	62.120	0.02021635	344.681
1/320	0.02021781	307.456	0.02021456	1842.532

Table 2.5: Problem 1, $p = 2$, Explicit and implicit schemes.

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

h	T^* by explicit scheme	CPUT	T^* by CBES	CPUT
1/40	0.00003347	0.0133	0.00031258	0.1915
1/80	0.000032854	0.035	0.00030986	1.932
1/160	0.00031258	0.501	0.00030758	3.4815
1/320	0.00030761	1.567	0.00030732	63.462

Table 2.6: Problem 2, $p = 3$, Explicit and implicit schemes.

h	T^* by explicit scheme	CPUT	T^* by CBES	CPUT
1/40	0.0000229157	0.0402	0.00001972346	0.1213
1/80	0.0000212346	0.0469	0.00001953125	1.29
1/160	0.00001972346	0.0485	0.0000194992	3.315
1/320	0.00001953125	0.497	0.0000194931	56.157

Table 2.7: Problem 3, $p = 4$, Explicit and implicit schemes.

Figures 2.3-2.5 display present the discrete graph of the numerical solution of problems 1-3 obtained using explicit and CBE schemes.

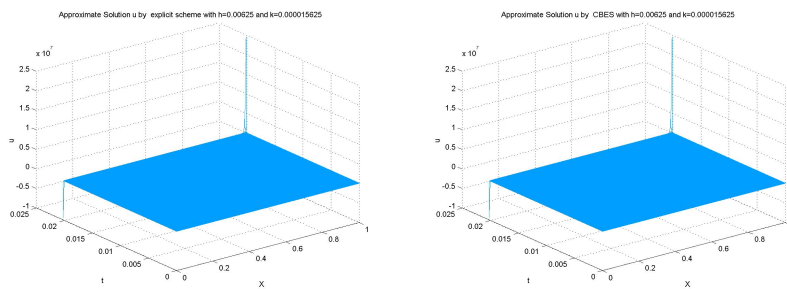


Figure 2.3: Numerical solution by (a) explicit and (b) CBE schemes for Problem 1.

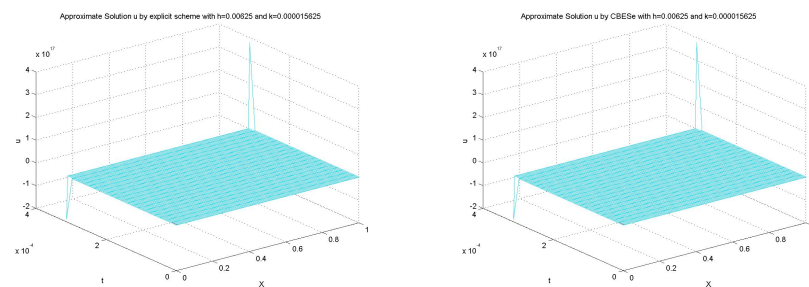


Figure 2.4: Numerical solution by (a) explicit and (b) CBE schemes for Problem 2.

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

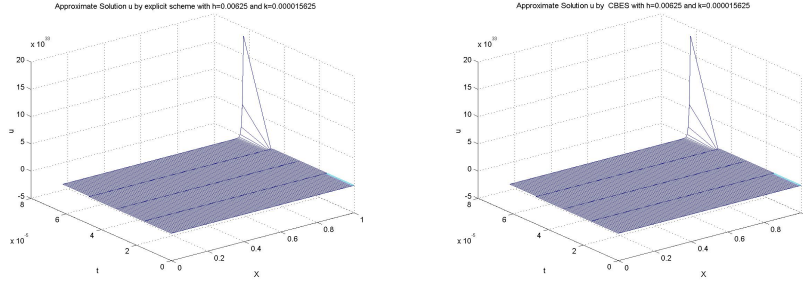


Figure 2.5: Numerical solution by (a) explicit and (b) CBE schemes for Problem 3.

2.3 Solvability of the problem for $b \neq 0, c = 0$:

In this section, we focus on the corresponding problem (2.1) to establish both the existence and uniqueness of solution in relation to the associated linear problem. This analysis involves demonstrating that under certain conditions, a solution to the linear problem not only exists but is also unique, thereby providing a foundational understanding for the broader context of the problem at hand.

Let $(0, T)$ and $\Omega = (0, 1)$ be open bounded intervals in \mathbb{R} . We consider the domain $Q_T = \Omega \times (0, T)$ in which we study a nonlinear parabolic problem of the form:

$$\begin{cases} u_t - a\Delta u + bu = f(x, t) & (x, t) \in Q_T \\ u(x, 0) = \varphi(x) & x \in \Omega \\ u_x(0, t) = \int_0^1 k_1(x, t)g(u(x, t))dx & t \in (0, T) \\ u_x(1, t) = \int_0^1 k_2(x, t)h(u(x, t))dx & t \in (0, T), \end{cases} \quad (2.49)$$

where φ is a given function, a and b positive integers, $p \geq 1$ and k_1, k_2 given bounded functions.

Proof First, the solution existence is proved by applying the Faedo-Galerkin method. Let $(w_k)_{k \in \mathbb{N}}$ be a sequence of linearly independent vectors in $V = H^1(\Omega)$ whose finite linear combinations are dense in V . By possibly using the Gram-Schmidt orthogonalization process we can assume $(w_k)_{k \in \mathbb{N}}$ to be orthonormal in $L^2(\Omega)$. Moreover, for any integer $m \geq 1$, a function u_m is defined as follows:

$$t \mapsto u_m(x, t) = \sum_{i=1}^m S_{im}(t) w_i(x).$$

Also, $\varphi_k \rightarrow \varphi$ in V . For a fixed $k \in \mathbb{N}^*$, we seek an approximate solution to the

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

problem (2.49), that is, for solutions of the finite-dimensional problem:

$$\begin{cases} ((u_k)_t, w_i) - a(\Delta u_k, w_i) + b(u_k, w_i) = (f, w_i), & i = \overline{1, k}, \\ u_k(0) = \varphi_k. \end{cases} \quad (2.50)$$

By simplifying the problem (2.50), it follows that

$$\begin{cases} ((u_k)_t, w_i) + a(\nabla u_k, \nabla w_i) + a \int_{\Omega} k_1 g(u_k) dx w_k(0) + b(u_k, w_i) \\ = (f, w_i) + a \int_{\Omega} k_2 h(u_k) dx w_k(1), & i = \overline{1, k}, \\ u_k(0) = \varphi_k. \end{cases} \quad (2.51)$$

By using the Picard–Lindelöf theorem, the problem (2.51) admits a unique local solution $u_k(t) \in C[0, t_k]$ et $u'_k(t) \in L^2[0, T]$. Then, by multiplying the equation (2.51) by $S_{i_k}(t)$ and summing up for $i = 1, \dots, k$, we obtain

$$\begin{aligned} ((u_k)_t, u_k) + a(\nabla u_k, \nabla u_k) + b(u_k, u_k) &= (f, u_k) + a \int_{\Omega} k_2 h(u_k) dx u_k(1, t) \\ &\quad - a \int_{\Omega} k_1 g(u_k) dx u_k(0, t), \end{aligned}$$

Assume that

$$\forall x \in \mathbb{R} : \int_{\Omega} h(x) dx \leq \int_{\Omega} g(x) dx$$

,
we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_k\|_2^2 + a \|\nabla u_k\|_2^2 + b \|u_k\|_2^2 &\leq \frac{1}{2\varepsilon} \|u_k\|_2^2 + \frac{\varepsilon}{2} \|f\|_2^2 + a\beta \int_{\Omega} h(u_k) dx \int_0^1 \nabla u_k dx, \\ &\leq \frac{1}{2\varepsilon} \|u_k\|_2^2 + \frac{\varepsilon}{2} \|f\|_2^2 + \frac{a\beta}{2\lambda} \|h(u_k)\|_2^2 + \frac{a\beta\lambda}{2} \|\nabla u_k\|_2^2. \end{aligned}$$

Then, we using that h is Lipschitzian function satisfying, we obtain

$$\frac{d}{dt} \|u_k\|_2^2 + 2 \left(a - \frac{a\beta\lambda}{2} - \frac{a\beta c}{2\lambda} \right) \|\nabla u_k\|_2^2 \leq 2 \left(\frac{1}{2\varepsilon} + \frac{a\beta c}{2\lambda} - b \right) \|u_k\|_2^2 + \varepsilon \|f\|_2^2.$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

Now, using the Gronwall's lemma, it follows that

$$\|u_k\|_2^2 + 2 \left(a - \frac{a\beta\lambda}{2} - \frac{a\beta c}{2\lambda} \right) \|\nabla u_k\|_2^2 \leq (\varepsilon \|f\|_2^2 + \|\varphi\|_2^2) \exp \left(2 \left(\frac{1}{2\varepsilon} + \frac{a\beta c}{2\lambda} - b \right) T \right). \quad (2.52)$$

Next, let us multiply (2.51) by $S'_{im}(t)$ and sum up for $i = 1, \dots, k$. Similarly, we get

$$\begin{aligned} & \|(u_k)_t\|_2^2 + \frac{a}{2} \frac{d}{dt} \|\nabla u_k\|_2^2 + \frac{b}{2} \frac{d}{dt} \|u_k\|_2^2 \\ & \leq \frac{1}{2\varepsilon} \|(u_k)_t\|_2^2 + \frac{\varepsilon}{2} \|f\|_2^2 \\ & \quad + a\beta \left(\int_{\Omega} h(u_k) dx (u_k)_t(1, t) - \int_{\Omega} g(u_k) dx (u_k)_t(0, t) \right) \end{aligned}$$

After that, by simplifying and integrating over $(0, T)$, it follows that

$$\begin{aligned} & \int_0^t \|(u_k)_t\|_2^2 + \frac{a}{2} \|\nabla u_k\|_2^2 + \frac{b}{2} \|u_k\|_2^2 \\ & \leq \frac{1}{2\varepsilon} \int_0^t \|(u_k)_t\|_2^2 + \frac{\varepsilon}{2} \int_0^t \|f\|_2^2 + \frac{a\beta\gamma}{2} \|u_k\|_{H^1(\Omega)}^2 \\ & \quad + \frac{a\beta}{\gamma} \|\nabla u_k\|_2^2 + \frac{a\beta}{\gamma} \|\nabla \varphi_k\|_2^2 + \frac{a}{2} \|\nabla \varphi_k\|_2^2 + \frac{b}{2} \|\varphi_k\|_2^2. \end{aligned}$$

By setting: $\forall \varepsilon > 0, \forall \gamma > 0$: $C_1 = 1 - \frac{1}{2\varepsilon}$, $C_2 = \frac{a}{2} - \frac{a\beta\gamma}{2}$, $C_3 = \frac{b}{2} - \frac{a\beta\gamma}{2}$, $C_4 = \max \left(\frac{a\beta}{\gamma} + \frac{a}{2}, \frac{b}{2} \right)$, the latter inequality implies

$$C_1 \int_0^T \|(u_k)_t\|_2^2 + C_2 \sup_{0 < t < T} \|\nabla u_k\|_2^2 + C_3 \sup_{0 < t < T} \|u_k\|_2^2 \leq \frac{\varepsilon}{2} \int_0^T \|f\|_2^2 + C_4 \|\varphi_k\|_{H^1(\Omega)}^2. \quad (2.53)$$

We subsequently obtain the following additional a priori estimates, that provide valuable bounds and insights into the behavior of the solutions under consideration:

$$\begin{cases} u_k \text{ is uniformly bounded in } L^\infty(0, T; H^1(\Omega)), \\ u_k \text{ is uniformly bounded in } L^2(0, T; H^1(\Omega)), \\ (u_k)_t \text{ is uniformly bounded in } L^2(0, T; L^2(\Omega)) \end{cases} \quad (2.54)$$

Thus, according to the Lemma 1.3 and 1.4, there exists a subsequence of u_k , still denoted by u_k , that retains the relevant properties necessary for our analysis.

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

Among many others, we can cite the following properties:

$$\begin{cases} u_k \longrightarrow u \text{ weakly}^* \text{ in } L^\infty(0, T; H^1(\Omega)), \\ u_k \longrightarrow u \text{ weakly in } L^2(0, T; H^1(\Omega)), \\ (u_k)_t \longrightarrow v \text{ weakly in } L^2(0, T; L^2(\Omega)), \\ \nabla u_k \longrightarrow w \text{ weakly in } L^2(0, T; L^2(\Omega)). \end{cases} \quad (2.55)$$

We deduce from lemma 1.5 there are subsequences denoted by (u_{k_n}) and $\left(\frac{\partial u_{k_n}}{\partial t}\right)$ of (u_k) and $(u_k)_t$ respectively, such that

$$u_{k_n} \rightharpoonup u \quad \text{in } L^2(0, T; H^1(\Omega)), \quad (2.56)$$

$$\frac{\partial u_{k_n}}{\partial t} \rightharpoonup w \quad \text{in } L^2(0, T; L^2(\Omega)). \quad (2.57)$$

We know that, according to the Rellich-Kondrachov's theorem, the injection of $H^1(Q)$ into $L^2(Q)$ is compact. Similar to the results derived from the Rellich's theorem, any weakly convergent sequence in $H^1(Q)$ possesses a subsequence that converges strongly in $L^2(Q)$. Therefore, we can conclude that the properties of weak convergence in $H^1(Q)$ ensure the existence of a strongly convergent subsequence in $L^2(Q)$, facilitating our analysis of the solutions in these function spaces. Thus,

$$u_{k_n} \longrightarrow u \quad \text{in } L^2(Q). \quad (2.58)$$

On the other hand, according to Lemma 1.4, which we will continue to denote as $(u_{k_n})_n$, that converges almost everywhere to u_{k_n} converges almost everywhere to u . Consequently, this subsequence exhibits the following property:

$$u_{k_n} \longrightarrow u \quad \text{almost everywhere in } Q. \quad (2.59)$$

By Lemma 10 and 11, there is a subsequence of u_k , still denoted by u_k , that u_k converges to u almost everywhere in $Q_T = \Omega \times [0, T]$. It remains to show that $w = \frac{\partial u}{\partial t}$. For that it suffices to prove

$$u(t) = \varphi + \int_0^t w(\tau) d\tau. \quad (2.60)$$

As

$$u_{k_n} \rightharpoonup u \quad \text{in } L^2(0, T; L^2(\Omega)),$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

the proof of (2.60) is equivalent to verify that

$$u_{k_n} \rightharpoonup \varphi + \chi \quad \text{in } L^2(0, T; L^2(\Omega)),$$

i.e.

$$\lim (u_{k_n} - \varphi - \chi, v)_{L^2(0, T; L^2(\Omega))} = 0, \quad \forall v \in L^2(0, T; L^2(\Omega)),$$

where

$$\chi(t) = \int_0^t w(\tau) d\tau.$$

Using the equality:

$$u_{k_n} - \varphi_{k_n} = \int_0^t \frac{\partial u_{k_n}}{\partial \tau} d\tau, \quad \forall t \in [0, T],$$

since $u_{k_n} \in L^2(0, T; V_{k_n})$ and $(u_{k_n})_t \in L^2(0, T; V_{k_n})$, it follows that

$$\begin{aligned} & \left(u_{k_n} - \varphi - \int_0^t w(\tau) d\tau, v \right)_{L^2(0, T; L^2(\Omega))} \\ &= \left(\int_0^t \left(\frac{\partial u_{k_n}}{\partial \tau} - w(\tau) \right) d\tau, v \right)_{L^2(0, T; L^2(\Omega))} + (\varphi_{k_n} - \varphi, v)_{L^2(0, T; L^2(\Omega))}, \quad \forall t \in [0, T]. \end{aligned}$$

By virtue of Lemma 1.5, we get

$$\begin{aligned} & \left(u_{k_n} - \varphi - \int_0^t w(\tau) d\tau, v \right)_{L^2(0, T; L^2(\Omega))} \\ &= \int_0^t \left(\frac{\partial u_{k_n}}{\partial \tau} - w(\tau), v \right)_{L^2(0, T; L^2(\Omega))} d\tau + (\varphi_{k_n} - \varphi, v)_{L^2(0, T; L^2(\Omega))}, \quad \forall t \in [0, T]. \end{aligned}$$

On the one hand, we have

$$\lim_{n \rightarrow \infty} \int_0^t \left(\frac{\partial u_{k_n}}{\partial \tau} - w(\tau), v \right)_{L^2(0, T; L^2(\Omega))} d\tau = 0, \quad \text{for } t \in [0, T]. \quad (2.61)$$

Likewise,

$$\lim_{k \rightarrow \infty} (\varphi_k - \varphi, v)_{L^2(0, T; L^2(\Omega))} = 0. \quad (2.62)$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

Therefore,

$$\lim_{k \rightarrow \infty} (u_{m_k} - \varphi - \chi, v)_{L^2(0,T; L^2(\Omega))} = 0, \quad \forall v \in L^2(0, T; L^2(\Omega)).$$

Finally, combinig (2.61) and (2.62) leads to

$$\lim_{k \rightarrow \infty} (u_{m_k} - \varphi - \chi, v)_{L^2(0,T; L^2(\Omega))} = 0, \quad \forall v \in L^2(0, T; L^2(\Omega)).$$

Now, passing to the limit in (P2), we obtain

$$((u_m(t))_t, w_k) + a(\Delta u_m(t), w_k) + b(u_m(t), w_k) = (f(t), w_k) \quad \forall k = \overline{1, m}. \quad (2.63)$$

Since $\{w_j, j \in \mathbb{N}\}$ is a basis of V , we get

$$u' - a\Delta u + bu = f. \quad (2.64)$$

As $u', \Delta u, u$ and f all belong to $L^2(0, T; L^2(\Omega))$, (2.64) also holds in $L^2(0, T; L^2(\Omega))$. ■

If u is a solution to problem (2.1) and w is a solution to problem (2.49), then $v = u - w$ satisfies

$$\left\{ \begin{array}{ll} \mathcal{L}y = \frac{\partial y}{\partial t} - a \frac{\partial^2 y}{\partial x^2} + by = F(x, t, y) & (x, t) \in Q_T \\ y(x, 0) = 0 & x \in (0, 1) \\ \frac{\partial y}{\partial x}(x, 0) = 0 & t \in (0, T) \\ \frac{\partial y}{\partial x}(1, t) = 0 & t \in (0, T). \end{array} \right. \quad (2.65)$$

where $G(x, t, y) = f(x, t, y)$.

It is clear that if the function f is Lipschitzian, then the function G is also Lipschitzian, i.e, there exists a positive constant k such that

$$\|G(x, t, v_1) - G(x, t, v_2)\|_{L^2(Q)} \leq k \|v_1 - v_2\|_{L^2(0,T;H^1(\Omega))}, \quad (2.66)$$

for any $(v_1, v_2) \in (L^2(Q))^2$.

Next, we will focus on proving that problem (2.1) has a unique solution which depends continuously on the data. So it remains to prove that the problem (2.65) admits a unique weak solution. First, we propose the studied concept of the

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

solution.

Let $v = v(x, t)$ be any function in $L^2(0, T; H^1(\Omega))$. Multiplying the equation in (2.65) by v and integrating the resulting one on Q , we obtain

$$\int_Q \frac{\partial y}{\partial t} v dx dt - a \int_Q \frac{\partial^2 v}{\partial x^2} v dx dt = \int_Q v G(x, t, v) dx dt.$$

By applying the integration by parts and the condition on y , we can derive the following expression:

$$\int_Q \frac{\partial y}{\partial t} v dx dt + a \int_Q \frac{\partial y}{\partial x} \frac{\partial v}{\partial x} dx dt = \int_Q v F(x, t, y) dx dt, \quad (2.67)$$

From (2.67), it follows

$$\Theta(h, v) = \int_Q v F(x, t, h) dx dt, \quad (2.68)$$

where

$$\Theta(y, v) = \int_Q \frac{\partial y}{\partial t} v dx dt + a \int_Q \frac{\partial y}{\partial x} \frac{\partial v}{\partial x} dx dt.$$

Definition 2.3

A function $y \in L^2(0, T; H^1(0, 1))$ is said to be a weak solution of problem (2.65) if y verifies (2.68) and $y_x(0, t) = y_x(1, t) = 0$.

Now, let us move on to the construction of a recursive sequence. Starting with $y^{(0)} = 0$, the sequence $(y^{(n)})_{n \in \mathbb{N}}$ is defined as follows: given $y^{(n-1)}$, for $n = 1, 2, 3, \dots$, y will be solution of the nonlinear parabolic problem in the domain Q_T

$$\begin{cases} \frac{\partial y^{(n)}}{\partial t} - a \frac{\partial^2 y^{(n)}}{\partial x^2} = F(x, t, y^{(n-1)}) \\ y^{(n)}(x, 0) = 0, \\ \frac{\partial y^{(n)}}{\partial x}(0, t) = \frac{\partial y^{(n)}}{\partial x}(1, t) = 0. \end{cases} \quad (2.69)$$

As stated in Theorem (2.1), for each fixed n , problem (2.69) admits a unique solution $y^{(n)}(x, t)$. Introduce the notation: $z^{(n)}(x, t) = y^{(n+1)}(x, t) - y^{(n)}(x, t)$,

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

yielding the following new problem:

$$\begin{cases} \frac{\partial z^{(n)}}{\partial t} - a \frac{\partial^2 z^{(n)}}{\partial x^2} = p^{(n-1)}(x, t), \\ z^{(n)}(x, 0) = 0, \\ \frac{\partial z^{(n)}}{\partial x}(0, t) = \frac{\partial z^{(n)}}{\partial x}(1, t) = 0. \end{cases} \quad (2.70)$$

where

$$p^{(n-1)}(x, t) = G\left(x, t, z^{(n)}, \frac{\partial z^{(n)}}{\partial x}\right) - G\left(x, t, z^{(n-1)}, \frac{\partial z^{(n-1)}}{\partial x}\right).$$

Lemma 2.2

Provided that condition (2.66) is verified, the following a priori estimate holds for problem (2.70):

$$\|z^n\|_{L^2(0,T; H^1(\Omega))} \leq c \|z^{n-1}\|_{L^2(0,T; H^1(\Omega))},$$

where

$$c = \sqrt{\frac{k^2 \exp\left(\frac{T}{2}\right)}{\min\left\{\frac{1}{2}, a\right\}}}.$$

Proof By multiplying the equation

$$\frac{\partial z^{(n)}}{\partial t} - a \frac{\partial^2 z^{(n)}}{\partial x^2} = p^{(n-1)}(x, t), \quad (2.71)$$

by $z^{(n)}(x, t)$, and then integrating on Q_τ , we get

$$I_1 = I_2,$$

where

$$\begin{aligned} I_1 &= \int_{Q_\tau} \frac{\partial z^{(n)}}{\partial t}(x, t) \cdot z^{(n)}(x, t) dx dt - a \int_{Q_\tau} \frac{\partial^2 z^{(n)}}{\partial x^2}(x, t) \cdot z^{(n)}(x, t) dx dt \\ &\quad + b \int_{Q_\tau} z^{(n)}(x, t) \cdot z^{(n)}(x, t) dx dt, \\ I_2 &= \int_{Q_\tau} p^{(n-1)}(x, t) \cdot z^{(n)}(x, t) dx dt. \end{aligned}$$

For determining I_1 , it suffices to integrate by parts and then consider the initial

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

and boundary conditions to obtain the following expression:

$$I_1 = \frac{1}{2} \int_0^1 (z^{(n)}(x, \tau))^2 dx + a \int_{Q_\tau} \left(\frac{\partial z^{(n)}}{\partial x}(x, t) \right)^2 dx dt + b \int_{Q_\tau} (z^{(n)}(x, t))^2 dx dt.$$

Regarding I_2 , the use of the Cauchy's inequality results in the following inequality:

$$I_2 \leq \frac{1}{2} \int_Q (z^{(n)})^2 + \frac{1}{2} \int_Q (p^{(n-1)})^2$$

Moreover, G satisfies the Lipschitz condition, we derive the following:

$$|p^{n-1}(x, t)|^2 \leq 2k^2 \left(|y^{(n-1)}|^2 + |y_x^{(n-1)}|^2 \right),$$

so that

$$\int_{Q_T} |p^{n-1}(x, t)|^2 dx dt \leq 2k^2 \|z^{n-1}\|_{L^2(0, T; H^1(0, 1))}^2. \quad (2.72)$$

Finally, by using the Gronwall's lemma and taking the supremum over the interval $(0, T)$, we can get

$$\|z^n\|_{L^\infty(0, T; L^2(0, 1))}^2 + \|z^n\|_{L^2(0, T; H^1(0, 1))}^2 \leq \frac{k^2 \exp\left(\frac{T}{2}\right)}{\min\left\{\frac{1}{2}, a, b\right\}} \|z^{n-1}\|_{L^2(0, T; H^1(0, 1))}^2,$$

Then, we obtain

$$\|z^n\|_{L^2(0, T; H^1(0, 1))} \leq c \|z^{n-1}\|_{L^2(0, T; H^1(0, 1))}.$$

where

$$c = \sqrt{\frac{k^2 \exp\left(\frac{T}{2}\right)}{\min\left\{\frac{1}{2}, a, b\right\}}}.$$

The proof of the lemma is completed.

We can apply various convergence tests to determine the behavior of $\sum_{n=1}^{\infty} z^{(n)}$, according to the convergence criterion for series. Specifically, this series is convergent if

$$k < \sqrt{\frac{\min\left\{\frac{1}{2}, a, b\right\}}{\exp\left(\frac{T}{2}\right)}}.$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

Since $z^{(n)}(x, t) = y^{(n+1)}(x, t) - y^{(n)}(x, t)$, and $y^{(0)}(x, t) = 0$, we have

$$\sum_{i=0}^{n-1} z^{(i)} = y^{(n)},$$

so that $(y^{(n)})_{n \in \mathbb{N}}$, defined by

$$y^{(n)}(x, t) = \sum_{i=0}^{n-1} J^{(i)},$$

converges to an element $y \in L^2(0, T; H^1(0, 1))$. ■

Next, we show that $\lim_{n \rightarrow \infty} y^{(n)}(x, t) = y(x, t)$ is a solution to problem (2.70) proving that

$$\Theta(y, v) = \int_Q v F \left(x, t, y, \frac{\partial y}{\partial x} \right) dx dt.$$

To derive the weak formulation, we multiply the equation by a test function v belonging to an appropriate function space of the subsequent problem (2.70). This leads to the following integral form:

$$\Theta(y^{(n)}, v) = \int_Q \frac{\partial y^{(n)}}{\partial t} v dx dt + a \int_Q \frac{\partial y^{(n)}}{\partial x} \frac{\partial v}{\partial x} dx dt + b \int_Q y^{(n)} v dx dt.$$

Due to the linearity of Θ , we can write

$$\begin{aligned} \Theta(y^{(n)}, v) &= \Theta(y^{(n)} - y, v) + \Theta(y, v) \\ &= \int_Q \frac{\partial(y^{(n)} - y)}{\partial t} v dx dt + a \int_Q \frac{\partial(y^{(n)} - y)}{\partial x} \frac{\partial v}{\partial x} dx dt + b \int_Q (y^{(n)} - y) v dx dt \\ &\quad + \Theta(y, v) \end{aligned} \tag{2.73}$$

This approach allows to apply the Cauchy-Schwartz inequality in $\Theta(y^{(n)} - y, v)$, and we get

$$\begin{aligned} &\int_Q \frac{\partial(y^{(n)} - y)}{\partial t} v dx dt + a \int_Q \frac{\partial(y^{(n)} - y)}{\partial x} \frac{\partial v}{\partial x} dx dt + b \int_Q v(y^{(n)} - y) dx dt \\ &\leq 2 \max\{1, a\} \cdot \|v_x\|_{L^2(0, T; H^1(0, 1))} \left[\begin{aligned} &\| (y^{(n)} - y)_t \|_{L^2(0, T; H^1(0, 1))} \\ &+ \| (y^{(n)} - y) \|_{L^2(0, T; H^1(0, 1))} \end{aligned} \right]. \end{aligned}$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

On the other hand, since

$$y^{(n)} \longrightarrow y \quad \text{in } L^2(0, T; H^1(\Omega)),$$

we get

$$\begin{aligned} y^{(n)} &\longrightarrow y && \text{in } L^2(0, T; L^2(\Omega)), \\ y_t^{(n)} &\longrightarrow y_t && \text{in } L^2(0, T; L^2(\Omega)), \\ y_x^{(n)} &\longrightarrow y_x && \text{in } L^2(0, T; L^2(\Omega)). \end{aligned}$$

By passing to the limit when $n \rightarrow +\infty$ we get:

$$\lim_{n \rightarrow +\infty} \Theta(y^{(n)} - y, v) = 0. \quad (2.74)$$

According to (2.74), by passing to the limit in (2.73), we obtain

$$\lim_{n \rightarrow +\infty} \Theta(y^{(n)}, v) = \Theta(y, v).$$

Thus, we have just proved the following result.

Theorem 2.6

If the solution to the problem (2.70) satisfies (2.66), and

$$k < \sqrt{\frac{\min\left\{\frac{1}{2}, a, b\right\}}{\exp\left(\frac{T}{2}\right)}},$$

then problem (2.70) admits a weak solution in $L^2(0, T; L^2(\Omega))$.

Now, let us move to the solution uniqueness of the given nonlinear parabolic problem (2.70). This result is crucial for the reliability and applicability of the solution in various contexts.

Theorem 2.7

If condition (2.66) is satisfied, then the solution is unique.

Proof Let $y_1, y_2 \in L^2(0, T; H^1(\Omega))$ be two solutions of (2.70). Then

$$y = y_1 - y_2,$$

is also a solution in $L^2(0, T; H^1(\Omega))$ of the problem:

$$\begin{aligned} \frac{\partial y}{\partial t} - a \frac{\partial^2 y}{\partial x^2} + by &= \Psi(x, t), & \forall (x, t) \in Q_T \\ y(x, 0) &= 0, \\ \frac{\partial y}{\partial x}(0, t) &= \frac{\partial y}{\partial x}(1, t) = 0, \end{aligned}$$

where

$$\Psi(x, t) = G(x, t, y_1) - G(x, t, y_2).$$

By invoking the context of Lemma (2.3), we obtain

$$\|y\|_{L^2(0,T,H^1(\Omega))} \leq c \|y\|_{L^2(0,T,H^1(\Omega))},$$

i.e.

$$(1 - c) \|y\|_{L^2(0,T,H^1(\Omega))} \leq 0.$$

Since $c \leq 1$, it follows that

$$\|y\|_{L^2(0,T,H^1(\Omega))} = 0,$$

i.e.

$$y_1 = y_2,$$

■

2.4 Solvability of a semi-linear parabolic equation with second integral conditions

The solvability of semi-linear parabolic equations with integral conditions mainly involves analyzing the existence, uniqueness.

Let $\Omega = (0, l)$ be a bounded open of \mathbb{R} and $Q_T = \Omega \times (0, T)$. Consider the following problem:

$$\left\{ \begin{aligned} \frac{\partial u}{\partial t} - a \frac{\partial^2 u}{\partial x^2} + u^p - bu &= f(x, t) & \forall (x, t) \in Q_T, \\ u(x, 0) &= \varphi(x) & \forall x \in (0, l), \\ \frac{\partial u}{\partial x}(0, t) &= \int_0^l k_1(x, t) u(x, t) dx & \forall t \in [0, T], \\ \frac{\partial u}{\partial x}(l, t) &= \int_0^l k_2(x, t) u(x, t) dx & \forall t \in [0, T], \end{aligned} \right. \quad (PI_1)$$

where a, b and p are positive odd integers, and k_1, k_2 are bounded functions.

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

The aim of this section is to define a solution $u = u(x, t)$, $x \in \Omega$, $t \in [0, T]$; to problem (PI_1) based on specific assumptions (H) . In order to properly state the problem, and to have the tools to solve it, we need to introduce some functional concepts and spaces that we will use later. We define the space:

$$V = H^1(\Omega) \cap L^{p+1}(\Omega),$$

endowed with the norm $\|v\|_V = \|v\|_{H^1(\Omega)} + \|v\|_{L^{p+1}(\Omega)}$ and the scalar product of $H^1(\Omega)$. We are now able to formulate the problem precisely (PI_1) we will study, under the following hypotheses:

$$(H) : \begin{cases} f \in L^2(0, T; L^2(\Omega)) & (H.1) \\ \varphi \in H^1(\Omega) \cap L^{p+1}(\Omega) & (H.2) \\ k_i \in L^\infty((0, T) \times \Omega) \quad \forall i \in \{1, 2\}. \end{cases}$$

Definition 2.4:

A weak solution of the problem (PI_1) is a function u that satisfies the following conditions:

- (i) $u \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$,
- (ii) u admits a strong derivative $\frac{\partial u}{\partial t} \in L^2(0, T; H^1(\Omega))$,
- (iii) $u(0) = \varphi$,
- (iv) u verifies $(u_t, v) + a(u_x, v_x) + (u^p, v) - b(u, v) + av(t) \int_0^l k_2(x, t)u(x, t)dx = (f, v) + av(0) \int_0^l k_1(x, t)u(x, t)dx$
 $\forall v \in V$, and $\forall t \in [0, T]$.

2.4.1 Existence of the solution to the semi-linear problem (PI₁)

Variational formulation

By multiplying the equation

$$\frac{\partial u}{\partial t} - a \frac{\partial^2 u}{\partial x^2} + u^p - bu = f(x, t), \quad (2.75)$$

by an element $v \in V$ and then integrating on Ω , we obtain

$$\int_{\Omega} \frac{\partial u}{\partial t} \cdot v dx - a \int_{\Omega} \frac{\partial^2 u}{\partial x^2} \cdot v dx + \int_{\Omega} u^p \cdot v dx - b \int_{\Omega} u \cdot v dx = \int_{\Omega} f \cdot v dx,$$

leading to the equation:

$$\begin{aligned} & \int_{\Omega} \frac{\partial u}{\partial t} \cdot v dx + a \int_{\Omega} \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} dx + \int_{\Omega} u^p \cdot v dx \\ & - b \int_{\Omega} u \cdot v dx - av(l)u_x(l, t) + av(0)u_x(0, t) = \int_{\Omega} f \cdot v dx. \end{aligned} \quad (2.76)$$

Using the boundary conditions and Green's formula, (2.76) becomes

$$(u_t, v) + a(u_x, v_x) + (u^p, v) - b(u, v) - av(l)u_x(l, t) + av(0)u_x(0, t) = (f, v) \quad \forall v \in V, \quad (2.77)$$

where (\cdot, \cdot) denotes the scalar product in $L^2(\Omega)$.

Study of the solution existence to problem (PI₁)

The demonstration is based on the method of Faedo-Galerkin which consists in carrying out the following three steps.

Functional space separability: Since the space V is separable, there exists a sequence $\omega_1, \omega_2, \dots, \omega_m$, having the following properties:

$$\begin{cases} \omega_i \in V, & \forall i, \\ \text{for any } m, \omega_1, \omega_2, \dots, \omega_m & \text{are linearly independent,} \\ V_m = \langle \{\omega_1, \omega_2, \dots, \omega_m\} \rangle & \text{is dense in } V. \end{cases} \quad (2.78)$$

In particular, we have

$$\forall \varphi \in V, \exists (\alpha_{km})_m \in \mathbb{N}^*, \varphi_m = \sum_{k=1}^m \alpha_{km} \omega_k \longrightarrow \varphi \text{ when } m \longrightarrow +\infty. \quad (2.79)$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

Approximation process: The Faedo Galerkin's method consists in searching for any integer $m \geq 1$, a function:

$$t \mapsto u_m(x, t) = \sum_{i=1}^m g_{im}(t) \omega_i(x),$$

that satisfies the following identities:

$$\begin{cases} ((u_m(t))_t, \omega_k) + a(\Delta u_m(t), \omega_k) + (u_m^p(t) - bu_m(t), \omega_k) = (f(t), \omega_k) & \forall k = \overline{1, m} \\ (u_m(0), \omega_k) = \alpha_{km} \end{cases} \quad (P_2)$$

Therefore,

$$\begin{aligned} u_m(0) &= \sum_{i=1}^m g_{im}(0) \omega_i(x) \\ &= \varphi_m \\ &= \sum_{k=1}^m \alpha_{km} \omega_k(x). \end{aligned}$$

Existence of such α_{km} follows from $u_0 \in V \cap L^{P+1}(\Omega)$ and the fact that $\{\omega_k, k \in \mathbb{N}\}$ is the base in $V \cap L^{P+1}(\Omega)$. Thus, (PI_1) is reduced to the following initial-value problem for a system of first-order differential equations with respect to g_{im} :

$$\begin{cases} \sum_{i=1}^m (w_i, w_k) \frac{\partial g_{im}}{\partial t}(t) + a \sum_{i=1}^m ((w_i)_x, (w_k)_x) g_{im}(t) \\ -a \sum_{i=1}^m g_{im}(t) \frac{\partial w_i}{\partial x}(l) w_k(l) + a \sum_{i=1}^m g_{im}(t) \frac{\partial w_i}{\partial x}(0) w_k(0) + (u_m^p - bu_m, w_k) = (f(t), w_k), \\ g_{km}(0) = \alpha_{km} \quad \forall k = \overline{1, m}. \end{cases} \quad (P_3)$$

Consider the two vectors:

$$g_m = (g_{1m}(t), \dots, g_{mm}(t)), f_m = ((f, \omega_1), \dots, (f, \omega_m))$$

and the matrices:

$$B_m = ((\omega_i, \omega_j))_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}, A_m = \left(\left(\frac{\partial \omega_i}{\partial x}, \frac{\partial \omega_j}{\partial x} \right) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

$$C_m = \left(\frac{\partial \omega_i}{\partial x}(l) \cdot \omega_j(l) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}, D_m = \left(\frac{\partial \omega_i}{\partial x}(0) \cdot \omega_j(0) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}$$

$$G(g) = \left(\left(\left(\sum_{i=1}^m g_{im}(t) \omega_i \right)^p, \omega_j \right) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}.$$

Using these notations, problem (PI_1) can be written in the following matrix form:

$$\begin{cases} B_m \frac{\partial g_m}{\partial t}(t) + aA_m g_m - bB_m g_m + G(g) = f_m + aC_m g_m - aD_m g_m \\ g_m(0) = (\alpha_{im})_{1 \leq i \leq m} \end{cases} \quad (2.80)$$

Using the Carathéodory's existence theorem for ODEs, we can conclude that there exists a t_m depending only on $|\alpha_{im}|$ such that in the interval $[0, t_m]$, problem (2.80) admits a unique local solution $g_m(t) \in C[0, t_m]$ with $g'_m(t) \in L^2[0, T]$.

Now, let us study the a priori estimates for the approximate solution $u_m(x, t)$ obtained in the previous step.

Theorem 2.8

Let $m \in \mathbb{N}^*$, $\frac{p}{2} \geq b$. Suppose that $\varphi \in H^1(\Omega) \cap L^{p+1}(\Omega)$, $f \in L^2(0, T, L^2(\Omega))$. Then problem (PI_1) admits a solution u such that

$$u \in L^2(0, T; V) \cap L^\infty(0, T; H^1(\Omega)),$$

and

$$u' \in L^2(0, T; L^2(\Omega)).$$

Proof Multiplying both sides of the equations in (PI_1) by $g_{km}(t)$, and summing for k from 1 to m , follows that

$$\begin{aligned} & \sum_{k=1}^m ((u_m)_t, \omega_k) \cdot g_{km}(t) - a \sum_{k=1}^m (\Delta u_m, \omega_k) \cdot g_{km}(t) + \sum_{k=1}^m (u_m^p - bu_m, \omega_k) \cdot g_{km}(t) \\ &= \sum_{k=1}^m (f, \omega_k) \cdot g_{km}(t), \end{aligned}$$

i.e.

$$((u_m)_t, u_m) - a(\Delta u_m, u_m) + (u_m^p - bu_m, u_m) = (f, u_m)$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

Thus,

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial t} \|u_m\|_{L^2(\Omega)}^2 + a \left\| \frac{\partial u_m}{\partial x} \right\|_{L^2(\Omega)}^2 + \|u_m\|_{L^{p+1}(\Omega)}^{p+1} \\ &= (f, u_m) + b \|u_m\|_{L^2(\Omega)}^2 + a \frac{\partial u_m}{\partial x}(l, t) u_m(l, t) - a \frac{\partial u_m}{\partial x}(0, t) u_m(0, t). \end{aligned}$$

After integrating from 0 to t , the Cauchy inequality gives

$$\int_0^t \left(\frac{\partial u_m}{\partial x}(l, \tau) \cdot u_m(l, \tau) \right) d\tau < \frac{1}{2} \int_0^t u_m^2(l, \tau) d\tau + \frac{1}{2} \int_0^t u_{mx}^2(l, \tau) d\tau.$$

To obtain the estimate, we need the inequality:

$$u^2(l, t) \leq 2 \int_x^l u_x^2 dx + 2u^2,$$

which easily follows from the identity:

$$u(l, t) = \int_x^l u_x(x, t) dx + u(x, t).$$

Likewise, by the second-kind integral condition, we have

$$\begin{aligned} & \int_0^t u_x(l, t) u(l, t) dt \\ & \leq \frac{1}{4} \int_Q u_x^2 dx dt + \frac{1}{4} \int_0^t u^2 dt + 2K \int_Q u^2 dx dt, \end{aligned}$$

where $K = \max_{\tau \in [0, T]} \int_\Omega k_i^2(x, t) dx dt$, $i = 1, 2$.

Thus, by using the same method, we get

$$\begin{aligned} & \int_0^t u_x(0, t) u(0, t) dt \\ & \leq \frac{1}{4} \int_Q u_x^2 dx dt + \frac{1}{4} \int_0^t u^2 dt + 2K \int_Q u^2 dx dt. \end{aligned}$$

So,

$$\begin{aligned} & \|u_m\|_{L^2(\Omega)}^2 + a \int_0^t \left\| \frac{\partial u_m}{\partial x} \right\|_{L^2(\Omega)}^2 + 2 \int_0^t \|u_m\|_{L^{p+1}(\Omega)}^{p+1} \\ & \leq \int_0^t \|f\|_{L^2(\Omega)}^2 + \left(2b + 1 + \frac{a}{2l} + 4aK \right) \int_0^t \|u_m\|_{L^2(\Omega)}^2 + \|\varphi_m\|_{L^2(\Omega)}^2 \end{aligned}$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

Using the Gronwall's lemma, we get

$$\begin{aligned} & \|u_m\|_{L^\infty(0,T, L^2(\Omega))}^2 + \left\| \frac{\partial u_m}{\partial x} \right\|_{L^2(Q)}^2 + \|u_m\|_{L^{p+1}(0,T, L^{p+1}(\Omega))}^{p+1} \\ & \leq \frac{\exp\left(\left(2b + 1 + \frac{a}{2l} + 4aK\right)T\right)}{\min\{1, a\}} \left(\|f\|_{L^2(Q)}^2 + \|\varphi_m\|_{L^2(\Omega)}^2\right). \end{aligned}$$

By setting:

$$C_T = \frac{\exp\left(\left(2b + 1 + \frac{a}{2l} + 4aK\right)T\right)}{\min\{1, a\}} \left(\|f\|_{L^2(Q)}^2 + \|\varphi_m\|_{L^2(\Omega)}^2\right), \quad (2.81)$$

We obtain

$$\|u_m\|_{L^\infty(0,T, L^2(\Omega))}^2 + \left\| \frac{\partial u_m}{\partial x} \right\|_{L^2(Q)}^2 + \|u_m\|_{L^{p+1}(0,T, L^{p+1}(\Omega))}^{p+1} \leq C_T. \quad (2.82)$$

Of course, C_T is a positive constant depending only on $\int_0^T \|f(\tau)\|_{L^2(\Omega)}^2, \|\varphi_m\|_{L^2(\Omega)}^2$ and T . From (2.82), it follows that

$$\|u_m(t)\|_{L^2(\Omega)}^2 \leq C_T.$$

This implies that the solution to the initial-value problem for the ODEs system (2.80) can be extended to $[0, T]$, and on $[0, T]$, we have the following uniform a priori estimates:

$$\begin{cases} u_m \text{ is uniformly bounded in } L^\infty(0, T; L^2(\Omega)), \\ u_m \text{ is uniformly bounded in } L^2(0, T; H^1(\Omega)), \\ u_m \text{ is uniformly bounded in } L^{p+1}(0, T; L^{p+1}(\Omega)). \end{cases}$$

Now, we would like to get more a priori estimates. To this end, multiplying both sides of the equation in (2.80) by $g'_{km}(t)$, using the same variational formulation, respectively, and then summing over k , we get

$$\begin{aligned} & \sum_{k=1}^m ((u_m)_t, \omega_k) \cdot g'_{km}(t) + \sum_{k=1}^m a(\Delta u_m, \omega_k) \cdot g'_{km}(t) + \sum_{k=1}^m (u_m^p - bu_m, \omega_k) \cdot g'_{km}(t) \\ & = \sum_{k=1}^m (f, \omega_k) \cdot g'_{km}(t). \end{aligned}$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

So,

$$((u_m)_t, (u_m)_t) - a \left(\frac{\partial^2 u_m}{\partial x^2}, (u_m)_t \right) + (u_m^p - bu_m, (u_m)_t) = (f, (u_m)_t).$$

After an integration on $(0, t)$, it follows that:

$$\begin{aligned} & \min \left\{ \frac{1}{2}, \frac{a}{T} \left(\frac{1}{2} - \frac{l}{\varepsilon} \right), \frac{l}{(p+1)T} \right\} \left[\|(u_m)_t\|_{L^2(Q)}^2 + \|(u_m)_x\|_{L^2(Q)}^2 + \int_0^t \|u_m\|_{L^{p+1}(\Omega)}^{p+1} \right] \\ & \leq \max \left\{ \frac{1}{2}, \left(\frac{a}{2} + a \frac{l}{\varepsilon} \right), \frac{1}{p+1}, -\frac{b}{2} \right\} \left[\begin{aligned} & \|f\|_{L^2(Q)}^2 + \left(a \frac{\varepsilon}{2} K^2 + \frac{b}{2} \right) C_T \\ & + \|(\varphi_m)_x\|_{L^2(\Omega)}^2 + \|\varphi_m\|_{L^{p+1}(\Omega)}^{p+1} + \|\varphi_m\|_{L^2(\Omega)}^2 \end{aligned} \right]. \end{aligned}$$

Setting:

$$C = \frac{\max \left\{ \frac{1}{2}, \left(\frac{a}{2} + a \frac{l}{\varepsilon} \right), \frac{1}{p+1}, -\frac{b}{2} \right\}}{\min \left\{ \frac{1}{2}, \frac{a}{T} \left(\frac{1}{2} - \frac{l}{\varepsilon} \right), \frac{l}{(p+1)T} \right\}}$$

We deduce

$$\begin{aligned} & \|(u_m)_t\|_{L^2(Q)}^2 + \|(u_m)_x\|_{L^2(Q)}^2 + \int_0^t \|u_m\|_{L^{p+1}(\Omega)}^{p+1} \\ & \leq C \left[\|f\|_{L^2(Q)}^2 + \frac{bl}{2} C_T + \|(\varphi_m)_x\|_{L^2(\Omega)}^2 + \|\varphi_m\|_{L^{p+1}(\Omega)}^{p+1} + \|\varphi_m\|_{L^2(\Omega)}^2 \right] \end{aligned}$$

and therefore,

$$\|(u_m)_t\|_{L^2(Q)}^2 \leq C. \quad (2.83)$$

Then, we get the following other a priori estimates :

$$\begin{cases} u_m \text{ is uniformly bounded in } L^{p+1}(0, T; L^{p+1}(\Omega)), \\ u_m \text{ is uniformly bounded in } L^2(0, T; H^1(\Omega)), \\ (u_m)_t \text{ is uniformly bounded in } L^2(0, T; L^2(\Omega)). \end{cases} \quad (2.84)$$

Consequently, by Lemma (1.4), there is a subsequence of u_m , still denoted by u_m , such that

$$\begin{cases} u_m \longrightarrow u \text{ weakly in } L^{p+1}(0, T; L^{p+1}(\Omega)) \\ u_m \longrightarrow u \text{ weakly in } L^2(0, T; H^1(\Omega)) \\ u_m \longrightarrow u \text{ weakly in } L^2(0, T; L^2(\Omega)) \end{cases} \quad (2.85)$$

We deduce from lemma (1.4) that there are subsequences , still denoted by $(u_{m_k}), \left(\frac{\partial u_{m_k}}{\partial t} \right)$ of (u_m) and $(u_m)_t$ respectively, such that

$$u_{m_k} \rightharpoonup u \quad \text{in } L^2(0, T; H^1(\Omega)), \quad (2.86)$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

$$\frac{\partial u_{m_k}}{\partial t} \rightharpoonup w \quad \text{in } L^2(0, T; L^2(\Omega)). \quad (2.87)$$

We know that according to Rellich-Kondrachov's theorem that the injection of $H^1(Q)$ into $L^2(Q)$ is compact. As a consequence of this theorem, any weakly convergent sequence in $H^1(Q)$ has a subsequence which converges strongly in $L^2(Q)$. Thus, we can assume that

$$u_{m_k} \longrightarrow u \quad \text{in } L^2(Q). \quad (2.88)$$

On the other hand, from Lemma (10) there is a subsequence of $(u_{m_k})_k$, still denoted by u_{m_k} , that converges almost everywhere to u , such that

$$u_{m_k} \longrightarrow u \quad \text{almost everywhere } Q. \quad (2.89)$$

By Lemma (10), there is a subsequence of u_m , still denoted by u_m , that converges almost everywhere to u in $Q_T = \Omega \times [0, T]$. It turns out that

$$(u_m)^p \text{ converges almost everywhere } u^p \text{ in } Q_T.$$

On the other hand, (2.84) implies that $(u_m)^p$ is uniformly bounded in $L^{\frac{p+1}{p}}(Q_T)$. Therefore, we infer from Lemma (1.5) that

$$u_m^p \rightharpoonup u^p \quad \text{weakly in } L^{\frac{p+1}{p}}\left(0, T, L^{\frac{p+1}{p}}(\Omega)\right).$$

It remains to demonstrate that $w = \frac{\partial u}{\partial t}$. For that it suffices show that

$$u(t) = \varphi + \int_0^t w(\tau) d\tau. \quad (2.90)$$

Since

$$u_{m_k} \rightharpoonup u \quad \text{in } L^2(0, T; L^2(\Omega)),$$

the proof of (2.90) is equivalent to demonstrate that

$$u_{m_k} \rightharpoonup \varphi + \chi \quad \text{in } L^2(0, T; L^2(\Omega)),$$

which means

$$\lim (u_{m_k} - \varphi - \chi, v)_{L^2(0,T; L^2(\Omega))} = 0 \quad \forall v \in L^2(0, T; L^2(\Omega)),$$

where

$$\chi(t) = \int_0^t w(\tau) d\tau.$$

Using the equality:

$$u_{m_k} - \varphi_{m_k} = \int_0^t \frac{\partial u_{m_k}}{\partial \tau} d\tau \quad \text{for all } t \in [0, T],$$

it follows from $u_{m_k} \in L^2(0, T; V_{m_k})$ and $(u_{m_k})_t \in L^2(0, T; V_{m_k})$ that

$$\begin{aligned} & \left(u_{m_k} - \varphi - \int_0^t w(\tau) d\tau, v \right)_{L^2(0,T; L^2(\Omega))} \\ &= \left(\int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau) \right) d\tau, v \right)_{L^2(0,T; L^2(\Omega))} + (\varphi_{m_k} - \varphi, v)_{L^2(0,T; L^2(\Omega))}, \text{ for all } t \in [0, T]. \end{aligned}$$

Moreover, by virtue of Lemma (1.5), we get

$$\begin{aligned} & \left(u_{m_k} - \varphi - \int_0^t w(\tau) d\tau, v \right)_{L^2(0,T; L^2(\Omega))} \\ &= \int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau), v \right)_{L^2(0,T; L^2(\Omega))} d\tau + (\varphi_{m_k} - \varphi, v)_{L^2(0,T; L^2(\Omega))}, \text{ for all } t \in [0, T]. \end{aligned}$$

On the one hand, we have

$$\lim_{k \rightarrow \infty} \int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau), v \right)_{L^2(0,T; L^2(\Omega))} d\tau = 0, \text{ for } t \in [0, T]. \quad (2.91)$$

Also, we have

$$\lim_{k \rightarrow \infty} (\varphi_{m_k} - \varphi, v)_{L^2(0,T; L^2(\Omega))} = 0, \quad (2.92)$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

So that

$$\lim_{k \rightarrow \infty} (u_{m_k} - \varphi - \chi, v)_{L^2(0,T; L^2(\Omega))} = 0, \quad \forall v \in L^2(0, T; L^2(\Omega)).$$

Finally, from (2.91) and (2.92), we get

$$\lim_{k \rightarrow \infty} (u_{m_k} - \varphi - \chi, v)_{L^2(0,T; L^2(\Omega))} = 0, \quad \forall v \in L^2(0, T; L^2(\Omega)).$$

Now, passing to the limit in (P₃), since each term on the left side of (P₃) is weakly convergent in $L^{\frac{p+1}{p}}(\Omega)$, the following equations hold in $L^{\frac{p+1}{p}}(\Omega)$:

$$((u_m(t))_t, \omega_k) + a(u_m(t), \omega_k) + (u_m^p(t) - bu_m(t), \omega_k) = (f(t), \omega_k) \quad \forall k = \overline{1, m}. \tag{2.93}$$

$\{\omega_j; j \in \mathbb{N}\}$ being also a basis of $L^{\frac{p+1}{p}}(\Omega)$, we infer from (2.93) that the following equation holds in $L^{\frac{p+1}{p}}(0, T, L^{\frac{p+1}{p}}(\Omega))$:

$$u' - a\Delta u + u^p - bu = f. \tag{2.94}$$

Since $u', \Delta u$, and f all belong to $L^2(0, T; L^2(\Omega))$, u^p also belongs to $L^2(0, T; L^2(\Omega))$, and (2.94) also holds in $L^2(0, T; L^2(\Omega))$. Thus, we get the desired result. ■
 In summary, we have established a framework for studying the existence of solutions for the semi-linear problem (PI₁) through the variational formulation and the Faedo-Galerkin method. By carefully defining functional spaces and utilizing appropriate approximations, we aim to demonstrate the existence of solutions under the specified hypotheses.

Solution uniqueness

In the analysis of the semi-linear problem (PI₁), we focus on establishing the uniqueness of solution under a condition on the exponent p . This section will outline the rationale for this attention and the implications of the odd nature of p on the uniqueness of solution.

Theorem 2.9

Suppose that $\varphi \in H^1(\Omega) \cap L^{p+1}(\Omega)$ and $f \in L^2(0, T; L^2(\Omega))$. Then problem (PI_1) admits a unique solution u such that

$$u \in L^2(0, T, H^1(\Omega)) \cap L^{p+1}(0, T; L^{p+1}(\Omega)),$$

with

$$u' \in L^2(0, T, L^2(\Omega)).$$

Proof Suppose that p is odd. Multiplying the equation in problem (PI_1) by u and integrating it on the domain $\Omega = (0, l)$, we get

$$\int_{\Omega} u_t(x, t) u dx - a \int_{\Omega} \Delta u \cdot u dx + \int_{\Omega} u^p u dx - b \int_{\Omega} u^2 dx = \int_{\Omega} f(x, t) u dx.$$

Therefore,

$$\frac{1}{2} \frac{\partial}{\partial t} \|u\|_{L^2(\Omega)}^2 + a \int_{\Omega} \left(\frac{\partial u}{\partial x} \right)^2 dx + \int_{\Omega} u^{p+1} dx - b \int_{\Omega} u^2 dx = \int_{\Omega} f u dx,$$

Then, by integrating on $(0, \tau)$ with $\tau \in (0, T)$, we get

$$\begin{aligned} & \frac{1}{2} \|u\|_{L^2(\Omega)}^2 + (a - a\varepsilon) \int_0^t \left\| \frac{\partial u}{\partial x} \right\|_{L^2(\Omega)}^2 d\tau + \left(1 - \left(\frac{1}{2} + \frac{\varepsilon}{l} + \frac{K}{2\varepsilon} + b \right) \frac{2}{p+1} \right) \int_0^t \|u\|_{L^{p+1}(\Omega)}^{p+1} d\tau \\ & \leq \frac{1}{2} \int_0^T \|f(\tau)\|_{L^2(\Omega)}^2 d\tau + \frac{1}{2} \|\varphi\|_{L^2(\Omega)}^2 + \left(\frac{1}{2} + \frac{\varepsilon}{l} + \frac{K}{2\varepsilon} + b \right) \frac{(p+1) \text{mes}(\Omega) T}{p-1}. \end{aligned}$$

By setting:

$$C_T = \frac{\frac{1}{2} \int_0^T \|f(\tau)\|_{L^2(\Omega)}^2 d\tau + \frac{1}{2} \|\varphi_m\|_{L^2(\Omega)}^2 + \left(\frac{1}{2} + \frac{\varepsilon}{l} + \frac{K}{2\varepsilon} + b \right) \frac{(p+1) \text{mes}(\Omega) T}{p-1}}{\min \left(\frac{1}{2}, (a - a\varepsilon), \left(1 - \left(\frac{1}{2} + \frac{\varepsilon}{l} + \frac{K}{2\varepsilon} + b \right) \frac{2}{p+1} \right) \right)}, \quad (2.95)$$

we can write

$$\|u(t)\|_{L^2(\Omega)}^2 + \int_0^t \left\| \frac{\partial u}{\partial x} \right\|_{L^2(\Omega)}^2 d\tau + \int_0^t \|u\|_{L^{p+1}(\Omega)}^{p+1} d\tau \leq C_T. \quad (2.96)$$

We put

$$\|u\|_{L^\infty(0, T; L^2(\Omega))}^2 + \|u\|_{L^2(0, T; L^2(\Omega))}^2 + \|u\|_{L^{p+1}(0, T; L^{p+1}(\Omega))}^{p+1} \equiv \|u\|_B.$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

If u_1 and u_2 be two solutions to the problem (PI_1) , then

$$Lu_1 - Lu_2 = 0,$$

where L is the differential operator of the main semi-linear problem. Therefore,

$$L(u_1 - u_2) = 0,$$

so that

$$\|u_1 - u_2\|_B \leq c \|0\|_F = 0,$$

which gives

$$u_1 = u_2.$$

■

Finite-time blow-up solution

This behavior is particularly relevant in the study of nonlinear evolution equations, where nonlinearity often leads to complex dynamics. This section discusses the conditions under which finite-time blow-up occurs, the implications for solutions, and some examples illustrating this phenomenon.

By setting:

$$\Psi(t) = \int_{\Omega} u^2 dx.$$

and by integrating the equation over Ω with null source, we get

$$\begin{aligned} u_t - a\Delta u - bu &= u^p \\ \int_{\Omega} u_t u dx - a \int_{\Omega} \Delta u \cdot u dx - \int_{\Omega} bu^2 dx &= \int_{\Omega} u^{p+1} dx \end{aligned}$$

By using the Green's formula, and Pointcarré's inequality, it follows that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx + a \int_{\Omega} u_x^2 - \int_{\Omega} bu^2 dx &\leq a [u_x(l, t)u(l, t) - u_x(0, t)u(0, t)] + \int_{\Omega} u^{p+1} dx \\ &\leq a \max_{(x,t) \in Q_T} (k_1, k_2) \left[\frac{l}{2} \int_{\Omega} u^2 + \frac{l}{2} \int_{\Omega} u_x^2 \right] + \int_{\Omega} u^{p+1} dx. \end{aligned}$$

Thus,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx - \left(b + \frac{al \max_{(x,t) \in Q_T} (k_1, k_2)}{2} \right) \int_{\Omega} u^2 dx \leq \int_{\Omega} u^{p+1} dx.$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

Next, applying the Jensen's inequality gives

$$\Psi'(t) - 2 \left(b + \frac{al \max_{(x,t) \in Q_T} (k_1, k_2)}{2} \right) \Psi(t) + \frac{2}{(\text{mes}(\Omega))^{\frac{q+1}{2}-1}} (\Psi(t))^{\frac{p+1}{2}} = 0. \quad (2.97)$$

To solve this ODE, we use the following change of variable:

$$v = \Psi^{1-q}, \quad (2.98)$$

where

$$q = \frac{p+1}{2}$$

After combining [2.97](#) and [3.37](#), it follows that

$$\frac{1}{q-1} v' v^{\frac{q}{1-q}} - 2 \left(b + \frac{al \max_{(x,t) \in Q_T} (k_1, k_2)}{2} \right) v^{\frac{1}{1-q}} + \frac{c_1}{(\text{mes}(\Omega))^{q-1}} v^{\frac{q}{1-q}} = 0. \quad (2.99)$$

Multiplying [\(2.99\)](#) by $v^{\frac{-q}{1-q}}$ yields

$$v' - 2(q-1) \left(b + \frac{al \max_{(x,t) \in Q_T} (k_1, k_2)}{2} \right) v + (q-1) \frac{1}{(\text{mes}(\Omega))^{q-1}} = 0. \quad (2.100)$$

Before solving this ODE, let us solve the associated homogeneous equation. By setting:

$$K = 2(q-1) \left(b + \frac{al \max_{(x,t) \in Q_T} (k_1, k_2)}{2} \right)$$

the ODE:

$$v' + Kv = 0$$

Therefore, the problem clearly admits the solution

$$v_1(t) = Be^{-Kt}.$$

Now, move on to solving the non-homogeneous ODE [\(2.100\)](#) by the method of constants variation. We start by setting:

$$v_2(t) = B(t)e^{-Kt},$$

Chapter 2. Solvability and blow-up study of weak solutions for a nonlinear parabolic equation with a generalized nonlinear integral condition of second type

Therefore,

$$v_2(t) = \frac{1}{K} (q-1) \frac{1}{(\text{mes}(\Omega))^{q-1}}$$

and so,

$$v(t) = \left(B e^{-Kt} + \frac{1}{K} (q-1) \frac{c_1}{(\text{mes}(\Omega))^{q-1}} \right)^{\frac{1}{1-q}}$$

For $t = 0$, we get

$$B = (\Psi(0))^{1-q} - \frac{1}{K} (q-1) \frac{c_1}{(\text{mes}(\Omega))^{q-1}}$$

Like $\frac{1}{q-1} > 0$, then $\Psi \rightarrow \infty$, so we obtain

$$T = \frac{1}{K} \ln \left(\frac{\frac{1}{K} (q-1) \frac{1}{(\text{mes}(\Omega))^{q-1}}}{\left((\Psi(0))^{1-q} - \frac{1}{K} (q-1) \frac{1}{(\text{mes}(\Omega))^{q-1}} \right)} \right).$$

Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

This chapter explores the solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation subject to a mixed linear and nonlinear integral condition. The presence of singularities, degeneracies, and complex integral conditions introduces significant challenges to theoretical analysis and practical applications. The results presented here give rise to a rich mathematical structure, providing deeper insights into the behavior of solutions.

Let $t \in (0, T)$, $\{\Omega \subset \mathbb{R}^2, x \in \Omega = [0, 1], 0 < t < T\}$, This chapter deals with local existence, blow-up for the solutions of a singular degenerate parabolic equation subject to different cases of the boundary conditions. Let $Q = [0, 1] \times (0, T]$ with $T > 0$, the typical non-linear problem studied is:

$$\begin{cases} \frac{\partial u}{\partial t} - (x^\alpha u_x)_x = f(x, t, u), & (x, t) \in Q, \\ u(x, 0) = \varphi(x), & x \in [0, 1], \\ u_x(0, t) = \int_0^1 k_1(x)h(x, t, u)dx & t \in (0, T], \\ u_x(1, t) = \int_0^1 k_2(x)g(x, t, u)dx & t \in (0, T]. \end{cases} \quad (P_r)$$

Assume that $f \in L^2(Q)$, $\varphi \in L^2(0, 1)$, the weight functions k_1, k_2 are continuous on $[0, 1]$ and the function f is Lipschitzian, i.e. there exists a positive constant L such that:

$$\|f(x, t, u_1) - f(x, t, u_2)\|_{L^2(Q)} \leq L \left(\|u_1 - u_2\|_{L^2(Q)} \right), \quad \forall u_1, u_2 \in L^2(Q).$$

Moreover, the functions g and h verify respectively the following inequalities:

$$\|g(x, t, u)\|_{L^2(Q)} \leq C_0 \|u\|_{L^2(Q)}, \quad \|h(x, t, u)\|_{L^2(Q)} \leq C_1 \|u\|_{L^2(Q)}, \quad (3.1)$$

where C_0 and C_1 are positive constants. It is necessary to point out that problem (P_r) is singular and degenerate because the coefficients of u_x and u_{xx} tend to ∞ and 0 as $x \rightarrow 0$. There is a wide literature on problem (P_r) in different cases and under various assumptions on h, g .

3.1 Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with nonlinear integral condition

Here, we investigate the existence and uniqueness of weak solutions for a singular and degenerate parabolic equation. The presence of a nonlinear integral boundary condition further complicates the equation. This work delves into the delicate balance between the singular, degenerate nature of the equation and the nonlinearity of the boundary condition. To deduce the solvability of problem (P_r) , let us first consider the linear case.

3.1.1 Study of the linear problem

This study of the linear problem is mainly focused on the existence of a solution. We analyze the linear problem derived from the main problem of this chapter, by using the Faedo-Galerkin method.

In the specified domain $Q = \{(x, t) \in \mathbb{R}^2 : 0 < x < 1, 0 < t < T\}$, consider the following linear singular and degenerate problem:

$$\begin{cases} u_t - a(x^\alpha u_x)_x = f(x, t) & (x, t) \in Q, \\ u(x, 0) = \varphi(x) & x \in (0, 1), \\ u_x(0, t) = \int_0^1 k_1(x)h(x, t, u)dx & t \in (0, T), \\ u_x(1, t) = \int_0^1 k_2(x)g(x, t, u)dx & t \in (0, T). \end{cases} \quad (P_1)$$

where f, φ, k_1 and k_2 are given functions, and $\alpha \in (0, 1)$. Moreover, we assume $f \in L^2(Q)$, $\varphi \in L^2(0, 1)$, and the weight functions k_1, k_2 are continuous on $[0, 1]$.

Definition 3.1

We say that the function u is the weak solution of problem (P_1) if

- (i) $u \in L^2(0, T; H^1_{x^{\frac{\alpha}{2}}}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$,
- (ii) u admits a strong derivative $\frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega))$,
- (iii) $u(0) = \varphi$,
- (iv) verifies: $(u_t, v) - a((x^\alpha u_x)_x, v) = (f, v), \quad \forall v \in V, \quad \forall t \in [0, T]$.

Theorem 3.1

Suppose that $\varphi \in H^1(\Omega), f \in L^2(0, T, L^2(\Omega))$. and $\varepsilon, \delta > 0$ such that $(1 - (2\varepsilon + \frac{1}{2\delta})) > 0, (\frac{\alpha}{2} - 2\varepsilon) > 0$. Then, the problem (P_1) admits a solution

$$u \in L^2(0, T, H^1_{x^{\frac{\alpha}{2}}}(\Omega)) \cap L^\infty(0, T, L^2(\Omega))$$

with

$$\frac{\partial u}{\partial t} \in L^2(0, T, L^2(\Omega)).$$

Proof

Showing the existence of a solution to problem (P_1) utilizing the Faedo-Galerkin method is grounded in executing the following key steps: Formulation of the approximate problem, Derivation of a priori estimates, Convergence to a weak solution.

Step 1: Construction of the approximate solutions Since the space $V = L^2(0, T, H^1(\Omega)) \cap L(0, T; L^2(\Omega))$ is separable, there exists a countable sequence $\omega_1, \omega_2, \dots, \omega_m$, that possesses the following properties:

$$\begin{cases} \omega_i \in V & \forall i, \\ \forall m, \omega_1, \omega_2, \dots, \omega_m & \text{are linearly independent,} \\ V_m = \langle \{\omega_1, \omega_2, \dots, \omega_m\} \rangle & \text{is dense in } V. \end{cases} \quad (3.2)$$

In particular,

$$\forall \varphi \in V \implies \exists (\alpha_{km})_m \in \mathbb{N}^*, \varphi_m = \sum_{k=1}^m \alpha_{km} \omega_k \longrightarrow \varphi \text{ when } m \longrightarrow +\infty. \quad (3.3)$$

Faedo-Galerkin's approximation involves seeking, for any integer $m \geq 1$, a function u_m that can be expressed as a linear combination of basis functions from a finite-dimensional subspace of the function space under consideration. Specifically, we

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

write

$$u_m(x, t) = \sum_{i=1}^m g_{im}(t) \omega_i(x),$$

and verify

$$\begin{cases} u_m(t) \in V_m \quad \forall t \in [0, T], \\ ((u_m(t))_t, \omega_k) + \chi(u_m(t), \omega_k) = (f(t), \omega_k). \quad \forall k = 1, \dots, m \end{cases} \quad (P_3)$$

where

$$\begin{aligned} ((u_m(t))_t, \omega_k) &= \left(\left(\sum_{i=1}^m g_{im}(t) \omega_i \right)_t, \omega_k \right) = \left(\sum_{i=1}^m \frac{\partial g_{im}}{\partial t}(t) \omega_i(x), \omega_k \right) \\ &= \sum_{i=1}^m (\omega_i, \omega_k) \frac{\partial g_{im}}{\partial t}(t), \end{aligned}$$

$$\begin{aligned} \chi(u_m(t), \omega_k) &= \left(\sum_{i=1}^m g_{im}(t) (x^\alpha (\omega_i)_x)_x, \omega_k \right) \\ &= -a \sum_{i=1}^m \left(x_i^{\frac{\alpha}{2}} \omega, x_k^{\frac{\alpha}{2}} \omega \right) g_{im}(t) + a \sum_{i=1}^m g_{im}(t) \frac{\partial \omega_i}{\partial x}(1) \omega_k(1), \end{aligned}$$

and

$$u_m(0) = \sum_{i=1}^m g_{im}(0) \omega_i(x) = \varphi_m = \sum_{i=1}^m \alpha_{im} \omega_i(x).$$

We obtain a system of first-order nonlinear differential equations, which can be expressed in the following general form:

$$\begin{cases} \sum_{i=1}^m (\omega_i, \omega_k) \frac{\partial g_{im}}{\partial t}(t) + a \sum_{i=1}^m \left(x_i^{\frac{\alpha}{2}} \omega_i, x_k^{\frac{\alpha}{2}} \omega_k \right) g_{im}(t) \\ = (f(t), \omega_k) + a \sum_{i=1}^m g_{im}(t) \frac{\partial \omega_i}{\partial x}(1) \omega_k(1), \\ g_{im}(0) = \alpha_{im}, \quad \forall i = 1, \dots, m. \end{cases} \quad (P_4)$$

This vector allows us to succinctly express the system in a more compact form:

$$g_m = (g_{1m}(t), \dots, g_{mm}(t)), f_m = ((f, \omega_1), \dots, (f, \omega_m))$$

We also consider the matrix θ_m, χ_m, η_m that represents the coefficients of the linearized system or any relevant relationships among the components of the vector ω . This matrix can be expressed as follows:

$$\theta_m = ((\omega_i, \omega_j))_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}, \chi_m = \left(\left(x_i^{\frac{\alpha}{2}} \omega_i, x_j^{\frac{\alpha}{2}} \omega_j \right) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}$$

and

$$\eta_m = \left(\frac{\partial \omega_i}{\partial x}(1) \cdot \omega_j(1) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}.$$

Problem (P_4) in the following matrix form:

$$\begin{cases} \theta_m \frac{\partial g_m}{\partial t}(t) + a \chi_m g_m = f_m + a \eta_m g_m, \\ g_m(0) = (\alpha_{im})_{1 \leq i \leq m}. \end{cases}$$

Since the matrix entries θ_m are linearly independent (due to the fact that it is a diagonal matrix), we have $\det \theta_m \neq 0$, so it is invertible. Consequently, we can state that g_m is the solution of the system:

$$\begin{cases} \frac{\partial g_m}{\partial t}(t) + (a \theta_m^{-1} \chi_m - a \theta_m^{-1} \eta_m) g_m = \theta_m^{-1} f_m, \\ g_m(0) = (\alpha_{im})_{1 \leq i \leq m}. \end{cases} \quad (P_5)$$

The usual existence and uniqueness theorems applicable to ODE systems allow us to derive significant conclusions about the behavior of the solutions. By applying the results from ([?], Theorem 3.1, Chapter 3), we deduce that the matrix $(a \theta_m^{-1} A_m + b \theta_m^{-1} B_m - a \theta_m^{-1} \eta_m)$ has constant coefficients and the vector $\theta_m^{-1} f_m$ consists of continuous functions that are dominated by integrable functions on the interval $(0, T)$. Thus, we can conclude that there exists t_m which depends only on the values of $|\alpha_{im}|$ such that in the interval $[0, t_m]$, the nonhomogeneous problem (P_5) admits a unique local solution $g_m(t) \in C[0, t_m]$ with $g'_m(t) \in L^2[0, T]$. But, as the elements of the vector $\theta_m^{-1} f_m$ are majorized by integrable functions on $(0, T)$, the solution can be extended to $[0, T]$.

Step 2: Establishing a priori estimates: Consider the following theorem

Theorem 3.2

For each $m \in \mathbb{N}$, the approximate solution u_m satisfies the following uniform bounds as $m \rightarrow +\infty$:

$$\begin{cases} u_m \text{ is uniformly bounded in } L^\infty(0, T; L^2(\Omega)), \\ x^{\frac{\alpha}{2}} u_m \text{ is uniformly bounded in } L^2(0, T; H^1(\Omega)), \\ (u_m)_t \text{ is uniformly bounded in } L^2(0, T; L^2(\Omega)). \end{cases} \quad (3.4)$$

Proof Multiply the equation in (P_3) by $g_{km}(t)$ and sum up for k to obtain

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

$$((u_m(t))_t, u_m(t)) + a \left(\frac{\partial}{\partial x} \left(x^\alpha \frac{\partial u_m}{\partial x} \right), u_m \right) = (f(t), u_m(t)).$$

Thus, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u_m\|_{L^2(\Omega)}^2 + a \left\| x^{\frac{\alpha}{2}} \frac{\partial u_m}{\partial x} \right\|_{L^2(\Omega)}^2 \\ &= (f(t), u_m(t)) + a \sum_{i=1}^m g_{im}(t) \frac{\partial \omega_i}{\partial x}(1) \sum_{k=1}^m g_{km}(t) \omega_k(1). \end{aligned}$$

By using the Cauchy inequality with $\varepsilon/2$ we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|u_m\|_{L^2(\Omega)}^2 + a \left\| x^{\frac{\alpha}{2}} \frac{\partial u_m}{\partial x} \right\|_{L^2(\Omega)}^2 \\ & \leq \frac{1}{2\varepsilon} \|f\|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2} \|u_m\|_{L^2(\Omega)}^2 + a \sum_{i=1}^m g_{im}(t) \frac{\partial \omega_i}{\partial x}(1) \sum_{k=1}^m g_{km}(t) \omega_k(1). \quad (3.5) \end{aligned}$$

By integrating from 0 to t , we can get an estimate of the third term of right-hand side of (3.5) as follows:

$$\int_0^t \left(\sum_{i=1}^m g_{im}(\tau) \frac{\partial \omega_i}{\partial x}(1) \sum_{k=1}^m g_{km}(\tau) \omega_k(1) \right) d\tau = \int_0^t \left(\frac{\partial u_m}{\partial x}(1, \tau) \cdot u_m(1, \tau) \right) d\tau.$$

Again, by using the Cauchy inequality with ε , we have

$$\begin{aligned} & \int_0^t \left(\frac{\partial u_m}{\partial x}(1, \tau) \cdot u_m(1, \tau) \right) d\tau \\ & \leq \varepsilon \left[\left\| x^{\frac{\alpha}{2}} \frac{\partial u_m}{\partial x} \right\|_{L^2(Q)}^2 + \|u_m\|_{L^2(Q)}^2 \right] + \frac{C_0 K}{2\varepsilon} \|u_m\|_{L^2(Q)}^2, \end{aligned}$$

where $K = \max_{\Omega} \int_Q k^2(x, t) dx dt$ and C_0 is the constant in (3.1). By using the Gronwell's lemma and take $\delta = 1$, we get

$$\begin{aligned} & \|u_m\|_{L^\infty(0, T; L^2(\Omega))}^2 + \left\| x^{\frac{\alpha}{2}} \frac{\partial u_m}{\partial x} \right\|_{L^2(Q)}^2 \\ & \leq \exp\left(\left(\frac{1}{2} + 2\varepsilon + \frac{C_0 K}{\varepsilon}\right) T\right) \left[\|f\|_{L^2(0, T; L^2(\Omega))}^2 + \|\varphi_m\|_{L^2(\Omega)}^2 \right], \end{aligned}$$

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

$$\begin{aligned} & \|u_m\|_{L^\infty(0,T; L^2(\Omega))}^2 + 2(a - \varepsilon) \left\| x^{\frac{\alpha}{2}} \frac{\partial u_m}{\partial x} \right\|_{L^2(Q)}^2 \\ & \leq \frac{\exp\left(\left(\frac{1}{2} + 2\varepsilon + \frac{C_0 K}{\varepsilon}\right) T\right)}{\min\{1, 2(a - \varepsilon)\}} \left(\|f\|_{L^2(0,T; L^2(\Omega))}^2 + \|\varphi_m\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

So,

$$\|u_m\|_{L^\infty(0,T; L^2(\Omega))}^2 + \left\| x^{\frac{\alpha}{2}} \frac{\partial u_m}{\partial x} \right\|_{L^2(Q)}^2 \leq R_1 \left(\|f\|_{L^2(0,T; L^2(\Omega))}^2 + \|\varphi_m\|_{L^2(\Omega)}^2 \right), \quad (3.6)$$

where

$$R_1 = \frac{\exp\left(\left(\frac{1}{2} + 2\varepsilon + \frac{C_0 K}{\varepsilon}\right) T\right)}{\min\{1, 2(a - \varepsilon)\}}.$$

Now, using the same variational formulation (P_4) , by multiplying the new equation by $g'_{km}(t)$ and summing up for over k , we find

$$\begin{aligned} & \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(Q)}^2 + \frac{a}{2} \left\| x^{\frac{\alpha}{2}} \frac{\partial u_m}{\partial x}(\tau) \right\|_{L^2(\Omega)}^2 \\ & = \int_0^\tau \left(f(t), \frac{\partial u_m}{\partial t} \right) + a \int_0^\tau \frac{\partial u_m}{\partial x}(l, t) \cdot \frac{\partial u_m}{\partial t}(l, t) dt + \frac{a}{2} \left\| x^{\frac{\alpha}{2}} \frac{\partial \varphi_m}{\partial x} \right\|_{L^2(\Omega)}^2. \end{aligned} \quad (3.7)$$

Hence with (3.7), using the Cauchy inequality with ε , we get

$$\begin{aligned} & \left(1 - \left(2\varepsilon + \frac{1}{2\delta} \right) \right) \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(Q)}^2 + \left(\frac{a}{2} - 2\varepsilon \right) \left\| x^{\frac{\alpha}{2}} \frac{\partial u_m}{\partial x}(\tau) \right\|_{L^2(\Omega)}^2 \\ & \leq a \frac{K}{2\varepsilon} R_1 \left(\|f\|_{L^2(0,T; L^2(\Omega))}^2 + \|\varphi_m\|_{L^2(\Omega)}^2 \right) + \left(\frac{a}{2} + 2\varepsilon \right) \left\| \frac{\partial \varphi_m}{\partial x} \right\|_{L^2(\Omega)}^2 + \frac{\delta}{2} \|f\|_{L^2(Q)}^2 \\ & \leq R_2, \end{aligned}$$

where

$$R_2 = \frac{\max\left(a \frac{K}{2\varepsilon} R_1, \left(\frac{a}{2} + 2\varepsilon\right), \frac{\delta}{2}\right)}{\min\left(\left(1 - \left(2\varepsilon + \frac{1}{2\delta}\right)\right), \left(\frac{a}{2} - 2\varepsilon\right)\right)} \left(\|f\|_{L^2(Q)}^2 + \left\| \frac{\partial \varphi_m}{\partial x} \right\|_{L^2(\Omega)}^2 + \|\varphi_m\|_{L^2(\Omega)}^2 \right).$$

Thus,

$$\left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(0,T; L^2(\Omega))} \leq R_2. \quad (3.8)$$

From (4.11) it follows that the solution to the initial-value problem for the system

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

of ODE (P_4) can be extended to $[0, T]$. This confirms what we have demonstrated in the first step. More precisely, when $m \rightarrow +\infty$ in (3.8), we have obtained

$$\begin{cases} u_m \text{ is uniformly bounded in } L^\infty(0, T; L^2(\Omega)), \\ x^{\frac{\alpha}{2}} u_m \text{ is uniformly bounded in } L^2(0, T; H^1(\Omega)), \\ (u_m)_t \text{ is uniformly bounded in } L^2(0, T; L^2(\Omega)). \end{cases} \quad (3.9)$$

■

Step 3: Convergence and result of existence: Consider the following theorem

Theorem 3.3

There exist a function $u \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ with $\frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega))$, and a subsequence denoted by $(u_{m_k})_k \subseteq (u_m)_m$ such that

$$\begin{cases} x^{\frac{\alpha}{2}} u_{m_k} \rightharpoonup x^{\frac{\alpha}{2}} u & \text{in } L^2(0, T; H^1(\Omega)), \\ \frac{\partial u_{m_k}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} & \text{in } L^2(0, T; L^2(\Omega)), \end{cases}$$

when $m \rightarrow +\infty$.

Proof We deduce from lemma 1.3 that there are subsequences denoted by $(u_{m_k})_k, \left(\frac{\partial u_{m_k}}{\partial t}\right)_k$ of (u_m) and $(u_m)_t$ respectively, such that

$$x^{\frac{\alpha}{2}} u_{m_k} \rightharpoonup x^{\frac{\alpha}{2}} u \quad \text{in } L^2(0, T; H^1(\Omega)), \quad (3.10)$$

$$\frac{\partial u_{m_k}}{\partial t} \rightharpoonup w \quad \text{in } L^2(0, T; L^2(\Omega)). \quad (3.11)$$

We know, from Relikh-Kondrachov's theorem, that the injection of $H^1(Q)$ into $L^2(Q)$ is compact. Furthermore, any weakly convergent sequence in $H^1(Q)$ has a subsequence which converges strongly in $L^2(Q)$. So,

$$x^{\frac{\alpha}{2}} u_{m_k} \longrightarrow x^{\frac{\alpha}{2}} u \quad \text{in } L^2(Q). \quad (3.12)$$

On the other hand, from lemma 1.3 there is a subsequence of $(u_{m_k})_k$, still denoted by u_{m_k} , that converges almost everywhere to u , such that

$$x^{\frac{\alpha}{2}} u_{m_k} \longrightarrow x^{\frac{\alpha}{2}} u \quad \text{almost everywhere } Q. \quad (3.13)$$

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

It remains to show that $w = \frac{\partial u}{\partial t}$, and it suffices to prove

$$u(t) = \varphi + \int_0^t w(\tau) d\tau. \quad (3.14)$$

As

$$u_{m_k} \rightharpoonup u \quad \text{in } L^2(0, T; L^2(\Omega)),$$

the proof of (4.20), is equivalent to demonstrate that

$$u_{m_k} \rightharpoonup (\varphi + \gamma) \quad \text{in } L^2(0, T; L^2(\Omega)),$$

which means

$$\lim ((u_{m_k} - \varphi - \gamma), v)_{L^2(0, T; L^2(\Omega))} = 0, \quad \forall v \in L^2(0, T; L^2(\Omega)),$$

where

$$\gamma(t) = \int_0^t w(\tau) d\tau.$$

Using the equality:

$$u_{m_k} - \varphi_{m_k} = \int_0^t \frac{\partial u_{m_k}}{\partial \tau} d\tau \quad \text{for all } t \in [0, T],$$

which results from $u_{m_k} \in L^2(0, T; V_{m_k})$ and $(u_{m_k})_t \in L^2(0, T; V_{m_k})$ that by virtue of (ii), it comes

$$\begin{aligned} & \left(\left(u_{m_k} - \varphi - \int_0^t w(\tau) d\tau \right), v \right)_{L^2(0, T; L^2(\Omega))} \\ &= \int_0^t \left(\left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau) \right), v \right)_{L^2(0, T; L^2(\Omega))} d\tau + \left(x^{\frac{\alpha}{2}} (\varphi_{m_k} - \varphi), v \right)_{L^2(0, T; L^2(\Omega))}, \quad \forall t \in [0, T]. \end{aligned}$$

On the one hand, we have

$$\lim_{k \rightarrow \infty} \int_0^t \left(\left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau) \right), v \right)_{L^2(0, T; L^2(\Omega))} d\tau = 0, \quad \text{for } t \in [0, T]. \quad (3.15)$$

Moreover,

$$\lim_{k \rightarrow \infty} (\varphi_{m_k} - \varphi, v)_{L^2(0, T; L^2(\Omega))} = 0. \quad (3.16)$$

So, we get

$$\lim_{k \rightarrow \infty} ((u_{m_k} - \varphi - \gamma), v)_{L^2(0, T; L^2(\Omega))} = 0, \quad \forall v \in L^2(0, T; L^2(\Omega)).$$



Theorem 3.4

The function u of Theorem (4.1.3) is the weak solution to problem (P_2) in the sense of the definition 4.1.1

Proof From Theorem(4.1.3), we have shown that the limit function u satisfies the first two conditions Definition 4.1.1

Now, let us demonstrate (iii). According to Theorem 4.1.3, we have

$$u_{m_k}(0) \rightharpoonup u(0) \quad \text{in } L^2(\Omega).$$

On the other hand, we have

$$u_{m_k}(0) \longrightarrow \varphi \quad \text{in } L^2(\Omega),$$

so that

$$u_{m_k}(0) \rightharpoonup \varphi \quad \text{in } L^2(\Omega).$$

From the uniqueness of the limit, we get

$$u(0) = \varphi.$$

Still to demonstrate (iv) :

$$(u_t, v) + a(u, v) = (f, v) \quad \forall v \in V, \quad \forall t \in [0, T].$$

Integrating (P_3) on $(0, T)$, we find

$$\int_0^t ((u_m(t))_t, \omega_k) d\tau - a \int_0^t ((x^\alpha (u_m)_x)_x, \omega_k) d\tau = \int_0^t (f(t), \omega_k) d\tau, \\ \forall k = \overline{1, m}, \quad \forall t \in [0, T]. \quad (3.17)$$

By employing equation (3.8) and recognizing that V_m is densely embedded in V , we take the limit as expressed in equation (4.23). This leads us to the result:

$$\int_0^T (u_t, \omega_k) d\tau - a \int_0^T ((x^\alpha u_x)_x, \omega_k) d\tau = \int_0^T (f, \omega_k) d\tau, \quad \forall t \in [0, T],$$

Thus, condition (iv) holds.



Corollary 3.1

The uniqueness of the solution to problem (P_1) follows directly from the estimate (4.11).

■

3.2 Investigation of the existence and uniqueness of weak solutions to the non-linear problem

This section presents an analysis of the existence and uniqueness of weak solutions for nonlinear parabolic equations through the application of the linearization method. The study establishes a robust theoretical framework of the existence and uniqueness, tackling the intricate challenges associated with nonlinear problems. The problem under consideration is formulated as follows:

$$\begin{cases} u_t - (x^\alpha u_x)_x = f(x, t, u) & (x, t) \in Q, \\ u(x, 0) = \varphi(x) & x \in (0, 1), \\ u_x(0, t) = \int_0^1 k_1(x)h(u)(x, t)dx & t \in (0, T), \\ u_x(1, t) = \int_0^1 k_2(x)g(u)(x, t)dx & t \in (0, T). \end{cases} \quad (P_1)$$

Theorem 3.5

Let f , g and h be Lipschitz continuous in u , with the respective constants k , C_0 and C_1 . Then, problem (P_1) admits a weak solution in $L^2(0, T; L^2_{\sqrt{x}}(\Omega) \cap L^2(\Omega))$.

Proof

Set:

$$u = y + w,$$

such that w is a solution to the following nonlocal linear problem:

$$\begin{cases} w_t - (x^\alpha w_x)_x = 0 & (x, t) \in Q, \\ w(x, 0) = \varphi(x) & x \in (0, 1), \\ w_x(0, t) = \int_0^1 k_1(x)h(w(x, t))dx & t \in (0, T), \\ w_x(1, t) = \int_0^1 k_2(x)g(w(x, t))dx & t \in (0, T), \end{cases} \quad (P_4)$$

and

$$y = u - w$$

is a solution to the following nonlocal nonlinear problem:

$$\mathcal{L}y = y_t - (x^\alpha y_x)_x = G(x, t, y),$$

$$y(x, 0) = 0, \quad \forall x \in (0, 1),$$

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

$$y_x(0, t) = y_x(1, t) = 0, \quad \forall t \in (0, T),$$

where

$$G(x, t, y) = f(x, t, y + w).$$

Like the function f , the function G is also Lipschitzian, i.e. there is a positive constant k' such that:

$$\|G(x, t, u_1) - G(x, t, u_2)\|_{L^2(Q)} \leq k' (\|u_1 - u_2\|_{L^2(0, T, H^1(0, 1))}). \quad (3.18)$$

Let us build a recurring sequence starting with $y^{(0)} = 0$. This sequence $(y^{(n)})_{n \in \mathbb{N}}$ is defined as follows:

For $n = 1, 2, 3, \dots$, $y^{(n)}$ is solution of the following problem:

$$\begin{cases} \frac{\partial y^{(n)}}{\partial t} - (x^\alpha y_x^{(n)})_x = G(x, t, y^{(n-1)}), \\ y^{(n)}(x, 0) = 0, \\ y^{(n)}(0, t) = y^{(n)}(1, t) = 0. \end{cases} \quad (P_5)$$

According to the study of the previous linear problem, for fixed n , problem (P_5) admits a unique solution $y^{(n)}(x, t)$. Now, by setting:

$$z^{(n)}(x, t) = y^{(n+1)}(x, t) - y^{(n)}(x, t),$$

we get a new problem:

$$\begin{cases} \frac{\partial z^{(n)}}{\partial t} - (x^\alpha z_x^{(n)})_x = p^{(n-1)}(x, t), \\ z^{(n)}(x, 0) = 0, \\ z_x^{(n)}(0, t) = z_x^{(n)}(1, t) = 0, \end{cases} \quad (P_6)$$

where

$$p^{(n-1)}(x, t) = G(x, t, y^{(n)}) - G(x, t, y^{(n-1)}).$$

Next, multiply the equation:

$$\frac{\partial z^{(n)}}{\partial t} - (x^\alpha z_x^{(n)})_x = p^{(n-1)}(x, t)$$

by $z^{(n)}$, and integrate it on Q_τ , use an integration by parts, By taking account of

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

the initial and boundary conditions, we find

$$\begin{aligned} & \frac{1}{2} \int_0^1 (z^{(n)}(x, \tau))^2 dx + \int_{Q_\tau} x^\alpha \left(\frac{\partial z^{(n)}}{\partial x}(x, t) \right)^2 dx dt \\ &= \int_{Q_\tau} p^{(n-1)}(x, t) z^{(n)}(x, t) dx dt. \end{aligned}$$

The application of the Cauchy's inequality with ϵ on the second member of the equation yields

$$\begin{aligned} & \int_{Q_\tau} p^{(n-1)}(x, t) z^{(n)}(x, t) dx dt \\ &\leq \frac{1}{2\epsilon} \int_{Q_\tau} |G(x, t, y^{(n)}) - G(x, t, y^{(n-1)})|^2 dx dt + \frac{\epsilon}{2} \int_{Q_\tau} (z^{(n)}(x, t))^2 dx dt. \\ &\leq \frac{k'^2}{2\epsilon} \int_{Q_\tau} (|y^{(n)} - y^{(n-1)}|)^2 dx dt + \frac{\epsilon}{2} \int_{Q_\tau} (z^{(n)}(x, t))^2 dx dt \\ &\leq \frac{k'^2}{2\epsilon} \|z^{(n-1)}\|_{L^2(Q)}^2 + \frac{\epsilon}{2} \int_{Q_\tau} (z^{(n)}(x, t))^2 dx dt. \end{aligned}$$

Then multiply by 2 and apply Gronwell's lemma. By integrating with respect to t , we obtain:

$$\|z^{(n)}\|_{L^2(Q)}^2 + \left\| \frac{\partial z^{(n)}}{\partial x} \right\|_{L^2_{\sqrt{x^\alpha}}(Q)}^2 \leq \frac{Tk^2 \exp(\epsilon T)}{\epsilon \min(1, 2T)} \left(\|z^{(n-1)}\|_{L^2(Q)}^2 + \left\| \frac{\partial z^{(n-1)}}{\partial x} \right\|_{L^2_{\sqrt{x^\alpha}}(Q)}^2 \right).$$

Set:

$$c = \frac{Tk^2 \exp(\epsilon T)}{\epsilon \min(1, 2T)}.$$

Thus,

$$\|z^{(n)}\|_V^2 \leq c \|z^{(n-1)}\|_V^2, \quad (3.19)$$

where

$$V = \left\{ y : y \in L^2(Q), y_x \in L^2_{\sqrt{x^\alpha}}(Q) \right\}.$$

As we have

$$\sum_{i=1}^{n-1} z^{(i)} = y^{(n)}$$

according to the convergence criterion of the series, the series $\sum_{n=1}^{\infty} z^{(n)}$ converges if $|c| < 1$, which implies:

$$k < \sqrt{\frac{\epsilon \min(1, 2T)}{T \exp(\epsilon T)}}.$$

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

Therefore, $(y^{(n)})_n$ converges to an element y of V . Show that in V :

$$\lim_{n \rightarrow \infty} y^{(n)}(x, t) = y(x, t)$$

is a solution to the problem (P_5) showing that y checks:

$$A(y, v) = \int_{Q_\tau} G(x, t, y)v(x, t)dxdt \quad \forall v \in O, \quad (3.20)$$

where

$$O = \{v \in C^1(Q), v(0, t) = v(1, t) = 0, \forall t \in (0, T)\},$$

and

$$A(y^{(n)}, v) = \int_{Q_\tau} \frac{\partial y^{(n)}}{\partial t}(x, t)v(x, t)dxdt + \int_{Q_\tau} x^\alpha \frac{\partial y^{(n)}}{\partial x}(x, t) \frac{\partial v}{\partial x}(x, t)dxdt.$$

We have

$$A(y^{(n)} - y, v) = \int_{Q_\tau} \frac{\partial(y^{(n)} - y)}{\partial t}(x, t)v(x, t)dxdt + \int_{Q_\tau} x^\alpha \frac{\partial(y^{(n)} - y)}{\partial x}(x, t) \frac{\partial v}{\partial x}(x, t)dxdt.$$

By applying the Cauchy-Schwarz inequality, we find

$$A(y^{(n)} - y, v) \leq \|v\|_V \|(y^{(n)} - y)_t\|_V + \|v\|_V \|(y^{(n)} - y)_x\|_V.$$

On the other hand, as

$$y^{(n)} \longrightarrow y \quad \text{in } V,$$

we have

$$\begin{aligned} y^{(n)} &\longrightarrow y && \text{in } L^2(Q), \\ y_t^{(n)} &\longrightarrow y_t && \text{in } L^2(Q), \\ y_x^{(n)} &\longrightarrow y_x && \text{in } L^2_{\sqrt{x^\alpha}}(Q). \end{aligned}$$

By passing to the limit when $n \rightarrow +\infty$, we get

$$\lim_{n \rightarrow +\infty} A(y^{(n)} - y, v) = 0.$$

Next, to show the solution uniqueness, let y_1 and y_2 be two solutions in V . Then, we proceed as follows:

$$Z = y_1 - y_2$$

satisfies $Z \in V$ and

$$\begin{aligned} \mathcal{L}Z &= \frac{\partial Z}{\partial t} - (x^\alpha Z_x(x, t))_x = \beta(x, t), \\ Z(x, 0) &= 0, \quad \forall x \in (0, 1), \\ Z(1, t) &= Z(0, t) = 0, \quad \forall t \in (0, t), \end{aligned} \tag{3.21}$$

where

$$\beta(x, t) = G(x, t, u_1) - G(x, t, u_2).$$

Now multiply (3.34) by Z , we find:

$$\frac{1}{2} \int_0^1 (Z(x, \tau))^2 dx + \int_{Q_\tau} (Z_x(x, t))^2 dx dt = \int_{Q_\tau} \beta(x, t) Z(x, t) dx dt.$$

We use the Cauchy-Schwarz inequality and Gronwall's lemma to get

$$\int_{Q_T} (Z(x, \tau))^2 dx dt + \int_{Q_T} \left(x^{\frac{\alpha}{2}} Z_x(x, t)\right)^2 dx dt \leq c \|Z\|_V,$$

which gives

$$\|Z\|_V \leq c \|Z\|_V$$

and since $c < 1$, we have

$$(1 - c) \|Z\|_V \leq 0.$$

This implies that

$$Z = 0, \text{i.e. } y_1 = y_2.$$

We conclude the uniqueness of the weak solution. ■

3.3 Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a linear integral condition

This section provides an analysis of the solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with $g(u) = h(u) = u$. By employing the Galerkin method, energy estimates, and blow-up criteria, we establish conditions under which weak solutions exist, are unique, and blow up in finite time.

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

In the domain $\Pi = \{(x, t) \in \mathbb{R}^2 : 0 < x < 1 \text{ and } 0 < t < T\}$, consider the problem:

$$\begin{cases} u_t - (x^\alpha u_x)_x = f(x, t, u) & (x, t) \in \Pi, \\ u(x, 0) = \varphi(x) & x \in (0, 1), \\ u(0, t) = \int_0^1 k_1(x)u(x, t)dx & t \in (0, T), \\ u(1, t) = \int_0^1 k_2(x)u(x, t)dx & t \in (0, T). \end{cases} \quad (P_m)$$

3.3.1 Solvability of the linear problem

Here, we will first formulate the main linear problem related to (P_m) , discuss the problem's solvability in view of the variable separation method, and explore the existence and uniqueness of solution to that problem.

Position of the problem

Using the same domain mentioned above, consider the following linear problem:

$$\begin{cases} u_t - (x^\alpha u_x)_x = f(x, t) & (x, t) \in \Pi, \\ u(x, 0) = \varphi(x) & x \in (0, 1), \\ u(0, t) = \int_0^1 k_1(x)u(x, t)dx & t \in (0, T), \\ u(1, t) = \int_0^1 k_2(x)u(x, t)dx & t \in (0, T), \end{cases} \quad (P_{m1})$$

where $0 \leq \alpha < 1$ and the functions f , φ , k_1 and k_2 are known functions.

In order to go forward in our investigation, we intend to separate formally the problem (P_{m1}) into two further linear problems. These problems are stated as follows:

$$\begin{cases} v_t - (x^\alpha v_x)_x = 0 & \forall (x, t) \in \Pi \\ v(x, 0) = \varphi(x) & \forall x \in (0, 1) \\ v(0, t) = \int_0^1 k_1(x)v(x, t)dx & \forall t \in (0, T) \\ v(1, t) = \int_0^1 k_2(x)v(x, t)dx & \forall t \in (0, T), \end{cases} \quad (P_2)$$

and

$$\begin{cases} \mathcal{L}w = w_t - (x^\alpha w_x)_x = f(x, t) & \forall (x, t) \in \Pi \\ lw = w(x, 0) = 0 & \forall x \in (0, 1) \\ w(0, t) = w(1, t) = 0 & \forall t \in (0, T). \end{cases} \quad (P_3)$$

where $L = (\mathcal{L}, l)$ and $\mathcal{F} = f(x, t)$.

Solvability of problem (P_2)

The solvability of problem (P_2) is analytically treated using the method of separation of variables.

To this end, let us assume that a solution to the problem takes the following form:

$$v(x, t) = X(x)T(t). \quad (3.22)$$

For finding $X(x)$, we substitute the above solution in (P_2) . Consequently, for $\lambda = \omega^2 > 0$, we get:

$$\frac{(x^\alpha X')'}{X} = \frac{T'}{T} = -\omega^2.$$

This gives the following Sturm-Liouville problem:

$$\left\{ \begin{array}{l} x^\alpha X''(x) + \alpha x^{\alpha-1} X'(x) + \omega^2 X = 0, \\ X(0) = \int_0^1 k_1(x) X(x) dx, \\ X(1) = \int_0^1 k_2(x) X(x) dx. \end{array} \right.$$

Thus, the solution of this problem can then be given formally as:

$$X_n(x) = AJ + By.$$

A, B positive constant, J is a Bessel function. Alternatively, in order to obtain the expression for $T(t)$ as stated in (3.22) , we can use the principle of superpositions and assume:

$$v(x, t) = \sum_{n \geq 0} X_n(x) \cdot T_n(t). \quad (3.23)$$

This immediately implies

$$\begin{aligned} T_n(0) &= \varphi_n, \\ T_n(t) &= \varphi_n e^{-\omega_n^2 t}, \end{aligned}$$

where

$$\varphi_n = \int_0^1 \varphi(x) \cdot X_n(x) dx.$$

Solvability of problem (P_3)

In this subsection, we will use the energy inequality method to obtain a priori estimates for the solution problem (P_3) . To this aim, we introduce the next theoretical result that is related to the uniqueness of the solution to (P_3) .

Theorem 3.6

For any function w in the space $C(0, T, L^2_{\sqrt{\xi}}(0, 1))$, the following estimate holds:

$$\|w\|_E \leq k \|\mathcal{F}\|_F, \quad (3.24)$$

where $E = C(0, T, L^2_{\sqrt{\xi}}(0, 1))$ and $F = L^2(\Pi)$ are Banach and Hilbert spaces with finite norms and k is positive integer, respectively. The uniqueness of the solution to problem (P_3) , if it exists, is guaranteed by this theorem.

Proof To prove this result, we first multiply the PDE in problem (P_3) by the multiplier Mu , which has the following form:

$$Mu = w(x, t),$$

and then integrate the result over $\Pi_\tau = (0, 1) \times (0, \tau)$, where $\tau \in [0, T]$. This would give

$$\int_{\Pi_\tau} [w_t - (x^\alpha w_x)_x] \cdot w(x, t) dx dt = \int_{\Pi_\tau} f(x, t) \cdot w(x, t) dx dt.$$

Simplifying the last equation and applying the Cauchy inequality to the result yields:

$$\frac{1}{2} \int_{\Omega} w(x, t)^2 dx + \int_{\Pi} x^\alpha w(x, t)^2 dx dt \leq \frac{1}{2} \int_{\Pi} f(x, t)^2 dx dt + \frac{1}{2} \int_{\Pi} w(x, t)^2 dx dt.$$

Consequently, applying the Gronwall's Lemma to the above inequality gives:

$$\max_{0 < \tau < T} \int_{\Omega} w(x, \tau)^2 dx dt + 2 \int_{\Pi} x^\alpha w(x, \tau)^2 dx dt \leq e^T \left(\int_{\Pi} f(x, t)^2 dx dt \right)$$

where $C = e^T$. Accordingly, we obtain:

$$\|w\|_{C(0, T, L^2(0, 1))}^2 + \|\partial_x w\|_{L^2(0, T, L^2_{\sqrt{x^\alpha}}(0, 1))}^2 \leq C \|f\|_{L^2(\Pi)}^2.$$

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

This is equivalent to the following inequality:

$$\|w\|_E \leq k \|\mathcal{F}\|_F,$$

and this completes the proof. ■

Corollary 3.2

For any function $w \in D(L)$ satisfying $\|w\|_E \leq k\|\mathcal{F}\|_F$, Equation (5.29) provides an estimate that ensures the uniqueness of the solution to problem (P_3) , assuming such a solution exists.

Proof Assume w_1 and w_2 are two solutions of problem (P_3) , which means that

$$\begin{cases} Lw_1 = \mathcal{F} \\ Lw_2 = \mathcal{F} \end{cases}, \text{ so that } L(w_1 - w_2) = 0.$$

In accordance of (3.3.1), we can infer:

$$\|w_1 - w_2\|_E^2 \leq c \|0\|_F^2 = 0,$$

which asserts that $w_1 = w_2$, and this completes the proof. ■

In addition, we present the following theoretical result to explore the existence of a solution to problem (P_3) .

Theorem 3.7:

If $\vartheta \in L^2(\Pi)$ and for any $w \in D(L)$, the integral of $\mathcal{L}w$ and ϑ over Π is zero, then ϑ vanishes almost everywhere Π .

Proof The scalar product is given by $(Lu, \omega)_F = \int_Q \mathcal{L}w \cdot \vartheta dxdt$. Here, \mathcal{L} is a differential operator, and ϑ, w belong to appropriate function spaces. Moreover, equation (P_3) can be expressed as: $\int_{\Pi} \frac{\partial w}{\partial t} \cdot \vartheta dxdt - \int_{\Pi} \frac{\partial}{\partial x} \left(x^\alpha \frac{\partial w}{\partial x} \right) \cdot \vartheta dxdt = 0$, (3.25) where $w, \frac{\partial w}{\partial t}$ and $\frac{\partial w}{\partial x} \in L^2_{\sqrt{x^\alpha}}(\Pi)$ such that w satisfies the boundary conditions of problem (P_3) . From the last equality, we can choose the function ϑ as follows:

$$\vartheta = w.$$

This means that $\vartheta \in L^2(Q)$. Now, substituting for ϑ into (3.25) yields

$$\int_{\Pi} \frac{\partial w}{\partial t} \cdot \vartheta dxdt = - \int_{\Pi} x^\alpha \left(\frac{\partial w}{\partial x} \right)^2 dxdt \leq 0.$$

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

Accordingly, if one integrates by part in the above inequality and uses the given boundary conditions, then the following inequality is obtained:

$$\int_0^1 (w)^2 \Big|_{\tau=0}^{\tau=T} dx \leq 0,$$

which immediately gives $w = 0$ in Q . Therefore, we can infer that $w = 0$ implies $\overline{R(L)} = F$, which finishes the proof of this theorem. ■

Uniqueness of solution to problem (P_1)

Our upcoming goal is to investigate the uniqueness of the solution to problem (P_1) . To achieve this, we need the following result.

Theorem 3.8

Provided that for every function $u \in L^2(0, T; L^2(\Omega))$, the following estimate holds:

$$\|u\|_A \leq k \|\mathcal{F}\|_B, \quad (3.26)$$

where $A = L^2(0, T; L^2_{\sqrt{x^\alpha}}(\Omega)) \cap L^2(0, T; L^2(\Omega)) \cap L^\infty(0, T; L^2_{\sqrt{x^\alpha}}(\Omega))$ is a Banach space with a finite norm:

$$\|u\|_A^2 = \|u_x\|_{L^\infty(0, T; L^2_{\sqrt{x^\alpha}}(\Omega))}^2 + \|u_x\|_{L^2(0, T; L^2_{\sqrt{x^\alpha}}(\Omega))}^2 + \|u_t\|_{L^2(0, T; L^2(\Omega))}^2 + \|u\|_{L^2(0, T; L^2(\Omega))}^2,$$

and $B = L^2(0, T; L^2(\Omega))$ is a Hilbert space with a finite norm:

$$\|u\|_B^2 = \|f\|_{L^2(0, T; L^2(\Omega))}^2 + \|\varphi\|_{(L^2(\Omega))}^2 + \|\varphi_x\|_{L^2_{\sqrt{x^\alpha}}(\Omega)}^2,$$

The solution to problem (P_1) , if it exists, is unique.

Proof Let $\varepsilon, c, d, \delta, \beta$ be very small positive integer such that $d = 4k^2$, $\beta = \frac{1}{8k^2}$, $\delta = 4k^2$, $c = \frac{1}{4}$. By multiplying problem (P_1) by u , and then integrating the result over the domain Π_τ , we obtain:

$$\int_{\Pi_\tau} [u_t - (x^\alpha u_x)_x - f] \cdot M u dx = \int_{\Pi_\tau} f(x, t) u dx,$$

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

Now, using the Cauchy inequality with $\epsilon > 0$ leads us to the following inequality:

$$\begin{aligned} & \frac{1}{2} \int_0^1 u(x, \tau)^2 dx + \int_{\Pi_\tau} x^\alpha (u_x(x, t))^2 dx dt - \int_0^\tau u_x(1, t) u(1, t) dt \quad (3.27) \\ & \leq \frac{1}{2\epsilon} \int_{\Pi_\tau} f^2 dx dt + \frac{\epsilon}{2} \int_{\Pi_\tau} u(x, t)^2 dx dt + \frac{1}{2} \int_0^1 \varphi(x)^2 dx. \end{aligned}$$

The integral conditions therefore give

$$\int_0^\tau u_x(1, t) u(1, t) dt \leq \frac{\delta}{2} \int_0^\tau u_x(1, t)^2 + \frac{k'^2}{2\delta} \int_0^\tau \int_0^1 u(x, t)^2 dx dt, \quad (3.28)$$

where $k' = \max_{0 < \xi < 1} k_2(\xi)$. In addition, we have

$$\int_0^1 u_t(x, t) dx - \int_0^1 (x^\alpha u_x)_x dx = \int_0^1 f dx,$$

so that

$$\frac{\delta}{2} \int_0^\tau |u_x(1, t)|^2 dt \leq \frac{\delta \beta}{2} \int_0^\tau \int_0^1 u_t(x, t)^2 dx + \frac{\delta}{2} \frac{1}{2\beta} \int_0^\tau \int_0^1 f^2 dx. \quad (3.29)$$

Thus,

$$\begin{aligned} & \frac{1}{2} \int_0^1 u(x, t)^2 dx + \int_{\Pi_\tau} x^\alpha u_x(x, t)^2 + \int_{Q_\tau} u_t(x, t)^2 + \frac{1}{2} \int_0^1 x^\alpha u_x(x, t)^2 \\ & \leq \left(\frac{1}{2\epsilon} + \frac{1}{2c} + \frac{d}{4\beta} + \frac{\delta}{4\beta} \right) \int_{Q_\tau} f^2 + \frac{1}{2} \int_0^1 \varphi(x)^2 dx + \frac{1}{2} \int_0^1 x^\alpha \varphi_x(x)^2 \\ & \quad + \left(\frac{\epsilon}{2} + \frac{k^2}{2\delta} \right) \int_{\Pi_\tau} u(x, t)^2 + \left(\frac{c}{2} + \frac{d\beta}{4} + \frac{k^2}{2d} + \frac{\delta\beta}{4} \right) \int_{\Pi_\tau} u_t(x, t)^2. \end{aligned}$$

The Gronwall's Lemma can be used to yield

$$\begin{aligned} & \int_{\Pi} u(x, t)^2 dx + \int_{\Pi} x^\alpha u_x(x, t)^2 dx dt + \int_{\Pi} x^\alpha u_x(x, t)^2 dx + \int_{\Pi} u_t(x, t)^2 dx dt \\ & \leq C^* \left(\int_{\Pi} f^2 dx dt + \int_0^1 \varphi(x)^2 dx + \int_0^1 x^\alpha \varphi_x(x)^2 dx \right), \end{aligned}$$

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

where

$$C^* = \frac{\max \left\{ \left(\frac{1}{2\varepsilon} + \frac{1}{2c} + \frac{d}{4\beta} + \frac{\delta}{4\beta} \right), \frac{1}{2} \right\}}{\min \left\{ \left(1 - \left(\frac{c}{2} + \frac{d\beta}{4} + \frac{\delta\beta}{4} + \frac{k^2}{2d} \right) \right), \frac{1}{2} \right\}} \exp \left(\left(\frac{\varepsilon}{2} + \frac{k^2}{2\delta} \right) T \right).$$

Hence,

$$\begin{aligned} & \|u_x\|_{L^\infty(0,T, L^2_{\sqrt{x^\alpha}}(\Omega))}^2 + \|u_x\|_{L^2(0,T, L^2_{\sqrt{x^\alpha}}(\Omega))}^2 + \|u_t\|_{L^2(0,T, L^2(\Omega))}^2 + \|u\|_{L^2(0,T, L^2(\Omega))}^2 \\ & \leq C^* (\|f\|_{L^2(0,T, L^2(\Omega))}^2 + \|\varphi\|_{L^2(\Omega)}^2 + \|\varphi_x\|_{L^2_{\sqrt{x^\alpha}}(\Omega)}^2), \end{aligned}$$

which completes the proof. ■

3.3.2 Solvability of the nonlinear problem

The objective is to prove that there exists a unique solution to the following nonlinear problem (P_m) with the nonlinear function $f(x, t, u)$:

$$\begin{cases} u_t - (x^\alpha u_x)_x = f(x, t, u) & (x, t) \in \Pi, \\ u(x, 0) = \varphi(x) & x \in (0, 1), \\ u(0, t) = \int_0^1 k_1(x) u(x, t) dx & t \in (0, T), \\ u(1, t) = \int_0^1 k_2(x) u(x, t) dx & t \in (0, T). \end{cases} \quad (P_m)$$

Next, set:

$$u = y + w,$$

where w is the solution of problem:

$$\begin{cases} w_t - (x^\alpha w_x)_x = 0 & (x, t) \in \Pi, \\ w(x, 0) = \varphi(x) & \xi \in (0, 1), \\ w(0, t) = \int_0^1 k_1(x) w(x, t) dx & t \in (0, T), \\ w(1, t) = \int_0^1 k_2(x) w(x, t) dx & t \in (0, T). \end{cases} \quad (P_4)$$

Besides,

$$y = u - w$$

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

satisfies the following nonlocal nonlinear problem:

$$\mathcal{L}y = y_t - (x^\alpha y_x)_x = H(x, t, y), \quad (3.30)$$

$$y(x, 0) = 0 \quad x \in (0, 1), \quad (3.31)$$

$$y(0, t) = y(1, t) = 0 \quad t \in (0, t). \quad (3.32)$$

where

$$H(x, t, y) = f(x, t, y + w).$$

The Lipschitz continuity of H is obviously similar to that of f . This implies the existence of a positive constant k such that:

$$\| H(x, t, u_1, \cdot) - H(x, t, u_2, \cdot) \|_{L^2(\Pi)} \leq \mathcal{K} \| u_1 - u_2 \|_{L^2(0, T, H^1(0, 1))}. \quad (3.33)$$

To solve the nonlocal nonlinear problem, we construct a recursive sequence with $y^{(0)} = 0$ and defined as follows. For every $k = 1, 2, \dots$, we solve the problem:

$$\begin{cases} \frac{\partial y^{(k)}}{\partial t} - (x^\alpha y_x^{(k)})_x = H(x, t, y^{(k-1)}) \\ y^{(k)}(x, 0) = 0 \\ y^{(k)}(0, t) = y^{(k)}(1, t) = 0, \forall k = 1, 2, 3, \dots \end{cases}, \quad (P_5)$$

In accordance to the study of the previous linear problem, note that at each iteration we have fixed k . Thus, problem (P_5) will admit a unique solution $y^{(k)}(x, t)$. Now, by setting:

$$z^{(k)}(x, t) = y^{(k+1)}(x, t) - y^{(k)}(x, t),$$

we get a new problem:

$$\begin{cases} \frac{\partial z^{(k)}}{\partial t} - (x^\alpha z_x^{(k)})_x = p^{(k-1)}(x, t), \\ z^{(k)}(x, 0) = 0, \\ z^{(k)}(0, t) = z^{(k)}(1, t) = 0, \end{cases} \quad (P_6)$$

or

$$p^{(k-1)}(\xi, t) = H(x, t, y^{(k)}) - H(x, t, y^{(k-1)}).$$

Multiplying the PDE in problem (P_6) :

$$\frac{\partial z^{(k)}}{\partial t} - (x^\alpha z_x^{(k)})_x = p^{(k-1)}(x, t)$$

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

by $z^{(k)}$, and then integrating the result over Π_τ yield

$$\begin{aligned} & \int_{\Pi_\tau} \frac{\partial z^{(k)}}{\partial t}(x, t) \cdot z^{(k)}(x, t) dx dt - \int_{\Pi_\tau} (x^\alpha z_x^{(k)})_x \cdot z^{(k)}(x, t) dx dt \\ & \leq \frac{1}{2\varepsilon} \int_{\Pi_\tau} |p^{(k-1)}(x, t)|^2 dx dt + \frac{\varepsilon}{2} \int_{\Pi_\tau} (z^{(k)}(x, t))^2 dx dt, \\ & \leq \frac{1}{2\varepsilon} \int_{\Pi_\tau} |H(x, t, y^{(k)}) - H(x, t, y^{(k-1)})|^2 dx dt + \frac{\varepsilon}{2} \int_{\Pi_\tau} (z^{(k)}(x, t))^2 dx dt. \end{aligned}$$

Since H is a Lipschitzian function, we get

$$\int_{\Pi_\tau} \frac{\partial z^{(k)}}{\partial t}(x, t) \cdot z^{(k)}(x, t) dx dt - \int_{\Pi_\tau} (x^\alpha z_x^{(k)})_x \cdot z^{(k)}(x, t) dx dt \leq \frac{\mathcal{K}^2}{\varepsilon} \|z^{(k-1)}\|_{L^2(\Pi)}^2 + \frac{\varepsilon}{2} \int_{\Pi_\tau} (z^{(k)}(x, t))^2 dx dt.$$

Multiplying the above inequality by 2 and then applying Gronwall's Lemma lead to the following inequality:

$$\int_0^1 (z^{(k)}(x, \tau))^2 dx + 2 \int_{\Pi_\tau} x^\alpha \left(\frac{\partial z^{(k)}}{\partial x}(x, t) \right)^2 dx dt \leq 2 \frac{\mathcal{K}^2}{\varepsilon} \|z^{(k-1)}\|_{L^2(0, T, L^2(0, 1))}^2 \exp(\varepsilon T).$$

Next, by integrating the latter with respect to t , the following inequality holds:

$$\|z^{(k)}\|_{L^2(\Pi)}^2 + \left\| \frac{\partial z^{(k)}}{\partial x} \right\|_{L^2_{\sqrt{x^\alpha}}(\Pi)}^2 \leq \frac{2T\mathcal{K}^2 \exp(\varepsilon T)}{\varepsilon \min(1, 2T)} \|z^{(k-1)}\|_{L^2(\Pi)}^2 + \left\| \frac{\partial z^{(k-1)}}{\partial x} \right\|_{L^2_{\sqrt{x^\alpha}}(\Pi)}^2.$$

Consequently, if

$$c = \max \left\{ 1, \frac{2T\mathcal{K}^2 \exp(\varepsilon T)}{\varepsilon \min(1, 2T)} \right\},$$

then

$$\|z^{(k)}\|_V^2 \leq c \|z^{(k-1)}\|_V^2,$$

where

$$V = \left\{ y : y \in L^2(\Pi), y_x \in L^2_{\sqrt{x^\alpha}}(\Pi) \right\}$$

and

$$\sum_{i=1}^{k-1} z^{(i)} = y^{(k)}.$$

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

Based on the series convergence criterion, we can establish that the series $\sum_{k=1}^{\infty} z^{(k)}$ is convergent provided that $|c| < 1$. Hence,

$$k < \sqrt{\frac{\varepsilon \min(1, 2T)}{2T \exp(\varepsilon T)}}.$$

Thus, the sequence $y^{(k)}$ converges to an element y in V . In this regard, we have to prove

$$\lim_{k \rightarrow \infty} y^{(k)}(x, t) = y(x, t)$$

in V , where $y(x, t)$ represents the solution of problem (P_4) in which y should satisfy

$$M(y, v) = \int_{\Pi_\tau} H(x, t, y) \cdot v(x, t) dx dt \quad \forall v \in O,$$

where

$$O = \left\{ v \in C^1(Q) : v(0, t) = v(1, t) = 0, \forall t \in (0, T) \right\},$$

and

$$M(y^{(k)}, v) = \int_{\Pi_\tau} \frac{\partial y^{(k)}}{\partial t}(x, t) \cdot v(x, t) dx dt + \int_{\Pi_\tau} x^\alpha \frac{\partial y^{(k)}}{\partial x}(x, t) \cdot \frac{\partial v}{\partial x}(x, t) dx dt.$$

Note that we also have

$$M(y^{(k)} - y, v) = \int_{\Pi_\tau} \frac{\partial(y^{(k)} - y)}{\partial t}(x, t) \cdot v(x, t) dx dt + \int_{\Pi_\tau} x^\alpha \frac{\partial(y^{(k)} - y)}{\partial x}(x, t) \cdot \frac{\partial v}{\partial x}(x, t) dx dt.$$

It is possible to obtain the following equation by means of the Cauchy-Schwartz inequality:

$$M(y^{(k)} - y, v) \leq \|v\|_V \left\| (y^{(k)} - y)_t \right\|_V + \|v\|_V \left\| (y^{(k)} - y)_\xi \right\|_V.$$

As $y^{(k)}$ converges to y in V , it follows that

$$\begin{aligned} y^{(k)} &\longrightarrow y && \text{in } L^2(\Pi), \\ y_t^{(k)} &\longrightarrow y_t && \text{in } L^2(\Pi), \\ y_\xi^{(k)} &\longrightarrow y_\xi && \text{in } L^2_{\sqrt{\xi^\alpha}}(\Pi). \end{aligned}$$

Moreover,

$$\lim_{k \rightarrow +\infty} M(y^{(k)} - y, v) = 0.$$

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

For the uniqueness of the solution of problem (P_r) , let y_1 and y_2 be two solutions of the problem at hand in V . Therefore,

$$Z = y_1 - y_2$$

is in V . Consequently, by setting:

$$\eta(x, t) = H(x, t, u_1) - H(x, t, u_2),$$

Z will clearly satisfy the equation:

$$\mathcal{L}Z = \frac{\partial Z}{\partial t} - (x^\alpha Z_x(x, t))_x = \eta(x, t), \quad (3.34)$$

and

$$\begin{aligned} Z(x, 0) &= 0, \quad \forall x \in (0, 1), \\ Z(1, t) &= Z(0, t) = 0 \quad \forall t \in (0, t). \end{aligned}$$

Multiplying equation (3.34) by Z leads to the following assertion:

$$\frac{1}{2} \int_0^1 (Z(x, \tau))^2 dx + \int_{\Pi_\tau} (Z_x(x, t))^2 dx dt = \int_{\Pi_\tau} \eta(x, t) \cdot Z(x, t) dx dt.$$

By using the Cauchy-Schwartz inequality and the Gronwall's Lemma, we obtain

$$\int_{\Pi_T} (Z(x, \tau))^2 dx dt + \int_{\Pi_T} (Z_x(x, t))^2 dx dt \leq c \|Z\|_V,$$

which implies

$$\|Z\|_{L^2(0, T, H^1(\omega))} \leq c \|Z\|_{L^2(0, T, H^1(\omega))}.$$

Since $c < 1$, we get

$$(1 - c) \|Z\|_{L^2(0, T, H^1(\omega))} \leq 0.$$

Therefore,

$$Z = 0 \text{ i.e. } y_1 = y_2,$$

and hence the uniqueness follows.

3.3.3 Construction of a blow-up solution of Fujita type

This section is devoted to the construction of blow-up solutions in the sense of Fujita. Our primary goal is to provide a clear understanding of finite-time blow-up solutions and present effective numerical methods to approximate them. We will delve into two crucial schemes: the forward time centered space scheme and the backward time centered space scheme, which are well-suited for approximating finite-time blow-up solutions. To facilitate the analysis, we will assume that the function in (P_m) takes the form $f(x, t, u) = u^p$ throughout this section.

Finite-time blow-up solution

Let λ_1 be the first eigenvalue and $\psi(x)$ its corresponding eigenfunction of the following eigenvalue problem:

$$\begin{cases} -(x^\alpha \psi_x)_x = \lambda_1 \psi \\ \psi(0) = 0 \\ \psi_x(1) = 0. \end{cases} \quad (3.35)$$

It is worth noting that according to [?], λ_1 is obtained as the first zero of $J_{\frac{1-\alpha}{2-\alpha}}(\frac{2\sqrt{\lambda_1}}{2-\alpha}x^{\frac{2-\alpha}{2}})$, where $J_{\frac{1-\alpha}{2-\alpha}}$ denotes the Bessel function of the first kind of orders $\frac{1-\alpha}{2-\alpha}$. Moreover, $\psi(x)$ is a positive and smooth function on $(0, 1)$, and can be explicitly expressed as follows:

$$\psi(x) = dx^{\frac{1-\alpha}{2}} J_{\frac{1-\alpha}{2-\alpha}}(\frac{2\sqrt{\lambda_1}}{2-\alpha}x^{\frac{2-\alpha}{2}}). d > 0.$$

Suppose now that u is a positive function and

$$\rho(t) = \int_0^1 \psi(x)u(x, t)dx.$$

Multiplying the equation:

$$\frac{\partial u}{\partial t} - (x^\alpha u_x)_x = u^p$$

by ψ and integrating the resulting expression over the domain $\Omega = (0, 1)$ lead to

$$\frac{\partial}{\partial t} \int_0^1 u(x, t)\psi dx - \int_0^1 \lambda k_2 u(x, t)dx + \int_0^1 \lambda \psi u(x, t) = \int_0^1 u^p \psi dx.$$

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

Thus, as a result of utilizing the Jensen inequality to the aforementioned equality, we deduce that

$$\frac{\partial}{\partial t} \int_0^1 u(x, t) \psi dx - \lambda \frac{\min k_2}{\max \psi} \int_0^1 u(x, t) \psi dx + \int_0^1 \lambda u(x, t) \psi dx \geq \left(\int_0^1 u(x, t) \psi dx \right)^p.$$

It immediately follows:

$$\rho'(t) + c\rho(t) \geq \rho(t)^p,$$

where

$$c = -\lambda \frac{\min g_2}{\max \psi} + \lambda.$$

Now, if one solves the resulting Bernoulli equation:

$$\rho'(t) + c\rho(t) - \rho(t)^p = 0, \quad (3.36)$$

by assuming

$$v = \rho^{1-p}, \quad (3.37)$$

and $\frac{1}{p-1} > 0$, then

$$\rho \rightarrow \infty \text{ if } \left((\rho(0))^{1-p} - \frac{1}{(1-p)c} \right) e^{-(1-p)ct} + \frac{1}{(1-p)c} \rightarrow 0.$$

Hence,

$$T^* = \frac{1}{(1-p)c} \ln \left(\frac{\frac{1}{(1-p)c}}{\left((\rho(0))^{1-p} - \frac{1}{(1-p)c} \right)} \right).$$

Approximation of the finite-time blow-up solution

We employ the finite difference method to compute the numerical solution for problem (P_5) . Let us partition the intervals $[0, 1]$ and $[0, T]$ into M and N sub-intervals of uniform sizes $h = 1/M$ and $k = T/N$, respectively. As usual, u_i^n denotes the finite difference approximations of $u(x_i, t_n)$ at the i^{th} grid point and n^{th} time step. The grid points (x_i, t_n) are given by $x_i = ih$, for $i = 0, 1, 2, \dots, M$, and $t_n = nk$, $k = 0, 1, 2, \dots, N$.

Explicit forward-time centered-space scheme First, it should be noted that we can approximate the time derivative by the forward difference quotient.

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

Besides, we can use the 1st- and the 2nd-order centered difference formulas for approximating the 1st- and the 2nd-order spatial derivatives occurring in (P_5) , respectively. However, this would give the following explicit expression:

$$u_i^{n+1} = k(u_i^n)^p + (1 - 2rx_i^\alpha)u_i^n + (rx_i^\alpha + \frac{\alpha}{2i}rx_i^\alpha)u_{i+1}^n + (rx_i^\alpha - \frac{\alpha}{2i}rx_i^\alpha)u_{i-1}^n, \quad (3.38)$$

for $i = 1, 2, \dots, M - 1$, $n = 0, 1, \dots, N$, and $r = k/h^2$. Clearly, it remains to determine the two unknowns u_0^{n+1} and u_M^{n+1} . For this, we approximate the integrals arising in (P_m) using the trapezoidal rule. In so doing, we get

$$(1 - \frac{h}{2}k_1(0))u_0^{n+1} - \frac{h}{2}k_1(1)u_M^{n+1} = h \sum_{i=1}^{M-1} k_1(x_i)u_i^{n+1}, \quad (3.39)$$

and

$$-\frac{h}{2}k_2(0)u_0^{n+1} + (1 - \frac{h}{2}k_2(1))u_M^{n+1} = h \sum_{i=1}^{M-1} k_2(x_i)u_i^{n+1}. \quad (3.40)$$

Now, set: $a_1 = (1 - \frac{h}{2}k_1(0))$, $a_2 = -\frac{h}{2}k_2(0)$, $b_1 = -\frac{h}{2}k_1(1)$, $b_2 = (1 - \frac{h}{2}k_2(1))$, $c_1 = h \sum_{i=1}^{M-1} k_1(x_i)u_i^{n+1}$ and $c_2 = h \sum_{i=1}^{M-1} k_2(x_i)u_i^{n+1}$. So, the following condition:

$$a_1b_2 - a_2b_1 \neq 0$$

is satisfied for sufficiently small h . Finally, using the Cramer's rule, we can find u_0^{n+1} and u_M^{n+1} .

Implicit backward-time centered-space scheme In what follows, we will use the classical backward-time centered-space finite difference scheme to approximate the derivatives occurring in problem (P_5) . Thus,

$$(1 + 2rx_i^\alpha - kp(u_i^n)^{p-1})u_i^{n+1} + (\frac{\alpha}{2i}rx_i^\alpha - x_i^\alpha)u_{i-1}^{n+1} + (-\frac{\alpha}{2i}rx_i^\alpha - x_i^\alpha)u_{i+1}^{n+1} = u_i^n + k(1-p)(u_i^n)^p, \quad (3.41)$$

for $i = 1, 2, \dots, M - 1$, $n = 0, 1, \dots, N$, and $r = 3b_1k/h^2$. Note that there are $M - 1$ linear equations given by (3.41) with the $M + 1$ unknowns u_0, u_1, \dots, u_M . In order to find these unknowns, we need two more equations. For this purpose, let us formally approximate the integrals appearing in (P_1) using the trapezoidal rule. As a result, we get

$$u_0^{n+1} = \frac{h}{2}(k_1(0)u_0^{n+1} + k_1(M)u_M^{n+1} + 2 \sum_{i=1}^{M-1} k_1(x_i)u_i^{n+1}), \quad (3.42)$$

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

and

$$u_M^{n+1} = \frac{h}{2}(k_2(0)u_0^{n+1} + k_2(M)u_M^{n+1} + 2 \sum_{i=1}^{N-1} k_2(x_i)u_i^{n+1}). \quad (3.43)$$

Combining (3.42) and (3.43) with (3.41) yields a $(M + 1) \times (M + 1)$ system of linear equations. The latter can be written in matrix form as follows:

$$A^{n+1}U^{n+1} = B^{n+1} \quad (3.44)$$

where

$$A^{n+1} = \begin{pmatrix} (1 - \frac{h}{2}k_1(0)) & -hk_2(x_1) & -hk_2(x_2) & \dots & -hk_2(M-1) & -\frac{h}{2}k_2(1) \\ \frac{\alpha}{2}rx_1^\alpha - rx_1^\alpha & \psi_1 & -\frac{\alpha}{2}rx_1^\alpha - rx_1^\alpha & & & 0 \\ 0 & \frac{\alpha}{4}rx_2^\alpha - rx_2^\alpha & \psi_2 & & & 0 \\ \dots & & & & & \\ 0 & & & & -\frac{\alpha}{2}rx_{(M-1)}^\alpha - rx_{(M-1)}^\alpha & \\ -\frac{h}{2}k_2(M) & -hk_2(x_1) & -hk_2(x_2) & \dots & -hk_2(x_{M-1}) & -(1 - \frac{h}{2}k_2(0)) \end{pmatrix}$$

$$\psi_i = 1 + 2rx_i^\alpha - kp(u_i^n)^{p-1},$$

$$U^{n+1} = \begin{pmatrix} u_0^{n+1} \\ u_1^{n+1} \\ u_2^{n+1} \\ \dots \\ u_{M-1}^{n+1} \\ u_M^{n+1} \end{pmatrix}_{(M+1) \times 1}$$

$$B^{n+1} = \begin{pmatrix} 0 \\ u_1^n + k(1-p)(u_1^n)^p \\ u_2^n + k(1-p)(u_2^n)^p \\ \dots \\ u_{M-1}^n + k(1-p)(u_{M-1}^n)^p \\ u_M^n + k(1-p)(u_M^n)^p \\ 0 \end{pmatrix}.$$

Illustrative numerical experiments

Now, let us validate the accuracy and efficiency of the explicit and implicit schemes just derived through three numerical experiments. These test problems will demonstrate the practical usage of these schemes. The nonlinear problems are formulated as follows:

$$\begin{aligned}
 \text{Problem 1:} & \left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left(x^\alpha \frac{\partial u}{\partial x} \right) = u^2 \quad x \in [0.1], t > 0 \\ u(0, t) = \int_0^1 k_1(x) u(x, t) dx \quad \forall t \in (0, T), \\ u(1, t) = \int_0^1 k_2(x) u(x, t) dx \quad \forall t \in (0, T), \\ u(x, 0) = A \cos\left(\frac{\pi}{3}x\right) \quad x \in [0.1] \end{array} \right. \\
 \text{Problem 2:} & \left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left(x^\alpha \frac{\partial u}{\partial x} \right) = u^3 \quad x \in [0.1], t > 0 \\ u(0, t) = \int_0^1 k_1(x) u(x, t) dx \quad \forall t \in (0, T), \\ u(1, t) = \int_0^1 k_2(x) u(x, t) dx \quad \forall t \in (0, T), \\ u(x, 0) = A \cos\left(\frac{\pi}{3}x\right) \quad x \in [0.1] \end{array} \right. \\
 \text{Problem 3:} & \left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left(x^\alpha \frac{\partial u}{\partial x} \right) = u^4 \quad x \in [0.1], t > 0 \\ u(0, t) = \int_0^1 k_1(x) u(x, t) dx \quad \forall t \in (0, T), \\ u(1, t) = \int_0^1 k_2(x) u(x, t) dx \quad \forall t \in (0, T), \\ u(x, 0) = A \cos\left(\frac{\pi}{3}x\right) \quad x \in [0.1] \end{array} \right.
 \end{aligned}$$

where $k_1(x) = \frac{1}{\cos(\frac{\pi x}{3})}$, $k_2(x) = \frac{1}{2\cos(\frac{\pi x}{3})}$ and $A = 12$. It is important to note that we are unable to determine the exact analytical solutions for all three problems. As a result, we will rely on numerical estimation to determine their blow-up times T . The upcoming tables present the estimated blow-up times and their corresponding CPU time in seconds for varying space-step values utilizing both explicit and implicit schemes.

h	T^* by the explicit scheme	CPU time	T^* by the implicit scheme	CPU time
1/40	0.084931	1.18	0.084912	1.316
1/80	0.084392	18.2629	0.084356	33.229
1/160	0.08409375	152.368	0.08409143	544.681
1/320	0.0840813	2108.014	0.0840805	13069.686

Table 3.1: Comparison of the explicit and implicit schemes in obtaining T^* for Problem 1 with $p = 2$.

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

h	T^* by the explicit scheme	CPU time	T^* by the implicit scheme	CPU time
1/40	0.00368172	0.330	0.00367421	0.270
1/80	0.00360731	0.873	0.00360582	1.942
1/160	0.00358246	4.94	0.00358112	24.315
1/320	0.00353125	58.60	0.00353089	363.32

Table 3.2: Comparison of the explicit and implicit schemes in obtaining T^* for Problem 2 with $p = 3$.

In the same context, Figures 3.1, 3.2 and 3.3 present the corresponding graphs of

h	T^* by the explicit scheme	CPU time	T^* by the implicit scheme	CPU time
1/40	0.00025327	0.32	0.00025156	0.215
1/80	0.00023564	0.59	0.00023048	2.346
1/160	0.00021998	0.883	0.00021945	5.298
1/320	0.00021875	3.63	0.00021871	22.506

Table 3.3: Comparison of the explicit and implicit schemes in obtaining T^* for Problem 3 with $p = 4$.

the numerical solutions of the three problems 1, 2 and 3 obtained using the explicit and implicit schemes, respectively.

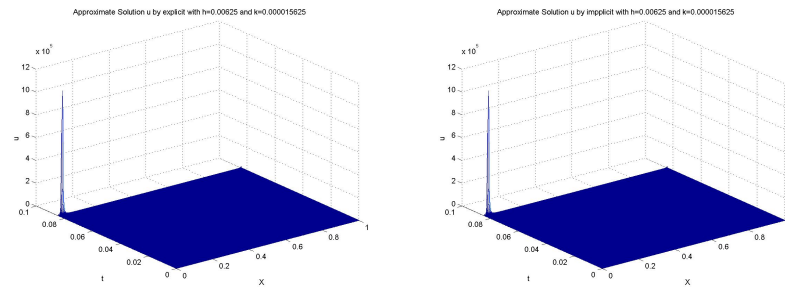


Figure 3.1: Numerical solution to Problem 1 by (a) explicit scheme and (b) implicit scheme

Chapter 3. Solvability and blow-up of weak solutions for a singular and degenerate semilinear parabolic equation with a mixed linear and nonlinear integral condition

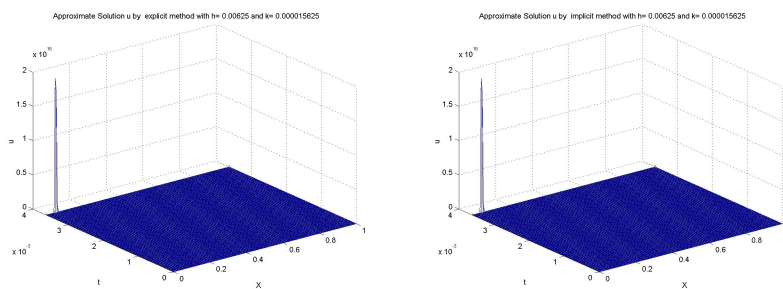


Figure 3.2: Numerical solution to Problem 2 by (a) explicit scheme and (b) implicit scheme

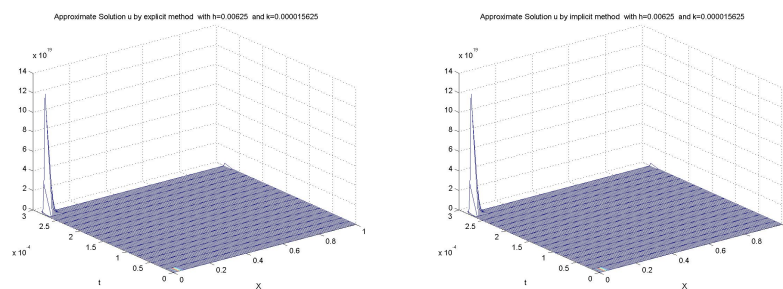


Figure 3.3: Numerical solution to Problem 3 by (a) explicit scheme and (b) implicit scheme

Exploring Solvability of Nonlinear Hyperbolic Equations with Mixed Nonlinear and Linear Integral Neumann Boundary Conditions

This chapter examines the solvability of linear hyperbolic problems with second-type integral conditions, using the Faedo-Galerkin method. This general framework establishes the foundational assumptions, boundary conditions, and key properties. From this, we derive specific problems as particular cases in next sections.

Let $Q = \{(x, t) \in \mathbb{R}^2, x \in \Omega =]0, l[\text{ and } 0 < t < T\}$. Consider the nonlinear problem [?], [8], given below:

$$\left\{ \begin{array}{l} \frac{\partial^2 u}{\partial t^2} - a \frac{\partial^2 u}{\partial x^2} + bu = f(x, t, u), \\ u(x, 0) = \varphi(x), \\ u_t(x, 0) = \psi(x) \\ \frac{\partial u}{\partial x}(0, t) = \int_0^l k(x, t) \alpha(u)(x, t) dx, \\ \frac{\partial u}{\partial x}(l, t) = \int_0^l k(x, t) \beta(u)(x, t) dx. \end{array} \right. \quad (P_1)$$

Under these assumptions, f, φ and $\psi \in L^2(Q)$ and $k \in L^\infty(Q)$. Also, α and β are a Lipschitzian function.

4.1 Analysis of the solvability of the solution of linear hyperbolic problem with integral condition of second type by the method of Fadeo-galarkin

4.1.1 Position of the linear problem (P_1)

In the rectangular domain $Q = \Omega \times (0, T)$, with T finite. Consider the following linear problem (P_2):

$$\left\{ \begin{array}{ll} \frac{\partial^2 u}{\partial t^2} - a \frac{\partial^2 u}{\partial x^2} + bu = f(x, t) & \forall (x, t) \in Q, \\ u(x, 0) = \varphi(x) & \forall x \in (0, l), \\ u_t(x, 0) = \psi(x) & \forall x \in (0, l), \\ \frac{\partial u}{\partial x}(0, t) = \int_0^l k(x, t)\alpha(u)(x, t)dx & \forall t \in (0, T), \\ \frac{\partial u}{\partial x}(l, t) = \int_0^l k(x, t)\beta(u)(x, t)dx & \forall t \in (0, T). \end{array} \right. \quad (P_2)$$

The hyperbolic equation associated with this problem is given by:

$$\mathcal{L}u = \frac{\partial^2 u}{\partial t^2} - a \frac{\partial^2 u}{\partial x^2} + bu = f(x, t), \quad (4.1)$$

The initial conditions can be formally expressed as:

$$\ell u = \left\{ \begin{array}{l} u(x, 0) = \varphi(x) \\ u_t(x, 0) = \psi(x) \end{array} \right., \quad x \in (0, l),$$

Along with the integral condition of the second kind:

$$\begin{aligned} \frac{\partial u}{\partial x}(0, t) &= \int_0^l k(x, t)\alpha(u)(x, t)dx, & t \in (0, T). \\ \frac{\partial u}{\partial x}(l, t) &= \int_0^l k(x, t)\beta(u)(x, t)dx, & t \in (0, T). \end{aligned}$$

Where

$$k(x, t) \geq 0, \quad \forall (x, t) \in Q \text{ and } \alpha(u)(x, t) \leq \beta(u)(x, t), \quad \forall (x, t) \in Q.$$

We define space V by:

$$V = \{u \in H^1(\Omega)\}.$$

Chapter 4. Exploring Solvability of Nonlinear Hyperbolic Equations with Mixed Nonlinear and Linear Integral Neumann Boundary Conditions

Where V , equipped with the norm $\|v\|_V = \|v\|_{H^1(\Omega)}$, this forms a Hilbert space. To effectively formulate and investigate problem (P_2) , it is essential to establish the following hypotheses that will guide our study:

$$(H) : \begin{cases} f \in L^2(0, T; L^2(\Omega)) & (H.1), \\ \varphi \in H^1(\Omega) & (H.2). \end{cases}$$

Definition 4.1

The weak solution of the problem (P_2) is a function allowing for a thorough exploration of its properties and behaviors:

- (i) $u \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$.
- (ii) u admits a strong derivative $\frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega))$.
- (iii) $u(0) = \varphi, u_t(0) = \psi$.
- (iv) Identity: $\forall v \in V, \forall t \in [0, T]$

$$(u_{tt}, v) + a(u_x, v_x) + b(u, v) = (f, v) + u_x(l, t)v(l) - u_x(0, t)v(0).$$

4.1.2 Variational formulation

This section, presents the Variational formulation of the problem (P_2) :

$$(u_{tt}, v) + a(u_x, v_x) + b(u, v) = (f, v) + u_x(l, t)v(l) - u_x(0, t)v(0) \quad \forall v \in V, \quad (4.2)$$

Where (\cdot, \cdot) denotes the scalar product $L^2(\Omega)$.

4.1.3 Study of the existence of weak solution of the problem (P_2)

The demonstration of the existence of the solution of the problem (P_2) is grounded in the Faedo-Galerkin method. Which consists of three critical steps, each designed to progressively build towards a rigorous proof of existence. Below is an elaboration on each step:

Step 1: Construction of a Galerkin Approximation:

The first step involves identifying a suitable finite-dimensional subspace V_m of the Hilbert space V in which we seek solutions. This subspace is spanned by a finite set of basis functions w_1, w_2, \dots, w_m . We express the approximate solution $u_m(t)$ in terms of these basis functions as follows:

$$\begin{cases} w_i \in V, & \forall i, \\ \forall m, w_1, w_2, \dots, w_m & \text{are linearly independent,} \\ V_m = \langle \{w_1, w_2, \dots, w_m\} \rangle & \text{is dense in } V. \end{cases} \quad (4.3)$$

In particular :

$$\forall \varphi \in V \implies \exists (\theta_{km})_m \in \mathbb{N}^*, \varphi_m = \sum_{k=1}^m \theta_{km} \omega_k \longrightarrow \varphi \text{ when } m \longrightarrow +\infty. \quad (4.4)$$

$$\forall \psi \in V \implies \exists (\lambda_{km})_m \in \mathbb{N}^*, \psi_m = \sum_{k=1}^m \lambda_{km} \omega_k \longrightarrow \psi \text{ when } m \longrightarrow +\infty. \quad (4.5)$$

Faedo Galerkin's approximation consists in searching for any integer $m \geq 1$, a function

$$t \mapsto u_m(x, t) = \sum_{i=1}^m \sigma_{im}(t) \omega_i(x),$$

verifies

$$\begin{cases} u_m(t) \in V_m, & \forall t \in [0, T], \\ ((u_m(t))_{tt}, \omega_k) + A(u_m(t), \omega_k) = (f(t), \omega_k) & \forall k = \overline{1, m}. \end{cases} \quad (P_3)$$

Where, we have

$$\begin{aligned} ((u_m(t))_{tt}, \omega_k) &= \left(\left(\sum_{i=1}^m \sigma_{im}(t) \omega_i \right)_{tt}, \omega_k \right), \\ &= \left(\sum_{i=1}^m \frac{\partial^2 \sigma_{im}}{\partial t^2}(t) \omega_i(x), \omega_k \right), \\ &= \sum_{i=1}^m (\omega_i, \omega_k) \frac{\partial^2 \sigma_{im}}{\partial t^2}(t). \end{aligned} \quad (4.6)$$

and

$$\begin{aligned}
 & A(u_m(t), \omega_k) \\
 &= A\left(\sum_{i=1}^m \sigma_{im}(t) \omega_i, \omega_k\right), \\
 &= a \sum_{i=1}^m \sigma_{im}(t) \left[\int_{\Omega} \frac{\partial \omega_i}{\partial x} \frac{\partial \omega_k}{\partial x} dx - \frac{\partial \omega_i}{\partial x}(l) \omega_k(l) + \frac{\partial \omega_i}{\partial x}(0) \omega_k(0) \right],
 \end{aligned} \tag{4.7}$$

$$\begin{aligned}
 &= a \sum_{i=1}^m \sigma_{im}(t) \int_{\Omega} \frac{\partial \omega_i(x)}{\partial x} \frac{\partial \omega_k(x)}{\partial x} dx - a \sum_{i=1}^m \sigma_{im}(t) \frac{\partial \omega_i}{\partial x}(l) \omega_k(l) + a \sum_{i=1}^m \sigma_{im}(t) \frac{\partial \omega_i}{\partial x}(0) \omega_k(0), \\
 &= \sum_{i=1}^m A(\omega_i, \omega_k) \sigma_{im}(t).
 \end{aligned} \tag{4.8}$$

Also, we have

$$\begin{aligned}
 u_m(0) &= \sum_{i=1}^m \sigma_{im}(0) \omega_i(x), \\
 &= \kappa_m = \sum_{i=1}^m \kappa_{im} w_i(x).
 \end{aligned}$$

and

$$\begin{aligned}
 u'_m(0) &= \sum_{i=1}^m \sigma'_{im}(0) w_i(x), \\
 &= \chi_m = \sum_{i=1}^m \chi_{im} w_i(x).
 \end{aligned}$$

We obtain a system of second order nonlinear differential equations :

$$\left\{ \begin{array}{l} \sum_{i=1}^m (\omega_i, \omega_k) \frac{\partial^2 \sigma_{im}}{\partial t^2}(t) + a \sum_{i=1}^m \left(\frac{\partial \omega_i}{\partial x}, \frac{\partial \omega_k}{\partial x} \right) \sigma_{im}(t) \\ = (f(t), \omega_k) + a \sum_{i=1}^m \sigma_{im}(t) \frac{\partial \omega_i}{\partial x}(l) \omega_k(l) - a \sum_{i=1}^m \sigma_{im}(t) \frac{\partial \omega_i}{\partial x}(0) \omega_k(0), \\ \sigma_{im}(0) = \kappa_{im} \quad \forall i = \overline{1, m}. \\ \sigma'_{im}(0) = \chi_{im} \quad \forall i = \overline{1, m}. \end{array} \right. \tag{P_4}$$

We consider the vector

$$\sigma_m = (\sigma_{1m}(t), \dots, \sigma_{mm}(t)), f_m = ((f, \omega_1), \dots, (f, \omega_m)),$$

Chapter 4. Exploring Solvability of Nonlinear Hyperbolic Equations with Mixed Nonlinear and Linear Integral Neumann Boundary Conditions

and the matrix

$$B_m = ((\omega_i, \omega_j))_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}, A_m = \left(\left(\frac{\partial \omega_i}{\partial x}, \frac{\partial \omega_j}{\partial x} \right) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}},$$

and

$$C_m = \left(\frac{\partial \omega_i}{\partial x}(l) \cdot \omega_j(l) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}, D_m = \left(\frac{\partial \omega_i}{\partial x}(0) \cdot \omega_j(0) \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}.$$

We write the problem (P_4) in the matrix form, we obtain :

$$\begin{cases} B_m \frac{\partial^2 \sigma_m}{\partial t^2}(t) + aA_m \sigma_m + aD_m \sigma_m = f_m + aC_m \sigma_m, \\ \sigma_m(0) = (\kappa_{im})_{1 \leq i \leq m}, \\ \sigma'_m(0) = (\chi_{im})_{1 \leq i \leq m}. \end{cases}$$

As the matrix entries B_m are linearly independent (because it is adiaagonal matrix) then $\det B_m \neq 0$, so it is invertible, then σ_m is the solution of

$$\begin{cases} \frac{\partial^2 \sigma_m}{\partial t^2}(t) + (aB_m^{-1}A_m + bB_m^{-1}D_m - aB_m^{-1}C_m) \sigma_m = B_m^{-1}f_m, \\ \sigma_m(0) = (\kappa_{im})_{1 \leq i \leq m}, \\ \sigma'_m(0) = (\chi_{im})_{1 \leq i \leq m}. \end{cases} \quad (P_5)$$

It is easy that we can verify this ordinary differential systems have a solution whare the matrix $(aB_m^{-1}A_m + bB_m^{-1}D_m - aB_m^{-1}C_m)$ with constant coefficients and the vector $B_m^{-1}f_m$ with continuous functions and majorized by integrable fuctions on $(0, T)$ then we can conclude that there exists a t_m depends only on $|\kappa_{im}|, |\chi_{im}|$.

Step 2: A priori estimate

Lemma 4.1

For all $m \in \mathbb{N} \setminus \{0\}$ and if

$$\frac{1}{8a} > k^2,$$

the solution $u_m \in L^2(0, T; V_m)$ of the problem (P_2) checks

$$\begin{aligned} \|u_m\|_{L^2(0,T;H^1(\Omega))} &\leq c_1, \\ \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(0,T;L^2(\Omega))} &\leq c_2. \end{aligned}$$

where c_1, c_2 are two positive constants independent of m .

Proof Multiply the equation of (P_3) by $\sigma'_{km}(t)$ and we sum over k , we find

$$\begin{aligned} &\sum_{k=1}^m ((u_m(t))_{tt}, \omega_k) \cdot \sigma'_{km}(t) + a \sum_{k=1}^m \left(\frac{\partial^2 u_m}{\partial x^2}(t), \omega_k \right) \cdot \sigma'_{km}(t) \\ = &\sum_{k=1}^m (f(t), \omega_k) \cdot \sigma'_{km}(t) + a \sum_{i=1}^m \sigma_{im}(t) \frac{\partial \omega_i}{\partial x}(l) \sum_{k=1}^m \sigma'_{km}(t) \omega_k(l) \\ &- a \sum_{i=1}^m \sigma_{im}(t) \frac{\partial \omega_i}{\partial x}(0) \sum_{k=1}^m \sigma'_{km}(t) \omega_k(0). \end{aligned}$$

So, we obtain

$$\begin{aligned} &((u_m(t))_{tt}, (u_m(t))_t) - a \left(\frac{\partial^2 u_m}{\partial x^2}(t), \frac{u_m}{\partial t}(t) \right) \\ = &(f(t), (u_m(t))_t) + a \sum_{i=1}^m \sigma_{im}(t) \frac{\partial \omega_i}{\partial x}(l) \sum_{k=1}^m \sigma'_{km}(t) \omega_k(l) \\ &- a \sum_{i=1}^m \sigma_{im}(t) \frac{\partial \omega_i}{\partial x}(0) \sum_{k=1}^m \sigma'_{km}(t) \omega_k(0). \end{aligned}$$

Thus; we get

$$\begin{aligned} &((u_m(t))_{tt}, (u_m(t))_t) + \frac{a}{2} \frac{d}{dt} \left\| \frac{\partial u_m}{\partial x} \right\|_{L^2(\Omega)}^2 \\ = &(f(t), u_m(t)) + a \sum_{i=1}^m \sigma_{im}(t) \frac{\partial \omega_i}{\partial x}(l) \sum_{k=1}^m \sigma'_{km}(t) \omega_k(l) \\ &- a \sum_{i=1}^m \sigma_{im}(t) \frac{\partial \omega_i}{\partial x}(0) \sum_{k=1}^m \sigma'_{km}(t) \omega_k(0). \end{aligned}$$

Chapter 4. Exploring Solvability of Nonlinear Hyperbolic Equations with Mixed Nonlinear and Linear Integral Neumann Boundary Conditions

integrating over 0 to t , by using the Cauchy inequality with ε , ($|ab| \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$), we get

$$\begin{aligned}
 & \frac{1}{2} \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 + \frac{a}{2} \|\nabla u_m\|_{L^2(\Omega)}^2 \\
 \leq & \frac{1}{2\varepsilon} \|f\|_{L^2(Q)}^2 + \left(\frac{\varepsilon C_T}{2} + \frac{(ak)^2}{2\delta} \right) \left\| \frac{\partial u_m}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega))}^2 + \delta \|\nabla u_m\|_{L^\infty(0,T;L^2(\Omega))}^2 \\
 & + \frac{a}{2} \|\nabla \chi_m\|_{L^2(\Omega)}^2 + \max\left(\frac{1}{2}, ak\delta\right) \|\kappa_m\|_{H^1(\Omega)}^2, \tag{4.9}
 \end{aligned}$$

where the constant $K = \max \int_Q k^2(x, t) dxdt$. Then, we obtain

$$\begin{aligned}
 & \frac{1}{2} \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)}^2 + \frac{a}{2} \|\nabla u_m\|_{L^2(\Omega)}^2 \\
 \leq & \frac{1}{2\varepsilon} \|f\|_{L^2(Q)}^2 + \left(\frac{\varepsilon C_T}{2} + \frac{(ak)^2}{2\delta} \right) \left\| \frac{\partial u_m}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega))}^2 + \delta \|\nabla u_m\|_{L^\infty(0,T;L^2(\Omega))}^2 \\
 & + \frac{a}{2} \|\nabla \psi_m\|_{L^2(\Omega)}^2 + \max\left(\frac{1}{2}, \delta\right) \|\varphi_m\|_{H^1(\Omega)}^2, \tag{4.10}
 \end{aligned}$$

$$\begin{aligned}
 & \left(\frac{1}{2} - \left(\frac{\varepsilon C_T}{2} + \frac{(ak)^2}{2\delta} \right) \right) \left\| \frac{\partial u_m}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega))}^2 + \left(\frac{a}{2} - \delta \right) \|\nabla u_m\|_{L^\infty(0,T;L^2(\Omega))}^2 \\
 \leq & \frac{1}{2\varepsilon} \|f\|_{L^2(Q)}^2 + \frac{a}{2} \|\nabla \psi_m\|_{L^2(\Omega)}^2 + \max\left(\frac{1}{2}, \delta\right) \|\kappa_m\|_{H^1(\Omega)}^2.
 \end{aligned}$$

setting $\varepsilon = \frac{1}{4C_T}$, $\delta = 4(ak)^2$, we get

$$\begin{aligned}
 & \left\| \frac{\partial u_m}{\partial t} \right\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|\nabla u_m\|_{L^\infty(0,T;L^2(\Omega))}^2 \\
 \leq & C_1 \left[\|f\|_{L^2(Q)}^2 + \|\nabla \chi_m\|_{L^2(\Omega)}^2 + \|\kappa_m\|_{H^1(\Omega)}^2 \right]. \tag{4.11}
 \end{aligned}$$

therefore,

$$C_1 = \frac{\max\left(2C_T, \frac{a}{2}, \max\left(\frac{1}{2}, (ak)^2\right)\right)}{\min\left\{\frac{1}{4}, \left(\frac{a}{2} - 4(ak)^2\right)\right\}}.$$

From (4.11), it follows that

$$\left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)} \leq \sqrt{C_1} \left[\|f\|_{L^2(Q)}^2 + \|\nabla \chi_m\|_{L^2(\Omega)}^2 + \|\kappa_m\|_{H^1(\Omega)}^2 \right]^{\frac{1}{2}}. \tag{4.12}$$

Chapter 4. Exploring Solvability of Nonlinear Hyperbolic Equations with Mixed Nonlinear and Linear Integral Neumann Boundary Conditions

Integrating (4.12) over $[0, T]$, we obtain

$$\begin{aligned}
 \left\| \int_0^\tau \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)} &\leq \int_0^\tau \left\| \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)} \\
 &\leq T\sqrt{C_1} \left[\|f\|_{L^2(Q)}^2 + \|\nabla\chi_m\|_{L^2(\Omega)}^2 + \|\kappa_m\|_{H^1(\Omega)}^2 \right]^{\frac{1}{2}}, \\
 \left\| \int_0^\tau \frac{\partial u_m}{\partial t} \right\|_{L^2(\Omega)} &\leq T\sqrt{C_1} \left[\|f\|_{L^2(Q)}^2 + \|\nabla\chi_m\|_{L^2(\Omega)}^2 + \|\kappa_m\|_{H^1(\Omega)}^2 \right]^{\frac{1}{2}}, \\
 \|u_m - \kappa_m\|_{L^2(\Omega)} &\leq T\sqrt{C_1} \left[\|f\|_{L^2(Q)}^2 + \|\nabla\chi_m\|_{L^2(\Omega)}^2 + \|\kappa_m\|_{H^1(\Omega)}^2 \right]^{\frac{1}{2}}, \\
 \left| \|u_m\|_{L^2(\Omega)} - \|\kappa_m\|_{L^2(\Omega)} \right| &\leq T\sqrt{C_1} \left[\|f\|_{L^2(Q)}^2 + \|\nabla\chi_m\|_{L^2(\Omega)}^2 + \|\kappa_m\|_{H^1(\Omega)}^2 \right]^{\frac{1}{2}}, \\
 \left(\|u_m\|_{L^2(\Omega)} - \|\kappa_m\|_{L^2(\Omega)} \right)^2 &\leq TC_1 \left[\|f\|_{L^2(Q)}^2 + \|\nabla\chi_m\|_{L^2(\Omega)}^2 + \|\kappa_m\|_{H^1(\Omega)}^2 \right], \\
 \|u_m\|_{L^2(\Omega)}^2 + \|\kappa_m\|_{L^2(\Omega)}^2 &\leq TC_1 \left[\|f\|_{L^2(Q)}^2 + \|\nabla\chi_m\|_{L^2(\Omega)}^2 + \|\kappa_m\|_{H^1(\Omega)}^2 \right] \\
 &\quad + 2 \|u_m\|_{L^2(\Omega)} \|\kappa_m\|_{L^2(\Omega)}.
 \end{aligned}$$

Applying Cauchy inequality with γ , it follows that

$$\begin{aligned}
 \|u_m\|_{L^2(\Omega)}^2 + \|\kappa_m\|_{L^2(\Omega)}^2 &\leq TC_1 \left[\|f\|_{L^2(Q)}^2 + \|\nabla\chi_m\|_{L^2(\Omega)}^2 + \|\kappa_m\|_{H^1(\Omega)}^2 \right] \\
 &\quad + \frac{1}{\gamma} \|u_m\|_{L^2(\Omega)}^2 + \gamma \|\kappa_m\|_{L^2(\Omega)}^2.
 \end{aligned} \tag{4.13}$$

Setting $\gamma = 2$, we get

$$\|u_m\|_{L^2(\Omega)}^2 \leq 4TC_1 \left[\|f\|_{L^2(Q)}^2 + \|\nabla\chi_m\|_{L^2(\Omega)}^2 + \|\kappa_m\|_{H^1(\Omega)}^2 \right]. \tag{4.14}$$

It follows from (4.11) and (5.9) that the solution to the initial value problem for the system of ODE (P₄) can be extended to $[0, T]$. We obtain

$$\begin{cases} u_m \text{ is uniformly bounded in } L^2(0, T; H^1(\Omega)), \\ (u_m)_t \text{ is uniformly bounded in } L^2(0, T; L^2(\Omega)). \end{cases} \tag{4.15}$$

■

Step 3: Convergence and result of existence

Theorem 4.1

There is a function $u \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ with $\frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega))$ and a subsequence denoted by $(u_{m_k})_k \subseteq (u_m)_m$, such that

$$\begin{cases} u_{m_k} \rightharpoonup u & \text{in } L^2(0, T; H^1(\Omega)) \\ \frac{\partial u_{m_k}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} & \text{in } L^2(0, T; L^2(\Omega)) \end{cases} ,$$

when $m \rightarrow +\infty$.

Proof We deduce from lemma 1.3,1.4 chapter 1, there are subsequences denoted by $(u_{m_k})_k, (\frac{\partial u_{m_k}}{\partial t})_k$ of $(u_m)_m$ and $(u_m)_t$ respectively, such that

$$u_{m_k} \rightharpoonup u \quad \text{in } L^2(0, T; H^1(\Omega)) . \quad (4.16)$$

$$\frac{\partial u_{m_k}}{\partial t} \rightharpoonup w \quad \text{in } L^2(0, T; L^2(\Omega)) . \quad (4.17)$$

We know that according to Relikh-Kondrachoff's theorem that the injection of $H^1(Q)$ into $L^2(Q)$ is compact. And like the results of Rellich's theorem, any weakly convergent sequence in $H^1(Q)$ has a subsequence which converges strongly in $L^2(Q)$. So

$$u_{m_k} \rightarrow u \quad \text{in } L^2(Q). \quad (4.18)$$

On the other hand, from lemma 1.5 chapter 1, there is a subsequence of $(u_{m_k})_k$ is still denoted by u_{m_k} converges almost everywhere to u , such that

$$u_{m_k} \rightarrow u \quad \text{almost everywhere } Q . \quad (4.19)$$

It remains to demonstrate that $w = \frac{\partial u}{\partial t}$, for that it suffices to prove :

$$u(t) = \varphi + \int_0^t w(\tau) d\tau. \quad (4.20)$$

As

$$u_{m_k} \rightharpoonup u \quad \text{in } L^2(0, T; L^2(\Omega)) ,$$

then, the proof of (4.20), is equivalent to demonstrate that

$$u_{m_k} \rightharpoonup \varphi + h \quad \text{in } L^2(0, T; L^2(\Omega)) ,$$

Chapter 4. Exploring Solvability of Nonlinear Hyperbolic Equations with Mixed Nonlinear and Linear Integral Neumann Boundary Conditions

which means

$$\lim (u_{m_k} - \varphi - h, v)_{L^2(0,T; L^2(\Omega))} = 0, \quad \forall v \in L^2(0, T; L^2(\Omega)),$$

as

$$h(t) = \int_0^t w(\tau) d\tau.$$

Using equality

$$u_{m_k} - \varphi_{m_k} = \int_0^t \frac{\partial u_{m_k}}{\partial \tau} d\tau, \quad \text{for all } t \in [0, T],$$

which results from $u_{m_k} \in L^2(0, T; V_{m_k})$ and $(u_{m_k})_t \in L^2(0, T; V_{m_k})$ that

$$\begin{aligned} & \left(u_{m_k} - \varphi - \int_0^t w(\tau) d\tau, v \right)_{L^2(0,T; L^2(\Omega))} \\ &= \left(\int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau) \right) d\tau, v \right)_{L^2(0,T; L^2(\Omega))} + (\varphi_{m_k} - \varphi, v)_{L^2(0,T; L^2(\Omega))}, \quad \text{for all } t \in [0, T], \end{aligned}$$

by virtue of Lemma 1.5 chapter 1, it comes

$$\begin{aligned} & \left(u_{m_k} - \varphi - \int_0^t w(\tau) d\tau, v \right)_{L^2(0,T; L^2(\Omega))} \\ &= \int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau), v \right)_{L^2(0,T; L^2(\Omega))} d\tau + (\varphi_{m_k} - \varphi, v)_{L^2(0,T; L^2(\Omega))}, \quad \text{for all } t \in [0, T]. \end{aligned}$$

On the one hand, we have

$$\lim_{k \rightarrow \infty} \int_0^t \left(\frac{\partial u_{m_k}}{\partial \tau} - w(\tau), v \right)_{L^2(0,T; L^2(\Omega))} d\tau = 0, \quad \text{for } t \in [0, T]. \quad (4.21)$$

Also, we have

$$\lim_{k \rightarrow \infty} (\varphi_{m_k} - \varphi, v)_{L^2(0,T; L^2(\Omega))} = 0. \quad (4.22)$$

So we get

$$\lim_{k \rightarrow \infty} (u_{m_k} - \varphi - h, v)_{L^2(0,T; L^2(\Omega))} = 0, \quad \forall v \in L^2(0, T; L^2(\Omega)).$$

■

Theorem 4.2

The function u of the theorem (4.1) is the weak solution to the problem (P_2) in the sense of the definition (4.1).

Proof From the theorem (4.1), we have shown that the limit function u satisfies the first two conditions of the definition (4.1). Now we will demonstrate **(iii)**. According to the theorem (4.1), we have

$$u_{m_k}(0) \rightharpoonup u(0) \quad \text{in } L^2(\Omega) .$$

On the other hand, we have

$$u_{m_k}(0) \longrightarrow \varphi \quad \text{in } L^2(\Omega) ,$$

so

$$u_{m_k}(0) \rightharpoonup \varphi \quad \text{in } L^2(\Omega) .$$

From the uniqueness of the limit, we get

$$u(0) = \varphi .$$

By using the same steps we demonstrate $u_t(0) = \psi$. Still to demonstrate **(iv)** :

$$(u_{tt}, v) + a(u, v) = (f, v) \quad \forall v \in V, \text{ and } \forall t \in [0, T] .$$

Integrating (P_3) on $(0, T)$, we find

$$\int_0^t ((u_m(t))_{tt}, \omega_k) d\tau + \int_0^t a(u_m(t), \omega_k) d\tau = \int_0^t (f(t), \omega_k) d\tau \quad \forall k = \overline{1, m}, \text{ and } \forall t \in [0, T] . \tag{4.23}$$

we know that V_m dense in V and by passing to the limit in (4.23) , we find

$$\int_0^T (u_{tt}, \omega_k) d\tau + \int_0^T a(u, \omega_k) d\tau = \int_0^T (f, \omega_k) d\tau, \quad \forall t \in [0, T],$$

so **(iv)** is verified. ■

Corollary 4.1

The uniqueness of the solution of problem (P_2) comes straight through the estimate (4.11).

4.2 Exploration of weak solutions in the context of nonlinear source with $b=0$ and homogeneous boundary conditions

To broaden the class of admissible solutions in the nonlinear problem (P_1) , the proof of the solvability of the weak solution concept becomes even more crucial to the nonlinear problem with $b = 0$, and homogeneous boundary conditions.

First, we propose the concept of the studied solution. Let $v = v(x, t)$ be any function of V , such that

$$V = \left\{ v \in C^1(Q), v_x(l, t) = v_x(0, t) = 0, t \in [0, T] \right\}.$$

Multiplying

$$\frac{\partial^2 y}{\partial t^2} - a \frac{\partial^2 y}{\partial x^2} = f(x, t, y, y_x).$$

By v and integrate it on Q_τ , we find

$$\begin{aligned} & \int_{Q_\tau} \frac{\partial^2 y}{\partial t^2}(x, t) \cdot \frac{\partial v}{\partial t}(x, t) dx dt - a \int_{Q_\tau} \Delta y(x, t) \cdot \frac{\partial v}{\partial t}(x, t) dx dt \\ &= \int_{Q_\tau} G(x, t, y, y_x) \cdot \frac{\partial v}{\partial t}(x, t) dx dt, \end{aligned}$$

and using an integration by parts and the conditions on y, v we get

$$\begin{aligned} & \int_{Q_\tau} \frac{\partial^2 y}{\partial t^2}(x, t) \cdot \frac{\partial v}{\partial t}(x, t) dx dt + a \int_{Q_\tau} \frac{\partial y}{\partial x}(x, t) \cdot \frac{\partial^2 v}{\partial x \partial t}(x, t) dx dt \\ &= \int_{Q_\tau} G(x, t, y, y_x) \cdot \frac{\partial v}{\partial t}(x, t) dx dt, \end{aligned} \tag{4.24}$$

Chapter 4. Exploring Solvability of Nonlinear Hyperbolic Equations with Mixed Nonlinear and Linear Integral Neumann Boundary Conditions

it then results from (4.24) that

$$\Phi(y, v) = \int_{Q_\tau} G(x, t, y, y_x) \cdot \frac{\partial v}{\partial t}(x, t) dx dt, \quad (4.25)$$

Where,

$$A(y, v) = \int_{Q_\tau} \frac{\partial^2 y}{\partial t^2}(x, t) \cdot \frac{\partial v}{\partial t}(x, t) dx dt + a \int_{Q_\tau} \frac{\partial y}{\partial x}(x, t) \cdot \frac{\partial^2 v}{\partial x \partial t}(x, t) dx dt.$$

Building a recurring sequence starting with $y^{(0)} = 0$. The sequence $(y^{(n)})_{n \in \mathbb{N}}$ is defined as follows: given the element $y^{(n-1)}$, then for $n = 1, 2, 3, \dots$ we will solve the following problem

$$\left\{ \begin{array}{l} \frac{\partial^2 y^{(n)}}{\partial t^2} - a \Delta y^{(n)} = G(x, t, y^{(n-1)}, y_x^{(n-1)}), \\ y^{(n)}(x, 0) = 0, \\ y_t^{(n)}(x, 0) = 0, \\ y_x^{(n)}(0, t) = 0, \\ y_x^{(n)}(l, t) = 0. \end{array} \right. \quad (P_4)$$

According to the study of the previous linear problem each time we fix the n , the problem (P_4) admits a unique solution $y^{(n)}(x, t)$, as established through the Faedo-Galerkin method. This unique solution is derived by constructing a finite-dimensional approximation of the infinite-dimensional problem, effectively demonstrating that the solution exists within the prescribed constraints of the method.

$$z^{(n)}(x, t) = y^{(n+1)}(x, t) - y^{(n)}(x, t),$$

so we get a new problem :

$$\left\{ \begin{array}{l} \frac{\partial^2 z^{(n)}}{\partial t^2} - a \Delta z^{(n)} = p^{(n-1)}(x, t), \\ z^{(n)}(x, 0) = 0, \\ z_t^{(n)}(x, 0) = 0, \\ z_x^{(n)}(0, t) = 0, \\ z_x^{(n)}(l, t) dx = 0. \end{array} \right. \quad (P_5)$$

or

$$p^{(n-1)}(x, t) = G(x, t, z^{(n)}, z_x^{(n)}) - G(x, t, z^{(n-1)}, z_x^{(n-1)}).$$

Chapter 4. Exploring Solvability of Nonlinear Hyperbolic Equations with Mixed Nonlinear and Linear Integral Neumann Boundary Conditions

Multiplying

$$\frac{\partial^2 z^{(n)}}{\partial t^2} - a\Delta z^{(n)} = p^{(n-1)}(x, t),$$

by $z^{(n)}$, and integrating it on Q_τ , we obtain :

$$\begin{aligned} & \int_{Q_\tau} \frac{\partial^2 z^{(n)}}{\partial t^2}(x, t) \cdot \frac{\partial z^{(n)}}{\partial t}(x, t) dxdt - a \int_{Q_\tau} \Delta z^{(n)}(x, t) \cdot \frac{\partial z^{(n)}}{\partial t}(x, t) dxdt \\ &= \int_{Q_\tau} p^{(n-1)}(x, t) \cdot \frac{\partial z^{(n)}}{\partial t}(x, t) dxdt. \end{aligned}$$

Let us perform integration by parts on each term of the relevant equation, carefully considering both the initial conditions and the boundary conditions. This approach allows us to manipulate the terms effectively and derive useful expressions for our analysis. Starting with the equation, we can express it as follows:

$$\frac{1}{2} \int_{\Omega} \left(\frac{\partial z^{(n)}}{\partial t}(x, \tau) \right)^2 dx + \frac{a}{2} \int_{\Omega} \left(\frac{\partial z^{(n)}}{\partial x}(x, t) \right)^2 dxdt = \int_{Q_\tau} p^{(n-1)}(x, t) \cdot \frac{\partial z^{(n)}}{\partial t}(x, t) dxdt.$$

We can apply the Cauchy-Schwarz inequality to the second part of the equation to provide an estimate for the integral involving the inner products of the functions. Recall that the Cauchy-Schwarz inequality states, we can express it as follows:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \left(\frac{\partial z^{(n)}}{\partial t}(x, \tau) \right)^2 dx + \frac{a}{2} \int_{\Omega} \left(\frac{\partial z^{(n)}}{\partial x}(x, t) \right)^2 dxdt \\ & \leq \frac{1}{2\varepsilon} \int_{Q_\tau} |G(x, t, y^{(n)}, y_x^{(n)}) - G(x, t, y^{(n-1)}, y_x^{(n-1)})|^2 dxdt \\ & \quad + \frac{\varepsilon}{2} \int_{Q_\tau} \left(\frac{\partial z^{(n)}}{\partial t}(x, t) \right)^2 dxdt, \end{aligned}$$

Like G is lipschitz function, we find :

$$\begin{aligned} & \leq \frac{k^2}{2\varepsilon} \int_{Q_\tau} (|y^{(n)} - y^{(n-1)}| + |y_x^{(n)} - y_x^{(n-1)}|)^2 dxdt + \frac{\varepsilon}{2} \int_{Q_\tau} \left(\frac{\partial z^{(n)}}{\partial t}(x, t) \right)^2 dxdt, \\ & \leq \frac{k^2}{\varepsilon} \|z^{(n-1)}\|_{L^2(0, T, H^1(0, l))} + \frac{\varepsilon}{2} \int_{Q_\tau} \left(\frac{\partial z^{(n)}}{\partial t}(x, t) \right)^2 dxdt. \end{aligned}$$

Chapter 4. Exploring Solvability of Nonlinear Hyperbolic Equations with Mixed Nonlinear and Linear Integral Neumann Boundary Conditions

We multiply by 2 and apply Grenwell's lemma, we get

$$\int_{\Omega} \left(\frac{\partial z^{(n)}}{\partial t}(x, \tau) \right)^2 dx + a \int_{\Omega} \left(\frac{\partial z^{(n)}}{\partial x}(x, t) \right)^2 dx \leq \frac{k^2}{\varepsilon} \exp(\varepsilon T) \|z^{(n-1)}\|_{L^2(0, T, H^1(0, l))}.$$

We integrate on t , we obtain

$$\int_{Q_T} \left(\frac{\partial z^{(n)}}{\partial t}(x, \tau) \right)^2 dx dt + \int_{Q_T} \left(\frac{\partial z^{(n)}}{\partial x}(x, t) \right)^2 dx dt \leq \frac{T k^2 \exp(\varepsilon T)}{\varepsilon \min(1, a)} \|z^{(n-1)}\|_{L^2(0, T, H^1(0, l))}.$$

Setting

$$c = \frac{T k^2 \exp(\varepsilon T)}{\varepsilon \min(1, a)}.$$

Therefore,

$$\left\| \frac{\partial z^{(n)}}{\partial t} \right\|_{L^2(0, T, H^1(0, l))}^2 + \left\| \frac{\partial z^{(n)}}{\partial x} \right\|_{L^2(0, T, H^1(\Omega))}^2 \leq c \|z^{(n-1)}\|_{L^2(0, T, H^1(\Omega))}^2,$$

applying poincaré inequality, we have

$$\|z^{(n)}\|_{L^2(0, T, H^1(\Omega))}^2 \leq T c \|z^{(n-1)}\|_{L^2(0, T, H^1(\Omega))}^2.$$

$$\sum_{i=1}^{n-1} z^{(i)} = y^{(n)}.$$

According to the convergence criterion of the series $\sum_{n=1}^{\infty} z^{(n)}$ converges if the absolute value of the ratio $|c| < 1$, This implies that the terms of the series diminish sufficiently fast as n increases, leading to a finite sum, then we can conclude that:

$$\left| \frac{(T k)^2 \exp(\varepsilon T)}{\varepsilon \min(1, a)} \right| < 1 \Rightarrow k < \sqrt{\frac{\varepsilon \min(1, a) \exp(-\varepsilon T)}{T}}.$$

Then $(y^{(n)})_n$ converges to an element of $L^2(0, T, H^1(\Omega))$, we call ρ . We will show that

$$\lim_{n \rightarrow \infty} y^{(n)}(x, t) = y(x, t)$$

is a solution to the problem (P₅) by showing that ρ satisfies:

$$\Phi(y, v) = \int_{Q_T} G(x, t, y, y_x) \cdot v(x, t) dx dt.$$

To establish the weak formulation, we begin by multiplying the governing equation

Chapter 4. Exploring Solvability of Nonlinear Hyperbolic Equations with Mixed Nonlinear and Linear Integral Neumann Boundary Conditions

of the problem (P₁). The weak formulation can be expressed in an integral form over the domain Q_τ :

$$\Phi(y^{(n)}, v) = \int_{Q_\tau} \frac{\partial^2 y^{(n)}}{\partial t^2}(x, t) \cdot \frac{\partial v}{\partial t}(x, t) dx dt + \frac{a}{2} \int_{Q_\tau} \frac{\partial y^{(n)}}{\partial x}(x, t) \cdot \frac{\partial^2 v}{\partial t \partial x}(x, t) dx dt.$$

From the linearity of Φ , we have :

$$\begin{aligned} \Phi(y^{(n)}, v) &= \Phi(y^{(n)} - y, v) + \Phi(y, v) \\ &= \int_{Q_\tau} \frac{\partial^2(y^{(n)} - y)}{\partial t^2}(x, t) \cdot \frac{\partial v}{\partial t}(x, t) dx dt \\ &\quad + \frac{a}{2} \int_{Q_\tau} \frac{\partial(y^{(n)} - y)}{\partial x}(x, t) \cdot \frac{\partial^2 v}{\partial t \partial x}(x, t) dx dt \\ &\quad - \int_{Q_\tau} \frac{\partial^2 y}{\partial t^2}(x, t) \cdot \frac{\partial v}{\partial t}(x, t) dx dt + \frac{a}{2} \int_{Q_\tau} \frac{\partial y}{\partial x}(x, t) \cdot \frac{\partial^2 v}{\partial t \partial x}(x, t) dx dt, \end{aligned}$$

So

$$\begin{aligned} \Phi(y^{(n)} - y, v) &= \int_{Q_\tau} \frac{\partial^2(y^{(n)} - y)}{\partial t^2}(x, t) \cdot \frac{\partial v}{\partial t}(x, t) dx dt \\ &\quad + \frac{a}{2} \int_{Q_\tau} \frac{\partial(y^{(n)} - y)}{\partial x}(x, t) \cdot \frac{\partial^2 v}{\partial t \partial x}(x, t) dx dt, \end{aligned}$$

We apply the Cauchy-Schwartz inequality, and we obtain

$$\begin{aligned} \Phi(y^{(n)} - y, v) &\leq \|v_t\|_{L^2(Q_\tau)} \left\| (y^{(n)} - y)_{tt} \right\|_{L^2(0, T, L^2(\Omega))} \\ &\quad + \frac{a}{2} \|v_{xt}\|_{L^2(Q_\tau)} \left\| (y^{(n)} - y)_x \right\|_{L^2(0, T, L^2(\Omega))}. \end{aligned}$$

Then, we find

$$\begin{aligned} \Phi(y^{(n)} - y, v) &\leq C \left(\left\| (y^{(n)} - y)_{tt} \right\|_{L^2(0, T, L^2(\Omega))} + \left\| (y^{(n)} - y)_x \right\|_{L^2(0, T, L^2(\Omega))} \right) (\|v_t\|_{L^2(Q_\tau)} \\ &\quad + \|v_{xt}\|_{L^2(Q_\tau)}), \end{aligned}$$

Where

$$C = \max\left(1, \frac{a}{2}\right).$$

As

$$y^{(n)} \longrightarrow y \quad \text{in } L^2(0, T, H^1(0, l)) \cong H^1(Q),$$

Chapter 4. Exploring Solvability of Nonlinear Hyperbolic Equations with Mixed Nonlinear and Linear Integral Neumann Boundary Conditions

So

$$\begin{aligned}y^{(n)} &\longrightarrow y && \text{in } L^2(Q), \\y_t^{(n)} &\longrightarrow y_t && \text{in } L^2(Q), \\y_x^{(n)} &\longrightarrow y_x && \text{in } L^2(Q),\end{aligned}$$

Then, when $n \longrightarrow +\infty$, we find

$$\lim_{n \rightarrow +\infty} \Phi(y^{(n)} - y, v) = 0.$$

Solvability of Nonlinear Fractional Problems with Second Kind Integral Conditions and Boundary Specifications

This chapter explores the solvability of nonlinear fractional partial differential equations with second kind integral conditions and specific boundary conditions. The study is divided into two sections: one focusing on time-fractional problems with integral conditions, and the other on fractional Fujita problems analyzed using the energy inequality method.

5.1 A nonlinear fractional problem with a second kind integral condition for time-fractional partial differential equation

In this section, our study insights into non-linear fractional problem with a second kind of integral condition. First, we explore the existence of a solution for the associated linear fractional case. Building on these findings, we then apply linearization methods to study whether solutions exist for the full non-linear problem.

5.1.1 Formulation of the non-linear problem (P₁)

Let $Q = \{(x, t) \in \mathbb{R}^2 \text{ with } : 0 < x < 1 \text{ and } 0 < t < T\}$. We intend to explore the following nonlinear problem:

$$\left\{ \begin{array}{ll} {}_0^c D_t^\alpha u(x, t) - a \frac{\partial^2 u(x, t)}{\partial x^2} + bu(x, t) = f\left(x, t, u, \frac{\partial u}{\partial x}\right) & \forall (x, t) \in Q, \\ u(x, 0) = \varphi(x) & \forall x \in (0, 1), \\ u(0, t) = 0 & \forall t \in (0, T), \\ \frac{\partial u}{\partial x}(1, t) = \int_0^1 u(x, t) dx & \forall t \in (0, T). \end{array} \right. \quad (P_1)$$

Chapter 5. Solvability of Nonlinear Fractional Problems with Second Kind Integral Conditions and Boundary Specifications

Where $0 < \alpha < 1$, and $a, b \in \mathbb{R}_+^*$. In the remainder of this section we assume $f \in L^2(Q)$ to be lipschitz, i.e, there is a constant $k > 0$ as :

$$\|f(x, t, u, u_x) - f(x, t, v, v_x)\|_{L^2(Q)} \leq k (\|u - v\|_{L^2(Q)} + \|u_x - v_x\|_{L^2(Q)}),$$

$$\forall u, u_x, v, v_x \in L^2(Q).$$

5.1.2 Study of the associated linear problem

In this section, we focus on investigating the existence and uniqueness of solutions for the linear problem. Moreover, the study existence of solutions obtained for the linear problem was derived through the method of separation of variables.

Position of the associated linear problem (P_2)

In $Q = (0, 1) \times (0, T)$, with $T < \infty$, we examine the following associated linear problem :

$$\left\{ \begin{array}{ll} {}_0^c D_t^\alpha u(x, t) - a \frac{\partial^2 u(x, t)}{\partial x^2} + bu(x, t) = f(x, t) & \forall (x, t) \in Q, \\ u(x, 0) = \varphi(x) & \forall x \in (0, 1), \\ u(0, t) = 0 & \forall t \in (0, T), \\ \frac{\partial u}{\partial x}(1, t) = \int_0^1 u(x, t) dx & \forall t \in (0, T). \end{array} \right. \quad (P_2)$$

Which can be written in this operational form :

$$\mathcal{L}u = {}_0^c D_t^\alpha u(x, t) - a \frac{\partial^2 u(x, t)}{\partial x^2} + bu(x, t) = f(x, t),$$

With the initial condition

$$\ell u = u(x, 0) = \varphi(x), \quad x \in (0, 1),$$

Dirichlet boundary condition

$$u(0, t) = 0, \quad t \in (0, T),$$

and the integral condition of the second kind

$$\frac{\partial u(1, t)}{\partial x} = \int_0^1 u(x, t) dx, \quad t \in (0, T).$$

Resolution of problem (P₂) by the variable separation method

Let the following associated homogeneous linear problem :

$$\left\{ \begin{array}{l} {}_0^c D_t^\alpha u(x, t) - a \frac{\partial^2 u(x, t)}{\partial x^2} + bu(x, t) = 0 \quad \forall (x, t) \in Q, \\ u(x, 0) = \varphi(x) \quad \forall x \in (0, 1), \\ u(0, t) = 0 \quad \forall t \in (0, T), \\ \frac{\partial u(1, t)}{\partial x} = \int_0^1 u(x, t) dx \quad \forall t \in (0, T). \end{array} \right. \quad (P_3)$$

We can certainly demonstrate that the problem (P₃) admits a unique solution. To achieve this, we will introduce the following approach:

$$u(x, t) = X(x)Y(t). \quad (5.1)$$

Substituting (5.1) into (P₃), we obtain

$$\left\{ \begin{array}{l} {}_0^c D_t^\alpha Y \cdot X - aX''Y + bXY = 0, \\ X(x)Y(0) = \varphi(x), \\ X(0)Y(t) = 0, \\ X'(1)Y(t) = \int_0^1 X(x)Y(t) dx. \end{array} \right.$$

Therefore, we get for $\lambda > 0$

$$\frac{{}_0^c D_t^\alpha Y}{Y} = a \frac{X''}{X} - b = -\lambda.$$

We begin by demonstrating that a Sturm-Liouville problem is well-defined and can be analyzed within the context of our study. A Sturm-Liouville problem typically takes the following standard form:

$$\left\{ \begin{array}{l} X''(x) + \frac{\mu}{a}X(x) = 0, \\ X(0) = 0, \\ X'(1) = \int_0^1 X(x) dx. \end{array} \right.$$

With $\mu = b - \lambda$, has a solution which given by

$$X(x) = A \cos \sqrt{\frac{\mu}{a}}x + B \sin \sqrt{\frac{\mu}{a}}x,$$

Or A and B are two real arbitrary.

Chapter 5. Solvability of Nonlinear Fractional Problems with Second Kind Integral Conditions and Boundary Specifications

The Dirichlet boundary conditions specify that specified values at the endpoints of the interval. Formally, we can express this as:

$$A = 0.$$

It follows that

$$X(x) = B \sin \sqrt{\frac{\mu}{a}} x.$$

Now, we use the integral condition to derive additional insights into the Sturm-Liouville problem. Specifically, the integral condition can be expressed as:

$$\begin{aligned} X'(1) &= B \sqrt{\frac{\mu}{a}} \cos \sqrt{\frac{\mu}{a}}, \\ &= \int_0^1 X(x) dx, \\ &= -\frac{B}{\sqrt{\frac{\mu}{a}}} \cos \sqrt{\frac{\mu}{a}} x \Big|_0^1, \\ &= -\frac{B}{\sqrt{\frac{\mu}{a}}} \cos \sqrt{\frac{\mu}{a}} + \frac{B}{\sqrt{\frac{\mu}{a}}}. \end{aligned} \tag{5.2}$$

This ordering allows for the systematic analysis of the eigenfunctions, which form a complete basis for representing solutions to a wide range of differential equations in the corresponding function space., we can establish that the eigenvalues are ordered such that:

$$\sqrt{\mu} = \sqrt{a} \arccos \frac{a}{\mu + a}.$$

Our next goal is to determine the explicite form of $\mathbf{Y}(\mathbf{t})$.

According to the superposition theorem, we pose :

$$u(x, t) = \sum_{n \geq 1} X_n(x) \cdot Y_n(t). \tag{5.3}$$

Replacing (5.3) by (P₂), leads to :

$$\sum_{n \geq 1} ({}^c D_t^\alpha Y_n + (b - \mu_n) Y_n) \cdot \sin \sqrt{\frac{\mu_n}{a}} x = \sum_{n \geq 1} \sin \left(\sqrt{\frac{\mu_n}{a}} x \right) \cdot f_n(t), \quad \forall n \in \mathbb{N},$$

Which implies

$${}^c D_t^\alpha Y_n(t) + (b - \mu_n) Y_n(t) = f_n(t). \tag{5.4}$$

As

$$\begin{aligned} u(x, 0) &= \sum_{n \geq 1} \sin \left(\sqrt{\frac{\mu_n}{a}} x \right) Y_n(0) \\ &= \varphi(x) \\ &= \sum_{n \geq 0} \varphi_n \cdot \sin \left(\sqrt{\frac{\mu_n}{a}} x \right), \end{aligned}$$

then

$$\varphi_n = \int_0^1 \varphi(x) \cdot \sin \left(\sqrt{\frac{\mu_n}{a}} x \right) dx,$$

hence

$$Y_n(0) = \varphi_n. \quad (5.5)$$

We can then solve in a simple way the fractional ordinary differential problem (5.4) – (5.5) in a straightforward manner using the Laplace transform method. By applying the Laplace transform to the fractional differential equations, we can convert them into an algebraic form that is more manageable for analysis, we can transform the fractional derivatives as follows:

$$L({}_0^c D_t^\alpha Y_n(t)) = \frac{sF(s) - Y_n(0)}{s^{1-\alpha}},$$

then

$$\frac{sF(s) - Y_n(0)}{s^{1-\alpha}} + (b - \mu_n) F(s) = L(f_n(t)),$$

So

$$F(s) = \frac{1}{s^\alpha + b - \mu_n} \left[s^{\alpha-1} \varphi_n + L(f_n(t)) \right],$$

Finally, we obtain

$$Y_n(t) = L^{-1} \left[\frac{1}{s^\alpha + b - \mu_n} \left(s^{\alpha-1} \varphi_n + L(f_n(t)) \right) \right].$$

Thus, we conclude that the explicit solution of the problem (P₃) obtained through the Laplace transform method provides a powerful tool for analyzing and understanding the dynamics of the fractional ordinary differential equations at hand :

$$\sum_{n \geq 0} \left(B_n \sin \left(\sqrt{\frac{\mu_n}{a}} x \right) \right) \cdot \left(L^{-1} \left[\frac{1}{s^\alpha + b - \mu_n} \left(s^{\alpha-1} \varphi_n + L(f_n(t)) \right) \right] \right).$$

Solvability of the weak solution of the non-linear problem Let us consider the homogeneous equation associated with our initial problem

$$\begin{cases} {}_0^c D_t^\alpha v(x, t) - a \frac{\partial^2 v(x, t)}{\partial x^2} + bv(x, t) = 0 & \forall (x, t) \in Q, \\ v(x, 0) = \varphi(x) & \forall x \in (0, 1), \\ v(0, t) = 0 & \forall t \in (0, T), \\ \frac{\partial v(1, t)}{\partial x} = \int_0^1 v(x, t) dx & \forall t \in (0, T). \end{cases} \quad (P_4)$$

If u is a solution of (P₁) and v is a solution to (P₄), then $w = u - v$ satisfies

$$\mathcal{L}y = {}_0^c D_t^\alpha w(x, t) - a \frac{\partial^2 w(x, t)}{\partial x^2} + bw(x, t) = G(x, t, w, w_x), \quad (5.6)$$

$$w(x, 0) = 0, \quad \forall x \in (0, 1), \quad (5.7)$$

$$w(0, t) = 0 \quad \forall t \in (0, t), \quad (5.8)$$

$$\frac{\partial w(1, t)}{\partial x} = 0 \quad \forall t \in (0, t), \quad (5.9)$$

or $G(x, t, w, w_x) = f(x, t, u + v, u_x + v_x)$. Going forward, we assume that the function G is Lipschitz, i.e., there exists a positive constant k such that:

$$\begin{aligned} \|G(x, t, w_1, (w_1)_x) - G(x, t, w_2, (w_2)_x)\|_{L^2(Q)} &\leq k \left(\|w_1 - w_2\|_{L^2(Q)} + \|(w_1)_x - (w_2)_x\|_{L^2(Q)} \right) \\ \forall w_1, (w_1)_x, w_2, (w_2)_x &\in L^2(Q). \end{aligned} \quad (5.10)$$

To derive the desired results, it is necessary to propose the concept of the solution under consideration. Let $\vartheta = \vartheta(x, t)$ represent an arbitrary function from V , defined as:

$$V = \left\{ \vartheta \in C^1(Q), \vartheta(0, t) = \frac{\partial}{\partial x} \vartheta(1, t) = 0, t \in [0, T] \right\}.$$

multiply (5.6) by ϑ and integrate it on Q_τ , we find

$$\begin{aligned} &\int_{Q_\tau} {}_0^c D_t^\alpha w(x, t) \cdot \vartheta(x, t) dx dt - \int_{Q_\tau} a \frac{\partial^2 w}{\partial x^2}(x, t) \cdot \vartheta(x, t) dx dt + \int_{Q_\tau} bw(x, t) \cdot \vartheta(x, t) dx dt \\ &= \int_{Q_\tau} G(x, t, w, w_x) \cdot \vartheta(x, t) dx dt, \end{aligned}$$

Chapter 5. Solvability of Nonlinear Fractional Problems with Second Kind Integral Conditions and Boundary Specifications

by the use integration by parts and the conditions on w, ϑ we get

$$\int_{Q_\tau} {}^R D_t^\alpha w(x, t) \cdot \vartheta(x, t) dx dt + \int_{Q_\tau} a \frac{\partial w}{\partial x}(x, t) \cdot \frac{\partial \vartheta}{\partial x}(x, t) dx dt + \int_{Q_\tau} bw(x, t) \cdot \vartheta(x, t) dx dt = \int_{Q_\tau} G(x, t, w, w_x) \cdot \vartheta(x, t) dx dt, \quad (5.11)$$

then, (5.11) establishes the following formula

$$A(w, \vartheta) = \int_{Q_\tau} G(x, t, w, w_x) \cdot \vartheta(x, t) dx dt, \quad (5.12)$$

or

$$A(y, \vartheta) = \int_{Q_\tau} {}^R D_t^\alpha y(x, t) \cdot \vartheta(x, t) dx dt + \int_{Q_\tau} a \frac{\partial y}{\partial x}(x, t) \cdot \frac{\partial \vartheta}{\partial x}(x, t) dx dt + \int_{Q_\tau} by(x, t) \cdot \vartheta(x, t) dx dt.$$

Definition 5.1

We say the solution to the problem (5.6) – (5.9) is weak, any function $w \in L^2(0, T; H^1(0, 1))$ such as (4.12), (5.8) and (5.9) are achieved.

Now, we build a recurring sequence defined as follows : from $w^{(n-1)}$ we can define $(w^{(n)})_{n \in \mathbb{N}}$, where the first element is given by $w^{(0)} = 0$, then for $n = 1, 2, 3, \dots$ the following problem is solved

$$\left\{ \begin{array}{l} {}^R D_t^\alpha w^{(n)}(x, t) - a \frac{\partial^2 w^{(n)}(x, t)}{\partial x^2} + bw^{(n)}(x, t) = G(x, t, w^{(n-1)}, (w)_x^{(n-1)}), \\ w^{(n)}(x, 0) = 0, \\ w^{(n)}(0, t) = 0, \\ \frac{\partial w^{(n)}(1, t)}{\partial x} = 0. \end{array} \right. \quad (P_5)$$

For any n fixed, based on the analysis of (P₂) where we explicitly provided the solution using the variable separation method, we can conclude that the problem (P₅) has a unique solution denoted as $w^{(n)}(x, t)$. Now, let's define a new function $z^{(n)}(x, t) = w^{(n+1)}(x, t) - w^{(n)}(x, t)$. This formulation leads us to a new problem that explores the relationship between consecutive solutions in the sequence defined

Chapter 5. Solvability of Nonlinear Fractional Problems with Second Kind Integral Conditions and Boundary Specifications

by our iterative method, This new problem can be expressed as:

$$\begin{cases} {}_0^R D_t^\alpha z^{(n)}(x, t) - a \frac{\partial^2 z^{(n)}(x, t)}{\partial x^2} + bz^{(n)}(x, t) = p^{(n-1)}(x, t), \\ z^{(n)}(x, 0) = 0, \\ z^{(n)}(0, t) = 0, \\ \frac{\partial z^{(n)}(1, t)}{\partial x} = 0. \end{cases} \quad (P_6)$$

or

$$p^{(n-1)}(x, t) = G(x, t, w^{(n)}, w_x^{(n)}) - G(x, t, w^{(n-1)}, w_x^{(n-1)}).$$

Lemma 5.1

We assume that (3.33) be satisfied. So we obtain for the problem (P₆) the following a priori estimate :

$$\|z^{(n)}\|_{L^2(0,T,H^1(0,1))} \leq c \|z^{(n-1)}\|_{L^2(0,T,H^1(0,1))},$$

or

$$c = \sqrt{\frac{k^2}{2\varepsilon \min\left(a, b - \frac{\varepsilon}{2}\right)}}.$$

Proof We multiply by $z^{(n)}$ this equation

$${}_0^R D_t^\alpha z^{(n)}(x, t) - a \frac{\partial^2 z^{(n)}(x, t)}{\partial x^2} + bz^{(n)}(x, t) = p^{(n-1)}(x, t),$$

and we integrate it on Q_τ , we get :

$$\begin{aligned} & \int_{Q_\tau} {}_0^R D_t^\alpha z^{(n)}(x, t) \cdot z^{(n)}(x, t) dxdt - \int_{Q_\tau} a \frac{\partial^2 z^{(n)}(x, t)}{\partial x^2} \cdot z^{(n)}(x, t) dxdt + \int_{Q_\tau} b \left(z^{(n)}(x, t)\right)^2 dxdt \\ &= \int_{Q_\tau} p^{(n-1)}(x, t) \cdot z^{(n)}(x, t) dxdt. \end{aligned}$$

Performing integration by part and using lemma 2, 3, 4, 5 in [?]-[?], we obtain :

$$\begin{aligned} & \int_0^1 \left({}_0^R D_t^{\frac{\alpha}{2}} z^{(n)}(x, \tau) \right)^2 dx + \int_{Q_\tau} a \left(\frac{\partial z^{(n)}}{\partial x}(x, t) \right)^2 dxdt + \int_{Q_\tau} b \left(z^{(n)}(x, t)\right)^2 dxdt \\ &= \int_{Q_\tau} p^{(n-1)}(x, t) \cdot z^{(n)}(x, t) dxdt. \end{aligned}$$

Chapter 5. Solvability of Nonlinear Fractional Problems with Second Kind Integral Conditions and Boundary Specifications

Using Cauchy with ε -inequality, we get :

$$\begin{aligned} & \int_0^1 \left({}^R D_t^{\frac{\alpha}{2}} z^{(n)}(x, \tau) \right)^2 dx + \int_{Q_\tau} a \left(\frac{\partial z^{(n)}}{\partial x}(x, t) \right)^2 dxdt + \int_{Q_\tau} b \left(z^{(n)}(x, t) \right)^2 dxdt \\ & \leq \frac{1}{2\varepsilon} \int_{Q_\tau} (p^{(n-1)})^2 dxdt + \frac{\varepsilon}{2} \int_{Q_\tau} (z^{(n)}(x, t))^2 dxdt. \end{aligned} \quad (5.13)$$

Exploiting the fact that

$$\left| p^{n-1}(x, t) \right|^2 = \left| G(x, t, y^{(n)}, y_x^{(n)}) - G(x, t, y^{(n-1)}, y_x^{(n-1)}) \right|^2,$$

we conclude from (3.33) that :

$$\begin{aligned} \int_{Q_\tau} \left| p^{n-1}(x, t) \right|^2 & \leq \int_{Q_\tau} k^2 \left(\left| y^{(n)} - y^{(n-1)} \right| + \left| y_x^{(n)} - y_x^{(n-1)} \right| \right)^2 \\ & \leq k^2 \|z^{(n-1)}(x, t)\|_{L^2(0, T; H^1(0, 1))}^2. \end{aligned}$$

Then, we can rewrite the equation as:

$$\begin{aligned} & b \int_{Q_\tau} (z^{(n)}(x, \tau))^2 dx + a \int_{Q_\tau} \left(\frac{\partial z^{(n)}}{\partial x}(x, t) \right)^2 dxdt \\ & \leq \int_0^1 \left({}^R D_t^{\frac{\alpha}{2}} z^{(n)}(x, \tau) \right)^2 dx + \int_{Q_\tau} a \left(\frac{\partial z^{(n)}}{\partial x}(x, t) \right)^2 dxdt + \int_{Q_\tau} b \left(z^{(n)}(x, t) \right)^2 dxdt \\ & \leq \frac{1}{2\varepsilon} \int_{Q_\tau} (p^{(n-1)})^2 dxdt. \end{aligned}$$

Finally,

$$\begin{aligned} & \|z^{(n)}(x, t)\|_{L^2(0, T; H^1(0, 1))}^2 \\ & \leq \frac{k^2}{2\varepsilon \min\left(a, b - \frac{\varepsilon}{2}\right)} \|z^{(n-1)}(x, t)\|_{L^2(0, T; H^1(0, 1))}^2. \end{aligned}$$

Our next concern will be the convergence of the sequence $\left(y^{(n)} \right)_n$, Therefore, we are particularly interested in the convergence of the series $\sum_{n=1}^{\infty} z^{(n)}$. To investigate this, we will utilize the criterion for convergence of series, the following conditions

Chapter 5. Solvability of Nonlinear Fractional Problems with Second Kind Integral Conditions and Boundary Specifications

are met:

$$\sqrt{\frac{k^2}{2\varepsilon \min\left(a, b - \frac{\varepsilon}{2}\right)}} < 1,$$

Then

$$k < \sqrt{2\varepsilon \min\left(a, b - \frac{\varepsilon}{2}\right)}.$$

As $z^{(n)}(x, t) = w^{(n+1)}(x, t) - w^{(n)}(x, t)$, and $w^{(0)}(x, t) = 0$ we have

$$\begin{aligned} \sum_{i=0}^{n-1} z^{(i)} &= \sum_{i=0}^{n-1} \left(w^{(i+1)}(x, t) - w^{(i)}(x, t) \right) \\ &= w^{(n)}. \end{aligned}$$

It is noteworthy that we have rigorously demonstrated the convergence of the sequence $(w^{(n)})_n$ based on the following key observation:

$$w^{(n)}(x, t) = \sum_{i=0}^{n-1} z^{(i)},$$

now established, our aim is to show that this limit function, $w(x, t)$, indeed represents a solution to the boundary value problem (5.6) – (5.9). So,

$$A(w, \vartheta) = \int_{Q_\tau} G(x, t, w, w_x) \cdot \vartheta(x, t) dx dt.$$

from (P₅), we get :

$$\begin{aligned} A(w^{(n)}, \vartheta) &= \int_{Q_\tau} {}^R D_t^\alpha w^{(n)}(x, t) \cdot \vartheta(x, t) dx dt + \int_{Q_\tau} a \frac{\partial w^{(n)}}{\partial x}(x, t) \cdot \frac{\partial \vartheta}{\partial x}(x, t) dx dt \\ &+ \int_{Q_\tau} b w^{(n)}(x, t) \cdot \vartheta(x, t) dx dt. \end{aligned}$$

Chapter 5. Solvability of Nonlinear Fractional Problems with Second Kind Integral Conditions and Boundary Specifications

As A is linear, we get

$$\begin{aligned} A(w^{(n)}, \vartheta) &= A(w^{(n)} - y, \vartheta) + A(w, \vartheta) \\ &= \int_{Q_\tau} {}^R D_t^\alpha (w^{(n)} - w)(x, t) \cdot \vartheta(x, t) dx dt \end{aligned} \quad (5.14)$$

$$\begin{aligned} &+ \int_{Q_\tau} a \frac{\partial (w^{(n)} - w)}{\partial x}(x, t) \cdot \frac{\partial \vartheta}{\partial x}(x, t) dx dt \\ &+ \int_{Q_\tau} b (w^{(n)} - w)(x, t) \cdot \vartheta(x, t) dx dt + \int_{Q_\tau} {}^R D_t^\alpha w(x, t) \cdot \vartheta(x, t) dx dt \\ &+ \int_{Q_\tau} a \frac{\partial w}{\partial x}(x, t) \cdot \frac{\partial \vartheta}{\partial x}(x, t) dx dt + \int_{Q_\tau} b w(x, t) \cdot \vartheta(x, t) dx dt, \end{aligned} \quad (5.15)$$

Applying the Cauchy Schwartz inequality on $A(w^{(n)} - w, \vartheta)$, we get

$$\begin{aligned} &\int_{Q_\tau} {}^R D_t^\alpha (w^{(n)} - w)(x, t) \cdot \vartheta(x, t) dx dt + \int_{Q_\tau} a \frac{\partial (w^{(n)} - w)}{\partial x}(x, t) \cdot \frac{\partial \vartheta}{\partial x}(x, t) dx dt \\ &+ \int_{Q_\tau} b (w^{(n)} - w)(x, t) \cdot \vartheta(x, t) dx dt \\ &\leq \max(1, a\gamma, b) \|\vartheta_x\|_{L^2(Q)} \left[\|{}^R D_t^\alpha (w^{(n)} - w)\|_{L^2(0, T, H^1(0, 1))} + \|(w^{(n)} - w)_x\|_{L^2(0, T, H^1(0, 1))} \right], \end{aligned}$$

Where γ is the Poincaré inequality constant, therefore

$$w^{(n)} \longrightarrow w \quad \text{in } L^2(0, T, H^1(0, 1)) \cong H^1(Q),$$

So

$$\begin{aligned} w^{(n)} &\longrightarrow w \quad \text{in } L^2(Q), \\ w_x^{(n)} &\longrightarrow w_x \quad \text{in } L^2(Q), \end{aligned}$$

Letting $n \longrightarrow +\infty$, we obtain

$$\lim_{n \rightarrow +\infty} A(w^{(n)} - w, \vartheta) = 0. \quad (5.16)$$

From (5.16) and going to the limit in (5.15), we obtain

$$\lim_{n \rightarrow +\infty} A(w^{(n)}, \vartheta) = A(y, \vartheta).$$

Theorem 5.1

If the condition (3.33) is satisfied and

$$k < \sqrt{2\varepsilon \min\left(a, b - \frac{\varepsilon}{2}\right)}.$$

So the problem (5.6) – (5.9) admits a weak solution which belongs to $L^2(0, T; H^1(0, 1))$. It remains to be proven that the problem (5.6) – (5.9) admits a unique solution.

Theorem 5.2

According to (3.33), the solution of the problem (5.6) – (5.9) is unique.

Proof Suppose that y_1 and y_2 in $L^2(0, T; H^1(0, 1))$ are two solutions of (5.6) – (5.9), then $Z = y_1 - y_2$ satisfied $Z \in L^2(0, T; H^1(0, 1))$ and

$$\mathcal{L}Z = {}^R_0D_t^\alpha Z(x, t) - a \frac{\partial^2 Z(x, t)}{\partial x^2} + bZ(x, t) = \psi(x, t), \quad \forall (x, t) \in \bar{Q}$$

$$\begin{aligned} Z(x, 0) &= 0, \quad \forall x \in (0, 1), \\ Z(0, t) &= 0 \quad \forall t \in (0, t), \\ \frac{\partial Z(1, t)}{\partial X} &= 0 \quad \forall t \in (0, t), \end{aligned}$$

where,

$$\psi(x, t) = G(x, t, y_1, (y_1)_x) - G(x, t, y_2, (y_2)_x).$$

Similar analysis to that in the proof of the lemma (5.1), shows that

$$\|Z\|_{L^2(0, T; H^1(0, 1))} \leq c \|Z\|_{L^2(0, T; H^1(0, 1))}, \quad (5.17)$$

then c is the same constant of lemma (5.1). As $c < 1$, so according to (5.17) it comes that

$$(1 - c) \|Z\|_{L^2(0, T; H^1(0, 1))} \leq 0.$$

We conclude that $y_1 = y_2$ in $L^2(0, T; H^1(0, 1))$, which is the desired conclusion. ■

5.2 Solvability of fractional Fujita problems with specified boundary conditions using the energy inequality method

5.2.1 Formulation of the problem

This section introduces the nonlinear fractional evolution problem of the general form expressed by:

Let $T > 0$, $\Omega \subset \mathbb{R}^n$ and $Q = \Omega \times (0, T) = \{(x, t) \in \mathbb{R}^{n+1} : x \in \Omega, 0 < t < T\}$. The nonlinear fractional parabolic problem we consider is as follows

$$\begin{cases} {}^C D_t^\alpha u - a\Delta u + b(x, t)u + c(x, t)u^p = f(x, t), \\ u(x, 0) = \varphi(x), \\ u(0, t) = u(l, t) = 0. \end{cases} \quad (\text{P}_1)$$

The formulation of the nonlinear fractional parabolic problem is expressed mathematically as follows:

$$\mathcal{L}u = {}^C D_t^\alpha u - a\Delta u + b(x, t)u + c(x, t)u^p = f(x, t). \quad (5.18)$$

Initial Condition:

$$lu = u(x, 0) = \varphi(x) \quad \forall x \in [0, l]. \quad (5.19)$$

Boundary conditions:

$$u(0, t) = u(l, t) = 0 \quad \forall t \in [0, T]. \quad (5.20)$$

where $f(x, t)$, and $\varphi(x)$ are given functions and $p > 1$. where $b(x, t)$ and $c(x, t)$ satisfy the following assumptions:

A1. $b_1 \leq b(x, t) \leq b_0$, $c_1 \leq c(x, t) \leq c_0$, $(x, t) \in \bar{Q}$.

We aim to establish a priori bounds and prove the existence of solutions for problems (5.18) – (5.20).

Let $Lu = F$, where $L = (\mathcal{L}, l)$, and $F = (f, u_0)$ corresponds to the operator equation associated with problems (5.18) – (5.20). The operator L acts from E to F defined as follows. The Banach space E consists of all functions $u(x, t)$ with the

Chapter 5. Solvability of Nonlinear Fractional Problems with Second Kind Integral Conditions and Boundary Specifications

finite norm

$$\|u\|_E^2 = \left\| {}^C D_t^{\frac{\alpha}{2}} u \right\|_{L^2(Q)}^2 + \|\nabla u\|_{L^2(Q)}^2 + \|u\|_{L^2(Q)}^2 + \max_{0 \leq k \leq p-2} \|u\|_{L^2(Q)}^{p-k}. \quad (5.21)$$

The Hilbert space F consists of the vector valued functions $F = (f, u_0)$ with the norm

$$\|\mathcal{F}\|_F^2 = \|f\|_{L^2(Q)}^2 \quad (5.22)$$

We assume that the data function φ satisfies the conditions of the form (5.20) :

$$\varphi(0) = \varphi(l) = 0.$$

Then, we using the following changement of variable :

$$v(x, t) = u(x, t) - \varphi(x) \Rightarrow u(x, t) = v(x, t) + \varphi(x),$$

So, we get the following new problem

$$\begin{cases} {}^C D_t^\alpha v - a\Delta v + b(x, t)v + c(x, t)(v + \varphi)^p = \tilde{f}(x, t), \\ v(x, 0) = 0, \\ v(0, t) = v(l, t) = 0. \end{cases} \quad (P_2)$$

whose nonlinear fractional parabolic equation is given as follows

$$\begin{aligned} \mathcal{L}v &= {}^C D_t^\alpha v - a\Delta v + b(x, t)v + c(x, t)(v + \varphi)^p, \\ &= f(x, t) + a\Delta\varphi + b(x, t)\varphi, \\ &= \tilde{f}(x, t). \end{aligned} \quad (5.23)$$

To properly analyze this equation, we need to define appropriate boundary and initial conditions:

Initial Condition:

$$lv = v(x, 0) = 0 \quad \forall x \in [0, l]. \quad (5.24)$$

These conditions are crucial for ensuring a well-posed problem and can take various forms:

the Dirichlet boundary conditions

$$v(0, t) = v(l, t) = 0 \quad \forall t \in [0, T]. \quad (5.25)$$

We first establish a priori estimate for the solution to problems (5.23) – (5.25) .

3.2 A priori bound

Theorem 5.3

If the assumptions $A1$ are satisfied then for any function $v \in D(L)$, there exists a positive constant c independent of u such that

$$\left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \|\nabla v\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2 + \max_{0 \leq k \leq p-1} \|v\|_{L^2(Q)}^{p-k} \leq \delta \|\tilde{f}\|_{L^2(Q)}^2, \quad (5.26)$$

and $D(L)$ is the domain of definition of the operator L defined by

$$D(L) = \{v \in L^2(Q) / {}^C D_t^{\frac{\alpha}{2}} v, \nabla u \in L^2(Q)\},$$

satisfying conditions (5.20).

Proof Taking the scalar product in $L^2(Q)$ of Eq. (5.23) and the operator

$$Mv = v,$$

where $Q^\tau = \Omega \times (0, T)$, we have

$$\begin{aligned} (\mathcal{L}u, Mu)_{L^2(Q^\tau)} &= ({}^C D_t^\alpha v, v)_{L^2(Q^\tau)} - a(\Delta v, v)_{L^2(Q^\tau)} + (bv, v)_{L^2(Q^\tau)} + (c(v + \varphi)^p, v)_{L^2(Q^\tau)} \\ &= (\tilde{f}, v)_{L^2(Q^\tau)}, \end{aligned} \quad (5.27)$$

The successive integration by parts of integrals on the right-hand side of (5.27), yields

$$\begin{aligned} ({}^C D_t^\alpha v, v)_{L^2(Q^\tau)} &= \left({}^C D_t^{\frac{\alpha}{2}} v, {}^C D_t^{\frac{\alpha}{2}} v \right)_{L^2(Q^\tau)}, \\ &= \cos\left(\frac{\alpha\pi}{2}\right) |v|_{C^{\frac{\alpha}{2}} H^{\frac{\alpha}{2}}(\Omega)}^2, \\ &= \cos\left(\frac{\alpha\pi}{2}\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q^\tau)}^2, \end{aligned} \quad (5.28)$$

Chapter 5. Solvability of Nonlinear Fractional Problems with Second Kind Integral Conditions and Boundary Specifications

and

$$-a(\Delta v, v)_{L^2(Q_\tau)} = a \|\nabla v\|_{L^2(Q_\tau)}^2, \quad (5.29)$$

and

$$(bv, v)_{L^2(Q_\tau)} = \int_{Q_\tau} b(x, t)v^2(x, t)dxdt, \quad (5.30)$$

Also

$$\begin{aligned} (c(v + \varphi)^p, v)_{L^2(Q_\tau)} &= \left(\sum_{k=0}^{p-1} C\varphi^k v^{p-k-1}, v \right)_{L^2(Q_\tau)}, \\ &\geq \sum_{k=0}^{p-2} c_1 \left\| \varphi^{\frac{k}{p-k}} v \right\|_{L^{p-k}(Q_\tau)}^{p-k}, \\ &\geq c_1 \max_{0 \leq k \leq p-2} (\varphi_0^k) \max_{0 \leq k \leq p-2} \|v\|_{L^2(Q_\tau)}^{p-k}. \end{aligned} \quad (5.31)$$

Substituting (5.28) – (5.31) into (5.27), we obtain

$$\begin{aligned} \cos\left(\frac{\alpha\pi}{2}\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q_\tau)}^2 + a \|\nabla v\|_{L^2(Q_\tau)}^2 + \int_{Q_\tau} b(x, t)v^2(x, t)dxdt + c_1 \max_{0 \leq k \leq p-2} (\varphi_0^k) \max_{0 \leq k \leq p-2} \|v\|_{L^2(Q_\tau)}^{p-k} \\ \leq (\tilde{f}, v), \end{aligned} \quad (5.32)$$

estimate the last term on the right-hand side of (5.31) by applying Cauchy inequality with ε , ($|ab| \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}$), then we get

$$\begin{aligned} \cos\left(\frac{\alpha\pi}{2}\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q_\tau)}^2 + a \|\nabla v\|_{L^2(Q_\tau)}^2 + b_1 \int_{Q_\tau} v^2(x, t)dxdt + c_1 \max_{0 \leq k \leq p-2} (\varphi_0^k) \max_{0 \leq k \leq p-2} \|v\|_{L^2(Q_\tau)}^{p-k} \\ \leq \frac{1}{2\varepsilon} \|\tilde{f}\|_{L^2(Q)}^2 + \frac{\varepsilon}{2} \|v\|_{L^2(Q)}^2, \end{aligned} \quad (5.33)$$

Finally, we get

$$\left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \|\nabla v\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2 + \max_{0 \leq k \leq p-1} \|v\|_{L^2(Q)}^{p-k} \leq \delta \|\tilde{f}\|_{L^2(Q)}^2,$$

Where

$$\delta = \frac{1}{2\varepsilon \min \left\{ \cos\left(\frac{\alpha\pi}{2}\right), a, \left(b_1 - \frac{\varepsilon}{2}\right), C_0 \max_{0 \leq k \leq p-1} (\varphi_0^k) \right\}}.$$

So, we have

$$\|v\|_E \leq \delta \|Lv\|_F \quad (5.34)$$

Let $R(L)$ be the range of the operator L . However, since we do not have any

Chapter 5. Solvability of Nonlinear Fractional Problems with Second Kind Integral Conditions and Boundary Specifications

information about $R(L)$, except that $R(L) \subset F$, we must extend L , so that estimate (5.34) holds for the extension and its range is the whole space F . We first state the following proposition. ■

Proposition 5.1

The operator $L : E \rightarrow F$ has a closure

Proof let $(u_n)_{n \in \mathbb{N}} \subset D(L)$ a sequence where :

$$v_n \rightarrow 0 \quad \text{in } E$$

and

$$Lv_n \rightarrow (\tilde{f}; 0) \quad \text{in } F \tag{5.35}$$

we must proof that

$$f \equiv 0 .$$

The convergence of v_n to 0 in E drives :

$$v_n \rightarrow 0 \quad \text{in } D'(Q) . \tag{5.36}$$

According to the continuity of the derivation of $D'(Q)$ in $D'(Q)$, and the continuity the distribution of the function v^q . The relation (5.36) involved

$$\mathcal{L}v_n \rightarrow 0 \quad \text{in } D'(Q) , \tag{5.37}$$

Moreover, The convergence of $\mathcal{L}v_n$ to f in $L^2(Q)$ give

$$\mathcal{L}v_n \rightarrow f \quad \text{in } D'(Q) \tag{5.38}$$

As we have the the uniqueness of the limit in $D'(Q)$, we conclude from (5.37) and (5.38) that

$$f = 0.$$

Then, we get L is closable of this operator, with domain of definition $D(L)$. ■

Definition 5.2

A function v is termed a strong solution of the operator equation

$$\bar{L}v = \mathcal{F}$$

is called a strong solution to problems (5.18) – (5.20) if it satisfies the equation in the strong sense, meaning that v lies in the domain $D(L)$ of the operator L and the equation holds in the relevant function space.

$$\left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + \|\nabla v\|_{L^2(Q)}^2 + \|v\|_{L^2(Q)}^2 + \max_{0 \leq k \leq p-1} \|v\|_{L^2(Q)}^{p-k} \leq \delta \|f\|_{L^2(Q)}^2, \quad \forall v \in D(\bar{L}) \quad (5.39)$$

We deduce from the estimate (5.39) :

Corollary 5.1

The range $R(\bar{L})$ of the operator \bar{L} is closed in F and is equal to the closure $\overline{R(L)}$ of $R(L)$, that is $R(\bar{L}) = \overline{R(L)}$.

Proof The proof is into two parts:

Part 1: closed Range: to demonstrate that $R(L)$ is closed in F .

Part 2: equality of the Range and Its Closure: We also need to show that $R(\bar{L}) = \overline{R(L)}$.

To start the proof, let $z \in \overline{R(L)}$, so there is a cauchy sequence $(z_n)_{n \in \mathbb{N}}$ in F constituted of the elements of the set $R(L)$ such as

$$\lim_{n \rightarrow +\infty} z_n = z.$$

There is then a corresponding sequence $v_n \in D(L)$ such as

$$z_n = Lv_n.$$

The estimate (5.34) , we get

$$\|v_p - v_q\|_E \leq C \|Lv_p - Lv_q\|_F \rightarrow 0,$$

Where p, q tend towards infinity. We can deduce that $(v_n)_{n \in \mathbb{N}}$ is a cauchy sequence

Chapter 5. Solvability of Nonlinear Fractional Problems with Second Kind Integral Conditions and Boundary Specifications

in E , so like E is a Banach space, it exists $v \in E$ such as

$$\lim_{n \rightarrow +\infty} v_n = v \text{ in } E.$$

By virtue of the definition of \bar{L} ($\lim_{n \rightarrow +\infty} v_n = v$ in E ; If $\lim_{n \rightarrow +\infty} Lv_n = \lim_{n \rightarrow +\infty} z_n = z$, then $\lim_{n \rightarrow +\infty} \bar{L}v_n = z$ as like \bar{L} and is closed, so $\bar{L}v = z$), the function u check :

$$v \in D(\bar{L}), \bar{L}v = z.$$

Then $z \in R(\bar{L})$, so

$$\overline{R(\bar{L})} \subset R(\bar{L})$$

Also we conclude here that $R(\bar{L})$ is closed because it is Banach (any complete subspace of a metric space (not necessarily complete) is closed). It remains to show the reverse inclusion

Either $z \in R(\bar{L})$ then it exists a Cauchy sequence $(z_n)_{n \in \mathbb{N}}$ in F constituted of the elements of the set $R(\bar{L})$ such that

$$\lim_{n \rightarrow +\infty} z_n = z$$

or $z \in R(\bar{L})$, because $R(\bar{L})$ is a closed subset a completed F , So $R(\bar{L})$ is complet. There is then a corresponding sequence $v_n \in D(\bar{L})$ such that

$$\bar{L}v_n = z_n.$$

We get from (5.34):

$$\|v_p - v_q\|_E \leq C \|\bar{L}v_p - \bar{L}v_q\|_F \rightarrow 0,$$

Where p, q tend towards infinity. We can deduce that $(v_n)_{n \in \mathbb{N}}$ is a cauchy sequence in E , so like E is a Banach space, it exists $v \in E$ such as

$$\lim_{n \rightarrow +\infty} v_n = v \text{ in } E.$$

Once again, there is a corresponding sequel $(Lv_n)_{n \in \mathbb{N}} \subset R(L)$ such as

$$\bar{L}v_n = Lv_n \text{ on } R(L), \forall n \in \mathbb{N}.$$

So

$$\lim_{n \rightarrow +\infty} Lv_n = z,$$

Consequently $z \in \overline{R(L)}$, then we conclude that

$$R(\bar{L}) \subset \overline{R(L)}.$$

■

4.3 Existence and uniqueness of solution

This section outlines the key principles and methods used to demonstrate these properties for the equations under consideration. This section is dedicated to illustrating the existence of the problem (P_2) and establishing the uniqueness of the primary problem.

Theorem 5.4

Under the assumption A_1 , for every $F = (f, 0) \in F$, there exists a unique strong solution $u = \bar{L}^{-1}\mathcal{F} = \bar{L}^{-1}\mathcal{F}$ to the problems defined in equations (5.18) – (5.20).

Proof We have

$$(Lv, W)_F = \int_Q \mathcal{L}v \cdot w dxdt \quad (5.40)$$

Where

$$W = (w, 0).$$

Si for $w \in L^2(Q)$ and and for all $v \in D_0(L) = \{v, v \in D(L) : \ell v = 0\}$, we have

$$\int_Q \mathcal{L}u \cdot w dxdt = 0$$

By putting $w = v$, we obtain

$$\begin{aligned} & \left({}^C D_t^\alpha v, v \right)_{L^2(Q^\tau)} - a (\Delta v, v)_{L^2(Q^\tau)} + (bv, v)_{L^2(Q^\tau)} + (c(v + \varphi)^p, v)_{L^2(Q^\tau)} = 0, \\ & \cos\left(\frac{\alpha\pi}{2}\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + b_1 \|v\|_{L^2(Q)}^2 + c_1 \max_{0 \leq k \leq p-2} (\varphi_0^k) \max_{0 \leq k \leq p-2} \|v\|_{L^2(Q)}^{p-k} \leq -a \int_{Q^\tau} (\nabla u)^2, \\ & \cos\left(\frac{\alpha\pi}{2}\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + b_1 \|v\|_{L^2(Q)}^2 + c_1 \max_{0 \leq k \leq p-2} (\varphi_0^k) \max_{0 \leq k \leq p-2} \|v\|_{L^2(Q)}^{p-k} \leq 0. \end{aligned}$$

we obtain

$$\|v\|_{L^2(Q)}^2 \leq 0.$$

So, it's give $u = w = 0$. ■

Corollary 5.2

If for every function $u \in D(L)$, we the following estimate holds:

$$\|u\|_E \leq C \|\mathcal{F}\|_F,$$

Then the solution of the problem (P_1) is unique, provided it exists.

Proof To demonstrate the uniqueness of the solution, we begin by assuming that there exist two distinct solutions, u_1 and u_2 to the problem (P_1) . We can define the difference between these two problems as:

$$\begin{cases} Lu_1 = \mathcal{F} \\ Lu_2 = \mathcal{F} \end{cases} \implies Lu_1 - Lu_2 = 0,$$

By substituting v into the corresponding equation of problem (P_1) and leveraging the linearity of the operator, we derive the following relationship: by multiplying the last homogen equation by v where $v = u_1 - u_2$, we then obtain

$$\cos\left(\frac{\alpha\pi}{2}\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + b_1 \|v\|_{L^2(Q)}^2 + a \|\nabla v\|_{L^2(Q)}^2 + \int_{Q_T} (u_1^p - u_2^p) v dx dt = 0,$$

for

$$u_1 - u_2 \leq 0 \text{ and } u_1^p - u_2^p \leq 0,$$

we can easily conclude that

$$\cos\left(\frac{\alpha\pi}{2}\right) \left\| {}^C D_t^{\frac{\alpha}{2}} v \right\|_{L^2(Q)}^2 + b_1 \|v\|_{L^2(Q)}^2 + a \|\nabla v\|_{L^2(Q)}^2 \leq 0$$

Consequently, applying the estimate provided in the statement, we obtain:

$$\|v_1 - v_2\|_E^2 \leq c \|0\|_F^2 = 0,$$

which leads us to conclude that v must be the zero function:

$$v_1 = v_2.$$

conclusion

The comprehensive study delves into the solvability and blow-up and decay phenomena of nonlinear parabolic equations, emphasizing their theoretical foundations and practical applications across diverse scientific fields.

The research offers a meticulous examination of nonlinear parabolic equations, focusing on generalized nonlinear integral conditions. It provides theoretical insights into the conditions that lead to the existence and uniqueness of weak solutions while exploring scenarios that cause blow-ups, where solutions become unbounded in finite time. These mathematical findings are crucial for applications in fields such as fluid dynamics, heat transfer, and biological modeling.

By employing a variety of mathematical techniques—including functional analysis, the Faedo-Galerkin method, energy methods, and fixed-point theorems—the study builds a solid theoretical framework. This foundation addresses complex behaviors like stability, decay, and blow-up in dynamic systems, offering practical methods for modeling and simulation. Moreover, the incorporation of fractional calculus and computational tools like the finite element method enhances the analysis of time-dependent and memory-influenced processes.

The investigation of blow-up phenomena, particularly through Kaplan's First Eigenvalue Method, uncovers the conditions under which solutions become unbounded. This understanding informs fields that rely on predictive modeling and safety considerations, such as reaction-diffusion systems and thermal engineering.

Where we present a detailed exploration of Banach, Hilbert, and Sobolev spaces, highlighting their importance in the formulation and solution of time-dependent partial differential equations.

Then, Critical insights into parameters and conditions that lead to blow-ups, with implications for engineering, physics, and ecology.

Also, the use of interpolation techniques for fractional derivatives and finite element methods to validate theoretical models and simulate real-world phenomena.

The research encourages interdisciplinary collaboration and highlights the potential for advancements in mathematical modeling, simulation techniques, and applications across various domains. It proposes future exploration into more sophisticated computational methods, the integration of machine learning, and new classes of

Chapter 5. Solvability of Nonlinear Fractional Problems with Second Kind Integral Conditions and Boundary Specifications

integral conditions to refine and extend the theoretical findings.

Bibliography

- [1] Ahmed, B., Taki-Eddine, O., Imad, R., and Zainouba, C., *Solvability of solution of singular and degenerate fractional nonlinear parabolic Dirichlet problems. Bulletin of the Institute of Mathematics*, 17(1), 2022, 105–123.
- [2] Anakira, N., Chebana, Z., Oussaeif, T.-E., Batiha, I. M., and Ouannas, A., *A study of a weak solution of a diffusion problem for a temporal fractional differential equation. Nonlinear Functional Analysis and Applications*, 27(3), 2022, 679–689.
- [3] Batten, G. W., Second-order correct boundary conditions for the numerical solution of the mixed boundary problem for parabolic equations. *Mathematics of Computation*, 17(83), 1963, 405–413.
- [4] Batiha, I. M., Chebana, Z., Oussaeif, T.-E., Ouannas, A., and Jebri, I. H., *On a weak solution of a fractional-order temporal equation. Mathematics and Statistics*, 10(5), 2022, 1116–1120.
- [5] Batiha, I. M., Ogilat, O., Chebana, Z., Oussaeif, T.-E., Anakira, N., Amourah, A., and Momani, S., *Finite-Time Blow-Up and Solvability for a Semilinear Parabolic Problem with Nonlinear Integral Conditions. Journal of Nonlinear Functional Analysis*, 2024, 2024, 29. doi:10.23952/jnfa.2024.29.
- [6] Batiha, I. M., Jebri, I. H., Chebana, Z., et al., Finite-time blow-up and solvability of a weak solution for a superlinear reaction-diffusion problem with integral conditions of the second type. *Advances in Fixed Point Theory*, 2024. <https://doi.org/10.28919/afpt/8708>
- [7] Batiha, I. M., Chebana, Z., Oussaeif, T.-E., et al., Solvability and dynamics of superlinear reaction diffusion problem with integral condition. *IAENG International Journal of Applied Mathematics*, 53(1), 2023, 1–9.
- [8] Batiha, I. M., Chebana, Z., Oussaeif, T.-E., et al., Solvability of nonlinear wave equation with nonlinear integral Neumann conditions. *International Journal of Analysis and Applications*, 2023.

Bibliography

- [9] Bellout, H., *Blow-up and global existence in reaction-diffusion systems with nonlocal terms. Nonlinear Analysis: Theory, Methods & Applications*, 185, 2019, 1–23. doi:10.1016/j.na.2019.02.012. (Addresses nonlocal reaction-diffusion blow-up.)
- [10] Berger, M. and Kohn, R. V., *A rescaling algorithm for the numerical calculation of blowing-up solutions. Communications on Pure and Applied Mathematics*, 40(6), 1987, 841–863. doi:10.1002/cpa.3160400606. (Numerical methods for tracking blow-up.)
- [11] Bebernes, J. and Bressan, A., *Similarity solutions and catastrophic behavior for a semilinear heat equation. Nonlinear Analysis: Theory, Methods & Applications*, 20(7), 1993, 755–769. doi:10.1016/0362-546X(93)90039-Y.
- [12] Beilin, S. A., Existence of solutions for one-dimensional wave equations with nonlocal conditions. *Electronic Journal of Differential Equations*, 2001, 1–8.
- [13] Benuar, N. E. and Yurchuk, N. I., A mixed problem with an integral condition for parabolic equations with a Bessel operator. *Differentsial'nye Uravneniya*, 27(12), 1991, 2094–2098.
- [14] Bouziani, A., Oussaeif, T. E., and Leila, B. A., *A mixed problem with an integral two-space-variables condition for parabolic equation with the Bessel operator. Journal of Mathematics*, 2013, 2013, 457631. doi:10.1155/2013/457631.
- [15] Boukhatem, Y., Benabderrahmane, B., Rahmoune, A., *Méthode de Faedo-Galerkin pour un problème aux limites non linéaire. Analele Universităţii din Oradea, Fascicola Matematică*, 16, 2009, 167–181.
- [16] Cannon, J. R., The solution of the heat equation subject to the specification of energy. *Quarterly of Applied Mathematics*, 21(1963), 155–160.
- [17] Cannon, J. R. and van der Hoek, J., The existence of and a continuous dependence result for the solution of the heat equation subject to the specification of energy. *Bollettino dell'Unione Matematica Italiana Supplemento*, 1981, 253–282.
- [18] Cannon, J. R. and van der Hoek, J., Diffusion subject to the specification of mass. *Journal of Mathematical Analysis and Applications*, 115(1986), 517–529.
- [19] Cao, J., Xiao, A., and Wang, W., *Reconstruction-Based a Posteriori Error Estimates for the L1 Method for Time Fractional Parabolic Problems. Journal*

Bibliography

- of *Computational Mathematics*, 2024. URL: <https://api.semanticscholar.org/CorpusID:267220883>.
- [20] Chebana, Z., Oussaeif, T.-E., Dehilis, S., et al., *On nonlinear Neumann integral condition for a semilinear heat problem with blowup simulation. Palestine Journal of Mathematics*, 12, 2023, 378–394.
- [21] Chebana, Z., Oussaeif, T.-E., Dehilis, S., and Benaoua, L., A priori estimate and numerical study of solution for a parabolic equation with nonlinear integral condition. *Bulletin of the Institute of Mathematics Academia Sinica (New Series)*, 17(1), 2022, 83–103.
- [22] Chebana, Z., Oussaeif, T.-E., and Benoua, L., *Solvability of the solution of class of fractional Fujita problems by the energy inequality. Eurasian Journal of Mathematical and Computer Applications*, 10(3), 2022, 4–14.
- [23] Chebana, Z. and Oussaeif, T.-E., *Solvability and dynamics study of a weak solution for a nonlinear parabolic problem with the Bessel operator. Innovative Journal of Mathematics (IJM)*, 1(1), 2022, 93–109.
- [24] Chebana, Z., Oussaeif, T.-E., and Ouannas, A., Solvability of Dirichlet problem for a fractional partial differential equation by using energy inequality and Faedo-Galerkin method. *Innovative Journal of Mathematics (IJM)*, 1(1), 2022, 34–44.
- [25] Chebana, Z., Oussaeif, T. E., *Study of Some Initial Boundary Value Non-Linear Parabolic Problems with Different Boundary Conditions*. Master's Thesis, Université Oum El Bouaghi, Oum El Bouaghi, Algeria, 2020.
- [26] Chen, Y. P., *The blow-up profile for a nonlocal nonlinear parabolic equation with a nonlocal boundary condition. Quarterly of Applied Mathematics*, 70(2012), 759–772.
- [27] Coddington, E. A. and Levinson, N., *Theory of Ordinary Differential Equations*. McGraw-Hill, New York, 1987.
- [28] Dautray, R. and Lions, J.-L., *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol. 1–6. Springer, Berlin, 1993.
- [29] Dehilis, S., Bouziani, A., and Oussaeif, T.-E., *Study of solution for a parabolic integro-differential equation with the second kind integral condi-*

Bibliography

- tion. *International Journal of Analysis and Applications*, 16(4), 2018, 569–593. doi:10.28924/2291-8639-16-2018-569.
- [30] Evans, L. C., *Partial Differential Equations*. American Mathematical Society, Graduate Studies in Mathematics, Volume 19, 1998.
- [31] Fujita, H., *On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$* . *Journal of the Faculty of Science, University of Tokyo*, 13(1), 1966, 109–124. doi:10.1080/00036816608833216
- [32] Fujita, H., *On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$* . *Journal of the Faculty of Science, University of Tokyo*, 13(1), 1966, 109–124. (Seminal work on blow-up in reaction-diffusion equations.)
- [33] Galaktionov, V. A. and Vázquez, J. L., *The problem of blow-up in nonlinear parabolic equations*. *Discrete and Continuous Dynamical Systems*, 8(2), 2002, 399–433. doi:10.3934/dcds.2002.8.399.
- [34] Galaktionov, V. A. and Vázquez, J. L., *A dynamical systems approach for the asymptotic analysis of nonlinear heat equations*. *Journal of Differential Equations*, 195(1), 2004, 1–35. doi:10.1016/j.jde.2003.07.002.
- [35] Hamadneh, T., Chebana, Z., Abu Falahah, I., AL-Khassawneh, Y. A., et al., *On Finite-Time Blow-Up Problem for Nonlinear Fractional Reaction Diffusion Equation: Analytical Results and Numerical Simulations*. *Fractal and Fractional*, 7(8), 2023, 589. doi:10.3390/fractalfract7080589.
- [36] Hu, B., *Remarks on the blowup estimate for solution of the heat equation with a nonlinear boundary condition*. *Differential and Integral Equations*, 9(5), 1996, 891–907.
- [37] Ionkin, N. I., *The stability of a problem in the theory of heat conduction with nonclassical boundary conditions*. *Differential Equations*, 15(1979), 911–914.
- [38] John, F., *Partial Differential Equations*, 4th ed. Springer, New York, 1991. (Classic textbook on PDEs with rigorous analysis.)
- [39] Kamynin, L. I., *A boundary-value problem in the theory of heat conduction with non-classical boundary conditions*. *U.S.S.R. Computational Mathematics and Mathematical Physics*, 4(1964), 33–59.

Bibliography

- [40] Kaplan, S., *On the growth of solutions of quasi-linear parabolic equations. Communications on Pure and Applied Mathematics*, 16(3), 1963, 305–330. doi:10.1002/cpa.3160160307.
- [41] Korpusov, M. O., Ovchinnikov, A. V., Sveshnikov, A. G., and Yushkov, E. V., *Blow-Up in Nonlinear Equations of Mathematical Physics*. Walter de Gruyter GmbH, Berlin/Boston, 2018.
- [42] Levine, H. A., *The role of critical exponents in blow-up theorems. SIAM Review*, 32(2), 1990, 262–288. doi:10.1137/1032046.
- [43] LeVeque, R. J., *Finite Difference Methods for Ordinary and Partial Differential Equations*. SIAM, Philadelphia, 2007.
- [44] Li, C. and Zeng, F., *Numerical Methods for Fractional Calculus*. Chapman & Hall/CRC, Boca Raton, FL, 2015.
- [45] Lions, J. L., *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris, 1969.
- [46] Logan, J. D., *Applied Partial Differential Equations*, 3rd ed. Springer, New York, 2015.
- [47] Momani, S., Batiha, I. M., Chebana, Z., Oussaeif, T.-E., Ouannas, A., and Dehilis, S., *Weak Solution, Blow-up and Numerical Study of a Nonlinear Reaction Diffusion Problem with an Integral Condition. Progress in Fractional Differentiation and Applications*, 11(1), 2025, 45–61. doi:10.18576/pfda/110104.
- [48] Oussaeif, T.-E., Chebana, Z., Dehilis, S., and Ouannas, A., *Solvability of nonlocal parabolic problem with numerical solution. Bulletin of the Institute of Mathematics*, 18(1), 2023.
- [49] Oussaeif, T.-E. and Chebana, Z., *Solvability of nonlinear hyperbolic equation with nonlinear integral Neumann conditions. Bulletin of the Institute of Mathematics*, 17(3), 2022, 333–354.
- [50] Oussaeif, T.-E. and Bouziani, A., *Inverse problem of a hyperbolic equation with an integral overdetermination condition. Electronic Journal of Differential Equations*, 2016(138), 2016, 1–7.
- [51] Oussaeif, T.-E. and Bouziani, A., *A priori estimates for weak solution for a time-fractional nonlinear reaction-diffusion equations with an integral condition. Chaos, Solitons & Fractals*, 103, 2017, 79–89. doi:10.1016/j.chaos.2017.05.035.

Bibliography

- [52] Podlubny, I., *Fractional Differential Equations*. Academic Press, San Diego, 1999.
- [53] Poláčik, P., *Parabolic equations: Asymptotic behavior and dynamics on invariant manifolds*. In *Handbook of Differential Equations: Evolutionary Equations*, Vol. 5, Elsevier, 2012, 453–558. doi:10.1016/B978-0-444-53053-2.00006-3.
- [54] Quittner, P. and Souplet, P., *Superlinear Parabolic Problems: Blow-up, Global Existence and Steady States*. Birkhäuser, Basel, 2007. doi:10.1007/978-3-7643-8442-5.
- [55] Samarskii, A. A., Galaktionov, V. A., Kurdyumov, S. P., and Mikhailov, A. P., *Blow-up in quasilinear parabolic equations*. *Differentsial'nye Uravneniya*, 16(11), 1980, 1925–1955.
- [56] Samarskii, A. A., *Some problems of the theory of differential equations*. *Differentsial'nye Uravneniya*, 16(11), 1980, 1925–1935.
- [57] Souplet, P., *Finite-time blow-up for solutions of nonlinear heat equations*. *Bulletin of the American Mathematical Society*, 58(2), 2021, 263–286. doi:10.1090/bull/1715.
- [58] Strauss, W. A., *Partial Differential Equations: An Introduction*, 2nd ed. Wiley, Hoboken, NJ, 2008.
- [59] Vitillaro, E., *Global existence for the wave equation with nonlinear boundary damping and source terms*. *Journal of Differential Equations*, 186(1), 2002, 259–298. doi:10.1006/jdeq.2001.4193.
- [60] Vitillaro, E., *Global existence for the heat equation with nonlinear dynamical boundary conditions*. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 135, 2005, 1231–1246.
- [61] World Scientific (Ed.), *Recent Advances in Applied Nonlinear Dynamics with Numerical Analysis: Fractional Dynamics, Network Dynamics, Classical Dynamics and Fractal Dynamics with their Numerical Simulations*. Interdisciplinary Mathematical Sciences, Vol. 1, World Scientific Publishing Co. Pte. Ltd, Singapore, 2013.
- [62] Yang, X. and Zhou, Z., *Blow-up problems for the heat equation with a local nonlinear Neumann boundary condition*. *Journal of Differential Equations*, 260(3), 2016, 2133–2158. doi:10.1016/j.jde.2015.09.042.

Bibliography

- [63] Yurchuk, N. I., Mixed problem with an integral condition for certain parabolic equations. *Differential Equations*, 22(12), 1986, 1457–1463.
- [64] Zheng, S., *Nonlinear Evolution Equations*. CRC Press, Boca Raton, FL, 2004. doi:10.1201/9780203492222.
- [65] Zeidler, E., *Applied Functional Analysis: Applications to Mathematical Physics*, Springer, New York, 1995.