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Noureddine Djenina

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N°	first and last name	Grade	University	Quality
01	Imad Rezzoug	Prof.	Oum El-Bouaghi	President
02	**Adel Ouannas**	Prof.	Oum El-Bouaghi	Supervisor
03	Oussaeif Taki Eddine	Prof.	Oum El-Bouaghi	Co-reporter
04	Billel Gheraibia	MCA	Oum El-Bouaghi	Examiner
05	Fareh hannachi	Prof.	Tebessa	Examiner
06	Moharned Dalah	Prof.	Constantine 1	Examiner

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Abstract

This thesis investigates discrete fractional calculus, which adapts fractional derivatives to discrete systems, enabling the modeling of processes with memory effects in discrete time intervals. The work is divided into two parts: foundational concepts and original research. The first part explores the definitions, properties, and stability of discrete fractional systems, along with their applications in fields such as epidemiology and control systems. The second part introduces novel contributions, including stability analysis of incommensurate-order systems, variable-order systems, and their applications to epidemic modeling, particularly COVID-19. These advancements highlight the theoretical and practical significance of discrete fractional systems in addressing real-world challenges.

Keywords: Fractional calculus, Discrete fractional calculus, Fractional difference systems, Stability analysis, Incommensurate orders, Variable-order systems, Epidemic modeling, COVID-19, Memory effects, Dynamical systems.

Contents

Introduction

Fractional calculus extends the idea of derivatives and integrals to non-integer (fractional) orders. It originated from a discussion between L'Hôpital and Leibniz in 1695, where the concept of a half-order derivative was first questioned. Over the centuries, mathematicians such as Riemann, Liouville, Caputo, and Grünwald made significant contributions, defining various approaches to fractional differentiation and integration [1, 2]. These developments allowed fractional calculus to model systems with memory and hereditary effects, which are common in fields like viscoelasticity, diffusion, and control theory. For example, fractional derivatives are used to capture anomalous diffusion, where particles move at irregular rates, a phenomenon seen in biological and physical systems [3, 4].

In parallel, the development of discrete fractional calculus aimed to extend fractional calculus to discrete settings, where changes occur in steps rather than continuously. Discrete fractional calculus uses fractional difference operators instead of derivatives, making it suitable for studying systems described by sequences or discrete data. This approach is particularly useful in modeling digital signal processing, population dynamics, and economic systems, where processes naturally evolve in discrete time intervals. Mathematicians such as Kenneth S. Miller and Bertram Ross contributed foundational theories in this area, introducing discrete analogs of fractional operators [5, 6]. These tools have enabled the study of stability, control, and dynamic behavior in discrete systems.

Modern researchers have further refined discrete fractional calculus by exploring numerical methods and applications in real-world problems. For instance, discrete fractional models have been applied to epidemiological studies [7], financial modeling [8], and control systems [9], providing insights into how fractional order differences influence system dynamics. Key references in this field include Podlubny's seminal book on fractional differential equations [1], Miller and Ross's foundational text on fractional calculus and difference equations [5], and recent advances documented in works by Atici and Eloe [10] and Anastassiou [11]. These texts and studies highlight the theoretical and applied aspects of this growing area of research.

Fractional calculus and discrete fractional calculus have found widespread applications in various complex systems, including chaos, synchronization, epidemics, and neural networks. In chaos theory, fractional-order systems offer more flexibility in modeling and analyzing chaotic behavior, as they can capture memory and hereditary properties inherent in chaotic processes, which traditional integer-order models often overlook [12]-[47]. These properties have been used to design more effective control and anti-synchronization schemes in chaotic systems. In synchronization, fractional models enable the coupling of systems with different dynamics, improving robustness and precision in applications such as secure communications and signal processing [48]-[89]. In epidemiology, fractional and discrete fractional models have been used to study the spread of infectious diseases like COVID-19, capturing non-local interactions and memory effects in population dynamics. This leads to more accurate predictions and insights into disease control strategies [90]-[96]. Furthermore, in neural networks, fractional calculus enhances the modeling of neural activity by incorporating memory-dependent dynamics, improving learning algorithms and signal processing tasks [97]-[104]. Discrete fractional calculus, in particular, is well-suited for digital implementations in these areas, enabling efficient computation and analysis in modern applications. These advancements highlight the versatility and importance of fractional calculus in addressing real-world challenges.

The first part of this thesis, Conceptual Investigation and Earlier Studies, provides a comprehensive background to the discrete fractional calculus and its existing body of research. It begins by introducing the basic concepts of discrete fractional calculus, including key definitions, properties, and operators such as fractional differences and sums. These elements form the foundation for analyzing fractional-order systems. Following this, the thesis delves into the stability of discrete fractional systems, a crucial aspect of ensuring the reliability and performance of such systems in engineering and scientific applications. Finally, a survey of recent advancements in the applications of fractional difference systems is presented, showcasing their use in various domains and underscoring their broad potential.

The second part, Novel Contributions and Applications, highlights the original research conducted in this work. A major focus is placed on stability analysis of fractional difference systems, particularly those with incommensurate orders, where system components operate with different fractional orders. This unique perspective expands the understanding of stability in complex dynamical systems. Additionally, the thesis investigates variable-order discrete-time systems, which introduce flexibility by allowing the fractional order to vary over time, enabling more accurate modeling of dynamic processes. The practical importance of discrete fractional systems is further demonstrated through their application to epidemic models, including the analysis and simulation of infectious diseases such as COVID-19. These applications showcase the ability of fractional models to account for memory effects and complex transmission dynamics, providing deeper insights into disease spread and control strategies.

The significance of this research lies in its dual contribution to the theoretical advancement of discrete fractional calculus and its application to pressing real-world challenges. By addressing open problems in stability analysis and demonstrating the utility of fractional models in epidemiology, this thesis bridges the gap between mathematical theory and practical application. It is hoped that this work will inspire further exploration of discrete fractional systems and promote their adoption across a wider range of scientific and engineering disciplines.

Part I

Conceptual Investigation and Earlier Studies

Chapter 1

Basic Concepts of Discrete Fractional Calculus

Discrete dynamical systems have a wide range of applications in various fields. Some common applications include: Physics and Engineering, Control Systems, Computer Science, Algorithms and Data Structures, Computer Networks, Biology and Ecology, Economics and Finance, Chaos Theory, Game Theory, Chemistry $\cdots$ ect.

These applications demonstrate the versatility and importance of discrete dynamical systems in understanding, analyzing, and predicting the behavior of complex systems across various disciplines.

A discrete dynamical system is a mathematical model that represents how a system evolves over time in discrete steps. Suppose we have a variable x that represents the state of the system at each time step, and let $f : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$ be a function that determines the evolution of the system. The discrete dynamical system can then be expressed mathematically as:

$$x(t+1) = f(t, x(t)), \tag{1.1}$$

Here, $x(t)$ represents the state of the system at time step t, and $x(t+1)$ represents the state at the next time step $t+1$. The function f describes how the state evolves from $x(t)$ to $x(t+1)$.

A dynamic system can be defined implicitly as follows

$$x(t) = f(t, x(t)), \tag{1.2}$$

1.1 Basics on the Delta Difference Operator

Delta Difference equations, serve as a distinctive subset within this mathematical framework, specifically addressing the immediate alterations in values over successive time intervals.

These equations find application in capturing the real-time evolution of variables, making them particularly valuable in the dynamic modeling of processes characterized by change.

1.1.1 Definitions and Som Properties of Delta Difference Operator

In the exploration that follows, we will delve into the definition and characteristics, shedding light on the unique attributes that distinguish the Delta Difference operator. For $t_0 \in \mathbb{R}$, we define the set $\mathbb{N}_{t_0}$ as $\{t_0, t_0+1, t_0+2, \cdots\}$,

Definition 1.1 *[105] Suppose $f : \mathbb{N}_0 \to \mathbb{R}$. In this context, we establish the Delta difference operator Δ as follows:*

$$\Delta f(t) := f(t+1) - f(t), \quad \forall t \in \mathbb{N}_0. \tag{1.3}$$

Furthermore, we define the operators Δ^N, $N = 1, 2, 3, \cdots$ recursively as $\Delta^N f(t) = \Delta(\Delta^{N-1} f(t))$ for $t \in \mathbb{N}_0$. Additionally, Δ^0 represents the identity operator, denoted as $\Delta^0 f(t) = f(t)$.

Here are some pertinent properties concerning this operator.

Theorem 1.1 *[105] Given functions f and g mapping from $\mathbb{N}$ to $\mathbb{R}$, and constants α and β belonging to $\mathbb{R}$, for any $t \in \mathbb{N}_0$, the following holds:*

(i) $\Delta\alpha = 0$.

(ii) $\Delta \alpha f(t) = \alpha \Delta f(t)$.

(iii) $\Delta [f+g](t) = \Delta f(t) + \Delta g(t)$.

(iv) $\Delta \alpha^{t+\beta} = (\alpha - 1)\alpha^{t+\beta}$.

(v) $\Delta [fg](t) = f(t+1)\Delta g(t) + \Delta f(t) g(t)$.

(vi) $\Delta \left(\frac{f}{g}\right)(t) = \frac{g(t)\Delta f(t) - f(t)\Delta g(t)}{g(t)g(t+1)}$,

*where in **(vi)** we assume $g(t) \neq 0$, $\forall t \in \mathbb{N}_0$.*

Proposition 1.1 *[105] Suppose $f : \mathbb{N}_0 \to \mathbb{R}$. Then, for a given $t_0 \in \mathbb{N}_0$, we have:*

$$\Delta \sum_{i=t_0}^{t-1} f(i) = f(t), \quad \forall t \in \mathbb{N}_0, \tag{1.4}$$

and

$$\sum_{i=t_0}^{t-1} \Delta f(i) = f(t) - f(t_0), \quad \forall t \in \mathbb{N}_0. \tag{1.5}$$

Remark 1.1 *When $t \leq t_0$, the usual convention is applied:*

$$\sum_{i=t_0}^{t-1} f(i) := 0. \tag{1.6}$$

Theorem 1.2 ***(Summation by parts)***[105] *Considering two functions u_1 and u_2 mapping from $\mathbb{N}_0$ to $\mathbb{R}$, and two natural numbers t_0 and t_1, where $t_0 < t_1$, we derive the summation by parts formulas:*

$$\sum_{i=t_0}^{t_1-1} u_1(i)\Delta u_2(i) = u_1(i)u_2(i)|_{t_0}^{t_1} - \sum_{i=t_0}^{t_1-1} u_2(i+1)\Delta u_1(i), \tag{1.7}$$

where $u_1(i)u_2(i)|_{t_0}^{t_1} = u_1(t_1)u_2(t_1) - u_1(t_0)u_2(t_0)$.

1.1.2 Falling Function

In continuous calculus, polynomial functions are considered among the most important functions, because they are characterized by a large set of properties, especially differentiation and integration.

In discrete calculus, polynomial functions lose their properties, but some functions maintain the same properties, which we call factorial functions.

In the case of the delta operator, these functions are called Falling function and are defined as follows:

Definition 1.2 *[105] For any $r \in \mathbb{R}$, we define the falling function, $t^{(r)}$, read t to the r falling, by:*

$$t^{(r)} := \frac{\Gamma(t+1)}{\Gamma(t-r+1)} = t(t-1)(t-2)\cdots(t-r+1). \tag{1.8}$$

For values of t and r where the right-hand side of this equation is meaningful, we extend this definition by adopting the common convention that $t^{(r)} := 0$, when $t-r+1$ is a nonpositive integer.

Remark 1.2 *When $r = 0$, we define $t^{(0)} := 1$.*

Theorem 1.3 ***(Power Rules)***[105] *For real numbers r, t_0, and t, the following power rules hold:*

$$\Delta(t+t_0)^{(r)} = r(t+t_0)^{(r-1)}, \tag{1.9}$$

and

$$\Delta(t_0-t)^{(r)} = -r(t_0-t-1)^{(r-1)}, \tag{1.10}$$

provided that the expressions in these two formulas are well-defined.

Theorem 1.4 ***(Repeated summation rule)****[105] Consider a function $f : \mathbb{N}_0 \to \mathbb{R}$ and let $t_0 \in \mathbb{N}_0$, then:*

$$\sum_{\tau_1=t_0}^{t-1}\sum_{\tau_2=t_0}^{\tau_1-1}\cdots\sum_{\tau_p=t_0}^{\tau_{p-1}-1} f(\tau_p) = \sum_{i=t_0}^{t-1}\frac{(t-i-1)^{(p-1)}}{(p-1)!}f(i), \tag{1.11}$$

for $t = p+t_0, p+t_0+1, \cdots$.

1.2 Basics on the Nabla Difference Operator

1.2.1 Definitions and Som Properties of Nabla Difference Operator

In the forthcoming exploration, we'll delve into the definition and features, elucidating the distinct attributes that set apart the Nabla Difference operator.

Definition 1.3 *[105] Suppose $f : \mathbb{N}_0 \to \mathbb{R}$. In this context, we establish the Nabla difference operator ∇ as follows:*

$$\nabla f(t) := f(t) - f(t-1), \quad \forall t \in \mathbb{N}, t \geq 1. \tag{1.12}$$

Furthermore, we define the operators ∇^N, $N = 1, 2, 3, \cdots$ recursively as $\nabla^N f(t) = \nabla(\nabla^{N-1} f(t))$. Additionally, ∇^0 represents the identity operator, denoted as $\nabla^0 f(t) = f(t)$.

Theorem 1.5 *[105] Given functions $f, g : \mathbb{N}_0 \to \mathbb{R}$, and constants $\alpha, \beta \in \mathbb{R}$, for any $t \in \mathbb{N}_0, t \geq 1$, the following holds:*

(i) $\nabla\alpha = 0$.

(ii) $\nabla\alpha f(t) = \alpha\nabla f(t)$.

(iii) $\nabla[f+g](t) = \nabla f(t) + \nabla g(t)$.

(iv) *if $\alpha \neq 0$, then $\nabla\alpha^{t+\beta} = \frac{(\alpha-1)}{\alpha}\alpha^{t+\beta}$.*

(v) $\nabla[fg](t) = f(t-1)\nabla g(t) + \nabla f(t)g(t)$.

(vi) $\nabla\left(\frac{f}{g}\right)(t) = \frac{g(t)\nabla f(t) - f(t)\nabla g(t)}{g(t)g(t-1)}$,

*where in **(vi)** we assume $g(t) \neq 0$, $\forall t$.*

Proposition 1.2 *[105] Suppose $f : \mathbb{N} \to \mathbb{R}$. Then, for a given $t_0 \in \mathbb{N}$, we have:*

$$\nabla \sum_{i=t_0+1}^{t} f(i) = f(t), \quad \forall t \in \mathbb{N}_0, \tag{1.13}$$

and

$$\sum_{i=t_0+1}^{t} \nabla f(i) = f(t) - f(t_0), \quad \forall t \in \mathbb{N}_1. \tag{1.14}$$

Theorem 1.6 ***(Summation by parts)***[105] *Suppose we have two functions $u_1, u_2 : \mathbb{N} \to \mathbb{R}$ and two natural numbers t_0 and t_1, where $t_0 < t_1$. Then, the summation by parts formulas are as follows:*

$$\sum_{i=t_0}^{t_1+1} u_1(i)\nabla u_2(i) = u_1(i)u_2(i)|_{t_0}^{t_1} - \sum_{i=t_0}^{t_1+1} u_2(i-1)\nabla u_1(i), \tag{1.15}$$

where $u_1(i)u_2(i)|_{t_0}^{t_1} = u_1(t_1)u_2(t_1) - u_1(t_0)u_2(t_0)$.

1.2.2 Rising Function

The factorial function, in the case of the nabla operator, is referred to as the Rising function and is defined as follows:

Definition 1.4 *[105] For any $r \in \mathbb{R}$, we define the falling function, $t^{\overline{r}}$, read t to the r rising, by:*

$$t^{\overline{r}} := \frac{\Gamma(t+r)}{\Gamma(t)} = t(t+1)(t+2)\cdots(t+r-1). \tag{1.16}$$

For values of t and r where the right-hand side of this equation is meaningful, we extend this definition by adopting the common convention that $t^{\overline{r}} := 0$, when t is a nonpositive integer, but $t+r$ is not a nonpositive integer.

Remark 1.3 *When $r = 0$, we define $t^{\overline{0}} := 1$.*

Theorem 1.7 ***(Power Rules)***[105] *For real numbers r, t_0, and t, the following power rules hold:*

$$\nabla(t+t_0)^{\overline{r}} = r(t+t_0)^{\overline{r-1}}, \tag{1.17}$$

and

$$\nabla(t_0-t)^{\overline{r}} = -r(t_0-t+1)^{\overline{r-1}}, \tag{1.18}$$

provided that the expressions in these two formulas are well-defined.

Theorem 1.8 ***(Repeated summation rule)***[105] *Consider a function $f : \mathbb{N}_0 \to \mathbb{R}$ and let $t_0 \in \mathbb{N}_0$, then:*

$$\sum_{\tau_1=t_0}^{t+1}\sum_{\tau_2=t_0}^{\tau_1+1}\cdots\sum_{\tau_p=t_0}^{\tau_{p-1}+1} f(\tau_p) = \sum_{i=t_0}^{t+1}\frac{(t-i+1)^{\overline{p-1}}}{(p-1)!}f(i), \tag{1.19}$$

for $t = t_0-p+1, t_0-p+2, t_0-p+3, \cdots$.

1.3 The h-Difference Operators

The h-difference operator serves as an extension of the conventional difference operator, offering a more versatile tool for analyzing changes between values. Unlike the fixed step of 1 employed by the traditional difference operator, the h-difference operator takes into account the length of the step, denoted by 'h'. This variable step size enables a more nuanced examination of the differences between values, allowing for a more flexible and detailed analysis in various mathematical contexts. For $t_0 \in \mathbb{R}$, we define the set $(h\mathbb{N})_{t_0}$ as $\{t_0, t_0 + h, t_0 + 2h, \cdots\}$,

1.3.1 The Delta h-Difference

Let's begin by introducing the delta h-difference operator.

Definition 1.5 *[106] For a function $f : \mathbb{N}_0 \to \mathbb{R}$, the Delta h-Difference operator is defined as*

$$\Delta_h f(t) := \frac{f(t+h) - f(t)}{h}, \quad \forall t \in (h\mathbb{N})_0 . \tag{1.20}$$

While the h-difference sum is given by

$$_{t_0}\Delta_h^{-1} f(t) := \sum_{i=\frac{t_0}{h}}^{\frac{t}{h}+1} f(th)h, \quad \forall t \in (h\mathbb{N})_0 . \tag{1.21}$$

Definition 1.6 *[106] For arbitrary $t, r \in \mathbb{R}$ the h-falling function is defined by*

$$t_h^{(r)} := \frac{\Gamma(\frac{t}{h} + 1)}{\Gamma(\frac{t}{h} - r + 1)}. \tag{1.22}$$

For values of t and r where the right-hand side of this equation is meaningful.

Proposition 1.3 *[106] For real numbers r, t, and $h > 0$, the following power rules hold:*

$$\Delta_h t_h^{(r)} = r t_h^{(r-1)}, \tag{1.23}$$

1.3.2 The Nabla h-Difference

Now we introduce the Nabla h-Difference operator.

Definition 1.7 *[107] For a function $f : \mathbb{N}_0 \to \mathbb{R}$, the Nabla h-Difference operator is defined as*

$$\nabla_h f(t) := \frac{f(t) - f(t-h)}{h}, \quad \forall t \in (h\mathbb{N})_1 . \tag{1.24}$$

while the h-difference sum is given by

$$_{t_0}\nabla_h^{-1} f(t) := \sum_{i=\frac{t_0}{h}+1}^{\frac{t}{h}} f(th)h, \quad \forall t \in (h\mathbb{N})_0 . \tag{1.25}$$

Definition 1.8 *[107] For arbitrary $t, r \in \mathbb{R}$ the h-rising function is defined by*

$$t_h^{\overline{r}} := h^r \frac{\Gamma(\frac{t}{h}+r)}{\Gamma(\frac{t}{h})}. \tag{1.26}$$

For values of t and r where the right-hand side of this equation is meaningful.

Proposition 1.4 *[107] For real numbers r, t, and $h > 0$, the following power rules hold:*

$$\nabla_h t_h^{\overline{r}} = r t_h^{\overline{r-1}}. \tag{1.27}$$

1.4 On The Z-transform

The Z-transform [108] is an important tool used in signal processing and discrete-time systems analysis. It converts discrete-time signals into functions of a complex variable called "z", facilitating analysis in the frequency domain. By transforming difference equations into algebraic equations in terms of z, it simplifies system analysis and design. The Z-transform enables operations like convolution, differentiation, and integration on discrete-time signals and systems. Overall, it's a powerful tool for analyzing and designing digital filters, control systems, and other discrete-time systems.

Definition 1.9 *[108] The Z-transform of a sequence $(x(t))_{t\in\mathbb{Z}}$ with values only for non-negative integers t (i.e., $x(t) = 0$ for $t = -1, -2, \cdots$), is defined as:*

$$\tilde{x}(z) = Z(x(t)) := \sum_{t=0}^{\infty} x(t) z^{-t}, \tag{1.28}$$

where $z \in \mathbb{C}$.

The region in the complex plane where the series (1.28) converges defines the region of convergence of x(z). Utilizing the ratio test is a prevalent approach to determine this region for the series (1.28). Let's assume:

$$\lim_{t\to\infty} \left| \frac{x(t+1)}{x(t)} \right| = R. \tag{1.29}$$

The series (1.28) converges within the region defined by $|z| > R$ and diverges outside this region, specifically for $|z| < R$.

1.4.1 Properties of the Z-transform

- **Linearity:**

  Consider $\tilde{x}(z)$ as the Z-transform of the sequence $(x(t))_{t\in\mathbb{Z}}$, possessing a radius of convergence R_1, and $\tilde{y}(z)$ as the Z-transform of the sequence $(y(t))_{t\in\mathbb{Z}}$, with a radius of convergence R_2. For any complex numbers λ_1 and λ_2, the following holds:

$$Z\left[\alpha x(t) + \beta y(t)\right] = \alpha\tilde{x}(z) + \beta\tilde{y}(z), \quad \text{for } |z| > max(R_1, R_2). \tag{1.30}$$

- **Shifting**:

  Suppose R represents the radius of convergence of $\tilde{x}(z)$

  - Regarding right-shifting: If $x(-i) = 0$ for $i = 1, 2, \cdots, i$, then

$$Z\left[x(t-i)\right] = z^{-i}\tilde{x}(z) \quad \text{for } |z| > R. \tag{1.31}$$

  - Left-shifting:

$$Z\left[x(t+i)\right] = z^{-i}\tilde{x}(z) - \sum_{r=0}^{i-1} x(r)z^{i-r} \quad \text{for } |z| > R. \tag{1.32}$$

- **Initial and final value:**

  - Initial value theorem:

$$\lim_{|z|\to\infty} \tilde{x}(z) = x(0). \tag{1.33}$$

  - Final value theorem:

$$x(\infty) = \lim_{t\to\infty} x(t) = \lim_{z\to 1}(z-1)\tilde{x}(z). \tag{1.34}$$

- **Convolution:**

  The convolution $*$ of two sequences $(x(t))_{t\in\mathbb{Z}}, (y(t))_{t\in\mathbb{Z}}$ is defined by

$$x(t) * y(t) = \sum_{r=0}^{t} x(t-r)y(r) = \sum_{r=0}^{t} x(r)y(t-r). \tag{1.35}$$

  And The Z-transform of $x(t) * y(t)$:

$$Z\left[x(t) * y(t)\right] = \tilde{x}(z)\tilde{y}(z). \tag{1.36}$$

- **Multiplication by $a^{\mathbf{t}}$:**

  Consider $\tilde{x}(z)$ as the Z-transform of the sequence $(x(t))_{t\in\mathbb{Z}}$, possessing a radius of convergence R.

$$Z\left[a^t x(t)\right] = \tilde{x}\left(\frac{z}{a}\right), \quad \text{for } |z| > |a|\, R. \tag{1.37}$$

1.4.2 The Inverse Z-transform Using the Integral Method

When we multiply both sides of equation (1.28) by z^{t-1}, where $t \in \mathbb{N}$, we obtain:

$$\tilde{x}(z)z^{t-1} = \sum_{i=0}^{\infty} x(i)z^{t-i-1}. \tag{1.38}$$

Equation (1.38) provides the Laurent series expansion of $\tilde{x}(z)z^{t-1}$ centered at $z = 0$.

Let's contemplate a circle C centered at the origin of the z-plane, encompassing all poles of $\tilde{x}(z)z^{t-1}$. Given that $x(t)$ represents the coefficient of z^{-1}, we deduce from the Cauchy integral formula that:and by

$$x(t) = \frac{1}{2\pi i} \oint_C \tilde{x}(z)z^{t-1}dz, \tag{1.39}$$

Applying the residue theorem, we obtain:

$$x(t) = \text{sum of residues of } \tilde{x}(z)z^{t-1}. \tag{1.40}$$

1.5 On Volterra Difference Equations of Convolution Type

Volterra difference equations of convolution type represent a class of difference equations that involve convolutions of sequences. These equations are named after the Italian mathematician Vito Volterra, who made significant contributions to integral equations and mathematical biology in the early 20th century. The equation essentially states that the value of $x(t)$ at time t depends on the convolution of past values of $x(t)$, with appropriate coefficients.

Definition 1.10 *[109] A Volterra difference equation of convolution type can be written in the form:*

$$x(t+1) = Ax(t) + \sum_{i=0}^{t} B(t-i)x(i), \tag{1.41}$$

where $x(t) \in \mathbb{R}^n, \forall t \in \mathbb{N}$, $A = (a_{ii})_{1 \le i, i \le n}$ is a $n \times n$ real matrix and $B(t)$ is a $n \times n$ real matrix defined on $\mathbb{N}$.

If $(B(t))_{t \in \mathbb{N}} \in \ell^1(\mathbb{N}^{n \times n})$, the resolvent matrix $R(t)$ of (1.41) is defined as the unique solution of the matrix equation:

$$R(t+1) = AR(t) + \sum_{i=0}^{t} B(t-i)R(i), \quad R(0) = I_n, t \in \mathbb{N}, \tag{1.42}$$

where I_n is the identity matrix. Let $(x(t))_{t\in\mathbb{N}}$ denote the solution of the equation

$$x(t+1) = Ax(t) + \sum_{i=0}^{t} B(t-i)x(i) + g(t). \tag{1.43}$$

Then by the variation of constants formula, we obtain

$$x(t) = R(t)x(0) + \sum_{i=0}^{t-1} R(t-i-1)g(i). \tag{1.44}$$

Solving Volterra difference equations of convolution type often involves techniques from discrete-time signal processing and linear algebra. Depending on the specific coefficients and input sequences involved, various methods such as z-transforms.

These equations find applications in diverse fields including control theory, signal processing, communication systems, and mathematical modeling in biology and economics. They are particularly useful in systems where the evolution of a sequence depends not only on its past values but also on the past values of another related sequence, as captured by the convolution operation.

1.6 Fractional Discrete Operators

In this section, we introduce fractional summation and discuss two key fractional difference operators: the Riemann-Liouville and Caputo operators. We explore their definitions, properties, and establish connections between them.

1.6.1 Fractional Delta Sum Operator

Firstly, we introduce the concept of fractional summation by leveraging the Repeated Summation Rule and properties of the Gamma function. By substituting a real positive number α for the natural number p, we arrive at the definition of fractional order summation.

Definition 1.11 *[110] Let $\alpha > 0$. We define the α-th fractional sum of a function $f : \mathbb{N}_a \to \mathbb{R}$ as follows:*

$$\Delta_{t_0}^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{i=t_0}^{t-\alpha} (t-i-1)^{(\alpha-1)} f(i), \tag{1.45}$$

$\forall t \in \mathbb{N}_{t_0+\alpha}$.

Remark 1.4 *It's worth noting that $\Delta_{t_0}^{-\alpha}$ extends functions from $\mathbb{N}_{t_0}$ to $\mathbb{N}_{t_0+\alpha}$.*

Lemma 1.1 *[105] Given that $\mu \geq 0$ and $\alpha > 0$, we have:*

$$\Delta_{t_0+\mu}^{-\alpha}(t-t_0)^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-t_0)^{(\mu+\alpha)}, \tag{1.46}$$

$\forall t \in \mathbb{N}_{t_0+\mu+\alpha}$.

Theorem 1.9 *[110] Let $f : \mathbb{N}_{t_0} \to \mathbb{R}$ be a given function, and let μ and ν be positive real numbers. Then:*

$$\Delta_{t_0+\mu}^{-\nu}[\Delta_{t_0}^{-\mu} f(t)] = \Delta_{t_0}^{-(\mu+\nu)} f(t) = \Delta_{t_0+\nu}^{-\mu}[\Delta_{t_0}^{-\nu} f(t)], \tag{1.47}$$

$\forall t \in \mathbb{N}_{t_0+\mu+\nu}$.

Theorem 1.10 *[110] Let $f : \mathbb{N}_{t_0} \to \mathbb{R}$ be given, for any $\alpha > 0$ and any positive integer p, we have:*

$$\Delta_{t_0}^{-\alpha}\Delta^p f(t) = \Delta^p \Delta_{t_0}^{-\alpha} f(t) - \sum_{i=0}^{p-1} \frac{(t-t_0)^{(\alpha-p+i)}}{\Gamma(\alpha+i-p+1)}\Delta^i f(t_0), \tag{1.48}$$

$\forall t \in \mathbb{N}_{t_0+\alpha}$.

1.6.2 The Fractional Delta Rimann Liouville Difference Operator

The Riemann-Liouville difference operator, is a fractional difference operator used in the context of fractional calculus. It generalizes the notion of discrete differencing to non-integer orders.

Definition 1.12 *[110] For $\alpha > 0$, the α-order Riemann-Liouville fractional difference of a function f defined on $\mathbb{N}_{t_0}$ is defined as follows:*

$$\Delta_{t_0}^{\alpha} f(t) := \Delta^p \Delta_{t_0}^{-(p-\alpha)} f(t) = \frac{1}{\Gamma(p-\alpha)}\Delta^p \sum_{i=t_0}^{t-(p-\alpha)} (t-i-1)^{(p-\alpha-1)} f(i), \quad \forall t \in \mathbb{N}_{t_0+p-\alpha}, \tag{1.49}$$

where $p = [\alpha] + 1$.

The Riemann-Liouville difference operator captures temporal memory in a time series by computing a weighted sum of differences between the current value $f(t)$ and past values. This operator is extensively utilized in fractional calculus to analyze systems exhibiting memory or long-range dependence.

Remark 1.5 *For $\alpha > 0$, and $p = [\alpha] + 1$*

$$\lim_{\alpha \to p} \Delta_{t_0}^{\alpha} f(t) = \Delta^p f(t), \text{ and } \lim_{\alpha \to p-1} \Delta_{t_0}^{\alpha} f(t) = \Delta^{p-1} f(t). \tag{1.50}$$

We can interpret the $\Delta_{t_0}^{\alpha}$ operator as an interpolation between the Δ^p operators. Furthermore, it's evident that $\Delta_{t_0}^{\alpha}$ maps functions defined on $\mathbb{N}_{t_0}$ to functions defined on $\mathbb{N}_{t_0+(p-\alpha)}$.

Remark 1.6 *Note that*

$$\Delta_{t_0}^{\alpha} 1 = \frac{1}{\Gamma(1-\alpha)}(t-t_0)^{(-\alpha)} \neq 0. \tag{1.51}$$

Theorem 1.11 *[110] Given $\alpha > 0$ and f defined on $\mathbb{N}_{t_0}$, we have:*

$$\Delta_{t_0+\alpha}^{\alpha} \Delta_{t_0}^{-\alpha} f(t) = f(t), \quad \forall t \in \mathbb{N}_{t_0+p}, \tag{1.52}$$

where $p = [\alpha] + 1$.

Theorem 1.12 *[110] If $\alpha > 0$ and f is defined on $\mathbb{N}_{t_0}$, then:*

$$\Delta_{t_0+p-\alpha}^{-\alpha} \Delta_{t_0}^{\alpha} f(t) = f(t) - \sum_{i=0}^{p-1} \frac{(t-t_0)^{(\alpha-p+i)}}{\Gamma(\alpha+i-p+1)} \Delta_{t_0}^{i-(p-\alpha)} f(t_0), \ \ \forall t \in \mathbb{N}_{t_0+p}, \tag{1.53}$$

where $p = [\alpha] + 1$.

Remark 1.7 *We see that*

$$\Delta_{t_0+\alpha}^{-\alpha} \Delta_{t_0}^{\alpha} f(t) \neq f(t) \tag{1.54}$$

generally.

1.6.3 The Fractional Delta Caputo Difference Operator

The Caputo difference operator is named after the Italian mathematician Michele Caputo, who introduced it as an alternative approach to fractional calculus. It has wide applications in various fields, including physics, engineering, and mathematical modeling, where systems exhibit non-local or memory-dependent behaviors. The Caputo operator is often preferred over the Riemann-Liouville operator in fractional calculus because it has the advantage of producing initial value problems with more physical relevance, particularly in the context of fractional difference equations. The Caputo operator takes into account the entire history of the function being differences, whereas the Riemann-Liouville operator includes contributions from negative time, which may not be physically meaningful in many applications. The Delta Caputo operator has found applications in various fields such as physics, engineering, signal processing, and finance, where systems with memory effects, non-local behaviors, or fractal properties are encountered.

Definition 1.13 *[110] For $\alpha > 0$ where α is not a natural number, the α-order Caputo fractional difference of a function f defined on $\mathbb{N}_{t_0}$ is given by:*

$$ {}^{C}\Delta_{t_0}^{\alpha} f(t) := \Delta_{t_0}^{-(p-\alpha)} \Delta^{p} f(t) = \frac{1}{\Gamma(p-\alpha)} \sum_{i=t_0}^{t-(p-\alpha)} (t-i-1)^{(p-\alpha-1)} \Delta^{p} f(i), \quad \forall t \in \mathbb{N}_{t_0+p-\alpha}, \tag{1.55} $$

where $p = [\alpha] + 1$.

Remark 1.8 *Note that*

$$ \lim_{\alpha \to p-1} {}^{C}\Delta_{t_0}^{\alpha} f(t) = \Delta^{p-1} f(t) - \Delta^{p-1} f(t_0). \tag{1.56} $$

So ${}^{C}\Delta_{t_0}^{\alpha}$ *is not an interpolation of* Δ^{p}*. However, it is used in modeling because it is consistent with the classical initial and boundary conditions. To avoid confusion, we define* ${}^{C}\Delta_{t_0}^{p}$ *as* ${}^{C}\Delta_{t_0}^{p} f(t) := \Delta^{p} f(t)$*, where* $p \in \mathbb{N}$*. Also, it is clear that* ${}^{C}\Delta_{t_0}^{\alpha}$ *maps functions defined on* $\mathbb{N}_{t_0}$ *to functions defined on* $\mathbb{N}_{t_0+(p-\alpha)}$.

Remark 1.9 *The fractional Caputo difference of a constant* $c \in \mathbb{R}$ *is*

$$ {}^{C}\Delta_{t_0}^{\alpha} c = \Delta_{t_0}^{-(p-\alpha)} \Delta^{p} c = 0. \tag{1.57} $$

Theorem 1.13 *[110] Suppose* $\alpha > 0$ *and* f *is defined on* $\mathbb{N}_{t_0}$*, then:*

$$ \Delta_{t_0+(p-\alpha)}^{-\alpha} \, {}^{C}\Delta_{t_0}^{\alpha} f(t) = f(t) - \sum_{i=0}^{p-1} \frac{(t-t_0)^{(i)}}{i!} \Delta^{i} f(t_0), \quad \forall t \in \mathbb{N}_{t_0}, \tag{1.58} $$

where $p = [\alpha] + 1$.
Specifically, if $0 < \alpha \leq 1$ *then:*

$$ \Delta_{t_0+(1-\alpha)}^{-\alpha} \, {}^{C}\Delta_{t_0}^{\alpha} f(t) = f(t) - f(t_0), \quad \forall t \in \mathbb{N}_{t_0}. \tag{1.59} $$

Theorem 1.14 *[110] Given* $\alpha > 0$*, and* f *defined on* $\mathbb{N}_{t_0}$*, then:*

$$ {}^{C}\Delta_{t_0}^{\alpha} f(t) = \Delta_{t_0}^{\alpha} f(t) - \sum_{i=0}^{p-1} \frac{(t-t_0)^{(i-\alpha)}}{\Gamma(i-\alpha+1)} \Delta^{i} f(t_0), \quad \forall t \in \mathbb{N}_{t_0+p-\alpha}, \tag{1.60} $$

where $p = [\alpha] + 1$.
Specifically, when $0 < \alpha < 1$*, we have*

$$ {}^{C}\Delta_{t_0}^{\alpha} f(t) = \Delta_{t_0}^{\alpha} f(t) - \frac{(t-t_0)^{(-\alpha)}}{\Gamma(1-\alpha)} f(t_0), \quad \forall t \in \mathbb{N}_{t_0-\alpha+1}. \tag{1.61} $$

Theorem 1.15 ***(Discrete Taylor's Formula)***[111] *Consider f defined on $\mathbb{N}_{t_0}$. Then, for all $t \in \mathbb{N}_{t_0}$ and $p \in \mathbb{N}$ with $p \geq 1$:*

$$f(t) = \sum_{i=0}^{p-1} \frac{(t-t_0)^{(i)}}{i!} \Delta^i f(t_0) + \frac{1}{(p-1)!} \sum_{i=t_0}^{t-p} (t-i-1)^{(p-1)} \Delta^p f(i). \tag{1.62}$$

Theorem 1.16 *[112] Assume $\alpha > 0$ and $\alpha \notin \mathbb{N}$, and f is defined on $\mathbb{N}_{t_0}$. Then:*

$$f(t) = \sum_{i=0}^{p-1} \frac{(t-t_0)^{(i)}}{i!} \Delta^i f(t_0) + \frac{1}{\Gamma(\alpha)} \sum_{i=t_0+p-\alpha}^{t-\alpha} (t-i-1)^{(\alpha-1)} \, {}^C\Delta_{t_0}^{\alpha} \, f(i), \quad \forall t \in \mathbb{N}_{t_0+p}, \tag{1.63}$$

where $p = [\alpha] + 1$.

1.6.4 The Fractional Nabla Sum Operator

The fractional Nabla sum is an operation used in fractional calculus, specifically in discrete fractional calculus. It extends the traditional discrete sum to non-integer orders, allowing for the analysis and manipulation of discrete-time signals and systems with fractional-order dynamics.

The fractional Nabla sum has applications in various fields, including signal processing, control theory, and time-series analysis, where systems with memory effects, non-local behaviors, or fractal properties are encountered. It provides a powerful tool for modeling and analyzing complex dynamic systems with fractional-order dynamics in discrete-time domains.

Definition 1.14 *[113] Let $\alpha > 0$, the Nabla fractional sum of the function $f : \mathbb{N}_{t_0} \to \mathbb{R}$ characterized by:*

$$\nabla_{t_0}^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{i=t_0+1}^{t} (t-i+1)^{\overline{\alpha-1}} f(i), \tag{1.64}$$

$\forall t \in \mathbb{N}_{t_0}$.

Lemma 1.2 *[113] Given that $\mu > -1$ and $\alpha > 0$, the following holds:*

$$\nabla_{t_0}^{-\alpha} (t-t_0)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} (t-t_0)^{\overline{\mu+\alpha}}, \tag{1.65}$$

for any $t \in \mathbb{N}_{t_0}$.

Theorem 1.17 *[113] Consider the function $f : \mathbb{N}_{t_0} \to \mathbb{R}$ be given, and suppose $\mu, \nu > 0$. Then:*

$$\nabla_{t_0}^{-\nu} [\nabla_{t_0}^{-\mu} f(t)] = \nabla_{t_0}^{-(\mu+\nu)} f(t) = \nabla_{t_0}^{-\mu} [\nabla_{t_0}^{-\nu} f(t)], \tag{1.66}$$

$\forall t \in \mathbb{N}_{t_0}$.

Theorem 1.18 *[113] Given a function $f : \mathbb{N}_{t_0} \to \mathbb{R}$, for any $\alpha > 0$ and any positive integer p, it follows that:*

$$\nabla_{t_0+p-1}^{-\alpha} \nabla^p f(t) = \nabla^p \nabla_{t_0+p-1}^{-\alpha} f(t) - \sum_{i=0}^{p-1} \frac{(t-t_0-p+1)^{\overline{\alpha-p+i}}}{\Gamma(\alpha+i-p+1)} \nabla^i f(t_0+p-1). \tag{1.67}$$

1.6.5 The Fractional Nabla Rimann Liouville Difference Operator

The fractional nabla Riemann-Liouville difference operator finds applications in various fields such as signal processing, physics, finance, and engineering, particularly in modeling systems with memory effects or long-range dependence.

Definition 1.15 *[105] When $\alpha > 0$, the α-order Riemann-Liouville fractional nabla differences of a function f defined on $\mathbb{N}_{t_0}$ are defined as follows:*

$$\nabla_{t_0}^{\alpha} f(t) := \nabla^p \nabla_{t_0}^{-(p-\alpha)} f(t) = \frac{1}{\Gamma(p-\alpha)} \nabla^p \sum_{i=t_0+1}^{t} (t-i+1)^{\overline{p-\alpha-1}} f(i), \quad \forall t \in \mathbb{N}_{t_0+p}, \tag{1.68}$$

where $p = [\alpha] + 1$.

Theorem 1.19 *[105] Assuming $\alpha > 0$ and f is defined on $\mathbb{N}_{t_0}$, we have:*

$$\nabla_{t_0}^{\alpha} \nabla_{t_0}^{-\alpha} f(t) = f(t), \quad \forall t \in \mathbb{N}_{t_0}, \tag{1.69}$$

where $p = [\alpha] + 1$.

Theorem 1.20 *[113] Given $\alpha > 0$ and f defined on $\mathbb{N}_{t_0}$, we have:*

$$\nabla_{t_0+p-1}^{-\alpha} \nabla_{t_0}^{\alpha} f(t) = f(t) - \sum_{i=0}^{p-1} \frac{(t-t_0-p+1)^{\overline{\alpha-p+i}}}{\Gamma(\alpha+i-p+1)} \nabla^{\alpha+i-p} f(t_0+p-1). \tag{1.70}$$

where $p = [\alpha] + 1$.

Remark 1.10 *We observe that*

$$\nabla_{t_0}^{-\alpha} \nabla_{t_0}^{\alpha} f(t) \neq f(t) \tag{1.71}$$

in general.

1.6.6 The Fractional Nabla Caputo Difference Operator

The Fractional Nabla Caputo Difference Operator is another important operator in fractional calculus, similar to the Fractional Nabla Riemann-Liouville Difference Operator but with a different definition. Compared to the Riemann-Liouville difference operator, the Caputo difference operator is more suitable for initial value problems because it eliminates the singular behavior at the initial point.

Definition 1.16 *[105] For $\alpha > 0$ where α is not a natural number, the α-order Caputo fractional difference of a function f defined on $\mathbb{N}_{t_0}$ is defined as follows:*

$$ {}^{C}\nabla_{t_0}^{\alpha} f(t) := \nabla_{t_0}^{-(p-\alpha)} \nabla^{p} f(t) = \frac{1}{\Gamma(p-\alpha)} \sum_{i=t_0+1}^{t} (t-i+1)^{\overline{p-\alpha-1}} \nabla^{p} f(i), \quad \forall t \in \mathbb{N}_{t_0+p}, \tag{1.72} $$

where $p = [\alpha] + 1$.

Remark 1.11 *The fractional Caputo difference of a constant $c \in \mathbb{R}$ is simply 0, as it remains constant regardless of the fractional difference operation:*

$$ {}^{C}\nabla_{t_0}^{\alpha} c = \nabla_{t_0}^{-(p-\alpha)} \nabla^{p} c = 0. \tag{1.73} $$

Theorem 1.21 *[113] Assuming $\alpha > 0$ and f is defined on $\mathbb{N}_{t_0}$, then:*

$$ \nabla_{t_0}^{-\alpha}\ {}^{C}\nabla_{t_0}^{\alpha}\ f(t) = f(t) - \sum_{i=0}^{p-1} \frac{(t-t_0)^{\overline{i}}}{i!} \nabla^{i} f(t_0), \quad \forall t \in \mathbb{N}_{t_0}, \tag{1.74} $$

where $p = [\alpha] + 1$.

Theorem 1.22 *[113] Let $\alpha > 0$, and f is defined on $\mathbb{N}_{t_0}$. Then:*

$$ {}^{C}\nabla_{t_0}^{\alpha}\ f(t) = \nabla_{t_0+p}^{\alpha} f(t) - \sum_{i=0}^{p-1} \frac{(t-t_0-p+1)^{\overline{i-\alpha}}}{\Gamma(i-\alpha+1)} \nabla^{i} f(t_0+p-1), \ \ \forall t \in \mathbb{N}_{t_0+p-\alpha}, \tag{1.75} $$

where $p = [\alpha] + 1$.

1.7 Fractional h-Difference Operators

Fractional h-difference operators are a generalization of the standard difference operators, such as the Delta and Nabla differences, to fractional orders. These operators are used in various fields including physics, engineering, and signal processing to model and analyze systems or phenomena that exhibit non-integer order behavior.

Fractional h-difference operators have been extensively studied due to their applications in modeling various phenomena exhibiting fractal or non-integer order behavior. They are used in fractional calculus, time series analysis, numerical methods for solving fractional differential equations, and in modeling anomalous diffusion processes, among other applications.

1.7.1 The Fractional h-Delta Operators

Definition 1.17 *[106] Let $\alpha > 0$. We define the α-th fractional h-Delta sum of a function $f : (h\mathbb{N})_{t_0} \to \mathbb{R}$ as follows:*

$$_{t_0}\Delta_h^{-\alpha} f(t) := \frac{h}{\Gamma(\alpha)} \sum_{i=\frac{t_0}{h}}^{\frac{t}{h}-\alpha} (t - ih - h)_h^{(\alpha-1)} f(ih), \tag{1.76}$$

$\forall t \in (h\mathbb{N})_{t_0+\alpha h}$.

Remark 1.12 *It's worth noting that $_{t_0}\Delta_h^{-\alpha}$ extends functions from $(h\mathbb{N})_{t_0}$ to $(h\mathbb{N})_{t_0+\alpha h}$.*

Definition 1.18 *[106] For $\alpha > 0$, the α-order Riemann-Liouville fractional Delta h-difference of a function f defined on $(h\mathbb{N})_{t_0}$ is defined as follows:*

$$_{t_0}\Delta_h^{\alpha} f(t) := \Delta_h^p \, {_{t_0}\Delta_h^{-(p-\alpha)}} f(t) = \frac{h}{\Gamma(p-\alpha)} \Delta_h^p \sum_{i=\frac{t_0}{h}}^{\frac{t}{h}-(p-\alpha)} (t-ih-h)_h^{(p-\alpha-1)} f(ih), \quad \forall t \in (h\mathbb{N})_{t_0+ph-\alpha h}, \tag{1.77}$$

where $p = [\alpha] + 1$.

Definition 1.19 *[114] For $\alpha > 0$ where α is not a natural number, the α-order Caputo fractional Delta h-difference of a function f defined on $(h\mathbb{N})_{t_0}$ is given by:*

$$^{C}_{t_0}\Delta_h^{\alpha} f(t) :={}_{t_0} \Delta_h^{-(p-\alpha)} \Delta_h^p f(t) = \frac{h}{\Gamma(p-\alpha)} \sum_{i=\frac{t_0}{h}}^{\frac{t}{h}-(p-\alpha)} (t-ih-h)_h^{(p-\alpha-1)} \Delta_h^p f(ih), \quad \forall t \in (h\mathbb{N})_{t_0+ph-\alpha h}, \tag{1.78}$$

where $p = [\alpha] + 1$.

1.7.2 The Fractional h-Nabla Operators

Definition 1.20 *[107] Let $\alpha > 0$, the Nabla fractional h-Nabla sum of the function $f : (h\mathbb{N})_{t_0} \to \mathbb{R}$ characterized by:*

$$_{t_0}\nabla_h^{-\alpha} f(t) := \frac{h}{\Gamma(\alpha)} \sum_{i=\frac{t_0}{h}+1}^{\frac{t}{h}} (t - ih + h)_h^{\overline{\alpha-1}} f(ih), \tag{1.79}$$

$\forall t \in (h\mathbb{N})_{t_0}$.

Definition 1.21 *[107] When $\alpha > 0$, the α-order Riemann-Liouville fractional nabla h-differences of a function f defined on $(h\mathbb{N})_{t_0}$ are defined as follows:*

$$\nabla_{t_0}^{\alpha} f(t) := \nabla_h^p \, {}_{t_0}\nabla_h^{-(p-\alpha)} f(t) = \frac{h}{\Gamma(p-\alpha)} \nabla_h^p \sum_{i=\frac{t_0}{h}+1}^{\frac{t}{h}} (t - ih + h)_h^{\overline{p-\alpha-1}} f(ih), \quad \forall t \in (h\mathbb{N})_{t_0+ph}\,, \tag{1.80}$$

where $p = [\alpha] + 1$.

Definition 1.22 *[107] For* $\alpha > 0$ *where* α *is not a natural number, the* α*-order Caputo fractional h-difference of a function* f *defined on* $\mathbb{N}_{t_0}$ *is defined as follows:*

$$^{C}\nabla_{t_0}^{\alpha} f(t) :=_{t_0} \nabla_h^{-(p-\alpha)} \nabla_h^p f(t) = \frac{h}{\Gamma(p-\alpha)} \sum_{i=t_0+1}^{t} (t - ih + h)_h^{\overline{p-\alpha-1}} \nabla_h^p f(ih), \quad \forall t \in (h\mathbb{N})_{t_0+ph}\,, \tag{1.81}$$

where $p = [\alpha] + 1$.

Chapter 2

On the Stability of Discrete Fractional Systems

2.1 Background on Stability of Integer Order Discrete Systems

Stability analysis of integer-order difference equations involves examining the behavior of solutions over time to determine whether they converge or diverge. Unlike differential equations, which involve derivatives with respect to continuous time, a difference equation deals with discrete time steps. The stability of a solution in this context is crucial for understanding the long-term behavior of a dynamic system.

Difference equations typically depict the progression of specific phenomena over time intervals. For instance, in scenarios involving discrete generations within a population, the size of the $(t+1)$th generation, denoted as $x(t+1)$, is determined by a function of the nth generation, $x(t)$. This dynamic relationship is formalized through a difference equation

$$x\left(t+1\right)=f\left(x\left(t\right)\right). \tag{2.1}$$

Alternatively, we can approach this issue from a different perspective. By commencing from an initial point x_0, we have the ability to generate a sequence:

$$x_0, f\left(x_0\right), f\left(f\left(x_0\right)\right), f\left(f\left(f\left(x_0\right)\right)\right), \cdots. \tag{2.2}$$

The concept of equilibrium points, or states, holds significant importance in analyzing the dynamics of various physical systems. Across diverse fields such as biology, economics, physics, and engineering, there's a common desire for all solutions of a system to converge towards its equilibrium state. This pursuit of stability constitutes a fundamental aspect of stability theory, which is of paramount importance to scientists and engineers alike. Now, let's delve into the formal definition of an equilibrium point.

Definition 2.1 (Equilibrium Points) *[108] A point x^* in the domain of function f is considered an equilibrium point of equation (2.1) if it satisfies the condition $f(x^*) = x^*$, indicating that x^* is a fixed point of f.*

2.1.1 Notions of Stability

Let us consider the difference system

$$x\left(t+1\right) = f\left(t, \left(x\left(t\right)\right)\right), \tag{2.3}$$

where $x\left(t\right) \in \mathbb{R}^n, f : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$. We assume that $f(n, x)$ is continuous in x. Recall that (2.3) is said to be autonomous or time-invariant if the variable t does not appear explicitly in the right-hand side of the equation

$$f(t, x\left(t\right)) \equiv f(x\left(t\right)), \tag{2.4}$$

A point x^* in $\mathbb{R}^n$ is called an equilibrium point of (2.3) if $f(t, x^*) = x^*$, for all $t \geq t_0$. In most of the literature x^* is assumed to be the origin 0 and is called the zero solution. The justification for this assumption is as follows:

Let $y(t) = x(t) - x^*$. Then (2.3) becomes

$$y(t+1) = f(t, y(t) + x^*) - x^* = g(t, y(t)). \tag{2.5}$$

Notice that $y = 0$ corresponds to $x = x^*$. Since in many cases it is not convenient to make this change of coordinates, we will not assume that $x^* = 0$ unless it is more convenient to do so.

Definition 2.2 *[108] A norm, denoted by $\|.\|$, is a real-valued function defined on a vector space V, satisfying the following properties:*
(i) $\|x\| \geq 0$ and $\|x\| = 0$ only if $x = 0$.
(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for all $x \in V$ and scalars α.
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Definition 2.3 *[115] The equilibrium point x^* of (2.3) is said to be:*
• stable if, for each $\epsilon > 0$ and $t_0 \in \mathbb{N}_a$ there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that for any solution $x(t) = x(t, t_0, x_0)$, with $\|x_0 - x^*\| < \delta$ one has $\|x(t, t_0, x_0) - x^*\| < \epsilon$, for all $t \in \mathbb{N}_{t_0} \subseteq \mathbb{N}_a$,
• uniformly stable if it is stable and δ depends solely on ϵ,
Attracting if there exists $\delta = \delta(t_0)$ such that $\|x_0 - x^*\| < \delta$ implies $\lim\limits_{t\to\infty} \to x\left(t, t_0, x_0\right) = x^*$,

• *uniformly attracting if the choice of δ is independent of t_0. The condition for uniform attractivity may be paraphrased by saying that there exists $\delta > 0$ such that for every ϵ and t_0 there exists $N = N(\epsilon)$ independent of t_0 such that $\|x(t, t_0, x_0) - x^*\| < \epsilon$ for all $t \geq t_0 + N$ whenever $\|x_0 - x^*\| < \delta$.*

• *asymptotically stable if it is stable and uniformly attracting*

• *uniformly asymptotically stable if it is uniformly stable and uniformly attracting.*

• *Exponentially stable if there exist $\delta > 0, M > 0$ and $\varrho \in (0, 1)$ such that $\|x(t, t_0, x_0) - x^*\| < M\varrho^{t-t_0}$, whenever $\|x_0 - x^*\| < \delta$*

• *globally asymptotically stable if it is asymptotically stable for all $x_0 \in \mathbb{R}^n$,*

• *globally uniformly asymptotically stable if it is uniformly asymptotically stable for all $x_0 \in \mathbb{R}^n$.*

2.1.2 Stability of Linear Systems

In this subsection we specialize the results of the previous section to autonomous (time-invariant) systems of the form

$$x(t+1) = Ax(t), t \geq t_0 \geq 0, \tag{2.6}$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in \mathbb{R}$, and $A = (a_{ij})$ is a $n \times n$ real matrix. Here T indicates the transpose of a vector

In the next theorem we summarize the main stability results for the linear autonomous systems (2.6).

Theorem 2.1 *[108] The following statements hold:*
The zero solution of (2.6) is stable if and only if $\rho(A) \leq 1$ and the eigenvalues of unit modulus are semisimple.
The zero solution of (2.6) is asymptotically stable if and only if $\rho(A) < 1$.

2.1.3 Stability of non-Linear Systems

Stability by Linear Approximation

The linearization method, pioneered by mathematicians Liapunov and Perron, is a crucial tool in stability theory. Widely utilized in control system design, engineers and scientists favor this method for analyzing and designing feedback devices. Here, we specifically employ Perron's approach to study the stability of nonlinear systems of difference equations.

$$x(t+1) = Jx(t) + g(x(t)), \tag{2.7}$$

where $J = f'(0)$ is the Jacobian matrix of f at 0, and $g(x) = f(x) - Jx$. Since f is differentiable at 0, it follows that $g(x) = o(x)$ as $\|x\| \to 0$. Equivalently,

$$\lim_{\|x\|\to 0} \frac{\|g(x)\|}{\|x\|} = 0 \tag{2.8}$$

Theorem 2.2 *[108] Assume that $g(x) = o(x)$ uniformly as $\|x\| \to 0$. If the zero solution of the linear system (2.6) is asymptotically stable, then the zero solution of the nonlinear system (2.7) is exponentially stable.*

Corollary 2.1 *[108] If $\rho(J) < 1$, then the zero solution of (2.7) is exponentially stable.*

Corollary 2.2 *[108] If $\|f'(0)\| < 1$, then the zero solution of (2.7) is exponentially stable.*

Theorem 2.3 *[108] **(Hartman–Grobman)** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ a C^r-diffeomorphism with hyperbolic fixed point x^*. Then there exists neigh-borhoods V of x^* and W of 0 and a homeomorphism $h : W \to V$ such that $f(h(x)) = h(Ax)$, where $A = Df(x^*)$.*
In other words, f is topologically conjugate in a neighborhood of the fixed point x^* to the linear map induced by the derivative at the fixed point.

Remark 2.1 *The fixed point x^* is assumed to be hyperbolic, where none of the eigenvalues of A lie on the unit circle.*

Liapunov's Direct, or Second, Method

In 1892, A.M. Liapunov introduced the influential Liapunov's direct method for studying the stability of nonlinear differential equations. This method, a cornerstone in stability theory, enables the qualitative analysis of solutions without e$x$plicitly determining them. It relies on identifying specific real-valued functions, named after Liapunov, though a challenge remains in selecting the appropriate Liapunov function for a given equation.

Definition 2.4 *[108] The function $V : \mathbb{R}^n \to \mathbb{R}^+$ is said to be a Liapunov function on a subset H of $\mathbb{R}^n$ if:*
V is continuous on H, and $\Delta V(x) \leq 0$, whenever x and $f(x)$ belong to H.

Theorem 2.4 *[108] If V is a Liapunov function for (2.7) in a neighborhood H of the equilibrium point x^*, and V is positive definite with respect to x^*, then x^* is stable. If, in addition, $\Delta V(x) \leq 0$ whenever $x, f(x) \in H$ and $x \neq x^*$, then x^* is asymptotically stable. Moreover, if $G = H = \mathbb{R}^n$ and $V(x) \to \infty$ as $\|x\| \to \infty$, then x^* is globally asymptotically stable.*

Based on the above, in this section we will formulate the stability theorems in the case of difference (Delta difference, Nabla difference, Delta h-difference, Nabla h-difference)

2.1.4 Stability Analysis of Delta Difference Systems

Stability of Linear Systems

Consider the integer order difference system:

$$\begin{cases} \Delta x(t) = Ax(t), & t \in \mathbb{N}, \\ x(0) = x_0, & x_0 \in \mathbb{R}^n, \end{cases} \tag{2.9}$$

where $x(t) \in \mathbb{R}^n$ and A is an $n \times n$ constant matrix.
It has an equilibrium point at the origin $(x = 0)$. The solution of the linear system (2.9) starting from x_0 has the form

$$x(t) = (A + I_n)^t x_0, \quad \forall t \in \mathbb{N}, \tag{2.10}$$

where I_n is the identity matrix. We have the following result on the stability of linear system (2.9).

Theorem 2.5 *Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A.*
If all the eigenvalues λ_i of A satisfies $|\lambda_i + 1| < 1$, $1 \leq i \leq n$, then the trivial solution of (2.10) is globally asymptotically stable on $\mathbb{N}$.
If $|\lambda_i + 1| \leq 1$, $1 \leq i \leq n$, and whenever $|\lambda_i + 1| = 1$, then λ_i is a simple eigenvalue of A. Then the trivial solution of (2.9) is stable on $\mathbb{N}$.
Furthermore, if there is an eigenvalue λ of A withe $|\lambda + 1| > 1$, then the trivial solution of (2.9) is unstable on $\mathbb{N}$.

Proof. By writing the system in explicit form, we find:

$$\begin{cases} x(t+1) = (A + I_n)\, x(t), & t \in \mathbb{N}, \\ x(0) = x_0, & x_0 \in \mathbb{R}^n, \end{cases} \tag{2.11}$$

By applying the Theorem 2.1 to the system (2.11) noting that the eigenvalues of $(A + I_n)$ are of the form $\lambda + 1$ where λ is an eigenvalue of A we find:
The zero solution is asymptotically stable if and only if all the eigenvalues λ_i of A satisfies $|\lambda_i + 1| < 1$, $1 \leq i \leq n$
The zero solution stable if and only if and whenever $|\lambda_i + 1| = 1$, then λ_i is a simple eigenvalue of A.
If there is an eigenvalue λ of A withe $|\lambda + 1| > 1$, then the zero solution is unstable on $\mathbb{N}$.

■

Remark 2.2 *Assume* $|\lambda_i + 1| \leq 1, 1 \leq i \leq n$. *If there is a non simple eigenvalue* λ *of* A *satisfying* $|\lambda + 1| = 1$, *then we can't conclude.*

Example 2.1 *Consider the following linear system:*

$$\begin{cases} \Delta x_1(t) = 0.2x_1(t) - 4x_2(t), \\ \Delta x_2(t) = 0.3x_1(t) - 3x_2(t), \end{cases} \quad t \in \mathbb{N}. \tag{2.12}$$

The eigenvalues of $A = \begin{pmatrix} 0.2 & -0.8 \\ 0.3 & -1 \end{pmatrix}$ *are* $\lambda_1 = -5.3590 \times 10^{-2}$ *and* $\lambda_2 = -0.74641$ *Since*

$$\begin{array}{l} |\lambda_1 + 1| = 0.94641 < 1 \\ |\lambda_2 + 1| = 0.25359 < 1 \end{array},$$

then, by Theorem 2.5 the trivial solution of (2.12) is globally asymptotically stable on $\mathbb{N}$.

Example 2.2 *Consider the following system:*

$$\begin{cases} \Delta x_1(t) = x_2(t), \\ \Delta x_2(t) = 0.3x_1(t) - 2x_2(t), \end{cases} \quad t \in \mathbb{N}. \tag{2.13}$$

The eigenvalues are $\lambda_1 = 0, \lambda_2 = -2$. *Since*

$$\begin{array}{l} |\lambda_1 + 1| = 1 \\ |\lambda_2 + 1| = 1 \end{array},$$

and λ_1 *and* λ_2 *are simple eigenvalues then, by Theorem 2.5 the zero solution of (2.13) is stable*

Stability of Non-Linear Systems by Linearisation Method

Consider the following non-linear system:

$$\begin{cases} \Delta x(t) = f(x(t)), \quad t \in \mathbb{N}, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.14}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ a continuously differentiable function, and suppose $f(0) = 0$, that is $x = 0$ is an equilibrium point for system (2.14).

Theorem 2.6 *Let* J *be the Jacobian matrix of* f *at* 0. *If all the eigenvalues* $\lambda_i, 1 \leq i \leq n$, *of* J *satisfies* $|\lambda_i + 1| < 1$, *then the trivial solution of (2.14) is asymptotically (exponentially) stable on* $\mathbb{N}$.

Proof. By writing the system in explicit form, we find:

$$\begin{cases} x(t+1) = f(x(t)) + x(t), \quad t \in \mathbb{N}, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.15}$$

By applying the Theorem 2.2 to the system (2.15) noting that the eigenvalues of $(J + I_n)$ are of the form $\lambda + 1$ where λ is an eigenvalue of J we find: The zero solution is asymptotically stable if and only if all the eigenvalues λ_i of J satisfies $|\lambda_i + 1| < 1, 1 \leq i \leq n$.

■

Example 2.3 *Consider the following non-linear system:*

$$\begin{cases} \Delta x_1(t) = 0.2x_1(t)e^{x_2(t)} - 4\sin\left(x_2(t)\right), \\ \Delta x_2(t) = \cos\left(x_1(t)\right) - 4x_2(t) - 1, \end{cases} t \in \mathbb{N}. \tag{2.16}$$

Note that 0 *is a fixed point for the system. Let* $f = (f_1, f_2)^T$, *where* $f_1 = 0.2x_1(t)e^{x_2(t)} - 4\sin\left(x_2(t)\right)$ *and* $f_2 = \frac{x_1(k)}{(1+x_2^2(k))} - x_2(k)$, *then the Jacobian matrix is given by*

$$J = \begin{pmatrix} \frac{\partial f_1(0)}{\partial x_1} & \frac{\partial f_1(0)}{\partial x_2} \\ \frac{\partial f_2(0)}{\partial x_1} & \frac{\partial f_2(0)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -0.2 & -4 \\ 0 & -1.5 \end{pmatrix},$$

hence the eigenvalues of J *are* $\lambda_1 = -0.2, \lambda_1 = -1.5$. *Since*

$$\begin{array}{c} |\lambda_1 + 1| = 0.8 < 1 \\ |\lambda_2 + 1| = 0.5 < 1 \end{array},$$

then, by Theorem 2.6 the trivial solution of (2.16) is asymptotically stable.

2.1.5 Stability Analysis of Nabla Difference Systems

Stability of Linear Systems

Consider the integer order difference system:

$$\begin{cases} \nabla x(t) = Ax(t), \quad t \in \mathbb{N}_1, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.17}$$

where $x(k) \in \mathbb{R}^n$ and A is an $n \times n$ constant matrix. It has an equilibrium point at the origin $(x = 0)$. If $\det\left(I_n - A\right) \neq 0$, then the solution of the linear system (2.17) starting from x_0 existe and has the form

$$x(t) = (I_n - A)^{-t}x_0, \quad \forall t \in \mathbb{N}_1, \tag{2.18}$$

We have the following result on the stability of linear system (2.17).

Theorem 2.7 *Let* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *be the eigenvalues of* A.
If all the eigenvalues λ_i *of* A *satisfies* $|\lambda_i - 1| < 1, 1 \leq i \leq n$, *then the trivial solution of (2.17) is*

globally asymptotically stable on $\mathbb{N}$.
If $|\lambda_i - 1| \leq 1$, $1 \leq i \leq n$, *and whenever* $|\lambda_i - 1| = 1$, *then* λ_i *is a simple eigenvalue of* A. *Then the trivial solution of (2.17) is stable on* $\mathbb{N}$.
Furthermore, if there is an eigenvalue λ *of* A *withe* $|\lambda - 1| > 1$, *then the trivial solution of (2.17) is unstable on* $\mathbb{N}$.

Proof. By writing the system in explicit form, we find:

$$\begin{cases} x(t+1) = (I_n - A)^{-1} x(t), & t \in \mathbb{N}, \\ x(0) = x_0, & x_0 \in \mathbb{R}^n, \end{cases} \tag{2.19}$$

By applying the Theorem 2.1 to the system (2.19) noting that the eigenvalues of $(I_n - A)^{-1}$ are of the form $(1-\lambda)^{-1}$ where λ is an eigenvalue of A we find:
The zero solution is asymptotically stable if and only if all the eigenvalues λ_i of A satisfies $|\lambda_i - 1| > 1$, $1 \leq i \leq n$.
The zero solution stable if and only if and whenever $|\lambda_i - 1| = 1$, then λ_i is a simple eigenvalue of A.
If there is an eigenvalue λ of A withe $|\lambda - 1| < 1$, then the zero solution is unstable on $\mathbb{N}$.
■

Remark 2.3 *Assume* $|\lambda_i - 1| \geq 1$, $1 \leq i \leq n$. *If there is a non simple eigenvalue* λ *of* A *satisfying* $|\lambda - 1| = 1$, *then we can't conclude.*

Example 2.4 *Consider the following linear system:*

$$\begin{cases} \nabla x_1(t) = 3x_1(t) - 2x_2(t), \\ \nabla x_2(t) = x_1(t) - 3x_2(t), \end{cases} t \in \mathbb{N}. \tag{2.20}$$

The eigenvalues of $A = \begin{pmatrix} 3 & -2 \\ 1 & -3 \end{pmatrix}$ *are* $\lambda_1 = -\sqrt{7}$ *and* $\lambda_2 = \sqrt{7}$ *Since*

$$\begin{array}{l} |\lambda_1 - 1| = 3.6458 > 1 \\ |\lambda_2 - 1| = 1.6458 > 1 \end{array},$$

then, by Theorem 2.7 the trivial solution of (2.20) is globally asymptotically stable on $\mathbb{N}$.

Example 2.5 *Consider the following system:*

$$\begin{cases} \nabla x_1(t) = -0.3913x_1(t) - 0.3936x_2(t), \\ \nabla x_2(t) = 2.3773x_1(t) - 2.3913x_2(t), \end{cases} t \in \mathbb{N}. \tag{2.21}$$

The eigenvalues are $\lambda_1 = 0, \lambda_2 = 2$. *Since*

$$\begin{array}{c} |\lambda_1 - 1| = 1 \\ |\lambda_2 - 1| = 1 \end{array},$$

and λ_1 *and* λ_2 *are simple eigenvalues then, by Theorem 2.7 the zero solution of (2.21) is stable.*

Stability of non-Linear Systems by Linearisation Method

Consider the following non-linear system:

$$\begin{cases} \nabla x(t) = f(x(t)), \quad k \in \mathbb{N}_1, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.22}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ a continuously differentiable function, and suppose $f(0) = 0$, that is $x = 0$ is an equilibrium point for system (2.22).

Theorem 2.8 *Let* J *be the Jacobian matrix of* f *at* 0. *If all the eigenvalues* $\lambda_i, 1 \leq i \leq n$, *of* J *satisfies* $|\lambda_i - 1| > 1$, *then the trivial solution of (2.22) is locally asymptotically stable on* $\mathbb{N}$.

Proof. By writing the system in explicit form, we find:

$$\begin{cases} x(t+1) = f(x(t+1)) + x(t), \quad t \in \mathbb{N}, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.23}$$

According to Hartman–Grobman theorem if $|\lambda_i - 1| > 1$ (0 is hyperbolic fixed point) the system (2.23) equivalent to

$$\begin{cases} x(t+1) = Jx(t+1) + x(t), \quad t \in \mathbb{N}, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases}$$

in a neighborhood of the fixed point 0. So if $|\lambda_i - 1| > 1$, then the zero solution is locally asymptotically stable. ■

Example 2.6 *Consider the following non-linear system:*

$$\begin{cases} \nabla x_1(t) = 2\sin\left(x_2(t)\right), \\ \nabla x_2(t) = x_1(t) + x_1(t)x_2(k). \end{cases} \tag{2.24}$$

Let $f = (f_1, f_2)^T$, *where* $f_1 = \sin\left(x_1(t)\right) - x_2(t)$, *and* $f_2 = x_1(t) + x_1(t)x_2(k) - x_2(t)$, *then the Jacobian matrix is given by*

$$J = \begin{pmatrix} \frac{\partial f_1(0)}{\partial x_1} & \frac{\partial f_1(0)}{\partial x_2} \\ \frac{\partial f_2(0)}{\partial x_1} & \frac{\partial f_2(0)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix},$$

hence the eigenvalues of J are $\lambda_{1,2} =^{+}_{-} 2\sqrt{2}$. Since

$$\begin{aligned} |\lambda_1 - 1| = 2\sqrt{2} - 1 > 1 \\ |\lambda_2 - 1| = 2\sqrt{2} + 1 > 1 \end{aligned},$$

then, by Theorem 2.8 the trivial solution of (2.24) is unstable on $\mathbb{N}$.

2.1.6 Stability Analysis of Delta h-difference Systems

Stability of Linear Systems

Consider the integer order difference system:

$$\begin{cases} \Delta_h x(t) = Ax(t), \quad t \in \mathbb{N}, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.25}$$

where $x(t) \in \mathbb{R}^n$ and A is an $n \times n$ constant matrix.
It has an equilibrium point at the origin $(x = 0)$. The solution of the linear system (2.25) starting from x_0 has the form

$$x(t) = (hA + I_n)^t x_0, \quad \forall t \in \mathbb{N}, \tag{2.26}$$

where I_n is the identity matrix. We have the following result on the stability of linear system (2.26).

Theorem 2.9 *Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A.*
If all the eigenvalues λ_i of A satisfies $|h\lambda_i + 1| < 1$, $1 \le i \le n$, then the trivial solution of (2.25) is globally asymptotically stable on $\mathbb{N}$.
If $|h\lambda_i + 1| \le 1$, $1 \le i \le n$, and whenever $|h\lambda_i + 1| = 1$, then λ_i is a simple eigenvalue of A. Then the trivial solution of (2.25) is stable on $\mathbb{N}$.
Furthermore, if there is an eigenvalue λ of A withe $|h\lambda + 1| > 1$, then the trivial solution of (2.25) is unstable on $\mathbb{N}$.

Proof. The proof of this Theorem is exactly the same as the proof of the Theorem 2.5. ■

Example 2.7 *Consider the following linear system:*

$$\begin{cases} \Delta_{0.2} x_1(t) = -0.9688 x_1(t) - 0.2247 x_2(t), \\ \Delta_{0.2} x_2(t) = 0.9195 x_1(t) + 0.0688 x_2(t), \end{cases} \quad t \in \mathbb{N}. \tag{2.27}$$

The eigenvalues of $A = \begin{pmatrix} -0.9688 & -0.2247 \\ 0.9195 & 0.0688 \end{pmatrix}$ are $\lambda_1 = -0.199\,92$ and $\lambda_2 = -0.700\,08$ Since

$$\begin{aligned} |h\lambda_1 + 1| = 0.960\,02 < 1 \\ |h\lambda_2 + 1| = 0.859\,98 < 1 \end{aligned},$$

then, by Theorem 2.9 the trivial solution of (2.27) is globally asymptotically stable on $\mathbb{N}$.

Example 2.8 *Consider the following system:*

$$\begin{cases} \Delta_{0.3}x_1(t) = -15.0770x_1(t) - 15.7975x_2(t), \\ \Delta x_{0.3}(t) = 8.0268x_1(t) + 8.4104x_2(t), \end{cases} \quad t \in \mathbb{N}. \tag{2.28}$$

The eigenvalues are $\lambda_1 = -6.6667, \lambda_2 = 0$ *Since*

$$\begin{array}{c} |h\lambda_1 + 1| = 1 \\ |h\lambda_2 + 1| = 1 \end{array},$$

and λ_1 *and* λ_2 *are simple eigenvalues then, by Theorem 2.9 the zero solution of (2.28) is stable*

Stability of non-Linear Systems by Linearisation Method

Consider the following non-linear system:

$$\begin{cases} \Delta_h x(t) = f(x(t)), \quad t \in \mathbb{N}, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.29}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ a continuously differentiable function, and suppose $f(0) = 0$, that is $x = 0$ is an equilibrium point for system (2.29).

Theorem 2.10 *Let* J *be the Jacobian matrix of* f *at* 0. *If all the eigenvalues* $\lambda_i, 1 \leq i \leq n$, *of* J *satisfies* $|h\lambda_i + 1| < 1$, *then the trivial solution of (2.29) is asymptotically (exponentially) stable on* $\mathbb{N}$.

Proof. The proof of this Theorem is exactly the same as the proof of the Theorem 2.6 ■

Example 2.9 *Consider the following non-linear system:*

$$\begin{cases} \Delta_{0.1}x_1(t) = -0.0447x_1(t) - x_1(t)x_2(t) + 0.2124x_2(t), \\ \Delta_{0.1}x_2(t) = \sin\left(-1.4298x_1(t) - 2.1553x_2(t)\right) \end{cases} \quad t \in \mathbb{N}. \tag{2.30}$$

Note that 0 *is a fixed point for the system. Let* $f = (f_1, f_2)^T$, *where* $f_1 = -0.0447x_1(t) - x_1(t)x_2(t) + 0.2124x_2(t)$ *and* $f_2 = \sin\left(-1.4298x_1(t) - 2.1553x_2(t)\right)$, *then the Jacobian matrix is given by*

$$J = \begin{pmatrix} \frac{\partial f_1(0)}{\partial x_1} & \frac{\partial f_1(0)}{\partial x_2} \\ \frac{\partial f_2(0)}{\partial x_1} & \frac{\partial f_2(0)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -0.0447 & 0.2124 \\ -14298 & -2.1553 \end{pmatrix},$$

hence the eigenvalues of J *are* $\lambda_1 = -0.2, \lambda_1 = -2$. *Since*

$$\begin{array}{c} |h\lambda_1 + 1| = 0.98 < 1 \\ |h\lambda_2 + 1| = 0.8 < 1 \end{array},$$

then, by Theorem 2.10 the trivial solution of (2.30) is asymptotically stable.

2.1.7 Stability Analysis of Nabla Difference Systems

Stability of Linear Systems

Consider the integer order h-difference system:

$$\begin{cases} \nabla_h x(t) = Ax(t), & t \in \mathbb{N}_1, \\ x(0) = x_0, & x_0 \in \mathbb{R}^n, \end{cases} \tag{2.31}$$

where $x(t) \in \mathbb{R}^n$ and A is an $n \times n$ constant matrix. It has an equilibrium point at the origin $(x = 0)$. If $\det(I_n - A) \neq 0$, then the solution of the linear system (2.31) starting from x_0 existe and has the form

$$x(t) = (I_n - hA)^{-t} x_0, \quad \forall t \in \mathbb{N}_1, \tag{2.32}$$

We have the following result on the stability of linear system (2.31).

Theorem 2.11 *Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of A.*
If all the eigenvalues λ_i of A satisfies $|h\lambda_i - 1| < 1$, $1 \le i \le n$, then the trivial solution of (2.31) is globally asymptotically stable on $\mathbb{N}$.
If $|\lambda_i - 1| \le 1$, $1 \le i \le n$, and whenever $|h\lambda_i - 1| = 1$, then λ_i is a simple eigenvalue of A. Then the trivial solution of (2.31) is stable on $\mathbb{N}$.
Furthermore, if there is an eigenvalue λ of A withe $|h\lambda - 1| > 1$, then the trivial solution of (2.31) is unstable on $\mathbb{N}$.

Proof. The proof of this Theorem is exactly the same as the proof of the Theorem 2.7 ■

Remark 2.4 *Assume $|h\lambda_i - 1| \ge 1$, $1 \le i \le n$. If there is a non simple eigenvalue λ of A satisfying $|h\lambda - 1| = 1$, then we can't conclude.*

Example 2.10 *Consider the following linear system:*

$$\begin{cases} \nabla_{0.8} x_1(t) = 4x_1(t) - 5x_2(t), \\ \nabla_{0.8} x_2(t) = 2x_1(t) - x_2(t), \end{cases} \quad t \in \mathbb{N}. \tag{2.33}$$

The eigenvalues of A are $\lambda_1 = \frac{1}{2}i\sqrt{15} + \frac{3}{2}$ and $\lambda_2 = \frac{3}{2} - \frac{1}{2}i\sqrt{15}$ Since

$$|h\lambda_1 - 1| = |h\lambda_2 - 1| = 1.562 > 1,$$

then, by Theorem 2.11 the trivial solution of (2.33) is globally asymptotically stable on $\mathbb{N}$.

Stability of non-Linear Systems by Linearisation Method

Consider the following non-linear system:

$$\begin{cases} \nabla_h x(t) = f(x(t)), \quad t \in \mathbb{N}_1, \\ x(0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.34}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ a continuously differentiable function, and suppose $f(0) = 0$, that is $x = 0$ is an equilibrium point for system (2.34).

Theorem 2.12 *Let J be the Jacobian matrix of f at 0. If all the eigenvalues $\lambda_i, 1 \leq i \leq n$, of J satisfies $|h\lambda_i - 1| > 1$, then the trivial solution of (2.34) is asymptotically stable on $\mathbb{N}$.*

Proof. The proof of this Theorem is e$x$actly the same as the proof of the Theorem 2.8 ■

Example 2.11 *Consider the following non-linear system:*

$$\begin{cases} \nabla_{0.7} x_1(t) = 1.89 \cos\left(x_1(t) + \frac{\pi}{2}\right), \\ \nabla_{0.7} x_2(t) = \sin\left(0.77 x_1(t)\right) - 2x_2\left(t\right) + x_1(t) e^{x_2(t)}. \end{cases} \tag{2.35}$$

Let $f = (f_1, f_2)^T$, where $f_1 = 1.89 \cos\left(x_1(t) + \frac{\pi}{2}\right)$, and $f_2 = \sin\left(0.77 x_1(t) + x_1(t) x_2(t)\right)$, then the Jacobian matrix is given by

$$J = \begin{pmatrix} \frac{\partial f_1(0)}{\partial x_1} & \frac{\partial f_1(0)}{\partial x_2} \\ \frac{\partial f_2(0)}{\partial x_1} & \frac{\partial f_2(0)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -1.89 & 0 \\ 0.77 & -2 \end{pmatrix},$$

hence the eigenvalues of J are $\lambda_1 = -1.89$ and $\lambda_1 = -2$. Since

$$\begin{array}{c} |h\lambda_1 - 1| = 2.323 > 1 \\ |h\lambda_2 - 1| = 2.4 > 1 \end{array},$$

then, by Theorem 2.12 the trivial solution of (2.35) is unstable on $\mathbb{N}$.

Remark 2.5 *Note that Lyapunov's direct method is valid for all previous systems.*

2.2 Stability Analysis of Discrete Fractional Systems with Commensurate Orders

There are many efforts devoted to studying the stability of discrete fractional systems. But in this section, we will mention the most important results that were widely spread and had many contributions to many studies.

2.2.1 Stability of Linear Systems

Consider the fractional order difference system:

$$\begin{cases} {}^{C}\Delta_{t_0}^{\alpha} x(t) = Ax(t+\alpha-1), \quad t \in \mathbb{N}_{t_0+1-\alpha}, \\ x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.36}$$

where $0 < \alpha \leq 1$, $t_0 \in \mathbb{R}$, and A is an $n \times n$ constant matri$x$.

we introduce the set:

$$S^{\alpha} = \left\{ z \in \mathbb{C} : |z| < \left(2\cos\frac{|\arg z| - \pi}{2-\alpha}\right)^{\alpha} \text{ and } |\arg z| > \frac{\alpha\pi}{2} \right\}. \tag{2.37}$$

Theorem 2.13 *[116] Let $\alpha \in (0,1)$ and A is an $n \times n$ constant matrix. If $\lambda \in S^{\alpha}$ for all the eigenvalues λ of A, then the trivial solution of (2.36) is asymptotically stable. In this case, the solutions of (2.37) decay towards zero algebraically (and not exponentially), more precisely*

$$\|x(t)\| = O(t^{-\alpha}) \quad \text{as } t \to \infty,$$

for any solution x of (2.36).

Furthermore, if $\lambda \in \mathbb{C}\backslash\ cl(S^{\alpha})$ for an eigenvalue λ of A, the zero solution of (2.36) is not stable.

Example 2.12 *Consider the following system:*

$$\begin{cases} {}^{C}\Delta_{t_0}^{\frac{2}{3}} x_1(t) = -\frac{3}{8}x_1(t) + -\frac{3}{8}x_2(t), \\ {}^{C}\Delta_{t_0}^{\frac{2}{3}} x_2(t) = -\frac{17}{27}x_2(t), \end{cases} t \in \mathbb{N}_{t_0+1-\frac{2}{3}}. \tag{2.38}$$

The characteristic equation for $\begin{pmatrix} -\frac{3}{8} & \frac{3}{2} \\ 0 & -\frac{17}{27} \end{pmatrix}$ *is* $(\lambda + \frac{3}{8})(\lambda + \frac{17}{27}) = 0$ *and hence the eigenvalues of A are $\lambda_1 = -\frac{3}{8}$ and $\lambda_2 = -\frac{17}{27}$. Since*

$$|\lambda_1| = \frac{3}{8} < \left(2\cos\frac{|\arg\lambda_1| - \pi}{2-\frac{2}{3}}\right)^{\frac{2}{3}} \simeq 0.6666666666, \quad \textit{and} \quad |\arg\lambda_1| = \pi > \frac{2\pi}{6} \Leftrightarrow \lambda_1 \in S^{\frac{2}{3}},$$

and

$$|\lambda_2| = \frac{17}{27} < \left(2\cos\frac{|\arg\lambda_2| - \pi}{2-\frac{2}{3}}\right)^{\frac{2}{3}} \simeq 0.6666666666, \quad \textit{and} \quad |\arg\lambda_2| = \pi > \frac{2\pi}{6} \Leftrightarrow \lambda_2 \in S^{\frac{2}{3}}.$$

Then, by Theorem 2.13, the trivial solution of (2.38) is asymptotically stable.

2.2.2 Stability of Non-Linear Systems via Liapunov's Direct Method

Concerned in studying their stability are written in the general form as follow:

$$\begin{cases} {}^{C}\Delta_{t_0}^{\alpha}x(t) = f(t+\alpha-1, x(t+\alpha-1)), & t \in \mathbb{N}_{t_0+1-\alpha}, \\ x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.39}$$

where $n \in \mathbb{N}_1, t_0 \in \mathbb{N}_{t_0}, x(t) \in \mathbb{R}^n$, $f : \mathbb{N}_{t_0} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and $0 < \alpha \leq 1$. A point $x^* \in \mathbb{R}^n$ is a fixed (equilibrium) point of (2.1) if and only if

We will be concerned with the stability of the equilibrium point $x^* = 0$.

Theorem 2.14 *[115] If there exists a positive definite and decrescent scalar function $V(t,x) \in C\left[\mathbb{N}_{t_0} \times S_\rho, \mathbb{R}_+\right]$ such that*

$${}^{C}\Delta_{t_0}^{\alpha}V(t, x(t)) \leq 0, \tag{2.40}$$

for all $t_0 \in \mathbb{N}_{t_0}$ and $(t,x) \in \mathbb{N} \times S_\rho$, then the trivial solution of (2.39) is uniformly stable.

Theorem 2.15 *[115] If there exists a positive definite and decrescent scalar function $V(t,x) \in C\left[\mathbb{N}_{t_0} \times S_\rho, \mathbb{R}_+\right]$ such that*

$${}^{C}\Delta_{t_0}^{\alpha}V(t, x(t)) \leq -\psi(\|x(t+\alpha-1\|), \quad \forall t_0 \in \mathbb{N}_{t_0},\ (t,x) \in \mathbb{N} \times S_\rho, \tag{2.41}$$

where $\psi \in K$, then the trivial solution of (2.39) is uniformly asymptotically stable.

Theorem 2.16 *[115] If there exists a function $V(t,x) \in C\left[\mathbb{N}_{t_0} \times \mathbb{R}^n, \mathbb{R}_+\right]$ such that*

$$g(\|x(t)\|) \leq V(t,x) \leq f(\|x(t)\|) \quad \forall (t,x) \in \mathbb{N}_{t_0} \times \mathbb{R}^n,$$

$${}^{C}\Delta_{t_0}^{\alpha}V(t, x(t)) \leq -\psi(\|x(t+\alpha-1)\|) \quad \forall t_0 \in \mathbb{N}_{t_0},\ (t,x) \in \mathbb{N} \times \mathbb{R}^n,$$

where f, g, and $\psi \in KR$ hold for all $(t,x) \in \mathbb{N}_{t_0} \times \mathbb{R}^n$, then the trivial solution of (2.39) is globally uniformly asymptotically stable.

Remark 2.6 *Theorem 2.16 gives a sufficient condition to analyze stability of (2.39). But it's not easy to construct g, f and ψ functions directly. Yet, so far there is no direct way to know the stability of system (2.39).*

2.3 Stability Analysis of Nabla h-difference Systems

2.3.1 Stability of Linear Systems

Consider the fractional order difference system:

$$\begin{cases} {}_{t_0}^{C}\nabla_h^{\alpha} x(t) = Ax(t), \quad t \in \mathbb{N}_{t_0+h}, \\ x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.42}$$

where $0 < \alpha, h < 1$, $t_0 \in \mathbb{R}$, and A is an $n \times n$ constant matrix.

we introduce the set:

$$\tilde{S}_h^{\alpha} = \left\{ z \in \mathbb{C} : |\arg z| > \frac{\alpha\pi}{2} \text{ or } |z| > \left(\frac{2}{h} \cos \frac{|\arg z|}{\alpha} \right)^{\alpha} \right\}. \tag{2.43}$$

Theorem 2.17 *[117] Let $\det(I - h^{\alpha}A) \neq 0$. Then (2.42) has a unique solution for any initial vector $x_0 \in \mathbb{R}^n$. Moreover, the zero solution of (2.42) is asymptotically stable if all the eigenvalues of A are located in $\tilde{S}_h^{\alpha}$.*

Example 2.13 *Consider the following system:*

$$\begin{cases} {}_{t_0}^{C}\nabla_1^{\frac{2}{3}} x_1(t) = -3x_1(t) + x_2(t) \\ {}_{t_0}^{C}\nabla_1^{\frac{2}{3}} x_2(t) = 3x_2(t) \end{cases}, \quad t \in \mathbb{N}_{t_0+1-\frac{2}{3}}. \tag{2.44}$$

The eigenvalues of $A = \begin{pmatrix} -3 & 1 \\ 0 & 3 \end{pmatrix}$ *are* $\lambda_1 = -3$ *and* $\lambda_2 = 3$. *Since*

$$|\lambda_1| = 3 \Leftrightarrow \lambda_1 \in S^{\frac{2}{3}},$$

and

$$|\lambda_2| = 3 \Leftrightarrow \lambda_2 \in S^{\frac{2}{3}}.$$

Then, by Theorem 2.17, the trivial solution of (2.44) is asymptotically stable.

2.3.2 Stability of Non-Linear Nabla h-difference Systems

Consider the fractional order difference system:

$$\begin{cases} {}_{t_0}^{C}\nabla_h^{\alpha} x(t) = f\left(x(t)\right), \quad t \in \mathbb{N}_{t_0+h}, \\ x(t_0) = x_0, \quad x_0 \in \mathbb{R}^n, \end{cases} \tag{2.45}$$

where $0 < \alpha, h < 1$, $t_0 \in \mathbb{R}$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ a continuously differentiable function.

Theorem 2.18 *[117] Let x^* be an equilibrium point of (2.45). If all the eigenvalues of $f'(x^*)$ are located in $\tilde{S}_h^{\alpha}$, then (2.45) has a unique solution for all initial vectors close enough to x^* and, moreover, x^* is asymptotically stable.*

Example 2.14 *Consider the following system:*

$$\begin{cases} {}_{t_0}^{C}\nabla_1^{\frac{2}{3}} x_1(t) = -2x_1(t)e^{x_2(t)} \\ {}_{t_0}^{C}\nabla_1^{\frac{2}{3}} x_2(t) = x_1^3(t) - 4.7x_2(t)e^{x_1(t)} \end{cases}, \quad t \in \mathbb{N}_{t_0+1-\frac{2}{3}}. \tag{2.46}$$

The eigenvalues of $f'(x^*)$ are $\lambda_1 = -2$ and $\lambda_2 = -4.7$ Since $\lambda_1 \in S^{\frac{2}{3}}$, and $\lambda_2 \in S^{\frac{2}{3}}$. Then, by Theorem 2.18, the trivial solution of (2.45) is asymptotically stable.

Chapter 3

Survey Study About New Results in the Applications of Fractional Difference Systems

3.1 A Brief Introduction to Chaos

The mathematical concept of a dynamical system is based on the idea that the behavior of many real-world processes follows a set of specific rules. Chaos refers to a complex nonlinear phenomenon characterized by extreme sensitivity to initial conditions in dynamical systems. This sensitivity, often called the "butterfly effect," means that even the smallest changes in starting conditions can lead to vastly different outcomes over time.

Discrete chaotic systems, which have been studied extensively, exhibit this property of sensitivity to initial conditions. To better understand and analyze their behavior, researchers have developed various mathematical tools and techniques. In this chapter, we introduce fundamental characteristics and key concepts of chaos theory in the context of discrete fractional nonlinear dynamical systems.

Chaos theory is a branch of mathematics that focuses on deterministic nonlinear systems with high sensitivity to initial conditions. While chaotic systems may seem unpredictable at first glance, they often exhibit underlying patterns or structures that emerge over time and can be modeled mathematically.

Although there is no universally accepted definition of chaos, a widely referenced definition was proposed by Devaney in 1989. To understand this definition, we must first introduce some foundational concepts, which will be discussed in the following sections.

Definition 3.1 *[118] Let E be a metric space. The function $f : E \to E$ has sensitive dependence* *on initial conditions on E if there exists $\delta > 0$ such that, for any $x \in E$ and any neighborhood U of x, there exists $y \in U$ and $t \geq 0$ such that $|f^t(x) - f^t(y)| > \delta$.*

Definition 3.2 *[119] Let E be a metric space. The function $f : E \to E$ is said to be topologically transitive if for any pair of non-empty open sets $U, V \subset J$, there exists $t \in \mathbb{N}$ such that $g^t(U) \cap V = g$.*

Definition 3.3 *A subset $I \subset X$ is said to be dense in X if for every $x \in X$ there a exists sequence $(x_t)_{t\in\mathbb{N}} \in I$ such that $\lim_{t\to\infty} x_t = x$.*

In the following, we present the definition of chaos according to [118].

Definition 3.4 *[118] $f : E \to E$ is said to be chaotic on E if*

- *f possesses sensitive dependence on initial values.*
- *f is topologically transitive*
- *The set of periodic points is dense in E.*

3.1.1 Chaotic Systems

Over the years, many researchers have focused on studying chaotic systems and their associated maps. The advancement of technology and computational tools has greatly contributed to progress in this field. These developments have made it easier to track the trajectories of chaotic systems over extended periods and with greater precision.

As a result, theoretical studies of chaos have been significantly enhanced by numerical simulations. Various computational methods have been developed to investigate and understand chaotic behaviors, allowing researchers to explore these complex dynamics more effectively through the use of computer programs.

3.1.2 Characteristics of Chaotic Systems

Sensitivity to Initial Conditions

Sensitivity to initial conditions, as defined earlier, means that even a tiny perturbation in the state of a nonlinear deterministic system can result in significant differences in its future behavior. This property, often referred to as the "butterfly effect," was first observed by Lorenz (1963) in his study of atmospheric models [120], where small changes led to unexpected and dramatic outcomes. The term "butterfly effect" gained popularity from the title of Lorenz's 1972 academic paper, "Predictability: Does the Flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas?"

Mathematically, sensitivity to initial conditions is quantified using the Lyapunov exponents (LE). A positive Lyapunov exponent indicates that the system is highly sensitive to its starting conditions, with small differences in initial states diverging exponentially over time.

Strange Attractor

A strange attractor is a type of attractor that emerges from a chaotic system. Unlike simpler attractors, it exhibits such intricate complexity that it is impossible to precisely predict the exact location of the system's state on the attractor at any given time.

Strange attractors are characterized by sensitivity to initial conditions, meaning that points on the attractor that are initially close will diverge significantly over time. However, despite this divergence, the points will remain confined to the attractor's structure.

If the attractor is strange, the system's motion is non-periodic, meaning it does not repeat over time. This irregular yet deterministic motion is what defines chaotic behavior.

Bifurcation

Consider the non-linear discrete system

$$x\left(t+1\right)=f\left(x\left(t\right),\lambda\right) \tag{3.1}$$

where $\lambda \in \mathbb{R}$. When the parameter λ known as the bifurcation parameter, is varied, the dynamic behavior of the system may undergo significant changes. This phenomenon is called a bifurcation.

A bifurcation occurs when even a small change in the bifurcation parameter leads to a qualitative or topological shift in the system's behavior. These shifts might include changes in the number or stability of equilibrium points, periodic orbits, or the onset of chaotic dynamics.

Bifurcation diagrams are useful tools to visualize these changes. They plot the long-term behavior of the system's states as a function of the bifurcation parameter, allowing one to observe qualitative transitions in the dynamics.

There are various types of bifurcations, including: Saddle-node (or fold) bifurcation, Hopf bifurcation, Pitchfork bifurcation, Transcritical bifurcation,Period- doubling (or flip) bifurcation and Neimark–Sacker bifurcation.

These types reflect different ways in which the system's behavior can change as the parameter λ is varied.

Lyapunov Exponents

The Lyapunov exponents (LEs) method, introduced by Russian mathematician Aleksander Lyapunov, is a key tool for measuring chaos in dynamical systems. It quantifies the sensitive dependence on initial conditions, a hallmark of chaotic systems. Specifically, the presence of a positive Lyapunov exponent indicates that the system exhibits chaotic behavior.

Below, we define the Lyapunov exponents in the context of discrete-time systems and outline the Jacobian matrix algorithm for calculating the LE of fractional-order discrete systems.

Consider the discrete dynamical system:

$$x\,(t+1) = f\,(x\,(t)) = f^{t}\,(x\,(0)) \tag{3.2}$$

where $x \in \mathbb{R}^n$.

For a small perturbation δ_{x_0} in the initial values x_0, the sensitivity to initial conditions is measured as:

$$\|\delta x_t\| = \|\delta x_0\|\, e^{t\lambda}, \tag{3.3}$$

where λ is the maximum Lyapunov exponent (MLE). It is calculated as:

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \left(\frac{\|\delta x_t\|}{\|\delta x_0\|}. \right) \tag{3.4}$$

- $\lambda > 0$: The trajectories are chaotic.
- $\lambda < 0$: The trajectories converge.
- $\lambda = 0$: The trajectories exhibit periodic motion.

The Jacobian matrix J_t of the system is defined as:

$$J_t = J\,(x_{t-1})\, J\,(x_{t-2}) \cdots J\,(x_0)\,, \tag{3.5}$$

where $J\,(x_t) = D_x f(x)|_{x=x_0}$ is the Jacobian matrix of f at x_t.

The eigenvalues of J_t are denoted as:

$$\left|\lambda_1^{(t)}\right| \geq \left|\lambda_2^{(t)}\right| \geq \cdots \geq \left|\lambda_n^{(t)}\right|. \tag{3.6}$$

Then, according to [121], the i-th Lyapunov exponents is defined as

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln \left|\lambda_i^{(t)}\right|, i = 1, 2, \cdots, n. \tag{3.7}$$

Calculating the Lyapunov exponents analytically is challenging due to the difficulty of computing the Jacobian matrix J_t. Thus, numerical techniques are often employed. Common methods include:

- QR Decomposition Method [122].
- Wolf Method [123].

The Wolf method is widely used for integer-order dynamical systems. It repeatedly renormalizes vectors using the Gram-Schmidt procedure. However, it does not work for fractional-order discrete systems because fractional-order dynamics depend on all previous states.

This limitation led to the development of new algorithms tailored for fractional discrete systems. One such approach is the Jacobian matrix algorithm for discrete fractional maps, as described by Wu and Baleanu (2015) [124].

3.2 Synchronization

Synchronization describes a relationship between two systems where their states become coordinated over time through appropriate control mechanisms. This phenomenon is crucial in many applications, including secure communications, neuroscience, and control theory.

3.2.1 Master-Slave System

The master system is represented by:

$$\begin{cases} {}^C\Delta_0^{\alpha_1} x_1(t+1-\alpha_1) = f_1(X(t)), \\ {}^C\Delta_0^{\alpha_2} x_2(t+1-\alpha_2) = f_2(X(t)), \\ \vdots \\ {}^C\Delta_0^{\alpha_n} x_n(t+1-\alpha_n) = f_n(X(t)), \end{cases} \quad k = 0, 1, \cdots, \tag{3.8}$$

where ${}^C\Delta_0^{\alpha}$ denotes the Caputo fractional difference of order $\alpha, 0 < \alpha \leq 1$, for, $X(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in \mathbb{R}^n$ is the state of the system (3.8) and $(f_1, f_2, \cdots, f_n)^T : \mathbb{R}^n \to \mathbb{R}^n$. And the slave system is given by:

$$\begin{cases} {}^C\Delta_0^{\alpha_1} y_1(t+1-\alpha_1) = g_1(X(t)) + U_1, \\ {}^C\Delta_0^{\alpha_2} y_2(t+1-\alpha_2) = g_2(X(t)) + U_2, \\ \vdots \\ {}^C\Delta_0^{\alpha_n} y_n(t+1-\alpha_n) = g_n(X(t)) + U_n, \end{cases} \quad t = 0, 1, \cdots, \tag{3.9}$$

where $Y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^T \in \mathbb{R}^n$ is the state of the system (3.9), $(g_1, g_2, \cdots, g_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ and $U = (U_1, U_2, \cdots, U_n)^T \in \mathbb{R}^n$ is a control vector to be determined.

3.2.2 Types of Synchronization

Complete Synchronization (C.S)

Definition 3.5 *[125] Complete synchronization occurs when the control U ensures:*

$$\lim_{t\to\infty} \|Y(t) - X(t)\| = 0, \tag{3.10}$$

where $\|.\|$ denotes the Euclidean norm.

- If $f_i = g_i$ for all i, this is identical complete synchronization.
- If $f_i \neq g_i$, for some i, this is non-identical complete synchronization.

Anti-Synchronization

Definition 3.6 *[125] Anti-synchronization occurs when the control U ensures:*

$$\lim_{t\to\infty} \|Y(t) + X(t)\| = 0. \tag{3.11}$$

Projective Synchronization

Definition 3.7 *[126] Projective synchronization happens when there exists a diagonal matrix $H = diag(h_1, \cdots, h_n)$ such as:*

$$\lim_{t\to\infty} \|Y(t) - H \times X(t)\| = 0. \tag{3.12}$$

- If all $h_i = 1$, this represents complete synchronization.
- If all $h_i = -1$, this represents anti-synchronization.

Full-State Hybrid Projective Synchronization (FSHP.S)

Definition 3.8 *[127] FSHP synchronization occurs if there exist controls U_i and a constants γ_{ij} such as:*

$$\lim_{t\to\infty} \left| y_i(t) - \sum_{j=1}^{n} \gamma_{ij} x_j(t) \right| = 0, \quad i = 1, \cdots, n. \tag{3.13}$$

Inverse FSHP Synchronization (I.FSHP.S)

Definition 3.9 *[128] Inverse FSHP synchronization occurs if there exist controls U_i and a constants β_{ij} such that:*

$$\lim_{t\to\infty} \left| x_i(t) - \sum_{j=1}^{n} \beta_{ij} y_j(t) \right| = 0, \quad i = 1, \cdots, n. \tag{3.14}$$

Generalized Synchronization (G.S)

Definition 3.10 *[129] Generalized synchronization occurs if there exists a controller U and a function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that:*

$$\lim_{t\to\infty} \|Y(t) - f(X(t))\| = 0, \tag{3.15}$$

Remark 3.1 *Generalized synchronization encompasses all previously mentioned types of synchronization.*

Inverse Generalized Synchronization (I.G.S)

Definition 3.11 *[130] Inverse generalized synchronization occurs if there exists a controller U and a function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that:*

$$\lim_{t\to\infty} \|X(t) - f(Y(t))\| = 0, \tag{3.16}$$

Remark 3.2 *If $f(Y(t)) = BY(t)$ where $B = (\beta_{ij})$, this is referred to as inverse full-state hybrid projective synchronization.*

Q-S Synchronization

Definition 3.12 *[131] $Q - S$ synchronization occurs in dimension d if there exist a controller U and two functions $Q : \mathbb{R}^n \to \mathbb{R}^d, S : \mathbb{R}^n \to \mathbb{R}^d$ such that:*

$$\lim_{t\to\infty} \|Q(X(t)) - S(Y(t))\| = 0. \tag{3.17}$$

Remark 3.3 *$Q - S$ synchronization is a generalized form of all prior synchronization types.*

3.2.3 Analytical Results

Let us consider the master system defined as:

$$\begin{cases} {}^C\Delta_0^\alpha x_1(t+1-\alpha) = \sum\limits_{j=1}^{n} a_{1j}x_j(t) + f_1(X(t)), \\ {}^C\Delta_0^\alpha x_2(t+1-\alpha) = \sum\limits_{j=1}^{n} a_{2j}x_j(t) + f_2(X(t)), \\ \vdots \\ {}^C\Delta_0^\alpha x_n(t+1-\alpha) = \sum\limits_{j=1}^{n} a_{nj}x_j(t) + f_n(X(t)), \end{cases} \tag{3.18}$$

where $X(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ is the state vector, $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a coefficient matrix, and $(f_1, f_2, \cdots, f_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function.

Similarly, the slave system is given as:

$$\begin{cases} {}^{C}\Delta_0^{\alpha} y_1(t+1-\alpha) = \sum\limits_{j=1}^{n} b_{1j} y_j(t) + g_1(Y(t)) + U_1, \\ {}^{C}\Delta_0^{\alpha} y_2(t+1-\alpha) = \sum\limits_{j=1}^{n} b_{2j} y_j(t) + g_2(Y(t)) + U_2, \\ \vdots \\ {}^{C}\Delta_0^{\alpha} y_n(t+1-\alpha) = \sum\limits_{j=1}^{n} b_{nj} y_j(t) + g_n(Y(t)) + U_n, \end{cases} \tag{3.19}$$

where $Y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^T$ represents the states of the slave system, $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ is another coefficient matrix, and $(g_1, g_2, \cdots, g_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function. The control vector $U = (U_1, U_2, \cdots, U_n)^T$ is designed to synchronize the slave system with the master system.

Theorem 3.1 *The master-slave systems (3.18)–(3.19) achieve global complete synchronization when the control law is given by:*

$$U_i = -\left(\sum_{j=1}^{n} (c_{ij} - b_{ij}) e_j(t) + \sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) - \sum_{j=1}^{n} a_{ij} x_j(t) - f_i(X(t)) \right), \forall i = 1, 2, \cdots, n, \tag{3.20}$$

where $e_i(t) = y_i(t) - x_i(t)$ is the synchronization error. The control matrix C must satisfy:

$$c_{ij} = b_{ij}, \ \text{if } i \neq j \text{ and } -2^{\alpha} < b_{ii} - c_{ii} < 0, \quad 1 \leq i, j \leq n. \tag{3.21}$$

Proof. Define the synchronization error as $e_i(t) = y_i(t) - x_i(t)$. Its Caputo fractional difference is:

$$ {}^{C}\Delta_0^{\alpha} e_i(t+1-\alpha) = \sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) + U_i - \sum_{j=1}^{n} a_{ij} x_j(t) - f_i(X(t)). \tag{3.22}$$

Substituting the control law and defining:

$$R_i = \sum_{j=1}^{n} (c_{ij} - b_{ij}) e_j(t) + \sum_{j=1}^{n} b_{ij} y_j(t) + g_i(Y(t)) - \sum_{j=1}^{n} a_{ij} x_j(t) - f_i(X(t)), \tag{3.23}$$

the error dynamics simplify to:

$$ {}^{C}\Delta_0^{\alpha} e_i(t+\alpha-1) = \sum_{j=1}^{n} (b_{ij} - c_{ij}) e_j(t) + R_i + U_i(t), \quad i = 1, 2, \cdots, n. \tag{3.24}$$

Using the conditions on C and substituting into the error system, we get:

$$ {}^{C}\Delta_a^{\alpha} e_i(t+\alpha-1) = (b_{ii} - c_{ii}) e_i(t). \tag{3.25}$$

The error system is stable when $b_{ii} - c_{ii}$ satisfy $-2^{\alpha} < b_{ii} - c_{ii} < 0$. This ensures that the eigenvalues of $B - C$ lie within the stable region defined by Theorem 2.13, leading to asymptotic stability of the zero solution of the error system. Hence, the master and slave systems achieve global complete synchronization. ■

3.3 Synchronization of Fractional Discrete Chaotic Systems

In this section, we propose an adaptive control method to achieve synchronization in a fractional-order discrete-time Hopfield neural network (DTHNN). Let us define the master system as [132]:

$$\begin{cases} {}^{C}\Delta_{a}^{\alpha} x_{1m}(t) = & -x_{1m}(t-1+\alpha) - 1.4\tanh(x_{1m}(t-1+\alpha)) + 1.2\tanh(x_{2m}(t-1+\alpha)) \\ & -7\tanh(x_{3m}(t-1+\alpha)), \\ {}^{C}\Delta_{a}^{\alpha} x_{2m}(t) = & -x_{2m}(t-1+\alpha) - 1.1\tanh(x_{1m}(t-1+\alpha)) + 2.8\tanh(x_{3m}(t-1+\alpha)), \\ {}^{C}\Delta_{a}^{\alpha} x_{3}(t) = & -x_{3m}(t-1+\alpha) + P\tanh(x_{1m}(t-1+\alpha)) - 2\tanh(x_{2m}(t-1+\alpha)). \\ & +4\tanh(x_{3m}(t-1+\alpha)). \end{cases} \tag{3.26}$$

The slave system is defined as [132]:

$$\begin{cases} {}^{C}\Delta_{a}^{\alpha} x_{1r}(t) = & -x_{1r}(t-1+\alpha) - 1.4\tanh(x_{1r}(t-1+\alpha)) + 1.2\tanh(x_{2r}(t-1+\alpha)) \\ & -7\tanh(x_{3r}(t-1+\alpha)) + C_1(t-1+\alpha), \\ {}^{C}\Delta_{a}^{\alpha} x_{2r}(t) = & -x_{2r}(t-1+\alpha) - 1.1\tanh(x_{1r}(t-1+\alpha)) + 2.8\tanh(x_{3r}(t-1+\alpha)) \\ & +C_2(t-1+\alpha), \\ {}^{C}\Delta_{a}^{\alpha} x_{3}(t) = & -x_{3r}(t-1+\alpha) + P\tanh(x_{1r}(t-1+\alpha)) - 2\tanh(x_{2r}(t-1+\alpha)) \\ & +4\tanh(x_{3r}(t-1+\alpha)) + C_3(t-1+\alpha), \end{cases} \tag{3.27}$$

where C_1, C_2 and C_3 are the control functions ensuring synchronization. The master system states are denoted with the subscript m , while the slave system states are indicated by r.

The synchronization error is defined as:

$$\begin{cases} e_1 = x_{1r} - x_{1m}, \\ e_2 = x_{2r} - x_{2m}, \\ e_3 = x_{3r} - x_{3m}, \end{cases} \tag{3.28}$$

Synchronization between the master system (3.26) and the slave system (3.27) occurs if the errors satisfy:

$$\lim_{t\to\infty} |e_i(t)| = 0, \text{ for } i = 1, 2, 3. \tag{3.29}$$

The following theorem describes the control law required to achieve synchronization.

Theorem 3.2 *[132] Synchronization between the master system (3.26) and the slave system*

(3.27) is achieved with the control laws:

$$\begin{cases} C_1(t) = & 1.4\tanh(x_{1r}(t) - x_{1m}(t)) - 1.2\tanh(x_{2r}(t) - x_{2m}(t)) \\ & -7\tanh(x_{3r}(t) - x_{3m}(t)) - b_1 e_1(t), \\ C_2(t) = & 1.1\tanh(x_{1r}(t) - x_{1m}(t)) - 2.8\tanh(x_{3r}(t) - x_{3m}(t)) - b_2 e_2(t) \\ C_3(t) = & -P\tanh(x_{1r}(t) - x_{1m}(t)) + 2\tanh(x_{2r}(t) - x_{2m}(t)) \\ & -4\tanh(x_{3r}(t) - x_{3m}(t)) - b_3 e_3(t), \end{cases} \tag{3.30}$$

where $-1 < b_j < 2^{\alpha} - 1, j = 1, 2, 3.$

Proof. Using the error definitions and applying the Caputo-type fractional-order differences, the error system can be written as:

$$^{C}\Delta_a^{\alpha} e_i = -b_i e_i, i = 1, 2, 3. \tag{3.31}$$

By choosing $-1 < b_j < 2^{\alpha} - 1$, the eigenvalues of the corresponding matrix ensure stability. Thus, the errors converge to zero, and synchronization is achieved.

■

Numerical simulations in MATLAB validate this result. Figure 3.1 shows that the errors approach zero over time, confirming successful synchronization.

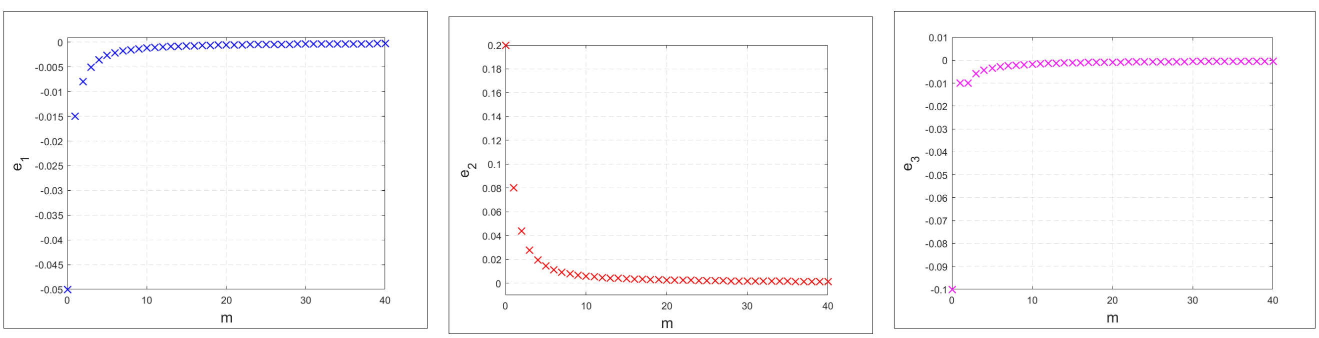

Figure 3.1: Numerical simulation of the error system using MATLAB.

Part II

Novel Contributions and Applications

Chapter 4

Stability Analysis of Fractional Difference Systems with Incommensurate Orders

To further explore the stability of linear commensurate Fractional-order Difference Systems (FoDSs), this chapter delves into their extended version involving incommensurate orders. New results are presented, providing simple and applicable conditions for assessing the stability of these systems. These results are achieved by converting the incommensurate fractional-order difference system into an equivalent system composed of fractional-order difference equations of the Volterra convolution type, utilizing properties of the Z-transform method.

4.1 Stability Analysis of Delta Difference Systems

4.1.1 Stability of Linear Systems

Consider the following linear incommensurate fractional order difference systems:

$$\begin{cases} {}^{C}\Delta_0^{\alpha_1} x_1(t+1-\alpha_1) = \sum\limits_{i=1}^{n} a_{1i} x_i(t), \\ {}^{C}\Delta_0^{\alpha_2} x_2(t+1-\alpha_2) = \sum\limits_{i=1}^{n} a_{2i} x_i(t), \\ \vdots \\ {}^{C}\Delta_0^{\alpha_n} x_n(t+1-\alpha_n) = \sum\limits_{i=1}^{n} a_{ni} x_i(t), \end{cases} \quad t = 0, 1, \cdots, \tag{4.1}$$

where ${}^{C}\Delta_0^{\alpha_i}$ is the Caputo fractional difference of order α_i, where $0 < \alpha_i \leq 1$, for $i = 1, 2, \cdots, n$, and $A = (a_{ij})_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$, subject to the initial vector condition $x(0) = x_0 \in \mathbb{R}^n$.

To follow the progress in the stability of linear incommensurate fractional-order difference systems, we will address these systems next. In this regard, we provide the following result:

Theorem 4.1 *[133] If all roots of the following characteristic equation:*

$$\det\left(dimg\left(z(1-z^{-1})^{\alpha_1}, z(1-z^{-1})^{\alpha_2}, \cdots, z(1-z^{-1})^{\alpha_n}\right) - A\right) = 0, \tag{4.2}$$

lie inside the unit disk, then the zero solution of system (4.1) is asymptotically stable.
If there exists a zero, say z^* of (4.2) such that $|z^*| > 1$, then the zero solution of system (4.1) is not stable.

Proof. As we observed that

$$({}^C\Delta_0^{\alpha_i} x_i)(t+1-\alpha) = \sum_{s=0}^{t} (-1)^{t-s+1} \binom{\alpha_i}{t-s+1} x(s) - (-1)^{t+1} \binom{\alpha_i - 1}{t+1} x(0) + x(t+1), \quad \forall i = 1, 2, \cdots, n.$$

So we can rewrite system (4.1) as follows:

$$\begin{cases} x_1(t+1) &= \sum\limits_{s=0}^{t} (-1)^{t-s} \binom{\alpha_1}{t-s+1} x_1(s) + (-1)^{t+1} \binom{\alpha_1 - 1}{t+1} x_1(0) + \sum\limits_{i=1}^{n} a_{1i} x_i(t), \\ x_2(t+1) &= \sum\limits_{s=0}^{t} (-1)^{t-s} \binom{\alpha_2}{t-s+1} x_2(s) + (-1)^{t+1} \binom{\alpha_2 - 1}{t+1} x_2(0) + \sum\limits_{i=1}^{n} a_{2i} x_i(t), \\ \vdots \\ x_n(t+1) &= \sum\limits_{s=0}^{t} (-1)^{t-s} \binom{\alpha_n}{t-s+1} x_n(s) + (-1)^{t+1} \binom{\alpha_n - 1}{t+1} x_n(0) + \sum\limits_{i=1}^{n} a_{ni} x_i(t). \end{cases} \quad t = 0, 1, \cdots . \tag{4.3}$$

One might take the Z-transform to (4.3). This yields the following system:

$$\begin{cases} zx_1(z) - zx_1(0) &= (z - z(1-\frac{1}{z})^{\alpha_1}) x_1(z) + (z(1-\frac{1}{z})^{\alpha_1 - 1} - z) x_1(0) + \sum\limits_{i=1}^{n} a_{1i} x_i(z), \\ zx_2(z) - zx_2(0) &= (z - z(1-\frac{1}{z})^{\alpha_2}) x_2(z) + (z(1-\frac{1}{z})^{\alpha_2 - 1} - z) x_2(0) + \sum\limits_{i=1}^{n} a_{2i} x(z), \\ \vdots \\ zx_n(z) - zx_n(0) &= (z - z(1-\frac{1}{z})^{\alpha_n}) x_n(z) + (z(1-\frac{1}{z})^{\alpha_n - 1} - z) x_n(0) + \sum\limits_{i=1}^{n} a_{nn} x(z), \end{cases} \tag{4.4}$$

where $x_i(z)$ indicates the Z-transform of $x_i(t)$, (i.e., $x_i(z) = Z[x_i(t)], 1 \leq i \leq n$). Consequently, we can rewrite system (4.4) as follows:

$$A(z). \begin{pmatrix} x_1(z) \\ x_2(z) \\ \vdots \\ x_n(z) \end{pmatrix} = \begin{pmatrix} z(1-\frac{1}{z})^{\alpha_1 - 1} x_1(0) \\ z(1-\frac{1}{z})^{\alpha_2 - 1} x_2(0) \\ \vdots \\ z(1-\frac{1}{z})^{\alpha_n - 1} x_n(0) \end{pmatrix}, \tag{4.5}$$

in which

$$A(z) = \begin{pmatrix} z(1-\frac{1}{z})^{\alpha_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & z(1-\frac{1}{z})^{\alpha_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & z(1-\frac{1}{z})^{\alpha_n} - a_{nn} \end{pmatrix}. \tag{4.6}$$

Multiplying both sides of (4.5) by $(z-1)$ gives:

$$A(z).\begin{pmatrix} (z-1)x_1(z) \\ (z-1)x_2(z) \\ \vdots \\ (z-1)x_n(z) \end{pmatrix} = \begin{pmatrix} z^2(1-\frac{1}{z})_1^{\alpha_1} x(0) \\ z^2(1-\frac{1}{z})_2^{\alpha_2} x(0) \\ \vdots \\ z^2(1-\frac{1}{z})^{\alpha_n} x_n(0) \end{pmatrix}. \tag{4.7}$$

Now, we should note that if all roots of $\det A(z) = 0$, lie inside the unit disk, then system (4.7) will be considered such that z satisfies $|z| \geq R$, withe $R \leq 1$ (R is the radius of convergence of $\tilde{X}(z)$). Actually, system (4.7) has a unique solution in this limited area represented by $((z-1)x_1(z), (z-1)x_2(z), \cdots, (z-1)x_n(z))$. Accordingly, we have:

$$\lim_{z\to 1}(z-1)x_i(z) = 0, \quad i = 1, 2, \cdots, n. \tag{4.8}$$

Based on the assumption stated in first part of this theorem, and based also on the Final-Value Theorem associated with Z-transform, we obtain:

$$\lim_{t\to\infty} x_i(t) = \lim_{z\to 1}(z-1)x_i(z) = 0, \quad i = 1, 2, \cdots, n. \tag{4.9}$$

On the other hand, considering the second part of this theorem implies that the convergence radius R of the series:

$$\sum_{t=0}^{\infty} X(t)z^{-t} = \tilde{X}(z), \tag{4.10}$$

is greater than 1 (i.e., $R > 1$). Therefore, there exist i_0, where $1 \leq i_0 \leq n$, which makes the convergence radius R_{i_0}, of the series:

$$\sum_{t=0}^{\infty} x_{i_0}(t)z^{-t} = x_{i_0}(z), \tag{4.11}$$

also be greater than 1 (i.e., $R_{i_0} > 1$). Thus, by using the Cauchy-Hadamard Theorem, we obtain:

$$R_{i_0} = \lim_{t\to\infty} \sup \sqrt[t]{|x_{i_0}(t)|} > 1. \tag{4.12}$$

Consequently, $\lim_{t\to\infty} \sup |x_{i_0}(t)| = \infty$. This, however, implies that x will be never bounded and hence (4.1) is not stable. ■

Corollary 4.1 *[133] Suppose that A is a triangular matrix with diagonal elements $\lambda_i, i = 1, \cdots, n$. If $-2^{\alpha_i} < \lambda_i < 0, \forall i$, then the zero solution of system (4.1) is asymptotically stable. Furthermore, such solution is not stable if either $\lambda_i > 0$ or $\lambda_i < -2^{\alpha_i}$, for some i.*

Proof. Consider the first part of Theorem 4.1 and A as assumed here. This will turn (4.2) into the form:

$$\prod_{i=1}^{n} \left(z(1 - z^{-1})^{\alpha_i} - \lambda_i \right) = 0. \tag{4.13}$$

It means that there exists i, where $0 \leq i \leq n$ such that:

$$\left(z(1 - z^{-1})^{\alpha_i} - \lambda_i \right) = 0. \tag{4.14}$$

Now, according to the assumption that supposes all the roots of (4.2) lie inside the unit disk, we deduce that all λ_i's belong to the set $\left\{ z(1 - z^{-1})^{\alpha_i},\ z \in \mathbb{C} \text{ and } |z| < 1 \right\}$, for $0 \leq i \leq n$. Based on:

$$\left\{ z(1 - z^{-1})^{\alpha_i},\ z \in \mathbb{C} \text{ and } |z| < 1 \right\} = \left\{ z \in \mathbb{C} : |z| < \left(2\cos\frac{|\arg z| - \pi}{2 - \alpha} \right)^{\alpha_i} \text{ and } |\arg z| > \frac{\alpha_i \pi}{2} \right\},$$

where $\lambda_i \in \mathbb{R}, 0 \leq i \leq n$, This means that $-2^{\alpha_i} < \lambda_i < 0$, as desired. ∎

Example 4.1 *Consider the vector difference equation*

$$\begin{cases} {}^{C}\Delta_0^{\sqrt{2}} x_1(t + 1 - \sqrt{2}) = \lambda_1 x_1(t) + 5x_2(t) + 6x_3(t), \\ {}^{C}\Delta_0^{\pi} x_2(t + 1 - \pi) = \lambda_2 x_2(t) + 8x_3(t), \\ {}^{C}\Delta_0^{e} x_3(t + 1 - e) = \lambda_3 x_3(t), \end{cases} \tag{4.15}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, by Corollary 4.1 the zero solution of (4.15) is asymptotically stable if

$$-2^{\sqrt{2}} < \lambda_1 < 0,$$

$$-2^{\pi} < \lambda_2 < 0,$$

and

$$-2^{e} < \lambda_3 < 0.$$

If either $\lambda_i > 0$ or $\lambda_i < -2^{\alpha_i}$, for some $i = 1, 2, 3$. Then, the zero solution of (4.15) is not stable.

As a matter of fact, the above result, represented by Corollary 4.1, is deemed one of the main significant results in this chapter. It can be easily implemented for exploring the stability of some linear incommensurate FoDSs which involve just a triangular matrix A in their forms. On the contrary, one can find it extremely hard to verify condition (4.2) in the proposed result represented by Theorem 4.1. Actually, this condition concerns with the full matrix that might be found in the linear incommensurate FoDSs. To deal with this problem, we present below another more practical result which is equivalent to Theorem 4.1.

Theorem 4.2 *[133] Consider $0 < \alpha_i < 1$, for $i = 1, 2, \cdots, n$, and M is the lowest common multiple of the denominators u_i of α_i's, in which $\alpha_i = \frac{v_i}{u_i}$ and $(u_i, v_i) = 1$, where $u_i, v_i \in \mathbb{Z}_+, \forall i$. Then, the zero solution of system (4.1), subject to the initial vector condition $X(0) = X_0 \in \mathbb{R}^n$, is:*

• Asymptotically stable if any zero solution of the polynomial:

$$\det\left(dimg\left(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, \cdots, \lambda^{M\alpha_n}\right) - (1-\lambda^M)A\right), \tag{4.16}$$

lie inside the set

$$\mathbb{C}\backslash K^{\gamma},$$

where $\gamma = \frac{1}{M}$ and where

$$K^{\gamma} = \left\{ z \in \mathbb{C} : |z| \leq \left(2\cos\frac{|\arg z|}{\gamma}\right)^{\gamma} \text{ and } |\arg z| \leq \frac{\gamma\pi}{2} \right\}. \tag{4.17}$$

• Not stable, furthermore, if there is a zero, say λ, of (4.16) such that $\lambda \in IntK^{\gamma}$.

Proof. In accordance with Theorem 4.1, system (4.1) is asymptotically stable if all the zeros of the following characteristic equation:

$$\det\left(diag\left(z(1-z^{-1})^{\alpha_1}, z(1-z^{-1})^{\alpha_2}, \cdots, z(1-z^{-1})^{\alpha_n}\right) - A\right) = 0,$$

are located inside the unit circle. However, setting

$$1 - \frac{1}{z} = \lambda^M \Leftrightarrow z = \frac{1}{1-\lambda^M}, \lambda^M \neq 1,$$

will turn the above characteristic equation to be in the form:

$$\det\left(dimg\left(\frac{\lambda^{M\alpha_1}}{1-\lambda^M}, \frac{\lambda^{M\alpha_2}}{1-\lambda^M}, \cdots, \frac{\lambda^{M\alpha_n}}{1-\lambda^M}\right) - A\right) = 0. \tag{4.18}$$

Multiplying both sides of (4.18) by $(1-\lambda^M)^n$ yields:

$$\det\left(diag\left(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, \cdots, \lambda^{M\alpha_n}\right) - (1-\lambda^M)A\right) = 0. \tag{4.19}$$

Now, one finds that it is necessary to prove the two assertions $z \in \{z \in \mathbb{C} : 0 < |z| < 1\} \Leftrightarrow \lambda \in \mathbb{C}\backslash K^{\gamma}$ and $z \in \{z \in \mathbb{C} : |z| > 1\} \Leftrightarrow \lambda \in IntK^{\gamma}$. For achieving those goals, consider the following steps:

• Step 1: (Defining the stability boundary). Consider the following curve:

$$L^{\gamma} = \left\{ \left(1 - \frac{1}{z}\right)^{\gamma} : |z| = 1 \right\}, \tag{4.20}$$

which defines the stability boundary for system (4.1) and also describes its structure. Suppose $z = e^{i\varphi}$ and $1 - z^{-1} = re^{i\omega}$ for $0 \leq \varphi < 2\pi$ and $r = r(\varphi) \geq 0$, and also suppose $\omega = \omega(\varphi)$, where $\omega \in [0, 2\pi)$. Then,

$$1 - e^{-i\varphi} = re^{i\omega}.$$

This equation, after the imaginary and real parts are equated, will be turned into the following two components:

$$\sin\varphi = r\sin\omega, \quad 1 - \cos\varphi = r\cos\omega.$$

Observe that, when $\varphi = 0$, then $r = 0$. Otherwise, we have:

$$\tan\omega = \frac{\sin\varphi}{1 - \cos\varphi}.$$

In view of the fact that:

$$\frac{\sin\varphi}{1 - \cos\varphi} = \frac{2\sin(\varphi/2)\cos(\varphi/2)}{2\sin^2(\varphi/2)} = \cot\frac{\varphi}{2} = \tan(\frac{\pi}{2} - \frac{\varphi}{2}),$$

we can write r and ω as $r = 2\sin\frac{\varphi}{2}$ and $\omega = \pi/2 - \varphi/2$, respectively. From here, we obtain:

$$L^\gamma = \left\{ \left(2\sin\frac{\varphi}{2}\right)^\gamma \exp(i\frac{(\pi-\varphi)\gamma}{2}) : 0 \leq \varphi < 2\pi \right\}.$$

Observe that setting $\theta = -\omega = \varphi/2 - \pi/2$, will turn L^γ to be as follows:

$$L^\gamma = \left\{ (2\cos\theta)^\gamma \exp(-i\gamma M) : -\frac{\pi}{2} \leq \theta < \frac{\pi}{2} \right\}.$$

One can use the polar form represented by ($|z| = (2\cos\theta)^\gamma$), where $\arg z = -\gamma\theta$, to obtain:

$$L^\gamma = \left\{ z \in \mathbb{C} : |z| = \left(2\cos\frac{|\arg z|}{\gamma}\right)^\gamma \text{ and } |\arg z| \leq \frac{\gamma\pi}{2} \right\}. \tag{4.21}$$

• Step 2: (Showing that $w_\gamma(z) = (1 - \frac{1}{z})^\gamma$ maps the set $D_E = \{z \in \mathbb{C} : |z| > 1\}$ onto $IntK^\gamma$, with noting that $w_\gamma(2) = 2^{-\gamma} \in IntK^\gamma$). In view of the Open mapping theorem, and since w_γ is nonconstant holomorphic on D_E, then it maps D_E to an open set. In other words, we have a neighborhood of $w_\gamma(z^*)$ included in $w_\gamma(D_E), \forall z^* \in D_E$. This implies that the boundary of $w_\gamma(D_E)$ can not be mapped by any point of D_E. This means that $w_\gamma(D_E) \subset IntK^\gamma$. Similarly, one can prove that $w_\gamma(D_I) \subset \mathbb{C}\backslash K^\gamma$, where $D_I = \{z \in \mathbb{C} : 0 < |z| < 1\}$. In view of $w_\gamma(\{z \in \mathbb{C} : |z| = 1\}) = L^\gamma$ (see the previous step), and also in view of the continuity of w_γ, the above arguments imply that $w_\gamma(D_E) = IntK^\gamma \backslash \{1\}$.

• Step 3: For the purpose of showing the other part this theorem, we first assume that there is a solution λ of (4.16) with $\lambda \in IntK^\gamma$. This implies that $z = \frac{1}{1-\lambda^\theta}$ is a solution of (4.2) with $|z| > 1$. Thus, we can deduce, in view of Theorem 4.1, that there is an instability of the zero solution of system (4.1). On the other hand, if each solution of (4.16) belongs to $\mathbb{C}\backslash K^\gamma$, then all solutions of (4.1) will belong to $\{z \in \mathbb{C} : |z| < 1\}$, which makes the zero solution of system (4.1), via Theorem 4.1, asymptotically stable, as required. ■

Example 4.2 *Consider the following linear incommensurate FoDS:*

$$\begin{cases} {}^C\Delta_0^{\frac{1}{2}} x_1\left(t+1-\frac{1}{2}\right) = -x_1(t) + x_2(t) \\ {}^C\Delta_0^{\frac{1}{4}} x_2\left(t+1-\frac{1}{4}\right) = -\frac{9}{16}x_1(t) + \frac{1}{2}x_2(t) \end{cases}, \quad t = 0, 1, 2, \cdots. \tag{4.22}$$

One can obtain M to be equal 4. This, however, implies:

$$\det\left(\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix} - (1-\lambda^4)\begin{pmatrix} -1 & 1 \\ -\frac{9}{16} & \frac{1}{2} \end{pmatrix}\right) = 0$$

$\Leftrightarrow$

$$\det\begin{pmatrix} \lambda^2 + (1-\lambda^4) & -(1-\lambda^4) \\ \frac{9}{16}(1-\lambda^4) & \lambda - \frac{1}{2}(1-\lambda^4) \end{pmatrix} = 0$$

$\Leftrightarrow$

$$\frac{1}{16}\lambda^8 + \frac{1}{2}\lambda^6 - \lambda^5 - \frac{1}{8}\lambda^4 + \lambda^3 - \frac{1}{2}\lambda^2 + \lambda + \frac{1}{16} = 0. \tag{4.23}$$

Accordingly, the solution of (4.23) will be in the following form:

$$\begin{cases} \lambda_1 = -1.1634, \\ \lambda_2 = -6.0451 \times 10^{-2}, \\ \lambda_3 = -0.78732 + 3.1894i, \\ \lambda_4 = -0.78732 - 3.1894i, \\ \lambda_5 = 1.3269 - 0.4875i, \\ \lambda_6 = 1.3269 + 0.4875i, \\ \lambda_7 = 7.2415 \times 10^{-2} + 0.80874i, \\ \lambda_8 = 7.2415 \times 10^{-2} - 0.80874i. \end{cases}$$

As per Theorem 4.2, and due to $\lambda_i \in \mathbb{C}\backslash K^{\frac{1}{4}}, \forall i = 1, 2, \cdots, 8$, then system (4.22) is asymptotically stable about its zero solution.

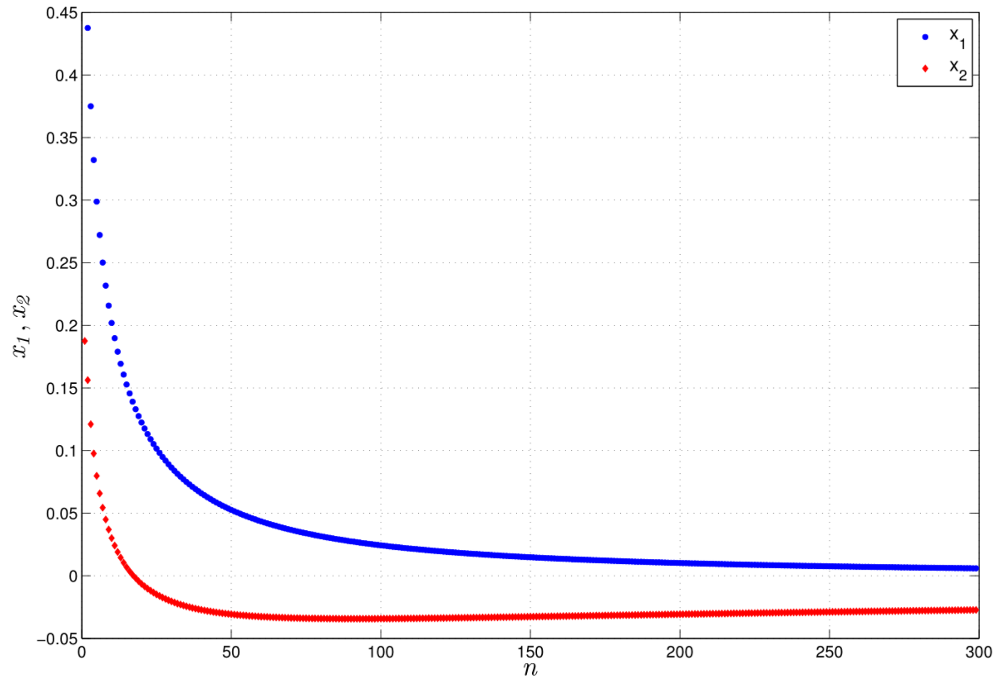

Figure 4.1: Numerical simulation of the system 4.22 using MATLAB.

Example 4.3 *Consider the following linear incommensurate FoDS:*

$$\begin{cases} {}^C\Delta_0^{\frac{1}{2}} x_1\left(t+1-\frac{1}{2}\right) = -x_1(t) - 0.2x_3(t), \\ {}^C\Delta_0^{\frac{1}{3}} x_2\left(t+1-\frac{1}{3}\right) = 3.4x_1(t) - x_2(t) + 0.2x_3(t), \quad t = 0, 1, 2, \cdots . \\ {}^C\Delta_0^{\frac{2}{3}} x_3\left(t+1-\frac{2}{3}\right) = -x_3(t), \end{cases} \tag{4.24}$$

One can find $M = 6$, which leads to:

$$\det\left(\begin{pmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^4 \end{pmatrix} - (1-\lambda^6)\begin{pmatrix} -1 & 0 & -0.2 \\ 3.4 & -1 & 0.2 \\ 0 & 0 & -1 \end{pmatrix}\right) = 0$$

$\Leftrightarrow$

$$\begin{gathered} -\lambda^{18} + \lambda^{16} + \lambda^{15} + \lambda^{14} - \lambda^{13} + 2\lambda^{12} - \lambda^{11} - 2\lambda^{10} - \lambda^9 \\ -2\lambda^8 + \lambda^7 - 2\lambda^6 + \lambda^5 + \lambda^4 + \lambda^3 + \lambda^2 + 1 = 0. \end{gathered} \tag{4.25}$$

Consequently, the solution of (4.25) will be in the form:

$$\begin{cases} \lambda_1 = -0.321\,31 - 0.874\,98i, \\ \lambda_2 = -0.321\,31 + 0.874\,98i, \\ \lambda_3 = 0.321\,31 - 0.874\,98i, \\ \lambda_4 = 0.321\,31 + 0.874\,98i, \\ \lambda_5 = -0.851\,80, \\ \lambda_6 = -0.544\,63 + 0.727\,6i, \\ \lambda_7 = -0.544\,63 - 0.727\,6i, \\ \lambda_8 = 0.544\,63 + 0.727\,6i, \\ \lambda_9 = 0.544\,63 - 0.727\,6i, \\ \lambda_{10} = 0.425\,90 - 0.737\,68i, \\ \lambda_{11} = 0.425\,90 + 0.737\,68i, \\ \lambda_{12} = -1.\,151\,0, \\ \lambda_{13} = 1.\,151\,0, \\ \lambda_{14} = -1.\,210\,6, \\ \lambda_{15} = 1.\,210\,6, \\ \lambda_{16} = -0.586\,99 - 1.\,016\,7i, \\ \lambda_{17} = -0.586\,99 + 1.\,016\,7i, \\ \lambda_{18} = 1.\,174\,0, \end{cases}$$

where

$$K^{\frac{1}{6}} = \left\{ z \in \mathbb{C} : |z| \leq (2\cos 6\,|\arg z|)^{\frac{1}{6}} \ \textit{and} \ |\arg z| \leq \frac{\pi}{12} \right\}.$$

Hence, in view of Theorem 4.2, system (4.24) is also asymptotically stable about its zero solution.

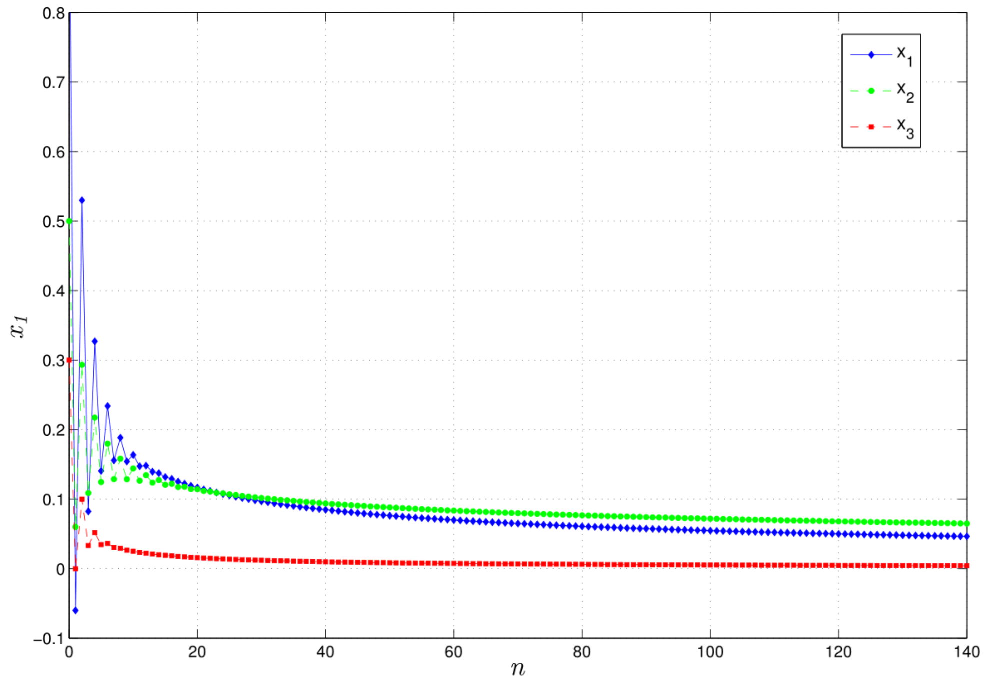

Figure 4.2: Numerical simulation of the system 4.24 using MATLAB.

In the present section, we report simple and applicable conditions for judging the stability of linear incommensurate Fractional-order Difference Systems. These results have been verified numerically by demonstrating the stability of solutions through several examples. The findings are applicable to various difference systems, such as the Duffing oscillator system, which has been successfully used to model a range of physical phenomena, including beam buckling, ionization waves in plasmas, nonlinear electronic circuits, stiffening springs, and superconducting Josephson parametric amplifiers.

4.1.2 Stability of non-Linear Systems

The absence of satisfactory results addressing the stability analysis of nonlinear incommensurate Fractional-order Difference Systems (FoDSs) and the difficulty in establishing a proper Lyapunov function to determine their stability are significant issues that need attention. To overcome these challenges, this subsection provides convenient conditions that can be used to ensure the asymptotic stability of these systems.

Consider the following non linear incommensurate FoDS:

$$\begin{cases} {}^C\Delta_0^{\alpha_1} x_1(t+1-\alpha_1) = f_1(x(t)) \\ {}^C\Delta_0^{\alpha_2} x_2(t+1-\alpha_2) = f_2(x(t)) \\ \vdots \\ {}^C\Delta_0^{\alpha_n} x_n(t+1-\alpha_n) = f_n(x(t)) \end{cases} \quad t = 0, 1, \cdots, \tag{4.26}$$

where ${}^C\Delta_0^{\alpha_i}$ is the Caputo fractional difference of order $\alpha_i, 0 < \alpha_i \leq 1$, for $i = 1, 2, ..., n$, $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$ and $f = (f_1, f_2, ..., f_n) : \mathbb{R}^n \to \mathbb{R}^n$ a continuously differentiable function, and suppose $f(0) = 0$.

Theorem 4.3 *[134] The zero solution of system (4.26), subject to the initial vector condition $x(0) = x_0 \in \mathbb{R}^n$, is locally asymptotically stable if all the roots of the characteristic equation:*

$$\det\left(diag\left(z(1-z^{-1})^{\alpha_1}, z(1-z^{-1})^{\alpha_2}, ..., z(1-z^{-1})^{\alpha_n}\right) - J\right) = 0, \tag{4.27}$$

where J is the jacobian matrix of f at 0, lie inside the unit disk.

Proof. The system (4.26) can be written as the following

$$\begin{cases} x_1(t+1) & = & \sum\limits_{s=0}^{t} (-1)^{t-s}\binom{\alpha_1}{t-s+1} x_1(s) + (-1)^{t+1}\binom{\alpha_1-1}{t+1} x_1(0) + f_1(x(t)) \\ & \vdots & \\ x_2(t+1) & = & \sum\limits_{s=0}^{t} (-1)^{t-s}\binom{\alpha_2}{t-s+1} x_2(s) + (-1)^{t+1}\binom{\alpha_2-1}{t+1} x_2(0) + f_2(x(t)) \\ x_n(t+1) & = & \sum\limits_{s=0}^{t} (-1)^{t-s}\binom{\alpha_n}{t-s+1} x_n(s) + (-1)^{t+1}\binom{\alpha_n-1}{t+1} x_n(0) + f_n(x(t)), \end{cases} \quad t = 0, 1, \cdots. \tag{4.28}$$

Using taylor development, (4.28) become

$$x(t+1) = Jx(t) + \sum_{s=0}^{t} B(t-s)x(s) + o(\|x(t)\|) + h(t), \quad t = 0, 1, \cdots. \tag{4.29}$$

Where

$$B(t) = \begin{pmatrix} (-1)^t\binom{\alpha_1}{t+1} & 0 & \cdots & 0 \\ 0 & (-1)^t\binom{\alpha_2}{t+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (-1)^t\binom{\alpha_n}{t+1} \end{pmatrix},$$

$$H(t) = \begin{pmatrix} (-1)^{t+1}\binom{\alpha_1-1}{t+1} \\ (-1)^{t+1}\binom{\alpha_2-1}{t+1} \\ \vdots \\ (-1)^{t+1}\binom{\alpha_n-1}{t+1} \end{pmatrix} x(0).$$

Taking the homogeneous part of (4.29)

$$x(t+1) = Jx(t) + \sum_{s=0}^{t} B(t-s)x(s), \tag{4.30}$$

We have $B(t) \in \left[\ell^1(\mathbb{N})\right]^{n\times n}$, the resolvent matrix $R(t)$ of (4.30) is defined as the unique solution of the matrix equation

$$R(t+1) = JR(t) + \sum_{s=0}^{t} B(t-s)R(s), \quad R(0) = I, t \in \mathbb{N},$$

where by Theorem 2 in [109] $R(t) \in \left[\ell^1(\mathbb{N})\right]^{n\times n}$ (because $\det(zI - J - \tilde{B}(z)) \neq 0$ for $|z| \geq 1$). Then by the variation of constants formula, we obtain

$$x(t) = R(t)x(0) + \sum_{s=0}^{t-1} R(t-s-1)\left(o(\|x(s)\|) + H(s)\right). \tag{4.31}$$

We have

$$\|x(t)\| \leq \|R(t)\| \, \|x(0)\| + \sum_{s=0}^{t-1} \|R(t-s-1)\| \, \|o(\|x(s)\|)\| + \sum_{s=0}^{t-1} \|R(t-s-1)\| \, \|H(s)\|, \tag{4.32}$$

for a given $\epsilon > 0$ there is $\delta > 0$ such that $o(\|x\|) < \epsilon \|x\|$ whenever $\|x\| < \delta$. So as long as $\|x(s)\| < \delta$:

$$\|x(t)\| \leq \|R(t)\| \, \|x(0)\| + \epsilon \sum_{s=0}^{t-1} \|R(t-s-1)\| \, \|x(s)\| + \sum_{s=0}^{t-1} \|R(t-s-1)\| \, \|H(s)\|,$$

we defined $y(t)$ as follow

$$y(t) = r(t)y(0) + \epsilon \sum_{s=0}^{t-1} r(t-s-1)y(s) + \sum_{s=0}^{t-1} r(t-s-1)h(s)$$

where

$$r(t) = \|R(t)\|, \quad h(s) = \|H(s)\|, \quad y(0) = \|x(0)\|.$$

$\Rightarrow$

$$y(t+1) = r(t+1)y(0) + \epsilon \sum_{s=0}^{t} r(t-s)y(s) + \sum_{s=0}^{t} r(t-s)h(s)$$

$\Rightarrow$

$$y(t+1) = r(t+1)y(0) + \epsilon r(t) * y(t) + r(t) * h(t). \tag{4.33}$$

We have

$$\|x(t)\| \leq y(t),$$

we see that $r(t) \in \ell^1(\mathbb{N})$. Taking the $Z-$transform on both sides of (4.33) gives

$$z\tilde{y}(z) - y(0)z = (z\tilde{r}(z) - z)y(0) + \epsilon\tilde{r}(z)\tilde{y}(z) + \tilde{r}(z)\tilde{h}(z),$$

with $R_r \leq 1$, and $R_h = 1$, where R_r is the convergence radius of $\tilde{r}(z)$ and R_r is the convergence radius of $\tilde{h}(z) \Rightarrow$

$$\tilde{y}(z) = (z - \epsilon\tilde{r}(z))^{-1}\left(z\tilde{r}(z)y(0) + \tilde{r}(z)\tilde{h}(z)\right),$$

for $|z| > \max\{R_r, 1, \epsilon\tilde{r}(1)\}$.

Choose $\epsilon < \frac{1}{\tilde{r}(1)}$, we get $\max\{R_r, 1, \epsilon\tilde{r}(1)\} = 1$, by final value theorem

$$\lim_{t\to\infty} y(t) = \lim_{z\to 1}(z-1)\tilde{y}(z) = \lim_{z\to 1}(z-1)((z - \epsilon\tilde{r}(z))^{-1}z\tilde{r}(z)y(0) + \tilde{r}(z)\tilde{h}(z))$$

we have

$$\begin{aligned}\sum_{s=0}^{t} \|R(t-s)\| \, \|H(s)\| &\leq C_1 \sum_{s=0}^{t} \tfrac{1}{(s+1)^\alpha} \|R(t-s)\| \, \|x(0)\| \\ &\leq C_1 \left(\sum_{s=0}^{\lfloor t/2 \rfloor} \tfrac{1}{(s+1)^\alpha} \|R(t-s)\| + \sum_{s=\lfloor t/2\rfloor+1}^{t} \tfrac{1}{(s+1)^\alpha} \|R(t-s)\| \right)\end{aligned}$$

where $C_1 > 0$ is a suitable real constant and the symbol $\lfloor . \rfloor$ stands for the floor function and $\alpha = \min_{1\leq i\leq n}\{\alpha_i\}$. Since $\|R(t)\|$ belongs to $\ell^1(\mathbb{N})$, we have $\|R(t)\| = O(t^{-1})$ as $t \to \infty$ and there exist $C_2, C_3 > 0$ such that

$$\sum_{s=0}^{\lfloor t/2 \rfloor} \frac{1}{(s+1)^\alpha} \|R(t-s)\| \leq \frac{C_2}{t+1} \sum_{s=0}^{\lfloor t/2 \rfloor} \frac{1}{(s+1)^\alpha} \leq \frac{C_3}{(t+1)^\alpha}$$

where we have used the inequality $\sum_{s=1}^{t}(s+1)^{-\alpha} \leq \int_0^t (x+1)^{-\alpha}dx$. Similarly, the second sum can be estimated as

$$\sum_{s=\lfloor t/2\rfloor+1}^{t} \frac{1}{(s+1)^\alpha} \|R(t-s)\| \leq \frac{C_4}{(t+1)^\alpha} \sum_{s=\lfloor t/2\rfloor+1}^{t} \|R(t-s)\| \leq \frac{C_5}{(t+1)^\alpha},$$

for suitable $C_4, C_5 > 0$.

In summary, we have

$$\sum_{s=0}^{t} \|R(t-s)\| \, \|H(s)\| \leq C_5 (t+1)^{-\alpha},$$

hence

$$\sum_{s=0}^{t} \|R(t-s)\| \, \|H(s)\| = O(t^{-\alpha}),$$

as $t \to \infty$. So by final value theorem

$$\lim_{z\to 1}(z-1)\tilde{r}(z)\tilde{h}(z) = 0,$$

and that's imply

$$\lim_{t\to\infty} y(t) = \lim_{z\to 1}(z-1)\tilde{y}(z) = \lim_{z\to 1}(z-1)((z-\epsilon\tilde{r}(z))^{-1}z\tilde{r}(z)y(0)+\tilde{r}(z)\tilde{h}(z)) = 0.$$

■

Corollary 4.2 *[134] Suppose that α_i's are rational numbers between 0 and 1, for $i=1,2,...,n$. Let M be the lowest common multiple (LCM) of the denominators u_i of α_i's, where $\alpha_i = \frac{v_i}{u_i}$, $(u_i,v_i) = 1, u_i, v_i \in \mathbb{Z}_+, i=1,2,...,n$, and set $\gamma = \frac{1}{M}$. Then the zero solution of system (4.26) with initial value $x_0 = x(0)$ is locally asymptotically stable if any zero solution of the polynomial equation*

$$\det\left(diag\left(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, ..., \lambda^{M\alpha_n}\right) - (1-\lambda^M)J\right) = 0, \tag{4.34}$$

lie inside the set

$$\mathbb{C}\backslash K^{\gamma},$$

where

$$K^{\gamma} = \left\{z\in\mathbb{C} : |z| \le \left(2\cos\frac{|\arg z|}{\gamma}\right)^{\gamma} \text{ and } |\arg z| \le \frac{\gamma\pi}{2}\right\}.$$

and J is the jacobian matrix of f at 0.

Proof. The proof came immediately from the equivalence between the coditions of Theorem 4.3 and conditions of Corollary 4.2. ■

Corollary 4.3 *[134] Let $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$. If $\lambda \in \left\{z\in\mathbb{C} : |z| < \left(2\cos\frac{|\arg z|-\pi}{2-\alpha}\right)^{\alpha} \text{ and } |\arg z| > \frac{\alpha\pi}{2}\right\}$ for all the eigenvalues λ of J the jacobian matrix of f at 0, then the trivial solution of (4.26) is locally asymptotically stable.*

Proof. The proof came immediately from the equivalence between the coditions of Theorem 4.3 and conditions of Corollary 4.3. ■

Example 4.4 *Consider the nonlinear systems*

$$\begin{cases} {}^{C}\Delta_0^{\frac{1}{2}} x_1(t) = x_2(t)e^{-x_1(t)} - x_1(t)e^{-x_2(t)}, \\ {}^{C}\Delta_0^{\frac{1}{4}} x_2(t) = \frac{1}{2}x_2(t)e^{-x_1(t)} - \frac{9}{16}x_1(t)e^{-\frac{1}{2}x_2(t)}, \end{cases} \tag{4.35}$$

where $(x_1(0), x_2(0)) = (1,1)$. It has an equilibrium point in the origin $(0,0)$. The Jacobian matrix is given by

$$\frac{\partial f(0,0)}{\partial x} = \begin{pmatrix} \frac{\partial f_1(0,0)}{\partial x_1} & \frac{\partial f_1(0,0)}{\partial x_2} \\ \frac{\partial f_2(0,0)}{\partial x_1} & \frac{\partial f_2(0,0)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -\frac{9}{16} & \frac{1}{2} \end{pmatrix},$$

we have $M = 4$, *so*

$$\det\left(\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix} - (1-\lambda^4)\begin{pmatrix} -1 & 1 \\ -\frac{9}{16} & \frac{1}{2} \end{pmatrix}\right) = 0$$

$\Leftrightarrow$

$$\frac{1}{16}\lambda^8 + \frac{1}{2}\lambda^6 - \lambda^5 - \frac{1}{8}\lambda^4 + \lambda^3 - \frac{1}{2}\lambda^2 + \lambda + \frac{1}{16} = 0 \tag{4.36}$$

the solution of (4.36) is

$$\begin{cases} \lambda_1 = -1.\,163\,4, \\ \lambda_2 = -6.\,045\,1 \times 10^{-2}, \\ \lambda_3 = -0.787\,32 + 3.\,189\,4i, \\ \lambda_4 = -0.787\,32 - 3.\,189\,4i, \\ \lambda_5 = 1.\,326\,9 - 0.487\,5i, \\ \lambda_6 = 1.\,326\,9 + 0.487\,5i, \\ \lambda_7 = 7.\,241\,5 \times 10^{-2} + 0.808\,74i, \\ \lambda_8 = 7.\,241\,5 \times 10^{-2} - 0.808\,74i. \end{cases}$$

we have $\lambda_i \in \mathbb{C}\backslash K^{\frac{1}{4}}, 1 \leq i \leq 8$, *by Corollary 4.2 the zero solution of (4.35) is locally asymptotically stable.*

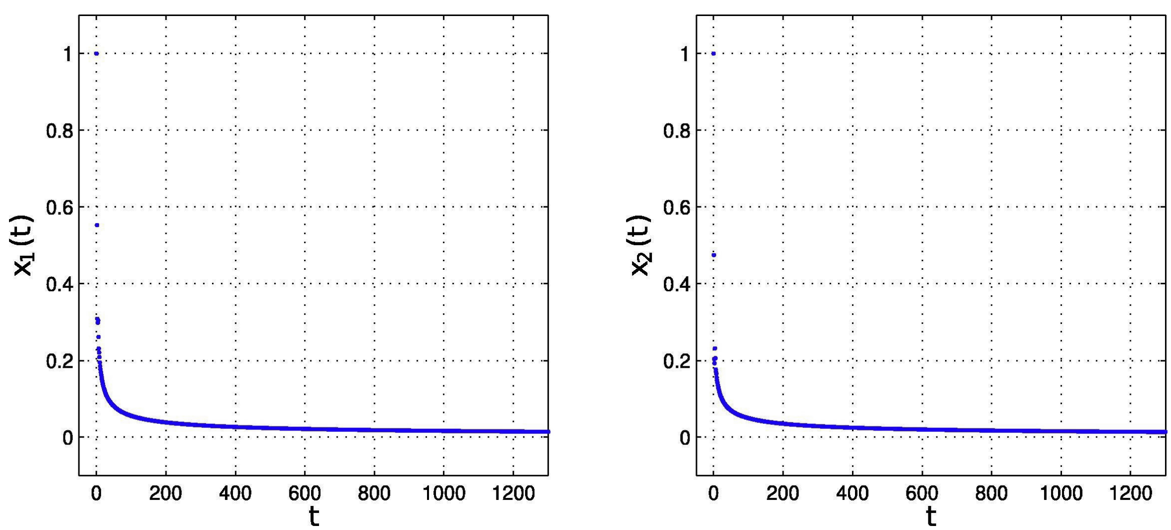

Figure 4.3: Numerical simulation of the system 4.35 using MATLAB.

Example 4.5 *Consider the non linear systems*

$$\begin{cases} {}^C\Delta_0^{\frac{1}{2}} x_1(t+1-\frac{1}{2}) = -x_1(t) + (x_3(t)-0.1)^2 - 0.01, \\ {}^C\Delta_0^{\frac{1}{3}} x_2(t+1-\frac{1}{3}) = 3.4x_1(t) - x_2(t) + 0.2x_3(t), \\ {}^C\Delta_0^{\frac{2}{3}} x_3(t+1-\frac{2}{3}) = x_1(t) + x_2^2(t) - x_3(t). \end{cases} \tag{4.37}$$

where $(x_1(0), x_2(0), x_3(0)) = (0.01, 0.05, 0.02)$. *It has an equilibrium point in the origin* $(0,0,0)$. *The Jacobian matrix is given by*

$$J = \begin{pmatrix} -1 & 0 & -0.2 \\ 3.4 & -1 & 0.2 \\ 1 & 0 & -1 \end{pmatrix}$$

we have $M = 6$, *so*

$$\det\left(\begin{pmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^4 \end{pmatrix} - (1-\lambda^6)\begin{pmatrix} -1 & 0 & -0.2 \\ 3.4 & -1 & 0.2 \\ 1 & 0 & -1 \end{pmatrix}\right) = 0$$

$\Leftrightarrow$

$$\begin{gathered} -1.2\lambda^{18} + \lambda^{16} + \lambda^{15} + 1.2\lambda^{14} - 1.0\lambda^{13} + 2.6\lambda^{12} - 1.0\lambda^{11} - 2.0\lambda^{10} \\ -1.0\lambda^9 - 2.4\lambda^8 + \lambda^7 - 2.6\lambda^6 + \lambda^5 + \lambda^4 + \lambda^3 + 1.2\lambda^2 + 1.2 = 0 \end{gathered} \tag{4.38}$$

the solution of (4.38) is

$$\begin{cases} \lambda_1 = -0.32131 - 0.87498i, \\ \lambda_2 = -0.32131 + 0.87498i, \\ \lambda_3 = 0.32131 + 0.87498i, \\ \lambda_4 = 0.32131 - 0.87498i, \\ \lambda_5 = 0.41947 - 0.75763i, \\ \lambda_6 = 0.41947 + 0.75763i, \\ \lambda_7 = 1.1553 + 7.5669 \times 10^{-2}i, \\ \lambda_8 = 1.1553 - 7.5669 \times 10^{-2}i, \\ \lambda_9 = -0.53846 + 0.73763i, \\ \lambda_{10} = -0.53846 - 0.73763i, \\ \lambda_{11} = -0.57297 + 1.0085i, \\ \lambda_{12} = -0.57297 - 1.0085i, \\ \lambda_{13} = -1.1872, \\ \lambda_{14} = -0.86379, \\ \lambda_{15} = -1.1510, \\ \lambda_{16} = 1.1510, \\ \lambda_{17} = 0.5622 + 0.74061i, \\ \lambda_{18} = 0.5622 - 0.74061i. \end{cases}$$

By Corollary 4.2 the zero solution of (4.37) is locally asymptotically stable.

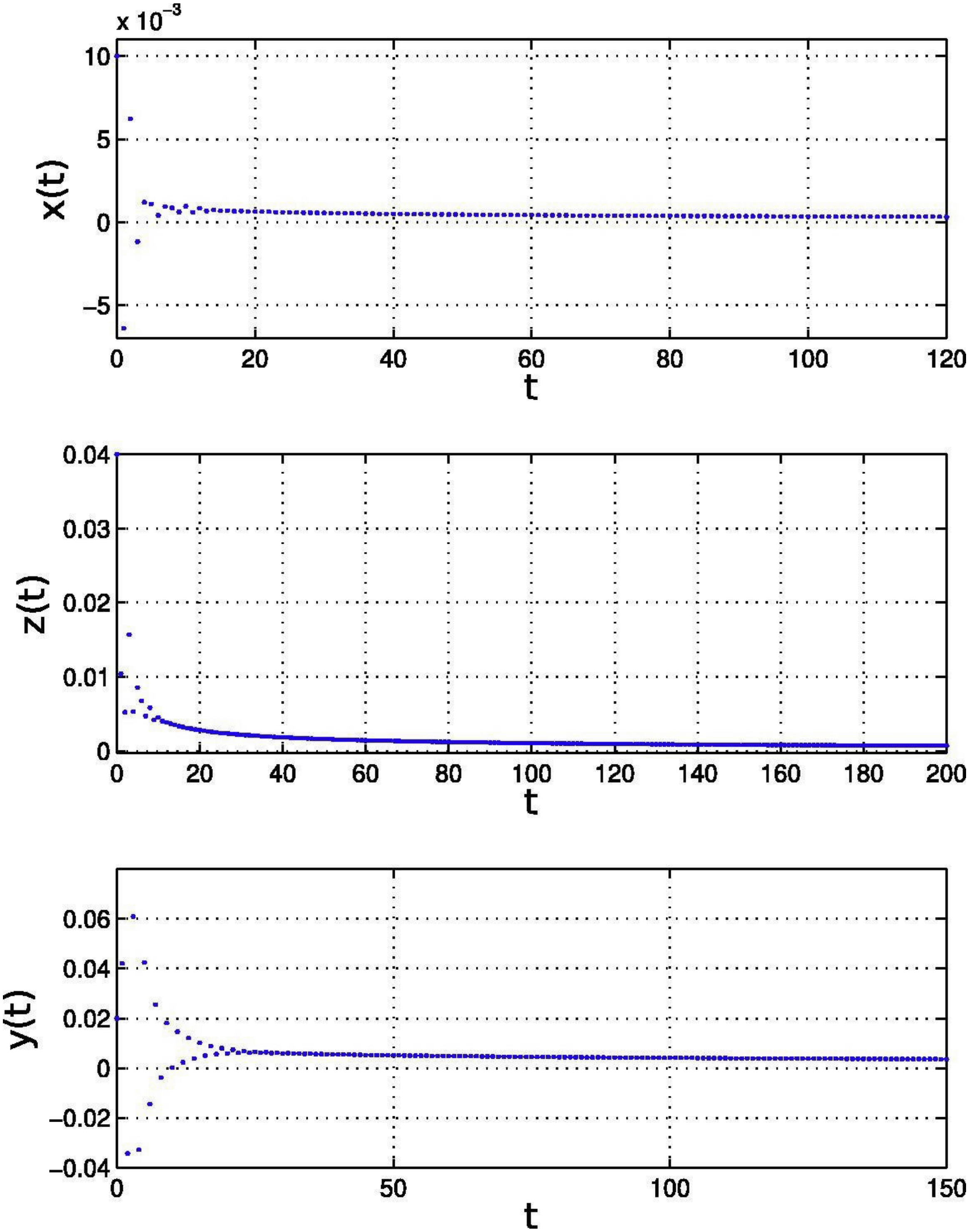

Figure 4.4: Numerical simulation of the system 4.37 using MATLAB.

4.2 Stability Analysis of Nabla Difference Systems

This section presents a study on the stability analysis of linear and nonlinear incommensurate h-nabla fractional-order difference systems. Theoretical results are derived using various theoretical approaches, including the Z-transform method, the Cauchy-Hadamard theorem, the Taylor series expansion, the final-value theorem, and the Banach fixed point theorem. These results are validated numerically through two illustrative examples that demonstrate the stability of the solutions of the systems under consideration.

Consider the following linear incommensurate FoDS:

$$ {}_{t_0}^{C}\nabla_h^{\alpha} x(t) = f(x(t)), \quad t \in \mathbb{N}_{t_0+h,h}, \tag{4.39}$$

where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in \mathbb{R}^n$, ${}_{t_0}^{C}\nabla_h^{\alpha} x(t) = \left({}_{t_0}^{C}\nabla_h^{\alpha_1} x_1(t), {}_{t_0}^{C}\nabla_h^{\alpha_2} x_2(t), ..., {}_{t_0}^{C}\nabla_h^{\alpha_n} x_n(t)\right)^T$, $0 < \alpha_i < 1$, for $i = 1, 2, \cdots, n$. $f = (f_1, f_2, \cdots, f_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ constantly differentiable twice function. To simplify, we choose $t_0 = 0$.

For $1 \leq i \leq n$ we have:

$$\frac{1}{h^{\alpha_i}} x_i(t) = \frac{1}{h^{\alpha_i}} \sum_{s=0}^{t/h-1} (-1)^{t/h-s+1} \binom{\alpha_i}{\frac{t}{h}-s} x_i(sh) + \frac{1}{h^{\alpha_i}} (-1)^{t/h} \binom{\alpha_i - 1}{\frac{t}{h}} x_i(0) + f_i(x(t)), t \in \mathbb{N}_{h,h},$$

so

$$x_i(t) = \sum_{s=0}^{t/h-1} (-1)^{t/h-s+1} \binom{\alpha_i}{\frac{t}{h}-s} x_i(sh) + (-1)^{t/h} \binom{\alpha_i - 1}{\frac{t}{h}} x_i(0) + h^{\alpha_i} f_i(x(t)), t \in \mathbb{N}_{h,h}.$$

$\Leftrightarrow$

$$x_i(th) = \sum_{s=0}^{t-1} (-1)^{t-s+1} \binom{\alpha_i}{t-s} x_i(sh) + (-1)^{t} \binom{\alpha_i - 1}{t} x_i(0) + h^{\alpha_i} f_i(x(th)), t = 1, 2, \cdots,$$

we put $y_i(t) = x_i(th)$ for $1 \leq i \leq n$ and $t = 0, 1, 2, \cdots$, we get

$$y_i(t+1) = \sum_{s=0}^{t} (-1)^{t-s} \binom{\alpha_i}{t+1-s} y_i(s) + (-1)^{t+1} \binom{\alpha_i - 1}{t+1} y_i(0) + h^{\alpha_i} f_i(y(t+1)), t = 0, 1, 2, \cdots.$$

This system is written in compact form as follow

$$y(t+1) = \sum_{s=0}^{t} B(t-s) y(s) + C(t) y(0) + H f(y(t+1)), t = 0, 1, 2, \cdots, \tag{4.40}$$

where

$$B(t) = diag\left((-1)^t \binom{\alpha_1}{t+1}, (-1)^t \binom{\alpha_2}{t+1}, \cdots, (-1)^t \binom{\alpha_n}{t+1}\right), \tag{4.41}$$

$$C(t) = diag\left((-1)^{t+1} \binom{\alpha_1 - 1}{t+1}, (-1)^{t+1} \binom{\alpha_2 - 1}{t+1}, \cdots, (-1)^{t+1} \binom{\alpha_n - 1}{t+1}\right), \tag{4.42}$$

and

$$H = diag(h^{\alpha_1}, h^{\alpha_2}, \cdots, h^{\alpha_n}). \tag{4.43}$$

We note that system (4.39) is equivalent to system (4.40). From now on, we will study the system (4.40)

4.2.1 Stability of Linear Systems

We assume that the function f is linear, this means that there is a matrix $M \in \mathbb{R}^{n\times n}$, so that the system (4.39) is written as follows

$$^{C}_{t_0}\nabla^{\alpha}_{h} x(t) = Ax(t), \quad t \in \mathbb{N}_{a+h,h}. \tag{4.44}$$

Lemma 4.1 *[135] Assume $(y(t))_{t\in\mathbb{N}}$, a sequence in $\mathbb{R}^n$, and let $\tilde{y}(z)$ be the $Z-$transform of $y(t)$ withe $\rho > 0$ the convergence radius. Then*
- *If $\rho < 1$, then $\lim\limits_{t\to\infty} y(t) = 0$.*
- *If $\rho > 1$, then $\lim\limits_{t\to\infty} y(t) = \infty$.*

Proof. Let $\rho < 1$. Based on the Final-Value Theorem associated with $Z-$transform, we obtain

$$\lim_{t\to\infty} y(t) = \lim_{z\to 1}(z-1)\tilde{y}(z) = 0. \tag{4.45}$$

On the other hand, considering the second part of this Lemma, then there exist i_0, where $1 \leq i_0 \leq n$, which makes the convergence radius of the series

$$\sum_{t=0}^{\infty} y_{i_0}(t) z^{-t} = \tilde{y}_{i_0}(z), \tag{4.46}$$

also be greater than 1 . Thus, by using the Cauchy-Hadamard Theorem, we obtain

$$\lim_{t\to\infty} \sup \sqrt[t]{|y_{i_0}(t)|} > 1. \tag{4.47}$$

Consequently, $\lim\limits_{t\to\infty} \sup |y_{i_0}(t)| = \infty$. ■

Theorem 4.4 *[135] Let $det(I - HA) \neq 0$. Then (4.44) has a unique solution for any initial vector $x_0 \in \mathbb{R}^n$. Moreover*
- *If all roots of the following characteristic equation:*

$$\det\left(diag\left((1-\frac{1}{z})^{\alpha_1}, (1-\frac{1}{z})^{\alpha_2}, ..., (1-\frac{1}{z})^{\alpha_n}\right) - HA\right) = 0, \tag{4.48}$$

lie inside the unit disk, then the zero solution is asymptotically stable.

• *If there exists a zero, say z^* of (4.48) such that $|z^*| > 1$, then the zero solution is not stable.*

Proof. System (4.44) equivalent to

$$y(t+1) = \sum_{s=0}^{t} B(t-s)y(s) + C(t)y(0) + HA(y(t+1)), t = 0, 1, 2, \cdots . \tag{4.49}$$

$\Leftrightarrow$

$$(I - HA)y(t+1) = \sum_{s=0}^{t} B(t-s)y(s) + C(t)y(0), t = 0, 1, 2, \cdots .$$

Now suppose that $\det(I - HA) \neq 0$ so we get

$$y(t+1) = \sum_{s=0}^{t} (I - HA)^{-1} B(t-s)y(s) + (I - HA)^{-1} C(t)y(0), t = 0, 1, 2, \cdots . \tag{4.50}$$

The resolvent matrix $R(t)$ of (4.50) is defined as the unique solution of the matrix equation

$$R(t+1) = \sum_{s=0}^{t} (I - HA)^{-1} B(t-s)R(s), R(0) = I, t = 1, 2, \cdots . \tag{4.51}$$

Then by the variation of constants formula [?] we obtain

$$y(t) = R(t)y(0) + \sum_{s=0}^{t-1} R(t-s-1)(I - HA)^{-1} C(s)y(0), t = 1, 2, \cdots , \tag{4.52}$$

hence the existence.

To study stability we will use $Z-$transform method. One might take the $Z-$transform to (4.52). This yields the following system

$$\begin{aligned} &\left(diag\left((1-\tfrac{1}{z})^{\alpha_1}, (1-\tfrac{1}{z})^{\alpha_2}, ..., (1-\tfrac{1}{z})^{\alpha_n}\right) - HA\right) \tilde{y}(z) = \\ &\qquad \left(diag\left((1-\tfrac{1}{z})^{\alpha_1 - 1}, (1-\tfrac{1}{z})^{\alpha_2 - 1}, ..., (1-\tfrac{1}{z})^{\alpha_n - 1}\right) - HA\right) y(0). \end{aligned} \tag{4.53}$$

where $\tilde{y}(z)$ indicates the $Z-$transform of $y(t)$.

Now, we should note that if all roots of $\det\left(diag\left((1-\frac{1}{z})^{\alpha_1}, (1-\frac{1}{z})^{\alpha_2}, ..., (1-\frac{1}{z})^{\alpha_n}\right) - HA\right) = 0$ lie inside the unit disk then

$$\begin{aligned} \tilde{y}(z) = \; & \left(diag\left((1-\tfrac{1}{z})^{\alpha_1}, (1-\tfrac{1}{z})^{\alpha_2}, ..., (1-\tfrac{1}{z})^{\alpha_n}\right) - HA\right)^{-1} \\ & \times \left(diag\left((1-\tfrac{1}{z})^{\alpha_1 - 1}, (1-\tfrac{1}{z})^{\alpha_2 - 1}, ..., (1-\tfrac{1}{z})^{\alpha_n - 1}\right) - HA\right) y(0), \end{aligned} \tag{4.54}$$

for $|z| \geq 1$. By Lemma 4.1

$$\lim_{t\to\infty} y(t) = 0.$$

If there exists a zero, say z^* of (4.48) such that $|z^*| > 1$, then

$$\lim_{t\to\infty} y(t) = \infty.$$

■

Corollary 4.4 *[135] Suppose that α_i's are rational numbers between 0 and 1, for $i = 1, 2, ..., n$. Let M be the lowest common multiple (LCM) of the denominators u_i of α_i's, where $\alpha_i = \frac{v_i}{u_i}, (u_i, v_i) = 1, u_i, v_i \in \mathbb{Z}_+, i = 1, 2, ..., n$, and set $\gamma = \frac{1}{M}$. Then the zero solution of (4.44) is*

- *asymptotically stable if and only if any zero solution of the polynomial*

$$\det\left(diag\left(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, ..., \lambda^{M\alpha_n}\right) - HA\right) = 0, \tag{4.55}$$

lie inside the set

$$\mathbb{C}\backslash K^\gamma,$$

where

$$K^\gamma = \left\{z \in \mathbb{C} : |z| \leq \left(2\cos\frac{\arg z}{\gamma}\right)^\gamma \text{ and } |\arg z| \leq \frac{\gamma\pi}{2}\right\}. \tag{4.56}$$

- *Furthermore, If there is a zero λ of (4.55) with $\lambda \in IntK^\gamma$, the zero solution is not stable.*

4.2.2 Stability of Non-Linear Systems

Theorem 4.5 *[135] Let 0 be an equilibrium point of (4.39). If all roots of the characteristic equation*

$$\det\left(diag\left((1-\frac{1}{z})^{\alpha_1}, (1-\frac{1}{z})^{\alpha_2}, ..., (1-\frac{1}{z})^{\alpha_n}\right) - HJ\right) = 0, \tag{4.57}$$

where J is the jacobian matrix of f at 0, lie inside the unit disk, then (4.39) has a unique solution for all initial vectors close enough to 0 and, moreover, 0 is asymptotically stable.

Proof. Let's start with the study of existence and uniqueness. System (4.39) equivalent to

$$y(t+1) = \sum_{s=0}^{t} B(t-s)y(s) + C(t)y(0) + Hf(y(t+1)), t = 0, 1, 2, \cdots. \tag{4.58}$$

Using taylor development:

$$y(t+1) = \sum_{s=0}^{t} B(t-s)y(s) + C(t)y(0) + HJy(t+1) + Hg(y(t+1)), \tag{4.59}$$

where $\|g(y(t+1))\| = o(\|y(t+1)\|)$, $\|.\|$ is an norm in $\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$ (as needed) with $\|I\| = 1$, and $g(0) = g'(0) = 0$ with $g'(0)$ is the jacobian matrix of g at 0 . Now suppose that $\det(I - HJ) \neq 0$ so we get

$$y(t+1) = \sum_{s=0}^{t}(I-HJ)^{-1}B(t-s)y(s)+(I-HJ)^{-1}C(t)y(0)+(I-HJ)^{-1}Hg(y(t+1)), t = 0, 1, 2, \cdots. \tag{4.60}$$

Taking the homogeneous part of (4.60)

$$y(t+1) = \sum_{s=0}^{t} (I - HJ)^{-1} B(t-s) y(s). \tag{4.61}$$

We have $(I - HJ)^{-1} B(t) \in \left[\ell^1(\mathbb{N})\right]^{n\times n}$, the resolvent matrix $R(t)$ of (4.61) is defined as the unique solution of the matrix equation

$$R(t+1) = \sum_{s=0}^{t} (I - HJ)^{-1} B(t-s) R(s), R(0) = I, t \in \mathbb{N},$$

where by Theorem 2 in [109] $R(t) \in \left[\ell^1(\mathbb{N})\right]^{n\times n}$. Then by the variation of constants formula we obtain

$$y(t) = R(t)y(0) + \sum_{s=0}^{t-1} R(t-s-1)(I - HJ)^{-1} \left(Hg(y(s+1)) + C(s)y(0)\right),\ t = 1, 2, \cdots, \tag{4.62}$$

$\Rightarrow$

$$y(t) = T_{y_0} y(t),\ t = 1, 2, \cdots,$$

where T_{y_0} is an operator defined for any initial condition $y(0) = y_0$ as follow

$$T_{y_0} y(t) = R(t)y(0) + \sum_{s=0}^{t-1} R(t-s-1)(I - HJ)^{-1} \left(Hg(y(s+1)) + C(s)y(0)\right).$$

To prove existence and uniqueness, it is sufficient to prove that the operator T_{y_0} is a contraction on a closed ball $\bar{B}(0, u)$ in a banach space for a $u > 0$, and $T_{y_0}\bar{B}(0, u) \subseteq \bar{B}(0, u)$, (banach fixed point Theorem).

Let $y_1, y_2 \in [\ell^\infty(\mathbb{N})]^n$ we have

$$\begin{aligned} \|T_{y_0} y_1(t) - T_{y_0} y_2(t)\| &= \left\| \sum_{s=0}^{t-1} R(t-s-1)(I - HJ)^{-1} H \left(g(y_1(s+1)) - g(y_2(s+1))\right) \right\|, \\ &\leq C_0 \sum_{s=0}^{t-1} \|R(t-s-1)\| \, \|g(y_1(s+1)) - g(y_2(s+1))\|, \\ &\leq C_0 \|R\|_{\ell^1(\mathbb{R}^{n\times n})} \|(g(y_1) - g(y_2))\|_{[\ell^\infty(\mathbb{N})]^n}, \end{aligned}$$

where $C_0 = \left\|(I - HJ)^{-1} H\right\|$. Note that for $y_1, y_2 \in \bar{B}(0, u), u > 0$

$$\|(g(y_1) - g(y_2))\|_{[\ell^\infty(\mathbb{N})]^n} \leq \max_{y \in \bar{B}(0,u)} \left\|g'(y)\right\| \|y_1 - y_2\|_{[\ell^\infty(\mathbb{N})]^n},$$

and since $g'(y) \to 0$ when $y \to 0$, we can choose u achived

$$\max_{y \in \bar{B}(0,u)} C_0 \left\|g'(y)\right\| \|R\|_{[\ell^1(\mathbb{N})]^{n\times n}} = L < 1,$$

$\Rightarrow$

$$\|T_{y_0} y_1 - T_{y_0} y_2\|_{[\ell^\infty(\mathbb{N})]^n} \leq L \|y_1 - y_2\|_{[\ell^\infty(\mathbb{N})]^n},$$

so T_{y_0} is a contraction on $\bar{B}(0,u) \subset [\ell^\infty(\mathbb{N})]^n$.
On the other hand for $y \in \bar{B}(0,u)$

$$\begin{aligned}
\|T_{y_0}y(t)\| &\leq \|R(t)y(0)\| + \sum_{s=0}^{t-1} \left\|R(t-s-1)(I-HJ)^{-1}\left(Hg(y(s+1)) + C(s)y(0)\right)\right\|, \\
&\leq \|R\|_{[\ell^1(\mathbb{N})]^{n\times n}} \|y(0)\| + \sum_{s=0}^{t-1} C_0 \|R(t-s-1)g(y(s+1))\| + M\|R(t-s-1)C(s)y(0)\|, \\
&\leq \|R\|_{[\ell^1(\mathbb{N})]^{n\times n}} \|y(0)\| + L\|y\|_{[\ell^\infty(\mathbb{N})]^n} + M\|R\|_{[\ell^1(\mathbb{N})]^{n\times n}} \|C(s)\|_{[\ell^\infty(\mathbb{N})]^{n\times n}} \|y(0)\|, \\
&\leq \left(1 + M\|C\|_{[\ell^\infty(\mathbb{N})]^{n\times n}}\right) \|R\|_{[\ell^1(\mathbb{N})]^{n\times n}} \|y(0)\| + L\|y\|_{[\ell^\infty(\mathbb{N})]^n}, \\
&\leq \left(1 + M\|C\|_{[\ell^\infty(\mathbb{N})]^{n\times n}}\right) \|R\|_{[\ell^1(\mathbb{N})]^{n\times n}} \|y(0)\| + Lu,
\end{aligned}$$

where $M = \left\|(I-HJ)^{-1}\right\|$. Since $L<1$,we can note that for any y_0 close enough to 0

$$\|T_{y_0}y\|_{[\ell^\infty(\mathbb{N})]^n} \leq u,$$

so

$$TB(0,u) \subseteq B(0,u),$$

for any y_0 close enough to 0. That's complete the proof of existence and uniqueness.
Now we move to study the stability. We have from (4.62)

$$\|y(t)\| \leq \|R(t)\|\,\|y(0)\| + C_0 \sum_{s=0}^{t-1} \|R(t-s-1)\|\, o(\|y(s+1)\|) + M\sum_{s=0}^{t-1} \|R(t-s-1)\|\,\|C(s)\|, \tag{4.63}$$

for a given $1 > \epsilon > 0$ there is $\delta > 0$ such that $o(\|y\|) < \epsilon\|y\|$ whenever $\|y\| < \delta$. So as long as $\|y(s+1)\| < \delta$, we have

$$\|y(t)\| \leq \|R(t)\|\,\|y(0)\| + \epsilon C_0 \sum_{s=0}^{t-1} \|R(t-s-1)\|\,\|y(s+1)\| + M\sum_{s=0}^{t-1} \|R(t-s-1)\|\,\|C(s)\|,$$

$\Rightarrow$

$$\|y(t)\| \leq \begin{array}{l} \frac{\|R(t)\|\|y(0)\|}{(1-\epsilon C_0)} + \frac{\epsilon C_0}{(1-\epsilon C_0)} \sum_{s=0}^{t-2} \|R(t-s-1)\|\,\|y(s+1)\| \\ + \frac{M}{(1-\epsilon C_0)} \sum_{s=0}^{t-1} \|R(t-s-1)\|\,\|C(s)\| \end{array}, \quad t = 1,2,\cdots,$$

we defined $w(t)$ as follow

$$w(t+1) = r(t+1)w(0) + \epsilon C_0 \sum_{s=0}^{t-1} r(t-s)w(s+1) + M\sum_{s=0}^{t} r(t-s)c(s), t = 0,1,2,\cdots, \tag{4.64}$$

where

$$r(t) = \frac{\|R(t)\|}{(1-\epsilon C_0)}, \quad c(t) = \|C(t)\|, \quad w(0) = \|y(0)\|.$$

We have

$$\|y(t)\| \leq w(t), t = 0,1,2,\cdots.$$

We see that $r(t) \in \ell^1(\mathbb{N})$. Taking the $Z-$transform on both sides of (4.64) gives

$$z\left(\frac{1}{(1-\epsilon C_0)} - \epsilon C_0 \tilde{r}(z)\right)\tilde{w}(z) = (1-\epsilon C_0)\, z\tilde{r}(z)w(0) + M\tilde{r}(z)\tilde{c}(z),$$

with $R_r \leq 1$, and $R_c = 1$, where R_r is the convergence radius of $\tilde{r}(z)$ and R_c is the convergence radius of $\tilde{c}(z)$.

We choose ϵ achieve

$$\left(\frac{1}{(1-\epsilon C_0)} - \epsilon C_0 \tilde{r}(z)\right) \neq 0, |z| \geq 1, \tag{4.65}$$

This option is possible because $\tilde{r}(z)$ is bonded when $|z| \geq 1$. From this we get

$$\tilde{w}(z) = z^{-1}\left(\frac{1}{(1-\epsilon C_0)} - \epsilon C_0 \tilde{r}(z)\right)^{-1}\left((1-\epsilon)\, z\tilde{r}(z)w(0) + M\tilde{r}(z)\tilde{c}(z)\right),$$

for $|z| > 1$. We have

$$\lim_{z\to 1}(z-1)z^{-1}\left(\frac{1}{(1-\epsilon C_0)} - \epsilon C_0 \tilde{r}(z)\right)^{-1}(1-\epsilon C_0)\, z\tilde{r}(z)w(0) = 0, \tag{4.66}$$

and from the proof of Theorem 4.5 we have

$$\lim_{z\to 1}(z-1)\tilde{r}(z)\tilde{c}(z) = 0. \tag{4.67}$$

By final value theorem

$$\begin{aligned}
\lim_{t\to\infty} w(t) &= \lim_{z\to 1}(z-1)\tilde{w}(z) \\
&= \lim_{z\to 1}(z-1)z^{-1}\left(\tfrac{1}{(1-\epsilon C_0)} - \epsilon C_0 \tilde{r}(z)\right)^{-1}\left((1-\epsilon C_0)\, z\tilde{r}(z)w(0) + M\tilde{r}(z)\tilde{c}(z)\right) \\
&= 0.
\end{aligned}$$

and that's imply

$$\lim_{t\to\infty} y(t) = 0.$$

■

Corollary 4.5 *[135] Suppose that α_i's are rational numbers between 0 and 1, for $i = 1, 2, ..., n$. Let M be the lowest common multiple (LCM) of the denominators u_i of α_i's, where $\alpha_i = \frac{v_i}{u_i}$, $(u_i, v_i) = 1, u_i, v_i \in \mathbb{Z}_+, i = 1, 2, ..., n$, and set $\gamma = \frac{1}{M}$. Then the zero solution of system (7) with initial value $x_0 = x(0)$ is locally asymptotically stable if any zero solution of the polynomial equation*

$$\det\left(diag\left(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, ..., \lambda^{M\alpha_n}\right) - HJ\right) = 0, \tag{4.68}$$

lie inside the set

$$\mathbb{C}\backslash K^{\gamma},$$

where

$$K^{\gamma} = \left\{ z \in \mathbb{C} : |z| \leq \left(2\cos\frac{\arg z}{\gamma} \right)^{\gamma} \text{ and } |\arg z| \leq \frac{\gamma\pi}{2} \right\}.$$

and J is the jacobian matrix of f at 0.

Example 4.6 *[135] Consider the following nonlinear incommensurate FoDS:*

$$\begin{cases} {}_{0}^{C}\nabla_{1}^{\frac{1}{2}} x_1(t) = -1.01\sin(x_1(t)) + 0.98\sin(x_2(t)), \\ {}_{0}^{C}\nabla_{1}^{\frac{1}{4}} x_2(t) = 0.48x_2(t)\cos(x_1(t)) - 0.56x_1(t)\cos(x_2(t)). \end{cases} \tag{4.69}$$

Note that the origin represents an equilibrium point of system (4.69). Furthermore, the Jacobian matrix J has the following form:

$$\frac{\partial f(0,0)}{\partial x} = \begin{pmatrix} \frac{\partial f_1(0,0)}{\partial x_1} & \frac{\partial f_1(0,0)}{\partial x_2} \\ \frac{\partial f_2(0,0)}{\partial x_1} & \frac{\partial f_2(0,0)}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -1.01 & 0.98 \\ -0.56 & 0.48 \end{pmatrix},$$

Here $M = 4$, and therefore

$$\det\left(\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} -1.01 & 0.98 \\ -0.56 & 0.48 \end{pmatrix} \right) = 0,$$

$\Leftrightarrow$

$$\lambda^3 - 0.48\lambda^2 + 1.01\lambda + 0.064 = 0. \tag{4.70}$$

Consequently, the solution of this characteristic equation will be as follows:

$$\begin{cases} \lambda_1 = -6.1349 \times 10^{-2}, \\ \lambda_2 = 0.27067 - 0.98486i, \\ \lambda_3 = 0.27067 + 0.98486i. \end{cases}$$

Obviously, one can deduce that $\lambda_i \in \mathbb{C}\backslash K^{\frac{1}{4}}, 1 \leq i \leq 3$. Hence, in view of Corollary 4.5, one can deduce that the trivial solution of system (4.69) is locally asymptotically stable.

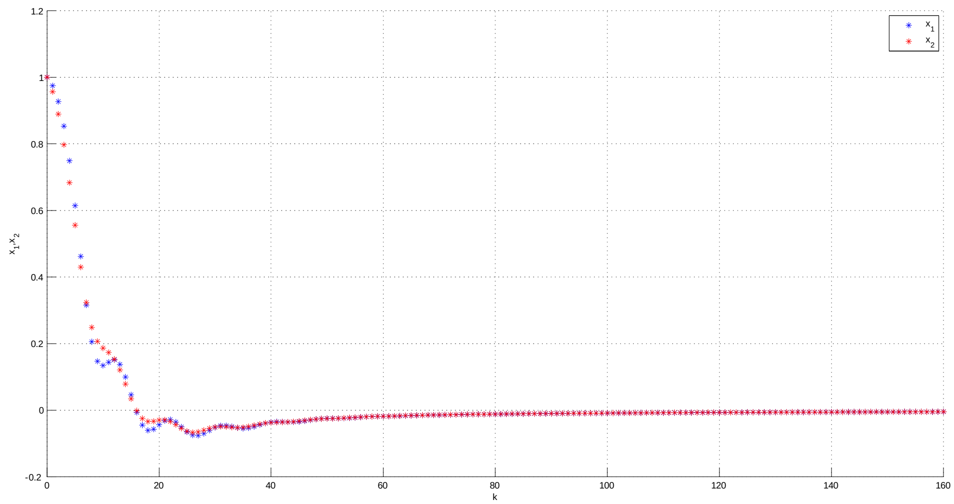

Figure 4.5: Numerical simulation of the system 4.69 using MATLAB.

Example 4.7 *[135] Consider the nonlinear incommensurate FoDS:*

$$\begin{cases} {}_0^C\nabla_1^{\frac{1}{2}} x_1(t) = -0.98\sin\left(x_1(t)\right) - 0.2\sin(x_3(t)), \\ {}_0^C\nabla_1^{\frac{1}{3}} x_2(\nu) = 0.2x_1(t) - 1.01x_2(t) + 0.02x_3(t), \\ {}_0^C\nabla_1^{\frac{2}{3}} x_3(\nu) = 0.4\sin\left(x_1(t)\right) + \frac{x_2^2(t)}{1+x_2^2(t)} - \sin\left(x_3(t)\right). \end{cases} \tag{4.71}$$

In order to handle this system, it is of course necessary to realize that the origin $(0,0,0)$ is an equilibrium point. Besides, the Jacobian matrix J is of the form:

$$J = \begin{pmatrix} -0.98 & 0 & -0.2 \\ 0.2 & -1.01 & 0.02 \\ 0.4 & 0 & -1 \end{pmatrix}.$$

Observe that $M = 6$, and so

$$\det\left(\begin{pmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^4 \end{pmatrix} - \begin{pmatrix} -0.98 & 0 & -0.2 \\ 0.2 & -1.01 & 0.02 \\ 0.4 & 0 & -1 \end{pmatrix}\right) = 0,$$

$\Leftrightarrow$

$$\lambda^9 + 1.01\lambda^7 + 0.98\lambda^6 + \lambda^5 + 0.989\,8\lambda^4 + 1.01\lambda^3 + 1.06\lambda^2 + 1.070\,6 = 0. \tag{4.72}$$

Accordingly, the solution of (4.72) will be as follows:

$$\begin{cases} \lambda_1 = -1.006\,5, \\ \lambda_2 = -1.005\,0i, \\ \lambda_3 = 1.005\,0i, \\ \lambda_4 = 0.485\,24 - 0.881\,98i, \\ \lambda_5 = 0.485\,24 + 0.881\,98i, \\ \lambda_6 = 0.729\,16 + 0.698\,26i, \\ \lambda_7 = 0.729\,16 - 0.698\,26i, \\ \lambda_8 = -0.711\,18 - 0.716\,88i, \\ \lambda_9 = -0.711\,18 + 0.716\,88i. \end{cases}$$

Obviously, one can deduce that $\lambda_i \in \mathbb{C} \backslash K^{\frac{1}{6}}, 1 \leq i \leq 9$. *Hence, in view of Corollary 4.5, one can deduce that the trivial solution of system (4.72) is locally asymptotically stable.*

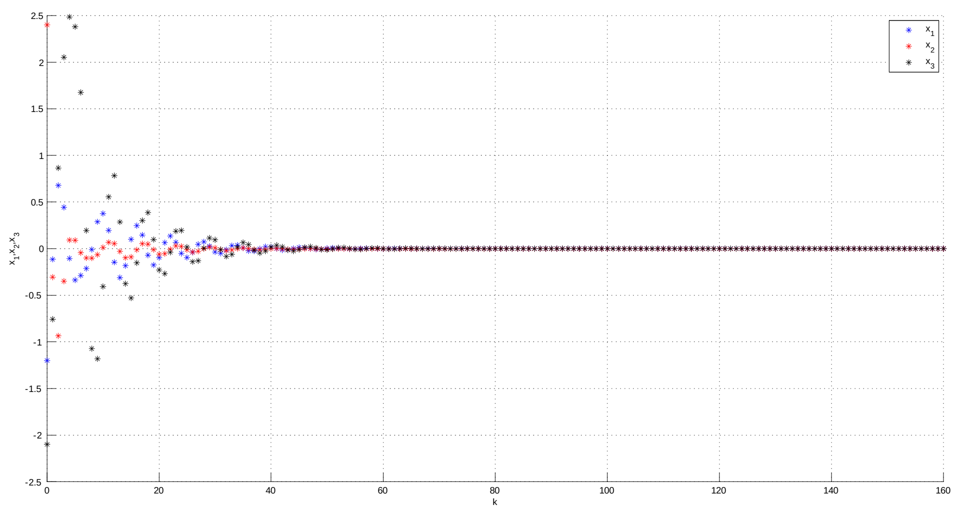

Figure 4.6: Numerical simulation of the system 4.71 using MATLAB.

4.3 Stability Exploration in Fractional h-Difference Equations with Incommensurate Orders

This section conducts a comprehensive stability analysis for h-Fractional Order Differences (h-FoDs) in both linear and nonlinear states, considering commensurate and incommensurate orders. The stability findings are encapsulated in relevant theorems. Numerical examples are provided to validate the theoretical results, demonstrating their applicability across various domains.

To initiate our exploration, let's examine an incommensurate h-FoDS, where for the sake of simplicity, we set $t_0 = 0$:

$$\begin{array}{c} ({}_0^C\Delta_h^{\alpha_1})x(t+h-\alpha_1 h) = f_1(x(t)) \\ ({}_0^C\Delta_h^{\alpha_2})x(t+h-\alpha_2 h) = f_2(x(t)) \\ \vdots \\ ({}_0^C\Delta_h^{\alpha_n})x(t+h-\alpha_n h) = f_n(x(t)) \end{array}, \quad t \in \mathbb{N}_{t_0,h}, \tag{4.73}$$

where $x(t) \in \mathbb{R}^n$, $0 < \alpha_i < 1$, for $1 \leq i \leq n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ constantly differentiable twice function.

As is known, the fractional system (4.73) can be written as a sum equation, S. Elaydi's extensive contributions to the stability analysis of Volterra difference systems of convolution type have laid a solid foundation [109, 108], serving as a valuable framework for our own research endeavors. Next we will present the Volterra difference equation equivalent to (4.73).

Lemma 4.2 *[136] The system (4.73) is equivalent to the Volterra system*

$$\begin{cases} x_1(th+h) &= \sum_{i=0}^{t}(-1)^{t-i}\binom{\alpha_1}{t-i+1}x_1(i) + (-1)^{t+1}\binom{\alpha_1-1}{t+1}x_1(0) + h^{\alpha_1}f_1(x(t)), \\ x_2(th+h) &= \sum_{i=0}^{t}(-1)^{t-i}\binom{\alpha_2}{t-i+1}x_2(i) + (-1)^{t+1}\binom{\alpha_2-1}{t+1}x_2(0) + h^{\alpha_2}f_2(x(t)), \\ \vdots & \\ x_n(th+h) &= \sum_{i=0}^{t}(-1)^{t-i}\binom{\alpha_n}{t-i+1}x_n(i) + (-1)^{t+1}\binom{\alpha_n-1}{t+1}x_n(0) + h^{\alpha_n}f_n(x(t)) \end{cases} \quad t = 0, 1, \cdots. \tag{4.74}$$

Proof. We have by (1.55)

$$\begin{aligned} ({}_0^C\Delta_h^{\alpha}x)((t+1-\alpha)h) &= \frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{t}(th-\alpha h-ih)_h^{(-\alpha)}(\Delta_h x)(ih)h, \\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{t}h^{-\alpha}\frac{\Gamma(t-\alpha-i+1)}{\Gamma(t-i+1)}(\Delta_h x)(ih)h, \\ &= \sum_{i=0}^{t}h^{-\alpha}\binom{t-\alpha-i}{t-i}(\Delta_h x)(ih)h, \\ &= h^{-\alpha}\sum_{i=1}^{t}\left[\binom{t-\alpha-i+1}{t-i+1}-\binom{t-\alpha-i}{t-i}\right]x(ih) - h^{-\alpha}\binom{t-\alpha}{t}x(0) + h^{-\alpha}\binom{-\alpha}{0}x(th+h). \end{aligned}$$

Using the Pascal rule

$$\begin{aligned} ({}_0^C\Delta_h^{\alpha}x)((t+1-\alpha)h) &= h^{-\alpha}\sum_{i=1}^{t}\binom{t-\alpha-i}{t-i+1}x(ih) - h^{-\alpha}\binom{t-\alpha}{t}x(0) + h^{-\alpha}x(th+h), \\ &= h^{-\alpha}\sum_{i=0}^{t}\binom{t-\alpha-i}{t-i+1}x(ih) - h^{-\alpha}\left[\binom{t-\alpha}{t}+\binom{t-\alpha}{t+1}\right]x(0) + h^{-\alpha}x(th+h), \\ &= h^{-\alpha}\sum_{i=0}^{t+1}(-1)^{t-i+1}\binom{\alpha}{t-i+1}x(ih) - h^{-\alpha}(-1)^{t+1}\binom{\alpha-1}{t+1}x(0). \end{aligned}$$

So (4.73) equivalent to

$$\left\{\begin{array}{l} h^{-\alpha_1}\sum\limits_{i=0}^{t+1}(-1)^{t-i+1}\binom{\alpha_1}{t-i+1}x_1(ih) - h^{-\alpha_1}(-1)^{t+1}\binom{\alpha_1-1}{t+1}x_1(0) = f_1(x(t)),\\ h^{-\alpha_2}\sum\limits_{i=0}^{t+1}(-1)^{t-i+1}\binom{\alpha_2}{t-i+1}x_2(ih) - h^{-\alpha_2}(-1)^{t+1}\binom{\alpha_2-1}{t+1}x_2(0) = f_2(x(t)),\\ \vdots\\ h^{-\alpha_n}\sum\limits_{i=0}^{t+1}(-1)^{t-i+1}\binom{\alpha_n}{t-i+1}x_n(ih) - h^{-\alpha_n}(-1)^{t+1}\binom{\alpha_n-1}{t+1}x_n(0) = f_n(x(t)), \end{array}\right. \quad t \in \mathbb{N},$$

$\Leftrightarrow$

$$\left\{\begin{array}{l} x_1(th+h) = \sum\limits_{i=0}^{t}(-1)^{t-i}\binom{\alpha_1}{t-i+1}x_1(ih) + h^{\alpha_1}f_1(x(t)) + (-1)^{t+1}\binom{\alpha_1-1}{t+1}x_1(0),\\ x_2(th+h) = \sum\limits_{i=0}^{t}(-1)^{t-i}\binom{\alpha_2}{t-i+1}x_2(ih) + h^{\alpha_2}f_2(x(t)) + (-1)^{t+1}\binom{\alpha_2-1}{t+1}x_2(0),\\ \vdots\\ x_n(th+h) = \sum\limits_{i=0}^{t}(-1)^{t-i}\binom{\alpha_n}{t-i+1}x_n(ih) + h^{\alpha_n}f_n(x(t)) + (-1)^{t+1}\binom{\alpha_n-1}{t+1}x_n(0), \end{array}\right. \quad t \in \mathbb{N}.$$

■

Next, we will present the key findings of our research, subsequently validated through numerical verification. These findings will unequivocally demonstrate the stability of the h-FoDS under both commensurate and incommensurate orders, elucidated through the formulation of useful conditions expressed as theorems.

4.4 Stability of Linear h-Difference Systems

Linear systems are considered very important in any theoretical framework. All stability results must be studied first Linear equations due to their simplicity and their ability to approximate nonlinear systems. For this purpose, we will consider the system:

$$\left\{\begin{array}{l} ({}_0^C\Delta_h^{\alpha_1})x(t+h-\alpha_1 h) = h^{\alpha_1}a_{11}x_1(t) + h^{\alpha_1}a_{12}x_2(t) + \cdots + h^{\alpha_1}a_{1n}x_n(t),\\ ({}_0^C\Delta_h^{\alpha_2})x(t+h-\alpha_2 h) = h^{\alpha_2}a_{21}x_1(t) + h^{\alpha_2}a_{22}x_2(t) + \cdots + h^{\alpha_2}a_{2n}x_n(t),\\ \vdots\\ ({}_0^C\Delta_h^{\alpha_n})x(t+h-\alpha_n h) = h^{\alpha_n}a_{n1}x_1(t) + h^{\alpha_n}a_{n2}x_2(t) + \cdots + h^{\alpha_n}a_{nn}x_n(t), \end{array}\right. , \quad t \in \mathbb{N}_{a,h}, \tag{4.75}$$

where $x(0) = x_0 \in \mathbb{R}^n$, $0 < \alpha_i \leq 1$, $1 \leq i \leq n$. If $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$, then system (4.75) can be written as

$${}_0^C\Delta_h^{\alpha}x(t+1-\alpha) = Ax(t), \quad t = 0, 1, \cdots, \tag{4.76}$$

where $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$. Let

$$H = diag\left(h^{\alpha_1}, h^{\alpha_2}, \cdots, h^{\alpha_n}\right). \tag{4.77}$$

First, we will present the theoretical framework the stability of system (4.75).

Lemma 4.3 *[136] If the solutions of the following nonlinear equation*

$$\det\left(diag\left(z(1-z^{-1})^{\alpha_1}, z(1-z^{-1})^{\alpha_2}, ..., z(1-z^{-1})^{\alpha_n}\right) - HA\right) = 0, \tag{4.78}$$

lie in the interior of unit disk. Then the zero solution of (4.75) is asymptotically stable.
If there exists a solution outside the unit disk for (4.78), then the zero solution of the linear system (4.75) is unstable.

Proof. Applying Lemma 4.2 and performing $Z-$transform

$$C(z).\begin{pmatrix} x_1(z) \\ x_2(z) \\ \vdots \\ x_n(z) \end{pmatrix} = \begin{pmatrix} z(1-\frac{1}{z})^{\alpha_1-1}x_1(0) \\ z(1-\frac{1}{z})^{\alpha_2-1}x_2(0) \\ \vdots \\ z(1-\frac{1}{z})^{\alpha_n-1}x_n(0) \end{pmatrix}, \tag{4.79}$$

where $x_i(z)$ is the $Z-$transform of $x_i(th), 1 \leq i \leq n$. In which

$$C(z) = diag\left(z(1-z^{-1})^{\alpha_1}, z(1-z^{-1})^{\alpha_2}, ..., z(1-z^{-1})^{\alpha_n}\right) - HA. \tag{4.80}$$

Multiplying $(z-1)$ on both sides of (4.79) gives

$$C(z)\begin{pmatrix} (z-1)x_1(z) \\ (z-1)x_2(z) \\ \vdots \\ (z-1)x_n(z) \end{pmatrix} = \begin{pmatrix} z^2(1-\frac{1}{z})^{\alpha_1}x_1(0) \\ z^2(1-\frac{1}{z})^{\alpha_2}x_2(0) \\ \vdots \\ z^2(1-\frac{1}{z})^{\alpha_n}x_n(0) \end{pmatrix}.$$

Assuming that all roots of the characteristic equation $\det C(z) = 0$ and using the final-value theorem of $Z-$transform, we get:

$$\lim_{t\to\infty} x_i(th) = \lim_{z\to 1}(z-1)x_i(z) = 0, \quad \forall i.$$

If there is a solution at least for (4.78) outside the unit disk by Cauchy-Hadamard

$$R_{i_0} = \lim_{t\to\infty} \sup \sqrt[k]{|x_{i_0}(th)|} > 1,$$

for some $1 \leq i_0 \leq n$, and consequently, $\lim\limits_{t\to\infty} \sup |x_{i_0}(th)| = \infty$. ■

The solutions of (4.78) can be found using numerical methods and MATLAB program, but in the following we will extract a result from the previous Lemma, which will make the equation a polynomial.

Theorem 4.6 *[136] Suppose that $\alpha_i = \frac{v_i}{u_i}, (u_i, v_i) = 1, u_i, v_i \in \mathbb{Z}_+, 1 \leq i \leq n$. Consider M as the least common multiple (LCM) of u_i.*
• If the solutions of the following equation

$$\det\left(diag\left(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, ..., \lambda^{M\alpha_n}\right) - (1-\lambda^M)HA\right) = 0, \tag{4.81}$$

lie inside the set

$$\left\{z \in \mathbb{C} : |\arg z| > \frac{\pi}{2M} \text{ or } |z| > (2\cos c\,|\arg z|)^{\frac{1}{M}}\right\}, \tag{4.82}$$

Then the zero solution of the linear system (4.75) is asymptotically stable.
• If there is a solution at least for (4.81) belongs to

$$\left\{z \in \mathbb{C} : |\arg z| < \frac{\pi}{2M} \text{ and } |z| < (2\cos c\,|\arg z|)^{\frac{1}{M}}\right\}, \tag{4.83}$$

the zero solution of (4.75) is not stable.

Proof. Lemma 4.3 implies that (4.75) achieves asymptotic stability when every zero of:

$$\det\left(diag\left(z(1-z^{-1})^{\alpha_1}, z(1-z^{-1})^{\alpha_2}, ..., z(1-z^{-1})^{\alpha_n}\right) - HA\right) = 0$$

are located in the interior of unit disk
Let

$$1 - \frac{1}{z} = \lambda^M \Leftrightarrow z = \frac{1}{1-\lambda^M}, \lambda^M \neq 1,$$

the characteristic equation become

$$\det\left(diag\left(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, ..., \lambda^{M\alpha_n}\right) - (1-\lambda^M)HA\right) = 0.$$

And acording to [?]

$$z \in \{z \in \mathbb{C} : 0 < |z| < 1\} \Leftrightarrow \lambda \notin \left\{z \in \mathbb{C} : |\arg z| \leq \frac{\pi}{2M} \text{ and } |z| \leq (2\cos M\,|\arg z|)^{\frac{1}{M}}\right\}.$$

and

$$z \in \{z \in \mathbb{C} : |z| > 1\} \Leftrightarrow \lambda \in \left\{z \in \mathbb{C} : |\arg z| < \frac{\pi}{2M} \text{ and } |z| < (2\cos M\,|\arg z|)^{\frac{1}{M}}\right\}.$$

■

We can get a simple result depending on the eigenvalues if the matrix is triangular or the α_i values are equal.

Corollary 4.6 *[136] Since A is a triangular matrix with diagonal elements $\lambda_i, 1 \leq i \leq n$, we have: the zero solution of (4.75) achieves asymptotic stability if:*

$$-\frac{2^{\alpha_i}}{h^{\alpha_i}} < \lambda_i < 0, i = 1, ..., n. \tag{4.84}$$

Furthermore, if $\lambda_i > 0$ or $\lambda_i < -\frac{2^{\alpha_i}}{h^{\alpha_i}}$ for some $0 \leq i \leq n$, the zero solution of (4.75) is not stable.

Proof. Applying Theorem 4.6 since A is a triangular (4.78) become

$$\prod_{i=1}^{n}\left(z(1-z^{-1})^{\alpha_i}-h^{\alpha_i}\lambda_i\right)=0.$$

All solutions of (4.78) lie in the interior of unit disk, mean all the $h^{\alpha_i}\lambda_i$'s belongs to

$$\left\{z(1-z^{-1})^{\alpha_i},\ z\in\mathbb{C}\text{ and }|z|<1\right\},$$

for $0\le i\le n$, by [116]

$$\left\{z(1-z^{-1})^{\alpha_i},\ z\in\mathbb{C}\text{ and }|z|<1\right\}=\left\{z\in\mathbb{C}:|\arg z|>\frac{\alpha_i\pi}{2}\text{ and }|z|<\left(2\cos\frac{|\arg z|-\pi}{2-\alpha}\right)^{\alpha_i}\right\},$$

where $\lambda_i\in\mathbb{R}$, this means $-2^{\alpha_i}<h^{\alpha_i}\lambda_i<0$. ■

Theorem 4.7 *[136] Suppose that*

$$\alpha_1=\alpha_2=...=\alpha_n=\alpha,\alpha\in(0,1)$$

If

$$\lambda\in\left\{z\in\mathbb{C}:|\arg z|>\frac{\alpha\pi}{2}\textit{ and }|z|<\left(2h^{-\alpha}\cos\frac{|\arg z|-\pi}{2-\alpha}\right)^{\alpha}\right\}, \tag{4.85}$$

for all the eigenvalues λ of A, then the zero solution of (4.73) is asymptotically stable. In this case

$$\|x(t)\|=O(t^{-\alpha})\quad\textit{as } t\to\infty,$$

Furthermore, if

$$\lambda\in\left\{z\in\mathbb{C}:|\arg z|<\frac{\alpha\pi}{2}\textit{ or }|z|>\left(2h^{-\alpha}\cos\frac{|\arg z|-\pi}{2-\alpha}\right)^{\alpha}\right\},$$

for an eigenvalue λ of A, the zero solution of (4.75) is not stable.

Proof. By Lemma 4.3 the system (4.75) is asymptotically stable if all the zeros of:

$$\det\left(diag\left(z(1-z^{-1})^{\alpha},z(1-z^{-1})^{\alpha},...,z(1-z^{-1})^{\alpha}\right)-HA\right)=0,$$

are located inside the unit circle.
we put

$$z(1-z^{-1})^{\alpha}=\lambda.$$

And acording to [116]

$$z\in\{z\in\mathbb{C}:0<|z|<1\}\Leftrightarrow\lambda\in\left\{z\in\mathbb{C}:|\arg z|>\frac{\alpha\pi}{2}\text{ and }|z|<\left(2h^{-\alpha}\cos\frac{|\arg z|-\pi}{2-\alpha}\right)^{\alpha}\right\}.$$

and

$$z\in\{z\in\mathbb{C}:|z|>1\}\Leftrightarrow\lambda\in Int\left\{z\in\mathbb{C}:|\arg z|<\frac{\alpha\pi}{2}\text{ or }|z|>\left(2h^{-\alpha}\cos\frac{|\arg z|-\pi}{2-\alpha}\right)^{\alpha}\right\}.$$

This complet the proof. ■

4.5 Stability of Non-Linear h-difference systems

In this section, we will generalize the previous results in the nonlinear case (system 4.73). Of course, the results do not generalize completely due to the complex nature of nonlinear systems.

Lemma 4.4 *[136] If the solutions of the following equation*

$$\det\left(diag\left(z(1-z^{-1})^{\alpha_1}, z(1-z^{-1})^{\alpha_2}, ..., z(1-z^{-1})^{\alpha_n}\right) - Hf'(0)\right) = 0, \tag{4.86}$$

lie in the interior of unit disk. Where $f'(0)$ is the jacobian matrix of f at 0. Then the zero solution of (4.73) is locally asymptotically stable.

Proof. The system (4.73) is equivalent to the Volterra system (4.74). Using taylor development, (4.74) become

$$x(th+h) = Hf'(0)\, x(t) + \sum_{j=0}^{t} B(t-j)x(jh) + o(\|x(th)\|) + Q(t), \quad t = 0, 1, \cdots.$$

Where

$$B(t) = \begin{pmatrix} (-1)^t\binom{\alpha_1}{t+1} & 0 & \cdots & 0 \\ 0 & (-1)^t\binom{\alpha_2}{t+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (-1)^t\binom{\alpha_n}{t+1} \end{pmatrix},$$

$$Q(t) = \begin{pmatrix} (-1)^{t+1}\binom{\alpha_1-1}{t+1} \\ (-1)^{t+1}\binom{\alpha_2-1}{t+1} \\ \vdots \\ (-1)^{t+1}\binom{\alpha_n-1}{t+1} \end{pmatrix} x(0),$$

and

$$H = diag\left(h^{\alpha_1}, h^{\alpha_2}, \cdots, h^{\alpha_n}\right).$$

We have $B(t) \in \left[\ell^1(\mathbb{N})\right]^{n\times n}$, and $R(t)$ is the resolvent matrix defined as:

$$R(t+1) = Hf'(0)\, R(t) + \sum_{j=0}^{t} B(t-j)R(j), \quad R(0) = I, t \in \mathbb{N},$$

where by Theorem 2 in [109] $R(t) \in \left[\ell^1(\mathbb{N})\right]^{n\times n}$. Then by the variation of constants

$$x(t) = R(t)x(0) + \sum_{j=0}^{t-1} R(t-j-1)\left(o(\|x(j)\|) + Q(j)\right).$$

And acording to [109], (4.73) is locally asymptotically stable. ■

Theorem 4.8 *[136] Suppose that* $\alpha_i = \frac{v_i}{u_i}, (u_i, v_i) = 1, u_i, v_i \in \mathbb{Z}_+, 1 \leq i \leq n$. *Let* M *be the lowest common multiple of* u_i. *If any solution of:*

$$\det\left(diag\left(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, ..., \lambda^{M\alpha_n}\right) - (1-\lambda^M)Hf'(0)\right) = 0, \tag{4.87}$$

lie inside the set (4.82) then the zero solution of (4.73) is locally asymptotically stable.

Corollary 4.7 *[136] Suppose*

$$f'(0) = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

then the zero solution of (4.73) is locally asymptotically stable if (4.84) hold.

Theorem 4.9 *[136] Suppose that*

$$\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha, \alpha \in (0,1)$$

If (4.85) hold for all the eigenvalues of $f'(0)$, *then the zero solution of (4.73) is locally asymptotically stable.*

Remark 4.1 *The proof of the last two results is trivial because we have proven the equality of conditions in the linear case.*

4.6 Illustrating Numerical Examples

In this section we provide numerical examples to give the reader insight into applying the results obtained. The application is simple and does not require complex or additional calculations.

Example 4.8 *[136] Let us consider the linear h-FoDs:*

$$\begin{cases} {}_0^C\Delta_h^\alpha x_1(t+h-\alpha h) = -0.4x_1(t) - 1.02x_2(t), \\ {}_0^C\Delta_h^\alpha x_2(t+h-\alpha h) = 0.04x_1(t) - x_2(t) + 0.3x_3(t), \\ {}_0^C\Delta_h^\alpha x_3(t+h-\alpha h) = -1.046x_2(t) - x_3(t), \end{cases}, t \in \mathbb{N}_0, \tag{4.88}$$

where $\alpha = 0.7, h = 0.5, x(0) = (0.2, 0.5, 0.3), x = (x_1, x_2, x_3) : \mathbb{N}_0 \to \mathbb{R}^3$.

$$A = \begin{pmatrix} -0.4 & -1.02 & 0 \\ 0.04 & -1 & 0.3 \\ 0 & -1.046 & -1 \end{pmatrix},$$

so the eigenvalues of the matrix are

$$(\lambda_1, \lambda_2, \lambda_3) = (-0.436\,42, -0.981\,79 - 0.577\,71i, -0.981\,79 + 0.577\,71i).$$

We notice the condition (4.80) is fulfilled, and therefore the system (4.88) is asymptotically stable around the origin. The following numerical simulations show the stability which agree with theoretical result

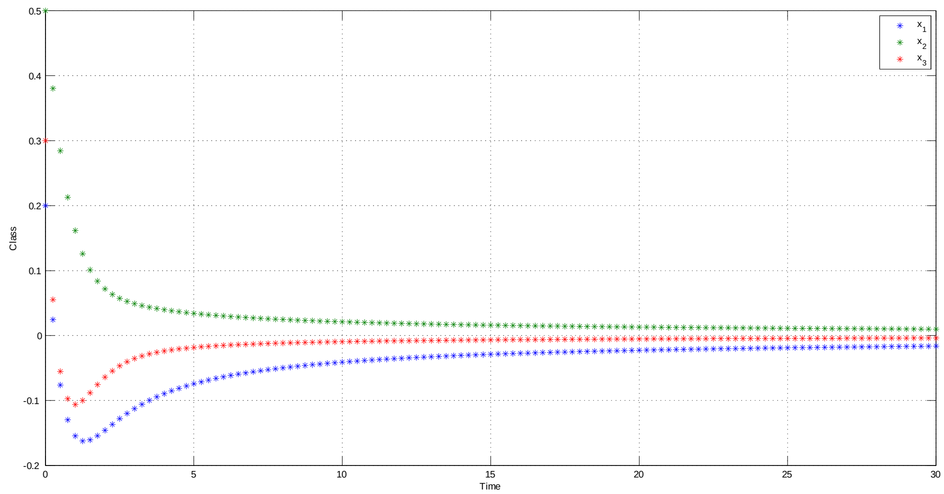

Figure 4.7: Numerical simulation of the system 4.88 using MATLAB.

Example 4.9 *[136] Let us consider the linear h-FoDs:*

$$\begin{cases} {}_0^C\Delta_h^{\alpha_1} x_1\left(t+h-\alpha_1 h\right) = -0.8307 x_1\left(t\right) + 0.00923 x_2\left(t\right) - 0.25844 x_3\left(t\right), \\ {}_0^C\Delta_h^{\alpha_2} x_2\left(t+h-\alpha_2 h\right) = 3.2305 x_1\left(t\right) - 0.923 x_2\left(t\right) + 0.20306 x_3\left(t\right), \\ {}_0^C\Delta_h^{\frac{1}{4}} x_2\left(t+h-\alpha_3 h\right) = -0.01846 x_1\left(t\right) + 0.02769 x_2\left(t\right) - 0.93223 x_3\left(t\right), \end{cases} , t \in \mathbb{N}_0, \quad (4.89)$$

where $(\alpha_1, \alpha_2, \alpha_3) = \left(\frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right), x(0) = (-0.3, 0.2, 0.1), h = 0.5, x = (x_1, x_2, x_3) : \mathbb{N}_0 \to \mathbb{R}^3$.

$$A = \begin{pmatrix} -0.8307 & 0.00923 & -0.25844 \\ 3.2305 & -0.923 & 0.20306 \\ -0.01846 & 0.02769 & -0.93223 \end{pmatrix}$$

we have $M = 4$, *so*

$$\det\left(\begin{pmatrix} \lambda^3 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^1 \end{pmatrix} - (1-\lambda^4)\begin{pmatrix} 0.5^{\frac{3}{4}} & 0 & 0 \\ 0 & 0.5^{\frac{2}{4}} & 0 \\ 0 & 0 & 0.5^{\frac{1}{4}} \end{pmatrix}\begin{pmatrix} -0.8307 & 0.00923 & -0.25844 \\ 3.2305 & -0.923 & 0.20306 \\ -0.01846 & 0.02769 & -0.93223 \end{pmatrix}\right) = 0,$$

$\Leftrightarrow$

$$\begin{aligned} -0.24786\lambda^{12} + 0.50828\lambda^{11} + 0.38482\lambda^{10} - 0.47407\lambda^9 + 9.0923\times 10^{-2}\lambda^8 - 1.5105\lambda^7 + 0.23037\lambda^6 \\ +0.16424\lambda^5 - 9.0923\times 10^{-2}\lambda^4 + 1.0022\lambda^3 + 0.38482\lambda^2 + 0.30984\lambda + 0.24786 \quad = 0, \end{aligned} \quad (4.90)$$

the solution of (4.90) is

$$\begin{pmatrix} 2.1451 \\ 1.2615 \\ -0.66065 \\ -0.70241 \\ -1.4279 \\ 1.4476 \\ -0.31256+1.0569i \\ -0.31256-1.0569i \\ 3.9644\times 10^{-3}-0.70082i \\ 3.9644\times 10^{-3}+0.70082i \\ 0.30232+0.74452i \\ 0.30232-0.74452i \end{pmatrix},$$

by Theorem 4.6 the zero solution of (4.89) is asymptotically stable. The following numerical simulations show the stability which agree with theoretical result

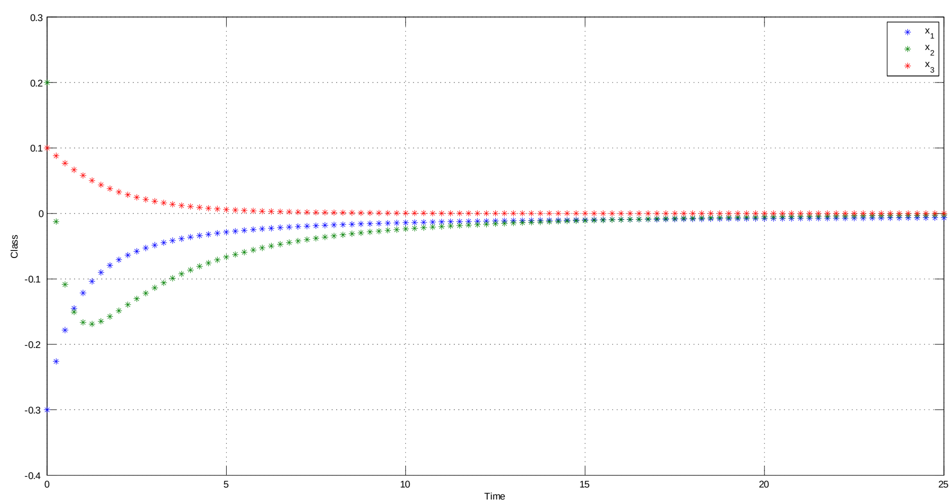

Figure 4.8: Numerical simulation of the system 4.89 using MATLAB.

Example 4.10 *[136] Let us consider the non-linear h-FoDs::*

$$\begin{cases} {}_0^C\Delta_h^\alpha x_1\left(t+h-\alpha h\right)=\sin\left(-1.98x_1\left(t\right)-1.01x_2\left(t\right)\right)+x_3^2(t) \\ {}_0^C\Delta_h^\alpha x_2\left(t+h-\alpha h\right)=2.02x_1\left(t\right)e^{-x_2(t)}+0.75\sin\left(x_2\left(t\right)\right) \\ {}_0^C\Delta_h^\alpha x_3\left(t+h-\alpha h\right)=-0.975x_1\left(t\right)e^{-x_2(t)}-1.023x_2\left(t\right)-0.99\sin\left(x_3\left(t\right)\right) \end{cases},t\in\mathbb{N}_0, \tag{4.91}$$

where $\alpha = \frac{3}{7}, h = 0.5, x(0) = (-0.05, 0.05, 0.03), x = (x_1, x_2, x_3) : \mathbb{N}_0 \to \mathbb{R}^3$. *The jacobian matrix of* f *at* 0 :

$$f'(0) = \begin{pmatrix} -1.98 & -1.01 & 0 \\ 2.02 & 0.75 & 0 \\ 0.975 & -1.023 & -0.99 \end{pmatrix},$$

so the eigenvalues of the matrix are

$$(\lambda_1, \lambda_2, \lambda_3) = (-0.615 + 0.420\,68i, -0.615 - 0.420\,68i, -0.99).$$

We notice the condition (4.80) is fulfilled, and therefore the system (4.90) is locally asymptotically stable around the origin. The following numerical simulations show the stability which agree with theoretical result

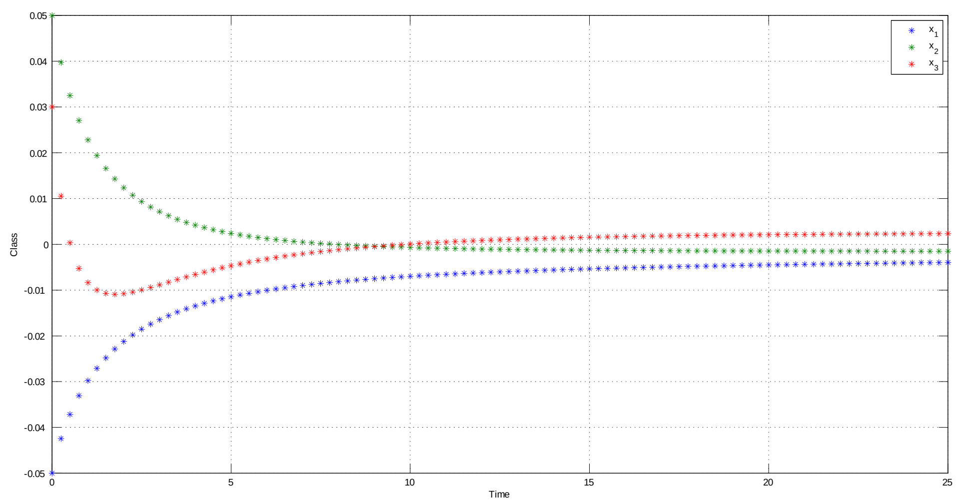

Figure 4.9: Numerical simulation of the system 4.91 using MATLAB.

Example 4.11 *[136] Let us consider the linear h-FoDs:*

$$\begin{cases} {}_0^C\Delta_h^{\alpha_1} x_1(t + h - \alpha_1 h) = -0.9\sin(x_1(t)) - 0.01\sin(x_2(t)) + 0.28\sin(x_3(t)), \\ {}_0^C\Delta_h^{\alpha_2} x_2(t + h - \alpha_2 h) = 3.5x_1(t)\, e^{-x_3(t)} + \sin(-x_2(t)) + 0.22x_3(t), \qquad , t \in \mathbb{N}_0, \\ {}_0^C\Delta_h^{\alpha_3} x_2(t + h - \alpha_3 h) = \sin(-0.02x_1(t) + 0.03x_2(t) - 1.01x_3(t)), \end{cases} \tag{4.92}$$

where $(\alpha_1, \alpha_2, \alpha_3) = \left(\frac{2}{3}, \frac{1}{2}, \frac{1}{3}\right), x(0) = (-0.4, 0.4, 0.04), h = 0.5, x = (x_1, x_2, x_3) : \mathbb{N}_0 \to \mathbb{R}^3$. *The jacobian matrix of* f *at* 0 :

$$f'(0) = \begin{pmatrix} -0.9 & 0.01 & -0.28 \\ 3.5 & -1 & 0.22 \\ -0.02 & 0.03 & -1.01 \end{pmatrix}$$

we have $M = 6$, *so*

$$\det\left(\begin{pmatrix} \lambda^4 & 0 & 0 \\ 0 & \lambda^3 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix} - (1-\lambda^6)\begin{pmatrix} 0.5^{\frac{2}{3}} & 0 & 0 \\ 0 & 0.5^{\frac{1}{2}} & 0 \\ 0 & 0 & 0.5^{\frac{2}{6}} \end{pmatrix}\begin{pmatrix} -0.9 & 0.01 & -0.28 \\ 3.5 & -1 & 0.22 \\ -0.02 & 0.03 & -1.01 \end{pmatrix}\right) = 0,$$

$\Leftrightarrow$

$$\begin{aligned} &-0.315\,21\lambda^{18} + 0.563\,14\lambda^{16} + 0.451\,7\lambda^{15} + 0.385\,31\lambda^{14} - 0.801\,64\lambda^{13} + 0.238\,53\lambda^{12} \\ &-0.566\,96\lambda^{11} - 1.126\,3\lambda^{10} + 0.096\,6\lambda^{9} - 0.770\,63\lambda^{8} + 0.801\,64\lambda^{7} - 0.238\,53\lambda^{6} \\ &+0.566\,96\lambda^{5} + 0.563\,14\lambda^{4} + 0.451\,7\lambda^{3} + 0.385\,31\lambda^{2} + 0.315\,21 = 0, \end{aligned} \tag{4.93}$$

the solution of (4.93) is

$$\begin{pmatrix} -1.419\,4 \\ 1.408\,6 \\ -0.269\,81 + 0.878\,50i \\ -0.269\,81 - 0.878\,50i \\ -0.800\,59 \\ -0.625\,12 - 1.079\,5i \\ -0.625\,12 + 1.079\,5i \\ 1.277\,9 \\ -0.540\,45 - 0.641\,47i \\ -0.540\,45 + 0.641\,47i \\ 1.169\,8 \\ 0.534\,87 - 0.639\,23i \\ 0.534\,87 + 0.639\,23i \\ -1.185\,1 \\ 0.406\,34 + 0.694\,3i \\ 0.406\,34 - 0.694\,3i \\ 0.268\,52 - 0.880\,95i \\ 0.268\,52 + 0.880\,95i \end{pmatrix},$$

by Theorem 4.8 the zero solution of (4.92) is locally asymptotically stable. The following numer-

ical simulations show the stability which agree with theoretical result

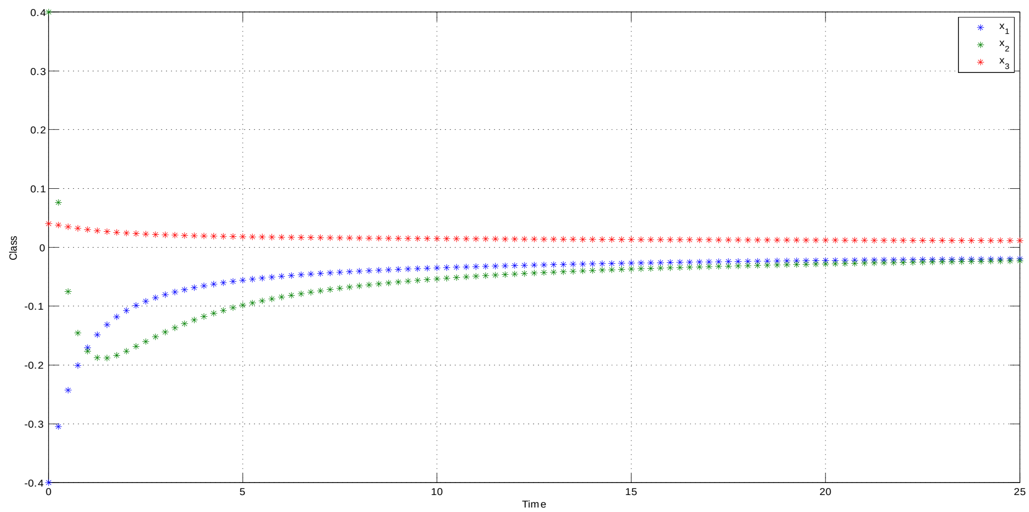

Figure 4.10: Numerical simulation of the system 4.92 using MATLAB.

The findings can be extended to broader scenarios involving variable orders in future research. This study enables the prediction of stability in various systems, including cryptography, epidemiology, engineering, and other applied h-FoDs models.

Chapter 5

Stability Analysis of Variable Order Discrete time Systems

Variable-order fractional operators have become essential tools across a wide range of disciplines, including physics, engineering, and signal processing. Their importance stems from their ability to model systems and phenomena characterized by varying levels of complexity and memory. These operators provide a highly adaptable and versatile approach to capturing the nuanced dynamics of systems where the fractional order changes in response to varying conditions and inputs. Consequently, they have become invaluable for exploring and understanding dynamic systems with variable or adaptive characteristics. Their flexible nature has expanded research possibilities, enabling more precise and responsive modeling in complex and dynamic scenarios.

It is well-known that there are numerous definitions of fractional variable-order operators, with differences arising from how the order is manipulated through various functions and modifications. In this paper, our primary focus is on a specific operator derived by simplifying the difference fractional order operator, representing it explicitly, and then modifying the order using a function.

In this chapter, we conduct a stability analysis of Fractional Variable-order Dynamic Systems. This investigation is motivated by the recent increase in models that fall into this category. Since stability is one of the most critical dynamic behaviors, developing theoretical foundations for its examination has become crucial. Our goal is to generalize the results found in constant-order cases to variable-order systems.

In our analysis, we consider the case where the parameter "a" is set to zero, a choice made for the sake of simplicity and without any loss of generality. This particular setting simplifies the mathematical expressions while still preserving the generality and relevance of our findings. With "a" set to zero, we transition to a notation where we represent the fractional discrete time operator as ${}^{C}\Delta_0^t$ by ${}^{C}\Delta^t$. By further simplifying the expressions and making use of Pascal's rule, we can express this fractional discrete time operator in a more concise and structured form as in [116]:

$$\begin{aligned}({}^{C}\Delta_0^{\alpha} f)(t+1-t) &= f(t+1) + \sum_{s=0}^{t} (-1)^{t-s+1} \binom{\alpha}{t-s+1} f(s) + (-1)^t \binom{\alpha-1}{t+1} f(0), \\ &= f(t+1) + (-1)^{t+1} \binom{\alpha}{t+1} * f(t) + (-1)^t \binom{\alpha-1}{t+1} f(0).\end{aligned}$$

This simplification is a crucial step in our analysis, as it brings us to a point where we can express the fractional discrete time operator in an explicit form. This explicit representation is pivotal for our research, as it enables us to formulate a precise and well-defined definition for the variable-order discrete time operator, which is a central concept in our study.

This simplification is a crucial step in our analysis, as it brings us to a point where we can express the fractional discrete time operator in an explicit form. This explicit representation is pivotal for our research, as it enables us to formulate a precise and well-defined definition for the variable-order discrete time operator, which is a central concept in our study.

Definition 5.1 *[137] Let $\alpha : \mathbb{N} \to (0,1]$. Then, the Caputo fractional variable-order discrete time operator with order function $t\,(.)$ is defined by:*

$$\begin{aligned}{}^{C}\Delta^{\alpha(t)} f\,(t+1-t(t)) &= f(t+1) + (-1)^{t+1} \binom{\alpha(t)}{t+1} * f(t) + (-1)^t \binom{\alpha(t)-1}{t+1} f(0), \\ &= f(t+1) + \sum_{s=0}^{t} (-1)^{t-s+1} \binom{\alpha(t-s)}{t-s+1} f(s) + (-1)^t \binom{\alpha(t)-1}{t+1} f(0),\end{aligned} \tag{5.1}$$

In this work, our focus turns to the examination of the stability characteristics of discrete systems featuring variable orders. To accomplish this, we will embark on a detailed analysis of a specific system. To set the stage for our investigation, let us consider the following system as our subject of study:

$${}^{C}\Delta^{\alpha(t)} x\,(t+1-\alpha(t)) = f\,(x\,(t))\,, t \in \mathbb{N}_0, \tag{5.2}$$

with initial condition $x(0) = x_0 \in \mathbb{R}^n$, $x : \mathbb{N}_0 \to \mathbb{R}^n$ is a state function and $f = (f_1, f_2, ..., f_n)^t : \mathbb{R}^n \to \mathbb{R}^n$, continuously differentiable function, and suppose $x(0) = 0$ (all cases can be transferred to be 0 the equilibrium point).

This system will serve as the foundation upon which we will build our exploration of the intricate dynamics and stability properties associated with variable-order discrete systems. By delving into the stability of this particular system, we aim to unravel essential insights that can be applied to a broader understanding of systems with varying orders. This endeavor is fundamental in our quest to gain a comprehensive understanding of the dynamics and behaviors of such systems.

Using 5.1 and taylor devlopment we get

$$x(t+1) = Jx(t) + \sum_{s=0}^{t} B(t-s)x(s) + g(t) + o(\|f(t)\|), \quad t = 0, 1, \cdots, \tag{5.3}$$

where J is the jacobian matrix of f at 0, $B(t) = (-1)^t \binom{\alpha(t)}{t+1} I_n$ and $g(t) = (-1)^{t+1}\binom{\alpha(t)-1}{t+1}x(0)$. so $x(t)$ is a solution of 5.2 if and only if it is a solution of 5.3. first analyze its homogeneous part

$$x(t+1) = Jx(t) + \sum_{s=0}^{t} B(t-s)x(s), \quad t = 0, 1, \cdots. \tag{5.4}$$

When $(B(k))_{k\in\mathbb{N}} \in \ell^1(\mathbb{N}^{n\times n})$, the resolvent matrix $R(k)$ of 5.4 is defined as:

$$R(t+1) = JR(t) + \sum_{s=0}^{t} B(t-s)R(s), \quad R(0) = I_n, t \in \mathbb{N}.$$

By the variation of constants formula, we obtain

$$x(t) = R(t)x(0) + \sum_{s=0}^{t-1} R(t-s-1)\left(g(s) + o(\|x(t)\|)\right).$$

We perform the necessary notation related to the Z-transform of the kernel $b(t) = (-1)^t\binom{\alpha(t)}{t+1}$. (with the radius of convergence $R = 1$) we have the following notation:

$$\tilde{b}(z) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha(k)}{k+1} z^{-k}. \tag{5.5}$$

5.1 Stability in Linear Variable Order Systems

In this section, our focus centers on the examination of the stability characteristics inherent to discrete linear systems with variable order. To facilitate our analysis, we shall delve into a specific system model featuring a variable-order structure. This chosen system takes on the following form:

$$^{C}\Delta^{\alpha(t)} x\left(t+1-\alpha(t)\right) = Ax\left(t\right), t \in \mathbb{N}_0, \tag{5.6}$$

$x(0) = x_0 \in \mathbb{R}^n$,$\alpha : \mathbb{N}_0 \to (0, 1]$ and $A \in \mathbb{R}^{n\times n}$.

By dissecting and scrutinizing this system, we aim to gain a comprehensive understanding of the stability aspects in the context of variable-order systems. This exploration will shed light on the behavior of such systems under different conditions, providing insights into their dynamics and potential applications.

Theorem 5.1 *[137] System 5.6 exhibits asymptotic stability solutions of:*

$$\det\left(zI_n - A - \tilde{B}(z)\right) = 0, \tag{5.7}$$

are inside the unit circle. Furthermore, if ther is a solution of 5.7 located outside the unit circle then the solution of 5.6 is not stable.

Proof. Suppose that the solutions of 5.7 are located inside the unit circle, by the variation of constants formula, we obtain

$$x(t) = \sum_{s=0}^{t} R(t-s)(-1)^s \binom{\alpha(s)-1}{s} x(0).$$

We have

$$\begin{aligned} \|x(t)\| &= \left\| \sum_{s=0}^{t} R(t-s)(-1)^s \tbinom{\alpha(s)-1}{s} x(0) \right\| \\ &\leq \left\| \sum_{s=0}^{t} R(t-s)(-1)^s \tbinom{\alpha(s)-1}{s} \right\| \|x(0)\| \\ &\leq \zeta_1 \sum_{s=0}^{t} \tfrac{1}{(s+1)^{\alpha_{\min}}} \|R(t-s)\| \\ &= \zeta_1 \left(\sum_{s=0}^{\lfloor t/2 \rfloor} \tfrac{1}{(s+1)^{\alpha_{\min}}} \|R(t-s)\| + \sum_{s=\lfloor t/2 \rfloor+1}^{t} \tfrac{1}{(s+1)^{\alpha_{\min}}} \|R(t-s)\| \right), \end{aligned}$$

where $\zeta_1 > 0$ is a real constant and $\lfloor . \rfloor$ is the floor function. Since the components of R belongs to $\ell^1(\mathbb{N}_0)$, we have $\|R(t)\| = O(t^{-1})$ as $t \to \infty$ and there exist $\zeta_2, \zeta_3 > 0$ such that

$$\sum_{s=0}^{\lfloor t/2 \rfloor} \frac{1}{(s+1)^{\alpha_{\min}}} \|R(t-s)\| \leq \frac{\zeta_2}{t+1} \sum_{s=0}^{\lfloor t/2 \rfloor} \frac{1}{(s+1)^{\alpha_{\min}}} \leq \frac{\zeta_3}{(t+1)^{\alpha_{\min}}},$$

where we have used the inequality $\sum_{s=1}^{t} (s+1)^{-\alpha_{\min}} \leq \int_0^t (x+1)^{-\alpha_{\min}} dx$. Similarly, the second sum can be estimated as

$$\sum_{s=\lfloor t/2 \rfloor+1}^{t} \frac{1}{(s+1)^{\alpha_{\min}}} \|R(t-s)\| \leq \frac{\zeta_4}{(t+1)^{\alpha_{\min}}} \sum_{s=\lfloor t/2 \rfloor+1}^{t} \|R(t-s)\| \leq \frac{\zeta_5}{(t+1)^{\alpha_{\min}}},$$

for suitable $\alpha_4, \alpha_5 > 0$. In summary, we have $\|x(t)\| \leq \zeta_5 (t+1)^{-\alpha_{\min}}$, hence $\|x(t)\| = O(t^{-t})$ as $t \to \infty$.

Hence, if there is a zero z with $|z| > 1$, then the radius of convergence of at least one component x_i of x satisfies $R > 1$. So

$$r = \lim_{t \to \infty} \sup \sqrt[t]{|x_i(t)|} > 1,$$

according Cauchy-Hadamard theorem, consequently, $\lim\limits_{t \to \infty} \sup |x_i(t)| = +\infty$ which proves that x is not bounded and thus 5.6 is not stable. ■

Now, we proceed to unveil a practical result that emerges from our analysis, a result that serves as a valuable addition to the understanding and application of the concepts under examination. To do this, we introduce a fundamental set, a concept that will play a pivotal role in our ongoing exploration:

$$\begin{aligned} S^{\alpha(0)} &= \left\{ z - \sum_{k=0}^{\infty} (-1)^k \binom{\alpha(0)}{k+1} z^{-k}, z \in \mathbb{C}, |z| < 1 \right\} \\ &= \left\{ z \in \mathbb{C} : |z| < \left(2\cos\frac{|\arg z| - \pi}{2-\alpha(0)} \right)^{\alpha(0)} \text{ and } |\arg z| > \frac{\alpha(0)\pi}{2} \right\} \end{aligned}. \tag{5.8}$$

This introduced set encapsulates key elements that underpin the practical implications of our study. It paves the way for the application of our theoretical insights in real-world scenarios and problem-solving. Through a closer examination of this set, we aim to highlight its importance and its role in bridging the gap between theoretical concepts and their practical relevance, ultimately contributing to the advancement of knowledge and the solution of real-world challenges.

Theorem 5.2 *[137] Let*

$$\rho := \max\left\{ \left(1 - \frac{\alpha_{\min}}{\alpha_{\max}} + \alpha_{\min} - \alpha(0)\right), \left(\frac{\alpha_{\max}}{\alpha_{\min}} - \alpha_{\max} - 1 + \alpha(0)\right) \right\},$$

where

$$\alpha_{\min} = \inf \alpha(t), \alpha_{\max} = \sup \alpha(t),$$

and

$$d\left(\lambda, \mathbb{C}\backslash S^{\alpha(0)}\right) = \inf\left\{ |\lambda - z|, z \in \mathbb{C}\backslash S^{\alpha(0)} \right\}.$$

If

$$d\left(\lambda, \mathbb{C}\backslash S^{\alpha(0)}\right) > \rho, \tag{5.9}$$

for any eigenvalues λ of A, then system 5.6 is asymptotically stable.

Proof. We have

$$\left|z - \lambda - \tilde{b}(z)\right| \geq \left|\left|z - \lambda - \sum_{k=0}^{\infty}(-1)^k \binom{\alpha(0)}{k+1} z^{-k}\right| - \left|\sum_{k=0}^{\infty}(-1)^k \binom{\alpha(0)}{k+1} z^{-k} - \sum_{k=0}^{\infty}(-1)^k \binom{\alpha(k)}{k+1} z^{-k}\right|\right|$$

This means if

$$\left|z - \lambda - \sum_{k=0}^{\infty}(-1)^k \binom{\alpha(0)}{k+1} z^{-k}\right| > \left|\sum_{k=0}^{\infty}(-1)^k \binom{\alpha(0)}{k+1} z^{-k} - \sum_{k=0}^{\infty}(-1)^k \binom{\alpha(k)}{k+1} z^{-k}\right|,$$

we get for any eigenvalues λ of A :

$$z - \lambda - \tilde{b}(z) \neq 0, \forall z \in \mathbb{C}, |z| \geq 1. \tag{5.10}$$

We have

$$\begin{aligned} &\left|\sum_{k=0}^{\infty}(-1)^k \binom{\alpha(0)}{k+1} z^{-k} - \sum_{k=0}^{\infty}(-1)^k \binom{\alpha(k)}{k+1} z^{-k}\right| = \\ &\qquad \left|\sum_{k=1}^{\infty} \left(\frac{(\alpha(0))(1-\alpha(0))\cdots(k-\alpha(0))}{\Gamma(k+2)} - \frac{(\alpha(k))(1-\alpha(k))\cdots(k-\alpha(k))}{\Gamma(k+2)} \right) z^{-k}\right|. \end{aligned}$$

So

$$\left|\sum_{k=1}^{\infty}(-1)^k\binom{\alpha(0)}{k+1}z^{-k}-\sum_{k=1}^{\infty}(-1)^k\binom{\alpha(k)}{k+1}z^{-k}\right|<\max\left\{\sum_{k=1}^{\infty}\left(\frac{(\alpha(0))(1-\alpha(0))\cdots(k-\alpha(0))}{\Gamma(k+2)}-\frac{(\alpha_{\min})(1-\alpha_{\max})\cdots(k-\alpha_{\max})}{\Gamma(k+2)}\right),\right.$$
$$\left.\sum_{k=1}^{\infty}\left(\frac{(\alpha_{\max})(1-\alpha_{\min})\cdots(k-\alpha_{\min})}{\Gamma(k+2)}-\frac{(\alpha(0))(1-\alpha(0))\cdots(k-\alpha(0))}{\Gamma(k+2)}\right)\right\},$$

$\Leftrightarrow$

$$\left|\sum_{k=1}^{\infty}(-1)^k\binom{\alpha(0)}{k+1}z^{-k}-\sum_{k=1}^{\infty}(-1)^k\binom{\alpha(k)}{k+1}z^{-k}\right|<$$
$$\max\left\{\sum_{k=1}^{\infty}\left(\frac{(\alpha(0))(1-\alpha(0))\cdots(k-\alpha(0))}{\Gamma(k+2)}-\frac{(\alpha_{\min})}{(\alpha_{\max})}\frac{(\alpha_{\max})(1-\alpha_{\max})\cdots(k-\alpha_{\max})}{\Gamma(k+2)}\right),\right.$$
$$\left.\sum_{k=1}^{\infty}\left(\frac{(\alpha_{\max})}{(\alpha_{\min})}\frac{(\alpha_{\min})(1-\alpha_{\min})\cdots(k-\alpha_{\min})}{\Gamma(k+2)}-\frac{(\alpha(0))(1-\alpha(0))\cdots(k-\alpha(0))}{\Gamma(k+2)}\right)\right\},$$

$\Leftrightarrow$

$$\left|\sum_{k=1}^{\infty}(-1)^k\binom{\alpha(0)}{k+1}z^{-k}-\sum_{k=1}^{\infty}(-1)^k\binom{\alpha(k)}{k+1}z^{-k}\right|<$$
$$\max\left\{\sum_{k=1}^{\infty}(-1)^k\binom{\alpha(0)}{k+1}-\frac{(\alpha_{\min})}{(\alpha_{\max})}\sum_{k=1}^{\infty}(-1)^k\binom{\alpha_{\max}}{k+1},\right.$$
$$\left.\frac{(\alpha_{\max})}{(\alpha_{\min})}\sum_{k=1}^{\infty}(-1)^k\binom{\alpha_{\min}}{k+1}-\sum_{k=1}^{\infty}(-1)^k\binom{\alpha(0)}{k+1}\right\},$$

$\Leftrightarrow$

$$\left|\sum_{k=1}^{\infty}(-1)^k\binom{\alpha(0)}{k+1}z^{-k}-\sum_{k=1}^{\infty}(-1)^k\binom{\alpha(k)}{k+1}z^{-k}\right|<$$
$$\max\left\{\left(1-\frac{\alpha_{\min}}{\alpha_{\max}}+\alpha_{\min}-\alpha(0)\right),\left(\frac{\alpha_{\max}}{\alpha_{\min}}-\alpha_{\max}-1+\alpha(0)\right)\right\},$$

If $\left|z-\lambda-\sum_{k=0}^{\infty}(-1)^k\binom{\alpha(0)}{k+1}z^{-k}\right|>\max\left\{\left(1-\frac{\alpha_{\min}}{\alpha_{\max}}+\alpha_{\min}-\alpha(0)\right),\left(\frac{\alpha_{\max}}{\alpha_{\min}}-\alpha_{\max}-1+\alpha(0)\right)\right\}$ then $\left|z-\lambda-\tilde{b}(z)\right|>0$ therefore the condition 5.7 is fulfilled. ■

5.1.1 Stability in Non-Linear Variable Order Systems

Now, we turn our attention to the examination of the stability properties of the nonlinear system 5.2 in a more comprehensive context. To achieve this, we introduce a critical component to our analysis, a Lemma that will aid in our understanding of the system's stability under various conditions and scenarios:

Theorem 5.3 *[137] System 5.2 exhibits local asymptotic stability if all solutions of:*

$$\det\left(zI_n-J-\tilde{B}(z)\right)=0, \tag{5.11}$$

Are contained within the unit disk.

Proof. Suppose that all the solutions of 5.11 are contained within the unit disk, by the variation of constants, we obtain

$$x(t)=R(t)x(0)+\sum_{s=0}^{t-1}R(t-s-1)\left(g(s)+o(\|x(s)\|\right).$$

where $R(t)$ is the resolvent matrix. We have

$$\|x(t)\| \leq \|R(t)\| \|x(0)\| + \sum_{s=0}^{t-1} \|R(t-s-1)\| \|o(\|x(s)\|)\| + \sum_{s=0}^{t-1} \|R(t-s-1)\| \|g(s)\|, \quad (5.12)$$

for a given $\epsilon > 0$ there is $\delta > 0$ such that $o(\|x\|) < \epsilon \|x\|$ whenever $\|x\| < \delta$. So as long as $\|x(s)\| < \delta$, 5.12 becomes

$$\|x(t)\| \leq \|R(t)\| \|x(0)\| + \epsilon \sum_{s=0}^{t-1} \|R(t-s-1)\| \|x(s)\| + \sum_{s=0}^{t-1} \|R(t-s-1)\| \|g(s)\|,$$

we defined $y(t)$ as follow

$$y(t) = r(t)y(0) + \epsilon \sum_{s=0}^{t-1} r(t-s-1)y(s) + \sum_{s=0}^{t-1} r(t-s-1)f(s)$$

where

$$r(t) = \|R(t)\|, \quad f(s) = \|g(s)\|, \quad y(0) = \|x(0)\|.$$

$\Rightarrow$

$$y(t+1) = r(t+1)y(0) + \epsilon \sum_{s=0}^{t} r(t-s)y(s) + \sum_{s=0}^{t} r(t-s)f(s)$$

$\Rightarrow$

$$y(t+1) = r(t+1)y(0) + \epsilon r(t) * y(t) + r(t) * f(t). \quad (5.13)$$

We have

$$\|x(t)\| \leq y(t),$$

we see that $r(t) \in \ell^1(\mathbb{N})$. Taking the $Z-$transform on 5.13 gives:

$$z\tilde{y}(z) - y(0)z = (z\tilde{r}(z) - z)y(0) + \epsilon \tilde{r}(z)\tilde{y}(z) + \tilde{r}(z)\tilde{f}(z),$$

with $R_r \leq 1$, and $R_f = 1$, where R_r is the convergence radius of $\tilde{r}(z)$ and R_r is the convergence radius of $\tilde{f}(z)$ $\Rightarrow$

$$\tilde{y}(z) = (z - \epsilon \tilde{r}(z))^{-1} \left(z\tilde{r}(z)y(0) + \tilde{r}(z)\tilde{f}(z) \right),$$

for $|z| > \max\{R_r, 1, \epsilon \tilde{r}(1)\}$.

Choose $\epsilon < \frac{1}{\tilde{r}(1)}$, we get $\max\{R_r, 1, \epsilon \tilde{r}(1)\} = 1$, by final value theorem

$$\lim_{t\to\infty} y(t) = \lim_{z\to 1}(z-1)\tilde{y}(z) = \lim_{z\to 1}(z-1)((z - \epsilon\tilde{r}(z))^{-1} z\tilde{r}(z)y(0) + \tilde{r}(z)\tilde{f}(z))$$

we have

$$\begin{aligned} \sum_{s=0}^{t} \|R(t-s)\| \|H(s)\| &\leq \zeta_1 \sum_{s=0}^{t} \tfrac{1}{(s+1)^t} \|R(t-s)\| \|x(0)\| \\ &\leq \zeta_1 \left(\sum_{s=0}^{\lfloor t/2 \rfloor} \tfrac{1}{(s+1)^\alpha} \|R(t-s)\| + \sum_{s=\lfloor t/2 \rfloor + 1}^{t} \tfrac{1}{(s+1)^\alpha} \|R(t-s)\| \right) \end{aligned}$$

where $\zeta_1 > 0$ is a suitable real constant and the symbol $\lfloor . \rfloor$ stands for the floor function and $\alpha = \min\limits_{1 \le i \le n} \{\alpha_i\}$. Since $\|R(t)\|$ belongs to $\ell^1(\mathbb{N})$, we have $\|R(t)\| = O(t^{-1})$ as $t \to \infty$ and there exist $\zeta_2, \zeta_3 > 0$ such that

$$\sum_{s=0}^{\lfloor t/2 \rfloor} \frac{1}{(s+1)^\alpha} \|R(t-s)\| \le \frac{\zeta_2}{t+1} \sum_{s=0}^{\lfloor t/2 \rfloor} \frac{1}{(s+1)^\alpha} \le \frac{\zeta_3}{(t+1)^\alpha}$$

where we have used the inequality $\sum_{s=1}^{t} (s+1)^{-\alpha} \le \int_0^t (x+1)^{-\alpha} dx$. Similarly, the second sum can be estimated as

$$\sum_{s=\lfloor t/2 \rfloor + 1}^{t} \frac{1}{(s+1)^\alpha} \|R(t-s)\| \le \frac{\zeta_4}{(t+1)^\alpha} \sum_{s=\lfloor t/2 \rfloor + 1}^{t} \|R(t-s)\| \le \frac{\zeta_5}{(t+1)^\alpha},$$

for suitable $\zeta_4, \zeta_5 > 0$. In summary, we have

$$\sum_{s=0}^{t} \|R(t-s)\| \, \|H(s)\| \le \zeta_5 (t+1)^{-\alpha},$$

hence

$$\sum_{s=0}^{t} \|R(t-s)\| \, \|H(s)\| = O(t^{-t}) \text{ as } t \to \infty.$$

So by final value theorem

$$\lim_{z \to 1} (z-1) \tilde{r}(z) \tilde{f}(z) = 0,$$

and that's imply

$$\lim_{t \to \infty} y(t) = \lim_{z \to 1} (z-1) \tilde{y}(z) = \lim_{z \to 1} (z-1)((z - \epsilon \tilde{r}(z))^{-1} z \tilde{r}(z) y(0) + \tilde{r}(z) f(z)) = 0.$$

■

Theorem 5.4 *[137] Suppose that for any eigenvalues λ of J :*

$$d\left(\lambda, \mathbb{C} \backslash S^{t(0)}\right) > \rho. \tag{5.14}$$

Then System 5.2 is locally asymptotically stable. Where

$$\rho := \max\left\{ \left(1 - \frac{\alpha_{\min}}{\alpha_{\max}} + \alpha_{\min} - \alpha(0)\right), \left(\frac{\alpha_{\max}}{\alpha_{\min}} - \alpha_{\max} - 1 + t(0)\right) \right\},$$

Proof. With the same steps of prove Theorem 8: If

$$\left| z - \lambda - \sum_{k=0}^{\infty} (-1)^k \binom{\alpha(0)}{k+1} z^{-k} \right| > \max\left\{ \left(1 - \frac{\alpha_{\min}}{\alpha_{\max}} + \alpha_{\min} - \alpha(0)\right), \left(\frac{\alpha_{\max}}{\alpha_{\min}} - \alpha_{\max} - 1 + \alpha(0)\right) \right\},$$

then

$$\left| z - \lambda - \tilde{b}(z) \right| > 0,$$

therefore the condition 5.11 is fulfilled. ■

5.2 Numerical Simulations

What we will do in this section is to test the obtained result numerically through two numerical examples regarding the linear case and two other regarding the nonlinear case and to carry out numerical simulations that support the theoretical results.

5.2.1 Linear Systems

Example 5.1 *[137] Let us consider the system with a variable-order the following form:*

$$\begin{cases} {}^{C}\Delta^{\alpha(t)}x_1\left(t+1-\alpha(t)\right) = -0.45x_1\left(t\right) + 1.5x_2\left(t\right) \\ {}^{C}\Delta^{\alpha(t)}x_2\left(t+1-\alpha(t)\right) = 0.01x_1\left(t\right) - 0.67x_2\left(t\right) \end{cases}, t \in \mathbb{N}_0, \tag{5.15}$$

with initial condition $(x_1, x_2) = (0.2, -0.3)$*, where* $\alpha : \mathbb{N}_0 \to (0, 1], t\left(t\right) = 0.4 + 0.1e^{-t}$*. We have*

$$\alpha_{\min} = 0.4, \alpha_{\max} = 0.5$$

$$\rho = 0.25.$$

The eigenvalues of the matrix are $\lambda_1 = -0.395\,38, \lambda_2 = -0.724\,62$
. We notice from the following figure obtained by MATLAB that the condition 5.14 is fulfilled, and therefore the system 5.15 is asymptotically stable around the origin.

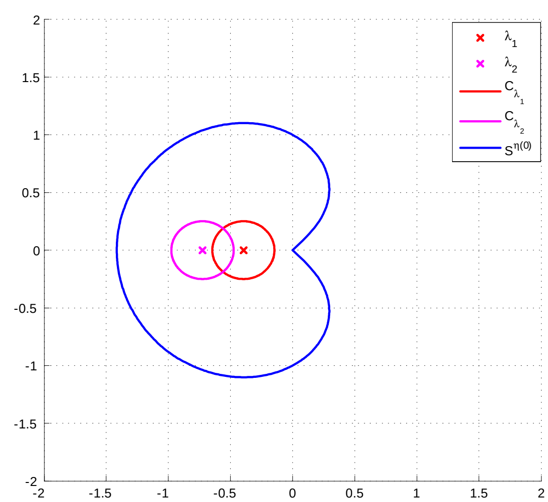

Figure 5.1: The location of eigenvalues and disks whose center eigenvalues and radius is ρ*.*

The following numerical simulations show the stability which agree with theoretical result

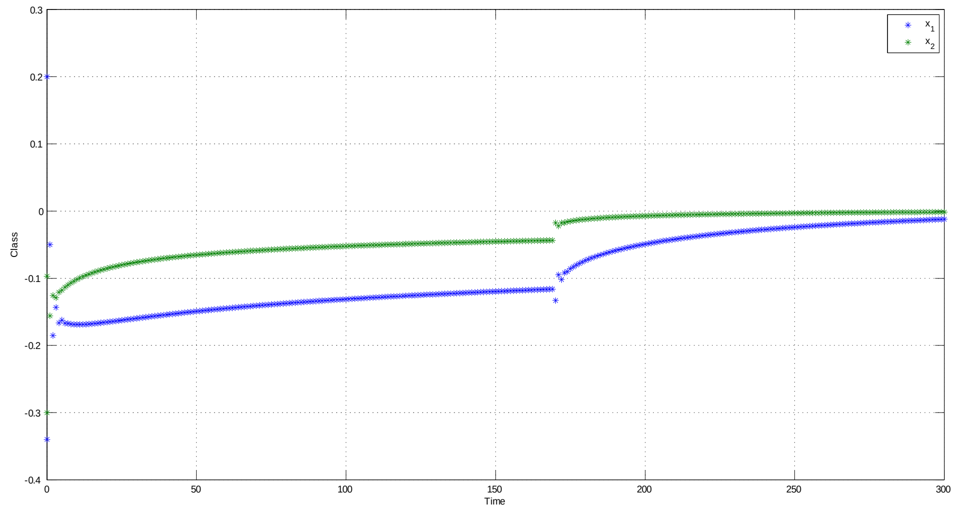

Figure 5.2: Numerical simulation of the system 5.15 using MATLAB.

Example 5.2 *[137] Let us consider the system with a variable-order the following form:*

$$\begin{cases} {}^{C}\Delta^{\alpha(t)}x_1\left(t+1-\alpha(t)\right) = -1.9x_1\left(t\right) - 0.9x_2\left(t\right) \\ {}^{C}\Delta^{\alpha(t)}x_2\left(t+1-\alpha(t)\right) = 2.01x_1\left(t\right) + 0.8x_2\left(t\right) \quad ,t\in\mathbb{N}_0, \\ {}^{C}\Delta^{\alpha(t)}x_3\left(t+1-\alpha(t)\right) = x_1\left(t\right) - x_2\left(t\right) - x_3\left(t\right) \end{cases} \tag{5.16}$$

with initial condition $(x_1, x_2, x_3) = (2.2, 1.01, -1, 4)$*, where* $\alpha : \mathbb{N}_0 \to (0,1], t\left(t\right) = 0.6 + 0.1\cos\frac{\pi}{4}t$*. We have*

$$\alpha_{\min} = 0.5, \alpha_{\max} = 0.7 \Rightarrow \ \rho = 0.4.$$

The eigenvalues of the matrix are $\lambda_1 = -0.433\,81, \lambda_2 = -0.666\,19, \lambda_3 = -1$*. We notice from the following figure obtained by MATLAB that the condition 5.14 is fulfilled, and therefore the system*

5.16 is asymptotically stable around the origin.

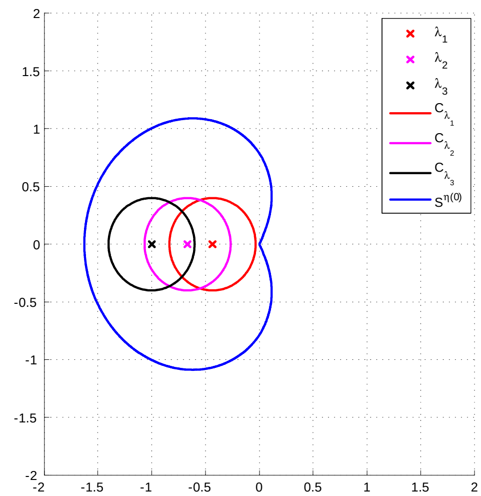

Figure 5.3: The location of eigenvalues and disks whose center eigenvalues and radius is ρ.

The following numerical simulations show the stability which agree with theoretical result

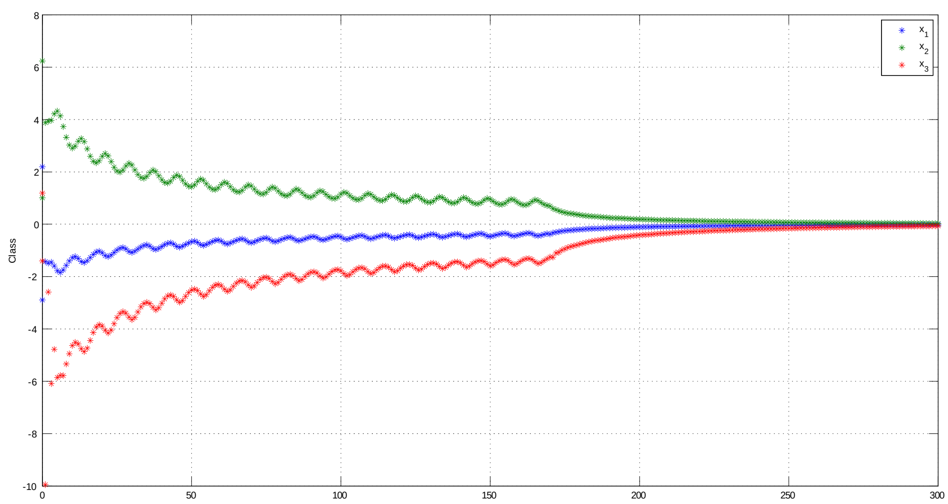

Figure 5.4: Numerical simulation of the system 5.16 using MATLAB.

5.2.2 Non-Linear Systems

Example 5.3 *[137] Let us consider the system with a variable-order the following form:*

$$\begin{cases} {}^{C}\Delta^{\alpha(t)}x_1\left(t+1-\alpha(t)\right) = -0.41\sin\left(x_1\left(t\right)\right) + 1.4\sin\left(x_2\left(t\right)\right) \\ {}^{C}\Delta^{\alpha(t)}x_2\left(t+1-\alpha(t)\right) = 0.5x_1^2\left(t\right) - 0.6x_2\left(t\right) \end{cases}, t \in \mathbb{N}_0, \tag{5.17}$$

with initial condition $x(0) = (0.5, -0.5)$*, where* $\alpha : \mathbb{N}_0 \to (0, 1], \alpha\left(t\right) = 0.3 + 0.05\cos(t)$*. We have*

$$\alpha_{\min} = 0.25, \alpha_{\max} = 0.35 \Rightarrow \rho = 0.4.$$

The eigenvalues of the Jacobian matrix of $\left(-0.41\sin\left(x_1\left(t\right)\right) + 1.4\sin\left(x_2\left(t\right)\right), 0.5x_1^2\left(t\right) - 0.6x_2\left(t\right)\right)^t$ *,at* $(0, 0)$ *are* $\lambda_1 = -0.41, \lambda_2 = -0.6$*. We notice from the following figure obtained by MATLAB that the condition 5.14 is fulfilled, and therefore the system 5.17 is asymptotically stable around the origin.*

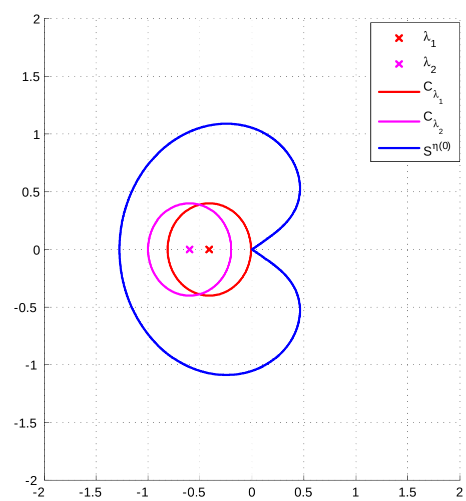

Figure 5.5: The location of eigenvalues and disks whose center eigenvalues and radius is ρ*.*

The following numerical simulations show the stability which agree with theoretical result

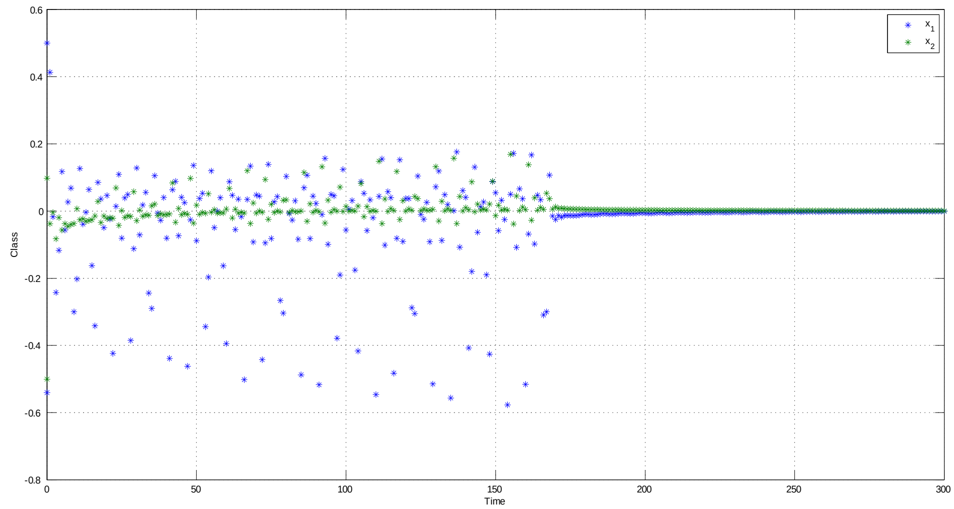

Figure 5.6: Numerical simulation of the system 5.17 using MATLAB.

Example 5.4 *[137] Let us consider the system with a variable-order the following form:*

$$\begin{cases} {}^{C}\Delta^{\alpha(t)}x_1\left(t+1-\alpha(t)\right) = -0.3\sin\left(x_1\left(t\right)\right) - \sin\left(x_2\left(t\right)\right) \\ {}^{C}\Delta^{\alpha(t)}x_2\left(t+1-\alpha(t)\right) = 0.09x_1\left(t\right) - 0.8\sin\left(x_2\left(t\right)\right) \\ {}^{C}\Delta^{\alpha(t)}x_3\left(t+1-\alpha(t)\right) = \sin\left(x_1\left(t\right) - x_2\left(t\right) - x_3\left(t\right)\right) \end{cases}, t \in \mathbb{N}_0, \tag{5.18}$$

with initial condition $x(0) = (2, 3, -4)$*, where* $\alpha : \mathbb{N}_0 \rightarrow (0, 1], \alpha\left(t\right) = 0.6 + 0.1\sin\frac{\pi}{2}t$*. We have*

$$\alpha_{\text{min}} = 0.5, \alpha_{\text{max}} = 0.7 \Rightarrow \rho = 0.4.$$

The eigenvalues of the Jacobian matrix of

$$\left(-0.3\sin\left(x_1\left(t\right)\right) - \sin\left(x_2\left(t\right)\right), 0.09x_1\left(t\right) - 0.8\sin\left(x_2\left(t\right)\right), \sin\left(x_1\left(t\right) - x_2\left(t\right) - x_3\left(t\right)\right)\right)^t,$$

at $(0, 0, 0)$ *are* $\lambda_1 = -0.55 + 0.165\,83i, \lambda_2 = -0.55 - 0.165\,83i, \lambda_3 = -1$*. We notice from the following figure obtained by MATLAB that the condition 5.14 is fulfilled, and therefore the system*

5.18 is asymptotically stable around the origin.

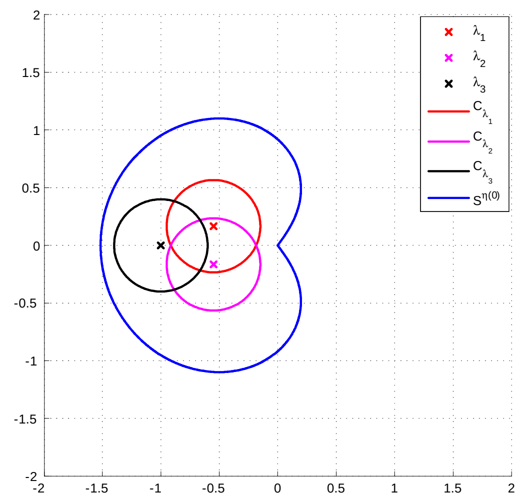

Figure 5.7: The location of eigenvalues and disks whose center eigenvalues and radius is ρ.

The following numerical simulations show the stability which agree with theoretical result

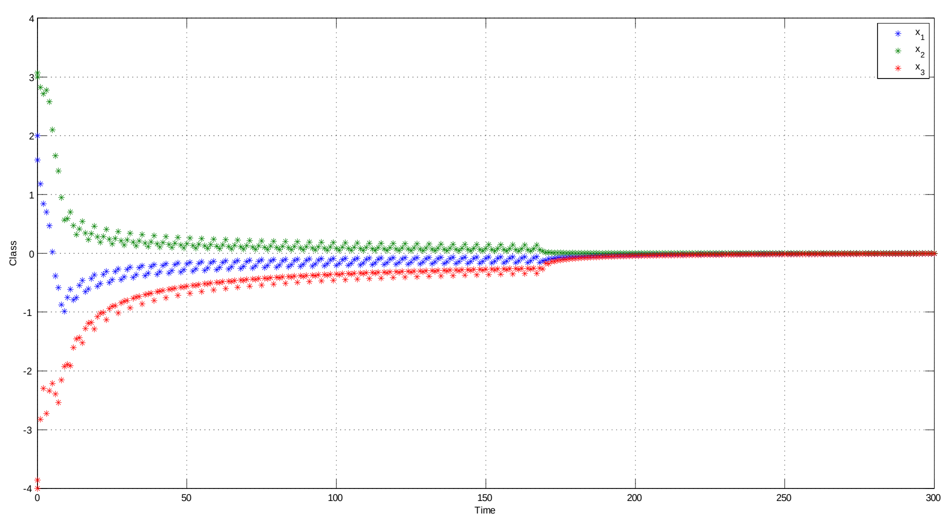

Figure 5.8: Numerical simulation of the system 5.18 using MATLAB.

In this chapter, we have introduced novel, rigorously validated stability criteria for Fractional Variable Order Discrete time Systems. This work considers a generalization of the results obtained in the past years regarding the stability of discrete-time systems with constant order. These criteria were confirmed through numerical demonstrations with multiple illustrative examples, offering valuable insights for a wide range of discrete-time systems, thus contributing significantly to the field. This work motivates us to study many models in the case of variable orders and to apply this type of model in many applied aspects such as biology and engineering.

Chapter 6

Application of Discrete Fractional Calculus in Epidemic Models

The mathematical modeling of epidemics has a rich history, beginning in the 18th century as scientists sought to understand and predict the spread of diseases. Here's a more detailed overview of its development [138, 139, 140, 141, 143]:

18th Century – The Beginnings with Bernoulli:

The idea of mathematical modeling for diseases originated in the mid-18th century. In 1760, Daniel Bernoulli developed a mathematical model to study the impact of smallpox on populations, with the main goal of evaluating the benefits of vaccination. Bernoulli's work demonstrated how vaccination could reduce mortality and used equations to determine the proportion of the population that needed to be vaccinated to control the disease. This marked one of the first instances where mathematics was used to understand disease dynamics.

19th Century – Population Distribution Laws:

In the 19th century, attempts were made to understand statistical relationships between disease outbreaks and population groups. Researchers like William Farr, who studied cholera outbreaks in Britain, used statistical data to describe trends in disease transmission. Though these efforts predated formal mathematical models, they laid the groundwork for the use of demographic data in epidemiology.

20th Century – The Emergence of the SIR Model:

A significant breakthrough came in 1927 when William Kermack and Anderson McKendrick developed the famous SIR model (Susceptible, Infectious, Recovered). This model divided populations into three key groups:

- Susceptible (S): Individuals who are not yet infected but are vulnerable to the disease.

- Infectious (I): Individuals who are currently infected and can transmit the disease to others.

- Recovered (R): Individuals who have recovered from the infection and have gained immunity.

The SIR model, based on differential equations, became a fundamental tool for understanding how diseases spread and how the number of infections changes over time. It allowed scientists to make predictions about the course of an epidemic and introduced key concepts like the basic reproduction number (R_0).

Mid-20th Century – Development of Complex Models:

By the mid-20th century, models became more sophisticated, incorporating additional variables like:

- Viral mutations: To study how genetic changes in viruses affect transmission (e.g., seasonal flu).

- Geographical differences: Factoring in how urban and rural settings influence the spread of diseases.

- Social behaviors: Including human behavior like social distancing, travel, and contact patterns to enhance predictions.

These advancements enabled models to account for real-world complexities and provided better insights into how diseases spread across diverse populations.

Late 20th Century – The Computing Revolution:

The 1980s and 1990s saw a leap in modeling capabilities with the rise of computing technology. Computer simulations allowed researchers to model large-scale disease outbreaks in greater detail. A prominent example is the use of mathematical models during the HIV/AIDS crisis, which helped understand the transmission dynamics of the virus and guided public health interventions like antiretroviral therapies and education campaigns.

21st Century – Global Epidemic Challenges:

As the 21st century progressed, outbreaks like SARS (2003), H1N1 (2009), Ebola (2014–2016), and Zika (2015) challenged the global community. Mathematical models played a crucial role in these crises, helping predict the spread of these diseases and guiding strategies like quarantine, vaccination campaigns, and public health responses. These models also helped determine the effectiveness of interventions in real-time as new data became available.

COVID-19 Pandemic (2020) – Modeling During a Global Crisis:

The COVID-19 pandemic marked a turning point in the visibility and reliance on epidemic modeling. Models were essential in predicting case numbers, hospital capacities, and the impacts of interventions like lockdowns and vaccination. During COVID-19, models such as the SEIR model (which adds an “Exposed” category for individuals in the incubation period) became vital tools for understanding the spread of a virus with a long latency period. These models helped governments and public health agencies make informed decisions in real-time, showing the practical value of mathematical modeling in pandemic response.

The Future – Artificial Intelligence and Machine Learning in Modeling:

As artificial intelligence (AI) and machine learning (ML) technologies advance, they are increasingly being applied to epidemic modeling. AI and ML allow the analysis of massive datasets, including epidemiological, clinical, and behavioral data, to create more accurate and realistic models. These techniques enhance the ability to predict future disease outbreaks and assess the effectiveness of public health measures in real-time.

Mathematical epidemic models have been widely used to estimate future trends of disease transmission, gain insight into disease dynamics, and plan control strategies to mitigate outbreaks though possess limitations, such as identifiability in data fitting, case specificity, and robustness. Recently, researchers have been applying advanced statistical methods, and deep neural networks to overcome the limitations of fitting and achieve more accurate forecast. However, these do not provide insight to disease transmission mechanism and implications to control. Time series forecasting is a well-known challenge for infectious diseases [138, 145, 146, 147].

It is now increasingly common for infectious disease epidemics to be analysed with mathematical models. Modelling is possible because epidemics involve relatively simple processes occurring within large populations of individuals. Modelling aims to explain and predict trends in disease incidence, prevalence, morbidity or mortality. Epidemic models give important insight into the development of an epidemic. Following disease establishment, epidemic growth is approximately exponential. The rate of growth in this phase is primarily determined by the basic reproduction number, R0, the number of secondary cases per primary case when the population is susceptible. R0 also determines the ease with which control policies can control an epidemic. Once a significant proportion of the population has been infected, not all contacts of an infected individual will be with susceptible people. Infection can now continue only because new births replenish the susceptible population. Eventually an endemic equilibrium is reached where every infected person infects one other individual on average. Heterogeneity in host susceptibility, infectiousness, human contact patterns and in the genetic composition of pathogen populations introduces substantial additional complexity into this picture, however e and into the models required to model real diseases realistically. This chapter concludes with a brief review of the recent application of mathematical models to a wide range of emerging human or animal epidemics, most notably the spread of HIV in Africa, the 2001 foot and mouth epidemic in British livestock, bioterrorism threats such as smallpox, the SARS epidemics in 2003 and most recently the use of modelling as a tool for influenza pandemic preparedness planning.

An epidemic is a chain reaction of disease spread within a population. Epidemics can sometimes be described and predicted by mathematical models because they involve relatively simple processes occurring within large populations. "Simple" means that infection and disease progression can be characterized by the transition of an individual from one state to another; for example, from being uninfected and susceptible to infection to becoming infected after contact with an infectious individual, or from an infected state to a recovered state following recovery and the acquisition of immunity. Defining a model involves classifying the possible infection states of an individual and identifying the processes that cause movement between these states. The aim is to predict changes over time in the proportions of the population in different infection states and the incidence of disease-related events [148].

Establishment: The epidemic grows from the first infected individual until enough individuals are infected that the random extinction of the disease becomes unlikely. Many epidemics may die out after infecting only one or two individuals if the last infected person recovers (or dies) before transmitting the infection to others. This randomness makes the establishment phase relatively long and variable in duration, which may explain the decades-long gap between HIV entering the human population before 1950 and the large-scale epidemic recognized in the early 1980s.

Exponential Growth: After establishment, epidemic growth tends to follow an approximately exponential pattern. The speed of this growth is largely determined by two factors: the number of secondary cases generated by a primary case at the onset of the epidemic (the basic reproduction number, R_0) and the average time for secondary cases to be infected by a primary case (generation time, TG). R_0 measures the infection's inherent transmissibility and depends on disease biology (which affects infectiousness) and host population structure (which affects contact rates). TG is mainly determined by the incubation period and the duration of infectiousness. $R_0 > 1$ is essential for epidemic growth because, for an infection to sustain itself, each infected person must infect at least one other. Vaccination campaigns aim to reduce R_0 below 1, thus driving the disease toward extinction [149].

Endemicity: Once a significant portion of the population becomes immune, dies, or remains chronically infected, not all contacts with infected individuals will involve susceptible people. As a result, the number of new infections declines, and once the susceptible population is exhausted, the primary epidemic ends. Disease extinction is again possible by chance, especially in small populations. Otherwise, the infection persists, with new births replenishing the susceptible population until an endemic equilibrium is reached, where each infected person infects one other individual on average. Based on the definition of R_0, this occurs when $\frac{1}{R_0}$ of an individual's contacts are susceptible, which means that R_0 determines not only the epidemic growth rate but also the proportion of the population infected and the steady-state incidence of the disease.

In theory, knowledge of the basic reproduction number (R_0) and the duration of infectiousness allows for simple predictions of key aspects of disease transmission. However, estimating R_0 is often challenging, except for some childhood diseases like measles, where serological data is available. Epidemics are complex due to factors such as the mode of transmission (e.g., insect- or food-borne), pathogen heterogeneity (e.g., different strains), variations in host contact patterns (e.g., age or geographic groups), long incubation periods (e.g., HIV), and seasonal contact rate changes (e.g., measles). Incorporating these complexities is necessary for creating models that can make accurate predictions, rather than just reproducing basic trends.

Realistic models often include many parameters that require data from epidemiology, clinical studies, and behavioral patterns, but estimating these parameters remains difficult due to the non-linear nature of transmission and the dynamic complexity it creates, such as chaotic behaviors and random variations in epidemic timing. This complexity limits the predictability of disease trends. While short-term predictions may be possible in some cases, models often provide probabilistic forecasts, such as estimating the likelihood of epidemics of a certain size over a decade, rather than precise weekly predictions. Distinguishing between these predictive situations is a major challenge in epidemic modeling.

6.1 A Novel Fractional-Order Discrete SIR Model for Predicting COVID-19 Behavior

During the broadcast of Coronavirus across the globe, many mathematicians made several mathematical models. This was, of course, in order to understand the forecast and behavior of this epidemic's spread precisely. Nevertheless, due to the lack of much information about it, the application of many models has become difficult in reality and sometimes impossible, unlike the simple SIR model. In this work [150], a simple, novel fractional-order discrete model is proposed in order to study the behavior of the COVID-19 epidemic. Such a model has shown its ability to adapt to the periodic change in the number of infections. The existence and uniqueness of the solution for the proposed model are examined with the help of the Picard Lindelöf method. Some theoretical results are established in view of the connection between the stability of the fixed points of this model and the basic reproduction number. Several numerical simulations are performed to verify the gained results.

In fact, in order to formulate the target mathematical model, which will enable us later to predict the COVID-19 behavior according to the available information, we suppose that the society is divided into four different classes; susceptible class S, infected class I, recovered class R and death class D. In addition, we suppose that the transition between these classes takes place according to the scheme exhibited in Figure 6.1.

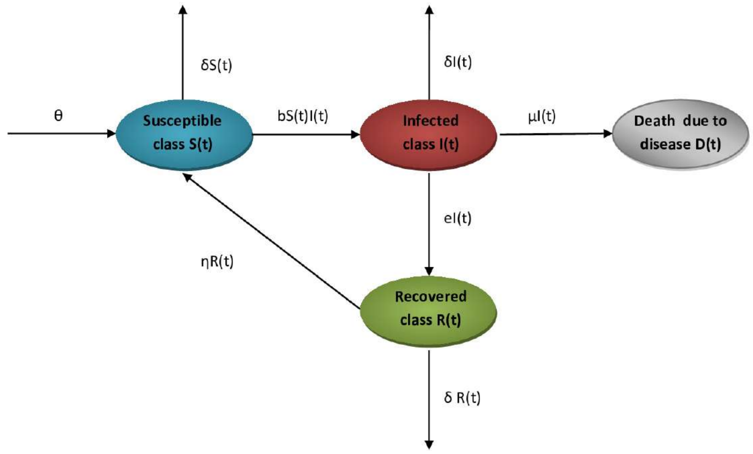

Figure 6.1: Disease transmission flow of the proposed model.

In light of this figure, the values of all parameters reported in the model at hand can be explained, as shown in the following Table:

Parameters	Description
μ	Corona death rate
δ	Natural death
θ	The number of new births
b	Infection rate
e	Recovery rate
t	The rate at which a recovering person is at risk of infection

As a matter of fact, the disease transmission flow of the proposed fractional-order discrete SIR model, which is shown in its compartmental form in 6.1, can be furthermore described by the following nonlinear system:

$$\begin{cases} S(n+1) = S(n) + \theta - bS(n)I(n) + \eta R(n) - \delta S(n), \\ I(n+1) = I(n) + bS(n)I(n) - (\mu + \delta + e)\, I(n), \\ R(n+1) = R(n) + eI(n) - (\delta + \eta)\, R(n), \\ D\,(n+1) = D\,(n) + \mu I\,(n)\,, \end{cases} \tag{6.1}$$

subject to the following initial conditions:

$$S(0), I(0), R(0), D(0) \geq 0. \tag{6.2}$$

Since the first three equations in the proposed system (6.1) are not related to class D, and because this class can be studied alone from using the equation:

$$N(n) = S(n) + I(n) + R(n) + D(n),$$

then the last equation of system (6.1) might be ignored. This would actually facilitate our examination by focusing only on the first three equations of that system. At the same time, it should be noted that because the value of the parameter b is very small ($b = \frac{pk}{N}$, where k is the rate of contacts per unit of time so $0 \leq k \leq 1$, p is the probabilities of contagion so $0 \leq p \leq 1$, and N is the number of the total population in the tens of thousands or more), the two terms $bS(n)$ and $bI(n)$ are also very small. The fact that we did not mention the age groups and vaccination does not mean that they were neglected, but the high life expectancy in a society can be seen as one of the causes of the high death rate due to the virus. As for vaccination, we know that it does not provide 100 percent immunity, but it reduces the possibility of infection (which reduces b), reduces virus mortality (η), and increases the recovery rate (e).

In accordance with the nonlinear model given in (6.1), we can now define the fractional-order model, which will be taken into consideration in this work from now on. This model has the following form:

$$\begin{cases} {}^{C}\Delta_0^{\alpha} S(t+1-\alpha) = \theta - \delta S(t) - bS(t)I(t) + \eta R(t), \\ {}^{C}\Delta_0^{\alpha} I(t+1-\alpha) = bS(t)I(t) - (\mu + \delta + e)\, I(t), \\ {}^{C}\Delta_0^{\alpha} R(t+1-\alpha) = eI(t) - (\delta + \eta)\, R(t), \end{cases} \tag{6.3}$$

where $0 < \alpha < 1$ and $t \in \mathbb{N}$. Continuing to move Delta in our examination, we recall below an important result that clarifies the stability of the solution of system (6.3).

6.1.1 Fixed Points and Stability Analysis

In this subsection, we will be concerned with analyzing the stability of the disease-free fixed point by finding sufficient conditions connected with the parameters of system (6.3) to ensure the stability of this point. To this aim, we will first find those fixed points by equating the right-hand side of system (6.3) with zero. In other words, we have:

$$\begin{cases} \theta - \delta S^* - bS^*I^* + tR^* = 0, \\ bS^*I^* - (\mu + \delta + e)\, I^* = 0, \\ eI^* - (\delta + \eta)\, R^* = 0, \end{cases} \tag{6.4}$$

It can be then seen that the above system has two fixed points at most. The first one is called the disease-free fixed point (or simply DFF point), which can be obtained by assuming $I^* = 0$, i.e.,

$$E_0 = \left(\frac{\theta}{\delta}, 0, 0\right). \tag{6.5}$$

On the other hand, if we let $I^* \neq 0$, then we obtain:

$$\begin{aligned} R^* &= \tfrac{e}{(\delta+t)} I^* \\ S^* &= \tfrac{(\mu+\delta+e)}{b}, \\ I^* &= \left(\tfrac{\delta(\mu+\delta+e)-b\theta}{b}\right)\left(\tfrac{(\delta+\eta)}{te-(\delta+\eta)(\mu+\delta+e)}\right), \end{aligned} \tag{6.6}$$

This, actually, yields the second fixed point of the considered system, which is called the pandemic fixed point E^*. In other words, this point can be yielded only when $I^* > 0$, and it has the form $E^* = (S^*, I^*, R^*)$, where S^*, I^* and $R^*$are defined above.

In the same regard, the so-called basic reproduction number R_0 can be defined as the number of infections caused by the first disease case. This 'infections' number typically appears in an appointed population where everyone is assumed to be susceptible to infection [149]. The importance of R_0 lies in knowing the rapidity of the spread of the emerging disease among the inhabitants and the proportion of the population to be immunized [149]. To be precise, the population spread of the epidemic will occur when $R_0 > 1$, where it is difficult to control. The method to enumerate the basic reproduction number can be performed by finding the spectral radius of the next generation matrix Y (i.e., $R_0 = \rho(Y)$). The matrix Y is a multiplication of F by V^{-1}, where:

$$F = \left[\frac{\partial F_i(E_0)}{\partial t_j}\right] \text{ and } F = \left[\frac{\partial V_i(E_0)}{\partial t_j}\right], \tag{6.7}$$

where F_i refers to the stream of freshly infected cases into compartment t_j, and V_i refers to the entering/leaving streams connected with t_j, for $i, j = 1, 2, 3, \cdots, m$ such that m is the total of compartments demonstrated in the model. Based on the aforesaid argument, one might calculate R_0 for the fractional-order model (6.3) by obtaining the two primary matrices F and V. To this aim, we can note that the infected compartment I can yield the following assertion:

$${}^{C}\Delta_a^{\alpha} I(t+1-\alpha) = bS(t)I(t) - (\mu + \delta + e)\, I(t) \tag{6.8}$$

This consequently implies the following Jacobian matrix:

$$J = \left(\frac{b\theta}{\delta}\right) - (\mu + \delta + e) \tag{6.9}$$

Accordingly, the above matrix can be decomposed in terms of the two matrices F and V so that $J = F - V$, where:

$$F = \left(\frac{b\theta}{\delta}\right) \tag{6.10}$$

and

$$V = (\mu + \delta + e) \tag{6.11}$$

Therefore, the basic reproduction number R_0 can be then calculated to be given as:

$$R_0 = \rho(Y) = \rho\left(FV^{-1}\right) = \frac{b\theta}{\delta(\mu + \delta + e)}. \tag{6.12}$$

Remark 6.1 *It should be noted that:*

$$I^* = \frac{(\delta + \eta)\,\theta}{(\mu + \delta + e)(\delta + \eta) - \eta e}\left(1 - \frac{1}{R_0}\right) \tag{6.13}$$

which asserts that the pandemic fixed point E^* will hold only when $R_0 > 1$.

In what follows, we will analyze the stability of the DFF point. This will be implemented by establishing some sufficient conditions related to the parameters of system (6.3) to ensure the stability of this point. In order to achieve this objective, we introduce the next theoretical result.

Theorem 6.1 *[150] In case of $R_0 < 1$.The the DFE point (E_0) of the system (6.3) is locally asymptotically stable if*

$$\max\left\{(\delta+\mu+e)(1-R_0),(\delta+\eta)\right\} < 2^{\alpha}. \tag{6.14}$$

Proof. The Jacobian matrix of F at E_0 can be given as:

$$J(E_0) = \begin{pmatrix} -\delta & -\frac{b\theta}{\delta} & \eta \\ 0 & \frac{b\theta}{\delta} - (\mu+\delta+e) & 0 \\ 0 & e & -(\delta+\eta) \end{pmatrix}, \tag{6.15}$$

Consequently, the characteristic polynomial will be as:

$$\det(\lambda Id - J(E_0)) = (\lambda+\delta)(\lambda+\delta+\eta)\left(\lambda + \frac{\delta e - b\theta + \delta^2 + \mu\delta}{\delta}\right),$$

$\Leftrightarrow$

$$\det(\lambda Id - J(E_0)) = (\lambda+\delta)(\lambda+(\delta+\eta))(\lambda-(\delta+\mu+e)(R_0-1)).$$

This implies:

$$\det(\lambda Id - J(E_0)) = 0,$$

$\Leftrightarrow$

$$\lambda_1 = -\delta < 0 \tag{6.16}$$

$$\lambda_2 = -(\delta+\eta) < 0 \tag{6.17}$$

and

$$\lambda_3 = (\delta+\mu+e)(R_0-1) < 0 \tag{6.18}$$

Thus, when $R_0 < 1$, we have:

$$\begin{aligned} -2^{\alpha} &< \lambda_1 < 0 \\ -2^{\alpha} &< \lambda_2 < 0 \end{aligned} \tag{6.19}$$

Hence, according to Theorem 2.13 , the DFE point is locally asymptotically stable if $R_0 < 1$.

■

6.1.2 Application to Predict the Behavior of COVID-19 in Germany

In this section, we will perform several numerical simulations to verify the results inferred in the previous sections. For this purpose, we will apply our study to predict the behavior of the virus in Germany. We will take all the statistics of a million people, which means we will take the initial population size $N(0)$ as $1,000,000$. According to [151], we can easily find: the number of new births per day for a million people $\theta = 26.3308$ and the death rate $\delta = 3.15 \times 10 - 5$. We can also obtain the stats described in Table 2 from the site [?].

$26 - Apr$	$27 - Apr$	$28 - Apr$	$29 - Apr$	$30 - Apr$	$1 - May$	$2 - May$
$30,791$	$29,637$	$28,642$	$28,126$	$27,845$	$27,375$	$26,262$
$3 - May$	$4 - May$	$5 - May$	$6 - May$	$7 - May$	$8 - May$	$9 - May$
$25,884$	$25,693$	$24,826$	$24,234$	$23,968$	$23,565$	$22,531$
$10 - May$	$11 - May$	$12 - May$	$13 - May$	$14 - May$	$15 - May$	$16 - May$
$21,817$	$21,253$	$20,707$	$20,664$	$20,557$	$20,250$	$19,914$
$17 - May$	$18 - May$	$19 - May$	$20 - May$	$21 - May$	$22 - May$	$23 - May$
$19,200$	$18,254$	$17,411$	$16,687$	$16,435$	$16,151$	$15,092$

Using Table 2 and [?], we find the contact rate $b = 5.1160 \times 10 - 7$, the recovery rate $e = 0.0789$, the rate at which a recovering person is at risk of infection $\eta = 0.87$, and the corona death rate $\mu = 0.1$. We also find that the initial number of the susceptible people is $S(0) = 686,400$, while the initial number of the exposed people is $I(0) = 33,966$, and finally, the initial number of the infected people is $R(0) = 279,630$. In order to apply Theorem 6.1, we first calculate the basic reproduction number:

$$R_0 = 0.73013 < 1, \tag{6.20}$$

and

$$\max\left\{(\delta + \mu + e)(1 - R_0), (\delta + \eta)\right\} = 0.87003 < 2^{\alpha}. \tag{6.21}$$

Note that the conditions of Theorem 6.1 are validated, then the DFE point is locally asymptotically stable. Anyhow, based on these, we plot Figures 6.2 and 6.3, which represents a numerical simulation, confirming the stability of the system (6.3) in this case.

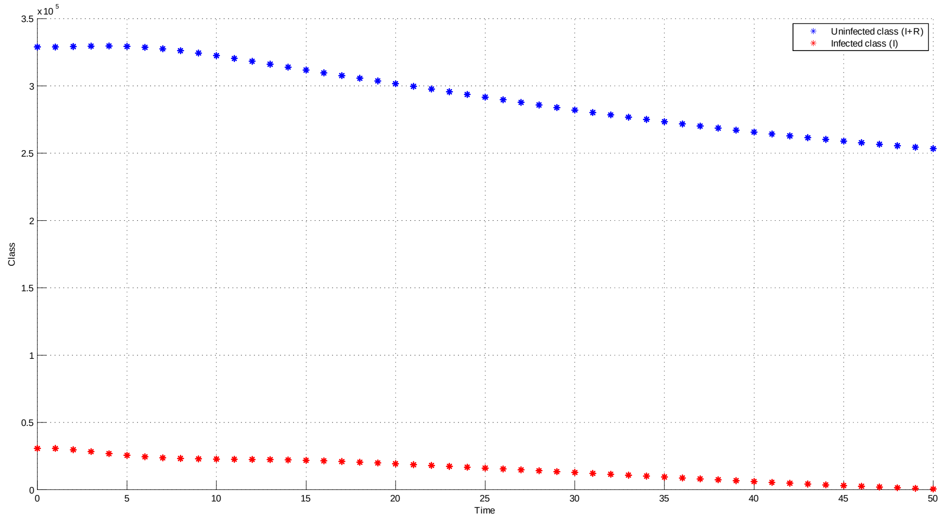

Figure 6.2: Numerical simulation of the system 6.3 using MATLAB.

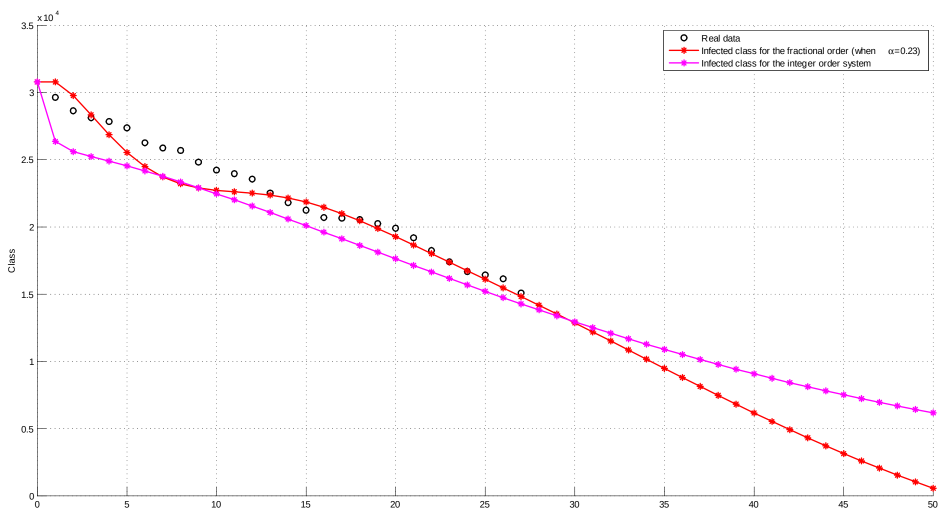

Figure 6.3: Numerical simulation of infected class with integer and commensurate fractional-order and comparison with the real data.

In order to obtain a better modeling use the incommensurate order system

$$\begin{cases} {}^{C}\Delta_0^{\alpha_1} S(t+1-\alpha) = \theta - \delta S(t) - bS(t)I(t) + \eta R(t), \\ {}^{C}\Delta_0^{\alpha_2} I(t+1-\alpha) = bS(t)I(t) - (\mu+\delta+e)\, I(t), \\ {}^{C}\Delta_0^{\alpha_3} R(t+1-\alpha) = eI(t) - (\delta+\eta)\, R(t), \end{cases} \tag{6.22}$$

where $0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1$. In this case, the existence of a direct and simple condition that deals with the stability of the system (6.22) is difficult and may be impossible in comparison with establishing a simple condition that deals with the stability of the commensurate system (6.3). However, it can indeed study the stability of the fixed points for system (6.22) by knowing the values of α_1,$1 \leq i \leq 3$. In such a case, we need to report the next corollary.

Overall we will consider $\alpha_1 = 0.223, \alpha_2 = 0.2, \alpha_3 = 0.232$. In this case, we obtain a more accurate approximation than the commensurate order case. Figure 6.4 shows a numerical simulation of this case, and Figure 6.5 represents the comparison of this case with the real data, while Figure 6.6 represents the comparison between the integer order case, the commensurate order case, the incommensurate order case, and the real data. It can be seen that each time, the system is more accurate.

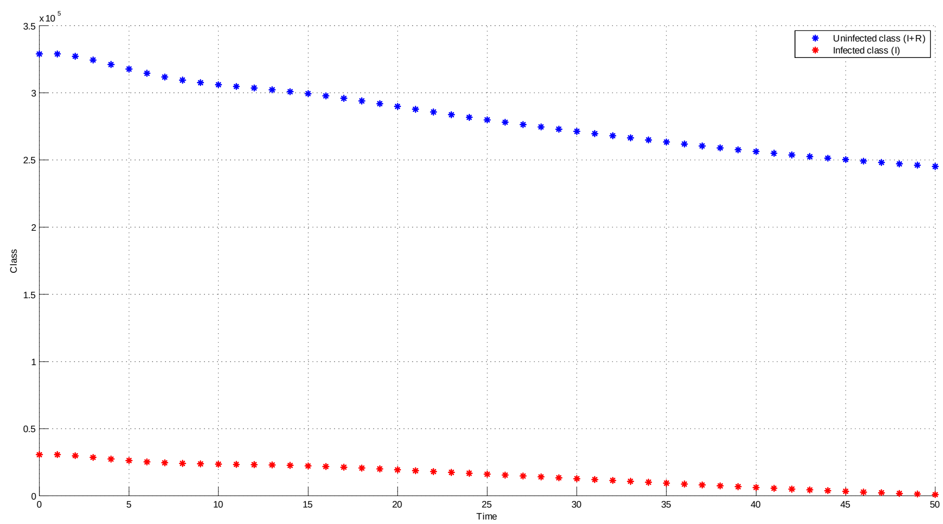

Figure 6.4: Numerical simulation of system (6.22) with $(\alpha_1 = 0.223, \alpha_2 = 0.2, \alpha_3 = 0.232)$.

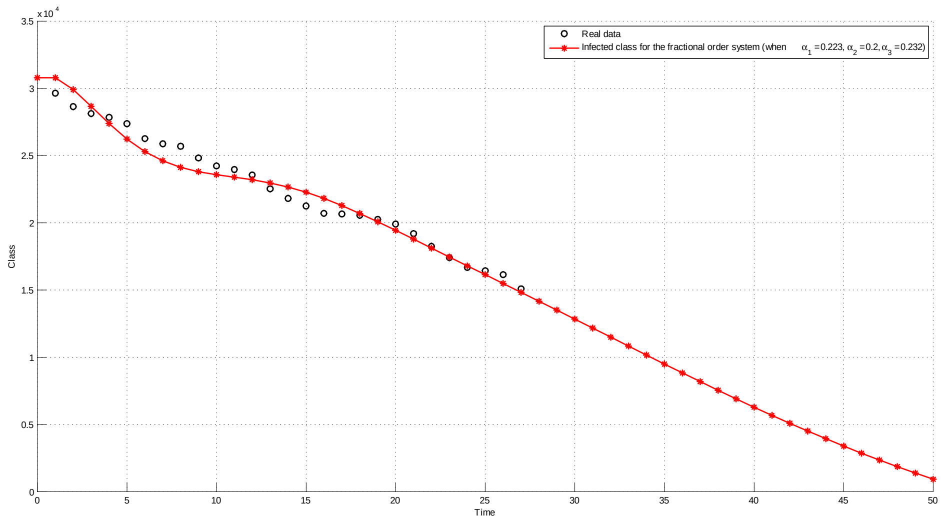

Figure 6.5: The comparison between the results of (6.22) and the real data.

It can be seen that system (6.22) is more accurate than system (6.1), and this reduces the error rate in prediction.

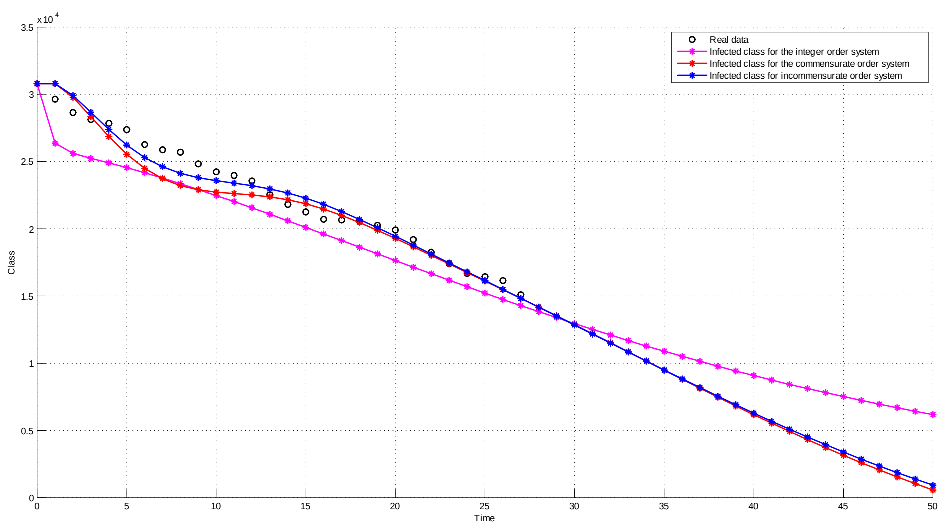

Figure 6.6: The comparison between the integer order, the commensurate order, and the incommensurate order systems with the real data.

We can assert that the presented model in this section can be applied to study the COVID-19 pandemic in many regions around the world, even if this model is formulated with the fractional-order Nabla operator in its commensurate or incommensurate cases.

6.2 The SEIR Covid-19 model described by fractional-order difference equations: analysis and application with real data in Brazil

One of the other Covid-19 models have been constructed in [154] as follow:

$$\begin{cases} \Delta S(n) = & \Lambda - r_1 S(n)E(n) - r_2 S(n)I(n) - \mu S(n) + \tau R(n), \\ \Delta E(n) = & r_1 S(n)E(n) + r_2 S(n)I(n) - (\mu + \rho)E(n), \\ \Delta I(n) = & \rho E(n) - (\gamma + \mu)I(n), \\ \Delta R(n) = & \gamma I(n) - (\mu + \tau)R(n), \end{cases} \quad n \in \mathbb{N}. \tag{6.23}$$

The proposed model's flowchart and parameters description are well explained in:

Variable/Parameter	Description	Real data[?]
$S(t)$	Susceptible class	17300532
$E(t)$	Exposed class	169017000
$I(t)$	Infected	1027023
$R(t)$	Recovered class	30921318
Λ	Recruitment rate into susceptible population	2837456.349
r_1 and r_2	Incidence rates	$4.3726 \times 10^{-11}, 1.3726 \times 10^{-11}$
μ	Natural mortality rate	0.014
τ	relapse rate	0.009
ρ	Progression rate from exposed to infectious class	0.0005
γ	Treatment rate for infectious individuals	0.158

Where $r_i = k_i p_i(1 - v)$, where k is the rate of contact, p is the probability of infection, $1 - v$ is the effect of vaccination on the studied population.

The total population N can be gained by adding up all classes announced in model (6.23), i.e.

$$N = S + E + I + R.$$

We will pay attention to the Caputo fractional order system (6.23), which is written as follows:

$$\begin{cases} {}^C\Delta_a^\alpha S(t+1-\alpha) = & \Lambda - r_1 S(t)E(t) - r_2 S(t)I(t) - \mu S(t) + \tau R(t), \\ {}^C\Delta_a^\alpha E(t+1-\alpha) = & r_1 S(t)E(t) + r_2 S(t)I(t) - (\mu + \rho)E(t), \\ {}^C\Delta_a^\alpha I(t+1-\alpha) = & \rho E(t) - (\gamma + \mu)I(t), \\ {}^C\Delta_a^\alpha R(t+1-\alpha) = & \gamma I(t) - (\mu + \tau)R(t), \end{cases} \tag{6.24}$$

where $0 < \alpha < 1$. For simplicity, we consider N to be constant. It is true that this will reduce the accuracy of the system, but it gives us simplicity in the study. As in [154] we can consider that the study time is short, which allows this hypothesis, we can put $R = N - S - E - I$ so the system 6.24 becomes:

$$\begin{cases} {}^{C}\Delta_a^{\alpha} S(t+1-\alpha) = & \Lambda + \tau N - r_1 S(t)E(t) - r_2 S(t)I(t) - (\mu+\tau)\, S(t) - \tau E(t) - \tau I(t), \\ {}^{C}\Delta_a^{\alpha} E(t+1-\alpha) = & r_1 S(t)E(t) + r_2 S(t)I(t) - (\mu+\rho)E(t), \\ {}^{C}\Delta_a^{\alpha} I(t+1-\alpha) = & \rho E(t) - (\gamma+\mu)I(t), \end{cases} \tag{6.25}$$

In addition, the following initial conditions are taken in consideration:

$$S(0), E(0), I(0) \geq 0. \tag{6.26}$$

Remark 6.2 *In [154] the presence of fixed points was studied and the authors gave a sufficient condition for the disappearance of the disease and also a sufficient condition for the positivity of S was given. Fixed point stability has not been comprehensively studied.*

In the following, we will prove the positivity of the solution to the fractional order system in all cases, and we will also give a sufficient condition for the stability of the disease-free stationary cat, i.e. we set a condition that guarantees the disappearance of the disease (absence of S) and also guarantees the absence of E, then we will study a model close to reality in the case of commensurate and incommensurate order and give notes on the fundamental changes between the two systems [155].

To find the fixed points of such system, we need to put:

$$\begin{cases} \Lambda + \tau N - r_1 S(t)E(t) - r_2 S(t)I(t) - (\mu+\tau)\, S(t) - \tau E(t) - \tau I(t) = 0, \\ r_1 S(t)E(t) + r_2 S(t)I(t) - (\mu+\rho)E(t) = 0, \\ \rho E(t) - (\gamma+\mu)I(t) = 0, \end{cases} \tag{6.27}$$

This would assert, with some simple calculations, that this system will have two fixed points atmost; the first one is called the disease-free fixed point $e_0 = (\frac{\Lambda+\tau N}{(\mu+\tau)}, 0, 0)$, and the second one, if it exists, is called the endemic fixed point.

6.2.1 Basic reproduction number

We have (E, I) the infected compartment. Then, we obtain we consider the following two equations that are taken out from system (6.25):

$$\begin{cases} {}^{C}\Delta_a^{\alpha} E(t+1-\alpha) = & r_1 S(t)E(t) + r_2 S(t)I(t) - (\mu+\rho)E(t), \\ {}^{C}\Delta_a^{\alpha} I(t+1-\alpha) = & \rho E(t) - (\gamma+\mu)I(t), \end{cases} \tag{6.28}$$

In accordance with the above equations, we can determine the matrix J^* that can be established based on the leaving fluxes from the infected compartment and the newly fluxes to such compartment. That is:

$$J^* = \begin{pmatrix} r_1 \frac{\Lambda+\tau N}{(\mu+\tau)} - (\mu+\rho) & r_2 \frac{\Lambda+\tau N}{(\mu+\tau)} \\ \rho & -(\gamma+\mu) \end{pmatrix} \tag{6.29}$$

Now, we can decompose the matrix J in terms of another two matrices F and V such that $J = F - V$. This, actually, yields:

$$F = \begin{pmatrix} r_1 \frac{\Lambda+\tau N}{(\mu+\tau)} & r_2 \frac{\Lambda+\tau N}{(\mu+\tau)} \\ 0 & 0 \end{pmatrix} \tag{6.30}$$

and

$$V = \begin{pmatrix} (\mu+\rho) & 0 \\ -\rho & (\gamma+\mu) \end{pmatrix} \tag{6.31}$$

Consequently, we can generate the next generation matrix FV^{-1} to be in the following form:

$$FV^{-1} = \begin{pmatrix} \frac{r_1}{(\tau+\mu)(\mu+\rho)} (\Lambda + N\tau) + \rho \frac{r_2}{(\tau+\mu)(\gamma+\mu)(\mu+\rho)} (\Lambda + N\tau) & \frac{r_2}{(\tau+\mu)(\gamma+\mu)} (\Lambda + N\tau) \\ 0 & 0 \end{pmatrix} \tag{6.32}$$

Therefore, the basic reproduction number R_0 will then be given by:

$$R_0 = \frac{(\gamma r_1 + \mu r_1 + \rho r_2)(\Lambda + N\tau)}{(\mu+\rho)(\tau+\mu)(\gamma+\mu)} \tag{6.33}$$

It is worth noting that if $R_0 > 1$, the system has an endemic fixed point of the form $e^* = (S^*, E^*, I^*)$, where

$$\begin{aligned} S^* &= \frac{(\gamma+\mu)(\mu+\rho)}{\gamma r_1 + \mu r_1 + \rho r_2} \\ E^* &= \frac{(\gamma+\mu)}{\rho} I^* \\ I^* &= \frac{R_0 - 1}{\left(\frac{\tau(\gamma+\mu)+\tau\rho}{\rho(\Lambda+N\tau)}\right) R_0 + \frac{\rho r_2 + r_1(\gamma+\mu)}{\rho(\tau+\mu)}} \end{aligned} \tag{6.34}$$

6.2.2 Stability analysis of the disease-free fixed point

In this subsection, we will be concerned with analyzing the stability of the disease-free fixed point by finding sufficient conditions for the parameters in System 6.25 to ensure the stability of this point.

Theorem 6.2 *[155] Suppose that $R_0 < 1$. then the disease-free fixed point of system (6.25) is locally asymptotically stable if*

$$\max\left\{(\mu+\tau), \gamma + 2\mu + \rho - \frac{r_1}{\tau+\mu}(\Lambda + N\tau)\right\} < 2^\alpha. \tag{6.35}$$

Proof. In fact, the Jacobian matrix J of system (6.25) at the fixed point e_0 can be obtained to be as:

$$J(e_0) = \begin{pmatrix} -(\mu+\tau) & -r_1\frac{\Lambda+\tau N}{(\mu+\tau)} - \tau & -r_2\frac{\Lambda+\tau N}{(\mu+\tau)} - \tau \\ 0 & r_1\frac{\Lambda+\tau N}{(\mu+\tau)} - (\mu+\rho) & r_2\frac{\Lambda+\tau N}{(\mu+\tau)} \\ 0 & \rho & -(\gamma+\mu) \end{pmatrix}, \tag{6.36}$$

Then, we obtain

$$\det(\lambda Id - J(E_F)) = 0 \tag{6.37}$$

$\Leftrightarrow$

$$((\mu+\tau)+\lambda)\left(-\lambda^2 - B\lambda + C\right) = 0 \tag{6.38}$$

where $B = \left(\gamma + 2\mu + \rho - \frac{r_1}{\tau+\mu}(\Lambda + N\tau)\right), C = \rho\frac{r_2}{\tau+\mu}(\Lambda+N\tau) - (\gamma+\mu)\left(\mu+\rho-\frac{r_1}{\tau+\mu}(\Lambda+N\tau)\right)$. Consequently, we have:

$$(\mu+\tau)+\lambda = 0 \Rightarrow \lambda_0 = -(\mu+\tau) \tag{6.39}$$

or

$$\left(-\lambda^2 - B\lambda + C\right) = 0. \tag{6.40}$$

Now, if $R_0 < 1$ then we have:

$$B^2 + 4C > 0,$$

and hence the two roots λ_1 and λ_2 of

$$\left(-\lambda^2 - B\lambda + C\right) = 0 \tag{6.41}$$

will be negative real numbers. Therefore, if $\lambda_1 + \lambda_2 = \gamma + 2\mu + \rho - \frac{r_1}{\tau+\mu}(\Lambda + N\tau) > -2^\alpha$ then we have:

$$\begin{aligned} -2^\alpha &< \lambda_1 < 0 \\ -2^\alpha &< \lambda_2 < 0 \end{aligned} \tag{6.42}$$

Hence, we can confirm, according to Theorem 2.13, that the disease-free fixed point is locally asymptotically stable if $R_0 < 1$.

■

From another point of view, we intend in the following content to present a simple condition that can be employed to ensure the global stability of system (6.25) at the disease-free fixed point.

Theorem 6.3 *[155] Supposethet $R_0 < 1$, if*

$$\frac{r_2 N}{(\gamma+\mu)} < \frac{(\mu+\rho) - r_1 N}{\rho}, \tag{6.43}$$

then the pandemic will disappear.

Proof. To prove this result, we intend to consider the Lyapunov function of the following form:

$$L(t) = E(t) + \theta I(t), \tag{6.44}$$

where $\frac{p_2\kappa_2}{(\gamma+\mu)} < \theta < \frac{(\mu+\rho)-p_1\kappa_1}{\rho}$. By differentiating L with respect to time, we obtain:

$$\begin{aligned}
{}^{C}\Delta_a^\alpha L(t+1-\alpha) &= {}^{C}\Delta_a^\alpha E(t+1-\alpha) + \theta\, {}^{C}\Delta_a^\alpha I(t+1-\alpha), \\
&= \left(r_1 S(t)E(t) + r_2 S(t)I(t) - (\mu+\rho)E(t)\right) + \theta\left(\rho E(t) - (\gamma+\mu)I(t)\right), \\
&= \left(r_1 S(t) - (\mu+\rho) + \theta_2\rho\right)E + \left(r_2 S(t) - \theta_2(\gamma+\mu)\right)I, \\
&\leq \left(r_1 N - (\mu+\rho) + \theta_2\rho\right)E + \left(r_2 N - \theta_2(\gamma+\mu)\right)I \\
&< 0.
\end{aligned}$$

Consequently, if condition (6.43) fulfilled, then ${}^{C}\Delta_a^\alpha L(t+1-\alpha)$ will be negative defined, and hence with the help of Theorem 2.14 we can confirm that the pandemic will disappear.

■

Remark 6.3 *We note that each of the fixed points and the basic reproduction number R_0 are not related to the order α. This makes the fractional order system (6.28) effective because it is a generalisation of integer order system and does not require new conditions.*

6.2.3 Application in Brazil

In this subsection ,we see the effectiveness of this system by applying it directly in Brazil, before that using the website described in [?], we get the number of active infections in Brazil in the period 23 July to 15 August 2022. We have

$$\frac{r_2 N}{(\gamma+\mu)} = 0.0182 \text{ and } \frac{(\mu+\rho) - r_1 N}{\rho} = 0.2230. \tag{6.45}$$

So the condition of the Theorem 6.3 is achieved and the disease disappears with the passage of time.

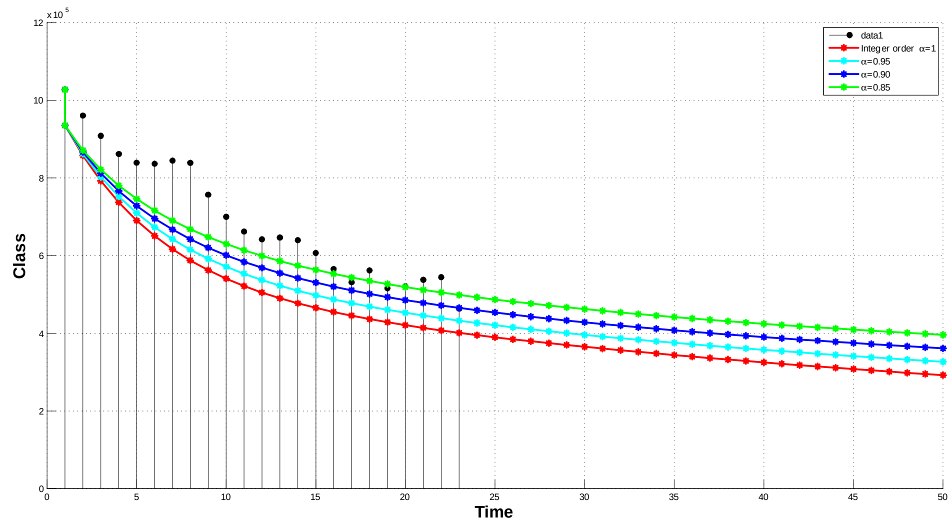

Figure 6.7: Numerical simulation of infected class and comparison withthe real data.

Figure 6.7 shows a comparison between the real data and the results obtained using the integer-order model (with order 1) and the fractional-order model with different values of α

Remark 6.4 *We note that changing the order allows the system to be improved and made more accurate.*

6.3 A New Incommensurate Fractional-Order Discrete COVID-19 Model with Vaccinated Individuals Compartment

To understand the behavior of the spread of the epidemic, we will divide the studied population into five classes: people who are exposed to infection and were not previously infected and did not receive vaccination, people who were previously infected and recovered from the disease and are at risk of being infected again, vaccinated people against the epidemic. As for the class of infected persons, it is divided into two secondary classes: The people with good immunity and for whom infection does not pose a great risk and suppose that their ratio in the society is λ, $(\lambda \leq 1)$, the infected persons for whom the infection is dangerous and consists of the elderly, pregnant women and people with chronic diseases (whose their ratio in the society is $(1-\lambda)$). Now we will explain the immigration in each class to the other[156]:

Susceptible class: This class acquires a number of people, which is the number of people entering the studied area, and in the event that the studied area is isolated, then represents the birth rate in this area, this class loses, people who are exposed to infection, as well as loses, people who have been vaccinated against the disease and natural deaths.

Recovered class: This class is acquired at the rate of new recovred persons and loses people who are exposed to infection, as well as losing people who have been vaccinated against the disease and natural deaths.

Vaccinated class: This class is acquired at the rate of new vaccinated persons, this class loses, people who are infected and natural deaths.

Infection class: This class is acquired new infection, this class loses, people who are recovered, and natural deaths. (We assume that in this class there are no deaths due to the epidemic).

Infection dangerous class: This class is acquired new infected persons, this class loses, people who are recovering, natural deaths and deaths due to infection.

We now summarize all the previous in the following diagram:

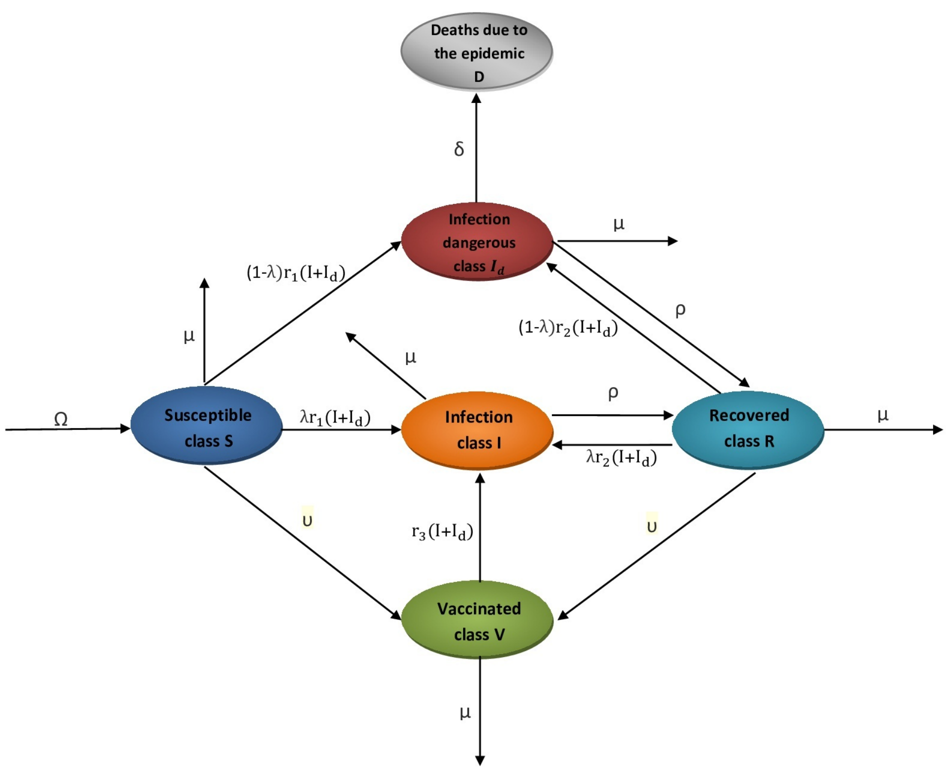

Figure 6.8: Diagram showing the transition between categories.

The proposed model's flowchart and parameters description are well explained in (6.46).

Variable	Description
S	Susceptible class
R	Recovered class
V	Vaccinated class
I	Infection class
I_d	Infection dangerous class
Ω	The birth rate
μ	Natural death rate
r_1, r_2, r_3	Infection rates
ρ	Recovered rate
υ	Vaccinated rate
δ	Death due to infection rate

(6.46)

where $r_i = \frac{p_i k}{N}, i = 1, 2, 3$, k is the average numbers of contacts per capita (per unit of time), p_i is the probabilities of contagion ($p_1 > p_2 > p_3$) and N is the total population (It can be considered as the maximum value of the population. Sometimes we take $N = \frac{\Omega}{\mu}$).

The mathematical translation of the above in the system of differential equations is as follows[156]:

$$\begin{cases} \frac{dS}{dt} = \Omega - r_1 \left(I\left(t\right) + I_d\left(t\right)\right) S\left(t\right) - \left(\mu + \upsilon\right) S\left(t\right), \\ \frac{dR}{dt} = \rho \left(I\left(t\right) + I_d\left(t\right)\right) - r_2 \left(I\left(t\right) + I_d\left(t\right)\right) R\left(t\right) - \left(\upsilon + \mu\right) R\left(t\right), \\ \frac{dV}{dt} = \upsilon \left(S\left(t\right) + R\left(t\right)\right) - r_3 \left(I + I_d\right) V\left(t\right) - \mu V\left(t\right), \\ \frac{dI}{dt} = \left(\lambda \left(r_1 S\left(t\right) + r_2 R\left(t\right)\right) + r_3 V\left(t\right)\right) \left(I\left(t\right) + I_d\left(t\right)\right) - \left(\mu + \rho\right) I\left(t\right), \\ \frac{dI_d}{dt} = \left(1 - \lambda\right) \left(r_1 S\left(t\right) + r_2 R\left(t\right)\right) \left(I\left(t\right) + I_d\left(t\right)\right) - \left(\mu + \delta + \rho\right) I_d\left(t\right). \end{cases} \quad t \in \mathbb{R}^+. \tag{6.47}$$

The total population is given as follow:

$$N = S + R + V + I + I_d.$$

In addition, the following initial conditions are taken in consideration:

$$S(0), R(0), V(0), I(0), I_d(0) \geq 0. \tag{6.48}$$

In this paper, we will study the general case of the discrete system, which is given by the Caputo incommensurate fractional differences we choose the order α for the healthy class and the order β for the infected class. The incommensurate order gives us more flexibility in modeling and making a more realistic model.

When converting system 6.47 to a system of fractional difference system, we have two types of systems: **The Delta difference system** and **The** Nabla **difference system.**

The fractional incommensurate order Delta difference system is given by[157]:

$$\begin{cases} {}^{C}\Delta_{0}^{\alpha}S(t+1-\alpha)=\Omega-r_{1}\left(I\left(t\right)+I_{d}\left(t\right)\right)S\left(t\right)-\left(\mu+\upsilon\right)S\left(t\right), \\ {}^{C}\Delta_{0}^{\alpha}R(t+1-\alpha)=\rho\left(I\left(t\right)+I_{d}\left(t\right)\right)-r_{2}\left(I\left(t\right)+I_{d}\left(t\right)\right)R\left(t\right)-\left(\upsilon+\mu\right)R\left(t\right), \\ {}^{C}\Delta_{0}^{\alpha}V(t+1-\alpha)=\upsilon\left(S\left(t\right)+R\left(t\right)\right)-r_{3}\left(I+I_{d}\right)V\left(t\right)-\mu V\left(t\right), & t\in\mathbb{N}_{1}. \\ {}^{C}\Delta_{0}^{\beta}I(t+1-\alpha)=\left(\lambda\left(r_{1}S\left(t\right)+r_{2}R\left(t\right)\right)+r_{3}V\left(t\right)\right)\left(I\left(t\right)+I_{d}\left(t\right)\right)-\left(\mu+\rho\right)I\left(t\right), \\ {}^{C}\Delta_{0}^{\beta}I_{d}(t+1-\alpha)=\left(1-\lambda\right)\left(r_{1}S\left(t\right)+r_{2}R\left(t\right)\right)\left(I\left(t\right)+I_{d}\left(t\right)\right)-\left(\mu+\delta+\rho\right)I_{d}\left(t\right). \end{cases} \tag{6.49}$$

where $0<\alpha,\beta<1$. Using the Theorem 1.13, System 6.49 can be written as follow:

$$X\left(t+1\right)=X(0)+\frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-s-1)^{(\alpha-1)}F\left(X\left(s\right)\right),t\in\mathbb{N}, \tag{6.50}$$

where

$$X(t)=(S(t),R(t),V(t),I(t),I_{d}(t)),$$

and

$$F\left(X\left(t\right)\right)=(f_{1}(X(t)),f_{2}(X(t)),f_{3}(X(t)),f_{4}(X(t)),f_{5}(X(t)))^{t},$$

$$\begin{aligned} f_{1}(X(t))&=\Omega-r_{1}\left(I\left(t\right)+I_{d}\left(t\right)\right)S\left(t\right)-\left(\mu+\upsilon\right)S\left(t\right), \\ f_{2}(X(t))&=\rho\left(I\left(t\right)+I_{d}\left(t\right)\right)-r_{2}\left(I\left(t\right)+I_{d}\left(t\right)\right)R\left(t\right)-\left(\upsilon+\mu\right)R\left(t\right), \\ f_{3}(X(t))&=\upsilon\left(S\left(t\right)+R\left(t\right)\right)-r_{3}\left(I+I_{d}\right)V\left(t\right)-\mu V\left(t\right), \\ f_{4}(X(t))&=\left(\lambda\left(r_{1}S\left(t\right)+r_{2}R\left(t\right)\right)+r_{3}V\left(t\right)\right)\left(I\left(t\right)+I_{d}\left(t\right)\right)-\left(\mu+\rho\right)I\left(t\right), \\ f_{5}(X(t))&=\left(1-\lambda\right)\left(r_{1}S\left(t\right)+r_{2}R\left(t\right)\right)\left(I\left(t\right)+I_{d}\left(t\right)\right)-\left(\mu+\delta+\rho\right)I_{d}\left(t\right). \end{aligned} \tag{6.51}$$

Thus, it is defined by a regression relationship. We can notice that existence and uniqueness are trivial in this case. It cannot be shown that the solution is positive for system 6.49. In fact, the solution to System 6.49 is not always positive even when the initial conditions are positive.

In the rest of the paper, we will study the fractional incommensurate order Nabla difference system who writes as follows:

$$\begin{cases} {}^{C}\nabla_{0}^{\alpha}S(t)=\Omega-r_{1}\left(I\left(t\right)+I_{d}\left(t\right)\right)S\left(t\right)-\left(\mu+\upsilon\right)S\left(t\right), \\ {}^{C}\nabla_{0}^{\alpha}R(t)=\rho\left(I\left(t\right)+I_{d}\left(t\right)\right)-r_{2}\left(I\left(t\right)+I_{d}\left(t\right)\right)R\left(t\right)-\left(\upsilon+\mu\right)R\left(t\right), \\ {}^{C}\nabla_{0}^{\alpha}V(t)=\upsilon\left(S\left(t\right)+R\left(t\right)\right)-r_{3}\left(I+I_{d}\right)V\left(t\right)-\mu V\left(t\right), & t\in\mathbb{N}_{1}. \\ {}^{C}\nabla_{0}^{\beta}I(t)=\left(\lambda\left(r_{1}S\left(t\right)+r_{2}R\left(t\right)\right)+r_{3}V\left(t\right)\right)\left(I\left(t\right)+I_{d}\left(t\right)\right)-\left(\mu+\rho\right)I\left(t\right), \\ {}^{C}\nabla_{0}^{\beta}I_{d}(t)=\left(1-\lambda\right)\left(r_{1}S\left(t\right)+r_{2}R\left(t\right)\right)\left(I\left(t\right)+I_{d}\left(t\right)\right)-\left(\mu+\delta+\rho\right)I_{d}\left(t\right). \end{cases} \tag{6.52}$$

When $\alpha=\beta=1$. By adding the equations from the system 6.52, we obtain:

$$\nabla N\left(t\right)=\Omega-\mu N(t)-\delta I_{d}\left(t\right),$$

since I_d is positive, we get:

$$\nabla N\left(t\right)\leq\Omega-\mu N(t),$$

so

$$N(t) \leq \frac{\Omega + N(t-1)}{1+\mu}.$$

Let $N(0) \leq \frac{\Omega}{\mu}$, and suppose that $N(t) \leq \frac{\Omega}{\mu}$ for t, we get

$$N(t+1) \leq \frac{\Omega + N(t)}{1+\mu} \leq \frac{\Omega + \frac{\Omega}{\mu}}{1+\mu} = \frac{\Omega}{\mu},$$

then by induction for all t when the solution existe

$$0 \leq N(t) \leq \frac{\Omega}{\mu}.$$

Therefore, the solution belongs to the invariant region:

$$\Psi = \left\{ (S, R, V, I, I_d) \in \mathbb{R}_+^5 \text{ and } S + R + V + I + I_d \leq \frac{\Omega}{\mu} \right\},$$

where $\mathbb{R}_+^5 = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \text{ and } x_i \geq 0 \text{ for } i = 1..5\}$, as invariant region.

6.3.1 Fixed Points and Basic Reproduction Number

To study the dynamics of (6.52), one must first find the fixed points, and to find the fixed points, the equation must be solved:

$$\begin{cases} \Omega - r_1 (I^* + I_d^*) S - (\mu + \upsilon) S^* = 0, \\ \rho (I^* + I_d^*) - r_2 (I^* + I_d^*) R^* - (\upsilon + \mu) R^* = 0, \\ \upsilon (S^* + R^*) - r_3 (I^* + I_d^*) V^* - \mu V^* = 0, \\ (\lambda (r_1 S^* + r_2 R^*) + r_3 V^*) (I^* + I_d^*) - (\mu + \rho) I^* = 0 \\ (1 - \lambda) (r_1 S^* + r_2 R^*) (I^* + I_d^*) - (\mu + \delta + \rho) I_d^* = 0. \end{cases} \tag{6.53}$$

The previous equation has the point $E_0 = \left(\frac{\Omega}{(\mu+\upsilon)}, 0, \frac{\upsilon\Omega}{\mu(\mu+\upsilon)}, 0, 0 \right)$ as a solution. It can be seen that the disease at this point is non-exist and therefore it is called the disease-free fixed point who are interested in studying its stability later.

If we suppose that $(I^* + I_d^*) \neq 0$, So we will get:

$$\begin{array}{c} \frac{\Omega}{r_1 \left(I^* + I_d^*\right) + (\mu + \upsilon)} = S^*, \\ \frac{\rho\left(I^* + I_d^*\right)}{r_2 \left(I^* + I_d^*\right) + (\upsilon + \mu)} = R^*, \\ \frac{\upsilon(S^* + R^*)}{r_3 \left(I^* + I_d^*\right) + \mu} = V^*, \\ \frac{(\lambda(r_1 S^* + r_3 R^*) + r_2 V^*)}{(\mu + \rho)} = \frac{I^*}{\left(I^* + I_d^*\right)}, \\ \frac{(1-\lambda)(r_1 S^* + r_2 R^*)}{(\mu + \delta + \rho)} = \frac{I_d^*}{\left(I^* + I_d^*\right)}, \end{array} \tag{6.54}$$

this system is complex and difficult to solve in the abstract case. Even if we were able to solve it, studying the stability of this fixed point contains many obstacles. Over all, these point is called the epidimic equilibrioum point $E^* = (S^*, R^*, V^*, I^*, I_d^*)$.

The basic reproduction number R_0 is very important in the study of stability for the disease-free fixed point, which represents the rate of new people being infected by one sick person until his recovery. We will follow the steps described in [149], we find :

$$R_0 = \frac{\Omega}{(\mu+\upsilon)}\left(\frac{(1-\lambda)\,r_1}{(\mu+\delta+\rho)} + \frac{\upsilon r_3 + \lambda\mu r_1}{\mu\,(\mu+\rho)}\right). \tag{6.55}$$

6.3.2 Stability analysis of the disease-free fixed point

One of the most important dynamic behaviors in the system is the stability of fixed points. Therefore, we always resort to studying it, and for the purpose of knowing whether the disease disappears or not, we will be interested in studying the asymptotic stability of the disease-free fixed point of the system.

Theorem 6.4 *[157] Suppose that $R_0 < 1$.Then the disease-free fixed point E_0 of the system 6.52 is locally asymptotically stable.*

Proof. To apply Theorem 2.17, we must first calculate the Jacobian matrix for the right-hand side of system (6.52) at fixed point E_0, by simple calculation we obtain

$$J = \begin{pmatrix} -(\mu+\upsilon) & 0 & 0 & -\frac{\Omega r_1}{(\mu+\upsilon)} & -\frac{\Omega r_1}{(\mu+\upsilon)} \\ 0 & -(\upsilon+\mu) & 0 & \rho & \rho \\ \upsilon & \upsilon & -\mu & -\frac{\Omega \upsilon r_3}{\mu(\mu+\upsilon)} & -\frac{\Omega \upsilon r_3}{\mu(\mu+\upsilon)} \\ 0 & 0 & 0 & \left(\frac{\lambda r_1\Omega}{(\mu+\upsilon)} + \frac{\upsilon r_3\Omega}{\mu(\mu+\upsilon)}\right) - (\mu+\rho) & \left(\frac{\lambda r_1\Omega}{(\mu+\upsilon)} + \frac{\upsilon r_3\Omega}{\mu(\mu+\upsilon)}\right) \\ 0 & 0 & 0 & \frac{(1-\lambda)r_1\Omega}{(\mu+\upsilon)} & \frac{(1-\lambda)r_1\Omega}{(\mu+\upsilon)} - (\mu+\delta+\rho) \end{pmatrix}, \tag{6.56}$$

The characteristic equation of J is:

$$\left(X^{M\alpha}+\mu\right)\left(X^{M\alpha}+\mu+\upsilon\right)^2\left(X^{2M\beta}+AX^{M\beta}+B\right),$$

where

$$A = (\mu+\rho) + (\mu+\delta+\rho)(1-R_0) + \tfrac{\Omega\delta(\upsilon r_3+\lambda\mu r_1)}{\mu(\mu+\rho)(\mu+\upsilon)},$$
$$B = (\mu+\rho)(\mu+\delta+\rho)(1-R_0).$$

So, we have

$$X^{M\alpha} = -\mu \text{ or } X^{M\alpha} = -\mu-\upsilon,$$

this means that

$$\arg\ \left|X^{M\alpha}\right| = \pi,$$

and therefore

$$\arg\ |X| = \frac{\pi}{M\alpha} > \frac{\pi}{2M}.$$

And if $R_0 < 1$ then $A, B > 0$. So according to Routh Hurwitz criterion the both roots of

$$Y^2 + AY + B,$$

is in the open left half plane so

$$\left| X^{M\beta} \right| > \frac{\pi}{2} \Rightarrow |X| > \frac{\pi}{2M\beta} > \frac{\pi}{2M}$$

from it the condition of Theorem 2.17 fulfilled. Accordingly the disease-free fixed point E_0 of system 6.52 is locally asymptotically stable. ■

Remark 6.5 *[157] We note that this result is very logical and compatible with the definition of the number R_0. As it is clear that if every infected person leaves less than one infection, the disease will disappear.*

6.3.3 A condition for the disappearance of the pandemic

The primary objective of the study of the mathematical model of the epidemic is to study the possibility of finding solutions to get rid of the epidemic. By adding the last two equations into the system 6.52 we find that the infection is described by the next equation:

$${}^{C}\nabla_0^{\beta} (I + I_d)(n) = (r_1 S(n) + r_2 R(n) + r_3 V(n) - (\mu + \rho)) (I(n) + I_d(n)) - \delta I_d(n),$$

Since I is positive and $r_i = \frac{p_i k}{N}, i = 1, 2, 3$:

$${}^{C}\nabla_0^{\beta} (I + I_d)(n) \leq (p_1 k - (\mu + \rho)) (I(n) + I_d(n)),$$

Applying the comparison Theorem we get

$$(I + I_d)(t) \leq Y(t),$$

where $Y(t)$ is the solution of

$${}^{C}\nabla_0^{\beta} Y(t) = (p_1 k - (\mu + \rho)) Y(t), Y(0) = (I + I_d)(0).$$

It is according to proposition 10:

$$Y(t) = (I + I_d)(0) F_\beta((p_1 k - (\mu + \rho)), (t)^{\overline{\beta}}),$$

then

$$(I + I_d)(t) \leq (I + I_d)(0) F_\beta((p_1 k - (\mu + \rho)), (t)^{\overline{\beta}}).$$

We note that if $(p_1 k - (\mu + \rho)) < 0$, then according to proposition 9 $(I + I_d)(t) \to 0$ when $t \to \infty$. So we get the following result:

Theorem 6.5 *[157] If*

$$\frac{(\mu+\rho)-1}{p_1} < k < \frac{(\mu+\rho)}{p_1}, \tag{6.57}$$

then the disease will disappear.

Remark 6.6 *[157] While the condition of Theorem 6.5 is considered to be higher than the condition of Theorem 6.5, it guarantees global stability.*

In real applications, it is always the case that $(\mu+\rho) < 1$, and the condition (6.57) becomes $k < \frac{(\mu+\rho)}{p_1}$.

The authorities in all countries of the world have imposed preventive measures such as protective masks and the use of sterilizer in public places, which affects the probabilities of contagion (p_1, p_2, p_3). The authorities can also impose a quarantine, which affects the friction as in all cases (k), where is the only possible control is to reduce infection rates (r_1, r_2, r_3).

Let $x(t) : [0, +\infty[\to [0, 1]$ be the control function, which is representing the percentage of quarantine (quarantine cannot be 100% because that is impossible). The controlled system is written as follows:

$$\begin{cases} {}^{C}\nabla_0^{\alpha} S(t) = & \Omega - (1-u(t))\, r_1 \left(I\left(t\right) + I_d\left(t\right)\right) S\left(t\right) - \left(\mu + \upsilon\right) S\left(t\right), \\ {}^{C}\nabla_0^{\alpha} R(t) = & \rho \left(I\left(t\right) + I_d\left(t\right)\right) - (1-u(t))\, r_2 \left(I\left(t\right) + I_d\left(t\right)\right) R\left(t\right) - \left(\upsilon + \mu\right) R\left(t\right), \\ {}^{C}\nabla_0^{\alpha} V(t) = & \upsilon \left(S\left(t\right) + R\left(t\right)\right) - (1-u(t))\, r_3 \left(I + I_d\right) V\left(t\right) - \mu V\left(t\right), \\ {}^{C}\nabla_0^{\beta} I(t) = & \left(\lambda \left((1-u(t))\, r_1 S\left(t\right) + (1-u(t))\, r_2 R\left(t\right)\right) + (1-u(t))\, r_3 V\left(t\right)\right) \left(I\left(t\right) + I_d\left(t\right)\right) \\ & - \left(\mu + \rho\right) I\left(t\right), \\ {}^{C}\nabla_0^{\beta} I_d(t) = & (1-\lambda) \left((1-u(t))\, r_1 S\left(t\right) + (1-u(t))\, r_2 R\left(t\right)\right) \left(I\left(t\right) + I_d\left(t\right)\right) - \left(\mu + \delta + \rho\right) I_d\left(t\right). \end{cases} \tag{6.58}$$

When the initial condition is close enough to the disease-free fixed point It suffices to choose a controller to ensure the asymptotic stability of the disease-free fixed point and this is what we did in the following theorem:

Theorem 6.6 *[157] Suppose that*

$$\frac{R_0 - 1}{R_0} < u(t). \tag{6.59}$$

Then the disease-free fixed point E_0 is locally asymptotically stable.

Proof. The characteristic equation become

$$\left(X^{M\alpha} + \mu\right) \left(X^{M\alpha} + \mu + \upsilon\right)^2 \left(X^{2M\beta} + A' X^{M\beta} + B'\right),$$

where

$$\begin{aligned} A' &= (\mu+\rho) + (\mu+\delta+\rho)\left(1 - (1-u(t))\, R_0\right) + \tfrac{\Omega\delta(1-u(t))(\upsilon r_3 + \lambda\mu r_1)}{\mu(\mu+\rho)(\mu+\upsilon)}, \\ B' &= (\mu+\rho)(\mu+\delta+\rho)\left(1 - (1-u(t))\, R_0\right). \end{aligned}$$

if $\frac{R_0-1}{R_0} < u(t)$, then $A', B' > 0$. So according to Routh Hurwitz criterion the both roots of

$$Y^2 + A'Y + B',$$

is in the open left half plane so

$$\left|X^{M\beta}\right| > \frac{\pi}{2} \Rightarrow |X| > \frac{\pi}{2M\beta} > \frac{\pi}{2M}.$$

By it the condition of Theorem 2.17 fulfilled. Accordingly the disease-free fixed point E_0 is locally asymptotically stable. ■

When the initial condition is far from the disease-free fixed point it is not possible to guarantee stability. To reduce the spread of the epidemic in this case, we present the following result:

Theorem 6.7 *[157] Suppose that*

$$1 - \frac{(\mu+\rho)}{p_1 k} < u(t) \tag{6.60}$$

then the disease will disappear.

Proof. Let $\min u(t) = \chi$. Adding the last two equations in the system 6.58 we find that infection is described by the following equation

$$\begin{aligned} {}^{C}\nabla_0^{\beta}(I+I_d) = \; & ((1-u(t))\, r_1 S(t) + (1-u(t))\, r_2 R(t) + (1-u(t))\, r_3 V(t) - (\mu+\rho))\,(I(t)+I_d(t)) \\ & -\delta I_d(t), \end{aligned}$$

since I is positive

$${}^{C}\nabla_0^{\beta}(I+I_d) \le ((1-u(t))\, r_1 S(t) + (1-u(t))\, r_2 R(t) + (1-u(t))\, r_3 V(t) - (\mu+\rho))\,(I(t)+I_d(t)),$$

Since $r_i = \frac{p_i k}{N}, i = 1, 2, 3$:

$$\begin{aligned} {}^{C}\nabla_0^{\beta}(I+I_d) \; & \le \left((1-u(t))\, \tfrac{p_1 k}{N} S(t) + (1-u(t))\, \tfrac{p_2 k}{N} R(t) + (1-u(t))\, \tfrac{p_3 k}{N} V(t) - (\mu+\rho)\right)(I(t)+I_d(t)), \\ & \le \left((1-u(t))\, \tfrac{p_1 k}{N}(S(t)+R(t)+V(t)) - (\mu+\rho)\right)(I(t)+I_d(t)), \\ & \le ((1-\chi)\, p_1 k - (\mu+\rho))\,(I(t)+I_d(t)). \end{aligned}$$

Applying the comparison Theorem 10 we get

$$(I+I_d)(t) \le Y(t),$$

where $Y(t)$ is the solution of

$${}^{C}\nabla_0^{\beta} Y(t) = ((1-\chi)\, p_1 k - (\mu+\rho))\, Y(t), Y(0) = (I+I_d)(0).$$

It is according to propostion 10

$$Y(t) = (I+I_d)(0) F_{\beta}(((1-\chi)\, p_1 k - (\mu+\rho)), (t)^{\overline{\beta}})$$

then

$$(I + I_d)(t) \leq (I + I_d)(0) F_\beta(((1-\chi) p_1 k - (\mu + \rho)), (t)^{\overline{\beta}}).$$

We note that if $((1-\chi) p_1 k - (\mu + \rho)) < 0$, then $(I + I_d)(t) \to 0$ when $t \to \infty$. ■

Remark 6.7 *[157] We note that the control given in Theorem 6.7 is more expensive than the control given in Theorem 6.6. Therefore, it is better to impose quarantine and other procedures before the spread of the disease, that is, when the initial condition is very close to the disease-free fixed point.*

6.3.4 Numerical Simulations and Application

To review the effectiveness of the studied system, in this section we apply it to the state of Germany and compare the results of the integer order system with the fractional incommensurate system to determine their compatibility with realistic results. All statistics are taken as $N(0) = 1,000,000$. According to [**?**], the initial population can be divided as follows

$$S(0) = 170145, R(0) = 260270, V(0) = 538794, I(0) = 27712, I_d(0) + 3079. \tag{6.61}$$

On the other hand, according to the same source [**?**] and according to [151], we can calculate the values of the system parameters and find

$$\begin{array}{ccc} \Omega = 32.3308; & \mu = 3.15 \times 10^{-5}; & \lambda = 0.9; \\ r_1 = 1.9008 \times 10^{-7}; & r_2 = 1.5849 \times 10^{-7}; & r_3 = 1.1405 \times 10^{-7}; \\ \rho = 0.016; & \upsilon = 0.0037; & \delta = 0.06. \end{array} \tag{6.62}$$

We take real data on active cases in Germany during the period from 26 April to 30 May 2022, represented in Figure 6.9.

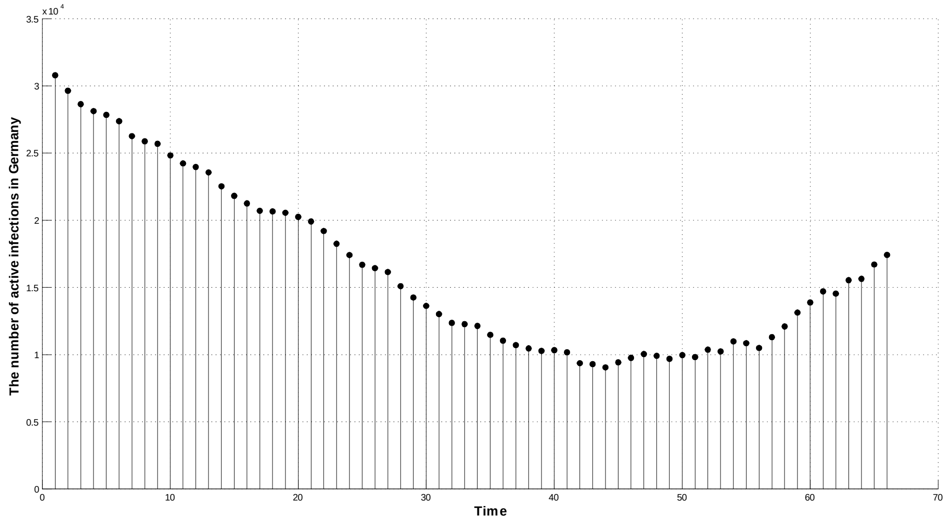

Figure 6.9: The number of active infections in Germany in the period 26 April to 30 May 2022 [**?**].

To apply Theorem 6.4, we must first calculate R_0:

$$R_0 = \frac{\Omega}{(\mu + \upsilon)} \left(\frac{(1-\lambda)\, r_1}{(\mu + \delta + \rho)} + \frac{\upsilon r_3 + \lambda \mu r_1}{\mu\,(\mu + \rho)} \right) = 0.5988228.$$

We note that $R_0 < 1$, according to Theorem 13 the disease-free fixed point E_0 is locally asymptotically stable. While we thus know that the number of active cases is decreasing with time, we do not know what the nature of this decrease is. First, when applying the integer order system ($\alpha = \beta = 1$) it can be seen from the simulations in Figure 6.10 that the model provides a good result at first, then completes an exponential decrease in contrast to the real results, which increase after a period of time.

We now apply the fractional incommensurate model by taking $\alpha = 0.99$ and $\beta = 0.09$. Note from Figure 6.11 that the results are very compatible and realistic, and in the end the number of active cases is decreasing, although is should be noted that there is an epidemic wave before this decrease as well. In particular, Figures 6.10 and 6.11 show numerical comparisons between real data which represents the number of active infections in Germany for the period 26 April to 30 May 2022 along with the results obtained from the integer-order system and fractional-order systems. Based on these two figures, we can clearly conclude that the fractional-order system is closer than the integer-order system to the real data, enabling us to rely on the construction of the fractional-order system to set a more suitable effective prediction than the integer-order system.

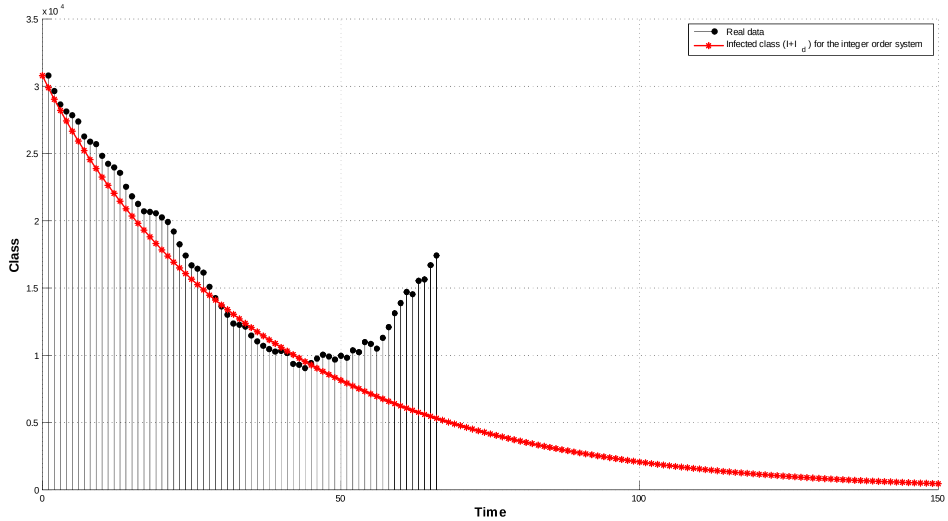

Figure 6.10: Numerical simulation of infected class with integer order and comparison with the real data.

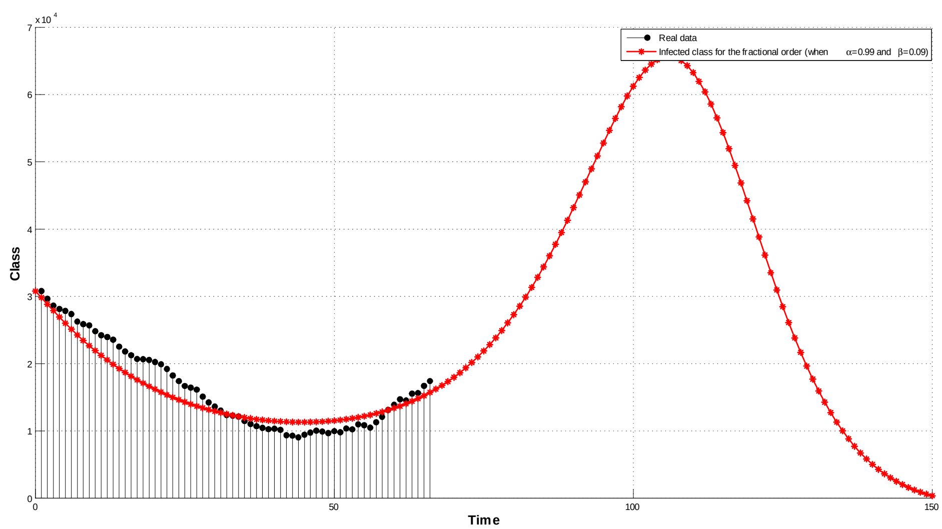

Figure 6.11: Numerical simulation of infected class with fractional-order system and comparison with real data.

In the context of the COVID-19 pandemic, this section introduces a novel discrete fractional-order compartment model that incorporates the number of vaccinated individuals as an additional state variable to describe system dynamics. In this manuscript, we demonstrated that the proposed fractional-order model, governed by fractional difference equations, has an asymptotically stable disease-free fixed point. Specifically, we proved a new theorem that establishes a condition for the pandemic's elimination when certain epidemic parameters meet an inequality. This finding is a key contribution of our approach and could help policymakers better understand the long-term epidemiological behavior of COVID-19. Additionally, we conducted multiple numerical simulations to highlight the effectiveness of discrete fractional calculus in capturing the dynamics of the COVID-19 pandemic. This is one of the major contributions of this work, alongside the application of fractional-order difference operators, which have rarely been used in previous COVID-19 modeling studies.

Conclusion and Perspectives

This thesis has made significant contributions to the field of discrete fractional calculus, exploring its theoretical foundations, stability analysis, and diverse applications. By extending classical discrete-time models with fractional-order operators, the research has addressed key challenges in modeling complex systems that exhibit memory effects, hereditary properties, and intricate dynamics. Through the introduction of fundamental operators and the analysis of their properties, alongside the development of new stability criteria for systems with fixed, incommensurate, and variable fractional orders, this work has filled critical gaps in the existing literature. These advancements enhance the theoretical understanding of discrete fractional calculus and demonstrate its practical relevance in addressing real-world problems.

A notable focus of this thesis was the application of discrete fractional systems to epidemic modeling. The study showcased their ability to capture memory effects and nonlinear dynamics inherent in infectious disease transmission, providing valuable tools for understanding and controlling outbreaks like COVID-19. By leveraging fractional-order dynamics, this thesis has not only improved modeling accuracy but also laid the groundwork for innovations in control theory, signal processing, and biological systems analysis.

The findings presented here open up several promising avenues for future research. One perspective involves the exploration of adaptive discrete fractional systems that can dynamically adjust their fractional orders in response to changing system conditions, enabling more robust and flexible models. Additionally, multi-scale modeling, integrating fractional calculus across different temporal or spatial scales, holds great potential for advancing applications in biology, finance, and engineering. The intersection of discrete fractional calculus with artificial intelligence also presents an exciting frontier, particularly in neural networks and machine learning, where memory effects and complex dynamics could enhance algorithmic performance and decision-making.

In conclusion, this thesis establishes discrete fractional systems as a powerful framework for studying and solving complex problems in diverse scientific and engineering domains. By bridging the gap between theory and practice, it is hoped that these contributions will inspire further research and innovation, promoting the adoption of fractional calculus in understanding and addressing the challenges of modern complex systems.

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