الجمهورية الجزائرية الديمقراطية الشعبية
PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
وزارة التعليم العالي و البحث العلمي
Ministry of Higher Education and Scientific Research
جامعة العربي بن مهيدي أم البواقي
Larbi Ben M'hidi University of Oum El Bouaghi

Faculty of Exact Sciences and Sciences of Nature and Life
Department of Mathematics and Computer Sciences

Order number: ……………
Serial number: ……………

THESIS

Submitted for obtain

Science Doctoral Degree in Mathematics

Option: ***Applied Mathematics***

Entitled

Common Fixed Point Theorems of Several Functions, with Applications to Integral Equations

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June 23, 2020

Dedication

I dedicate this work...

To **my father**, who encouraged me all my life and sacrificed himself for my well-being, and to his deep compassion for my success.

To **my mother**, her affection, her gentleness, her tenderness, and her eternal encouragement, without her nothing would have been possible.

To **my wife** for her attachment, her warm encouragement, especially for her understanding and patience.

To **my children**, **my brothers**, **my sisters**, **my friends** and **my colleagues**

Finally, to all those who -from nearest to the farthest- have collaborated in the realization of this work...

Acknowledgments

At the end of this work, I'm very grateful...

Praise to **Allah** Who gives us the power, the will and the comfort during these years of studying to finish this modest work.

To **Pr. A. Aliouche**, for time and support to accomplish this work.

To **Dr. A. Diar**, for accepting to preside over our work.

To **Pr. A. Zarai**, for accepting to examine this work.

To **Dr. K. Saoudi**, for accepting to examine this work.

To **Dr. K. Akrout**, for accepting to examine this work.

I especially thank my friend **Dr. T. Oussaeif** for his support, his competence in his field, his valuable advice and the useful discussions that we had together.

Abstract

In this work, we study several problems of a common fixed point in different metric spaces.

We start by the recalls of some fundamental preliminary concepts and necessary tools.

The second chapter is devoted to the study of a common fixed point for pairs of weakly compatible mappings by cancelling the continuity condition in metric space, and application on non-linear integral equation.

Then, the third chapter is intended to achieve common fixed point theorems for mappings satisfying generalized contractive condition in a b-metric space. The result applied to prove the existence of a common solution of system of Urysohn non-linear integral equations and unique solution for linear equations system.

Finally, the fourth chapter is devoted to prove some common coupled fixed point theorems for contractive mappings in Menger metric spaces. Also, we prove common fixed point theorems for pairs of weakly compatible mappings. The main result is supported by a suitable example and application to integral equations.

Keywords: b-metric space; Menger metric space; common fixed point; contractive condition, weakly compatible mapping, non-linear integral equations.

ملخص

يهدف هذا العمل إلى دراسة العديد من المسائل المختلفة حول النقطة الثابتة المشتركة في فضاءات مترية مختلفة.

نبدأ بالتذكير ببعض المفاهيم الأساسية، الأولية والأدوات المهمة المستعملة في هذا العمل.

الفصل الثاني يهدف الى دراسة النقطة المشتركة الثابتة من اجل زوجين من التطبيقات الضعيفة توافقيا مع إلغاء خاصية الاستمرار في الفضاء المتري. وتطبيقها على المعادلات التكاملية غير الخطية.

الفصل الثالث يرمي إلى التوصل الى نظريات تخص النقطة المشتركة الثابتة من اجل التطبيقات التي تحقق خاصية الشرط التقلصي المعمم في فضاءات ب- مترية، مع تطبيق النتائج لإثبات وجود حل مشترك لنظام معادلات أوريزون التكاملية غير الخطية والحل الوحيد لنظام المعادلات الخطية.

أخيرًا، يكرس الفصل الرابع لإثبات بعض نظريات النقطة الثابتة المزدوجة المشتركة لتطبيقات في فضاءات منجر المترية. أيضًا ، نثبت نظريات النقطة الثابتة المشتركة لأزواج التطبيقات الضعيفة توافقيا. يتم دعم النتيجة الرئيسية من خلال أمثلة مناسبة وتطبيقها على المعادلات التكاملية.

الكلمات المفتاحية: الفضاء ب- متري، فضاء منجر المتري؛ النقطة الثابتة المشتركة. شرط التقلص ، التطبيقات المتوافقة بضعف، المعادلات التكاملية غير الخطية.

Résumé

Dans ce travail on a étudié divers problèmes de point fixe commun dans différents espaces métriques.

On a débuté dans un premier chapitre par des rappels de certaines notions préliminaires fondamentales, et les outils nécessaires qui nous aident dans la démonstration des résultats obtenus dans ce travail.

Le deuxième chapitre est consacré à l'étude d'un point fixe commun pour deux paires d’applications faiblement compatibles en annulant la condition de continuité dans l'espace métrique, et application sur une équation intégrale non linéaire.

Ensuite, le troisième chapitre est destiné à réaliser des théorèmes de point fixe commun pour des applications satisfaisant la condition de contraction généralisée, dans un espace b-métrique. Le résultat appliqué pour prouver l'existence d'une solution commune de système d'équations intégrales non linéaires d'Urysohn, et d'une solution unique pour le système d'équations linéaires.

Enfin, le quatrième chapitre est consacré à la démonstration de quelques théorèmes de points fixes couplés communs pour des applications dans les espaces métriques de Menger. De plus, nous démontrons des théorèmes de points fixes communs pour des paires d’applications faiblement compatibles. Le résultat principal est soutenu par un exemple approprié et une application aux équations intégrales.

Mots clés : Espace b-métrique, espace métrique de Menger, point fixe commun, condition contractive, application faiblement compatible, équations intégrales non linéaires.

Contents

General introduction

What do you know about this equation $f(x) = x$?

We all know what the line (Δ) " defined by the equation $y = x$ " represent, or the points that remain stationary, by an application or a transformation. All these concepts are oriented towards the fixed point theory which is a very important branch of mathematics, and is one of the most powerful tools of modern mathematics. It is a combination of analysis, topology and geometry. In the last decades, fixed point theory has been revealed as a very powerful and important tool in the study of non-linear phenomena.

L. Brouwer (1881-1966) is the father of this theorie which states that if a continuous function f satisfies certain properties, then there is a fixed point. But the most widely theorem used is **Banach 1922** [7], which is the product of previous works in particularly those of **E. Picard.**

Around **1922, Banach** recognized the fundamental role of metric completeness, a property shared by all spaces commonly used in analysis. For many years, activity in fixed-point theory was limited to **Banach**'s mirror extensions of contraction principle and its multiple applications. Much of the acquired theory attributes to new results of **Browder**'s pioneering work in the mid-sixties, as well as the development of non-linear functional analysis as an active and vital branch of mathematics, central to this evolution were **Browder's** theorems.

The quality as well as the amount of research in fixed point theory in metric space increased greatly in the **1970**'s. Descriptions of important developments in this period prove the existence of fixed point theorems using more generalized contractive applications than previous contractive applications. More generalized contractive applications were designed by **Bianchini, Caristi**. In the **1980's, Sessa** and **Jungck** introduced the notion of low commutativity and compatible applications. Subsequently, a set of common fixed point theorems was introduced by **Sessa, Jungck** [49]**.**

The idea of b-metric was initiated from the works of **Bourbaki** and **Bakhtin**. **Czerwik** [16] gave an axiom which was weaker than the triangular inequality and formally defined a b-metric space with a view of generalizing the **Banach** contraction mapping theorem. A b- metric space was also called a metric type spaces in [17].

In **1942, Menger** [35] generalized the metric by associating a distribution function with each pair of points (x, y) of a non-empty set. This idea played a major role in the probabilistic development of metric space, and led to the birth of Menger spaces, and consequently probabilistic metric spaces as a generalization of metric space and which are of interest in several fields such as physics. It is also very important in probabilistic functional analysis, in non-linear analysis and their applications. Such spaces have been widely developed by **Schweizer**, **Sklar**, **Serstnev**, **Tardiff**, **Thorp** and later by many authors in various directions.

In this work, we present the theorem of fixed point in three different spaces, metric space, b-metric space and **Menger** metric space. By these selected contractive properties, we applied to solve some integral non-linear equations.

The thesis is organized as follows:

In **Chapter 1**, contains the essential elements that will be needed for the following chapters.

Chapter 2 is devoted to develop and generalize the work of **Pathak** and al. [42]; where we prove a common fixed point theorems for a pair of weakly compatible mappings by cancelling the continuity condition. Also, to illustrate and applied our results, an application on non-linear integral equation is given.

Chapter 3 is intended to establish some common fixed point results having rational type contraction conditions for mappings satisfying generalized contractive condition in a b-metric space. The result its applied to prove the existence of a common solution of system of **Urysohn** non-linear integral equations and unique solution for linear equations system.

Finally, in the **last chapter,** we present my work in [3] , where we prove some common coupled fixed point theorems for contractive mappings in **Menger** metric spaces under geometrically convergent t-norms. Also, we prove common fixed point theorems for pairs of weakly compatible mappings, which generalize the results of **Jian-Zhong Xiao** and all (**2011**). The main results are supported by a suitable example and application to integral equations.

Chapter 1

Preliminaries

In this chapter, we will recall the main notions and the concepts we will need, as well as the definitions concerning this work.

The plan of this chapter is as follows:

1. **Compatible applications**
2. **A b-metric space**
3. **Menger metric spaces**
4. **Coupled fixed point**

1.1 Compatible applications

Throughout this section, suppose (X, d) denotes a metric space.

Definition 1.1 ([49]) *Let* $A, S : X \longrightarrow X$ *be mappings. Then the pair* (A, S) *is said to be commutative if*

$$SAx = ASx,$$

for some $x \in X$.

The pair (A, S) *is said to be weakly commutative if*

$$d\left(ASx, SAx\right) \leq d\left(Ax, Sx\right),$$

for some $x \in X$.

Definition 1.2 ([28]) *Let* $A, S : X \longrightarrow X$ *be mappings. Then the pair* (A, S) *is said to be compatible if*

$$\lim_{n\to\infty} d\left(ASx_n, SAx_n\right) = 0,$$

whenever $\{x_n\}$ *is a sequence in* X *such that for some* $t \in X$:

$$\lim_{n\to\infty} Sx_n = \lim_{n} Ax_n = t.$$

It is easy to show that weak commutativity implies compatibility, but the converse is not true in general as it has been proved in [28].

Definition 1.3 ([40]) *Let* $A, S : X \longrightarrow X$ *be mappings. Then the pair* (A, S) *is said to be compatible of type* (I) *if* $d\left(t, St\right) \leq \lim_{n\to\infty} \sup d\left(t, ASx_n\right),$ *whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t$, *for some* $t \in X$.

Definition 1.4 ([40]) *Let* $A, S : X \longrightarrow X$ *be mappings. Then the pair* (A, S) *is said to be compatible of type* (II) *if and only if* (S, A) *is compatible of type* (I).

Definition 1.5 ([41]) *Let* $A, S : X \longrightarrow X$ *be mappings. Then the pair* (A, S) *is said to be weakly compatible if they commute at theire coincidence points, that is* $ASx = SAx$, *whenever* $Ax = Sx$ *for* $x \in X$.

It is well known that a compatible pair of maps is weakly compatible pair but converse need not be true [27]. However it is interesting to note that the concepts of weakly compatible maps and compatible maps of type (I) are independent from each other, To show this, we illustrate the following examples:

Example 1.1 *Let* $X = [0, \infty)$ *and* $A, S : X \longrightarrow X$ *defined by*

$$Ax = \begin{cases} \cos x & when\ x \neq 1 \\ 0 & when\ x = 1 \end{cases} \quad and \quad Sx = \begin{cases} e^x & when\ x \neq 1 \\ 0 & when\ x = 1 \end{cases}$$

Then, is clear that $Ax = Sx$ *iff* $x = 0$ *and* $x = 1$. *Also at these points* $ASx = SAx$. *It means the mappings* (A, S) *are weakly compatible.*

Now, we suppose that $\{x_n\}$ *is a sequence in* X *such that* $Ax_n, Sx_n \longrightarrow t \in X$. *Here* $t = 1$ *by definition of* A *and* S.

Now $d(t, St) = t$ *and* $\lim d(t, ASx_n) = (1 - \cos 1) < 1$. *Therefore, the pair* (A, S) *is not compatible of type* (I).

Example 1.2 *Let* $X = [0, \infty)$ *and* $A, S : X \longrightarrow X$ *defined by* $Ax = 2x + 1$ *and* $Sx = x^2 + 1$. *Then at* $x = 0$, $Ax = Sx$. *Also at* $x = 0, ASx = 3$ *and* $SAx = 2$, *which shows that the pair* (A, S) *is not weakly compatible. Now suppose that* $\{x_n\}$ *be a sequence in* X *such that* $\lim Ax_n = \lim Sx_n = t \in X$. *by definition of* A *and* S, $t = 1$. *For this value, we have* $d(t, St) = 1$ *and* $\lim d(t, ASx_n) = 2$, *which shows that the pair* (A, S) *is compatible mappings of type* (I).

1.2 A b-metric space

Definition 1.6 ([16]) *Let* X *be a (non-empty) set and* $s \geq 1$ *be a given real number.*

A function $d : X \times X \rightarrow R^+$ *is a b-metric if, for all* $x, y, z \in X$, *the following conditions are satisfied:*

1/ $d(x,y)=0$ *iff* $x=y$,

2/ $d(x,y)=d(y,x)$,

3/ $d(x,z)\leq s[d(x,y)+d(y,z)]$.

The pair (X,d) *is called a b-metric space.*

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric if (and only if) $s=1$.

We present an easy example to show that in general a b-metric need not be a metric.

Example 1.3 *Let* (X,ρ) *be a metric space, and* $d(x,y)=(\rho(x,y))^p$, *where* $p\geq 1$ *is a real number. Then* d *is a b-metric with* $s=2^{p-1}$.

However, (X,d)*is not necessarily a metric space. For example, if* $X=\mathbb{R}$ *is the set of real numbers and* $\rho(x,y)=|x-y|$ *is the usual Euclidean metric, then* $d(x,y)=(x-y)^2$ *is a b-metric on* $\mathbb{R}$ *with* $s=2$*, but it is not a metric on* $\mathbb{R}$.

Definition 1.7 ([10]) *Let* (X,d) *be a b-metric space. Then a sequence* $\{x_n\}$ *in* X *is called:*

(a) *convergent if and only if there exists* $x\in X$ *such that* $d(x_n,x)\to 0$, *as* $n\to+\infty$.

In this case, we write $\lim_{n\to\infty}x_n=x$.

(b) *Cauchy if and only if* $d(x_n,x_m)\to 0$, *as* $n,m\to+\infty$.

Proposition 1.1 *(See, remark 2.1 in* [10] *) In a b-metric space* (X,d) *the following assertions hold:*

(i) a convergent sequence has a unique limit,

(ii) each convergent sequence is ***Cauchy****,*

(iii) in general, a b-metric is not continuous.

Definition 1.8 ([10]) *The b-metric space* (X,d) *is complete if every* ***Cauchy*** *sequence in* X *converges.*

1.3 Menger metric spaces

Definition 1.9 *A probabilistic metric space is a generalization of metric spaces where the distance has no longer values in non-negative real numbers, but in distribution functions.*

Suppose that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{Z}^+$ be the set of positive integers.

A function $F : \overline{\mathbb{R}} \to [0, 1]$: is called a distribution function if it is non-decreasing and left continuous with $F(-\infty) = F(+\infty) = 1$.

The class of all distribution functions is denoted by D_∞.

Suppose that $D = \{F \in D_{\infty.} : \inf F(t) = 0, \sup F(t) = 1\}$, $D_\infty^+ = \{F \in D_\infty : F(0) = 0\}$ and $D^+ = D \cap D_\infty^+$ (cf $[10], [17]$).

A special element of D^+ is the heaviside function H defined by:

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

Definition 1.10 ([45]) *A function $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied for any $a, b, c, d \in [0, 1]$;*

(Δ-1) $\Delta(a, 1) = a$

(Δ-2) $\Delta(a, b) = \Delta(b, a)$;

(Δ-3) $\Delta(a, b) \geq \Delta(c, d)$, *for* $a \geq c, b \geq d$;

(Δ-4) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

Two exemples of t-norm are $\Delta_M(a, b) = \min\{a, b\}$ *and* $\Delta_P(a, b) = ab$.

It is evident that, as regards the point wise ordering, $\Delta \leq \Delta_M$ *for each t-norm* Δ.

Definition 1.11 $([45], [34])$ *A triplet (X, F, Δ) is called a **Menger** probabilistic metric space (for short, a **Menger** PM-space), if X is a non-empty set, Δ is t-norm and F is a mapping from $X \times X$ into D^+ satisfying the following condition ($F(x, y)$ for $x, y \in X$ is denoted by $F_{x,y}$):*

(MS-1) $F_{x,y}(t) = H(t)$ *for all* $t \in \mathbb{R}$ *if and only if* $x = y$;

(MS-2) $F_{x,y}(t) = F_{y,x}(t)$ *for all* $x, y \in X$ *and* $t \in \mathbb{R}$

(MS-3) $F_{x,y}(t + s) \geq \triangle(F_{x,z}(t), F_{z,y}(s))$ *for all* $x, y, z \in X$ *and* $t, s \in \mathbb{R}^+$.

A generalized **Menger** probabilistic metric space is a **Menger** space with $F(X \times X) \in D_\infty^+$.

Schweizer and al. $[45], [46]$ point out that if the t-norm $\triangle$ of a **Menger** PM-space satisfies the condition $\sup_{0<a<1} \Delta(a, a) = 1$, then (X, F, Δ) is a first countable Hausdroff topological space in the (ε, λ) $-$topology τ, i.e, the family of sets

$$\left\{U_x(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in [0,1], \ \ (x \in X) \ \right\}$$

is the base of neighborhoods of point x for τ, where $U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}$.

By virtue of this topology τ a sequence $\{x_n\}$ in (X, F, Δ) is said to be convergent to x (we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$; $\{x_n\}$ is called a **Cauchy** sequences in (X, F, Δ) if for any given $\varepsilon > 0$ and $\lambda \in [0,1]$, there exists $N = N(\varepsilon, \lambda) \in \mathbb{Z}^+$ such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$, whenever $n, m \geq N$, (X, F, Δ) is said to be complete if each **Cauchy** sequence in X is convergent to some point in X.

In the sequel, we will always assume that (X, F, Δ) is a **Menger** space with the (ε, λ) $-$topology.

Lemma 1.1 [3] *Let (X, d) be a usual metric space. Define a mapping $F : X \times X \to D^+$ by*

$$F_{x,y}(t) = H(t - d(x,y)), \quad \textit{for} \ \ x, y \in X \ \textit{and} \ t > 0.$$

Then (X, F, Δ_M) is a ***Menger*** *PM-space; it is called the induced* ***Menger*** *PM-space by (X, d) and it is complete if (X, d) is complete.*

An arbitrary t-norme can be extended (by(Δ-3)) in a unique way to an n-ary operation. For $(a_1, a_2, ..., a_n) \in [0,1]^n \, (n \in \mathbb{Z}^+)$, the valued $\Delta^n(a_1, a_2, ..., a_n)$ is defined by $\Delta^1(a_1) = a_1$ and $\Delta^n(a_1, a_2, ..., a_n) = \Delta\left(\Delta^{n-1}(a_1, a_2, ..., a_{n-1}), a_n\right)$.

For each $a \in [0,1]$, the sequence $\{\Delta^n(a)\}_{n=1}^{\infty}$ is defined by $\Delta^1(a) = a$ and $\Delta^n(a) = \Delta\left(\Delta^{n-1}(a), a\right)$.

Definition 1.12 [3] *A t-norm Δ is said to be of H-type if the sequence of functions $\{\Delta^n(a)\}_{n=1}^{\infty}$ is equi-continuous at $a = 1$.*

The t-norm Δ_M is a trivial example of a t-norm of H-type, but there are t-norms Δ of H-type with $\Delta \neq \Delta_M$. It is easy to see that if Δ is of H-type, then Δ satisfies $\sup_{0<a<1} \Delta(a, a) = 1$.

Lemma 1.2 [3] *Let (X, F, Δ) be a* ***Menger*** *PM-space.For each $\lambda \in (0,1]$, define a function $d_\lambda : X \times X \to \mathbb{R}^+$ by*

$$d_\lambda(x, y) = \inf\{t > 0 : F_{x,y}(t) > 1 - \lambda\}.$$

Then the following statements hold:

(1) $d_\lambda(x,y) < t$ *if and only if* $F_{x,y}(t) > 1-\lambda$;

(2) $d_\lambda(x,y) = d_\lambda(y,x)$ *for all* $x, y \in X$ *and* $\lambda \in (0,1]$;

(3) $d_\lambda(x,y) = 0$ *for all* $\lambda \in (0,1]$ *if and only if* $x = y$.

Lemma 1.3 [3] *Let* (X, F, Δ) *be a* ***Menger*** *PM-space, and let* $\{d_\lambda\}_{\lambda\in(0,1]}$ *be a family of pseudo-metrics on* X *defined in lemma 1.2.*

If Δ *is a t-norm of H-type, then for each* $\lambda \in (0,1]$, *there exists* $\mu \in (0,\lambda]$ *such that for each* $m \in \mathbb{Z}^+$,

$$d_\lambda(x_0, x_m) \le \textstyle\sum_{i=0}^{m-1} d_\mu(x_i, x_{i+1}), \quad \textit{for all } x_0, x_1, ..., x_m \in X.$$

Lemma 1.4 [3] *Suppose that* $F \in D^+$. *For each* $n \in \mathbb{Z}^+$, *let* $F_n : \mathbb{R} \to [0,1]$ *be non-decreasing, and* $g_n : (0,+\infty) \to (0,+\infty)$ *satisfy* $lim_{n\to+\infty} g_n(t) = 0$ *for any* $t > 0$.*If*

$$F_n(g_n(t)) \ge F(t), \forall t > 0$$

then $lim_{n\to+\infty} F_n(t) = 1$, *for any* $t > 0$.

1.4 Coupled fixed point

Definition 1.13 ([18]) *An element* $x \in X$ *is called a common fixed point of the mappings* $f : X \times X \to X$ *and* $g : X \to X$ *if*

$$x = f(x,x) = g(x)$$

Definition 1.14 ([33]) *An element* $(x,y) \in X \times X$ *is called*

(i) a coupled fixed point of the mapping $f : X \times X \to X$ *if*

$$f(x,y) = x; \quad f(y,x) = y.$$

(ii) a coupled coincidence point of the mappings $f : X \times X \to X$ *and* $g : X \to X$ *if*

$$f(x, y) = g(x)\,; \quad f(y, x) = g(y)$$

(iii) a common coupled fixed point of the mappings $f : X \times X \to X$ *and* $g : X \to X$ *if*

$$x = f(x, y) = g(x)\,; \quad y = f(y, x) = g(y)$$

In [1], ***Aamri*** *and al. Introduced the concept of weakly compatible mappings, Here we give a similar concept in* ***Menger*** *metric spaces as follows:*

Definition 1.15 *Let* (X, F, Δ) *be a* ***Menger*** *metric space and let* $f : X \times X \to X$ *and* $g : X \to X$ *be two mappings.* f *and* g *are said to be weakly compatible (or w-compatible) if they commute at their coupled coincidence points,i.e;if* (x, y) *is a coupled coincidence point of* f *and* g, *then*

$$g(f(x, y)) = f(g(x), g(y)), \quad g(f(y, x)) = f(g(y), g(x))$$

Definition 1.16 ([25]) *Let* $A : X \times X \to X,\ B : X \times X \to X,\ T : X \to X,\ S : X \to X$ *be four mappings. Then, the pairs* (B, S) *and* (A, T) *are said to have common coupled coincidence point if there exists* a, b *in* X *such that*

$$B(a, b) = S(a) = T(a) = A(a, b) \text{ and } B(b, a) = S(b) = T(b) = A(b, a)\,.$$

Chapter 2

Common Fixed Point Theorem in Metric Spaces for Weakly Compatible Mappings without Continuity, with Application to Non-linear Integral Equations

In this chapter, will develop and generalize the work of **Pathak** and al. [42]; where we prove a common fixed point theorems for a pair of weakly compatible mappings by cancelling the continuity condition. Also, to illustrate and applied our results, an application on non-linear integral equation is given.

The plan of this chapter is as follows:

1. **Introduction**
2. **Main results**
3. **Application to integral equations**

2.1 Introduction

Many common fixed point theorems for mappings, satisfying certain contractive conditions can be found in the recent literature and has been an interesting field of mathematics from **1922**, when **Banach** stated and proved his famous result (**Banach** contraction principle). The field of fixed point theory - that is involving four single valued maps -, began with the assumption that all of the maps are commuted. In **1998**, the concept of weakly compatible pairs of mappings has been introduced by **Jungck** [29]. Several common fixed point theorems have been proved on the basis of compatible mappings introduced by **Jungck** [28]. **Pathak** and al. [40] introduced notions of compatible mappings of type (I) and type (II) and utilized them in complete metric space for establishing a common fixed point theorems of a quadruple of self mappings.

In recent years, several authors have obtained common fixed point results for different classes of mappings on various metric spaces such as complete metric spaces. Applications of this theorem to **Voltera-Hammerstein** non-linear integral equations with infinite delay have also been given in [40]. Also, **Chugh** and **Kumar** in [13] , the authors established some results about common fixed point theorems for weakly compatible mappings. Motivated by these, we prove the same previous results without continuity of mappings.

2.2 Main results

Throughout this section, suppose ϕ denotes the collection of mappings $\varphi : [0,\infty) \longrightarrow [0,\infty)$ which are upper semi-continuous, non decreasing in each co-ordinate variables and $\varphi(t) < t$ for all $t > 0$.

To prove our main theorem, we need the following lemmas :

Lemma 2.1 ([41]) *. If $\varphi_i \in \phi$ and $i \in I$, where $I \subset \mathbb{N}$, then there exists some $\varphi \in \phi$ such that :*

$\max\{\varphi_i(t) : i \in I\} \leq \varphi(t) \quad$ *for all* $t > 0\}$

Let A, B, S and T be self mappings of a metric space (X, d) such that :

$$A(X) \subset T(X) \ \ \mathit{and} \ \ B(X) \subset S(X) \tag{2.1}$$

$$\begin{aligned} & d^{2p}(Ax, By) \\ \leq \; & a\,\varphi_0\left(d^{2p}(Sx, Ty)\right) + (1-a)\max\left\{\varphi_1\left(d^{2p}(Sx, Ty)\right),\right. \\ & \varphi_2\left(d^{q}(Sx, Ax)\,d^{q'}(Ty, By)\right), \varphi_3\left(d^{r}(Sx, By)\,d^{r'}(Ty, Ax)\right), \\ & \left.\varphi_4\left(\frac{1}{2}\left[d^{s}(Sx, Ax)\,d^{s'}(Ty, Ax)\right]\right), \varphi_5\left(\frac{1}{2}\left[d^{l}(Sx, By)\,d^{l'}(Ty, By)\right]\right)\right\} \end{aligned} \tag{2.2}$$

for all $x, y \in X$,where $\varphi_i \in \phi, i = 0, 1, 2, 3, 4, 5,\ 0 \leq a \leq 1, 0 < p, q, q', r, r', s, s', l, l' \leq 1$,

such that $2p = q + q' = r + r' = s + s' = l + l'$.Then for arbitrary x_0 in X . by (2.1) we choose a point $x_1 \in X$ such that $Tx_1 = Ax_0$ and for this point x_1, there exists a point x_2 in X such that $Sx_2 = Bx_1$ and so on. Continuing in this manner, we can define a sequence $\{y_n\}$ in X, for $n = 1, 2, 3, ...$such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n} \ \text{ and } \ y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \ , n \in \mathbb{N} \tag{2.3}$$

(see, for instance, [13])

Lemma 2.2 ([12]) . *Let* $\varphi \in \phi$ *and* $\{\xi_n\}$ *be a sequence of non-negative real numbers. if* $\xi_{n+1} \leq \varphi\{\xi_n\}$ *for* $n \in \mathbb{N}$, *then* $\{\xi_n\}$ *converges to* 0.

Now we will prove the following lemmas which help us to establish our results:

Lemma 2.3 *If we denote* $d_n = d(y_n, y_{n+1})$, *where* y_n *the sequence define in* (2.3) , *then*

$$\lim_{n \longrightarrow \infty} d_n = 0.$$

Proof The inequality (2.2) implies

$$\begin{aligned}
& d^{2p}\left(y_{2n}, y_{2n+1}\right) \\
\leq\; & a\ \varphi_0\left(d^{2p}\left(y_{2n-1}, y_{2n}\right)\right) + (1-a)\max\left\{\varphi_1\left(d^{2p}\left(y_{2n-1}, y_{2n}\right)\right),\right. \\
& \varphi_2\left(d^{q}\left(y_{2n-1}, y_{2n}\right) d^{q'}\left(y_{2n}, y_{2n+1}\right)\right), \varphi_3\left(d^{r}\left(y_{2n-1}, y_{2n+1}\right) d^{r'}\left(y_{2n}, y_{2n}\right)\right), \\
& \left.\varphi_4\left(\frac{1}{2}\left[d^{s}\left(y_{2n-1}, y_{2n}\right) d^{s'}\left(y_{2n}, y_{2n}\right)\right]\right), \varphi_5\left(\frac{1}{2}\left[d^{l}\left(y_{2n-1}, y_{2n+1}\right) d^{l'}\left(y_{2n}, y_{2n+1}\right)\right]\right)\right\}
\end{aligned}$$

or

$$\begin{aligned}
& d_{2n}^{2p} \\
\leq\; & a\ \varphi_0\left(d_{2n-1}^{2p}\right) + (1-a)\max\left\{\varphi_1\left(d_{2n-1}^{2p}\right), \varphi_2\left(d_{2n-1}^{q} d_{2n}^{q'}\right),\right. \\
& \left.\varphi_3(0),\ \varphi_4(0), \varphi_5\left(\frac{1}{2}\left[\left(d_{2n-1}^{l} + d_{2n}^{l}\right) d_{2n}^{l'}\right]\right)\right\} \\
\leq\; & a\ \varphi_0\left(d_{2n-1}^{2p}\right) + (1-a)\max\left\{\varphi_1\left(d_{2n-1}^{2p}\right), \varphi_2\left(d_{2n-1}^{q} d_{2n}^{q'}\right),\right. \\
& \left.\varphi_3(0),\ \varphi_4(0), \varphi_5\left(\frac{1}{2}\left[d_{2n-1}^{l} d_{2n}^{l'} + d_{2n}^{l} d_{2n}^{l'}\right]\right)\right\}
\end{aligned}$$

if $d_{2n} \geq d_{2n-1}$, then we have

$$\begin{aligned}
& d_{2n}^{2p} \\
\leq\; & a\ \varphi_0\left(d_{2n}^{2p}\right) \\
& + (1-a)\max\left\{\varphi_1\left(d_{2n}^{2p}\right), \varphi_2\left(d_{2n}^{q+q'}\right), \varphi_3(0),\ \varphi_4(0), \varphi_5\left(\frac{1}{2}\left[d_{2n}^{l+l'} + d_{2n}^{l+l'}\right]\right)\right\}.
\end{aligned}$$

or

$$\begin{aligned}
& d_{2n}^{2p} \\
\leq\; & a\ \varphi_0\left(d_{2n}^{2p}\right) + (1-a)\max\left\{\varphi_1\left(d_{2n}^{2p}\right), \varphi_2\left(d_{2n}^{2p}\right), \varphi_3(0),\ \varphi_4(0), \varphi_5\left(d_{2n}^{2p}\right)\right\} \\
\leq\; & \varphi\left(d_{2n}^{2p}\right) \\
<\; & d_{2n}^{2p}\ ,
\end{aligned}$$

a contradiction. Thus, we must have $d_{2n} < d_{2n-1}$.

Then using this inequality the condition (2.2) yields

$$d_{2n} \leq \varphi\left(d_{2n-1}\right) \tag{2.4}$$

Similary taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in (2.2) , we get

$$\begin{aligned}
& d^{2p}\left(y_{2n+1}, y_{2n+2}\right) \\
\leq\ & a\ \varphi_0\left(d^{2p}\left(y_{2n}, y_{2n+1}\right)\right) + (1-a)\max\left\{\varphi_1\left(d^{2p}\left(y_{2n}, y_{2n+1}\right)\right),\right. \\
& \varphi_2\left(d^{q}\left(y_{2n+1}, y_{2n+2}\right) d^{q'}\left(y_{2n}, y_{2n+1}\right)\right), \varphi_3\left(d^{r}\left(y_{2n+1}, y_{2n+1}\right) d^{r'}\left(y_{2n}, y_{2n+1}\right)\right), \\
& \left.\varphi_4\left(\frac{1}{2}\left[d^{s}\left(y_{2n+1}, y_{2n+2}\right) d^{s'}\left(y_{2n}, y_{2n+2}\right)\right]\right), \varphi_5\left(\frac{1}{2}\left[d^{l}\left(y_{2n+1}, y_{2n+1}\right) d^{l'}\left(y_{2n}, y_{2n+1}\right)\right]\right)\right\}
\end{aligned}$$

or

$$\begin{aligned}
& d_{2n+1}^{2p} \\
\leq\ & a\ \varphi_0\left(d_{2n}^{2p}\right) + (1-a)\max\left\{\varphi_1\left(d_{2n}^{2p}\right), \varphi_2\left(d_{2n+1}^{q} d_{2n}^{q'}\right), \varphi_3\left(0\right),\right. \\
& \left.\varphi_4\left(\frac{1}{2}\left[d_{2n+1}^{s}\left(d_{2n}^{s'} + d_{2n+1}^{s'}\right)\right]\right), \varphi_5\left(0\right)\right\} \\
\leq\ & a\ \varphi_0\left(d_{2n}^{2p}\right) + (1-a)\max\left\{\varphi_1\left(d_{2n}^{2p}\right), \varphi_2\left(d_{2n+1}^{q} d_{2n}^{q'}\right), \varphi_3\left(0\right),\right. \\
& \left.\varphi_4\left(\frac{1}{2}\left[d_{2n+1}^{s} d_{2n}^{s'} + d_{2n+1}^{s} d_{2n+1}^{s'}\right]\right), \varphi_5\left(0\right)\right\}
\end{aligned}$$

if $d_{2n+1} \geq d_{2n}$, then we have

$$\begin{aligned}
& d_{2n+1}^{2p} \\
\leq\ & a\ \varphi_0\left(d_{2n+1}^{2p}\right) \\
& + (1-a)\max\left\{\varphi_1\left(d_{2n+1}^{2p}\right), \varphi_2\left(d_{2n+1}^{q+q'}\right), \varphi_3\left(0\right),\ \varphi_4\left(\frac{1}{2}\left[d_{2n+1}^{s+s'} + d_{2n+1}^{s+s'}\right]\right), \varphi_5\left(0\right)\right\}.
\end{aligned}$$

which implies

$$\begin{aligned} & d_{2n+1}^{2p} \\ \leq\ & a\ \varphi_0\left(d_{2n+1}^{2p}\right) \\ & +(1-a)\max\left\{\varphi_1\left(d_{2n+1}^{2p}\right),\varphi_2\left(d_{2n+1}^{2p}\right),\varphi_3\left(0\right),\ \varphi_4\left(d_{2n+1}^{2p}\right),\varphi_5\left(0\right)\right\} \\ \leq\ & \varphi\left(d_{2n+1}^{2p}\right) \\ <\ & d_{2n+1}^{2p}, \end{aligned}$$

contradiction, Thus, we must have $d_{2n+1} \leq d_{2n}$.

Again from (2.2), we obtain

$$d_{2n+1} \leq \varphi\left(d_{2n}\right) \tag{2.5}$$

From eqs. (2.4) et (2.5), we obtain

$$d_{n+1} \leq \varphi\left(d_n\right), \quad \text{for } n=0,1,2,3,...$$

And so, by Lemma 2.2, we get $\lim\limits_{n\longrightarrow\infty} d_n = 0$. ■

Lemma 2.4 *The sequence* $\{y_n\}$ *defined by* (2.3)

$$y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \textit{and} \quad y_{2n+1} = Sx_{2n+2} \ = Bx_{2n+1} \quad, n \in \mathbb{N} \tag{2.6}$$

*is a **Cauchy** sequence.*

Proof Suppose that the subsequence $\{y_{2n}\}_{n\in\mathbb{N}}$ is not a **Cauchy** sequence. Then there exists an $\varepsilon > 0$ such that fo each even integer $2k$, there exist even integers $2m\left(k\right)$ and $2n\left(k\right)$ with $2m\left(k\right) > 2n\left(k\right) > k$ such that

$$d^{2p}(y_{2n(k)}, y_{2m(k)}) > \varepsilon. \tag{2.7}$$

For each even integer $2k$, Let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (2.7), that is:

$$d^{2p}(y_{2m(k)-2}, y_{2n(k)}) \leq \varepsilon \ \text{ and } \ d^{2p}(y_{2n(k)}, y_{2m(k)}) > \varepsilon. \tag{2.8}$$

Then for each even integer $2k$, we have:

$$\begin{aligned} \varepsilon &< d^{2p}(y_{2n(k)}, y_{2m(k)}) \\ &\leq d^{2p}(y_{2n(k)}, y_{2m(k)-2}) + d^{2p}(y_{2m(k)-2}, y_{2m(k)-1}) + d^{2p}(y_{2m(k)-1}, y_{2m(k)}) \\ &\leq \varepsilon + d^{2p}_{2m(k)-2} + d^{2p}_{2m(k)-1}. \end{aligned}$$

Hence from (2.8) and lemma 2.3 it follows that

$$\lim_{n \longrightarrow \infty} d\left(y_{2n(k)}, y_{2m(k)}\right) = \varepsilon \tag{2.9}$$

By triangular inequality, we have:

$$\left|d^{2p}\left(y_{2n(k)}, y_{2m(k)-1}\right) - d^{2p}\left(y_{2n(k)}, y_{2m(k)}\right)\right| \leq d^{2p}\left(y_{2m(k)-1}, y_{2m(k)}\right)$$

and

$$\begin{aligned} &\left|d^{2p}\left(y_{2n(k)+1}, y_{2m(k)-1}\right) - d^{2p}\left(y_{2n(k)}, y_{2m(k)}\right)\right| \\ \leq\ & d^{2p}\left(y_{2m(k)}, y_{2m(k)-1}\right) + d^{2p}\left(y_{2n(k)}, y_{2n(k)+1}\right) \end{aligned}$$

$$\left|d^{2p}\left(y_{2n(k)+1}, y_{2m(k)-1}\right) - d^{2p}\left(y_{2n(k)}, y_{2m(k)}\right)\right| \leq d^{2p}_{2m(k)-1} + d^{2p}_{2n(k)}.$$

By (2.9) and lemma 2.2, we obtain:

$$\begin{aligned} \lim_{n \longrightarrow \infty} d\left(y_{2n(k)}, y_{2m(k)-1}\right) &= \varepsilon \\ \lim_{n \longrightarrow \infty} d\left(y_{2m(k)+1}, y_{2m(k)-1}\right) &= \varepsilon. \end{aligned} \tag{2.10}$$

Now using (2.2) , we get

$$\begin{aligned}
& d^{2p}\left(Ax_{2n(k)}, Bx_{2m(k)-1}\right) \\
\leq\ & a\ \varphi_0\left(d^{2p}\left(Sx_{2n(k)}, Tx_{2m(k)-1}\right)\right) + (1-a)\max\left\{\varphi_1\left(d^{2p}\left(Sx_{2n(k)}, Tx_{2m(k)-1}\right)\right),\right. \\
& \varphi_2\left(d^{q}\left(Sx_{2n(k)}, Ax_{2n(k)}\right) d^{q'}\left(Tx_{2m(k)-1}, Bx_{2m(k)-1}\right)\right), \\
& \varphi_3\left(d^{r}\left(Sx_{2n(k)}, Bx_{2m(k)-1}\right) d^{r'}\left(Tx_{2m(k)-1}, Ax_{2n(k)}\right)\right), \\
& \varphi_4\left(\frac{1}{2}\left[d^{s}\left(Sx_{2n(k)}, Ax_{2n(k)}\right) d^{s'}\left(Tx_{2m(k)-1}, Ax_{2n(k)}\right)\right]\right), \\
& \left.\varphi_5\left(\frac{1}{2}\left[d^{l}\left(Sx_{2n(k)}, Bx_{2m(k)-1}\right) d^{l'}\left(Tx_{2m(k)-1}, Bx_{2m(k)-1}\right)\right]\right)\right\}
\end{aligned}$$

or

$$\begin{aligned}
& d^{2p}\left(y_{2n(k)}, y_{2m(k)-1}\right) \\
\leq\ & a\ \varphi_0\left(d^{2p}\left(y_{2n(k)-1}, y_{2n(k)}\right)\right) + (1-a)\max\left\{\varphi_1\left(d^{2p}\left(y_{2n(k)-1}, y_{2n(k)}\right)\right),\right. \\
& \varphi_2\left(d^{q}\left(y_{2n(k)-1}, y_{2n(k)}\right) d^{q'}\left(y_{2m(k)-2}, y_{2m(k)-1}\right)\right), \\
& \varphi_3\left(d^{r}\left(y_{2n(k)-1}, y_{2m(k)-1}\right) d^{r'}\left(y_{2m(k)-2}, y_{2n(k)}\right)\right), \\
& \varphi_4\left(\frac{1}{2}\left[d^{s}\left(y_{2n(k)-1}, y_{2n(k)}\right) d^{s'}\left(y_{2m(k)-2}, y_{2n(k)}\right)\right]\right), \\
& \left.\varphi_5\left(\frac{1}{2}\left[d^{l}\left(y_{2n(k)-1}, y_{2m(k)-1}\right) d^{l'}\left(y_{2m(k)-2}, y_{2m(k)-1}\right)\right]\right)\right\} \\
\leq\ & a\ \varphi_0\left(d^{2p}_{2n(k)-1}\right) + (1-a)\max\left\{\varphi_1\left(d^{2p}_{2n(k)-1}\right),\ \varphi_2\left(d^{q}_{2n(k)-1} d^{q'}_{2m(k)-2}\right),\right. \\
& \varphi_3\left(d^{r}\left(y_{2n(k)-1}, y_{2m(k)-1}\right) d^{r'}\left(y_{2m(k)-2}, y_{2n(k)}\right)\right), \\
& \varphi_4\left(\frac{1}{2}\left[d^{s}_{2n(k)-1} d^{s'}\left(y_{2m(k)-2}, y_{2n(k)}\right)\right]\right), \\
& \left.\varphi_5\left(\frac{1}{2}\left[d^{l}\left(y_{2n(k)-1}, y_{2m(k)-1}\right) d^{l'}_{2m(k)-2}\right]\right)\right\}
\end{aligned}$$

Letting $k \longrightarrow \infty$ and using Lemma 3.3, (2.9) and (2.10), we obtain

$$\varepsilon^{2p} \leq a\ \varphi_0(0) + (1-a)\max\left\{\varphi_1(0), \varphi_2(0), \varphi_3\left(\varepsilon^{r+r'}\right), \varphi_4(0), \varphi_5(0)\right\}$$

or

$$\begin{aligned} \varepsilon^{2p} &\leq (1-a)\,\varphi_3\left(\varepsilon^{r+r'}\right) \\ &\leq (1-a)\,\varphi\left(\varepsilon^{2p}\right) \\ &< \varepsilon^{2p}, \end{aligned}$$

which is a contradiction. Hence $\{y_{2n}\}_{n\in\mathbb{N}}$ is a **Cauchy** sequence. ■

Theorem 2.1 *Let A, B, S and T self-maps of a metric space (X, d) satisfying* (2.1) *and* (2.2) *and suppose any one of the subset $S(X)$ or $T(X)$ is complete. If the pairs (A, S) and (B, T) are weakly compatible, then, A, B, S and T have a unique common fixed point in X .*

Proof From Lemma 2.4, the sequence $\{y_{2n+1}\} = \{Sx_{2n+2}\} \subset S(X)$ is a **Cauchy** sequence in $S(X)$. Now suppose that $S(X)$ is complete, then the sequence $\{Sx_{2n+2}\}$ converges to a point $z = Su$ for $u \in X$. On the other hand, the subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}_{n\in\mathbb{N}}$ also converges to the point z. If $Au \neq z$ using (2.2), we obtain:

$$\begin{aligned} & d^{2p}(Au, Bx_{2n+1}) \\ \leq\ & a\ \varphi_0\left(d^{2p}(Su, Tx_{2n+1})\right) + (1-a)\max\left\{\varphi_1\left(d^{2p}(Su, Tx_{2n+1})\right),\right. \\ & \varphi_2\left(d^{q}(Su, Au)\, d^{q'}(Tx_{2n+1}, Bx_{2n+1})\right), \varphi_3\left(d^{r}(Su, Bx_{2n+1})\, d^{r'}(Tx_{2n+1}, Au)\right), \\ & \left.\varphi_4\left(\frac{1}{2}\left[d^{s}(Su, Au)\, d^{s'}(Tx_{2n+1}, Au)\right]\right), \varphi_5\left(\frac{1}{2}\left[d^{l}(Su, Bx_{2n+1})\, d^{l'}(Tx_{2n+1}, Bx_{2n+1})\right]\right)\right\} \end{aligned}$$

Letting $n \longrightarrow \infty$ we have

$$d^{2p}(Au, z) \leq (1-a)\max\left\{\varphi_4\left(\frac{1}{2}\left[d^{s}(z, Au)\, d^{s'}(z, Au)\right]\right)\right\}$$

$$\begin{aligned}
& d^{2p}(Au,z) \\
\leq\ & (1-a)\max\left\{\varphi_4\left(\frac{1}{2}d^{s+s'}(z,Au)\right)\right\} \\
\leq\ & \varphi\left(\frac{1}{2}d^{2p}(z,Au)\right) \\
<\ & \frac{1}{2}d^{2p}(z,Au) \\
<\ & d^{2p}(Au,z),
\end{aligned}$$

a contradiction. Hence $z = Au = Su$.

As $A(X) \subset T(X)$ there exist $v \in X$ such as $z = T\ v$.

If $Bv \neq z$ using (2.2), we obtain:

$$\begin{aligned}
& d^{2p}(Au,Bv) \\
\leq\ & a\ \varphi_0\left(d^{2p}(Su,Tv)\right) + (1-a)\max\left\{\varphi_1\left(d^{2p}(Su,Tv)\right),\right. \\
& \varphi_2\left(d^{q}(Su,Au)\,d^{q'}(Tv,Bv)\right), \varphi_3\left(d^{r}(Su,Bv)\,d^{r'}(Tv,Au)\right), \\
& \left.\varphi_4\left(\frac{1}{2}\left[d^{s}(Su,Au)\,d^{s'}(Tv,Au)\right]\right), \varphi_5\left(\frac{1}{2}\left[d^{l}(Su,Bv)\,d^{l'}(Tv,Bv)\right]\right)\right\}
\end{aligned}$$

or

$$\begin{aligned}
& d^{2p}(z,Bv) \\
\leq\ & (1-a)\max\{\ \varphi_5\left(\frac{1}{2}\left[d^{l}(Su,Bv)\,d^{l'}(Tv,Bv)\right]\right)\} \\
\leq\ & \varphi\left(\frac{1}{2}d^{l+l'}(z,Bv)\right) \\
<\ & \frac{1}{2}d^{2p}(z,Bv) \\
<\ & d^{2p}(z,Bv),
\end{aligned}$$

a contradiction. Hence $Tv = Bv = z$ because (A,S) is weakly compatible we have $Au = Su \Longrightarrow SAu = ASu$, we get $Sz = Az$.

If $Az \neq z$ using (2.2), we get

$$\begin{aligned}
& d^{2p}\left(Az, Bv\right) \\
\leq\; & a\ \varphi_0\left(d^{2p}\left(Sz, Tv\right)\right) + (1-a)\max\left\{\varphi_1\left(d^{2p}\left(Sz, Tv\right)\right),\right. \\
& \varphi_2\left(d^{q}\left(Sz, Az\right) d^{q'}\left(Tv, Bv\right)\right), \varphi_3\left(d^{r}\left(Sz, Bv\right) d^{r'}\left(Tv, Az\right)\right), \\
& \left.\varphi_4\left(\frac{1}{2}\left[d^{s}\left(Sz, Az\right) d^{s'}\left(Tv, Az\right)\right]\right), \varphi_5\left(\frac{1}{2}\left[d^{l}\left(Sz, Bv\right) d^{l'}\left(Tv, Bv\right)\right]\right)\right\}
\end{aligned}$$

$$\begin{aligned}
& d^{2p}\left(Az, z\right) \\
\leq\; & a\ \varphi_0\left(d^{2p}\left(Az, z\right)\right) + (1-a)\max\left\{\varphi_1\left(d^{2p}\left(Az, z\right)\right), \varphi_3\left(d^{r}\left(Az, z\right) d^{r'}\left(z, Az\right)\right)\right\} \\
\leq\; & \varphi\left(d^{2p}\left(Az, z\right)\right) \\
<\; & d^{2p}\left(Az, z\right),
\end{aligned}$$

which is a contradiction, Hence $Az = Sz = z$. By analogy we have $Tz = Bz = z$.

Finally, we have $Az = Sz = Bz = Tz = z$, Thus z is a common fixed point of A, B, S and T.

For uniqueness of common fixed point. Let A, B, S and T have another fixed point w , such a $w \neq z$.

Then from using (2.2), we obtain

$$\begin{aligned}
& d^{2p}\left(Az, Bw\right) \\
\leq\; & a\ \varphi_0\left(d^{2p}\left(Sz, Tw\right)\right) + (1-a)\max\left\{\varphi_1\left(d^{2p}\left(Sz, Tw\right)\right),\right. \\
& \varphi_2\left(d^{q}\left(Sz, Az\right) d^{q'}\left(Tw, Bw\right)\right), \varphi_3\left(d^{r}\left(Sz, Bw\right) d^{r'}\left(Tw, Az\right)\right), \\
& \left.\varphi_4\left(\frac{1}{2}\left[d^{s}\left(Sz, Az\right) d^{s'}\left(Tw, Az\right)\right]\right), \varphi_5\left(\frac{1}{2}\left[d^{l}\left(Sz, Bw\right) d^{l'}\left(Tw, Bw\right)\right]\right)\right\}
\end{aligned}$$

we get $d^{2p}\left(z, w\right) \leq \varphi\left(d^{2p}\left(z, w\right)\right) < d^{2p}\left(z, w\right)$, which is impossible, Hence $z = w$. ■

Remark 2.1 *By setting* $a = 1$ *and taking* $\varphi_0\left(t\right) = \alpha t$, $0 < \alpha < 1$ *in theorem* 2.1, *we obtain the following rsults:*

Corollary 2.1 *Let* A, B, S *and* T *self-maps of a metric space* (X, d) *satisfying* (2.1) *and* (2.2) *and*

$$d(Ax, By) \leq \alpha d(Sx, Ty), \tag{2.11}$$

For all $x, y \in X$, *where* $0 < \alpha < 1$ *and suppose the pairs* (A, S) *and* (B, T) *are weakly compatible and any one of the subset* $S(X)$ *or* $T(X)$ *is complete. Then,* A, B, S *and* T *have a unique common fixed point in* X.

2.3 Application to integral equations

Consider the following pair of non-linear integral equations (**Voltera** – **Hammerstein** equation):

$$x(t) = \omega(t, x(t)) + \mu \int_a^t m(t, s) g_i(s, x(s)) ds + \lambda \int_a^\infty k(t, s) h_j(s, x(s)) ds \tag{2.12}$$

For all $t \in [0, \infty)$, $\omega(t, x(t)) \in L^1([0, \infty))$ is known, $m(t, s)$, $k(t, s)$, $g_i(s,x(s))$ and $h_j(s, x(s))$, $i, j = 1, 2$ and $i \neq j$ are real or complex valued functions tha are measurable both in t and on $[0, \infty)$ and λ, μ are real or complex numbers. The previous functions satisfy the following conditions:

(d_0) $\int_0^\infty |\omega(s, x(s))|$ is bounded for all $x(s) \in L^1([0, \infty))$, and there exists $K_0 > 0$ such that for all $s \in [0, \infty)$ we have

$$|\omega(s, x(s)) - \omega(s, y(s))| \leq K_0 |x(s) - y(s)| \quad \text{for all } x, y \in [0, \infty) \tag{2.13}$$

(d_1) $\displaystyle\int_0^\infty \sup_{0 \leq s < \infty} |m(t, s)| dt = M_1 < +\infty$,

(d_2) $\displaystyle\int_0^\infty \sup_{0 \leq s < \infty} |k(t, s)| dt = M_2 < +\infty$,

(d_3) $g_i(s, x(s)) \in L^1([0, \infty))$ for each $x \in L^1([0, \infty))$, and there exists $K_1 > 0$ such that for all $s \in [0, \infty)$,

$$|g_1(s, x(s)) - g_2(s, y(s))| \leq K_1 |x(s) - y(s)| \quad \text{for all } x, y \in L^1([0, \infty)) \tag{2.14}$$

(d_4) $h_i(s, x(s)) \in L^1([0, \infty))$ for each $x \in L^1([0, \infty))$, and there exists $K_2 > 0$ such that for all $s \in [0, \infty)$,

$$|h_1(s, x(s)) - h_2(s, y(s))| \leq K_2 |x(s) - y(s)| \text{ for all } x, y \in L^1([0, \infty)). \tag{2.15}$$

The existence theorem can be formulated as follows:

Theorem 2.2 *Let* $\psi = \min\{\varphi_0, \varphi_1\}$ *where functions* φ_i, $i = 0, ..., 5$ *are defined in Lemma* 2.1 *with the assumptions* $(d_0), (d_1), (d_2), (d_3)$ *and* (d_4). *Also, If the following conditions are satisfied:*

(e_1) $\lambda \int_0^\infty k(t, s) h_i\left(s, \omega(s, x(s)) + \mu \int_0^s m(s, \tau) g_j(\tau, x(\tau)) d\tau\right) ds = 0,\ i, j = 1, 2, i \neq j;$

(e_2) *For some* $x \in L^1([0, \infty))$,

$$\begin{aligned} \mu \int_0^t m(t, s) g_i(s, x(s)) ds &= x(t) - \omega(t, x(t)) - \lambda \int_0^\infty k(t, s) h_i(s, x(s)) ds \\ &= \Gamma_i(t) \in L^1([0, \infty)). \end{aligned} \tag{2.16}$$

(e_3) *If for some* $\Gamma_i(t) \in L^1([0, \infty))$, *there exists* $\theta_i(t) \in L^1([0, \infty))$, *such that*

$$\begin{aligned} \mu \int_0^t m(t, s) g_i(s, x(s) - \Gamma_i(s)) ds &= \omega(t, x(t)) + \lambda \int_0^\infty k(t, s) h_i(s, x(s) - \Gamma_i(s)) ds \\ &= \theta_i(t), \quad i = 1, 2. \end{aligned} \tag{2.17}$$

then the systeme (2.12) *has a unique solution in* $L^1([0, \infty))$ *for each pair of real or complex numbers* λ *and* μ *with*

$$K_0 + |\mu| K_1 M_1 + |\lambda| K_2 M_2 < 1$$

and

$$(|\mu| K_1 M_1)^{2p} c \leq \psi((1 - (K_0 + |\lambda| K_2 M_2))^{2p} c). \quad c \geq 0,\ 0 < p \leq 1 \tag{2.18}$$

Proof We define for every $x \in L^1([0,\infty))$

$$\begin{aligned} Ax(t) &= \mu \int_0^t m(t,s)\, g_1(s,x(s))\, ds & (2.19)\\ Bx(t) &= \mu \int_0^t m(t,s)\, g_2(s,x(s))\, ds & (2.20)\\ Cx(t) &= \omega(t,x(t)) + \lambda \int_0^\infty k(t,s)\, h_1(s,x(s))\, ds & (2.21)\\ Dx(t) &= \omega(t,x(t)) + \lambda \int_0^\infty k(t,s)\, h_2(s,x(s))\, ds & (2.22)\\ Sx(t) &= (I-C)\,x(t), \qquad Tx(t) = (I-D)\,x(t), & (2.23) \end{aligned}$$

Were $\omega(t,x(t)) \in L^1([0,\infty))$ ara known, I is the identity operator on $L^1([0,\infty))$.

Then A, B, C, D, S and T are operator from $L^1([0,\infty))$ into itself.

Indeed, we have:

$$\begin{aligned} |A(x)| &\le \left| \mu \int_0^t m(t,s)\, g_1(s,x(s))\, ds \right| \\ &\le |\mu| \int_a^\infty |m(t,s)| \,.\, |g_1(s,x(s))|\, ds \\ &\le |\mu| \sup_{0\le s<\infty} |m(t,s)| \int_0^\infty |g_1(s,x(s))|\, ds. \end{aligned}$$

We apply conditions (d_1) and (d_3) and thus, we have:

$$\int_0^\infty |A(x)|\, dt \le |\mu| \int_0^\infty \sup_{0\le s<\infty} |m(t,s)|\, dt \int_0^\infty |g_1(s,x(s))|\, ds < +\infty,$$

hence $Ax \in L^1([0,\infty))$. Similary $Bx \in L^1([0,\infty))$.

We apply conditions (d_2) and (d_4) in the following manner:

$$\int_0^\infty |C(x)|\, dt \le \int_0^\infty \omega(t,x(t))\, dt + |\lambda| \int_0^\infty \sup_{0\le s<\infty} |k(t,s)|\, dt \int_0^\infty |h_1(s,x(s))|\, ds < +\infty. \quad (2.24)$$

As $\int_0^\infty |\omega(s,x(s))|$ is bounded so C is a self operator on $L^1([0,\infty))$. Similarly D is also a self operator on $L^1([0,\infty))$.

Hence S, T are operator from $L^1([0,\infty))$.

Now we prove that $A\left(L^1\left([0,\infty)\right)\right)\subseteq T\left(L^1\left([0,\infty)\right)\right)$. So let $x\in L^1\left([0,\infty)\right)$ be arbitrary. Then we have

$$\begin{aligned}
T\left(Ax\left(t\right)+\omega\left(t,x\left(t\right)\right)\right) &= \left(I-D\right)\left(Ax\left(t\right)+\omega\left(t,x\left(t\right)\right)\right)\\
&= Ax\left(t\right)-\lambda\int_0^\infty k\left(t,s\right)h_2\left(s,Ax\left(s\right)+\omega\left(s,x\left(s\right)\right)\right)ds\\
&= Ax\left(t\right)-\lambda\int_0^\infty k\left(t,s\right)h_2\left[s,\mu\int_0^s m\left(s,\tau\right)g_1\left(\tau,x\left(\tau\right)\right)d\tau+\omega\left(s,x\left(s\right)\right)\right]ds\\
&= Ax\left(t\right)
\end{aligned}$$

By assumption (e_1) of the theorem 2.2. Similarly one can prove $B\left(L^1\left([0,\infty)\right)\right)\subseteq S\left(L^1\left([0,\infty)\right)\right)$. With the help of (d_1) and (d_3), We have for all $x,y\in L^1\left([0,\infty)\right)$ that

$$\begin{aligned}
\left\|Ax-By\right\|^{2p} &= \left(\int_0^\infty\left|Ax\left(t\right)-By\left(t\right)\right|dt\right)^{2p}\\
&= \left(\int_0^\infty\left|\mu\int_0^t m\left(t,s\right)g_1\left(s,x\left(s\right)\right)ds-\mu\int_0^t m\left(t,s\right)g_2\left(s,y\left(s\right)\right)ds\right|dt\right)^{2p}\\
&= \left(\int_0^\infty\left|\mu\int_0^t m\left(t,s\right)\left[g_1\left(s,x\left(s\right)\right)-g_2\left(s,y\left(s\right)\right)\right]ds\right|dt\right)^{2p}\\
&\le \left(\int_0^\infty\left|\mu\int_0^\infty m\left(t,s\right)\left[g_1\left(s,x\left(s\right)\right)-g_2\left(s,y\left(s\right)\right)\right]ds\right|dt\right)^{2p}\\
&\le \left(\int_0^\infty\left|\mu\right|\sup_{0\le s<\infty}\left|m\left(t,s\right)\right|dt\int_0^\infty\left|g_1\left(s,x\left(s\right)\right)-g_2\left(s,y\left(s\right)\right)\right|ds\right)^{2p}\\
&\le \left(\left|\mu\right|K_1M_1\right)^{2p}\left(\int_0^\infty\left|x\left(s\right)-y\left(s\right)\right|ds\right)^{2p}\\
&= \left(\left|\mu\right|K_1M_1\right)^{2p}\left\|x-y\right\|^{2p}. \qquad (2.25)
\end{aligned}$$

Similarly, by $(d_0), (d_2)$ and (d_4), we get

$$\begin{aligned}
\|Cx - Dy\|^{2p} &= (\int_0^\infty |\omega(t,x(t)) - \omega(t,y(t)) \\
&\quad + \lambda \int_0^\infty k(t,s)\left[h_1(s,x(s)) - h_2(s,y(s))\right] ds \Big| \, dt)^{2p} \\
&\le (\int_0^\infty |\omega(t,x(t)) - \omega(t,y(t))| \, dt \\
&\quad + |\lambda| \int_0^\infty \sup_{0\le s<\infty} |k(t,s)| \, dt \int_0^\infty \left[h_1(s,x(s)) - h_2(s,y(s))\right] ds)^{2p} \\
&\le (K_0 + |\lambda| M_2 K_2)^{2p} \|x-y\|^{2p}. \qquad (2.26)
\end{aligned}$$

Consequently

$$\begin{aligned}
\|Sx - Ty\|^{2p} &= \|(I-C)x - (I-D)y\|^{2p} \\
&= \|x - y - (Cx - Dy)\|^{2p} \\
&\ge (\|x-y\| - \|(Cx-Dy)\|)^{2p} \\
&\ge (\|x-y\| - (K_0 + |\lambda| K_2 M_2)\|x-y\|)^{2p} \\
&\ge (1 - (K_0 + |\lambda| K_2 M_2))^{2p} \|x-y\|^{2p}, \qquad (2.27)
\end{aligned}$$

From (2.27), we obtain

$$\begin{aligned}
&(a\,\varphi_0 \|Sx - Ty\|^{2p} + (1-a)\max\{\varphi_1 \|Sx - Ty\|^{2p}, \\
&\varphi_2\left(\|Sx - Ax\|^q \|Ty - By\|^{q'}\right), \varphi_3\left(\|Sx - By\|^r \|Ty - Ax\|^{r'}\right), \\
&\varphi_4\left(\frac{1}{2}\left[\|Sx - Ax\|^s \|Ty - Ax\|^{s'}\right]\right), \varphi_5\left(\frac{1}{2}\left[\|Sx - By\|^l \|Ty - By\|^{l'}\right]\right)\}) \\
\ge\ & a\,\varphi_0 \|Sx - Ty\|^{2p} + (1-a)\varphi_1 \|Sx - Ty\|^{2p} \ \text{, with } \psi = \min\{\varphi_0, \varphi_1\} \\
\ge\ & a\,\psi \|Sx - Ty\|^{2p} + (1-a)\psi \|Sx - Ty\|^{2p} \\
=\ & \psi \|Sx - Ty\|^{2p} \qquad (2.28)
\end{aligned}$$

As ψ is non-decreasing, from $(2.18), (2.27)$ and (2.28) we obtain

$$\begin{aligned}
\psi \left\| Sx - Ty \right\|^{2p} &\geq \psi\left(\left(1 - (K_0 + |\lambda| K_2 M_2)\right)^{2p} \left\| x - y \right\|^{2p} \right) \\
&\geq \left(\left(|\mu| K_1 M_1 \right)^{2p} \left\| x - y \right\|^{2p} \right) \\
&\geq \left\| Ax - By \right\|^{2p}.
\end{aligned}$$

Thus condition (2.2) of theorem 2.1 is satisfied.

Now we prove that the pair (A, S) is weakly compatible. For this we have

$$\begin{aligned}
\left\| SAx(t) - ASx(t) \right\| &= \left\| (I - C) Ax(t) - A(I - C) x(t) \right\| \\
&= \left\| Ax(t) - CAx(t) - Ax(t) + ACx(t) \right\| \\
&= \left\| ACx(t) - CAx(t) \right\|. \qquad (2.29)
\end{aligned}$$

Whenever $Ax(t) = Sx(t)$. We have

$$\mu \int_0^t m(t,s)\, g_1(s, x(s))\, ds = x(t) - \omega(t, x(t)) - \lambda \int_0^\infty k(t,s)\, h_1(s, x(s))\, ds. \qquad (2.30)$$

Therefore from (2.29), we get

$$\begin{aligned}
\left\| SAx(t) - ASx(t) \right\| &= \left\| ACx(t) - CAx(t) \right\| \\
&= \Bigg\| A\left(\omega(t, x(t)) + \lambda \int_0^\infty k(t,s)\, h_1(s, x(s) - \Gamma_1(s))\, ds \right) \\
&\quad - C\left(\mu \int_0^t m(t,s)\, g_1(s, x(s) - \Gamma_1(s))\, ds \right) \Bigg\| \\
&= \Bigg\| \mu \int_0^t m(t,s)\, g_1\left(s, \omega(s, x(s)) + \lambda \int_0^\infty k(s,\tau)\, h_1(\tau, x(\tau) - \Gamma_1(\tau))\, d\tau \right) ds \\
&\quad - \omega(t, x(t)) - \lambda \int_0^\infty k(t,s)\, h_1\left(s, \mu \int_0^s m(s,\tau)\, g_1(\tau, x(\tau) - \Gamma_1(\tau))\, d\tau \right) ds \Bigg\| \\
&= 0.
\end{aligned}$$

This shows that the pair (A, S) is weakly compatible. Similarly we can show that the pair (B, T) is also weakly compatible.

Thus all the conditions of theorem 2.1 are satisfied. Therfore, there exists a unique $u \in L^1([0,\infty))$ such that $Au = Bu = Su = Tu = u$ and consequently, u is the unique solution of (2.12). ■

Remark 2.2 *For uniqueness of solution. Let* $v \in L^1([0,\infty))$ *another solution of* (2.12) *, such a* $w \neq z$.

Then from using the conditions $(d_0), (d_1), (d_2), (d_3)$ *and* (d_4), *we obtain*

$$\begin{aligned} \|u(t) - v(t)\| &\leq \int_0^\infty \Big| \omega(t, u(t)) - \omega(t, v(t)) + \mu \int_0^t m(t,s)\left(g_1(s, u(s)) - g_2(s, v(s))\right) ds \\ &\quad + \lambda \int_0^\infty k(t,s)\, h_1((s, u(s)) - h_2(s, v(s)))\, ds \Big| \, dt \\ &\leq (K_0 + |\mu| K_1 M_1 + |\lambda| K_2 M_2) \|u(t) - v(t)\| < \|u(t) - v(t)\|, \end{aligned}$$

Which is a contradiction. Hence $u(t) = v(t)$.

Chapter 3

New Results in Common Fixed Point Theorem in b-metric Spaces, with Applications to Linear System and Non-linear Integral Equations

In this chapter, we will obtain some common fixed point results having rational type contraction conditions for mappings satisfying generalized contractive condition in a b-metric space. The result is applied to prove the existence of a common solution of system of **Urysohn** non-linear integral equations and unique solution for linear equations system.

The plan of this chapter is as follows:

1. **Introduction**
2. **Main results**
3. **Applications:**
    - 3.1. Application to linear equations
    - 3.2. Application to integral equations

3.1 Introduction

The concept of b-metric space was introduced in **1989** by **Bakhtin**. In **1993**, **Czerwik** [16] extended the results of b-metric spaces that generalized the famous **Banach** contraction principle in metric space. **Akkouchi** [4], **Boriceanu** [11], **kir** et al. [31], **Olatinwo** and al. [37], **Pacurar** [38] extended the fixed point theorem in b-metric space. A b- metric space was also called a metric type spaces in [17]. Many articles generalized and extended the study of common fixed point in b-metric spaces see ([17], [36]).

The aim of this work is to present some common fixed point results having rational type contraction conditions for mappings, satisfying generalized contractive condition in a b-metric space.

3.2 Main results

Theorem 3.1 *Let* (X, d) *is a complete b-metric space and* $A, B : X \longrightarrow X$ *be two non-decreassings mappings, and suppose there exist non-negative real numbers* $\alpha, \beta, \gamma, \delta$

with $\alpha + \beta + \gamma + 2s\delta < \frac{1}{s}$ *such that, for all* $x, y \in X$

$$\begin{aligned} d(Ax, By) \leq\ & \alpha d(x,y) + \beta \cdot \frac{d(y,By)\cdot[1+d(x,Ax)]}{1+d(x,y)} \\ & +\gamma \cdot \frac{d(y,Ax)\cdot[1+d(x,By)]}{1+d(x,y)} \\ & +\delta \cdot [d(y,Ax)+d(x,By)] \end{aligned} \tag{3.1}$$

Then, A and B have a unique common fixed point in X.

Proof Let x_0 an arbitrary in X, we can define a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X verifying :

$$x_{2n+1} = Ax_{2n} \text{ et } x_{2n+2} = Bx_{2n+1} \quad , n \in \mathbb{N}$$

for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
d_{2n} &= d\left(x_{2n+1}, x_{2n+2}\right) \\
&= d\left(Ax_{2n}, Bx_{2n+1}\right) \\
&\leq \alpha.d(x_{2n}, x_{2n+1}) \ + \beta.\frac{d(x_{2n+1}, Bx_{2n+1}).\left[1 + d\left(x_{2n}, Ax_{2n}\right)\right]}{1 + d\left(x_{2n}, x_{2n+1}\right)} \\
&\quad +\gamma.\frac{d(x_{2n+1}, Ax_{2n}).\left[1 + d\left(x_{2n}, Bx_{2n+1}\right)\right]}{1 + d\left(x_{2n}, x_{2n+1}\right)} \\
&\quad +\delta.\left[d(x_{2n+1}, Ax_{2n}) + d\left(x_{2n}, Bx_{2n+1}\right)\right] \\
&\leq \alpha.d(x_{2n}, x_{2n+1}) \ + \beta.\frac{d(x_{2n+1}, x_{2n+2}).\left[1 + d\left(x_{2n}, x_{2n+1}\right)\right]}{1 + d\left(x_{2n}, x_{2n+1}\right)} \\
&\quad +\gamma.\frac{d(x_{2n+1}, x_{2n+1}).\left[1 + d\left(x_{2n}, x_{2n+2}\right)\right]}{1 + d\left(x_{2n}, x_{2n+1}\right)} + \delta.\left[d(x_{2n+1}, x_{2n+1}) + d\left(x_{2n}, x_{2n+2}\right)\right] \\
&\leq \alpha.d(x_{2n}, x_{2n+1}) \ + \beta.d(x_{2n+1}, x_{2n+2}) + \gamma.\frac{d(x_{2n+1}, x_{2n+1}).\left[1 + d\left(x_{2n}, x_{2n+2}\right)\right]}{1 + d\left(x_{2n}, x_{2n+1}\right)} \\
&\quad +\delta.d\left(x_{2n}, x_{2n+2}\right) \\
&\leq \alpha.d(x_{2n}, x_{2n+1}) \ + \beta.d(x_{2n+1}, x_{2n+2}) + \delta.d\left(x_{2n}, x_{2n+2}\right) \\
&\leq \alpha.d(x_{2n}, x_{2n+1}) + \beta.d(x_{2n+1}, x_{2n+2}) + \delta.d\left(x_{2n}, x_{2n+2}\right) \\
&\leq \alpha.d(x_{2n}, x_{2n+1}) \ + \beta.d(x_{2n+1}, x_{2n+2}) + s\delta.\left[d(x_{2n}, x_{2n+1}) + d\left(x_{2n+1}, x_{2n+2}\right)\right] \\
&\leq \frac{\alpha + s\delta}{1 - \beta - s\delta}.d(x_{2n}, x_{2n+1}) \\
&\leq h.d(x_{2n}, x_{2n+1}) \ \text{ tq : } h = \frac{\alpha + s\delta}{1 - \beta - s\delta} < \frac{1}{s}
\end{aligned}$$

This follows immediately

$$\begin{aligned}
d\left(x_{2n+1}, x_{2n+2}\right) &\leq h.d(x_{2n}, x_{2n+1}) \leq h^2.d(x_{2n-1}, x_{2n}) \\
&\leq h^3.d(x_{2n-2}, x_{2n-1}) \leq ... \leq h^{2n+1}.d(x_0, x_1)
\end{aligned}$$

for $m > n$

$$\begin{aligned}
d\left(x_{2n}, x_{2m}\right) &\leq s\left[d\left(x_{2n}, x_{2n+1}\right)+d\left(x_{2n+1}, x_{2m}\right)\right] \\
&\leq sd\left(x_{2n}, x_{2n+1}\right)+s^{2}\left[d\left(x_{2n+1}, x_{2n+2}\right)+d\left(x_{2n+2}, x_{2m}\right)\right] \\
&\leq sd\left(x_{2n}, x_{2n+1}\right)+s^{2} d\left(x_{2n+1}, x_{2n+2}\right) \\
&\quad +s^{3} d\left(x_{2n+2}, x_{2n+3}\right)+\ldots+s^{2m-2n} d\left(x_{2m-1}, x_{2m}\right) \\
&\leq \left(sh^{2n}+s^{2} h^{2n+1}+\ldots+s^{2m-2n} h^{2m-1}\right) .d(x_0, x_1) \\
&\leq sh^{2n}\left(1+(sh)+(sh)^{2}+\ldots+(sh)^{2m-2n-1}\right) .d(x_0, x_1) \\
&\leq \frac{(sh)^{2n}}{1-sh} .d(x_0, x_1)
\end{aligned}$$

because $0 \leq h < \frac{1}{s}$, we have $\frac{(sh)^{2n}}{1-sh} \longrightarrow 0$ if $n \longrightarrow \infty$.

Hence $\left\{x_{2n}\right\}_{n \in \mathbb{N}}$ is a Cauchy sequence then $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ is a **Cauchy** sequence.

By the completeness of X, there exists a point $z \in X$ such that $x_{n} \longrightarrow z$ as $n \longrightarrow \infty$

Next, we claim that $Az = z$. If $Az \neq z$ then $d\left(z, Az\right) > 0$ By (3.1) we have

$$\begin{aligned}
d\left(z, Az\right) &\leq s\left[d\left(z, x_{2n+2}\right)+d\left(x_{2n+2}, Az\right)\right] \\
&= s\left[d\left(z, x_{n+2}\right)+d\left(Bx_{2n+1}, Az\right)\right] \\
&= s\left[d\left(z, x_{n+2}\right)+d\left(Az, Bx_{2n+1}\right)\right] \\
&\leq s\left[d\left(z, x_{2n+2}\right)+\left(\alpha .d(z, x_{2n+1}) \right.\right. +\beta . \frac{d(x_{2n+1}, Bx_{2n+1}) .\left[1+d\left(z, Az\right)\right]}{1+d\left(z, x_{2n+1}\right)} \\
&\quad \left.\left. +\gamma . \frac{d(x_{2n+1}, Az) .\left[1+d\left(z, Bx_{2n+1}\right)\right]}{1+d\left(z, x_{2n+1}\right)}+\delta .\left[d(x_{2n+1}, Az)+d\left(z, Bx_{2n+1}\right)\right]\right)\right] \\
&= s\left[d\left(z, x_{2n+2}\right)+\left(\alpha .d(z, x_{2n+1}) \right.\right. +\beta . \frac{d(x_{2n+1}, x_{2n+2}) .\left[1+d\left(z, Az\right)\right]}{1+d\left(z, x_{2n+1}\right)} \\
&\quad \left.\left. +\gamma . \frac{d(x_{2n+1}, Az) .\left[1+d\left(z, x_{2n+2}\right)\right]}{1+d\left(z, x_{2n+1}\right)}+\delta .\left[d(x_{2n+1}, Az)+d\left(z, x_{2n+2}\right)\right]\right)\right]
\end{aligned}$$

Letting $n \longrightarrow \infty$

$$\begin{aligned} d(z,Az) &\leq s\Bigg[d(z,z)+\Bigg(\alpha.d(z,z) \ +\beta.\frac{d(z,z).\left[1+d(z,Az)\right]}{1+d(z,z)} \\ &\qquad +\gamma.\frac{d(z,Az).\left[1+d(z,z)\right]}{1+d(z,z)}+\delta.\left[d(z,Az)+d(z,z)\right]\Bigg)\Bigg] \\ &= s\left(\gamma.d(z,Az)+\delta.d(z,Az)\right) \end{aligned}$$

which implies that $1 \leq s(\gamma+\delta) \Longrightarrow (\gamma+\delta) \geq \frac{1}{s}$ which is a contradiction.

Thus, we get $d(z,Az)=0$ then $Az=z$.

Similarly $Bz \neq z$ then $d(z,Bz)>0$. By (3.1) we have

$$\begin{aligned} d(z,Bz) &\leq s\left[d(z,x_{2n+1})+d(x_{2n+1},Bz)\right] \\ &= s\left[d(z,x_{n+1})+d(Ax_{2n},Bz)\right] \\ &\leq s\Bigg[d(z,x_{2n+1})+\Bigg(\alpha.d(x_{2n},z) \ +\beta.\frac{d(z,Bz).\left[1+d(x_{2n},Ax_{2n})\right]}{1+d(x_{2n},z)} \\ &\qquad +\gamma.\frac{d(z,Ax_{2n}).\left[1+d(x_{2n},Bz)\right]}{1+d(x_{2n},z)}+\delta.\left[d(z,Ax_{2n})+d(x_{2n},Bz)\right]\Bigg)\Bigg] \\ &= s\Bigg[d(z,x_{2n+1})+\Bigg(\alpha.d(x_{2n},z) \ +\beta.\frac{d(z,Bz).\left[1+d(x_{2n},x_{2n+1})\right]}{1+d(x_{2n},z)} \\ &\qquad +\gamma.\frac{d(z,x_{2n+1}).\left[1+d(x_{2n},Bz)\right]}{1+d(x_{2n},z)}+\delta.\left[d(z,x_{2n+1})+d(x_{2n},Bz)\right]\Bigg)\Bigg] \end{aligned}$$

Letting $n \longrightarrow \infty$

$$\begin{aligned} d(z,Bz) &\leq s\Bigg[d(z,z)+\Bigg(\alpha.d(z,z) \ +\beta.\frac{d(z,Bz).\left[1+d(z,z)\right]}{1+d(z,z)} \\ &\qquad +\gamma.\frac{d(z,z).\left[1+d(z,Bz)\right]}{1+d(z,z)}+\delta.\left[d(z,z)+d(z,Bz)\right]\Bigg)\Bigg] \\ &= s\left(\beta.d(z,Bz)+\delta.d(z,Bz)\right) \end{aligned}$$

which implies that $1 \leq s(\beta+\delta) \Longrightarrow (\beta+\delta) \geq \frac{1}{s}$ which is a contradiction.

Thus, we get $d(z,Bz)=0$ then $Bz=z$.

Therefore, $z=Az=Bz$, that is, z is a common fixed point of A and B.

Finally, we show that z is a unique common fixed point of A and B.

Assume that there exists another point z' such that $z' = Az' = Bz'$. Now, we have

$$\begin{aligned}
d(z,z') &= \\
d(Az,Bz') &\leq \alpha.d(z,z') \ +\beta.\frac{d(z',Bz').\left[1+d(z,Az)\right]}{1+d(z,z')} \\
&\quad +\gamma.\frac{d(z',Az).\left[1+d(z,Bz')\right]}{1+d(z,z')}+\delta.\left[d(z',Az)+d(z,Bz')\right] \\
&= \alpha.d(z,z')+\beta.\frac{d(z',z').\left[1+d(z,Az)\right]}{1+d(z,z')} \\
&\quad +\gamma.\frac{d(z',z).\left[1+d(z,z')\right]}{1+d(z,z')}+\delta.\left[d(z',z)+d(z,z')\right] \\
&= \alpha.d(z,z')+\gamma.d(z,z')+2\delta.d(z,z')
\end{aligned}$$

which implies that $1 \leq \alpha+\gamma+2\delta$ which is a contradiction.

Thus, we get $d(z,z') = 0$ then $z = z'$.

Therefore, z is a unique common fixed point of A and B. ■

Now if we set $\beta = \gamma = \delta = 0$ and $A = B$ in inequality (3.1) of Theorem 3.1 then we get following corollary

Corollary 3.1 *Let (X,d) is a complete b-metric space and $A : X \longrightarrow X$ be non-decreassing mapping.*

Suppose that there exist non-negative real number α with $\alpha < \frac{1}{s}$ such that, for all $x,y \in X$

$$d(Ax,Ay) \leq \alpha d(x,y)$$

Then, A have a unique fixed point in X.

3.3 Applications

3.3.1 Application to linear equations

In this section we give an application using Corollary 3.1

Theorem 3.2 *Let $X = \mathbb{R}^n$ be a $b-$metric space with the metric*

$$d_\infty(x, y) = \max_{1 \le i \le n} |x_i - y_i|$$

where $x, y \in X$.

Consider the system of n linear equations in n unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{cases} \tag{3.2}$$

If

$$\begin{aligned} \alpha &= \max\left\{|a_{ii} + 1|, \left|a_{ij_{i\neq j}}\right|;\ \alpha_{ij} \in \mathbb{R};\ \forall\ 1 \le i; j \le n\right\} \\ &< \frac{1}{n} \end{aligned}$$

We have, the system of n linear equations in n unknowns (3.2) *has a unique solution.*

Proof *It is easy to show that (X, d_∞) is a complete $b-$metric space.*

So for proving that the system (3.2) *has a unique solution, we need to prove that the mapping $H : X \longrightarrow X$ given by*

$$H(X) = AX + IX - D$$

is a contraction.

Where $X = (x_1, x_2, ..., x_n)^t \in \mathbb{R}^n$, $D = (b_1, b_2, ..., b_n)^t \in \mathbb{R}^n$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Since

$$\begin{aligned} d_\infty (Hx, Hy) &= \max_{1\le i\le n} \left| \sum_{j=1}^{n} (Hx_j - Hy_j) \right| \\ &\leqslant \max_{1\le i\le n} \left| (\alpha_{ii}+1)(x_1-y_1) + \sum_{j=2}^{n} \alpha_{ij}(x_j-y_j) \right| \\ &\leqslant n \max \{ |\alpha_{ii}+1|, |\alpha_{ij}|;\ \forall\ 1\le i; j \le n \} \max_{1\le j\le n} |x_j - y_j| \\ &\leqslant \alpha n \max_{1\le j\le n} |x_j-y_j| \\ &\le \alpha n d_\infty (x,y). \end{aligned}$$

We conclude that H is a contraction mapping. By Corollary 1*, the linear equation system* (3.2) *has a unique solution.* ■

3.3.2 Application to integral equations

In this section, we show that Theorem 3.1 can be applied to the existence of a common solution of the system of the **Urysohn** integral equations.

Let $X = C([a,b], \mathbb{R}^n), a > 0$, and $d : X \times X \longrightarrow \mathbb{R}$ be defined by

$$d(x,y) = \max_{t\in[a,b]} \|x(t) - y(t)\|_\infty$$

Consider the **Urysohn** integral equations

$$\begin{cases} x(t) = \int\limits_a^b K_1(t,s,x(s))\, ds + g(t); \\ x(t) = \int\limits_a^b K_2(t,s,x(s))\, ds + h(t); \end{cases} \tag{3.3}$$

Where $t \in [a,b] \subseteq \mathbb{R}, x, g$ and $h \in X$ and $K_1, K_2 : [a,b]\times[a,b]\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$

Suppose that K_1, K_2 are such that $F_x, G_x \in X$ for all $x \in X$, where

$$\begin{aligned} F_x &= \int_a^b K_1(t, s, x(s))\, ds \\ G_x &= \int_a^b K_2(t, s, x(s))\, ds \end{aligned}$$

for all $t \in [a, b]$.

Theorem 3.3 *If there exist non-negative real numbers* α, β, γ *and* δ *such that for all inequality holds:*

(a) $\alpha + \beta + \gamma + 2s\delta < \frac{1}{s}$;

(b)

$$\begin{aligned} & \|F_x(t) - G_y(t) + g(t) - h(t)\|_\infty \\ \leq\ & \alpha \max_{t \in [a,b]} S(x, y)(t) + \beta \max_{t \in [a,b]} T(x, y)(t) \\ & + \gamma \max_{t \in [a,b]} V(x, y)(t) + \delta \max_{t \in [a,b]} W(x, y)(t) \end{aligned}$$

where

$$\begin{aligned} S(x, y)(t) &= \|x(t) - y(t)\|_\infty, \\ T(x, y)(t) &= \frac{\|G_y(t) + h(t) - y(t)\|_\infty + \|F_x(t) + g(t) - x(t)\|_\infty \|G_y(t) + h(t) - y(t)\|_\infty}{1 + d(x, y)}, \\ V(x, y)(t) &= \frac{\|F_x(t) + g(t) - y(t)\|_\infty + \|F_x(t) + g(t) - y(t)\|_\infty \|G_y(t) + h(t) - x(t)\|_\infty}{1 + d(x, y)} \\ W(x, y)(t) &= \|F_x(t) + g(t) - y(t)\|_\infty + \|G_y(t) + h(t) - x(t)\|_\infty \end{aligned}$$

then the system of integral equation (3.3) *has a unique common solution.*

Proof Define two mappings $S, T : X \longrightarrow X$ by $Ax = F_x + g$ and $Bx = G_x + h$. Then we have

$$\begin{aligned}
d(Ax,By) &= \max_{t\in[a,b]} \left\|F_x(t)-G_y(t)+g(t)-h(t)\right\|_{\infty},\\
d(y,By) &= \max_{t\in[a,b]} \left\|G_y(t)+h(t)-y(t)\right\|_{\infty},\\
d(y,Ax) &= \max_{t\in[a,b]} \left\|F_x(t)+g(t)-y(t)\right\|_{\infty},\\
d(x,By) &= \max_{t\in[a,b]} \left\|G_y(t)+h(t)-x(t)\right\|_{\infty},
\end{aligned}$$

and

$$d(x,Ax)=\max_{t\in[a,b]} \left\|F_x(t)+g(t)-x(t)\right\|_{\infty},$$

we can show easily that for all $x,y\in X$,

$$\begin{aligned}
d(Ax,By) \leq\ & \alpha d(x,y)+\beta\cdot\frac{d(y,By)\cdot[1+d(x,Ax)]}{1+d(x,y)}\\
&+\gamma\cdot\frac{d(y,Ax)\cdot[1+d(x,By)]}{1+d(x,y)}\\
&+\delta\cdot[d(y,Ax)+d(x,By)]
\end{aligned}$$

Now, we can apply Theorem 3.1. Therefore, we get the **Urysohn** integral equations (3.3) have a unique common solution. ■

Chapter 4

Common Coupled Fixed Point Theorems for Two Pairs of Weakly Compatible Mappings in Menger Metric Spaces, and Application to Fredholm Non-linear Integral Equation

In this chapter, we present my work in [3] . When we will prove some common coupled fixed point theorems for contractive mappings in **Menger** metric spaces under geometrically convergent t-norms. Also, we prove common fixed point theorems for pairs of weakly compatible mappings, which generalize the results of **Jian-Zhong Xiao** and al. (**2011**). The main results is supported by a suitable example and application to integral equations.

The plan of this chapter is as follows:

1. **Introduction**
2. **Main results**
3. **Application to integral equations**

4.1 Introduction

Many common coupled fixed point theorems for contractions in probabilistic metric spaces under either a t-norm of **Hadzic**-type or the t-norm $\triangle_p = prod$ can be found in the recent literature, see [48], [57] , [58] , [14] , [18] , [15] , [59]. The aim of this chapter is to obtain similar results in a larger class of **Menger** metric spaces, namely in **Menger** metric spaces endowed with geometrically convergent t-norms.

We only recall that a t-norm $\triangle$ is said to be of **Hadzic**-type (denoted $\triangle \in H$) if the family $\{\triangle^n(t)\}_{n=1}^{\infty}$ defined by

$$\triangle^1(t) = t, \quad \triangle^{n+1}(t) = \triangle(t, \triangle^n(t)) \qquad n = 1, 2, ... \quad t \in [0,1]$$

is equi-continuous at $t = 1$, and that a t-norm $\triangle$ is called geometrically convergent (or g-convergent) [18] if for all $q \in (0,1)$,

$$\lim_{n\to\infty} \triangle_{i=n}^{\infty}\left(1 - q^i\right) = 1.$$

It is worth noting (see [22]) that if for a t-norm there exists $q_0 \in (0,1)$ such that

$$\lim_{n\to\infty} \triangle_{i=n}^{\infty}\left(1 - q_0^i\right) = 1$$

then:

$$\lim_{n\to\infty} \triangle_{i=1}^{\infty}\left(1 - q^i\right) = 1.$$

The well known t-norms $\triangle_M = \min$, $\triangle_p = prod$ and $\triangle_L$ (**Lukasiewiez** t-norm) are g-convergent.

Also, every member of **Domby** family $\left(\triangle_\lambda^D\right)_{\lambda\epsilon(0;\infty)}$, **Aczel-Alsina** family $\left(\triangle_\lambda^{AA}\right)_{\lambda\epsilon(0;\infty)}$ and **Sugeno-Weber** family $\left(\triangle_\lambda^{SW}\right)_{\lambda\epsilon(-1;\infty)}$ is g-convergent [22].

A large class of g-convergent t-norms, in terms of the generators of strict t-norms is described in [22] (also see [21], Ch.1.8).

Bhaskar and **Lakshmikantham** [8] , **Lakshmikantham** and **Ćirić** [33] gave some coupled fixed point theorems. Coupled fixed point theorem under contraction conditions given by **Sedghi** and al. [48] are of great importance in the theory of fixed points in **fuzzy** metric spaces. **Fang** [18] proved a result for compatible and weakly compatible mappings under φ-contractive conditions in **fuzzy** metric spaces which provide a tool to **Xin-Qi Hu** [57] to prove a result, which is actually a generalization of the result of **Sedghi** [48] .

In this chapter, firstly we prove some common coupled fixed point theorems for contractive mappings in **Menger** metric spaces under geometrically convergent t-norms. At the end, we prove common fixed point theorems for pairs of weakly compatible mappings, which generalize the results of **Jian-Zhong Xiao** and al (**2011**). The main result is supported by a suitable example and application.

4.2 Main results

We now give our main result which provides a common coupled fixed point theorems for contractive mappings in **Menger** metric spaces under geometrically convergent t-norms.

Theorem 4.1 *Let (X, F, Δ) be a* ***Menger*** *metric space with Δ is a g-convergent t-norm. Let $A : X \times X \to X$, $B : X \times X \to X$,*

$T : X \to X$ and $S : X \to X$ be four mappings satisfying the following condition

(1) $A(X \times X) \subseteq T(X), B(X \times X) \subseteq S(X)$

(2) *There exists $k \in (0, 1)$ such that*

$$F_{A(x,y),B(u,v)}(kt) \geq Min\left(F_{Sx,Tu}(t),\ F_{Sy,Tv}(t)\right) \tag{4.1}$$

for all $x, y, u, v \in X$ and $t > 0$

(3) *The pairs (A, S) and (B, T) are weakly compatible.*

(4) *One of the subspaces $A(X \times X)$ or $T(X)$ and one of $B(X \times X)$ or $S(X)$ are complete.*

If there exists $\alpha > 0$ *and* $x_0, y_0 \in X$ *such that*

$$\sup_{t>0} t^{\alpha} \left(1 - F_{Sx_0, A(x_0,y_0)}(t)\right) < \infty,$$

and

$$\sup_{t>0} t^{\alpha} \left(1 - F_{Sy_0, A(y_0,x_0)}(t)\right) < \infty.$$

Then there exists a unique point a *in* X *such that* $A(a,a) = S(a) = T(a) = B(a,a) = a$.

Proof For arbitrary x_0, y_0 in X, by **(1)**, we can choose x_1, y_1 in X such that $T(x_1) = A(x_0, y_0), T(y_1) = A(y_0, x_0)$.

Again by **(1)**, we can choose x_2, y_2 in X such that $S(x_2) = B(x_1, y_1), S(y_2) = A(y_1, x_1)$.

Continuing in this way, we can construct two sequences $\{Z_n\}$ and $\{Z_n'\}$ in X such that

$$\begin{cases} Z_{2n+1} = A(x_{2n}, y_{2n}) = T(x_{2n+1}) \\ Z_{2n+2} = B(x_{2n+1}, y_{2n+1}) = S(x_{2n+2}) \end{cases}$$

$$\begin{cases} Z'_{2n+1} = A(y_{2n}, x_{2n}) = T(y_{2n+1}) \\ Z'_{2n+2} = B(y_{2n+1}, x_{2n+1}) = S(y_{2n+2}) \end{cases}$$

We divide the proof into 6 steps.

Step 1:

We show that $\{Z_n\}$ and $\{Z_n'\}$ are **Cauchy** sequences.

Indeed, let $\beta > 0.$ be such that

$$t^{\alpha} \left(1 - F_{Sx_0, A(x_0,y_0)}(t)\right) \leq \beta$$

and

$$t^{\alpha} \left(1 - F_{Sy_0, A(y_0,x_0)}(t)\right) \leq \beta$$

for all $t > 0$. Then

$$F_{Sx_0, Tx_1}\left(\frac{1}{t^n}\right) > 1 - \beta \left(t^{\alpha}\right)^n \text{ and } F_{Sy_0, Ty_1}\left(\frac{1}{t^n}\right) > 1 - \beta \left(t^{\alpha}\right)^n \text{ for every } t > 0 \text{ and } n \in N.$$

If $t > 0$ and $\varepsilon \in (0,1)$ are given, we choose μ in interval $(k,1)$ such that

$$\Delta_{i=n+1}^{\infty}\left(1-(\mu^{\alpha})^{i}\right) > 1-\varepsilon \text{ and } \delta = \frac{k}{\mu}.$$

As $\delta \in (0,1)$, we can find $n_1\left(=n_1(t)\right)$ such that $\sum_{n=n_1}^{\infty}\delta^{n} < t$.

Condition (4.1) implies that, for all $s > 0$,

$$F_{Tx_1,Sx_2}(ks) = F_{A(x_0,y_0),B(x_1,y_1)}(ks) \geq Min\left(F_{Sx_0,Tx_1}(s),\ F_{Sy_0,Ty_1}(s)\right),$$

and

$$F_{Ty_1,Sy_2}(ks) = F_{A(y_0,x_0),B(y_1,x_1)}(ks) \geq Min\left(F_{Sx_0,Tx_1}(s), F_{Sy_0,Ty_1}(s)\right).$$

It follows by induction that

$$F_{Z_n,Z_{n+1}}(k^{n}s) \geq Min\left(F_{Sx_0,Tx_1}(s), F_{Sy_0,Ty_1}(s)\right)$$

and

$$F_{\acute{Z}_n,\acute{Z}_{n+1}}(k^{n}s) \geq Min\left(F_{Sx_0,Tx_1}(s), F_{Sy_0,Ty_1}(s)\right)$$

for all $n \in N$. Then for all $n \geq n_1$ and $m \in N$ we obtain

$$\begin{aligned}
F_{Z_n,Zn+m}(t) &\geq F_{Z_n,Zn+m}\left(\sum_{i=n_1}^{\infty}\delta^{i}\right) \\
&\geq F_{Z_n,Zn+m}\left(\sum_{i=n}^{n+m-1}\delta^{i}\right) \\
&\geq \Delta_{i=n}^{n+m-1}\left(F_{Z_i,Z_{i+1}}(\delta^{i})\right) \\
&\geq \Delta_{i=n}^{n+m-1}\left(\Delta\left(F_{Sx_0,Tx_1}\left(\frac{1}{\mu^{i}}\right), F_{Sy_0,Ty_1}\left(\frac{1}{\mu^{i}}\right)\right)\right) \\
&\geq \Delta_{i=n}^{n+m-1}\left(1-\beta\mu^{\alpha i}\right).
\end{aligned}$$

If we choose $l_0 \in N$ such that $\beta\mu^{\alpha l_0} \leq \mu^{\alpha}$, then $1-\beta\left(\mu^{\alpha}\right)^{n+l_0} \geq 1-\left(\mu^{\alpha}\right)^{n+1}$, for all n. Thus,

$$F_{Z_{n+l_0},Z_{n+l_0+m}}(t) \geq \Delta_{i=n+1}^{\infty}\left(1-(\mu^{\alpha})^{i}\right) > 1-\varepsilon,$$

for every $n \geq n_1$ and $m \in N$, hence $\left\{ Z'_{2n+1} \right\}$ is a Cauchy sequence. Similarly one can show that $\left\{ Z'_n \right\}$ is a **Cauchy** sequence.

Step 2:

We show that

$$T(a) = B(a,b), T(b) = B(a,b) \text{ and } S(a) = A(a,b),\ S(b) = A(b,a).$$

Without loss of generality, assume that $T(X)$ and $S(X)$ are complete. Now, $\{Z_{2n+1}\}, \{Z_{2n+2}\}$ and $\left\{ Z'_{2n+1} \right\}, \left\{ Z'_{2n+2} \right\}$ being respective subsequence of **Cauchy** sequences$\{Z_n\}$ and $\left\{ Z'_n \right\}$ are also **Cauchy**.

Since $T(X)$ is complete, so there exists a, b in X such that:

$$\{Z_{2n+1}\} \to a \text{ and } \left\{ Z'_{2n+1} \right\} \to b$$

Again convergence of the subsequences $\{Z_{2n+1}\}$ and $\left\{ Z'_{2n+1} \right\}$ implies the convergence of original **Cauchy** sequences$\{Z_n\}$ and $\left\{ Z'_n \right\}$ respectively such that

$$\{Z_n\} \to a \text{ and } \left\{ Z'_n \right\} \to b$$

It follows that the sequences $\{Z_n\}$, $\{Z_{2n+1}\}$, $\{Z_{2n+2}\}$ converge to a and $\left\{ Z'_n \right\}$, $\left\{ Z'_{2n+1} \right\}$, $\left\{ Z'_{2n+2} \right\}$ converge to b.

Now $a, b \in T(X)$ implies the existence of $p, q \in X$ such that $T(p) = a$, $T(q) = b$, so that we have

$$\begin{aligned} \lim_{n\to\infty} Z_{2n+1} &= \lim_{n\to\infty} A(x_{2n}, y_{2n}) = \lim_{n\to\infty} T(x_{2n+1}) = a = T(p) \\ \lim_{n\to\infty} Z_{2n+2} &= \lim_{n\to\infty} B(x_{2n+1}, y_{2n+1}) = \lim_{n\to\infty} S(x_{2n+2}) = a = T(p), \end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} \acute{Z}_{2n+1} &= \lim_{n\to\infty} A\left(y_{2n}, x_{2n}\right) = \lim_{n\to\infty} T\left(y_{2n+1}\right) = b = T\left(q\right) \\
\lim_{n\to\infty} \acute{Z}_{2n+2} &= \lim_{n\to\infty} B\left(y_{2n+1}, x_{2n+1}\right) = \lim_{n\to\infty} S\left(y_{2n+2}\right) = b = T\left(q\right),
\end{aligned}$$

From condition **(2)**,

$$F_{A(x_{2n},y_{2n}),B(p,q)}\left(kt\right) \geq Min\left(F_{S(x_{2n}),T(p)}\left(t\right), F_{S(y_{2n}),T(q)}\left(t\right)\right).$$

Letting $n \to \infty$, we get

$$F_{T(p),B(p,q)}\left(kt\right) \geq 1 \text{ this implies } T\left(p\right) = B\left(p,q\right) = a.$$

Similarly $T\left(q\right) = B\left(q,p\right) = b$.

Since the pair $\left(B,T\right)$ are weakly compatible, so that $T\left(p\right) = B\left(p,q\right) = a$ implies $T\left(a\right) = B\left(a,b\right)$, similarly, $T\left(b\right) = B\left(b,a\right)$.

Again, since $S\left(X\right)$ is complete, so that $a, b \in S\left(X\right)$, which implies the existence of r, s in X, so that: $S\left(r\right) = a$, $S\left(s\right) = b$.

from inequality (4.1), we get

$$F_{A(r,s),B(x_{2n+1},y_{2n+1})}\left(kt\right) \geq Min\left(F_{S(r),T(x_{2n+1})}\left(t\right), F_{S(s),T(y_{2n+1})}\left(t\right)\right)$$

Letting $n \to \infty$, we get

$$F_{A(r,s),a}\left(kt\right) \geq 1 \text{ implies } A\left(r,s\right) = a = S\left(r\right)$$

Similarly, $A\left(s,r\right) = b = S\left(s\right)$.

Since the pair $\left(A,S\right)$ are weakly compatible, it follows that:

$$A\left(a,b\right) = S\left(a\right) \ and \ A\left(b,a\right) = S\left(b\right).$$

Step 3:

Next we show $S\left(a\right)=T\left(a\right)$ and $S\left(b\right)=T\left(b\right).$

From inequality (4.1):

$$F_{A(a,b),B(a,b)}\left(kt\right)\geq Min\left(F_{S(a),T(a)}\left(t\right),F_{S(b),T(b)}\left(t\right)\right)$$

and similarly

$$F_{A(b,a),B(b,a)}\left(kt\right)\geq Min\left(F_{S(a),T(a)}\left(t\right),F_{S(b),T(b)}\left(t\right)\right)$$

Thus

$$Min\left\{F_{S(a),T(a)}\left(t\right),F_{S(a),T(a)}\left(t\right)\right\}\geq Min\left\{F_{S(a),T(a)}\left(\frac{t}{k^{n}}\right),F_{S(b),T(b)}\left(\frac{t}{k^{n}}\right)\right\}$$

for all $n\in N$ implying

$$Min\left\{F_{S(a),T(a)}\left(t\right),F_{S(b),T(b)}\left(t\right)\right\}=1.$$

It follows that

$$F_{S(a),T(a)}\left(t\right)=1=F_{S(b),T(b)}\left(t\right)\text{ for all }t>0.$$

Whence $S\left(a\right)=T\left(a\right)$ and $S\left(b\right)=T\left(b\right)$, as claimed.

Therefore,

$$S\left(a\right)=A\left(a,b\right)=B\left(a,b\right)=T\left(a\right)\text{ and }S\left(b\right)=A\left(b,a\right)=B\left(b,a\right)=T\left(b\right).$$

Step 4:

We next show that

$$S\left(a\right)=a\text{ and }S\left(b\right)=b.$$

Indeed, letting $n\to\infty$ in the inequality

$$F_{A(x_{2n},y_{2n}),B(a,b)}\left(kt\right)\geq Min\left(F_{S(x_{2n}),T(a)}\left(t\right),F_{S(y_{2n}),T(b)}\left(t\right)\right).$$

We get

$$F_{a,S(a)}\left(kt\right)\geq Min\left(F_{a,S(a)}\left(t\right),F_{b,S(b)}\left(t\right)\right)$$

and similarly

$$F_{b,S(b)}\left(kt\right) \geq Min\left(F_{a,S(a)}\left(t\right), F_{b,S(b)}\left(t\right)\right).$$

Thus

$$Min\left\{F_{a,S(a)}\left(t\right), F_{b,S(b)}\left(t\right)\right\} \geq Min\left\{F_{a,S(a)}\left(\frac{t}{k^n}\right), F_{b,S(b)}\left(\frac{t}{k^n}\right)\right\},$$

for all $n \in N$, implies

$$Min\left\{F_{a,S(a)}\left(t\right), F_{b,S(b)}\left(t\right)\right\} = 1.$$

It follows that

$$F_{a,S(a)}\left(t\right) = 1 = F_{b,S(b)}\left(t\right) \text{ for all } t > 0.$$

Whence $S\left(a\right) = a$ and $S\left(b\right) = b$.There are :

$$B\left(a,b\right) = S\left(a\right) = a = T\left(a\right) = A\left(a,b\right) \text{ and } B\left(b,a\right) = S\left(b\right) = b = T\left(b\right) = A\left(b,a\right).$$

Step 5:

We shall that $a = b$. Indeed, letting $n \to \infty$ in the inequality

$$F_{A(x_{2n},y_{2n}),B(y_{2n+1},x_{2n+1})}\left(kt\right) \geq Min\left(F_{S(x_{2n}),T(y_{2n+1})}\left(t\right), F_{S(y_{2n}),T(x_{2n+1})}\left(t\right)\right)$$

We get

$$F_{a,b}\left(kt\right) \geq Min\left(F_{a,b}\left(t\right), F_{b,a}\left(t\right)\right)$$

It follows that $F_{a,b}\left(kt\right) \geq F_{a,b}\left(t\right)$ for all $t > 0$, and so $a = b$.

Hence:

$$B\left(a,a\right) = S\left(a\right) = a = T\left(a\right) = A\left(a,a\right).$$

Step 6:

We show that the fixed point is unique.

Let z, w be common fixed points for A, B, S and T, then from (4.1), we obtain

$$F_{A(z,z),B(w,w)}\left(kt\right) \geq Min\left(F_{Sz,Tw}\left(t\right), F_{Sz,Tw}\left(t\right)\right).$$

That is $F_{z,w}(kt) \geq F_{z,w}(t)$ for all $t > 0$ implying $z = w$. ■

If the t-norm Δ is of **Hadžić**-type, then the conditions

$$\sup_{t>0} t^{\alpha} \left(1 - F_{Sx_0,A(x_0,y_0)}(t)\right) < \infty,$$

and

$$\sup_{t>0} t^{\alpha} \left(1 - F_{Sy_0,A(y_0,x_0)}(t)\right) < \infty.$$

can be dropped, so we have the following result :

Corollary 4.1 *Let (X, F, Δ) be a **Menger** metric space with Δ is a t-norm of H-type. Let $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$ and $S : X \to X$ be four mappings satisfying the following condition*

(1) $A(X \times X) \subseteq T(X)$, $B(X \times X) \subseteq S(X)$

(2) *There exists $k \in (0, 1)$ such that*

$$F_{A(x,y),B(u,v)}(kt) \geq Min\left(F_{Sx,Tu}(t),\ F_{Sy,Tv}(t)\right)$$

for all $x, y, u, v \in X$ and $t > 0$

(3) *The pairs (A, S) and (B, T) are w-compatible.*

(4) *One of the subspaces $A(X \times X)$ or $T(X)$ and one of $B(X \times X)$ or $S(X)$ are complete.*

Then, there exists a unique point a in X such that $A(a, a) = S(a) = T(a) = B(a, a) = a$.

Corollary 4.2 *Let (X, F, Δ) be a **Menger** metric space with Δ is a t-norm of H-type. Let $A : X \times X \to X$ be a mapping, and assume that for any $t > 0$; there exists $k \in (0, 1)$ such that*

$$F_{A(x,y),A(u,v)}(kt) \geq Min\left(F_{x,u}(t),\ F_{y,v}(t)\right)$$

for all $x, y, u, v \in X$.

Suppose that $A(X \times X)$ is complete. Then A has a unique coupled fixed point $a \in X$ such that $A(a, a) = a$.

Jian-Zhong Xiao in [26] proved the following result:

Theorem 4.2 *Let* (X, F, Δ) *be a complete* ***Menger*** *metric space with* Δ *is a t-norm of H-type and* $\Delta \geq \Delta_p$. *Let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a gauge function such that* $\varphi^{-1}(\{0\}) = \{0\}$ *and* $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ *for any* $t > 0$.*Let* $A : X \times X \to X$, $T : X \to X$ *be two mappings such that*

$$F_{A(x,y),A(u,v)}(\varphi(t)) \geq [\Delta(F_{Tx,Tu}(t),\ F_{Ty,Tv}(t))]^{1/2}$$

for all $x, y, u, v \in X$ *and* $t > 0$,*where* $A(X \times X) \subseteq T(X)$, T *is continuous and commutative with* A.*Then, there exists a unique* $u \in X$ *such that* $u = Tu = A(u, u)$.

Now, we prove the following result which generalizes the previous theorem for two pairs of weakly compatible mappings:

Theorem 4.3 *Let* (X, F, Δ) *be a* ***Menger*** *metric space with* Δ *is a t-norm of H-type and* $\Delta \geq \Delta_p$. *Let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a gauge function such that* $\varphi^{-1}(\{0\}) = \{0\}$ *and* $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ *for any* $t > 0$.*Let* $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$ *and* $S : X \to X$ *be four mappings satisfying the following condition*

(1) $A(X \times X) \subseteq T(X)$, $B(X \times X) \subseteq S(X)$

(2)

$$F_{A(x,y),B(u,v)}(\varphi(t)) \geq [\Delta(F_{Sx,Tu}(t),\ F_{Sy,Tv}(t))]^{1/2} \tag{4.2}$$

for all $x, y, u, v \in X$ *and* $t > 0$

(3) *The pairs* (A, S) *and* (B, T) *are w-compatible.*

(4) *One of the subspaces* $A(X \times X)$ *or* $T(X)$ *and one of* $B(X \times X)$ *or* $S(X)$ *are complete. Then, there exists a unique point* $a \in X$ *such that* $A(a, a) = S(a) = T(a) = B(a, a) = a$.

Proof Step 1: We show that $\{Z_n\}$ and $\{Z_n'\}$ are **Cauchy** sequences.

By condition **(1)**,we can construct two sequences $\{Z_n\}$ and $\{Z_n'\}$ in X such that

$$\begin{cases} Z_{2n+1} = A(x_{2n}, y_{2n}) = T(x_{2n+1}) \\ Z_{2n+2} = B(x_{2n+1}, y_{2n+1}) = S(x_{2n+2}) \end{cases}$$

$$\begin{cases} Z'_{2n+1} = A\left(y_{2n}, x_{2n}\right) = T\left(y_{2n+1}\right) \\ Z'_{2n+2} = B\left(y_{2n+1}, x_{2n+1}\right) = S\left(y_{2n+2}\right) \end{cases}$$

Suppose that $t > 0$. From (4.2), we have

$$\begin{aligned} F_{Z_n,Z_{n+1}}\left(\varphi\left(t\right)\right) &= F_{Tx_n,Sx_{n+1}}\left(\varphi\left(t\right)\right) \\ &= F_{A(x_{n-1},y_{n-1}),B(x_n,y_n)}\left(\varphi\left(t\right)\right) \\ &\geq \left[\Delta\left(F_{Sx_{n-1},Tx_n}\left(t\right), F_{Sy_{n-1},Ty_n}\left(t\right)\right)\right]^{1/2} \\ &= \left[\Delta\left(F_{Z_{n-1},Z_n}\left(t\right), F_{Z'_{n-1},Z'_n}\left(t\right)\right)\right]^{1/2} \end{aligned} \tag{4.3}$$

$$\begin{aligned} F_{Z'_n,Z'_{n+1}}\left(\varphi\left(t\right)\right) &= F_{Ty_n,Sy_{n+1}}\left(\varphi\left(t\right)\right) \\ &= F_{A(y_{n-1},x_{n-1}),B(y_n,x_n)}\left(\varphi\left(t\right)\right) \\ &\geq \left[\Delta\left(F_{Sy_{n-1},Ty_n}\left(t\right), F_{Sx_{n-1},Tx_n}\left(t\right)\right)\right]^{1/2} \\ &= \left[\Delta\left(F_{Z'_{n-1},Z'_n}\left(t\right), F_{Z_{n-1},Z_n}\left(t\right)\right)\right]^{1/2} \end{aligned} \tag{4.4}$$

Suppose that $G_n\left(t\right) = \left[\Delta\left(F_{Z'_{n-1},Z'_n}\left(t\right), F_{Z_{n-1},Z_n}\left(t\right)\right)\right]^{1/2}$. Then,operating by t-norm Δ on (4.3) and (4.4), from the condition $\Delta \geq \Delta_p$, we obtain

$$G_{n+1}\left(\varphi\left(t\right)\right) \geq \left[\Delta\left(G_n\left(t\right), G_n\left(t\right)\right)\right]^{1/2} \geq \left[G_n\left(t\right) G_n\left(t\right)\right]^{1/2} = G_n\left(t\right) \tag{4.5}$$

Thus, it follows from (4.3) – (4.5) that

$$F_{Z_n,Z_{n+1}}\left(\varphi^n\left(t\right)\right) \geq G_n\left(\varphi^{n-1}\left(t\right)\right) \geq ... \geq G_1\left(t\right) \quad \text{and.} \tag{4.6}$$

$$F_{Z'_n,Z'_{n+1}}\left(\varphi^n\left(t\right)\right) \geq G_n\left(\varphi^{n-1}\left(t\right)\right) \geq ... \geq G_1\left(t\right). \tag{4.7}$$

In the next step we show that $\{Z_n\}$ is a **Cauchy** sequence. For each $\lambda \in (0, 1]$, suppose that

$$D_\lambda = \inf\left\{t > 0 : G_1\left(t\right) > 1 - \lambda\right\}.$$

Then, $G_1\left(D_\lambda + 1\right) > 1 - \lambda$. From (4.6) we see that $F_{Z_n,Z_{n+1}}\left(\varphi^n\left(D_\lambda + 1\right)\right) > 1 - \lambda$. By lemma 1.3, we have

$$d_\lambda\left(Z_n, Z_{n+1}\right) < \varphi^n\left(D_\lambda + 1\right), \quad \text{for each } \lambda \in (0, 1]. \tag{4.8}$$

By Lemma 1.3, for each $\lambda \in (0,1]$ there exists $\mu \in (0,\lambda]$ such that

$$d_\lambda (Z_n, Z_m) \leq \sum_{i=0}^{m-1} d_\mu (Z_i, Z_{i+1}), \quad \text{for all } m, n \in \mathbb{Z}^+ \text{ with } m > n. \tag{4.9}$$

Suppose that $\varepsilon > 0$ and $\lambda \in (0,1]$ are given. Since $\sum_{n=1}^{\infty} \varphi^n (D_\lambda + 1) < +\infty$, there exists $N \in \mathbb{Z}^+$ such that $\sum_{i=0}^{m-1} \varphi^i (D_\mu + 1) < \varepsilon$ for all $m > n \geq N$. Thus, by (4.8) and (4.9), we have $d_\lambda (Z_n, Z_{n+1}) < \varepsilon$. Using Lemma 1.3, we obtain that $F_{Z_n,Z_m} (\varepsilon) > 1 - \lambda$ for all $m > n \geq N$, i.e.,$\{Z_n\}$ is a Cauchy sequence. Similarly, from (4.7) we can show that $\left\{Z_n^{'}\right\}$ is a **Cauchy** sequence.

Step 2: We show that $T(a) = B(a,b), T(b) = B(b,a)$ and $S(a) = A(a,b), S(b) = A(b,a)$

wirhout loss of generality,assume that $T(X)$ and $S(X)$ are complete. Now $\{Z_{2n+1}\}, \{Z_{2n+2}\}$ and $\left\{Z_{2n+1}^{'}\right\}, \left\{Z_{2n+2}^{'}\right\}$ being respective subsequence of **Cauchy** sequences $\{Z_n\}$ and $\left\{Z_n^{'}\right\}$ are also **Cauchy**.

Since $T(X)$ is complete, so there exists a, b in X such that:

$$\{Z_{2n+1}\} \to a \text{ and } \left\{Z_{2n+1}^{'}\right\} \to b$$

Again convergence of the subsequence $\{Z_{2n+1}\}$ and $\left\{Z_{2n+1}^{'}\right\}$ implies the convergence of original **Cauchy** sequences $\{Z_n\}$ and $\left\{Z_n^{'}\right\}$ respectively such that

$$\{Z_n\} \to a \text{ and } \left\{Z_n^{'}\right\} \to b.$$

It follows that the sequences $\{Z_n\}$,$\{Z_{2n+1}\}$,$\{Z_{2n+2}\}$ converge to a and $\left\{Z_n^{'}\right\}$,$\left\{Z_{2n+1}^{'}\right\}$,$\left\{Z_{2n+2}^{'}\right\}$ converge to b.

Now $a, b \in T(X)$ implies the existence of $p, q \in X$ such that $T(p) = a$, $T(q) = b$, so that we have

$$\lim_{n\to\infty} Z_{2n+1} = \lim_{n\to\infty} A(x_{2n}, y_{2n}) = \lim_{n\to\infty} T(x_{2n+1}) = a = T(p) \tag{4.10}$$

$$\lim_{n\to\infty} Z_{2n+2} = \lim_{n\to\infty} B(x_{2n+1}, y_{2n+1}) = \lim_{n\to\infty} S(x_{2n+2}) = a = T(p) \tag{4.11}$$

and

$$\begin{aligned}\lim_{n\to\infty} \acute{Z}_{2n+1} &= \lim_{n\to\infty} A\left(y_{2n}, x_{2n}\right) = \lim_{n\to\infty} T\left(y_{2n+1}\right) = b = T\left(q\right)\\ \lim_{n\to\infty} \acute{Z}_{2n+2} &= \lim_{n\to\infty} B\left(y_{2n+1}, x_{2n+1}\right) = \lim_{n\to\infty} S\left(y_{2n+2}\right) = b = T\left(q\right).\end{aligned}$$

Since $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$, we have $\lim_{n\to\infty} \varphi^n(t) = 0$, so there exists $n_0 \in \mathbb{Z}^+$ such that $\varphi^{n_0}(t) < t$. Thus, from (4.2) we have

$$\begin{aligned}F_{T(x_{2n+1}),B(p,q)}(t) &\geq F_{T(x_{2n+1}),B(p,q)}\left(\varphi^{n_0}(t)\right) = F_{A(x_{2n},y_{2n}),B(p,q)}\left(\varphi^{n_0}(t)\right)\\ &\geq \left[\Delta\left(F_{S(x_{2n}),T(p)}\left(\varphi^{n_0-1}(t)\right), F_{S(y_{2n}),T(q)}\left(\varphi^{n_0-1}(t)\right)\right)\right]^{1/2}\\ &\geq \left[F_{S(x_{2n}),T(p)}\left(\varphi^{n_0-1}(t)\right) F_{S(y_{2n}),T(q)}\left(\varphi^{n_0-1}(t)\right)\right]^{1/2} \qquad (4.12)\end{aligned}$$

Letting $n \to \infty$ in (4.12), we have $\lim_{n\to\infty} T\left(x_{2n+1}\right) = B\left(p,q\right)$. By (4.10), $T\left(p\right) = B\left(p,q\right) = a$. Similarly, we can show that $T\left(q\right) = B\left(q,p\right) = b$.

Since the pair (B,T) are weakly compatible, so that $T\left(p\right) = B\left(p,q\right) = a$ implies $T\left(a\right) = B\left(a,b\right)$, similarly, $T\left(b\right) = B\left(b,a\right)$.

Again, since $S\left(X\right)$ is complete, so that $a, b \in S\left(X\right)$, which implies the existence of r, s in X, so that: $S\left(r\right) = a$, $S\left(s\right) = b$.

Since $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$,we have $\lim_{n\to\infty} \varphi^n(t) = 0$,and so there exists $n_0 \in \mathbb{Z}^+$ such that $\varphi^{n_0}(t) < t$.Thus,from (4.2) we get

$$\begin{aligned}F_{A(r,s),S(x_{2n+2})}(t) &\geq F_{A(r,s),S(x_{2n+2})}\left(\varphi^{n_0}(t)\right) = F_{A(r,s),B(x_{2n+1},y_{2n+1})}\left(\varphi^{n_0}(t)\right)\\ &\geq \left[\Delta\left(F_{S(r),T(x_{2n+1})}\left(\varphi^{n_0-1}(t)\right), F_{S(s),T(y_{2n+1})}\left(\varphi^{n_0-1}(t)\right)\right)\right]^{1/2}\\ &\geq \left[F_{S(r),T(x_{2n+1})}\left(\varphi^{n_0-1}(t)\right) F_{S(s),T(y_{2n+1})}\left(\varphi^{n_0-1}(t)\right)\right]^{1/2} \qquad (4.13)\end{aligned}$$

Letting $n \to \infty$ in (4.13), we have $\lim_{n\to\infty} S\left(x_{2n+2}\right) = A\left(r,s\right)$. By (4.11), $S\left(r\right) = A\left(r,s\right) = a$. Similarly, we can show that $S\left(s\right) = A\left(s,r\right) = b$.

Since the pair (A,S) are weakly compatible, it follows that: $A\left(a,b\right) = S\left(a\right)$ and $A\left(b,a\right) = S\left(b\right)$.

Step 3: We claim that $Ta = b, Tb = a$ and $Sa = b, Sb = a$

In fact, from (4.2) we have

$$\begin{aligned} F_{T(y_{2n+1}),Ta}\left(\varphi\left(t\right)\right) &= F_{A(y_{2n},x_{2n}),B(a,b)}\left(\varphi\left(t\right)\right) \\ &\geq \left[\Delta\left(F_{S(y_{2n}),T(a)}\left(t\right),F_{S(x_{2n}),T(b)}\left(t\right)\right)\right]^{1/2} \\ &\geq \left[F_{S(y_{2n}),T(a)}\left(t\right)F_{S(x_{2n}),T(b)}\left(t\right)\right]^{1/2}. \end{aligned} \tag{4.14}$$

Similarly, we have

$$F_{T(x_{2n+1}),Tb}\left(\varphi\left(t\right)\right) \geq \left[F_{S(x_{2n}),T(b)}\left(t\right)F_{S(y_{2n}),Ta}\left(t\right)\right]^{1/2} \tag{4.15}$$

Suppose that $Q_n\left(t\right) = F_{S(y_{2n}),T(a)}\left(t\right)F_{S(x_{2n}),T(b)}\left(t\right)$. By (4.14) and (4.15), we have $Q_n\left(\varphi\left(t\right)\right) \geq Q_{n-1}\left(t\right)$, hence

$$Q_n\left(\varphi^n\left(t\right)\right) \geq Q_{n-1}\left(\varphi^{n-1}\left(t\right)\right) \geq \ldots \geq Q_0\left(t\right). \tag{4.16}$$

Furthermore, from (4.14) – (4.16) it follows that

$$F_{T(y_{2n+1}),Ta}\left(\varphi^n\left(t\right)\right) \geq \left[Q_0\left(t\right)\right]^{1/2}; \quad \text{and} \quad F_{T(x_{2n+1}),Tb}\left(\varphi^n\left(t\right)\right) \geq \left[Q_0\left(t\right)\right]^{1/2} \tag{4.17}$$

It is evident that $\left[Q_0\left(t\right)\right]^{1/2} \in D^+$. Since $\lim_{n\to\infty}\varphi^n\left(t\right) = 0$, from (4.17) and lemma 1.4 we have

$$\lim_{n\to\infty} T\left(y_{2n+1}\right) = Ta \quad \text{and} \quad \lim_{n\to\infty} T\left(x_{2n+1}\right) = Tb.$$

This shows that $Ta = b$ and $Tb = a$. Hence, $B\left(a,b\right) = b$ and $B\left(b,a\right) = a$.

Similarly, we can show that $Sa = b$ and $Sb = a$. Hence, $A\left(a,b\right) = b$ and $A\left(b,a\right) = a$.

Step 4: Now we prove that $a = b$.

By (4.2) we have

$$\begin{aligned} F_{a,b}\left(\varphi\left(t\right)\right) &= F_{A(b,a),B(a,b)}\left(\varphi\left(t\right)\right) \\ &\geq \left[\Delta\left(F_{S(b),T(a)}\left(t\right),F_{S(a),T(b)}\left(t\right)\right)\right]^{1/2} \\ &\geq F_{a,b}\left(t\right). \end{aligned} \tag{4.18}$$

From (4.18), we have $F_{a,b}(\varphi^n(t)) \geq F_{a,b}(t)$. Using lemma 1.4 , we have $F_{a,b}(t) = 1$,i.e., $a = b$. The uniqueness of a follows from (4.2). So, the proof of theorem 4.3 is finished. ■

Theorem 4.4 *Let (X, F, Δ) be a* ***Menger*** *metric space with Δ is a t-norm of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}, \varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$.Let $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$, $S : X \to X$ be four mappings satisfying the following condition*

(1) $A(X \times X) \subseteq T(X)$, $B(X \times X) \subseteq S(X)$

(2)

$$F_{A(x,y),B(u,v)}(\varphi(t)) \geq [F_{Sx,Tu}(t)\, F_{Sy,Tv}(t)]^{1/2} \tag{4.19}$$

for all $x, y, u, v \in X$ and $t > 0$

(3)*The pairs (A, S) and (B, T) are weakly compatible.*

(4) *One of the subspaces $A(X \times X)$ or $T(X)$ and one of $B(X \times X)$ or $S(X)$ are complete.*

Then, there exists a unique point $a \in X$ such that $A(a, a) = S(a) = T(a) = B(a, a) = a$.

Proof Step1: We show that $\{Z_n\}$ and $\{Z'_n\}$ are **Cauchy** sequences.

By condition **(1)**, we can construct two sequences $\{Z_n\}$ and $\{Z'_n\}$ in X such that

$$\begin{cases} Z_{2n+1} = A(x_{2n}, y_{2n}) = T(x_{2n+1}) \\ Z_{2n+2} = B(x_{2n+1}, y_{2n+1}) = S(x_{2n+2}) \end{cases}$$

$$\begin{cases} Z'_{2n+1} = A(y_{2n}, x_{2n}) = T(y_{2n+1}) \\ Z'_{2n+2} = B(y_{2n+1}, x_{2n+1}) = S(y_{2n+2}) \end{cases}$$

Suppose that $t > 0$.From (4.19),we have

$$\begin{aligned} F_{Z_n,Z_{n+1}}(\varphi(t)) &= F_{Tx_n,Sx_{n+1}}(\varphi(t)) \\ &= F_{A(x_{n-1},y_{n-1}),B(x_n,y_n)}(\varphi(t)) \\ &\geq \left[F_{Sx_{n-1},Tx_n}(t)\, F_{Sy_{n-1},Ty_n}(t)\right]^{1/2} \\ &= \left[F_{Z_{n-1},Z_n}(t)\, F_{Z'_{n-1},Z'_n}(t)\right]^{1/2} \end{aligned} \tag{4.20}$$

$$\begin{aligned} F_{Z'_n,Z'_{n+1}}(\varphi(t)) &= F_{Ty_n,Sy_{n+1}}(\varphi(t)) \\ &= F_{A(y_{n-1},x_{n-1}),B(y_n,x_n)}(\varphi(t)) \\ &\geq \left[F_{Sy_{n-1},Ty_n}(t)\,F_{Sx_{n-1},Tx_n}(t)\right]^{1/2} \\ &= \left[F_{Z'_{n-1},Z'_n}(t)\,F_{Z_{n-1},Z_n}(t)\right]^{1/2} \end{aligned} \tag{4.21}$$

Suppose that $P_n(t) = \left[F_{Z_{n-1},Z_n}(t)\,F_{Z'_{n-1},Z'_n}(t)\right]^{1/2}$ .Then, (4.20) and (4.21) we obtain $P_{n+1}(\varphi(t)) \geq P_n(t)$ .This implies that

$$F_{Z_n,Z_{n+1}}(\varphi^n(t)) \geq P_n\left(\varphi^{n-1}(t)\right) \geq ... \geq P_1(t) \quad \text{and.} \tag{4.22}$$

$$F_{Z'_n,Z'_{n+1}}(\varphi^n(t)) \geq P_n\left(\varphi^{n-1}(t)\right) \geq ... \geq P_1(t). \tag{4.23}$$

Since $P_1(t) = \left[F_{Z_0,Z_1}(t)\,F_{Z'_0,Z'_1}(t)\right]^{1/2} \in D^+$ and $\lim_{n\to\infty}\varphi^n(t) = 0$ for each $t > 0$,by lemma 1.4 we have

$$\lim_{n\to\infty} F_{Z_n,Z_{n+1}}(t) = 1 \text{ and } \lim_{n\to\infty} F_{Z'_n,Z'_{n+1}}(t) = 1 \tag{4.24}$$

Thus,by (4.24),we have

$$\lim_{n\to\infty} P_n(t) = 1 \quad \text{for all } t > 0 \tag{4.25}$$

We claim that,for any $k \in \mathbb{Z}^+$,

$$F_{Z_n,Z_{n+k}}(t) \geq \Delta^k\left(P_n(t-\varphi(t))\right) \text{ and } F_{Z'_n,Z'_{n+k}}(t) \geq \Delta^k\left(P_n(t-\varphi(t))\right). \tag{4.26}$$

In fact,this is obvious for $k = 1$ by (4.20) and (4.21). Assume that (4.26) holds for some k.

Since $\varphi(t) < t$.By (4.20), we have

$F_{Z_n,Z_{n+1}}(t) \geq F_{Z_n,Z_{n+1}}(\varphi(t)) \geq P_n(t)$. By (4.19) and (4.26), we have

$$\begin{aligned} F_{Z_{n+1},Z_{n+k+1}}(\varphi(t)) &\geq \left[F_{Z_n,Z_{n+k}}(t)\,F_{Z'_n,Z'_{n+k}}(t)\right]^{1/2} \\ &\geq \Delta^k\left(P_n(t-\varphi(t))\right). \end{aligned}$$

Hence, by the monotonicity of Δ, we have

$$\begin{aligned} F_{Z_n,Z_{n+k+1}}(t) &= F_{Z_n,Z_{n+k+1}}(t-\varphi(t)+\varphi(t)) \\ &\geq \Delta\left(F_{Z_n,Z_{n+1}}(t-\varphi(t)), F_{Z_{n+1},Z_{n+k+1}}(\varphi(t))\right) \\ &\geq \Delta\left(P_n(t-\varphi(t)), \Delta^k(P_n(t-\varphi(t)))\right) = \Delta^{k+1}(P_n(t-\varphi(t))). \end{aligned}$$

Similarly, we have $F_{Z'_n,Z'_{n+k+1}}(t) \geq \Delta^{k+1}(P_n(t-\varphi(t)))$. Therefor, by induction, (4.26) holds for all $k \in \mathbb{Z}^+$. Suppose that $\varepsilon > 0$ and $\lambda \in (0,1]$ are given.

By hypothesis, Δ is a t-norm of H-type; there exists $\delta > 0$ such that

$$\Delta^k(s) > 1-\lambda, \quad \text{for all } s \in (1-\delta, 1] \text{ and } k \in \mathbb{Z}^+. \tag{4.27}$$

By (4.25) ,there exists $N \in \mathbb{Z}^+$ such that $P_n(\varepsilon - \varphi(\varepsilon)) > 1-\delta$ for all $n \geq N$.

Hence, from (4.26) and (4.27) we get

$$F_{Z_n,Z_{n+k}}(\varepsilon) > 1-\lambda \text{ and } F_{Z'_n,Z'_{n+k}}(\varepsilon) > 1-\lambda, \text{ for all } n \geq N;\ k \in \mathbb{Z}^+.$$

Therefor, $\{Z_n\}$ and $\{Z'_n\}$ are all **Cauchy** sequences.

Step 2: We show that $T(a) = B(a,b), T(b) = B(b,a)$ and $S(a) = A(a,b), S(b) = A(b,a)$

wirhout loss of generality,assume that $T(X)$ and $S(X)$ are complete.Now $\{Z_{2n+1}\}, \{Z_{2n+2}\}$ and $\{Z'_{2n+1}\}, \{Z'_{2n+2}\}$ being respective subsequences of **Cauchy** sequences $\{Z_n\}$ and $\{Z'_n\}$ are also **Cauchy.**

Since $T(X)$ is complete, so there exists a, b in X such that:

$$\{Z_{2n+1}\} \to a \text{ and } \{Z'_{2n+1}\} \to b$$

Again convergence of the subsequences$\{Z_{2n+1}\}$ and $\{Z'_{2n+1}\}$ implies the convergence of original **Cauchy** sequences$\{Z_n\}$ and $\{Z'_n\}$ respectively such that

$$\{Z_n\} \to a \text{ and } \{Z'_n\} \to b.$$

It follows that the sequences $\{Z_n\}$,$\{Z_{2n+1}\}$,$\{Z_{2n+2}\}$ converge to a and $\{Z'_n\}$,$\{Z'_{2n+1}\}$,$\{Z'_{2n+2}\}$ converge to b.

Now $a, b \in T(X)$ implies the existence of $p, q \in X$ such that $T(p) = a$, $T(q) = b$, so that we have

$$\lim_{n\to\infty} Z_{2n+1} = \lim_{n\to\infty} A(x_{2n}, y_{2n}) = \lim_{n\to\infty} T(x_{2n+1}) = a = T(p) \tag{4.28}$$

$$\lim_{n\to\infty} Z_{2n+2} = \lim_{n\to\infty} B(x_{2n+1}, y_{2n+1}) = \lim_{n\to\infty} S(x_{2n+2}) = a = T(p) \tag{4.29}$$

and

$$\begin{aligned}\lim_{n\to\infty} Z'_{2n+1} &= \lim_{n\to\infty} A(y_{2n}, x_{2n}) = \lim_{n\to\infty} T(y_{2n+1}) = b = T(q)\\ \lim_{n\to\infty} Z'_{2n+2} &= \lim_{n\to\infty} B(y_{2n+1}, x_{2n+1}) = \lim_{n\to\infty} S(y_{2n+2}) = b = T(q),\end{aligned}$$

From (4.19) and $\varphi(t) < t$,we obtain

$$\begin{aligned}F_{T(x_{2n+1}),B(p,q)}(t) &\geq F_{T(x_{2n+1}),B(p,q)}(\varphi(t)) = F_{A(x_{2n},y_{2n}),B(p,q)}(\varphi(t))\\ &\geq \left[F_{S(x_{2n}),T(p)}(t)\, F_{S(y_{2n}),T(q)}(t)\right]^{1/2}\end{aligned} \tag{4.30}$$

Letting $n \to \infty$ in (4.30), we have $\lim_{n\to\infty} T(x_{2n+1}) = B(p, q)$. Hence,$T(p) = B(p, q) = a$.

Similarly, we can show that $T(q) = B(q, p) = b$.

Since the pair (B, T) are weakly compatible, so that $T(p) = B(p, q) = a$ implies $T(a) = B(a, b)$, similarly, $T(b) = B(b, a)$.

Again, since $S(X)$ is complete, so that $a, b \in S(X)$, which implies the existence of r, s in X, so that: $S(r) = a$, $S(s) = b$.

Similarly,we can show that $S(s) = A(s, r) = b$ and $S(r) = A(r, s) = a$.

Since the pair (A, S) are weakly compatible, it follows that: $A(a, b) = S(a)$ and $A(b, a) = S(b)$.

Step 3: We claim that $Ta = b, Tb = a$ and $Sa = b, Sb = a$

In fact, from (4.19) we have

$$\begin{aligned} F_{T(y_{2n+1}),Ta}\left(\varphi\left(t\right)\right) &= F_{A(y_{2n},x_{2n}),B(a,b)}\left(\varphi\left(t\right)\right) \\ &\geq \left[F_{S(y_{2n}),T(a)}\left(t\right)F_{S(x_{2n}),T(b)}\left(t\right)\right]^{1/2}. \end{aligned} \tag{4.31}$$

Similarly, we have

$$F_{T(x_{2n+1}),Tb}\left(\varphi\left(t\right)\right) \geq \left[F_{S(x_{2n}),T(b)}\left(t\right)F_{S(y_{2n}),Ta}\left(t\right)\right]^{1/2} \tag{4.32}$$

Suppose that $Q_n\left(t\right) = F_{S(y_{2n}),T(a)}\left(t\right)F_{S(x_{2n}),T(b)}\left(t\right).$

By (4.31) and (4.32), we have

$$Q_n\left(\varphi^n\left(t\right)\right) \geq Q_{n-1}\left(\varphi^{n-1}\left(t\right)\right) \geq \ldots \geq Q_0\left(t\right).$$

$$F_{T(y_{2n+1}),Ta}\left(\varphi^n\left(t\right)\right) \geq \left[Q_0\left(t\right)\right]^{1/2}; \text{ and } F_{T(x_{2n+1}),Tb}\left(\varphi^n\left(t\right)\right) \geq \left[Q_0\left(t\right)\right]^{1/2}$$

Since $\left[Q_0\left(t\right)\right]^{1/2} \in D^+$. Since $\lim_{n\to\infty}\varphi^n\left(t\right) = 0$, by lemma 1.4 we conclude that

$$\lim_{n\to\infty} T\left(y_{2n+1}\right) = Ta \text{ and } \lim_{n\to\infty} T\left(x_{2n+1}\right) = Tb.$$

This shows that $Ta = b$ and $Tb = a$. Hence, $B\left(a,b\right) = b$ and $B\left(b,a\right) = a$.

Similarly, we can show that $Sa = b$ and $Sb = a$. Hence, $A\left(a,b\right) = b$ and $A\left(b,a\right) = a$.

Step 4: Finally, we prove that $a = b$.

By (4.19) we have

$$\begin{aligned} F_{a,b}\left(\varphi\left(t\right)\right) &= F_{A(b,a),B(a,b)}\left(\varphi\left(t\right)\right) \\ &\geq \left[F_{S(b),T(a)}\left(t\right)F_{S(a),T(b)}\left(t\right)\right]^{1/2} = F_{a,b}\left(t\right). \end{aligned} \tag{4.33}$$

From (4.33), we have $F_{a,b}\left(\varphi^n\left(t\right)\right) \geq F_{a,b}\left(t\right)$. Using lemma 1.4, we have $F_{a,b}\left(t\right) = 1$,i.e., $a = b$.The uniqueness of a follows from (4.19). So, the proof of Theorem 4.4 is finished. ■

Theorem 4.5 *Let* (X, F, Δ) *be a* ***Menger*** *metric space with* Δ *is a t-norm of H-type. Let*

$\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}, \varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = +\infty$ for any $t > 0$.Let $A : X \times X \to X,\ B : X \times X \to X,\ T : X \to X,\ S : X \to X$ be four mappings satisfying the following condition

(1) *$A(X \times X) \subseteq T(X),\ B(X \times X) \subseteq S(X)$*

(2)

$$F_{A(x,y),B(u,v)}(t) \geq \min\{F_{Sx,Tu}(\varphi(t)), F_{Sy,Tv}(\varphi(t))\} \tag{4.34}$$

for all $x, y, u, v \in X$ and $t > 0$

(3)*The pairs (A, S) and (B, T) are w-compatible.*

(4) *One of the subspaces $A(X \times X)$ or $T(X)$ and one of $B(X \times X)$ or $S(X)$ are complete. Then, there exists a unique point $a \in X$ such that $A(a, a) = S(a) = T(a) = B(a, a) = a$.*

Proof Step1: We show that $\{Z_n\}$ and $\{Z'_n\}$ are **Cauchy** sequences.

By condition **(1)**,we can construct two sequences $\{Z_n\}$ and $\{Z'_n\}$ in X such that

$$\begin{cases} Z_{2n+1} = A(x_{2n}, y_{2n}) = T(x_{2n+1}) \\ Z_{2n+2} = B(x_{2n+1}, y_{2n+1}) = S(x_{2n+2}) \end{cases}$$

$$\begin{cases} Z'_{2n+1} = A(y_{2n}, x_{2n}) = T(y_{2n+1}) \\ Z'_{2n+2} = B(y_{2n+1}, x_{2n+1}) = S(y_{2n+2}) \end{cases}$$

Suppose that $t > 0$.From (4.34),we have

$$\begin{aligned} F_{Z_n,Z_{n+1}}(t) &= F_{Tx_n,Sx_{n+1}}(t) \\ &= F_{A(x_{n-1},y_{n-1}),B(x_n,y_n)}(t) \\ &\geq \min\left\{F_{Sx_{n-1},Tx_n}(\varphi(t)), F_{Sy_{n-1},Ty_n}(\varphi(t))\right\} \\ &= \min\left\{F_{Z_{n-1},Z_n}(\varphi(t)), F_{Z'_{n-1},Z'_n}(\varphi(t))\right\} \end{aligned} \tag{4.35}$$

$$\begin{aligned} F_{Z'_n,Z'_{n+1}}(t) &= F_{Ty_n,Sy_{n+1}}(t) \\ &= F_{A(y_{n-1},x_{n-1}),B(y_n,x_n)}(t) \\ &\geq \min\left\{F_{Sy_{n-1},Ty_n}(\varphi(t)), F_{Sx_{n-1},Tx_n}(\varphi(t))\right\} \\ &= \min\left\{F_{Z'_{n-1},Z'_n}(\varphi(t)), F_{Z_{n-1},Z_n}(\varphi(t))\right\} \end{aligned} \tag{4.36}$$

Suppose that $E_n(t) = \min\left\{F_{Z'_{n-1},Z'_n}(\varphi(t)), F_{Z_{n-1},Z_n}(\varphi(t))\right\}$.Then, (4.35) and (4.36) we obtain $E_{n+1}(t) \geq E_n(\varphi(t))$.This implies that

$$E_{n+1}(t) \geq E_n(\varphi(t)) \geq E_{n-1}\left(\varphi^2(t)\right) \geq ... \geq E_1(\varphi^n(t)) \quad \text{and.} \tag{4.37}$$

Since $\lim_{t\to+\infty} E_1(t) = \lim_{t\to+\infty} \min\left\{F_{Z'_0,Z'_1}(\varphi(t)), F_{Z_0,Z_1}(\varphi(t))\right\} = 1$ and $\lim_{n\to\infty} \varphi^n(t) = +\infty$ for each $t > 0$,we have

$\lim_{t\to+\infty} E_1(\varphi^n(t)) = 1$.Moreover, by $(4.35) - (4.37)$,we have $F_{Z_n,Z_{n+1}}(t) \geq E_1(\varphi^n(t))$ and $F_{Z'_n,Z'_{n+1}}(t) \geq E_1(\varphi^n(t))$.Hence,$\lim_{t\to+\infty} F_{Z_n,Z_{n+1}}(t) = 1$ and $\lim_{t\to+\infty} F_{Z'_n,Z'_{n+1}}(t) = 1$.This implies that

$$\lim_{t\to+\infty} E_n(t) = 1 \text{ for all } t > 0. \tag{4.38}$$

In the next we show that,for any $k \in \mathbb{Z}^+$,

$$F_{Z_n,Z_{n+k}}(\varphi(t)) \geq \Delta^k(E_n(\varphi(t)-t)) \text{ and } F_{Z'_n,Z'_{n+k}}(\varphi(t)) \geq \Delta^k(E_n(\varphi(t)-t)) \tag{4.39}$$

In fact,this is obvious for $k = 1$ by (4.35) and (4.36). Assume that (4.39) holds for some k.Since $\varphi(t) > t$,by (4.35),we have

$F_{Z_n,Z_{n+1}}(t) \geq E_n(\varphi(t)) \geq E_n(t)$.By (4.34) and (4.39),we have

$$F_{Z_{n+1},Z_{n+k+1}}(t) \geq \min\left\{F_{Z_n,Z_{n+k}}(\varphi(t)), F_{Z'_n,Z'_{n+k}}(\varphi(t))\right\} \geq \Delta^k(E_n(\varphi(t)-t))$$

Hence,by the monotonicity of Δ,we have

$$\begin{aligned} F_{Z_n,Z_{n+k+1}}(\varphi(t)) &= F_{Z_n,Z_{n+k+1}}(\varphi(t)-t+t) \\ &\geq \Delta\left(F_{Z_n,Z_{n+1}}(\varphi(t)-t), F_{Z_{n+1},Z_{n+k+1}}(t)\right) \\ &\geq \Delta\left(E_n(\varphi(t)-t), \Delta^k(E_n(\varphi(t)-t))\right) = \Delta^{k+1}(E_n(\varphi(t)-t)) \end{aligned}$$

Similarly,we have $F_{Z'_n,Z'_{n+k+1}}\left(\varphi\left(t\right)\right)\geq\Delta^{k+1}\left(E_n\left(\varphi\left(t\right)-t\right)\right)$.Therefore, by induction, (4.39) holds for all $k\in\mathbb{Z}^+$.Furthermore, by (4.34) and (4.39) we have

$$F_{Z_n,Z_{n+k}}\left(\varphi\left(t\right)\right)\geq\Delta^{k}\left(E_{n-1}\left(\varphi\left(t\right)-t\right)\right)\text{ and }F_{Z'_n,Z'_{n+k}}\left(\varphi\left(t\right)\right)\geq\Delta^{k}\left(E_{n-1}\left(\varphi\left(t\right)-t\right)\right)\qquad(4.40)$$

Suppose that $\varepsilon>0$ and $\lambda\in(0,1]$ are given.Since Δ is a t-norm of H-type; there exists $\delta>0$ such that

$$\Delta^{k}\left(s\right)>1-\lambda,\quad\text{for all }s\in(1-\delta,1]\text{ and }k\in\mathbb{Z}^+.\qquad(4.41)$$

By (4.38), there exists $N\in\mathbb{Z}^+$ such that $E_{n-1}\left(\varphi\left(\varepsilon\right)-\varepsilon\right)>1-\delta$ for all $n\geq N$.

Hence, from (4.40) and (4.41) we get

$F_{Z_n,Z_{n+k}}\left(\varepsilon\right)>1-\lambda$ and $F_{Z'_n,Z'_{n+k}}\left(\varepsilon\right)>1-\lambda$, for all $n\geq N$ and $k\in\mathbb{Z}^+$.This shows that,$\{Z_n\}$ and $\left\{Z'_n\right\}$ are all **Cauchy** sequences.

Step 2: We show that $T\left(a\right)=B\left(a,b\right),T\left(b\right)=B\left(b,a\right)$ and $S\left(a\right)=A\left(a,b\right),S\left(b\right)=A\left(b,a\right)$

wirhout loss of generality,assume that $T\left(X\right)$ and $S\left(X\right)$ are complete. Now $\left\{Z_{2n+1}\right\},\left\{Z_{2n+2}\right\}$ and $\left\{Z'_{2n+1}\right\},\left\{Z'_{2n+2}\right\}$ being respective subsequences of **Cauchy** sequences $\{Z_n\}$ and $\left\{Z'_n\right\}$ are also **Cauchy**.

Since $T\left(X\right)$ is complete, so there exists a,b in X such that:

$$\left\{Z_{2n+1}\right\}\rightarrow a\ \text{ and }\ \left\{Z'_{2n+1}\right\}\rightarrow b$$

Again convergence of the subsequence $\left\{Z_{2n+1}\right\}$ and $\left\{Z'_{2n+1}\right\}$ implies the convergence of original **Cauchy** sequences$\{Z_n\}$ and $\left\{Z'_n\right\}$ respectively such that

$$\{Z_n\}\rightarrow a\text{ and }\ \left\{Z'_n\right\}\rightarrow b$$

It follows that the sequences $\{Z_n\}$,$\{Z_{2n+1}\}$,$\{Z_{2n+2}\}$ converge to a and $\left\{Z'_n\right\}$,$\left\{Z'_{2n+1}\right\}$,$\left\{Z'_{2n+2}\right\}$ converge to b.

Now $a,b\in T\left(X\right)$ implies the existence of $p,q\in X$ such that $T\left(p\right)=a$, $T\left(q\right)=b$, so that we have

$$\lim_{n\rightarrow\infty}Z_{2n+1}=\lim_{n\rightarrow\infty}A\left(x_{2n},y_{2n}\right)=\lim_{n\rightarrow\infty}T\left(x_{2n+1}\right)=a=T\left(p\right)\qquad(4.42)$$

$$\lim_{n\to\infty} Z_{2n+2} = \lim_{n\to\infty} B\left(x_{2n+1}, y_{2n+1}\right) = \lim_{n\to\infty} S\left(x_{2n+2}\right) = a = T\left(p\right) \tag{4.43}$$

and

$$\begin{aligned} \lim_{n\to\infty} \acute{Z}_{2n+1} &= \lim_{n\to\infty} A\left(y_{2n}, x_{2n}\right) = \lim_{n\to\infty} T\left(y_{2n+1}\right) = b = T\left(q\right) \\ \lim_{n\to\infty} \acute{Z}_{2n+2} &= \lim_{n\to\infty} B\left(y_{2n+1}, x_{2n+1}\right) = \lim_{n\to\infty} S\left(y_{2n+2}\right) = b = T\left(q\right), \end{aligned}$$

From (4.34) we obtain

$$F_{T(x_{2n+1}),B(p,q)}\left(t\right) = F_{A(x_{2n},y_{2n}),B(p,q)}\left(t\right) \geq \min\left\{F_{S(x_{2n}),T(p)}\left(\varphi\left(t\right)\right), F_{S(y_{2n}),T(q)}\left(\varphi\left(t\right)\right)\right\} \tag{4.44}$$

Letting $n \to \infty$ in (4.42), we have $\lim_{n\to\infty} T\left(x_{2n+1}\right) = B\left(p,q\right)$ .Hence,$T\left(p\right) = B\left(p,q\right) = a$ .In the same manner we can show that $T\left(q\right) = B\left(q,p\right) = b$.

Since the pair $\left(B,T\right)$ are weakly compatible, so that $T\left(p\right) = B\left(p,q\right) = a$ implies $T\left(a\right) = B\left(a,b\right)$, similarly, $T\left(b\right) = B\left(b,a\right)$.

Again, since $S\left(X\right)$ is complete, so that $a, b \in S\left(X\right)$, which implies the existence of r, s in X, so that: $S\left(r\right) = a$, $S\left(s\right) = b$.

Similarly,we can show that $S\left(s\right) = A\left(s,r\right) = b$ and $S\left(r\right) = A\left(r,s\right) = a$.

Since the pair $\left(A,S\right)$ are weakly compatible, it follows that: $A\left(a,b\right) = S\left(a\right)$ and $A\left(b,a\right) = S\left(b\right)$.

Step 3: We claim that $Ta = b, Tb = a$ and $Sa = b, Sb = a$

In fact, from (4.34) we have

$$\begin{aligned} F_{T(y_{2n}),Ta}\left(t\right) &= F_{A(y_{2n},x_{2n}),B(a,b)}\left(t\right) \\ &\geq \min\left\{F_{S(y_{2n-1}),T(a)}\left(\varphi\left(t\right)\right), F_{S(x_{2n-1}),T(b)}\left(\varphi\left(t\right)\right)\right\} \end{aligned} \tag{4.45}$$

Similarly,we have

$$F_{T(x_{2n}),Tb}\left(t\right) \geq \min\left\{F_{S(x_{2n-1}),T(b)}\left(t\right), F_{S(y_{2n-1}),Ta}\left(t\right)\right\} \tag{4.46}$$

Suppose that $M_n(t) = \min\left\{F_{T(x_{2n}),Tb}(t), F_{T(y_{2n}),Ta}(t)\right\}$. From (4.43) and (4.44) it follows that

$$M_n(t) \geq M_{n-1}(\varphi(t)) \geq \ldots \geq M_0(\varphi^n(t)).$$

Since $\lim_{n\to\infty} \varphi^n(t) = +\infty$, we have

$$M_0(\varphi^n(t)) = \min\left\{F_{T(x_0),Tb}(t), F_{T(y_0),Ta}(t)\right\} \to 1 \text{ as } n \to \infty$$

This shows that $M_n(t) \to 1$ as $n \to \infty$,and so

$$\lim_{n\to\infty} T(y_{2n+1}) = Ta \quad \text{and} \quad \lim_{n\to\infty} T(x_{2n+1}) = Tb.$$

Hence, $Ta = b$ and $Tb = a$.

Similarly,we can show that $Sa = b$ and $Sb = a$. Hence, $A(a, b) = b$ and $A(b, a) = a$.

Step 4: Finally, we prove that $a = b$.

By (4.34) we have

$$F_{a,b}(t) = F_{A(b,a),B(a,b)}(t) \geq \min\left\{F_{S(b),T(a)}(\varphi(t)), F_{S(a),T(b)}(\varphi(t))\right\} = F_{a,b}(\varphi(t)). \tag{4.47}$$

From (4.45), we have $F_{a,b}(t) \geq F_{a,b}(\varphi^n(t))$.

Letting $n \to \infty$, we have $F_{a,b}(t) = 1$, i.e., $a = b$. Since the uniqueness of a follows from (4.34), the proof of Theorem 4.5 is completed. ■

We give an exemple to illustrate the validity of Theorem 4.5.

Example 4.1 *Let* $X = [0, 1)$ *and*

$$F_{x,y}(t) = \frac{t}{t + |x - y|}.$$

Let $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$ *and* $S : X \to X$ *mappings, such that:*

$$A(x, y) = \begin{cases} \frac{x^2-y^2}{6}, & \textit{if} \quad x \geq y \\ 0, & \textit{if} \quad x < y \end{cases}$$

$$B(x,y)=\begin{cases}\frac{x-y}{6}, & if \quad x\geq y\\ 0, & if \qquad x<y\end{cases}$$

and

$$S(x)=\frac{x^2}{2},\ T(x)=\frac{x}{2}.$$

Clearly, $A(X\times X)\subseteq T(X)$ *and* $B(X\times X)\subseteq S(X)$ *and* $S(X)$ *and* $T(X)$ *are complete subspace of* X*.*

Next we show that our results can be used for this case .

Let us prove that the pairs (A,S) *and* (B,T) *are w-compatible. It is obtained that:*

$$A(x,y)=S(x)\ \ and: A(y,x)=S(y)\ \ if\ and\ only\ if\ x=y=0.$$

Since $A(S(0),S(0))=S(A(0,0))$*, the mappings* A *and* S *are wekly compatible. And*

$$B(x,y)=T(x)\ \ and\ B(y,x)=T(y)\quad if\ and\ only\ if\ \ x=y=0.$$

Since $B(T(0),T(0))=T(B(0,0))$*.*

Finally, we prove that for $x,y,u,v\in X$*,*

$$F_{A(x,y),B(u,v)}(\varphi(t))\geq\Delta(F_{Sx,Tu}(t),F_{Sy,Tv}(t)).$$

Let $\varphi:(0,\infty)\to(0,\infty)$ *by* $\varphi(t)=\frac{3}{2}t$*, Then* $\lim_{n\to+\infty}\varphi^n(t)=+\infty$ *for any* $t>0$*.we distinguish the following cases:*

Case 1: $x\geq y$ *and* $u\geq v$

$$\begin{aligned}F_{A(x,y),B(u,v)}(t) &= \frac{t}{t+\left|\frac{x^2-y^2}{6}-\frac{u-v}{6}\right|}\\ &= \frac{3t}{3t+\left|\left(\frac{x^2}{2}-\frac{u}{2}\right)-\left(\frac{y^2}{2}-\frac{v}{2}\right)\right|}\\ &\geq Min\left\{\frac{\frac{3}{2}t}{\frac{3}{2}t+|x^2-u|},\frac{\frac{3}{2}t}{\frac{3}{2}t+|y^2-v|}\right\}\\ &= Min\{F_{Sx,Tu}(\varphi(t)),F_{Sy,Tv}(\varphi(t))\}.\end{aligned}$$

Case 2: *$x \geq y$,and $u \leq v$*

$$\begin{aligned}
F_{A(x,y),B(u,v)}(t) &= \frac{t}{t+\left|\frac{x^2-y^2}{6}-0\right|} \\
&= \frac{3t}{3t+\left|\frac{x^2-y^2}{2}\right|} \\
&= \frac{3t}{3t+\left|\left(\frac{x^2}{2}-\frac{u}{2}\right)-\left(\frac{y^2}{2}-\frac{u}{2}\right)\right|} \\
&\geq \frac{\frac{3}{2}t}{\frac{3}{2}t+\left|\frac{x^2}{2}-\frac{u}{2}\right|} \\
&\geq Min\left\{F_{Sx,Tu}(\varphi(t)), F_{Sy,Tv}(\varphi(t))\right\}
\end{aligned}$$

Case 3: *$x \leq y$, and $u \geq v$*

$$\begin{aligned}
F_{A(x,y),B(u,v)}(t) &= \frac{t}{t+\left|0-\frac{u-v}{6}\right|} \\
&= \frac{3t}{3t+\left|\left(\frac{x^2}{2}-\frac{u}{2}\right)-\left(\frac{v}{2}-\frac{x^2}{2}\right)\right|} \\
&\geq \frac{\frac{3}{2}t}{\frac{3}{2}t+\left|\frac{x^2}{2}-\frac{u}{2}\right|} \\
&\geq Min\left\{F_{Sx,Tu}(\varphi(t)), F_{Sy,Tv}(\varphi(t))\right\}
\end{aligned}$$

Case 4: *$x \leq y$, and $u \leq v$*

$$\begin{aligned}
F_{A(x,y),B(u,v)}(t) &= \frac{t}{t+0} \\
&= 1 \\
&\geq Min\left\{F_{Sx,Tu}(\varphi(t)), F_{Sy,Tv}(\varphi(t))\right\}
\end{aligned}$$

Hence, all the hypotheses of theorem 4.5 hold. Clearly $(0,0)$ is the unique common coupled fixed point of A, B, S and T.

4.3 Application to integral equations

As an application of the coupled fixed point theorems established in section 2 of this chapter, we study the existence and uniqueness of the solution to a **Fredholm** non-linear integral equation.

We shall consider the following integral equation,

$$x(p)=\int_a^b \left(K_1(p,q)+K_2(p,q)\right)\left[f(q,x(q))+g(q,x(q))\right]dq+h(p), \tag{4.48}$$

for all $p \in I=[a,b]$.

Let Θ denote the set of all functions $\theta:[0,1]\to[0,1]$ satisfying

(i_θ) θ is non-decreasing,

(ii_θ) $\theta(p)\leq p$.

We assume that the functions K_1, K_2, f, g fulfill the following conditions:

Assumption 4.1

(**i**) $K_1(p,q)\geq 0$ and $K_2(p,q)\leq 0$ for all $p,q\in I$,

(**ii**) There exists $\theta\in\Theta$ such that for all $x,y\in\mathbb{R}$ with $x\geq y$, the following conditions hold:

$$0\leq f(q,x)-f(q,y)\leq \lambda\theta(x-y) \tag{4.49}$$

and

$$-\mu\theta(x-y)\leq g(q,x)-g(q,y)\leq 0, \tag{4.50}$$

(**iii**)

$$\max\{\lambda,\mu\}\sup_{p\in I}\int_a^b\left[K_1(p,q)-K_2(p,q)\right]dq\leq\frac{1}{4} \tag{4.51}$$

Consider the integral equation (4.48) with $K_1, K_2\in C(I\times I,\mathbb{R})$ and $h\in C(I,\mathbb{R})$. Suppose that Assumption 4.1 is satisfied. Then the integral equation (4.48) has a unique solution in $C(I,\mathbb{R})$.

Proof Consider $X=C(I,\mathbb{R})$. It is easy to check that $(X,F,*)$ is a complete **Menger** metric space with respect to the distribution distance

$$F_{x,y}(t) = \frac{t}{t + |x - y|}, \text{ for all } x, y \in X \text{ and } t > 0$$

$$\text{with } x * y = \min(x, y) \text{ for all } x, y \in X.$$

Define now the mapping $T : X \times X \to X$ by

$$T(x,y)(p) = \int_a^b K_1(p,q)\left[f(q,x(q)) + g(q,y(q))\right] dq + \int_a^b K_2(p,q)\left[f(q,y(q)) + g(q,x(q))\right] dq + h(p) \tag{4.52}$$

for all $p \in I$ and $k = \frac{1}{2}$ for all $t > 0$. Now,for all $x, y, u, v \in X$, using (4.49) and (4.50), we have

$$\begin{aligned}
& T(x,y)(p) - T(u,v)(p) \qquad (4.53)\\
= & \int_a^b K_1(p,q)\left[f(q,x(q)) + g(q,y(q))\right] dq \\
& + \int_a^b K_2(p,q)\left[f(q,y(q)) + g(q,x(q))\right] dq \\
& - \int_a^b K_1(p,q)\left[f(q,u(q)) + g(q,v(q))\right] dq \\
& - \int_a^b K_2(p,q)\left[f(q,v(q)) + g(q,u(q))\right] dq \\
= & \int_a^b K_1(p,q)\left[f(q,x(q)) - f(q,u(q)) + g(q,y(q)) - g(q,v(q))\right] dq \\
& + \int_a^b K_2(p,q)\left[f(q,y(q)) - f(q,v(q)) + g(q,x(q)) - g(q,u(q))\right] dq \\
= & \int_a^b K_1(p,q)\left[(f(q,x(q)) - f(q,u(q))) - (g(q,v(q)) - g(q,y(q)))\right] dq \\
& - \int_a^b K_2(p,q)\left[(f(q,v(q)) - f(q,y(q))) - (g(q,x(q)) - g(q,u(q)))\right] dq \\
\leq & \int_a^b K_1(p,q)\left[\lambda\theta(x(q) - u(q)) + \mu\theta(v(q) - y(q))\right] dq \\
& - \int_a^b K_2(p,q)\left[\lambda\theta(v(q) - y(q)) + \mu\theta(x(q) - u(q))\right] dq
\end{aligned}$$

Since the function θ is non-decreasing and so we have

$$\theta\left(x\left(q\right)-u\left(q\right)\right)\leq\theta\left(\left|x\left(q\right)-u\left(q\right)\right|\right)$$

and

$$\theta\left(v\left(q\right)-y\left(q\right)\right)\leq\theta\left(\left|v\left(q\right)-y\left(q\right)\right|\right),$$

hence by (4.53), in view of the fact $K_2\left(p,q\right)\leq 0$, we get

$$\begin{aligned}
& \left|T\left(x,y\right)\left(p\right)-T\left(u,v\right)\left(p\right)\right| \qquad (4.54)\\
\leq & \int_a^b K_1\left(p,q\right)\left[\lambda\theta\left(\left|x\left(q\right)-u\left(q\right)\right|\right)+\mu\theta\left(\left|v\left(q\right)-y\left(q\right)\right|\right)\right]dq\\
& -\int_a^b K_2\left(p,q\right)\left[\lambda\theta\left(\left|v\left(q\right)-y\left(q\right)\right|\right)+\mu\theta\left(\left|x\left(q\right)-u\left(q\right)\right|\right)\right]dq\\
\leq & \int_a^b K_1\left(p,q\right)\left[\max\left\{\lambda,\mu\right\}\theta\left(\left|x\left(q\right)-u\left(q\right)\right|\right)+\max\left\{\lambda,\mu\right\}\theta\left(\left|v\left(q\right)-y\left(q\right)\right|\right)\right]dq\\
& -\int_a^b K_2\left(p,q\right)\left[\max\left\{\lambda,\mu\right\}\theta\left(\left|v\left(q\right)-y\left(q\right)\right|\right)+\max\left\{\lambda,\mu\right\}\theta\left(\left|x\left(q\right)-u\left(q\right)\right|\right)\right]dq
\end{aligned}$$

as all the quantities on the right hand side of (4.53) are non-negative. Now by using (4.54), we get

$$\begin{aligned}
& \left|T\left(x,y\right)-T\left(u,v\right)\right| \qquad (4.55)\\
\leq & \max\left\{\lambda,\mu\right\}\int_a^b\left[K_1\left(p,q\right)-K_2\left(p,q\right)\right]dq.\left[\theta\left(\left|x\left(q\right)-u\left(q\right)\right|\right)+\theta\left(\left|v\left(q\right)-y\left(q\right)\right|\right)\right]\\
\leq & \max\left\{\lambda,\mu\right\}\sup_{p\in I}\int_a^b\left[K_1\left(p,q\right)-K_2\left(p,q\right)\right]dq.\left[\theta\left(\left|x\left(q\right)-u\left(q\right)\right|\right)+\theta\left(\left|v\left(q\right)-y\left(q\right)\right|\right)\right]\\
\leq & \frac{\theta\left(\left|x-u\right|\right)+\theta\left(\left|v-y\right|\right)}{4},
\end{aligned}$$

Thus

$$2\left|T\left(x,y\right)-T\left(u,v\right)\right|\leq\frac{\theta\left(\left|x-u\right|\right)+\theta\left(\left|v-y\right|\right)}{2} \qquad (4.56)$$

Now, since θ is nondecreasing, we have

$$\begin{aligned}
\theta\left(|x-u|\right) &\leq \theta\left(|x-u|\right)+\theta\left(|y-v|\right), \qquad (4.57)\\
\theta\left(|y-v|\right) &\leq \theta\left(|x-u|\right)+\theta\left(|y-v|\right),
\end{aligned}$$

which, by using (ii_θ), this implies

$$\begin{aligned}
\frac{\theta\left(|x-u|\right)+\theta\left(|y-v|\right)}{2} &\leq \theta\left(|x-u|+|y-v|\right) \qquad (4.58)\\
&\leq |x-u|+|y-v|\\
&\leq 2\max\left\{|x-u|,|y-v|\right\},
\end{aligned}$$

and so

$$\frac{\theta\left(|x-u|\right)+\theta\left(|y-v|\right)}{2}\leq \max\left\{|x-u|,|y-v|\right\} \qquad (4.59)$$

Thus, by (4.56) and (4.59) , we get

$$2\left|T\left(x,y\right)-T\left(u,v\right)\right|\leq \max\left\{|x-u|,|y-v|\right\} \qquad (4.60)$$

Now, it follows that

$$\begin{aligned}
F_{T(x,y),T(u,v)}\left(kt\right) &= F_{T(x,y),T(u,v)}\left(\frac{t}{2}\right)\\
&= \frac{\frac{t}{2}}{\frac{t}{2}+\left|T\left(x,y\right)-T\left(u,v\right)\right|}\\
&= \frac{t}{t+2\left|T\left(x,y\right)-T\left(u,v\right)\right|}\\
&\geq \frac{t}{t+\max\left\{|x-u|,|y-v|\right\}}\\
&\geq \min\left\{\frac{t}{t+|x-u|},\frac{t}{t+|y-v|}\right\}\\
&\geq \min\left\{F_{x,u}\left(t\right),F_{y,v}\left(t\right)\right\},
\end{aligned}$$

Thus

$$F_{T(x,y),T(u,v)}\left(kt\right)\geq F_{x,u}\left(t\right)*F_{y,v}\left(t\right)$$

show that all hypotheses of corollary 4.2 are satisfied.

This proves that T has a unique fixed point $a \in X$, that is, $a = T\left(a, a\right)$ and therefore $a \in C\left(I, \mathbb{R}\right)$ is the unique solution of the integral equation (4.48). ■

Conclusion

Fixed-point theory is one of the important fields in mathematics that contribute to the solution of many problems, especially non-linear.

In this work, we attempted to present the theory in three different spaces: metric spaces, b-metric spaces and **Menger** metric spaces, with selected contractive properties and applied into solve some integral non-linear equations, several works have been carried out previously in this field, the differentiation between these works can be found: either in the properties of the applications, or in the conditions of contraction, or in the spaces used, with applications in various fields of mathematics and other technical sciences.

Where, after deep research into new spaces for problems, we were able to achieve these results:

1. Weakly compatible maps and common fixed point theorem in metric spaces with application to nonlinear integral equations.

2. New results in common fixed theorem in b-metric spaces with application to linear system and nonlinear integral equations..

3. Common coupled fixed point theorems for two pairs of weakly compatible mappings in **Menger** metric spaces.

From this stand point, this theory remains in need of further development and search for more important and diverse fields of application in mathematics and other sciences.

Bibliography

[1] **M. Aamri**, **D**. **El Moutawakil**, some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl., 270 (2002), 181-188.

[2] **M**. **Abbas**, **M.Ali Khan**, **S. Radenovic.** Common coupled fixed point theorems in cone metric spaces for w-compatible mappings, Applied Mathematics and Computation 217 (2010) 195-202.

[3] **A. Abdelkrim**, **A. Aliouche**, **L. Ben Aoua**, **T. E. Oussaeif**. Common coupled fixed point theorems for two pairs of weakly compatible mappings in Menger metric spaces, Moroccan J. of Pure and Appl. Anal. 5(2), 2019, 197–224.

[4] **M. Akkouchi**, A Common fixed point theorem for expansive mappings under strict implicit conditions on b metric spaces,Acta Univ. Palacki, Olomac Fac. rer. Mathematica, 50, 5-15. (2011).

[5] **A. Aliouche**, **B. Fisher**, A related fixed point theorem for two pairs of mappings in two complete metric spaces, Hacet. J. Math. Stat., 34 (2005), 39-45.

[6] **I. A. Bakhtin**, The contraction mapping principal in quasi-metric spaces, Funct. Anal. Uniatowski Gis. Ped. Inst. 30, 26-37, (1989).

[7] **S. Banach**, On operators in abstract sets and their application to integral equations. Fundam. Math. 3, 133-181 (1922).

[8] **T. G. Bhaskar, V. Lakshmikantham**, Fixed point theorems in partially ordered metric spaces and applications,Non-linear Anal. 65(7) (2006) 1379-1393.

[9] **S. Bhatt, S. Chaukiyal, R. C. Dimri,** A common fixed point theorem for weakly compatible maps in complex-valued metric spaces. Int. J. Math.Sci. Appl. 1(3), 13851389 (2011).

[10] **M. Boriceanu**, **M. Bota**, **A. Petrusel**, Multivalued fractals in b-metric spaces, Cent. Eur. J. Math, 8 (2), 367-377, 2010.

[11] **M. Boriceanu**, Fixed point theory for multivalued generalized contraction on a set with two b- metrics, Studia Univ Babes, Bolya: Math. LIV (3) 1-14. (2009).

[12] **S. S. Chang**, A common fixed point theorem for commuting mappings, Math. Japan 26 (1981) 121–129.

[13] **R. Chugh**, **S. Kumar**, Common fixed points for weakly compatible maps, Proc. Indian Acad. Sci. Math. Sci. 111 (2001) 241–247.

[14] **L.Ciric**, **R.Agarwal**, **B.Samet**, Mixed monotone-generalized contractions in partially ordered probabilistic metric spaces, Fixed Point Theory and Applications 2011,2011:56 doi:10.1186/1687-1812-2011-56. 1.

[15] **L.Ciric**, **D.Mihet**, **R.Saadati**, Monotone generalized contractions in partially ordered probabilistic metric spaces, Topology and its Applications 156(2009), 2838-2844. 1.

[16] **S. Czerwik**, Contraction mappings in b-metric space, Acta Maths. Inf. Univ.Ostraviensis1, 5-11. (1993).

[17] **A. K. Dubey, R. Shukla**, Some fixed point results in b- metric spaces, Asian Journal of Mathematics and applications, vol.2014, Article ID ama0147, 6 pages.

[18] **J. X. Fang**, Common fixed point theorems of compatible and weakly compatible maps in Menger spaces, Nonlinear Analysis.Theory, Methods & Applications 71 (2009),1833-1843. 1.

[19] **L. Gholizadeh**, A fixed point theorem in generalized ordered metric spaces with application. J. Nonlinear Sci. Appl. 6, 244251 (2013).

[20] **S. Gyaz, E. Karapinar, V. Rakocevic, P. Salimi,** Existence of a solution of integral equations via fixed point theorem. J. Inequal. Appl. 2013, 529 (2013).

[21] **O.Hadzic**, **E.Pap**, Fixed Point Theory in Probabilistic Metric Spaces, Kluwer Academic Publishers, Dordrecht,2001. 1.

[22] **O.Hadzic**, **E.Pap**, **M.Budincevic**, Countable extension of triangular norms and their applications to the fixed point theory in probabilistc metric spaces, Kybernetika 38 (3) (2002),363-381. 1.

[23] **R. H. Haghi, S. Rezapour, N. Shahzadb**, Some fixed point generalizations are not real generalizations. Nonlinear Anal. 74, 17991803 (2011).

[24] **N. Hussain, D. Doric, Z. Kadelburg, S. Rednovic,** Suzuki-type fixed point results in metric spaces, Fixed point theory and applications 2012:126(2012).

[25] **M.Jain, S.Kumar, R.Chugh**, Coupled fixed point theorems for weak compatible mappings in fuzzy metric spaces, Annals of fuzzy Mathematics and Informatics (2013) 321-336.

[26] **Jian-Zhong Xiao, Xing-Hua Zhu,Yin-Fang Cao**, Common coupled fixed point results for probabilistic φ-contractions in Menger spaces.Nonlinear Analysis 74 (2011) 4589-4600.

[27] **G. Jungck**, Common fixed points for non-continuous, non-self mappings on a non-numeric spaces, Far East J. Math. Sci. 4 (2) (1996) 199–212.

[28] **G. Jungck**, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9 (1986) 771–779.

[29] **G. Jungck**, **B. E. Rhoades**, Fixed point for set valued functions without continuity, Indian J. Pure Appl. Math., 29, No. 3 (1998), 227-238.

[30] **T. Kamran, M. Samreen**,**Q. UL Ain**, A Generalization of b-Metric Space and Some Fixed Point Theorems, Mathematics 2017, 5, 19; doi:10.3390/math5020019.

[31] **M. Kir, H. ukmi Kiziltunc**, On some well known fixed point theorems in b- metric spaces, Turkish Jj. Of Analysis and Number theory, 1, 13-16. (2013).

[32] **I.Kramosil, J.Michalek**, Fuzzy metrics and statistical metric spaces, Kybernetika 11 (1975),336-344. 1.

[33] **V. Lakshmikantham, L.Ciric**, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis. Theory,Methods & Applications 70 (2009), 4341-4349. 2.

[34] **Y. Liu, Z. Li**, Coincidence point theorems in probabilistic and fuzzy metric spaces, Fuzzy Sets and Systems 158 (2007) 58-70.

[35] **K. Menger**, Statistical metric. Proc. Nat. acad. Sci. U.S.A, 28 (1942), 535–537.

[36] **Z. Mustafa, JR. Roshan, V. Parvaneh, Z. Kadelburg**, Fixed point theorems for weakly T-Chatterjea and weakly T-Kannan contractions in b-metric spaces. J. Inequal. Appl. 2014, Article ID 46 (2014)

[37] **M. O. Olatinwo, C.O. Imoru**, A generalization of some results on multivalued weakly Picard mappings in b- metric space, Fasciculi – Mathematici 40, 45-56(2008).

[38] **M. Pacurar**, A fixed point result for φ – contractions on b- metric spaces without the boundedness assumption, Faasc. Math., 43, 127-137. (2010).

[39] **H. K. Pathak**, **YJ. Cho, S.M. Kang,** Common fixed points of type (A) and applications. Intern. J. Math. Math. Sci. 21, 681694 (1998).

[40] **H. K. Pathak, S.N. Mishra, A.K. Kalinde**, Common fixed point theorems with applications to non-linear integral equations, Demonstratio Math. 32 (3) (1999) 547–564.

[41] **H. K. Pathak, M.S. Khan, Z. Liu, J.S. Ume**, Fixed point theorems in metrically convex spaces and applications, J. Non-linear Convex Anal. 4 (2) (2003) 231–244.

[42] **H. K. Pathak, M.S. Khan, R. Tiwaric**, A common fixed point theorem and its application to non-linear integral equations, Computers and Mathematics with Applications 53 (2007) 961–971.

[43] **R. A. Rashwan, S. M. Saleh,** Solution of nonlinear integral equations via fixed point theorems in G-metric spaces. Int. J. Appl. Math. Res. 3(4), 561571 (2014).

[44] **JR. Roshan, N. Shobkolaei, S. Sedghi, M. Abbas**, Common fixed point of four maps in b-metric spaces. Hacet. J. Math. Stat. 2014 (4), 613 – 624.

[45] **B. Schweizer, A. Sklar**, Probabilistic Metric Spaces, North-Holland, Amsterdam, 1983.

[46] **B. Schweizer, A. Sklar**, Statistical metric spaces, Paci. . . c J.Math. 10 (1960) 313-334.

[47] **B. Schweizer, A. Sklar, E. Thorp**, The metrization of statistical metric spaces, Paci. . . c J.Math. 10 (1960) 673-675.

[48] **S. Sedghi**, **I. Altun,N. Shobe**, Coupled fixed point theorems for contractions in fuzzy metric spaces, Non-linear Analysis.Theory,Methods & Applications 72 (2010), 1298-1304. 1

[49] **S. Sessa**, On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math. Beograd., 32 (46) (1982), 149-153.

[50] **W. Shatanawi**, Some fixed point theorems in ordered G-metric spaces and applications. Abstr. Appl. Anal. 2011, 126205 (2011).

[51] **W. Shatanawi, A. Pitea, R. Lazovic**, Contraction conditions using comparison functions on b-metric spaces. Fixed Point Theory Appl. 2014, Article ID 135 (2014).

[52] **W. Sintunavarat, YJ. Cho, P. Kumam,** Urysohn integral equations approach by common fixed points in complex-valued metric spaces. Adv. Diff. Equ. 2013, 49 (2013).

[53] **W. Sintunavarat,P. Kumam**, Common fixed points for R-weakly commuting in fuzzy metric spaces., Ann.Univ.Ferrara 58,389-406 (2012).

[54] **W. Sintunavarat, P. Kumam,** Generalized common fixed point theorems in complex valued metric spaces and applications. J. Inequal. Appl. 2012 (2012) (article 84).

[55] **W.Sintunavarat, M. B. Zada2, M. Sarwar,** Common solution of Urysohn integral equations with the help of common fixed point results in complex valued metric spaces, RACSAM (2016) doi 10.1007/s13398-016-0309-z.

[56] **N. Wairojjana, W. Sintunavarat, P. Kuman**, Common tripled fixed point theorems for w-compatible mappings along with the CLR_g property in abstract metric spaces, Journal of Inequalities and Applications (2014),2014:133.

[57] **Xin-Qi Hu**, Common Coupled Fixed Point Theorems for Contractive Mapping in Fuzzy Metric Spaces, Fixed Point Theory and Applications Volume 2011, Article ID 363716, doi:10.1155/2011/363716. 1

[58] **Xing-Hua Zhu, Jian-Zhong Xiao**, Note on "Coupled fixed point theorems for contractions in fuzzy metric spaces", Nonlinear Analysis.Theory,Methods & Applications 74 (2011), 5475-5479.

[59] **Xin-Qi Hu, Xiao-Yan Ma**, Coupled coincidence point theorems under contractive conditions in partially ordered probabilistic metric spaces, Non-linear Analysis.Theory,Methods & Applications 74(2011), 6451-6458. 1.