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On New Fixed Point Theorems in Some Generalized Metric Spaces and Their Applications

Presented by:
Besma Laouadi

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In front of the following committee members:

N⁰	First and last name	Grade	University	Quality
01	Mohamed Saadi	M.C.A	University of OEB	President
02	Taki Eddine Oussaeif	M.C.A	University of OEB	Supervisor
03	Leila Benaoua	M.C.B	University of OEB	Co-reporter
04	Hichem Ramoul	M.C.A	University of Khenchela	Examiner
05	Khaled Berrah	M.C.A	University of Tebessa	Examiner
06	Fayçal Merghadi	Prof.	University of Tebessa	Examiner

The theory of Fixed Points is one of the most powerful tools of modern mathematics. Not" only is it used on a daily basis in pure and applied mathematics, but it also serves as a bridge between Analysis and Topology, and provides a very fruitful area of interaction ".between the two

Haim Brézis, Rutgers University

Abstract

This thesis presents new contributions to the study of both unique and non-unique fixed point theorems under rational contractive conditions within several frameworks, including complete metric spaces, complete Menger spaces, and complete b-metric spaces. Our findings advance and improve the results of Khojasteh [76], Demma [39] and Yildirim [126] by integrating various types of contractions introduced by Kannan, Chatterjea, Reich, and Ćirić with the rational contraction to provide weak rational contraction conditions that confirm the existence of fixed points for such mappings.

Additionally, we have developed a novel theorem that examines the distances between fixed points, providing dynamic insights into their other fixed points, if they exist, including the distance between two fixed points in metric spaces, b-metric spaces, and their equivalents in Menger probabilistic metric spaces. These contributions provide significant generalizations of previously established results, such as [9,76,126].

Several mathematical problems have been examined based on the theoretical results obtained, including integral equations, coupled fixed point theorems, and congruence problems.

key words: Metric space, b-metric space, Menger space, Fixed point, Rational type contraction condition, Picard sequence, Coupled fixed point.

[1]

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ملخص

تقدم هذه الأطروحة إسهامات جديدة في دراسة نظريات النقطة الصامدة الوحيدة وغير الوحيدة تحت شروط تقلصية كسرية ضمن عدة أطر تشمل الفضاء المتري التام، فضاء منجر التام، والفضاء البيمتري التام. تطور نتائجنا وتحسن النتائج التي قدمها كل من: 39] Demma ،[76] Khojasteh] و 126] Yildirim]، ، من خلال دمج التقلص الكسري مع أنواع مختلفة من التقلصات التي قدمها Reich، Chatterjea ، Kannan و Ćirić، بهدف وضع شروط كسرية أخف تضمن وجود النقاط الصامدة.

زيادة على ذلك، قمنا بتطوير نظرية جديدة تحاكي البعد بين النقاط الصامدة، وتوفر رؤى ديناميكية حول نقاطها الصامدة الأخرى إن وُجدت، بما في ذلك المسافة بين نقطتين صامدتين في حالة الفضاء المتري، الفضاء البيمتري، وما يعادلهما في فضاءات منجر المترية الاحتمالية. توفر هذه الإسهامات تعميمات هامة للنتائج التي تم إنشاؤها سابقًا، مثل [9، 76، 126].

تم دراسة العديد من المسائل الرياضياتية استنادًا إلى النتائج النظرية التي تم الحصول عليها، بما في ذلك المعادلات التكاملية، ونظريات النقطة الصامدة الثنائية، ومسائل التوافق.

الكلمات المفتاحية: فضاء متري، فضاء بيمتري ، فضاء منجر،النقطة الصامدة، شرط تقلصي من النوع الكسري، متتالية بيكار ، النقطة الصامدة الثنائية.

Résumé

Cette thèse apporte de nouvelles contributions à l'étude des théorèmes de points fixes uniques et non uniques sous des conditions contractives rationnelles, dans plusieurs cadres, y compris les espaces métriques complets, les espaces de Menger complets et les espaces b-métriques complets. Nos résultats développent et améliorent ceux de Khojasteh [76], Demma [39] et Yildirim [126], tout en intégrant diverses contractions introduites par Kannan, Chatterjea, Reich et Ćirić, afin d'établir des conditions contractives rationnelles plus légères garantissant l'existence de points fixes pour ces applications.

De plus, nous avons développé un théorème novateur qui examine les distances entre les points fixes, offrant des perspectives dynamiques sur leurs autres points fixes, s'ils existent, notamment la distance entre deux points fixes dans les espaces métriques, les espaces b-métriques, et leurs équivalents dans les espaces métriques probabilistes de Menger. Ces contributions fournissent des généralisations significatives aux résultats précédemment établis, tels que [9,76,126].

Plusieurs problèmes mathématiques ont été analysés à partir des résultats théoriques obtenus, en particulier les équations intégrales, les théorèmes des points fixes couplés et les problèmes de congruence.

Mots-clés: Espace métrique, espace b-métrique, espace de Menger, point fixe, condition contractive rationnelle, suite de Picard, point fixe couplé.

Dedication

To my beloved parents, whose unwavering support and encouragement have been my constant source of strength.

To my teachers and mentors, whose wisdom and encouragement have inspired me to pursue knowledge and excellence.

And to all my friends who have walked beside me and believed in me during this academic journey, this work is dedicated to you.

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I express my gratitude to all the teachers who contributed to my education, particularly those from the department of mathematics at Oum El Bouaghi University.

Finally, I extend my most sincere thanks to my parents, my family, and all my relatives and friends, who accompanied, helped, supported, and encouraged me throughout the completion of this thesis.

Table of Notations

Symbol	Description
$\mathbb{N}$	set of natural numbers.
$\mathbb{Z}$	set of integers.
$\mathbb{R}$	set of real numbers.
$\mathbb{R}^n$	n-dimensional Euclidean space.
$\mathbb{C}$	set of complex numbers.
$\mathbb{K}$	field (real or complex numbers).
E	vector space over $\mathbb{K}$.
$\dim(E)$	dimension of the vector space E.
σ	metric distance (standard metric).
τ	b-metric distance.
$\|\cdot\|$	norm on a vector space.
$\lvert\cdot\rvert$	absolute value of a scalar.
$\mathcal{A}$	mapping or operator.
$\mathcal{A}^n$	n-th iteration of the mapping $\mathcal{A}$.
$\dot{\xi}, \dot{\eta}$	fixed points.
$\text{Fix}(\mathcal{A})$	set of all fixed points of the mapping $\mathcal{A}$.
μ	measure of non-compactness.
$\alpha(\cdot)$	Kuratowski measure of non-compactness.
$\chi(\cdot)$	Hausdorff measure of non-compactness.
$\mu_s(\cdot)$	supremum measure of non-compactness.
$\text{diam}(W)$	diameter of the set W.
$B(u, r)$	open ball centered at u with radius r.
∂W	boundary of the set W.
$\overline{W}$	closure of the set W.
$\text{conv}(W)$	convex hull of the set W.
$\overline{\text{conv}}(W)$	closed convex hull of W.
$C([0, 1])$	set of all continuous $\mathbb{R}$-valued functions defined on $[0, 1]$.
$\mathcal{P}(W)$	set of all subset of W.
$B(\Upsilon)$	set of all bounded functions defined on Υ.
$L^p(\mathbb{R})$	Lebesgue space for real functions $(0 < p \leq \infty)$.
$l^p(\mathbb{R})$	Lebesgue space for real sequences $(0 < p \leq \infty)$.
$H(W)$	set of holomorphic functions on W.
$\mathcal{B}(E)$	set of bounded subsets of E.
$*$	t-norm (triangular norm in probabilistic spaces).
$F_{u,v}$	distribution function of the pair (u, v).
$\mathcal{F}$	set of all distribution functions.
$H(\cdot)$	Heaviside step function.
$\text{rang}(\mathcal{A})$	image (range) of the mapping $\mathcal{A}$.
$\mathbb{Z}/n\mathbb{Z}$	residue classes modulo n.
$\bar{u}$	equivalence class of u.
$a \equiv b\ [n]$	a is congruent to b modulo n.
Δ	family of all distribution functions on $[-\infty, +\infty]$.
Δ^+	family of all distance distribution functions.
$\mathcal{D}^+$	set of all distribution functions $F \in \Delta^+$ such that $\lim_{t \to +\infty} F(t) = 1$.
(Ω, Σ, P)	probability measure space.

Abbreviations

Abbreviation	Description
PM	Probabilistic Metric spaces.
MNC	Measure of Non-Compactness.
BCP	Banach Contraction Principle.

Contents

Introduction

Fixed point theory is a captivating area of research that intersects analysis, topology, and geometry, yielding significant results that offer substantial applications in various fields, including biology, chemistry, economics, engineering, game theory, and physics.

One of the fundamental contributions in this area is Banach's contraction principle, established by the great Polish mathematician Stefan Banach [11] in 1922, which states that any contraction self mapping on a complete metric space possesses a unique fixed point. This celebrated principle has widespread applications in both pure and applied mathematics, particularly in solving nonlinear problems.It has been generalized by various authors, significantly expanding its applicability across multiple mathematical structures and disciplines.

A compelling illustration of the behavior of fixed points can be seen when using a calculator to compute the cosine of a number repeatedly for radians mode. The output converge to a specific point approximately 0.739085133. This can be viewed as an example of a fixed point, which is the point that remains unchanged when a function is applied. Noted that these fixed point arise from the function $f(x) = \cos(x)$, which is a contraction mapping. A function is considered a contraction mapping if it satisfies the condition $|f(x) - f(y)| \leq k|x - y|$ for all x and y, where $k < 1$. The mean value theorem can be applied to demonstrate the contraction constants for this function is $k = \sin(1)$, then we can applied Banach's contraction principle to guaranty the uniqueness of this fixed point.

In higher dimensions, fixed point theorems continue to provide profound insights into both mathematical theory and practical applications. For example, stirring a cup of coffee or crumpling a sheet of paper reveal that at least one point remains fixed during the transformation.

Despite the widespread fame of Banach's contraction principle in the mathematical world and its diverse applications, it was not the first fixed point theorem in history. The earliest fixed point theorems is attributed to the Dutch mathematician Brouwer [20] and before him Poincaré and others. In 1912, Brouwer proved his famous theorem regarding fixed points for continuous functions on closed and convex sets in finit dimensional spaces. This groundbreaking result laid the foundation for future developments in fixed point theory. Later, Schauder [111] extended this to infinite-dimensional spaces, with his fixed point theorem being applied widely in functional analysis and partial differential equations. Further generalizations by mathematicians such as Darbo [35], Sadovskii [109, 110], Mönch [93], and Krasnosel'skii [79, 80] expanded the scope of fixed point theorems to more intricate spaces and mappings, which further expanded the applicability of fixed point theorems in functional analysis and partial differential equations. As an example, we mention the Schrodinger equation, which is a typical example of a nonlinear partial differential equation. It is often used to describe the propagation of light in nonlinear optical fibers and is of great importance in quantum mechanics. The existence and uniqueness of the solution of the nonlinear Schrödinger equation can be obtained using one of the main mathematical techniques: the fixed point theorems of Banach, Brouwer, Schauder, Schaefer, etc., where both local and global results are traced (see [7, 86, 87, 125]).

The concept of metric spaces, introduced by Fréchet [48] in 1906, has undergone various generalizations over the past century. These generalizations arose from the need to address more complex problems that couldn't be tackled using traditional metric spaces.

Our study focuses on two key generalizations: b-metric spaces, introduced by Bakhtin [10] in 1989, and Menger PM spaces, introduced by Menger [88] in 1942. b-metric spaces relax the strict conditions of standard metric spaces, providing greater flexibility when dealing with non-continuous mappings. Numerous studies have investigated the topological properties of these two spaces, clarifying their differences from traditional metric spaces. For instance, the b-metric function is not necessarily continuous, as demonstrated by a counterexample. On the other hand, Menger PM spaces generalize the concept of distance through probabilistic functions, which enable the analysis of random and probabilistic fluctuations in applications like biology and physics. The goal of these generalizations is to give us stronger and more adaptable tools to deal with complicated nonlinear phenomena. This will make it easier to solve problems that traditional metric spaces could not handle, especially in the study of fixed point problems.

The essence of the fixed point theory lies in its versatility, bridging various mathematical disciplines like analysis, topology, and geometry, also finding applications in numerous fields such as biology, chemistry, economics, engineering, and game theory. With the emergence of b-metric spaces and Menger PM spaces, fixed point theory has evolved further, accommodating non-traditional metric structures as the work of Czerwik [32, 33], Sehgal [114], and Schweizer-Sklar [113], and others. These structures have significantly expanded the applicability of fixed point results, allowing researchers to address nonlinear problems with more flexible frameworks and opening up new avenues for problem-solving in both pure and applied mathematics.

The concept of coupled fixed points, first introduced by Guo and Lakshmikantham [52], has proven instrumental in addressing complex mathematical problems, particularly in nonlinear systems. Bhaskar and Lakshmikantham [15] were among the first to apply coupled fixed point theorems to establish the existence of a unique solution for periodic boundary value problems. Since then, numerous researchers have explored the potential applications of coupled fixed point theorems, significantly broadening their scope in areas such as differential systems, optimization, integral systems, and dynamic systems (see [18, 30, 116]). In this thesis, we propose a new theorem within this framework, contributing further to the development of coupled fixed point theorems and their applications.

The fixed point theory is of great importance to other fields, including: optimization (e.g., Cebysev approximation, control of rockets, game theory, and dual problems); quantum statistics (the C*-algebra approach); quantum field theory (the Fock space); quarks in elementary particle physics; gauge field theory (the Yang-Mills-Dirac equations), string theory, mathematical physics (see Zeidler[130]), mathematical economics (see Zeidler[129]). Because of its effective role in dealing with linear and nonlinear problems, including: ordinary differential equations, linear and nonlinear integral equations; variational problems, partial differential equations, partial differential equations of mathematical physics (e.g., the Laplace equation, the heat equation, the wave equation, and the Schrodinger equation); time evolution equations and mild solutions [123]; boundary-value problems and obstacle problems in nonlinear elasticity (see [89, 131–133]), the N-body problem in celestial mechanics; minimal surfaces and harmonic maps; superfluids, superconductors, and phase transition (the Landau-Ginzburg model); viscous fluids (the Navier-Stokes equations) (see [24, 63, 65, 123] and references therein).

Banach's theorem was a major contribution to the field of fixed point theory, playing a significant role in solving many nonlinear problems. Over the years, these theorems have been generalized in various ways, thanks to contributions from mathematicians such as Kannan [67, 69], Reich [106], Chatterjea [25], Hardy and Rogers [55], Ćirić [26, 27], Berinde [13], Suzuki [119] and others, who extended the application of fixed point theorems to include certain discontinuous functions, unlike Banach's original theorem.

Over the years, these generalizations have continued to evolve. The most important ones were made by Dass and Gupta [37], Jaggi [60], and Khan [74] (updated by Fisher [46]) in 1975. They set up conditions for rational contractions and found some nonuniqueness fixed point results that allowed for multiple fixed points in certain cases. Over the past decade, several variations of rational contracting have been proposed within numerous frameworks. These can be found in literature such as [8, 49, 57, 70, 91, 92, 97, 103] and the accompanying references.

In 2014, Khojasteh [76] presented a non-uniqueness fixed point result under rational contractive conditions, incorporating dynamic relations within the set of fixed points for such mappings. Building on this foundation, Yildirim [126] and Demma [39] generalized Khojasteh's findings by modifying the rational contraction component to enhance the dynamic information regarding the separating distance between fixed points. Additionally, Demma [39] extended Khojasteh's result naturally to b-metric spaces. Our work aligns with this context, offering broader and more general results, along with more precise dynamic information. Furthermore, we integrate these findings with other contraction types, such as Ćirić and Bianchi, and extend them to Menger probabilistic metric spaces and b-metric spaces.

This thesis is divided into five chapters as follows:

The first chapter provides the necessary background concerning metric spaces, b-metric spaces, and Menger probabilistic metric spaces, along with key lemmas and properties that play a crucial role in the subsequent chapters.

The second chapter offers an overview of notable results in fixed point theory, tracing their development from foundational contributions to contemporary generalizations. We highlight key theorems, such as the Schauder fixed point theorem and Banach's contraction principle, along with their historical extensions. This chapter also presents several generalized contractions introduced between 1912 and 2017, focusing on the most significant findings, including Kannan's contraction, Reich's contraction, Hardy and Rogers' contraction, Ćirić's quasi-contraction, and Suzuki's contraction, among others. Furthermore, we discuss several nonlinear rational contractions and their corresponding unique and non-unique fixed point theorems.

In the third chapter, we present our own results on unique and non-unique fixed point theorems for self-maps on complete metric spaces and under specific rational contractive conditions, incorporating dynamic properties about additional fixed points for such mapping. Our work is inspired by contributions from Khojasteh [76], Yildirim [126], Demma [39], Kannan [67, 69], Ćirić [26, 27], Bianchini [16], and others. In parallel, several examples are provided to validate our assumptions and demonstrate the applicability of the theorems. Furthermore, we address congruence problems and employ dynamic results to analyze the characteristics of solutions, particularly in cases where the solution is not unique. These results were published in [82, 84].

In the fourth chapter, we present some of the most significant and well-known fixed point theorems within the field of Menger probabilistic spaces, extending Banach's theorem to Menger probabilistic metric spaces. In addition, we introduce a novel theorem that simulates nonunique fixed point theorems presented in chapter 3 as well as the equvalence of dynamic results for Menger spaces. This theorem explores the separation distance between fixed points within the framework of probabilistic Menger spaces, providing dynamic information about other fixed points, if they exist, and specifically integrating dynamic properties into Menger probabilistic metric spaces. This groundbreaking result, published in [82], is the first of its kind in this regard.

The fifth and final chapter summarizes the findings from our published work (Laouadi et al. [12, 83]) in the setting of b-metric spaces, focusing on establishing improved fixed point results under weaker rational contractive conditions. In this chapter, we extend several theorems to complete b-metric spaces, based on the results presented in the second chapter. A significant part of this study was devoted to the analysis of the distances between fixed points in b-metric spaces, offering dynamic insights into the existence and properties of additional fixed points. Our work enhances the results of Khojasteh [76], Yildirim [126], Demma [39] by introducing new generalizations by combining rational contractions with Kannan's and, in some cases, Ćirić's or Chatterjea's contractions. This approach allows us to formulate more generalized rational contraction conditions, which enabled us to derive new fixed point theorems with enhanced dynamic properties. These results have practical applications, including solving integral equations and establishing coupled fixed point theorems. Notably, the results published in [83] provide a true generalization of [9, 76], as our work identifies limitations in their findings when applied to specific examples discussed in this chapter.

Chapter 1 Preliminaries

In this chapter, we provide the necessary background concerning metric spaces, b-metric spaces, and Menger probabilistic metric spaces, along with key lemmas and properties that play a crucial role in the subsequent chapters.

1.1 Basic notions of metric spaces

This section is concerned with metric spaces and their topological properties.

Throughout this thesis, it is assumed that every set is non-empty.

1.1.1 The origins of the metric space

In 1906, Maurice René Frechét[48] stated the axioms of metric space in his thesis, which he referred to as "L-space".However, The concept of the metric could be interpreted as an extension of the distance between two points, which was originally proposed by Euclid. As well, the term "metric space" was first used by Felix Hausdorff[56].

Maurice René Frechét

Definition 1.1 *A function $\sigma : \Upsilon \times \Upsilon \to [0,\infty)$ is considered a metric if it satisfies the following axioms over the set Υ.(i.e. for every $u, v, w \in \Upsilon$)*

$(\sigma_1)\, \sigma(u,v) = 0 \Longrightarrow u = v$

$(\sigma_2)\, \sigma(u,u) = 0$

$(\sigma_3)\, \sigma(u,v) = \sigma(v,u)$

$(\sigma_4)\, \sigma(u,w) \leq \sigma(u,v) + \sigma(v,w)$

More precisely, the pair (Υ,σ) is referred to as a metric space.

1.1.2 Examples of metric spaces

Example 1.1 *Each of the following functions represents a metric on the set $\mathbb{R}$, for every $u, v \in \mathbb{R}$:*

1. $\sigma_0(u,v) = |u - v|$, *(standard metric or Euclidean metric)*

2. $\sigma_1(u,v) = |\arctan u - \arctan v|$,

3. $\sigma_2(u,v) = \frac{|u-v|}{1+|u-v|}$,

4. $\sigma_4(u,v) = \max\{|u|,|v|\}$,

5. $\sigma_5(u,v) = \begin{cases} 0 \text{ if } u = v, \\ 1 \text{ otherwise.} \end{cases}$

Listed below are some classic examples in the functional analysis frameworks.

Example 1.2 *The metric space $(B(\Upsilon, Y), \sigma_\infty)$ is the space of bounded functions $f : \Upsilon \to Y$ equipped with the uniform metric*

$$\sigma_\infty(f,g) = \sup\{\sigma(f(u), g(u)) : u \in \Upsilon\}.$$

It is straightforward to check that σ_∞ is a metric on $B(\Upsilon, Y)$; in particular, $\sigma_\infty(f,g) < \infty$ if f,g are bounded functions.

Example 1.3 *Let $\Upsilon = C[0,1]$ be the set of all continuous $\mathbb{R}$-valued functions that are defined on the interval $[0,1]$. We define some interesting metrics on Υ as follows:*

1. $\sigma_1(f,g) = \int_0^1 |f(u) - g(u)| du$. *In terms of geometry, it represents the area bounded by the graphs of the two functions.*

2. $\sigma_2(f,g) = \sqrt{\int_0^1 (f(u) - g(u))^2 du}$.

1.1.3 Topological Properties of the Metric Spaces

In this section, we provide a concise overview of fundamental topological concepts for the metric spaces. We consider (Υ, σ) to be a metric space. This space is endowed with a natural topology. It is essentially unique and may be defined by taking the sets

$$B(u,r) = \{v \in \Upsilon : \sigma(u,v) < r\}$$

for $u \in \Upsilon$ and $r > 0$ as neighborhoods.

Convergence-compactness-completeness-continuity

Definition 1.2 *Let $\{u_n\}_{n=1}^\infty$ be a sequence in a metric space (Υ, σ), We say that:*

(a) The sequence $\{u_n\}_{n=1}^\infty$ converges to $u^* \in \Upsilon$ iff $\lim_{n\to\infty} \sigma(u_n, u^*) = 0$, and it is denoted as $\lim_{n\to\infty} u_n = u^*$ or $u_n \to u^*$ as $n \to \infty$.

(b) The sequence $\{u_n\}_{n=1}^\infty$ is fundamental or Cauchy sequence iff for each $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$, such that $\sigma(u_n, u_m) \le \varepsilon$, for each $n, m \ge n_0$ i. e. $\lim_{n,m\to\infty} \sigma(u_n, u_m) = 0$.

(c) The metric space (Υ, σ) is complete iff every Cauchy (fundamental) sequence in Υ converges to a point within Υ.

Theorem 1.1 *Let Y be a complete metric space. Then $(B(\Upsilon, Y), \sigma_\infty)$ is a complete metric space.*

Lemma 1.1 *Consider a Cauchy sequence $\{u_n\}$ in a metric space (Υ, σ), which need not be complete. If the sequence $\{u_n\}$ includes a convergent sub-sequence, then the sequence $\{u_n\}$ is also convergent and tends to the same limit.*

Definition 1.3 *A function $\mathscr{A} : \Upsilon \to \Upsilon$ is sequentially continuous at $a \in \Upsilon$ if for every sequence $\{u_n\}$ converges to a, $\mathscr{A}(u_n) \to \mathscr{A}(a)$.(i.e. $\lim_{n\to\infty} \mathscr{A}(u_n) = \mathscr{A}(\lim_{n\to\infty} u_n)$).*

Theorem 1.2 *(Υ, σ) is a metric space. A function $\mathscr{A} : \Upsilon \to \Upsilon$ is continuous at $a \in \Upsilon$ if and only if it is sequentially continuous at a.*

Definition 1.4 *A function $\mathscr{A} : \Upsilon \to \Upsilon$ is continuous if $\mathscr{A}$ is continuous at any point $u \in \Upsilon$.*

Next, we give the formal definition of compact metric space.

Definition 1.5 *If every open cover of a metric space (Υ, σ) has a finite sub-cover, then the corresponding metric space (Υ, σ) is called compact. In addition, we say that metric space (Υ, σ) is sequentially compact if each sequence of points in Υ has a convergent sub-sequence converging to a point in Υ.*

Note that in metric space (Υ, σ), the compactness of Υ is equivalent to the following statement: Each sequence in Υ has a convergent sub-sequence.

Lemma 1.2 *Every compact metric space is complete.*

Proof according to Lemma 1.1. ■

Definition 1.6 *[75] We say that a self-mapping $\mathscr{A} : \Upsilon \to \Upsilon$ on a metric space (Υ, σ) is asymptotically regular at a point $u \in \Upsilon$ if $\lim_{n\to\infty} \sigma\left(\mathscr{A}^n(u), \mathscr{A}^{n+1}(u)\right) = 0$*

It is worth noting that the sequence being asymptotically regular does not suffice to guarantee that it is a Cauchy sequence, as illustrated in the following example:

Example 1.4 *Consider $(\mathbb{R}, \sigma)$ a metric space with $\sigma(u, v) = |u - v|$. Let $\{u_n\}$ be a sequence defined as $u_n = \log(n)$.*

The sequence $\{u_n\}$ is asymptotically regular since

$$\sigma\left(u_n, u_{n+1}\right) = \log\left(1 + \frac{1}{n}\right) \to 0 \text{ as } n \to \infty.$$

However, the sequence $\{u_n\}$ is not a Cauchy sequence, as $\lim_{n\to\infty} u_n = \infty$.

Compact maps - completely continuous maps

The term "**compact**" in this context of mappings refers to maps that transform any bounded set into a set that is relatively compact. Throughout the main text, we reserve the term "compact" for operators that not only exhibit this property but also are continuous. In the authors' terminology, these are referred to as "**completely continuous operators**".

Condensing maps - measures of non-compactness

Condensing maps The definition of a condensing operator appeared as a natural generalization of the definitions of compact and contracting operators. A condensing operator, also known as a densifying operator, is a type of mapping where the image of any set becomes more compact than the set itself. To quantify the extent of non-compactness of a set, we employ functions known as **measures of non-compactness**.(see the next section)

Definition 1.7 *[110]*

Let E be a Banach space associated with a measure of non-compactness μ. Let $\mathscr{A}$ be an operator on E.

(1) The operator $\mathscr{A}$ is called a (k, μ)-set contraction iff

$$\mu(\mathscr{A}(W)) \leq k\mu(W),$$

for any $W \subseteq D_{\mathscr{A}}$, where $k \in [0,1)$ is a constant.

(2)The operator $\mathscr{A}$ is called μ-condensing iff

$$\mu(\mathscr{A}(W)) < \mu(W),$$

for every bounded subset $W \subseteq D_{\mathscr{A}}$ with $\mu(W) > 0$ (i.e W is not relatively compact).

In other words, $\mathscr{A}$ is μ-condensing if and only if it decreases the measure of non-compactness of any set $W \subseteq D_{\mathscr{A}}$ whose closure is not compact.

Remark 1.1

- The concept of (k,μ)-set contraction and μ-condensing operator was initially introduced for the Kuratowski measure of non-compactness α. Later, it was generalized for an arbitrary measure of non-compactness.

- Every (k,μ)-set contraction operator is μ-condensing, as was shown in [96], the converse is not true. However, every condensing map is 1-set contraction.

Example 1.5

- $\mathscr{A}$ is compact operator, then, $\mathscr{A}$ is 0 -set-contraction.

- $\mathscr{A}$ is k-contraction, then $\mathscr{A}$ is k-set-contraction with the same constant.

- The contracting maps and the completely continuous maps, and also sums of operators of these two types are condensing operators.

- For contracting maps was a condensing operator one can take as measure of non-compactness the diameter of a set or the Kuratowski MNC α, while for completely continuous maps can take the indicator function of a family of non-relatively compact sets or the Kuratowski MNC α.

- Contracting operators are not necessarily condensing with respect to Hausdorff measure of non-compactness χ.

Many theorems about fixed points of completely continuous operators generalize to condensing operators. For more detail about condensing and k-set contraction operators and their relation to fixed point theory see [5, 6, 36, 96, 110].

Measures of non-compactness

In 1930, K. Kuratowski[81] introduce a quantitative characteristic, denoted as α, which measures the degree of non-compactness of subset of metric space. His work was closely tied to problems in general topology. In the 1950s, several researchers including G. Darbo, L. S. Gol'denshtein, I. Gohberg, A. S. Markus, W. V. Petryshyn, A. Furi, A. Vignoli, J. Danes, Yu. G. Borisovich, Yu. I. Sapronov, M. A. Krasnosel'skil, P. P. Zabrelko, and others extended these ideas by applying various measures of non-compactness in the fields of fixed-point theory, as well as the theory of differential and integral equations.

Here, we present several standard examples for measures of non-compactness.

For any subset W of a Banach space E:

(1) The Kuratowski measure of non-compactness:

$$\alpha(W) = \inf\left\{ r_j > 0 : W \subset \bigcup_{i=1}^{n} A_i, \operatorname{diam}(A_i) \le r_j \right\}$$

$\alpha(W)$ can be regarded as a characteristic of the extent to which W is not compact.

(2) The Hausdorff measure of non-compactness (which were introduced by Goldenstein et al. [51]) of W as:

$$\chi(W) = \inf\left\{ r > 0 : W \subset \bigcup_{i=1}^{n} \overline{B}(u_i, r), u_i \in E \right\}$$

(3) The diameter of W as : (here E is a normed pace)

$$\mu_d(W) = \text{diam}(W) = \sup\{\|u - v\|, u, v \in W\}$$

(4) The supremum of W as : (here E is a normed pace)

$$\mu_s(W) = \sup\{\|u\|, u \in W\}.$$

For more details about the examples of the measures of noncompactness mentioned above, as well as other examples and their basic properties, we refer the reader to [6, 110].

Next, we present a general definition of the measure of non-compactness along with mentioning some of its types.

Definition 1.8 *Let $(E, \|\cdot\|)$ be a Banach space. A map $\mu : \mathscr{B}(E) :\to [0, \infty)$ is called a measure of non-compactness (MNC for short) on E if it satisfies the following properties :*

(i) $\mu(W) = \mu(\overline{\text{conv}}(W))$, where $\overline{\text{conv}}(W)$ is the closed convex hull of W.

(ii) $\mu(W) = 0$ implies that W is relatively compact.

Let μ be a measure of non-compactness on E. Then,

(1) μ is monotone, if $W_1 \subset W_2$ implies $\mu(W_1) \le \mu(W_2)$.

(2) μ is satisfy the generalized Cantor intersection property if for any decreasing sequence $\{W_n\}$ of nonempty, closed and bounded subsets of E such that $\lim_{n\to\infty} \mu(W_n) = 0$, then the intersection W_∞ of all W_n is nonempty.

(3) μ is called nonsingular if $\mu(W \cup \{u_0\}) = \mu(W)$ for any bounded set $W \subset E$ and any $u_0 \in E$.

Remark 1.2 *- The most frequently used measures of non-compactness do indeed characterize the extent to which a set is not compact; this justifies the choice of the term **"measure of non-compactness"**,*

- If μ is a monotone MNC on a Banach space E, then for any bounded set $W \subset E$ we have

$$\mu(W) \le \mu(\bar{W}) \le \mu(\overline{\text{conv}}(W)) = \mu(W)$$

which leads to $\mu(\bar{W}) = \mu(W)$.

- The aforementioned classical examples of MNCs exhibit monotonicity and fulfill generalized Cantor intersection property, while the two final ones do not achieve nonsingularity.

1.1.4 Normed space

Semi-norms or norms

A semi-norm on a (real or complex) vector space E is a function $p : E \to \mathbb{R}$ such that

1. $p(u + v) \le p(u) + p(v)$,
2. $p(\lambda u) = |\lambda| p(u)$,

for all $u, v \in E$ and all scalar λ. A semi-norm p is easily seen to fulfill the following properties:

- $p(0) = 0$.
- $p(u) \ge 0$.
- $|p(u) - p(v)| \le p(u - v)$.
- The sets of the form $\{u \in E : p(u) < \varepsilon\}$ for some $\varepsilon > 0$ are convex.

As a consequence of the first two properties above, we deduce that if $p(u) \neq 0$ whenever $u \neq 0$, then p is actually a norm.

First, we recall the notion of vector space. Let E be a nonempty set. We say that E is a vector space if it is closed under scalar multiplication and finite vector addition. One of the well-known and remarkable examples of a vector space is n-dimensional Euclidean space $\mathbb{R}^n$. In this construction, each element is represented by a list of n real numbers, scalars are real numbers (in the general case, the scalars are members of a field $\mathbb{K}$), scalar multiplication is multiplication on each term separately, and addition is componentwise. We generally say that E is a vector space over $\mathbb{K}$.

In what follows, we recall the concept of a normed space.

Definition 1.9 *A normed space is a pair $(E, \|\cdot\|)$ of a vector space E and a function $\|\cdot\| : E \to [0,\infty)$, called norm on E, such that*

1. *For every $u \in E$, we have $\|u\| = 0$ if and only if $u = 0$;*
2. *For every scalars λ and every $u \in E$, we have that $\|\lambda u\| = |\lambda|\|u\|$;*
3. *$\|u+v\| \leq \|u\| + \|v\|$ for every $u, v \in E$ (the triangle inequality).*

Relation Between normed and metric spaces

Notice that for a normed space E, a function $\sigma E \times E \to [0,\infty)$, defined as,

$$\sigma(u, v) := \|u - v\|, \text{ for each } u, v \in E \tag{1.1.1}$$

forms a metric on E with two crucial properties:

1. Translation invariance: $\sigma(u, v) = \sigma(u + w, v + w)$ for each $w \in E$;
2. Homogeneity: $\sigma(cu, cv) = |c|\sigma(u, v)$ for each $c \in \mathbb{K}$, where $\mathbb{K}$ can be $\mathbb{R}$ or $\mathbb{C}$.

Thus, all the notions that are defined for metric spaces can be applied to normed linear spaces.

We shall also consider the following metric that is obtained by a norm:

$$\sigma_b(u, v) := \frac{\|u - v\|}{1 + \|u - v\|}, \text{ for each } u, v \in E$$

Note that the values of the new metric $\sigma_b(u, v) < 1$ for each $u, v \in E$.

Definition 1.10 *A subset $\mathscr{C}$ of a normed space E is called convex if $\lambda u + (1-\lambda)v \in \mathscr{C}$, for each $u, v \in \mathscr{C}$ and real number $\lambda \in [0,1]$. In other words, the subset $\mathscr{C}$ is convex if the line segment joining any two points in the set lies in the set.*

As a first observation, the unit ball is convex.

Example 1.6 *Let $E = \mathbb{R}^n$ be a vector space. Here, E forms a normed space equipped with the norm*

$$\|u\|_2 := \sqrt{\sum_{i=1}^{n} u_i^2}$$

where $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$. The pair $(\mathbb{R}^n, \|\cdot\|)$ is called n-dimensional Euclidean space with Euclidean norm $\|\cdot\|_2$.

Furthermore, the 1-norm and the maximum norm are defined respectively as

$$\|u\|_1 := \sum_{i=1}^{n} |u_i|$$

$$\|u\|_\infty := \max\{|u_1|, |u_2|, \ldots, |u_n|\}.$$

A complete normed vector space is called a Banach space. Notice that every finite-dimensional normed space is complete. In other words, every finite-dimensional normed space forms a Banach space.

Example 1.7 *Suppose $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. For a sequence $u : \mathbb{N} \to \mathbb{K}$, we define a norm*

$$\|u\|_p := \left(\sum_{i=1}^{\infty} (u(i))^p\right)^{\frac{1}{p}} \quad \text{if } 1 \le p < \infty$$
$$\|u\|_\infty := \sup_{i \in \mathbb{N}} |u(i)| \quad \text{if } p = \infty$$

Here, $\ell^p = \ell^p(\mathbb{K}) := \{u : \mathbb{N} \to \mathbb{K} : \|u\|_p < \infty\}$ forms a normed space, which is complete for each $p \in [1, \infty]$.

Note that $\|u\|_p$ does not form a norm for $0 < p < 1$.

1.1.5 Various famous contraction type mappings on metric space

In this section, we present several common types of contraction mappings in metric spaces. Additional types of contractions, including those in generalized spaces, are addressed in the chapters of this thesis.

Definition 1.11 *Let $\mathscr{A}$ be a self-mapping on the metric space (Υ, σ), we say that $\mathscr{A}$ is:*

*(i) Lipschitz continuous (k-**Lipschitzian**) if there is a constant $k \ge 0$ such that*

$$\sigma(\mathscr{A}(u), \mathscr{A}(v)) \le k\sigma(u, v) \quad \text{for each pair } (u, v) \in \Upsilon \times \Upsilon, \tag{1.1.2}$$

*(ii) **contraction** if it is k-Lipschitzian with $0 \le k < 1$;*

*(iii) **nonexpansive** if for any pair $(u, v) \in \Upsilon \times \Upsilon$, $\sigma(\mathscr{A}(u), \mathscr{A}(v)) \le \sigma(u, v)$*

*(iv) **isometry** if for any pair $(u, v) \in \Upsilon \times \Upsilon$, $\sigma(\mathscr{A}(u), \mathscr{A}(v)) = \sigma(u, v)$*

*(v) **noncontractive** if for any pair $(u, v) \in \Upsilon \times \Upsilon$, $\sigma(\mathscr{A}(u), \mathscr{A}(v)) \ge \sigma(u, v)$*

*(vi) **expansive** if, for any two different elements u and v in the set Υ, $\sigma(\mathscr{A}(u), \mathscr{A}(v)) > \sigma(u, v)$*

*(vii) **weak contraction** also named "Edelstein contraction" if, for any two different elements u and v in the set Υ, $\sigma(\mathscr{A}(u), \mathscr{A}(v)) < \sigma(u, v)$;*

Remark 1.3

- *The minimal constant k that satisfied the inequality (1.1.2) has been referred to as the Lipschitz constant for the mapping $\mathscr{A}$.*

- *Contraction mapping brings points closer together. In other words, for every $u \in \Upsilon$ and $r > 0$, all points within the ball $B_r(u)$ are transferred into another ball $B_{r'}(\mathscr{A}(u))$ with a smaller radius $r' < r$. That can be observed in Figure 1.2.*

- *It is essential to realize that Lipschitzian maps are necessarily continuous.*

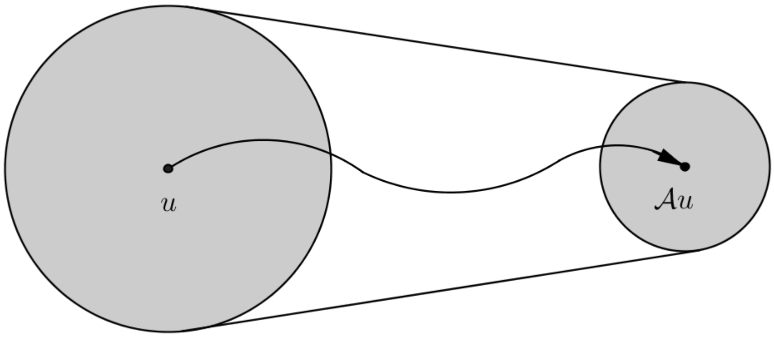

Figure 1.2: $\mathscr{A}$ is a contraction mapping.

1.1.6 Fixed point problem- coupled fixed point

Note that the following definition is independent of the type of space in which the mapping is defined.

Definition 1.12 *A point $\dot{\xi}$ is called a fixed point of a function $\mathscr{A}$ if $\mathscr{A}(\dot{\xi}) = \dot{\xi}$.*
The set of all fixed points of $\mathscr{A}$ is denoted by $\text{Fix}(\mathscr{A})$.

Definition 1.13 *Let (Υ, σ) be a metric space and $D \subset \Upsilon$ a nonempty subset. The fixed point problem is well posed for an operator $\mathscr{A} : D \to \Upsilon$ if and only if*

1 $\mathscr{A}$ has a unique fixed pint $\dot{\xi}$.

2 If $\{v_n\}$ in D and $\sigma(v_n, \mathscr{A}(v_n)) \to 0$ as $n \to \infty$, then $v_n \to \dot{\xi}$ as $n \to \infty$.

The condition 2 can be expressed as follows:

2' Let $\{\epsilon_n\}$ be a sequence of positive numbers so that $\lim_{n\to\infty} \epsilon_n = 0$ and $\{v_n\}$ in D. If $\sigma(v_{n+1}, \mathscr{A}(v_n)) \leq \epsilon_n$ as $n \in \mathbb{N}$, then $\lim_{n\to\infty} v_{n+1} = \dot{\xi}$.

Remark 1.4 *Note that* $\text{Fix}(\mathscr{A}) \subset \text{Fix}(\mathscr{A}^n)$. *Indeed, if $\dot{\xi} \in \text{Fix}(\mathscr{A})$, then*

$$\mathscr{A}^n(\dot{\xi}) = \mathscr{A}\left(\ldots \mathscr{A}\left(\dot{\xi}\right)\right) = \dot{\xi}.$$

Hence, $\dot{\xi} \in \text{Fix}(\mathscr{A}^n)$. The reverse direction is not true, in general (see example 1.8).
If $\text{Fix}(\mathscr{A}^n) = \text{Fix}(\mathscr{A}) = \{\dot{\xi}\}$ for each $n \in \mathbb{N}$, the operator $\mathscr{A}$ is called a Bessage operator.

Example 1.8 *For let $f : [0,2] \to [0,2]$ be defined by*

$$f(x) = \begin{cases} 2 & \text{if } x \in [0;1], \\ 1 & \text{if } x \in]1;2]. \end{cases} \tag{1.1.3}$$

Then

$$f^2(x) = \begin{cases} 1 & \text{if } x \in [0;1], \\ 2 & \text{if } x \in]1;2]. \end{cases} \tag{1.1.4}$$

We remark that f^2 have two fixed point while f does not have one.

Definition 1.14 *An element $(u, v) \in \Upsilon \times \Upsilon$ is called a coupled fixed point of a mapping $\mathscr{A}$: $\Upsilon \times \Upsilon \longrightarrow \Upsilon$ if $\mathscr{A}(u, v) = u$ and $\mathscr{A}(v, u) = v$.*

1.2 Basic notions of b-metric spaces

This section is concerned with b-metric spaces and their topological properties.

1.2.1 The origins of the b-metric space

It is well known that in certain functional spaces, the natural "norm" cannot satisfy the triangle inequality, for example, on $L^p(\mathbb{R}^n)$ when $p \in (0,1)$ the functional that is usually called the "p-norm" has this property. In light of this observation, the idea of the b-metric space was initiated from the works of Bakhtin [10] and Bourbaki [19].

In 1993, Czerwik [32] formalized a definition for the b-metric space by establishing axiom that was less sturdy than the triangular inequality in a way that allows the extension of fixed point theory to cover also these badly behaved function spaces. The definition is as follow:

Definition 1.15 *[32] Assume that Υ is a non-empty set. We define a mapping $\tau : \Upsilon \times \Upsilon \to [0,\infty)$ to be a b-metric if it satisfies the following conditions for any $u, v, w \in \Upsilon$,*
(b1) $\tau(u,v) = 0$ if and only if $u = v$;
(b2) $\tau(u,v) = \tau(v,u)$;
(b3) $\tau(u,w) \leq 2[\tau(u,v) + \tau(v,w)]$.
If (Υ, τ) satisfies the above conditions, it is known as a b-metric space.

After that, in 1998, Czerwik [33] generalized this notion where the constant 2 was replaced by a constant $s \geq 1$, also with the name b-metric. In 2010, Khamsi and Hussain [73] reintroduced the notion of a b-metric under the name metric-type and prove several topological properties in this space that was identical to the classical metric ones. The notion of b-metric spaces created a new direction in which fixed point theory could be developed. Czerwik [32] was the first to gave a generalization for the Banach's principle to this new space. Since then many authors contributed to this development and nowadays the field occupies a considerable position in fixed point theory.

In this section, we will look back on some preliminary definitions and properties about b-metric spaces; most of the concepts in this section come from [3, 33, 59].

Definition 1.16 *[33] Assume that Υ is a non-empty set and let s be a real number greater than or equal to 1. We define a mapping $\tau : \Upsilon \times \Upsilon \to [0,\infty)$ to be a b-metric if it satisfies the following conditions for any $u, v, w \in \Upsilon$,*
(b1) $\tau(u,v) = 0$ if and only if $u = v$;
(b2) $\tau(u,v) = \tau(v,u)$;
(b3) $\tau(u,w) \leq s[\tau(u,v) + \tau(v,w)]$.
If (Υ, τ, s) satisfies the above conditions, it is known as a b-metric space with a constant s ($s \geq 1$), and it is also called a "metric-type space" or "quasi metric space" in some references.

1.2.2 Examples of b-metric spaces

Some examples of b-metric spaces are given by Bakhtin, I. [10], S. Czerwik [33], etc. We give next some known examples of them.

Example 1.9 *[3] Let $(\Upsilon; \sigma)$ be a metric space and let the mapping $\tau_\sigma : \Upsilon \times \Upsilon \to [0,\infty)$ be defined by*

$$\tau_\sigma(u,v) = [\sigma(u,v)]^p \text{ for all } u, v \in \Upsilon; \tag{1.2.1}$$

where $p > 1$ is a fixed real number. Then (Υ, τ_σ) is a b-metric space with coefficient $s = 2^{\frac{1}{p}-1}$.

We show that τ_σ is a b-metric with $s = 2^{p-1}$.

Obviously conditions (b_1) and (b_2) of Definition 1.16 are satisfied. If $1 < p < \infty$, then the convexity of the function $f(x) = x^p (x > 0)$ implies

$$\left(\frac{a+b}{2}\right)^p \leq \frac{1}{2}\left(a^p + b^p\right)$$

and hence, $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ holds.

Thus, for each $u, v, z \in \Upsilon$ we obtain

$$\begin{aligned}\tau_\sigma(u,v) = (\sigma(u,v))^p &\leq (\sigma(u,z) + \sigma(z,v))^p \\ &\leq 2^{p-1}\left((\sigma(u,z))^p + (\sigma(z,v))^p\right) = 2^{p-1}(\tau_\sigma(u,z) + \tau_\sigma(z,v))\end{aligned}$$

Condition (b_3) of Definition 1.16 is satisfied, confirming that τ_σ qualifies as a b-metric. However, if (Υ, σ) is a metric space, it does not necessarily imply that (Υ, τ_σ) will also be a metric space.

For instance, consider the set $\Upsilon = \mathbb{R}$, which represents the set of real numbers. The metric defined as $\tau(u,v) = |u - v|$ corresponds to the standard Euclidean metric. Noted that $\tau(u,v) = (u-v)^2$. is a b-metric on $\mathbb{R}$ with $s = 2$; however, it is not a metric on $\mathbb{R}$.

Also the following example of a b-metric space is given in [127].

Example 1.10 *[127] The space $l^p (0 < p < 1)$,*

$$l^p = \left\{(u_n) \subset \mathbb{R} \mid \sum_{n=1}^{\infty} |u_n|^p < \infty\right\}$$

together with the function $\tau : l^p \times l^p \to \mathbb{R}$,

$$\tau(u,v) = \left(\sum_{n=1}^{\infty} |u_n - v_n|^p\right)^{1/p}$$

where $u = (u_n), v = (v_n) \in l^p$ is a b-metric space with $s = 2^{\frac{1}{p}-1} > 1$.

Example 1.11 *[127] The space $L^p(\mathbb{R})(0 < p < 1)$ of all real functions $u(t), t \in [0,1]$ such that:*

$$\int_0^1 |u(t)|^p dt < \infty$$

is a b-metric space if we take

$$\tau(u,v) = \left(\int_0^1 |u(t) - v(t)|^p dt\right)^{1/p}, \text{ for each } u, v \in L^p(\mathbb{R})$$

The constant s is as in the previous example equal to $2^{\frac{1}{p}-1}$.

It is important to remark that the two previous examples are also provided in [31] and several other sources (such as [10, 77]), but the coefficient of the b-metric specified in those sources is$2^{\frac{1}{p}}$. O. Zahi and H. Ramoul [127] have determined that the appropriate coefficient is $2^{\frac{1}{p}-1}$.

In 2018, Huang et al [58] gave an example of unusual b-metric space as follows.

Example 1.12 *Let $H^p(U) = \{f \in H(U) : \|f\|_{H^p} < \infty\}$ $(0 < p < 1)$ be H^p space defined on the unit disk U, where $H(U)$ is the set of all holomorphic functions on U and*

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}.$$

Denote $\Upsilon = H^p(U)$. Define a mapping $\tau : \Upsilon \times \Upsilon \to \mathbb{R}^+$ by

$$\tau(f,g) = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta}) - g(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}.$$

for all $f, g \in \Upsilon$. Then (Υ, τ) is a b-metric space with coefficient $s = 2^{\frac{1}{p}-1}$.

1.2.3 Comparison between metric and b-metric spaces

b-metric spaces are evidently broader than metric spaces. For $s = 1$, the b-metric space coincides with the metric space, thereby ensuring that every metric space is a b-metric space. However, the reverse does not hold, as demonstrated in example 1.9.

Furthermore, it is noteworthy that, unlike standard distance metrics, b-metrics are generally not continuous. This is demonstrated in the following example:

Example 1.13 *[59] Let $\Upsilon = \mathbb{N} \cup \{\infty\}$ and let $\tau : \Upsilon \times \Upsilon \to \mathbb{R}$ be defined by*

$$\tau(p,q) = \begin{cases} 0, & \text{if } p = q \\ \left|\frac{1}{p} - \frac{1}{q}\right|, & \text{if } p \text{ and } q \text{ are even or } pq = \infty \\ 5, & \text{if } p \text{ and } q \text{ are odd and } p \neq q \\ 2, & \text{otherwise} \end{cases}$$

Then it is easy to see that for all $p, q, r \in \Upsilon$, we have

$$\tau(p,r) \leq 3(\tau(p,q) + \tau(q,r))$$

Thus, $(\Upsilon, \tau, 3)$ is a metric-type space. Let $x_n = 2n$ for each $n \in \mathbb{N}$. Then

$$\tau(2n, \infty) = \frac{1}{2n} \to 0 \quad \text{as } n \to \infty$$

that is, $x_n \to \infty$, but $\tau(x_n, 1) = 2 \nrightarrow \tau(\infty, 1)$ as $n \to \infty$.

1.2.4 Topological properties of b-metric space

Fundamental topological concepts, such as open and closed sets, convergence, closedness, compactness, and completeness, in this space are discussed in [73]. The properties and proofs of these concepts are analogous to those in classical metric spaces. While the fact that these results extend to metric-type spaces is notable, it remains uncertain whether open or closed balls are truly open or closed with respect to the natural topology. The following concepts naturally arise as direct extensions of their metric counterparts:

Definition 1.17 *[3, 31, 73] Consider a b-metric space (Υ, τ, s). We define a sequence $\{u_n\}$ in Υ to be convergent if there is an element $u \in \Upsilon$ that satisfies the condition $\lim_{n\to+\infty} \tau(u_n, u) = 0$ and it is denoted as $\lim_{n\to\infty} u_n = u$;*

If the sequence $\{u_n\}$ is convergent, its limit is easily proven to be unique. Definitions of Cauchy sequences and completeness are presented below:

Definition 1.18 *[3, 31, 73] Consider a b-metric space (Υ, τ, s) and a sequence $\{u_n\}$ in Υ.*
We call that the sequence $\{u_n\}$ is a Cauchy sequence if $\lim_{n,m\to\infty} \tau(u_n, u_m) = 0$. Additionally, if every Cauchy sequence is convergent in Υ, then Υ is called a complete b-metric space.

Since a b-metric is typically not continuous, we require the following fundamental lemma concerning convergent sequences in the context of b-metric spaces to demonstrate our results.

Lemma 1.3 *[3] Let (Υ, τ, s) be a b-metric space, where $s \geq 1$. Consider the sequences (u_n) and (v_n), both of which converge in Υ to the points $u \in \Upsilon$ and $v \in \Upsilon$, respectively. Therefore, we have:*

$$\frac{1}{s^2}\tau(u,v) \leq \liminf_{n\to\infty} \tau(u_n, v_n) \leq \limsup_{n\to\infty} \tau(u_n, v_n) \leq s^2\tau(u,v) \tag{1.2.2}$$

Specifically, when $u = v$, it comes $\lim_{n\to\infty} \tau(u_n, v_n) = 0$.
Furthermore, for each $z \in \Upsilon$, the subsequent inequalities are satisfied:

$$\frac{1}{s}\tau(u,z) \leq \liminf_{n\to\infty} \tau(u_n, z) \leq \limsup_{n\to\infty} \tau(u_n, z) \leq s\tau(u,z) \tag{1.2.3}$$

Proof Employing the triangle inequality within a b-metric space indicates that

$$\tau(u,v) \leq s\tau(u,u_n) + s^2\tau(u_n,v_n) + s^2\tau(v_n,v)$$

and

$$\tau(u_n,v_n) \leq s\tau(u_n,u) + s^2\tau(u,v) + s^2\tau(v,v_n)$$

To derive the first desired result, we evaluate the limit as $n \to \infty$ applying the lower limit to the first inequality and the upper limit to the second.

Reapplying the triangle inequality once more, we derive the final statement. ■

1.3 Basic notions of Menger PM spaces

In this section, we will mention basic definitions and properties of Menger PM space that will be useful for the obtained fixed point results.

1.3.1 The origins of the Menger PM space

The metric space is based on the introduction of the function σ, which represents a positive real number written as $\sigma(u,v)$ (the distance between u and v) for each pair (u,v) of elements in a non-empty set Υ. However, for one reason or another, the exact value of $\sigma(u,v)$ may not be known, and only the probability of several possible values for this distance can be determined.

In 1942, K. Menger [88] proposed a probabilistic generalization of a metric space. He gave the first definition of a Statistical metric space (Menger space), which was later improved by Schweizer and Sklar [113]. Menger's idea was to replace the distance between two points $\sigma(u,v)$ with a distance distribution function $F_{u,v}$, which assigns to each positive real t the probability that the distance between the two points u and v is less than or equal to t, i.e. $F_{u,v}(t) = p(\sigma(u,v) \leq t)$. It follows that:

Karl Menger

$$F_{u,v}(t) = F_{v,u}(t) \quad \text{for all } t > 0 \text{ and } u, v \in \Upsilon, \tag{1.3.1}$$

$$F_{u,v}(t) = 1 \quad \text{for all } t > 0 \quad \Longleftrightarrow \quad u = v, \tag{1.3.2}$$

$$F_{u,v}(0) = 0. \tag{1.3.3}$$

Since the probability is between 0 and 1, we have:

$$0 \leq F_{u,v}(t) \leq 1. \quad \text{for all } t \geq 0. \tag{1.3.4}$$

Since the probability is non-decreasing (in the sense of inclusion), it follows that $F_{u,v}$ is non-decreasing function for all $u, v \in \Upsilon$.

1.3.2 Definition of PM spaces and Menger PM spaces

We recall some well-known notions and definitions concerning PM spaces and Menger PM spaces.

Definition 1.19 *[54]*

A distribution function (on $[-\infty,+\infty]$*) is a function* $F : [-\infty,+\infty] \to [0,1]$ *which is left-continuous on* $\mathbb{R}$*, non-decreasing and* $F(-\infty) = 0$, $F(+\infty) = 1$.

We denote by Δ *the family of all distribution functions on* $[-\infty,+\infty]$.

Definition 1.20 *[54]*

A distance distribution function $F : [-\infty,+\infty] \to [0,1]$ *is a distribution function with support contained in* $[0,+\infty]$.

The family of all distance distribution functions will be denoted by Δ^+*. We denote* $\mathscr{D}^+ = \{F \mid F \in \Delta^+, \lim_{t\to+\infty} F(t) = 1\}$.

Since any function from Δ^+ *is equal to* 0 *on* $[-\infty,0]$*, we can consider the set* Δ^+ *consisting of non-decreasing functions* F *defined on* $[0,+\infty]$ *that satisfy* $F(0) = 0$ *and* $F(+\infty) = 1$.

Moreover, $\mathscr{D}^+$ *then consists of non-decreasing functions* F *defined on* $[0,+\infty)$ *that satisfy* $F(0) = 0$ *and* $\lim_{t\to+\infty} F(t) = 1$.

The class $\mathscr{D}^+$ *will play an important role in the probabilistic fixed point theorems.*

H is a special element of $\mathscr{D}^+$ *defined by*

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

If Υ *is a non-empty set,* $\mathscr{F} : \Upsilon \times \Upsilon \to \mathscr{D}^+$ *is called a probabilistic distance on* Υ *and* $F(u,v)$ *is usually denoted by* $F_{u,v}$.

The following present examples of distribution functions.

Example 1.14

1. *For* $a \in \mathbb{R}^+ \cup \{+\infty\}$*, the unit element* ϵ_a *of* Δ^+ *is defined as:*

   If $a < +\infty$*,*
$$\epsilon_a(t) = \begin{cases} 0 & 0 \leq t \leq a, \\ 1 & a \leq t < +\infty. \end{cases}$$
   If $a = +\infty$
$$\epsilon_\infty(t) = \begin{cases} 0 & 0 \leq t \leq +\infty, \\ 1 & t = +\infty. \end{cases}$$

2. *For* $0 \leq a < b < +\infty$*, the uniform distribution on* $[a, b]$ *is the function* $U_{ab} \in \mathscr{D}^+$ *defined by*
$$U_{ab}(t) = \begin{cases} 0 & 0 \leq t < a, \\ \frac{t-a}{b-a} & a \leq t < b, \\ 1 & b \leq t < +\infty. \end{cases}$$

Definition 1.21 *[114] A probabilistic metric space (PM-space) is a pair* $(\Upsilon, \mathscr{F})$ *where* Υ *is a set and* $\mathscr{F}$ *is a function defined on* $\Upsilon \times \Upsilon$ *into the set of distribution functions* F *such that for all* u, v *and* z *in* Υ*, for all* $s, t > 0$

1. $F_{u,v}(0) = 0$*;*

2. $F_{u,v}(t) = H(t)$ *iff* $u = v$*;*

3. $F_{u,v}(t) = F_{v,u}(t)$*;*

4. *If* $F_{u,v}(t) = 1$ *and* $F_{v,z}(s) = 1$ *then* $F_{u,z}(s + t) = 1$.

Remark 1.5 Let (Υ, σ) be a metric space. The distribution function $F_{u,v}$ defined by the relation $F_{u,v}(t) = H(t - \sigma(u, v))$ for all u, v and $t > 0$, induces a PM-space.

Definition 1.22 *[54] A triangular norm* $*$ *(t-norm for short) is a binary operation on the unit interval* $[0, 1]$ *which is commutative, associative, non-decreasing in its second component and for all* $t \in [0, 1]$*,* $t * 1 = t$.

Remark 1.6 The monotonicity of a t-norm $*$ in its second component is, together with the commutativity, equivalent to the (joint) monotonicity in both components, i.e., to
$$t_1 * l_1 \leq t_2 * l_2 \text{ whenever } t_1 \leq t_2 \text{ and } l_1 \leq l_2.$$

Among the most well-known t-norms, the following examples:

Example 1.15 *For all* $a, b \in [0, 1]$*,*

1. *The minimum t-norm is defined by*
$$a *_M b = \min(a, b).$$

2. *The product t-norm $*_P$ is defined by*

$$a *_p b = a \cdot b.$$

3. *The Luckasiewicz t-norm $*_L$ is defined by*

$$a *_L b = \max(a + b - 1, 0)$$

4. *The Weakest t-norm $*_D$ is defined by*

$$a *_D b = \begin{cases} \min(a,b) & \text{if } \max(a,b) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.23 *[114] A Menger PM-space is a triplet $(\Upsilon, \mathscr{F}, *)$ where $(\Upsilon, \mathscr{F})$ is a PM-space and $*$ is a t-norm with the following condition:*

$$F_{u,z}(t+s) \geq F_{u,v}(t) * F_{v,z}(s),$$

for all $u, v, z \in \Upsilon$ and $s, t > 0$.

This inequality is known as Menger's triangle inequality.

1.3.3 Examples of Menger PM spaces

We give next some known examples of Menger PM spaces.

Example 1.16 *Let $\Upsilon = \mathbb{C}$, for all $u, v \in \Upsilon$ and $t > 0$, we define*

$$F_{u,v}(t) = \frac{1}{\exp\left(\frac{|u-v|}{t}\right)},$$

then $(\Upsilon, F, *_p)$ is a Menger PM space. Indeed, let $u, v \in \Upsilon$,

1. *It is obvious that $F_{u,v} = F_{v,u}$.*

2. *For all $t > 0$:*

$$\begin{aligned} u = v &\Leftrightarrow |u - v| = 0, \\ &\Leftrightarrow \frac{1}{\exp\left(\frac{|u-v|}{t}\right)} = 1, \\ &\Leftrightarrow F_{u,v}(t) = 1. \end{aligned}$$

3. *For $u, v, w \in \Upsilon$ and $t, s > 0$:*

$$\begin{aligned} |u-w| \leq \frac{t+s}{t}|u-v| + \frac{t+s}{s}|v-w| \Rightarrow \frac{|u-w|}{t+s} &\leq \frac{|u-v|}{t} + \frac{|v-w|}{s}, \\ \Rightarrow \exp\left(\frac{|u-w|}{t+s}\right) &\leq \exp\left(\frac{|u-v|}{t}\right)\exp\left(\frac{|v-w|}{s}\right), \\ \Rightarrow F_{u,w}(t+s) &\geq F_{u,v}(t)F_{v,w}(s). \end{aligned}$$

Example 1.17 *Let (Υ,σ) be a metric space. We consider the function F defined on $\Upsilon\times\Upsilon$ by*

$$F_{u,v}(t)=\frac{t}{t+\sigma(u,v)}.$$

Then $\left(\Upsilon,F,*_p\right)$ is a Menger PM space. Indeed, let $u,v\in\Upsilon$,

1. *It is evident that $F_{u,v}=F_{v,u}$.*
2. *For all $t>0$,*

$$\begin{aligned}u=v&\Leftrightarrow\sigma(u,v)=0,\\&\Leftrightarrow t+\sigma(u,v)=t,\\&\Leftrightarrow\frac{1}{t+\sigma(u,v)}=\frac{1}{t},\\&\Leftrightarrow\frac{t}{t+\sigma(u,v)}=\frac{t}{t},\\&\Leftrightarrow F_{u,v}(t)=1.\end{aligned}$$

3. *For $u,v,w\in\Upsilon$ and $t,s>0$,*

$$\begin{aligned}F_{u,w}(t)F_{w,v}(s)&=\frac{t}{t+\sigma(u,w)}\times\frac{s}{s+\sigma(w,v)},\\&=\frac{st}{st+t\sigma(w,v)+s\sigma(u,w)+\sigma(u,w)\sigma(w,v)},\\&\le\frac{st}{st+t\sigma(w,v)+s\sigma(u,w)},\\&\le\frac{st}{st+(\min(t,s))\sigma(w,v)+(\min(t,s))\sigma(u,w)},\\&\le\frac{st}{st+\min(t,s)\cdot\sigma(u,v)},\\&\le\frac{(s+t)}{(s+t)+\sigma(u,v)},\\&=F_{u,v}(s+t).\end{aligned}$$

Example 1.18 *[53] Let (Ω,Σ,P) be a probability measure space, and (M,σ) a complete separable metric space. Let S be the space of all classes $\widehat{X}$ of equivalence of measurable mappings $X:\Omega\to M$, i.e., $X,Y\in\widehat{X}$ if and only if $X=Y$ a.e. Then $(S,\mathscr{F},*_L)$ is a complete Menger space, where, for every $\widehat{X},\widehat{Y}\in S$,*

$$\mathscr{F}_{\widehat{X},\widehat{Y}}(t)=P(\{\omega\in\Omega\mid\sigma(X(\omega),Y(\omega))<t\})\quad(t>0).$$

1.3.4 Topological properties of Menger PM spaces

In a PM space, neighborhoods and corresponding topological structures may be defined in many non-equivalent ways (see [113]). The most effective approach is the one that closely resembles the standard construction of metric spaces.

The concept of a neighborhood in a PM-space was introduced by Schweizer and Sklar [112]. If $u\in\Upsilon$, and ϵ,λ are positive reals, then an (ϵ,λ)-neighborhood of u, denoted by $U_u(\epsilon,\lambda)$, is defined by

$$U_u(\epsilon,\lambda)=\{v\in\Upsilon:F_{v,u}(\epsilon)>1-\lambda\}.$$

The interpretation is that two points of a PM space are "near" when it is highly probable that the distance between them is small.

The (ϵ,λ)-topology in $(\Upsilon,\mathscr{F})$ is generated by the family of neighborhoods $(U_u(\epsilon,\lambda))_{(u,\epsilon,\lambda)\in\Upsilon\times\mathbb{R}^+\times(0,1)}$.

The following result is due to Schweizer and Sklar [112].

Theorem 1.3 *If $(\Upsilon,\mathscr{F},*)$ is a Menger space and $*$ is continuous, then $(\Upsilon,\mathscr{F},*)$ is a Hausdorff space in the topology induced by the family $\{U_u(\epsilon,\lambda) : u\in\Upsilon, \epsilon>0, \lambda>0\}$ of neighborhoods.*

Note that the above topology satisfies the first axiom of countability. In this topology, we define some topological notions as follow:

Definition 1.24 *[90] A sequence $(u_n)_{n\in\mathbb{N}}$ in a Menger PM-space $(\Upsilon,\mathscr{F},*)$ is said*

1. *to converge to a point u in Υ ($u_n\to u$) if and only if for every $\epsilon>0$ and $\lambda>0$, there is an integer n_0 such that $u_n\in U_u(\epsilon,\lambda)$, i.e., $F_{u_n,u}(\epsilon)>1-\lambda$, whenever $n\geq n_0$.*
2. *to be Cauchy sequence (fundamental) in Υ if for each $\epsilon>0$ and $\lambda>0$, there is an integer n_0 such that $F_{u_n,u_m}(\epsilon)>1-\lambda$ whenever $n,m\geq n_0$.*

In conformity with the completion concept in metric spaces, Schweizer introduce the notion of a complete PM-space.

Definition 1.25 *[90] A Menger PM-space $(\Upsilon,\mathscr{F},*)$ is said to be complete if every Cauchy sequence in Υ converges to a point in Υ.*

1.3.5 Relation between metric spaces and PM spaces

The following theorem is easy to prove and establishes a connection between metric spaces and Menger spaces.

Theorem 1.4 *[114] If (Υ,σ) is a metric space, then the metric σ induces a probabilistic distance $\mathscr{F}:\Upsilon\times\Upsilon\to\mathscr{L}$, where $\mathscr{F}(u,v)(u,v\in\Upsilon)$ is defined by*

$$\mathscr{F}(u,v)(t)=H(t-\sigma(u,v))\quad \text{for all } t\in\mathbb{R}.$$

Further, if the t-norm $*$ define by $a*b=\min\{a,b\}$, for all $a,b\in[0,1]$, then $(\Upsilon,\mathscr{F},*)$ is a Menger space.

It is complete if the metric σ is complete. The space $(\Upsilon,\mathscr{F},*)$ obtained called induced Menger space.

Chapter 2 Brief history of some prominent fixed point theorems

This chapter provides an overview of classical results in fixed point theory, as well as recently published interesting results in the context of (standard) metric spaces. The theorems are presented in chronological order.

2.1 Topological fixed point theorems

In this section, we aim to recollect and discuss classical fixed point theorems that are based on topological concepts such as compactness, boundedness, and convexity.

2.1.1 Brouwer's fixed point theorem

Brouwer's theorem has an extensive historical background. In 1886, Henri Poincaré discovered ideas that contributed to the proof of Brouwer's theorem. In 1909, Brouwer demonstrated the theorem for normed vector spaces of dimension $n = 3$. A year later, Hadamard presented the first proof of the theorem for arbitrary n. In 1912, Brouwer [20] established a proof of an enhanced version of the theorem. This is the well-known result that was later named after him, as stated below. Notably, the theorem is equivalent to a number seemingly unrelated propositions (see [129]). For instance, P. Bohl published a result in 1904 that is equivalent to Brouwer's theorem.

Theorem 2.1 ***(Brouwer's fixed-point theorem)***
Let $\mathscr{C}$ be a bounded, closed, convex, and nonempty set in a finite-dimensional normed space E over $\mathbb{K}$. If $\mathscr{A} : \mathscr{C} \to \mathscr{C}$ is a continuous operator, then $\mathscr{A}$ has at least one fixed point.

Proof Several techniques exist for proving Brouwer's theorem. For further details, refer to [128, 130]. ∎

Remark 2.1 *We demonstrate through counterexamples that each assumption of Brouwer's fixed point theorem is essential.*
(i) Let $\mathscr{C} := [0,1]$. The function $\mathscr{A} : \mathscr{C} \to \mathscr{C}$ defined by

$$\mathscr{A}(u) = \begin{cases} 1 & \text{if } u \in [0, 0.5[, \\ 0 & \text{if } u \in [0.5, 1]. \end{cases} \tag{2.1.1}$$

has no fixed point. The set $\mathscr{C}$ is bounded, closed, and convex, but $\mathscr{A}$ is not continuous.

(ii) Let $\mathscr{C} := \mathbb{R}$. The continuous function $\mathscr{A} : \mathscr{C} \to \mathscr{C}$ defined by $\mathscr{A}(u) := u+1$ has no fixed point. The set $\mathscr{C}$ is convex, but it is not bounded.

(iii) Let $\mathscr{C}$ be a closed annulus as shown in Figure 2.1.

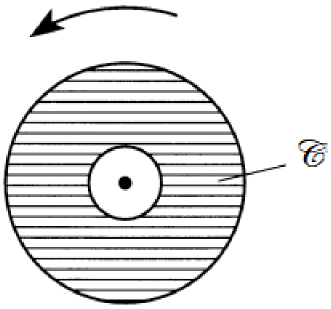

Figure 2.1: Domain of the function g.

Then, a proper rotation $g : \mathscr{C} \to \mathscr{C}$ of the annulus around its center is fixed point free. Here, the operator g is continuous and the set $\mathscr{C}$ is bounded and closed, but it is not convex.

The Brouwer fixed-point theorem is recognized as one of the most significant results in fixed point theory, owing to its wide-ranging applications. Zeidler's work, as cited in [129], illuminates the theorem's significance through compelling examples drawn from various disciplines. For instance, applications in game theory showcase how the theorem elucidates strategic equilibrium points, while its relevance in mathematical economics (see Zeidler [129]) underscores its utility in modeling market dynamics and equilibrium states. Additionally, in the field of numerical mathematics, the theorem serves as a fundamental tool for developing efficient algorithms for solving equations and optimization problems. This versatility underscores the theorem's pivotal role in advancing both theoretical understanding and practical applications across diverse domains.

This theorem extends the standard intermediate-value theorem for continuous real functions, which was first established by Bolzano in 1817. The following example illustrates this connection:

Example 2.1 *Consider the real interval $\mathscr{C} = [a,b]$, where $-\infty < a < b < \infty$. Within this context, for every continuous function $\mathscr{A} : [a,b] \to [a,b]$, there exists a fixed point $\dot{\xi}$, as illustrated in Figure 2.2.*

This particular case represents the elementary form of the Brouwer fixed-point theorem (Theorem 2.1). To demonstrate this assertion directly, we define $\wp(u) := \mathscr{A}(u) - u$ for all $u \in [a,b]$. Since $\mathscr{A}(a), \mathscr{A}(b) \in [a,b]$, it follows that $\mathscr{A}(a) \geq a$ and $\mathscr{A}(b) \leq b$, implying $\wp(a) \geq 0$ and $\wp(b) \leq 0$. By applying the intermediate-value theorem, we conclude that the continuous real function $\wp$ must have a zero, denoted by $\dot{\xi} \in [a,b]$, such that $\wp(\dot{\xi}) = 0$, which implies $\mathscr{A}(\dot{\xi}) = \dot{\xi}$.

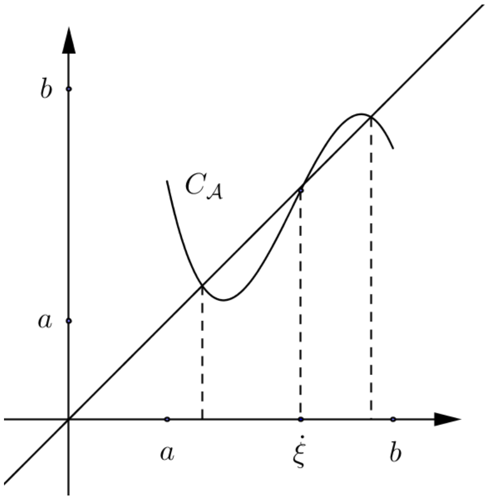

Figure 2.2: An arbitrary curve illustrating the fixed point of the function $\mathscr{A}$.

2.1.2 Schauder's fixed-point theorem

The following theorem was proved by Schauder [111] in 1930 as a generalization of the Brouwer fixed-point theorem.

Theorem 2.2 ***(Schauder's fixed-point theorem)***
If $\mathscr{C}$ is a bounded, closed, convex, and nonempty subset of a Banach space E over $\mathbb{K}$, then any continuous and compact operator (completely continuous operator) defined on $\mathscr{C}$ and mapping into itself possesses at least one fixed point.

Proof The proof of this theorem is based on the Brouwer fixed-point theorem and can be found in [130]. ■

Remark 2.2 - *The compactness condition of the operator $\mathscr{A}$ can be replaced with a more easily verifiable condition: "$\mathscr{A}(\mathscr{C})$ is relatively compact".*

- *If $\dim E < \infty$, then Schauder's fixed-point theorem coincides with the Brouwer fixed-point theorem.*

- *In 1935, Tychonoff [122] extended this theorem to locally convex topological vector spaces. A further extension of Tychonoff's theorem was given by Ky Fan [45].*

- *In 2012, López Pouso [85] presented another version of Schauder's fixed-point theorem, demonstrating its applicability to operators that are not necessarily continuous (see [85, Theorem 3.1]). Additionally, they showcased how this theorem can be applied to solve a class of discontinuous problems.*

- *Schauder's fixed-point theorem has numerous applications, including solving integral equations and ordinary, partial, and fractional differential equations. For further details, see [7, 130, 133].*

2.1.3 Leray-Schauder's fixed point theorem

In 1934, Leray and Schauder established a more general version of Schauder's theorem (Theorem 2.2), leading to the following result, which is regarded as more effective for applications in proving the existence of solutions to differential equations.

Theorem 2.3 ***(Leray-Schauder principle)***
Let $\mathscr{A}$ be a self-mapping on the Banach space E over $\mathbb{K}$. If $\mathscr{A}$ is completely continuous, then one of the following holds:

1. *The set $Q(\mathscr{A}) = \{u \in E \mid u = \lambda\mathscr{A}(u) \text{ for some } 0 < \lambda < 1\}$ is not bounded.*

2. *$\mathscr{A}$ possesses no less than one fixed point.*

In other words, if a self-mapping $\mathscr{A}$ on a Banach space E is completely continuous and the set $Q(\mathscr{A})$ is bounded, then $\mathscr{A}$ possesses no less than one fixed point.

Remark 2.3
- The set $Q(\mathscr{A})$ may be empty. (i.e., the equation $u = \lambda\mathscr{A}(u)$ may have no solution in E for some $\lambda \in (0, 1)$).
- The second condition can be replaced by " The set $\mathscr{A}(E)$ is bounded ".

Remark 2.4 *We illustrate through counterexamples that each assumption of the Leray-Schauder fixed point theorem is essential:*
(i) If a function is discontinuous or not compact, it does not necessarily have fixed points, as demonstrated in the examples provided in Remark 2.1.
(ii) Let $E := \mathbb{R}$. The completely continuous function $\mathscr{A} : E \to E$ defined as $\mathscr{A}(u) := u^2 + 1$ has no fixed point. However, the set $Q(\mathscr{A}) = \left\{\frac{1-\sqrt{1-4\lambda^2}}{2\lambda}, \frac{1+\sqrt{1-4\lambda^2}}{2\lambda} \text{ where } \lambda \in (0, \frac{1}{2}]\right\}$ is not bounded because $\lim_{\lambda\to 0} \frac{1+\sqrt{1-4\lambda^2}}{2\lambda} = \infty$.

Remark 2.5 *The Brouwer's fixed-point theorem (Theorem 2.1) implies Schauder's fixed-point theorem (Theorem 2.2), which in turn implies the Leray-Schauder principle (Theorem 2.3).*

2.1.4 Darbo's fixed point theorem and some of their generalizations

In 1955, Darbo [35] extended Schauder's fixed-point theorem and the Banach contraction principle by employing the concept of the measure of non-compactness. This result, referred to as Darbo's fixed-point theorem, is presented in the following theorem.

Theorem 2.4 ***(Darbo's Fixed Point Theorem)*** *Let E be a Banach space, and let $\mathscr{C} \subseteq E$ be a convex, closed, and bounded subset of E. Suppose that $\mathscr{A}$ is a continuous operator defined on $\mathscr{C}$ and maps $\mathscr{C}$ into itself. Then, $\mathscr{A}$ has at least one fixed point in $\mathscr{C}$, provided that $\mathscr{A}$ is an (α, k)-set contraction. Moreover, the set of fixed points, $\mathrm{Fix}(\mathscr{A})$, is relatively compact.*

Numerous generalizations and extensions of Darbo's fixed point theorem are available in the literature. We mentioned some of them in the sequel.
In 2015, Aghajani et al.[4] obtained the following generalization of Darbo's theorem.

Theorem 2.5 *Let E be a Banach space, and let $\mathscr{C} \subseteq E$ be a convex, closed, and bounded set. Suppose that $\mathscr{A}$ is a continuous operator that leaves $\mathscr{C}$ invariant, and for every $\varepsilon > 0$, there exists some $\delta_\varepsilon > 0$ such that*

$$W \in P(\mathscr{C}), \quad \varepsilon \le \mu(W) < \varepsilon + \delta_\varepsilon \quad \Longrightarrow \quad \mu(\mathscr{A}(W)) < \varepsilon, \tag{2.1.2}$$

where μ is any monotone measure of non-compactness on E that possesses the generalized Cantor intersection property. Then, $\mathscr{A}$ has at least one fixed point in $\mathscr{C}$.

Remark 2.6 *Darbo's fixed point theorem is a special case of the above theorem, since every (k,μ)-set contraction mapping satisfies the Aghajani condition (2.1.2).*

Indeed, let $\mathscr{A}:\mathscr{C}\to\mathscr{C}$ be a given (k,μ)-set contraction mapping.

Let $\varepsilon>0$. From the definition of a (k,μ)-set contraction mapping, there exists some $k\in(0,1)$ such that

$$\mu(\mathscr{A}(W))\le k\mu(W),$$

for any nonempty subset $W\subseteq\mathscr{C}$. Let $\delta_\varepsilon=\left(\frac{1}{k}-1\right)\varepsilon$. Then, for any nonempty subset $W\subseteq\mathscr{C}$, we have

$$\varepsilon\le\mu(W)<\varepsilon+\delta_\varepsilon=\frac{\varepsilon}{k}\quad\Longrightarrow\quad\mu(\mathscr{A}(W))\le k\mu(W)<\varepsilon,$$

which confirms that $\mathscr{A}$ satisfies the Aghajani condition.

In 2016, Jleli and co-authors [64] proposed a generalization of Darbo's fixed point theorem, stated as follows:

Theorem 2.6 *Let E be a Banach space, and let $\mathscr{A}$ be a continuous operator that maps a convex, closed, and bounded set $\mathscr{C}\subseteq E$ onto itself. Assume there exists a function $F:(0,\infty)\to\mathbb{R}$ that satisfies the following conditions:*

(F_1) *For any sequence $\{\alpha_n\}\subset(0,\infty)$,*

$$\lim_{n\to\infty}F(\alpha_n)=-\infty\quad\Longrightarrow\quad\lim_{n\to\infty}\alpha_n=0$$

(F_2) *There exists a constant $\tau>0$ such that, for any $W\in P(C)$,*

$$\mu(W)\mu(\mathscr{A}(W))>0\quad\Longrightarrow\quad\tau+F(\mu(\mathscr{A}(W)))\le F(\mu(W)),$$

where μ is any monotone measure of non-compactness on E that possesses the generalized Cantor intersection property. Then, $\mathscr{A}$ has at least one fixed point.

Proof The proof of this theorem is based on the concept of the measure of non-compactness and its properties and can be found in [64]. ■

Remark 2.7 *By taking $F(t)=\ln t$, where $t>0$, the condition (F_2) in Theorem 2.6 becomes equivalent to the condition for (α,k)-set contraction in Darbo's theorem. Thus, Theorem 2.6 encompasses Theorem 2.4 as a special case.*

Recently, Taoudi [121] generalized Darbo's fixed point theorem as follows:

Theorem 2.7 *Let E be a Banach space, and let μ denote a monotone measure of non-compactness on E that satisfies the generalized Cantor intersection property. Suppose $\mathscr{A}$ is a continuous operator that maps a convex, closed, and bounded set $\mathscr{C}\subseteq E$ onto itself. If, for every countable subset $W\subset\mathscr{C}$, the inequality*

$$\mu(\mathscr{A}(W))\le k\mu(W),$$

is satisfied for some constant k such that $0\le k<1$, then $\mathscr{A}$ has at least one fixed point.

Proof The proof of this theorem can be found in [121]. ■

2.1.5 Sadovskii's fixed point theorem

In 1967, Sadovskii [109] introduced the concept of a condensing map using measures of non-compactness, which are more general than (k,μ)-set contractions. Building on this concept, he provided a more extensive fixed point result compared to Darbo's theorem.

Theorem 2.8 ***(Sadovskii's fixed point theorem)*** *[72] Let E be a Banach space, and let μ denote a monotone measure of non-compactness on E that satisfies the generalized Cantor intersection property. Suppose $\mathscr{A}$ is a continuous operator that maps a convex, closed, and bounded set $\mathscr{C} \subseteq E$ onto itself. If $\mathscr{A}$ is a μ-condensing map, then $\mathscr{A}$ has at least one fixed point in $\mathscr{C}$.*

Proof The proof of this theorem is based on the Schauder fixed-point theorem. Details of the proof can be found in [109]. ■

Remark 2.8 *If, in the definition of a condensing operator, the strict inequality is replaced by "less than or equal to," the conclusion of the theorem may no longer hold, even in a Hilbert space. For instance, a counterexample illustrating this situation is provided in [109].*

The previous theorem has been extended to non-convex settings, as described in the next theorem. For additional results in this direction, see, for instance, [72].

Theorem 2.9 *[78]* ***Kirk and Shin fixed point theorem*** *Let H be a bounded hyper-convex metric space, and let $\mathscr{A} : H \longrightarrow H$ be a continuous condensing map. Then $\mathscr{A}$ has at least one fixed point.*

Remark 2.9

- Darbo's theorem can be seen as a particular case of Sadovskii's theorem since every (α,k)-set contraction mapping with $k \in (0,1)$ is μ-condensing.

- In their original formulation, Darbo's fixed point theorems exclusively address the Kuratowski measure of non-compactness α. However, upon careful examination of the proof, it becomes clear that other measures of non-compactness, such as the Hausdorff measure χ, the diameter measure diam, *or any monotone measure of non-compactness satisfying the generalized Cantor intersection property, can also be utilized. Similarly, Sadovskii's fixed point theorem, in its original form, deals only with the Hausdorff measure χ. Nonetheless, it is possible to use any abstract nonsingular measure of non-compactness (see Definition 1.8).*

- In 1977, De Blasi [38] introduced the concept of the measure of weak non-compactness. Subsequently, in 1981, G. Emmanuele [44] presented a fixed point result for condensing mappings with respect to the measure of weak non-compactness. Notably, Emmanuele's fixed point theorem requires the mapping to be weakly continuous.

2.1.6 Krasnosel'skii's fixed point theorem

In 1955, a unified approach to the main classical fixed point theorems: Banach Principle and Schauder's theorem, was put forward by M.A. Krasnosel'skii [79]. His result is widely applicable to solving equations of the form:

$$\mathscr{A}(u) + \mathscr{S}(u) = u, \quad u \in \mathscr{C} \tag{2.1.3}$$

where $\mathscr{C}$ is a subset of a Banach space E. It has been extensively used in the study of nonlinear integral equations of mixed type, as well as in differential and functional equations.

Theorem 2.10 ***(Krasnosel'skii's fixed point theorem)*** *The sum of a contraction $\mathscr{A}$ and a completely continuous mapping $\mathscr{S}$, both defined on a closed convex subset $\mathscr{C}$ of a Banach space E, admits at least one fixed point in $\mathscr{C}$, provided that $\mathscr{A}(u)+\mathscr{S}(v)\in\mathscr{C}$ for all $u, v \in \mathscr{C}$.*

Proof The proof is based on a combination of Banach's contraction principle and Schauder's fixed point theorem. ■

This theorem has been generalized in various ways. For instance, Burton [22] improved it by replacing the condition $\mathscr{A}(\mathscr{C})+\mathscr{S}(\mathscr{C})\subseteq\mathscr{C}$ with:

$$(u=\mathscr{S}(u)+\mathscr{A}(v), \quad v\in\mathscr{C}) \Rightarrow u\in\mathscr{C}.$$

Burton's result has applications in stability theory, integral equations, and covers cases where Theorem 2.10 can not be applied. More recently, the compactness assumption on $\mathscr{A}$ was relaxed in [124], leading to the development of several noncompact-type Krasnosel'skii fixed point theorems.

Remark 2.10 *[109] The fixed point theorems concerning an operator defined as the sum of a contraction and a completely continuous operator were initially established through an alternative method by R. L. Frum-Ketkov. This work generalizes Krasnosel'skii's result, particularly in cases where the set $\mathscr{C}$ is a solid angle.*

2.1.7 Mönch's fixed point theorem

In 1980, Harald Mönch [93] gave a variant of the result given by Sadovskii [110].

Theorem 2.11 ***Mönch's fixed point theorem*** *Every continuous self-map $\mathscr{A}$ defined on a closed convex subset $\mathscr{C}$ of a Banach space E has a fixed point, provided that for some $u_0\in\mathscr{C}$, the following condition holds:*

$$\overline{V}=\overline{\operatorname{conv}}(\{u_0\}\cup\mathscr{A}(V)) \quad\Rightarrow\quad V \text{ is relatively compact,} \tag{2.1.4}$$

for every countable subset V of $\mathscr{C}$.

Proof This is an immediate consequence of Schauder's fixed point theorem. ■

Additionally, Mönch established in the same manuscript a construction similar to the previous result. With a slight modification on the condition (2.1.4), we arrive at a fixed point result that is connected with the so-called Leray-Schauder boundary condition.

Theorem 2.12 *Let E be a Banach space, $D\subseteq E$ open, $0\in D$, and $\mathscr{A}:\overline{D}\longrightarrow E$ continuous, We assume further that $\mathscr{A}$ satisfies*

$$V\subset\overline{\operatorname{conv}}(\{0\}\cup\mathscr{A}(V)) \quad\Rightarrow\quad V \text{ is relatively compact,} \tag{2.1.5}$$

for every countable subset V of $\overline{D}$.

and the boundary condition

$$u\in\overline{D}, \quad \lambda\in(0,1),\ u=\lambda\mathscr{A}(u) \Rightarrow u\notin\partial D. \tag{2.1.6}$$

Then there exists a fixed point of the mapping $\mathscr{A}$ within the closure of the set D.

Proof The proof of this theorem is based on the Schauder principle, as outlined in [93]. ■

Remark 2.11 *The use of countable subsets in fixed point theorems was initiated by Daher [34] in 1978.*

2.2 The Banach Contraction Principle

This section focuses on one of the most significant metric fixed point theorems established by Banach [11], known as "The Banach Contraction Principle" (BCP). As previously stated, the Banach contraction mapping principle is not the earliest fixed point theorem but it was a cornerstone in the field of fixed point theory. This principle was initially introduced by Stefan Banach in his thesis in 1920 and published in 1922. However, a few years later, this principle was independently rediscovered by Caccioppoli [23]. In some older literature, this principle is referred to as the Picard-Banach theorem, while in other references, it is called the Banach-Caccioppoli theorem. It is a fundamental tool for proving the existence and uniqueness of solutions in various mathematical contexts. Originally formulated by Banach for normed spaces, it was subsequently extended by Caccioppoli to complete metric spaces in 1930. Notably, this theorem not only guarantees the existence and uniqueness of fixed points for specific self-maps in metric spaces but also provides an iterative method for constructing these fixed points.

Theorem 2.13 ***(Banach's fixed point theorem).*** *Define $\mathscr{A}$ as a contraction self-mapping on a complete metric space Υ. Then, there exists exactly one fixed point, denoted as $\dot{\xi}$, for $\mathscr{A}$. Specifically, the iterative sequence $(\mathscr{A}^n(u_0))_{n\in\mathbb{N}}$ converges to $\dot{\xi}$ for every value of u_0 in Υ (in other words, the fixed point problem is well posed). Furthermore, for any $n \in \mathbb{N}$, the following estimate holds:*

$$d(\mathscr{A}^n(u_0), \dot{\xi}) \leq \frac{k^n}{1-k} d(u_0, \mathscr{A}(u_0)), \tag{2.2.1}$$

where $k \in (0,1)$ is the smallest constant for the contraction condition.

Proof Banach's result and its extensions are typically demonstrated through the convergence of the geometric series $\sum_{n=0}^{\infty} k^n$. R. Kannan provides an alternative proof of Banach's theorem, as detailed in [68, theorem 3]. He examines the characteristics of subsets of Υ, defined as $S_a = \{u \in \Upsilon : d(u, \mathscr{A}(u)) \leq a\}, \quad 0 < a < +\infty$. ■

It is worth noting that this celebrated fixed point theorem has been generalized through various techniques:

- **Relaxation of Contraction Conditions:** This includes exploring more general formulations of contractivity conditions, such as Kannan's contraction, Hardy-Rogers's contraction, weak contraction, nonlinear contractions with altering distance functions, and control functions.

- **Expansion to Various Space Types:** This involves studying the validity of the fixed point theorem in a range of generalized spaces beyond metric spaces, including cone metric spaces, probabilistic metric spaces, G-metric spaces, Menger PM spaces, S-metric spaces, rectangular metric spaces, D-metric spaces, partially ordered metric spaces, b-metric spaces, intuitionistic metric spaces, etc.

- **Relaxation of Axioms on the Mapping and Space:** This includes relaxing continuity assumptions on the mapping or completeness assumptions on the metric space.

- **Extension to Multi-Valued Mappings:** Fixed point theorems have also been extended to multi-valued mappings, where a single point is mapped to a set of points rather than a single point. Notably, S. Nadler's result [94] is considered one of the most prominent theorems in this area.

- **Combination of Techniques:** Combining two or more of the aforementioned techniques to derive new results and insights.

These generalizations are crucial for advancing mathematical theory and discovering new applications in different fields such as differential equations, optimization, and dynamic systems.

2.3 Different types of contractive mappings and their relation to fixed point theory

The subsequent theorems exemplify several significant generalizations of Banach's principle. Despite their origins dating back to the early 20th century, these theorems have inspired our publications [12, 83, 84] and influenced numerous notable contemporary studies [3, 9, 40, 59, 76, 127]. This underscores their enduring relevance and profound impact on advancing mathematical research.

2.3.1 Nemytskiǐ and Edelstein's results

This section examines contractive mappings and the corresponding fixed point results established by Nemytskiǐ [95] and Edelstein [43].

Definition 2.1 *Let $\mathscr{A}$ be a self-mapping on the metric space (Υ, σ). We say that $\mathscr{A}$ is weak contraction, also referred to as "Edelstein contractive," if, for any two distinct elements u and v in the set Υ, the following inequality holds:*

$$\sigma(\mathscr{A}(u), \mathscr{A}(v)) < \sigma(u, v).$$

It is crucial to note that a weak contraction mapping does not necessarily have a fixed point. In other words, there exist weak contraction self-mappings that lack fixed points.

To illustrate this, consider the following examples.

Example 2.2 *Let the function $\mathscr{A} : \mathbb{R} \to \mathbb{R}$ be defined as $\mathscr{A}(u) = \sqrt{1+u^2}$.*

Taking the derivative of $\mathscr{A}$ yields:

$$\mathscr{A}'(u) = \frac{u}{\sqrt{1+u^2}}.$$

Notice that $|\mathscr{A}'(u)| < 1$ for all $u \in \mathbb{R}$. By the mean value theorem, for any two different real numbers u and v, there exists a $z \in \mathbb{R}$ compared between them, allowing:

$$\left|\frac{\mathscr{A}(u) - \mathscr{A}(v)}{u - v}\right| = |\mathscr{A}'(z)| < 1.$$

This implies $|\mathscr{A}(u) - \mathscr{A}(v)| < |u - v|$.

Next, we examine the existence of fixed points for $\mathscr{A}$. Consider the equation $\mathscr{A}(u) = u$, which simplifies to:

$$u^2 = 1 + u^2.$$

Subtracting u^2 from both sides results in: $0 = 1$,

indicating a contradiction. This implies that there are no fixed points for the function $\mathscr{A}$.

Therefore, the function $\mathscr{A}(u) = \sqrt{1+u^2}$ is a weak contraction map on a complete metric space without any fixed point.

Example 2.3 *[71] Let $\Upsilon = [1,\infty)$ be equipped with a usual metric $\sigma(u,v) = |u-v|$. Suppose that a self-mapping $\mathscr{A}$ from Υ into itself is defined by $\mathscr{A}(u) = u + \frac{1}{u}$. It is clear that $\mathscr{A}$ is a weak contraction. Indeed, for each distinct pair $u, v \in \Upsilon$, we observe that*

$$\begin{aligned}\sigma(\mathscr{A}(u),\mathscr{A}(v)) &= \left|u + \frac{1}{u} - v - \frac{1}{v}\right| \\ &= \left(1 - \frac{1}{uv}\right)|u-v| \\ &< \sigma(u,v),\end{aligned}$$

since $1 - \frac{1}{uv} < 1$. On the other hand $\mathscr{A}$ is fixed point free; that is, the fixed point equation $u = u + \frac{1}{u}$ has no solution in Υ.

In contrast, Nemytskiǐ [95] demonstrated in 1936 that a weak contraction on a compact metric space always has a fixed point. This result highlights the crucial role of compactness in ensuring the existence of fixed points, as illustrated in the previous examples.

Theorem 2.14 ***(Nemytskiǐ fixed point theorem [95])*** *In a compact metric space (Υ,σ), any weak contraction mapping $\mathscr{A}: \Upsilon \to \Upsilon$ possesses exactly one fixed point within Υ. Moreover, every Picard sequence (defined by the recurrence relation $u_{n+1} = \mathscr{A}(u_n)$ for $n \in \mathbb{N}$) converges to the fixed point of the mapping $\mathscr{A}$.*

Proof The proof of this theorem can be found in detail in [71, 95]. ■

Nemytskiǐ's work aimed to generalize the Banach contraction principle for $k = 1$. However, as demonstrated by the previous examples 2.2 and 2.3, the Banach contraction principle holds only for $k < 1$. To ensure the existence of a fixed point, Nemytskiǐ required a compact metric space rather than merely a complete one. In 1962, Edelstein [43] further refined the conditions by introducing a weaker criterion:

$$\exists u \in \Upsilon : \{\mathscr{A}^n(u)\} \supseteq \{\mathscr{A}^{n_i}(u)\}, \quad \lim_{i\to\infty} \mathscr{A}^{n_i}(u) = \xi \in \Upsilon, \tag{2.3.1}$$

(in words: there exists a point u such that its sequence of iterates contains a sub-sequence that converges to a point in Υ).

This relaxed condition is sufficient to ensure the existence of a fixed point for a weak contraction mapping. For more details, see [43, 71].

Comparison between Banach's contraction principle and Nemytskiǐ's theorem

Janos [62] also proved the equivalence between Banach's contraction principle and Nemytskiǐ's theorem under suitable conditions, as follows.

Theorem 2.15 *[62] Assuming that (Υ,σ) is a compact metric space and $\mathscr{A}: \Upsilon \to \Upsilon$ is continuous map, the following statements are equivalent:*

(i) $\mathscr{A}$ is a weak contraction relative to a suitable metric σ^* topologically equivalent to σ.

(ii) Given $k \in (0,1)$, there exists a metric σ^* topologically equivalent to σ, relative to which $\mathscr{A}$ is a contraction with Lipschitz constant k.

2.3.2 Browder's result

Banach's fixed point theorem does not apply when $k = 1$ (i.e., non-expensive mappings), as demonstrated by the counterexamples 2.2 and 2.3. In this context, Browder [21] proposed additional requirements in 1965 to ensure the existence of a fixed point. The next theorem addresses this matter.

Theorem 2.16 *Let Υ be a uniformly convex Banach space, and let $\mathscr{A}$ be a non-expansive mapping of a bounded closed convex subset $\mathscr{C}$ of Υ into itself. Then, $\mathscr{A}$ has no less than one fixed point in $\mathscr{C}$.*

Proof The proof of this theorem can be found in [21, 50]. ■

2.3.3 Kannan's result

In 1968, Kannan [67] proved the following theorem:

Theorem 2.17 *[67] Let $\mathscr{A}$ be a mapping of the complete metric space Υ into itself. If $\mathscr{A}$ is a Kannan contractive mapping, i.e.,*

$$\sigma[\mathscr{A}(u), \mathscr{A}(v)] \le \alpha\{\sigma[u, \mathscr{A}(u)] + \sigma[v, \mathscr{A}(v)]\}, \tag{2.3.2}$$

for all $u, v \in \Upsilon$, where $0 < \alpha < \frac{1}{2}$, then $\mathscr{A}$ leaves exactly one point of Υ fixed.

Furthermore, Kannan [67, 69] eliminated the compactness condition of the metric space and replaced it with more relaxed conditions (Condition 2.3.1 along with the continuity of the mapping $\mathscr{A}$ at the point $\dot{\xi}$), while still preserving the same results. The origin of this idea can be traced back to Edelstein [43]. Notably, these conditions together do not guarantee the completeness of the space. An illustrative example supporting this assertion is as follows:

Example 2.4 *[69] Let $\Upsilon = [0, 1)$, $\mathscr{A}(u) = \frac{u}{2}$, and let the distance function σ be the standard Euclidean distance on the real line.*

Comparison between Banach's contraction principle and Kannan's theorem

A comparison between Banach's fixed point theorem [11] and Kannan's theorem [67] reveals the distinction between the contraction condition and the Kannan contractive condition. At first glance, it is evident that a contraction mapping ensures the continuity of the map over the entire space, whereas the Kannan condition does not inherently require continuity.

To illustrate the independence of the contraction and Kannan contractive conditions, we present two examples demonstrating this distinction.

Example 2.5 *[69] Let $\Upsilon = [0, 1]$ be equipped with the ordinary Euclidean distance, and consider the mapping $\mathscr{A}$ defined as follows:*

$$\mathscr{A}(u) = \begin{cases} \dfrac{u}{4}, & \text{if } u \in [0, \frac{1}{2}), \\ \dfrac{u}{5}, & \text{if } u \in [\frac{1}{2}, 1]. \end{cases}$$

Here, $\mathscr{A}$ is discontinuous at $u = \frac{1}{2}$; consequently, $\mathscr{A}$ is not a contraction. However, it is easily verified that the Kannan condition is satisfied with $\alpha = \frac{4}{9}$.

Example 2.6 *[69] Let $\Upsilon = [0,1]$, and consider the mapping $\mathscr{A}(u) = \dfrac{u}{3}$ for $u \in [0,1]$, with the distance function given by the ordinary Euclidean distance. Here, $\mathscr{A}$ is a contraction; however, it is easily observed that the Kannan condition is not satisfied if we take $u = \frac{1}{3}$ and $v = 0$.*

Contrary to the conclusion drawn from the previous two examples, Janos [62], in 1976, investigated the relationship between the classical Banach contraction principle and its generalization given by Kannan (see [67, 69]). He demonstrated that every contraction possesses a fixed point, thereby satisfying the Kannan condition for a distance that is topologically equivalent to the original distance. Conversely, the Kannan condition implies the contraction condition under additional requirements (the continuity and compactness of the mapping).

Janos [62] established the equivalence between Kannan contractive and Banach contractive mappings under suitable conditions, as stated below.

Theorem 2.18 *[62] Let (Υ, σ) be a compact metric space, and let $\mathscr{A} : \Upsilon \to \Upsilon$ be a continuous mapping. Then, the following statements are equivalent:*

(i) $\mathscr{A}$ is Kannan contractive relative to a suitable metric σ^* that is topologically equivalent to σ.

(ii) For some $k \in (0,1)$, there exists a metric σ^*, topologically equivalent to σ, such that $\mathscr{A}$ is a contraction with Lipschitz constant k.

As a result, we can conclude the equivalence between Kannan contraction and weak contraction under the aforementioned conditions.

2.3.4 Chatterjea's result

In a manner similar to (2.3.2), Chatterjea [25] introduced the following contractive condition: There exists $\alpha \in [0, \frac{1}{2})$ such that

$$\sigma[\mathscr{A}(u), \mathscr{A}(v)] \le \alpha\{\sigma[u, \mathscr{A}(v)] + \sigma[v, \mathscr{A}(u)]\} \text{ for any } u, v \in \Upsilon, \tag{2.3.3}$$

which ensures the existence and uniqueness of a fixed point for such self-mappings in a complete metric space.

2.3.5 Reich's result

In 1971, Reich [106] introduced a significant generalization that encompasses both Banach's contraction theorem and Kannan's theorem.

Theorem 2.19 *Let Υ be a complete metric space endowed with the metric σ, and let $\mathscr{A} : \Upsilon \to \Upsilon$ be a Reich contractive mapping, i.e.,*

$$\sigma(\mathscr{A}(u), \mathscr{A}(v)) \le a\sigma(u, \mathscr{A}(u)) + b\sigma(v, \mathscr{A}(v)) + c\sigma(u, v), \quad u, v \in \Upsilon,$$

where the constants a, b, and c are nonnegative and satisfy the inequality $a + b + c < 1$. Under these conditions, it can be concluded that $\mathscr{A}$ has a unique fixed point within the space Υ.

Comparison of Banach's fixed point theorem, Kannan's fixed point theorem, and Reich's theorem

It is noteworthy that by setting $a = b = 0$, the above result reduces to Banach's fixed point theorem, while setting $a = b$ and $c = 0$ yields Kannan's fixed point theorem, as mentioned in [[69], p. 406]. Of course, we may always assume that $a = b$, although this is not essential.

To demonstrate that this theorem is more general than Banach's and Kannan's theorems, consider the following example:

Example 2.7 *Let $\Upsilon = [0,1]$, and define $\mathscr{A}(u) = \frac{u}{3}$ for $0 \le u < 1$ and $\mathscr{A}(1) = \frac{1}{6}$. The mapping $\mathscr{A}$ does not satisfy Banach's condition because it is not continuous at 1. Kannan's condition also cannot be satisfied because*

$$\sigma\left(\mathscr{A}(0), \mathscr{A}\left(\frac{1}{3}\right)\right) = \frac{1}{2}\left(\sigma(0, \mathscr{A}(0)) + \sigma\left(\frac{1}{3}, \mathscr{A}\left(\frac{1}{3}\right)\right)\right).$$

However, $\mathscr{A}$ satisfies Reich's contractive condition if we set $a = \frac{1}{6}$, $b = \frac{1}{9}$, and $c = \frac{1}{3}$ (these are not the smallest possible values).

2.3.6 Bianchini's result

In 1972, Bianchini [16] introduced a generalization of Kannan's contraction and established a more general fixed point result, stated in the following theorem.

Definition 2.2 *Let $\mathscr{A}$ be a map of the complete metric space Υ into itself. We say that $\mathscr{A}$ is a generalized Kannan mapping if and only if*

$$\sigma[\mathscr{A}(u), \mathscr{A}(v)] \le r \max\{\sigma[u, \mathscr{A}(u)], \sigma[v, \mathscr{A}(v)]\}, \quad \text{for all } u, v \in \Upsilon \text{ and some } 0 < r < 1. \tag{2.3.4}$$

Theorem 2.20 *[16] Every generalized Kannan mapping on a complete metric space has a unique fixed point on it.*

2.3.7 Hardy-Rogers's result

In 1973, Hardy and Rogers [55] introduced a generalization of Reich's theorem, which also serves as a generalization of both Banach's and Kannan's theorems. Before proceeding, we present the following definition.

Definition 2.3 *(See [55]) Let (Υ, σ) be a metric space. A self-mapping $\mathscr{A} : \Upsilon \longrightarrow \Upsilon$ is said to be generalized nonexpansive if, for all $u, v \in \Upsilon$, the following inequality holds:*

$$\sigma(\mathscr{A}(u), \mathscr{A}(v)) \le \alpha\sigma(u, v) + \beta\sigma(u, \mathscr{A}(u)) + \gamma\sigma(v, \mathscr{A}(v)) + \delta\sigma(u, \mathscr{A}(v)) + L\sigma(v, \mathscr{A}(u)), \tag{2.3.5}$$

where $\alpha, \beta, \gamma, \delta, L$ are nonnegative real numbers such that $\alpha + \beta + \gamma + \delta + L \le 1$.

If $\alpha + \beta + \gamma + \delta + L < 1$, we refer to $\mathscr{A}$ as a contraction of Hardy-Rogers type (or generalized contraction).

Remark 2.12

- The numbers $\alpha, \beta, \gamma, \delta$, and L may depend on both u and v, with $\sup\{\alpha + \beta + \gamma + \delta + L : u, v \in \Upsilon\} \le 1$.

- Without loss of generality, we may assume that $\delta = L$ and $\beta = \gamma$ due to the symmetry of the distance σ.

- Generalized nonexpansive mappings, or generalized contractions, may not be continuous in general, as demonstrated in Example 2.5 (where $\mathscr{A}$ is a generalized contraction mapping but not continuous, since it is discontinuous at $\frac{1}{2}$).

Theorem 2.21 *(See [55]) Let (Υ, σ) be a complete metric space, and consider a mapping $\mathscr{A}: \Upsilon \longrightarrow \Upsilon$. If $\mathscr{A}$ is classified as a contraction of Hardy-Rogers type, then it possesses a unique fixed point within the space Υ.*

Remark 2.13 *With an appropriate selection of coefficients, several well-known theorems can be recovered:*

1. *If $\beta = \gamma = \delta = L = 0$, we arrive at the famous Banach's theorem [11].*
2. *If $\alpha = \delta = L = 0$ and $\gamma = \beta$, we derive Kannan's theorem [67].*
3. *If $\alpha = \beta = \gamma = 0$, we arrive at Chatterjea's theorem [25].*
4. *If $\delta = L = 0$, we get Reich's theorem [106].*

Following the same approach as Nemytskiĭ, Hardy and Rogers [55] presented a generalization of his contraction for $\alpha + \beta + \gamma + \delta + L = 1$ under additional conditions, as stated in the following theorem.

Theorem 2.22 *(See [55]) Let (Υ, σ) be a compact metric space, and consider a mapping $\mathscr{A}: \Upsilon \longrightarrow \Upsilon$. If $\mathscr{A}$ is classified as a generalized nonexpansive continuous mapping, then $\mathscr{A}$ has a unique fixed point in Υ.*

In 1976, Bogin [17] established the following theorem for the case of generalized nonexpansive mappings.

Theorem 2.23 *(See [[17], Theorem 1]) Let $\mathscr{A}$ be a self-mapping on a complete metric space (Υ, σ). Suppose that for all $u, v \in \Upsilon$, the following inequality holds:*

$$\sigma(\mathscr{A}(u), \mathscr{A}(v)) \leq a\sigma(u, v) + b[\sigma(u, \mathscr{A}(u)) + \sigma(v, \mathscr{A}(v))] + c[\sigma(u, \mathscr{A}(v)) + \sigma(v, \mathscr{A}(u))]$$

where the parameters satisfy the condition $a + 2b + 2c = 1$, along with $a \geq 0$, $b > 0$, and $c > 0$. Under these conditions, the mapping $\mathscr{A}$ has a unique fixed point in Υ.

2.3.8 Ćirić's result

In 1971, Ćirić [26] introduced the notion of an $\mathscr{A}$-orbitally complete space, which was an extension of the concept of complete metric spaces.

Definition 2.4 *[27] Let $\mathscr{A}$ be a mapping of a metric space Υ into itself. For each $u \in \Upsilon$, define the following:*

$$O(u, n) = \{u, \mathscr{A}(u), \cdots, \mathscr{A}^n(u)\}, \quad n = 1, 2, \cdots$$
$$O(u, \infty) = \{u, \mathscr{A}(u), \cdots\}.$$

A metric space (Υ, σ) is called "$\mathscr{A}$-orbitally complete" if any Cauchy sequence within the orbit $O(u, \infty)$, for a given $u \in \Upsilon$, converges to a point in Υ.

It is clear that if Υ is a complete metric space, then Υ is $\mathscr{A}$-orbitally complete for any self-mapping $\mathscr{A}$. In [26, example 3], it was shown that a non-complete metric space may be orbitally complete relative to one mapping, but not to another.

In the same reference, Ćirić introduced a novel concept called "orbitally continuous mapping", which is a weaker condition than continuity and can replace the continuity requirement in several fixed-point theorems.

Definition 2.5 *[26] A mapping $\mathscr{A}$ of a space Υ into itself is said to be orbitally continuous if*

$$\mathscr{A}^{n_i}(u) \to z \Rightarrow \mathscr{A}(\mathscr{A}^{n_i}(u)) \to \mathscr{A}(z)$$

as $i \to \infty$.

Lemma 2.1 *[26] Every generalized contraction is orbitally continuous in the sense of Definition 2.5.*

One of the most general contraction conditions was obtained by Ćirić [27] in 1974.

Definition 2.6 *[27] A mapping $\mathscr{A}:\Upsilon \to \Upsilon$ on a metric space (Υ, σ) is called a quasi-contraction if there exists a constant q within the interval $[0,1)$ such that*

$$\sigma(\mathscr{A}(u),\mathscr{A}(v)) \leq q \cdot \max\{\sigma(u,v);\sigma(u,\mathscr{A}(u));\sigma(v,\mathscr{A}(v));\sigma(u,\mathscr{A}(v));\sigma(v,\mathscr{A}(u))\},$$

holds for every $u, v \in \Upsilon$.

Remark 2.14

- Any contraction map is a quasi-contraction.

- Any Kannan map is a quasi-contraction.

- Any Chatterjea map is a quasi-contraction.

- Any generalized contraction is a quasi-contraction, but the converse need not be true, as shown in the following example.

Example 2.8 *[27] Let*

$$\Upsilon_1 = \left\{ \frac{a}{b} \;\middle|\; a = 3^n,\ n \in \mathbb{N},\ b \equiv 1[3],\ b \in \mathbb{N} \right\} \cup \{0\}$$
$$\Upsilon_2 = \left\{ \frac{a}{b} \;\middle|\; a = 3^n,\ n \in \mathbb{N},\ b \equiv 2[3],\ b \in \mathbb{N} \right\}$$

and let $\Upsilon = \Upsilon_1 \cup \Upsilon_2$ with the usual metric. Define $\mathscr{A}:\Upsilon \to \Upsilon$ by

$$\mathscr{A}(u) = \begin{cases} \frac{3u}{5}, & \text{for } u \in \Upsilon_1, \\ \frac{u}{8}, & \text{for } u \in \Upsilon_2. \end{cases}$$

The mapping $\mathscr{A}$ is a quasi-contraction with $q = \frac{3}{5}$. Indeed, if both u and v are in Υ_1 or in Υ_2, then $\sigma(\mathscr{A}(u),\mathscr{A}(v)) \leq \frac{3}{5}\sigma(u,v)$.

Assume u belongs to Υ_1 and v is in Υ_2. Then we have:

If $u > \frac{5}{24}v$, it follows that:

$$\sigma(\mathscr{A}(u),\mathscr{A}(v)) = \frac{3}{5}\left(u - \frac{5}{24}v\right) \leq \frac{3}{5}\left(u - \frac{1}{8}v\right) = \frac{3}{5}\sigma(u,\mathscr{A}(v)).$$

Inversely, if $u < \frac{5}{24}v$, *then:*

$$\sigma(\mathscr{A}(u), \mathscr{A}(v)) = \frac{3}{5}\left(\frac{5}{24}v - u\right) \leq \frac{3}{5}(v-u) = \frac{3}{5}\sigma(u,v).$$

Therefore, $\mathscr{A}$ *on* Υ *satisfies the condition*

$$\sigma(\mathscr{A}(u), \mathscr{A}(v)) \leq \frac{3}{5}\max\{\sigma(u,v), \sigma(u,\mathscr{A}(v)), \sigma(v,\mathscr{A}(u))\}$$

and hence $\mathscr{A}$ *is a quasi-contraction.*

To establish that $\mathscr{A}$ *is not a generalized contraction on* Υ*, we set* $u = 1$ *and* $v = \frac{1}{2}$*. Thus, we have:*

$$\begin{aligned} &q\cdot\sigma(u,v) + r\cdot\sigma(u,\mathscr{A}(u)) + s\cdot\sigma(v,\mathscr{A}(v)) + t[\sigma(u,\mathscr{A}(v)) + \sigma(v,\mathscr{A}(u))] \\ &= q\cdot\frac{1}{2} + r\cdot\frac{2}{5} + s\cdot\frac{7}{16} + t\cdot\frac{83}{80} \\ &< (q+r+s+2t)\cdot\frac{83}{160} < \frac{83}{160} < \frac{43}{80} = \sigma(\mathscr{A}(u),\mathscr{A}(v)) \end{aligned}$$

as $q + r + s + 2t < 1$*, and we see that* $\mathscr{A}$ *is not a generalized contraction.*

Ćirić [27] used the notion of a quasi-contraction to prove the following theorem, which generalizes Banach's contraction principle, Hardy-Rogers's theorem, Kannan's theorem, generalized Kannan's theorem, Chatterjea's theorem, Reich's theorem, and others.

Theorem 2.24 *[27] Let* (Υ, σ) *be an orbitally complete metric space relative to a quasi-contraction mapping* $\mathscr{A}$ *of* Υ *into itself. Then:*

(a) $\mathscr{A}$ *leaves exactly one point of* Υ *fixed, denoted as* $\dot{\xi}$*,*

(b) $\lim_{n\to\infty} \mathscr{A}^n(u) = \dot{\xi}$*, and*

(c) $\sigma\left(\mathscr{A}^n(u), \dot{\xi}\right) \leq \left(\frac{q^n}{1-q}\right)\sigma(u, \mathscr{A}(u))$ *for every* $u \in \Upsilon$.

Note that [94, Theorem 2.5] is a special case of Theorem 2.24. Example 2.8 shows that Theorem 2.24 is more general than [94, Theorem 2.5] and Hardy-Rogers's theorem 2.21. In that example, Υ is $\mathscr{A}$-orbitally complete, and 0 is a fixed point under $\mathscr{A}$.

In 1979, Fisher [47] extended the definition of quasi-contraction mapping and provided a generalization of Ćirić's theorem. Later, Park S. and Rhoades B. E. [101] extended the fixed point theorems of Fisher [47] and Janos [62] in 1982.

Rhoades has authored numerous valuable research papers, including [107, 108], which classify various contractive definitions and their corresponding fixed point theorems. He also compared different contractive definitions, some of which were mentioned throughout this thesis, while others, such as Sehgal's and Zamfirescu's contractions, were not. Additionally, Park S. [100] introduced fourteen more contractive definitions.

2.3.9 Berinde's result

In 2004, Berinde [13] introduced the notion of an almost-contraction mapping, which unifies large classes of contractive-type mappings.

Definition 2.7 *A mapping $\mathscr{A} : \Upsilon \to \Upsilon$ defined on a metric space (Υ, σ) is called an almost-contraction if there exist constants k, with $0 \leq k < 1$, and $L \geq 0$, such that for every pair of points $u, v \in \Upsilon$, the following inequality holds:*

$$\sigma(\mathscr{A}(u), \mathscr{A}(v)) \leq k\sigma(u, v) + L\sigma(u, \mathscr{A}(v)). \tag{2.3.6}$$

Remark 2.15

- Any contraction map is an almost-contraction (for $L = 0$).
- Any Kannan map is an almost-contraction (see [13, Proposition 1]).
- Any Chatterjea map is an almost-contraction (see [13, Proposition 2]).
- Any quasi-contraction with $0 \leq q < \frac{1}{2}$ is an almost-contraction (see [13, Proposition 3]).

In the same manuscript, Berinde presented their fixed point theorem, which extends several well-known fixed point theorems, including those due to Banach, Kannan, Chatterjea, Zamfirescu, and others.

Theorem 2.25 *Every almost-contraction mapping $\mathscr{A} : \Upsilon \to \Upsilon$ on a complete metric space (Υ, σ) leaves at least one point of Υ fixed. Moreover, for any initial point $u_0 \in \Upsilon$, the Picard iteration sequence $\{u_n\}_{n=0}^{\infty}$ converges to some $\dot{\xi} \in \mathrm{Fix}(\mathscr{A})$, and the following estimations hold:*

$$\sigma(u_n, \dot{\xi}) \leq \frac{k^n}{1-k}\sigma(u_0, u_1), \quad n = 0, 1, 2, \ldots$$

$$\sigma(u_n, \dot{\xi}) \leq \frac{k}{1-k}\sigma(u_{n-1}, u_n), \quad n = 1, 2, \ldots$$

where k is the constant appearing in (2.3.6).

2.3.10 Suzuki's result

In 2008, Suzuki proved the following refinement of Banach's fixed point principle.

Definition 2.8 *A mapping $\mathscr{A} : \Upsilon \to \Upsilon$ defined on a metric space (Υ, σ) is called a Suzuki-contraction if there exists a constant r, with $0 \leq r < 1$, such that for every pair of points $u, v \in \Upsilon$,*

$$\theta_1(r)\sigma(u, \mathscr{A}(u)) \leq \sigma(u, v) \implies \sigma(\mathscr{A}(u), \mathscr{A}(v)) \leq r\sigma(u, v),$$

where $\theta_1 : [0, 1) \to (1/2, 1]$ is a real function defined by

$$\theta_1(r) = \begin{cases} 1, & 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2}, & \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Theorem 2.26 *([119, Theorem 2])*

Every Suzuki-contraction mapping $\mathscr{A} : \Upsilon \to \Upsilon$ on a complete metric space (Υ, σ) leaves exactly one point of Υ fixed, denoted as $\dot{\xi}$. Moreover, for any $u_0 \in \Upsilon$, the Picard iteration $\{u_n\}_{n=0}^{\infty}$ converges to $\dot{\xi}$.

In 2009, Suzuki also proved the following version of Edelstein's fixed point theorem.

Theorem 2.27 *([120, Theorem 3]) Let* (Υ, σ) *be a compact metric space. Consider a self-map* $\mathscr{A} : \Upsilon \to \Upsilon$ *that satisfies the following condition for every pair* $u, v \in \Upsilon$*, with* $u \neq v$*:*

$$\frac{1}{2}\sigma(u, \mathscr{A}(u)) < \sigma(u, v) \implies \sigma(\mathscr{A}(u), \mathscr{A}(v)) < \sigma(u, v).$$

Under these assumptions, the mapping $\mathscr{A}$ *guarantees the existence of a unique fixed point within* Υ.

This last theorem was generalized by Đoric et al. [42] in 2012.

2.4 Rational type contraction and uniqueness fixed point theorems

In 1975, Dass and Gupta [37] were the first to establish a fixed point theorem under rational contraction conditions in metric spaces.

Theorem 2.28 *Let* (Υ, σ) *denote a complete metric space, and let* $\mathscr{A} : \Upsilon \to \Upsilon$ *be a continuous self-mapping. If there exist constants* α *and* β *within the interval* $[0, 1)$ *such that* $\alpha + \beta < 1$*, and if the following inequality holds for every pair* $u, v \in \Upsilon$*,*

$$\sigma(\mathscr{A}(u), \mathscr{A}(v)) \leq \alpha\sigma(u, v) + \beta\frac{[1 + \sigma(u, \mathscr{A}(u))]\sigma(v, \mathscr{A}(v))}{1 + \sigma(u, v)}, \tag{2.4.1}$$

then $\mathscr{A}$ *possesses a unique fixed point within the space* Υ.

In 1976, Khan [74] established the following result in the metric context.

Theorem 2.29 *Consider a complete metric space* (Υ, σ)*, and let* $\mathscr{A} : \Upsilon \to \Upsilon$ *be a mapping that fulfills the condition:*

$$\sigma(\mathscr{A}(u), \mathscr{A}(v)) \leq K\frac{\sigma(u, \mathscr{A}(u))\sigma(u, \mathscr{A}(v)) + \sigma(v, \mathscr{A}(v))\sigma(v, \mathscr{A}(u))}{\sigma(u, \mathscr{A}(v)) + \sigma(v, \mathscr{A}(u))} \tag{2.4.2}$$

for all points u and v in Υ*, where K is a constant satisfying* $0 \leq K < 1$.

Under this assumption, the mapping $\mathscr{A}$ *admits a unique fixed point in* Υ*. That is, there exists a point* $\dot{\xi} \in \Upsilon$ *such that* $\mathscr{A}(\dot{\xi}) = \dot{\xi}$.

In 1978, Fisher [46] identified that condition (2.4.2) was not well-defined and subsequently presented a reformulation of Khan's result as follows:

Theorem 2.30 *Let* (Υ, σ) *be a complete metric space, and consider* $\mathscr{A} : \Upsilon \to \Upsilon$ *as a mapping that achieves*

$$\sigma(\mathscr{A}(u), \mathscr{A}(v)) \leq \begin{cases} K\frac{\sigma(u,\mathscr{A}(u))\sigma(u,\mathscr{A}(v))+\sigma(v,\mathscr{A}(v))\sigma(v,\mathscr{A}(u))}{\sigma(u,\mathscr{A}(v))+\sigma(v,\mathscr{A}(u))} & \text{if } \sigma(u, \mathscr{A}(v)) + \sigma(v, \mathscr{A}(u)) \neq 0, \\ 0, & \text{if } \sigma(u, \mathscr{A}(v)) + \sigma(v, \mathscr{A}(u)) = 0, \end{cases} \tag{2.4.3}$$

for any $u, v \in \Upsilon$*, where K is a constant in* $[0, 1)$.

Under this assumption, the mapping $\mathscr{A}$ *admits a unique fixed point* $\dot{\xi} \in \Upsilon$*. Moreover, for all* $u \in \Upsilon$*, the sequence* $\{\mathscr{A}^n(u)\}$ *converges to* $\dot{\xi}$.

In 1977, Jaggi [60] introduced a new type of rational contraction condition independent of Dass and Gupta's rational contraction.

Theorem 2.31 *Consider a complete metric space* (Υ,σ) *and a continuous mapping* $\mathscr{A}:\Upsilon\to\Upsilon$.

Assume that there exist constants α *and* β *within the interval* $[0,1)$ *such that* $\alpha+\beta<1$, *and that the following inequality holds for all distinct points* $u,v\in\Upsilon$:

$$\sigma(\mathscr{A}(u),\mathscr{A}(v))\leq\alpha\frac{\sigma(u,\mathscr{A}(u))\sigma(v,\mathscr{A}(v))}{\sigma(u,v)}+\beta\sigma(u,v). \tag{2.4.4}$$

Then $\mathscr{A}$ *admits a unique fixed point in* Υ.

A mapping $\mathscr{A}$ satisfying (2.4.4) is called a Jaggi contraction map.

In 1980, Jaggi and Dass [61] extended Banach's fixed point theorem using a rational expression. This result is presented in the following theorem.

Theorem 2.32 *Let* $\mathscr{A}$ *be a self-map of an orbitally complete metric space* (Υ,σ) *satisfying:*

$$\sigma(\mathscr{A}(u),\mathscr{A}(v))\leq\frac{\alpha\sigma(u,\mathscr{A}(u))\sigma(v,\mathscr{A}(v))}{\sigma(u,\mathscr{A}(v))+\sigma(v,\mathscr{A}(u))+\sigma(u,v)}+\beta\sigma(u,v),$$

for some $\alpha,\beta\in[0,1)$ *with* $\alpha+\beta<1$, *and for every pair of distinct elements* $u,v\in\Upsilon$.

Then $\mathscr{A}$ *has a unique fixed point in* Υ.

In 2021, Aouine and Aliouche [9] proved the following fixed point result with a rational contractive expression.

Theorem 2.33 *Let* $\mathscr{A}$ *be a self-map of a complete metric space* (Υ,σ) *satisfying:*

$$\sigma(\mathscr{A}(u),\mathscr{A}(v))\leq\frac{\sigma(u,\mathscr{A}(v))+\sigma(v,\mathscr{A}(u))}{\sigma(u,\mathscr{A}(u))+\sigma(v,\mathscr{A}(v))+1}\max\{\sigma(u,\mathscr{A}(u)),\sigma(v,\mathscr{A}(v))\} \tag{2.4.5}$$

for every pair $u,v\in\Upsilon$. *Under this condition, the following conclusions hold:*

(a) $\mathscr{A}$ *has a unique fixed point* $\dot{\xi}\in\Upsilon$,

(b) The fixed point problem for $\mathscr{A}$ *is well-posed, and*

(c) $\mathscr{A}$ *is continuous at the fixed point* $\dot{\xi}$.

The next section presents additional theorems on rational-type contractions in cases where fixed points are not necessarily unique.

2.5 Rational type contraction and non-uniqueness fixed point theorems

In the study of nonlinear equations, the concept of non-unique fixed points has emerged as a significant area of interest. Certain classes of mappings in metric spaces can possess more than one fixed point, naturally extending classical fixed point theorems. The first known result for finding non-unique fixed points for certain operators was proposed by Ćirić [28] in 1974.

Since then, many notable results have emerged regarding the existence and computation of non-unique fixed points for different operators in complete metric spaces, with key contributions from researchers such as Dhage [41], Pathak [102], Pachpatte [99], and Ćirić [28, 29]. These works have been fundamental in advancing the understanding of fixed point problems involving multiple solutions.

Theorem 2.34 ***(Ćirić's nonunique fixed point theorem)****[28] Assume that the metric space (Υ,σ) is orbitally complete and that $\mathscr{A}$ is a self-mapping on Υ.*

If there exists a real number $k \in [0,1)$ such that the following inequality holds for all $u, v \in \Upsilon$:

$$\min\{\sigma(v,\mathscr{A}(v)),\sigma(u,\mathscr{A}(u)),\sigma(\mathscr{A}(u),\mathscr{A}(v))\} - \min\{\sigma(\mathscr{A}(u),v),\sigma(u,\mathscr{A}(v))\} \le k\sigma(u,v).$$

Then, for any $u_0 \in \Upsilon$, the Picard iteration $\{u_n\}_{n=0}^{\infty}$ converges to some $\dot{\xi} \in \Upsilon$. Moreover, the limit $\dot{\xi}$ is the required fixed point of the self-mapping $\mathscr{A}$.

In 1979, Achari [1] studied the existence of fixed points for a mapping $\mathscr{A}$ using a symmetrical rational expression, as stated below.

Theorem 2.35 *(Achari's nonunique fixed point) [1]*

Assume that the metric space (Υ,σ) is orbitally complete and that $\mathscr{A}$ is an orbitally continuous self-mapping on Υ.

If there exists a real number $k \in [0,1)$ such that for every pair of points $u, v \in \Upsilon$, the following inequality holds:

$$\frac{A(u,v) - B(u,v)}{C(u,v)} \le k\sigma(u,v),$$

where

$$A(u,v) = \min\{\sigma(\mathscr{A}(u),\mathscr{A}(v))\sigma(u,v),\sigma(u,\mathscr{A}(u))\sigma(v,\mathscr{A}(v))\},$$

$$B(u,v) = \min\{\sigma(u,\mathscr{A}(u))\sigma(u,\mathscr{A}(v)),\sigma(v,\mathscr{A}(v))\sigma(\mathscr{A}(u),v)\},$$

and

$$C(u,v) = \min\{\sigma(u,\mathscr{A}(u)),\sigma(v,\mathscr{A}(v))\},$$

with $C(u,v) \neq 0$.

Then, for any $u_0 \in \Upsilon$, the Picard iteration $\{u_n\}_{n=0}^{\infty}$ converges to some $\dot{\xi} \in \Upsilon$. In addition, the limit $\dot{\xi}$ is the required fixed point of the self-mapping $\mathscr{A}$.

In 1998, Ćirić and Jotić [29] established a nonunique fixed point result under rational contraction condition, generalizing the main theorems of Pathak [102] and Dhage [41].

Theorem 2.36 ***(Ćirić-Jotić's nonunique fixed point)***

Assume that the metric space (Υ,σ) is complete, and let $\mathscr{A}$ be an orbitally continuous self-mapping defined on Υ.

If there exist non-negative constants a and $k \in [0,1)$ such that the following inequality holds for all pairs of points $u, v \in \Upsilon$,

$$A(u,v) - aB(u,v) \le kC(u,v),$$

where

$$A(u,v) = \min\left\{\begin{array}{c} \sigma(\mathscr{A}(u),\mathscr{A}(v)),\sigma(u,v),\sigma(u,\mathscr{A}(u)),\sigma(v,\mathscr{A}(v)), \\ \frac{\sigma(u,\mathscr{A}(u))[1+\sigma(v,\mathscr{A}(v))]}{1+\sigma(u,v)}, \frac{\sigma(v,\mathscr{A}(v))[1+\sigma(u,\mathscr{A}(u))]}{1+\sigma(u,v)}, \\ \frac{\min\{\sigma^2(\mathscr{A}(u),\mathscr{A}(v)),\sigma^2(u,\mathscr{A}(u)),\sigma^2(v,\mathscr{A}(v))\}}{\sigma(u,v)} \end{array}\right\},$$

$$B(u,v) = \min\{\sigma(v,\mathscr{A}(u)),\sigma(u,\mathscr{A}(v))\}, \text{ and } \quad C(u,v) = \max\{\sigma(u,v),\sigma(u,\mathscr{A}(u))\}.$$

Then, $\mathscr{A}$ leaves at least one point of Υ fixed.

In 2014, Khojasteh et al. [76] proved the following non-uniqueness theorem, which introduced novel forms of rational contractive mappings.

Theorem 2.37 *Let (Υ,σ) be a complete metric space, and let $\mathscr{A}$ be a self-mapping on Υ. Assume that $\mathscr{A}$ satisfies the following inequality:*

$$\sigma(\mathscr{A}(u),\mathscr{A}(v)) \le \left(\frac{\sigma(u,\mathscr{A}(v))+\sigma(v,\mathscr{A}(u))}{\sigma(u,\mathscr{A}(u))+\sigma(v,\mathscr{A}(v))+1}\right)\sigma(u,v), \tag{2.5.1}$$

for every pair of points $u,v \in \Upsilon$. Then, the following conclusions hold:

(a) $\mathscr{A}$ has at least one fixed point $\dot{\xi}$ in Υ;

(b) The sequence $\{\mathscr{A}^n(u)\}$ converges to a fixed point for any $u \in \Upsilon$;

(c) If $\dot{\xi}$ and $\dot{\eta}$ are two distinct fixed points of $\mathscr{A}$, then the distance between them is at least $\frac{1}{2}$, i.e., $\sigma(\dot{\xi},\dot{\eta}) \ge \frac{1}{2}$.

In 2017, Olatinwo [97] generalized Ćirić's and Jaggi's theorems (theorems 2.34 and 2.31 discussed previously). This generalization is detailed below.

Theorem 2.38 *Assume that the metric space (Υ,σ) is complete, and let $\mathscr{A}$ be an orbitally continuous self-mapping defined on Υ.*

If there exist non-negative numbers $\beta \ge 0$, $\alpha \ge 0$ and $k \in [0,1)$ such that the following inequality holds for all pairs of points $u,v \in \Upsilon$,

$$P(u,v) - Q(u,v) \le \alpha\frac{\sigma(u,\mathscr{A}(u))\cdot\sigma(v,\mathscr{A}(v))\cdot\sigma(v,\mathscr{A}(u))}{\beta\sigma(v,\mathscr{A}(u))+\sigma(u,v)} + k\sigma(u,v), \tag{2.5.2}$$

where

$$P(u,v) = \min\{\sigma(\mathscr{A}(u),\mathscr{A}(v)),\sigma(u,\mathscr{A}(u)),\sigma(v,\mathscr{A}(v))\},$$

$$Q(u,v) = \min\{\sigma(u,\mathscr{A}(v)),\sigma(v,\mathscr{A}(u))\},$$

then $\mathscr{A}$ leaves exactly one point of Υ fixed.

For some recent generalizations, as well as other fixed point theorems concerning unique and non-unique fixed points, see Agarwal, Meehan and O'Regan [2], Berinde [13, 14], Karapinar [70, 71], Redjel et al [104, 105], Olatinwo [97, 98], and Zeidler [128], along with the references therein.

Chapter 3 Fixed point theorems in dynamic property under rational contractive condition for metric spaces

This chapter aims to generalize and discuss several fixed point results in the literature related to rational contractions, as presented by Khojasteh [76], Demma [39], and Yildirim [126], for complete metric spaces. Additionally, we establish dynamic information regarding the existence of other fixed points, specifically the distance between two fixed points in the case of metric spaces. To strengthen our results, we provide several examples.

3.1 Fixed point results for rational contractive maps in metric spaces

In this section, we present some of the main results of our findings published in [84] and [82] regarding fixed points for mappings defined on complete metric spaces that satisfy rational contractive conditions. In light of the work of Khojasteh and its generalization by Yildirim, we obtain the following enhanced result.

Theorem 3.1 *Let* (Υ,σ) *be a complete metric space and let* $\mathscr{A}$ *be a self-mapping on* Υ*. Suppose there exist nonnegative constants* α,β,γ*, and* δ*, and a positive number* ϵ*, such that* $\alpha\le\min\{\gamma,\delta\}$*, and for all* $u,v\in\Upsilon$*, the following inequality holds:*

$$\sigma(\mathscr{A}(u),\mathscr{A}(v))\le\frac{\alpha\sigma(u,\mathscr{A}(v))+\beta\sigma(v,\mathscr{A}(u))}{\gamma\sigma(u,\mathscr{A}(u))+\delta\sigma(v,\mathscr{A}(v))+\epsilon}\sigma(u,v). \tag{3.1.1}$$

Then, the following conclusions hold:

1. $\mathscr{A}$ *has at least one fixed point* $\dot{\xi}\in\Upsilon$*;*
2. *Every Picard sequence* $(u_n)_{n\in\mathbb{N}}$ *converges to a fixed point;*
3. *If* $\mathscr{A}$ *has two distinct fixed points* $\dot{\xi}$ *and* $\dot{\eta}$ *in* Υ*, then* $\sigma(\dot{\xi},\dot{\eta})\ge\dfrac{\epsilon}{\alpha+\beta}$.

Proof Consider a Picard sequence $\{u_n\}_{n\in\mathbb{N}}$ generated by the recurrence relation $u_{n+1}=\mathscr{A}(u_n)$, initiated from a random element u_0 in Υ. If there exists an integer n_0 ensuring that $u_{n_0}=u_{n_0+1}$, then u_{n_0} represents a fixed point of the self-map $\mathscr{A}$, which concludes the proof.

However, if $u_n \neq u_{n+1}$ for each natural number n, we proceed as follows:
Claim 1: The sequence $(u_n)_{n\in\mathbb{N}}$ is asymptotically regular.
By setting $u = u_{n-1}$ and $v = u_n$ in inequality (3.1.1), we obtain:

$$\begin{aligned}
\sigma(u_n, u_{n+1}) &\leq \frac{\alpha\sigma(u_{n-1}, u_{n+1})}{\gamma\sigma(u_{n-1}, u_n) + \delta\sigma(u_n, u_{n+1}) + \epsilon}\sigma(u_{n-1}, u_n) \\
&\leq \frac{\alpha\sigma(u_{n-1}, u_n) + \alpha\sigma(u_n, u_{n+1})}{\gamma\sigma(u_{n-1}, u_n) + \delta\sigma(u_n, u_{n+1}) + \epsilon}\sigma(u_{n-1}, u_n) \\
&\leq \frac{\alpha\sigma(u_{n-1}, u_n) + \alpha\sigma(u_n, u_{n+1})}{\min\{\gamma,\delta\}\big(\sigma(u_{n-1}, u_n) + \sigma(u_n, u_{n+1})\big) + \epsilon}\sigma(u_{n-1}, u_n).
\end{aligned}$$

We denote that $\theta_n = \dfrac{\alpha\sigma(u_{n-1}, u_n) + \alpha\sigma(u_n, u_{n+1})}{\min\{\gamma,\delta\}\big(\sigma(u_{n-1}, u_n) + \sigma(u_n, u_{n+1})\big) + \epsilon}$ for all $n \in \mathbb{N}$.

Since $\alpha \leq \min\{\gamma,\delta\}$, it follows that $0 \leq \theta_n < 1$. Additionally, the sequence $(\theta_n)_{n\in\mathbb{N}}$ is decreasing since for all $n \in \mathbb{N}$:

$$\begin{aligned}
\theta_{n+1} - \theta_n &= \frac{\alpha\epsilon\big[\sigma(u_{n+1}, u_{n+2}) - \sigma(u_{n-1}, u_n)\big]}{\big[\min\{\gamma,\delta\}\big(\sigma(u_n, u_{n+1}) + \sigma(u_{n+1}, u_{n+2})\big) + \epsilon\big]} \\
&\quad \times \frac{1}{\big[\min\{\gamma,\delta\}\big(\sigma(u_{n-1}, u_n) + \sigma(u_n, u_{n+1})\big) + \epsilon\big]} \\
&< 0.
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
\sigma(u_n, u_{n+1}) &\leq \theta_n\sigma(u_{n-1}, u_n) \\
&\leq \theta_n\theta_{n-1}\sigma(u_{n-2}, u_{n-1}) \\
&\vdots \\
&\leq \theta_n\theta_{n-1}\cdots\theta_1\sigma(u_0, u_1) \\
&\leq \theta_1^n\sigma(u_0, u_1).
\end{aligned}$$

Taking the limit as $n \to +\infty$, we find

$$\lim_{n\to+\infty} \sigma(u_n, u_{n+1}) = 0.$$

Claim 2: The sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.
For any $n, m \in \mathbb{N}$ such that $m > n$, we obtain:

$$\begin{aligned}
\sigma(u_n, u_m) &\leq \sigma(u_n, u_{n+1}) + \sigma(u_{n+1}, u_{n+2}) + \cdots + \sigma(u_{m-1}, u_m) \\
&\leq \sum_{k=n}^{m-1} (\theta_k\theta_{k-1}\cdots\theta_1)\sigma(u_0, u_1).
\end{aligned}$$

Define $I_k = \theta_k\theta_{k-1}\cdots\theta_1$. Since $\lim_{n\to+\infty} \dfrac{I_{k+1}}{I_k} = 0$, the series $\sum_{k=1}^{+\infty} I_k$ is convergent. This means that

$$\lim_{n,m\to+\infty} \sum_{k=n}^{m-1} \theta_k\theta_{k-1}\cdots\theta_1 = 0. \tag{3.1.2}$$

Consequently, the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Υ.
Claim 3: The sequence $(u_n)_{n\in\mathbb{N}}$ is convergent.

Since the metric space (Υ,σ) is complete, the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent, and its limit is denoted by $\dot{\xi}\in\Upsilon$. That is,

$$\lim_{n\to+\infty} u_n = \dot{\xi}.$$

Claim 4: Check that $\dot{\xi}$ is a fixed point of $\mathscr{A}$.

By substituting $u=\dot{\xi}$ and $v=u_n$ into inequality (3.1.1), we find:

$$\sigma(\mathscr{A}(\dot{\xi}),u_{n+1}) \leq \frac{\alpha\sigma(\dot{\xi},u_{n+1})+\beta\sigma(u_n,\mathscr{A}(\dot{\xi}))}{\gamma\sigma(\dot{\xi},\mathscr{A}(\dot{\xi}))+\delta\sigma(u_n,u_{n+1})+\epsilon}\sigma(\dot{\xi},u_n). \tag{3.1.3}$$

Taking the limit as $n\to+\infty$ on both sides of inequality (3.1.3), we obtain:

$$\sigma(\mathscr{A}(\dot{\xi}),\mathscr{A}(\dot{\xi}))=0,$$

which implies that $\mathscr{A}(\dot{\xi})=\dot{\xi}$. Therefore, we conclude that $\dot{\xi}$ is a fixed point of $\mathscr{A}$.

Claim 5: We assume that the mapping $\mathscr{A}$ has two distinct fixed points $\dot{\xi}$ and $\dot{\eta}$ in Υ. We seek to determine the distance between these two fixed points.

By substituting $u=\dot{\xi}$ and $v=\dot{\eta}$ into inequality (3.1.1), we have:

$$\begin{aligned}\sigma(\dot{\xi},\dot{\eta}) &\leq \frac{\alpha\sigma(\dot{\xi},\dot{\eta})+\beta\sigma(\dot{\eta},\dot{\xi})}{\gamma\sigma(\dot{\xi},\dot{\xi})+\delta\sigma(\dot{\eta},\dot{\eta})+\epsilon}\sigma(\dot{\xi},\dot{\eta})\\ &\leq \frac{(\alpha+\beta)\sigma(\dot{\xi},\dot{\eta})}{\epsilon}\sigma(\dot{\xi},\dot{\eta}).\end{aligned}$$

From this inequality, it follows that $\sigma(\dot{\xi},\dot{\eta})\geq\dfrac{\epsilon}{\alpha+\beta}$. The proof is complete.

■

The following examples illustrate and support Theorem 3.1.

Example 3.1 *Let* $\Upsilon=\{0,1,2\}$ *and* $\sigma:\Upsilon\times\Upsilon\longrightarrow[0,+\infty)$ *be defined as follows:*

$$\sigma(1,0)=\sigma(0,1)=2,\quad \sigma(2,1)=\sigma(1,2)=3,\quad \sigma(2,0)=\sigma(0,2)=3.5,$$

$$\sigma(0,0)=\sigma(1,1)=\sigma(2,2)=0.$$

The pair (Υ,σ) *forms a complete metric space. Let* $\mathscr{A}:\Upsilon\longrightarrow\Upsilon$ *be a self-mapping defined by*

$$\mathscr{A}(0)=0,\quad \mathscr{A}(1)=1,\quad \mathscr{A}(2)=0.$$

We select the parameters $\alpha=1$, $\beta=1$, $\gamma=1$, $\delta=1$, *and* $\epsilon=4$. *Then, the following holds:*

$$\sigma(\mathscr{A}(0),\mathscr{A}(1))=2\leq\frac{\sigma(0,\mathscr{A}(1))+\sigma(1,\mathscr{A}(0))}{\sigma(0,\mathscr{A}(0))+\sigma(1,\mathscr{A}(1))+4}\sigma(0,1)=2,$$

$$\sigma(\mathscr{A}(1),\mathscr{A}(2))=2\leq\frac{\sigma(1,\mathscr{A}(2))+\sigma(2,\mathscr{A}(1))}{\sigma(1,\mathscr{A}(1))+\sigma(2,\mathscr{A}(2))+4}\sigma(1,2)=2,$$

$$\sigma(\mathscr{A}(0),\mathscr{A}(2))=0\leq\frac{\sigma(0,\mathscr{A}(2))+\sigma(2,\mathscr{A}(0))}{\sigma(0,\mathscr{A}(0))+\sigma(2,\mathscr{A}(2))+4}\sigma(0,2)=\frac{49}{30}.$$

Thus, the mapping $\mathscr{A}$ *satisfies all the conditions of Theorem 3.1. Additionally,* $\mathscr{A}$ *has two distinct fixed points* 0 *and* 1, *for which* $\sigma(0,1)=2\geq\frac{\epsilon}{\alpha+\beta}=2$.

Remark 3.1 *We note that this estimation is optimal due to the choice of the constants* $\alpha, \beta, \gamma, \delta$ *and* ϵ. *(See also Example 3.8).*

Example 3.2 *Let* $\Upsilon = \{0, 1, 2\}$ *be associated with a metric* σ *such that* $\sigma(1,0) = \sigma(0,1) = 0.75$, $\sigma(2,0) = \sigma(0,2) = 1$, $\sigma(2,1) = \sigma(1,2) = 0.25$, *and*

$$\sigma(0,0) = \sigma(1,1) = \sigma(2,2) = 0.$$

Let $\mathscr{A}$ *be a self-mapping on* Υ *such that* $\mathscr{A}(0) = 2$, $\mathscr{A}(1) = 1$, *and* $\mathscr{A}(2) = 2$.

It is straightforward to conclude that (Υ, σ) *is a complete metric space, and the inequality* (3.1.1) *is satisfied for all* $u, v \in \Upsilon$ *with constants* $\alpha = \beta = \frac{1}{2}$, $\gamma = \delta = 1$, *and* $\epsilon = \frac{1}{4}$. *According to Theorem 3.1, we conclude that* $\mathscr{A}$ *has at least one fixed point (specifically, it has two fixed points:* 1 *and* 2*). Furthermore, the distance between these fixed points is* $\sigma(1,2) \geq \frac{1}{4}$.

Remark 3.2 *It should be noted that the theorem by Khojasteh et al. [76] is not applicable in this example. However, the generalized Theorem 3.1 is applicable, as demonstrated in the example above, which underscores the robustness of our results.*

The following theorem combine two types of contractive mappings: the rational contraction represented by formula (3.1.1) and the generalized Chatterjea contraction to derive an improved result.

Theorem 3.2 *Let* (Υ, σ) *represent a complete metric space, and consider the self-map* $\mathscr{A}$ *defined on* Υ. *Assume there exist non-negative constants* $\alpha, \beta, \gamma, \delta$, *and* ϵ *with* $\epsilon > 0$, *satisfying either* $\alpha \leq \frac{1}{2}\min\{\gamma, \delta\}$ *or* $\beta \leq \frac{1}{2}\min\{\gamma, \delta\}$, *and the inequality* (3.1.4) *holds for each* $u, v \in \Upsilon$.

$$\sigma(\mathscr{A}(u), \mathscr{A}(v)) \leq \frac{\alpha\sigma(u, \mathscr{A}(v)) + \beta\sigma(v, \mathscr{A}(u))}{\gamma\sigma(u, \mathscr{A}(u)) + \delta\sigma(v, \mathscr{A}(v)) + \epsilon} \max\{\sigma(u, \mathscr{A}(v)), \sigma(v, \mathscr{A}(u))\}. \tag{3.1.4}$$

Under these conditions, the subsequent statements are valid:

1. *There is, at minimum, one fixed point of the mapping* $\mathscr{A}$, *denoted by* $\dot{\xi}$, *within the space* Υ;
2. *Each sequence of the form* $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ *tends to a fixed point of* $\mathscr{A}$;
3. *If* $\mathscr{A}$ *has more than one fixed point, then each pair of fixed points must satisfy the inequality* (3.1.5)*:*
$$\sigma(\dot{\xi}, \dot{\eta}) \geq \frac{\epsilon}{\alpha + \beta}. \tag{3.1.5}$$

Proof Consider a Picard sequence $\{u_n\}_{n\in\mathbb{N}}$ generated by the recurrence relation $u_{n+1} = \mathscr{A}(u_n)$, initiated from a random element u_0 in Υ. If there exists an integer n_0 ensuring that $u_{n_0} = u_{n_0+1}$, then u_{n_0} represents a fixed point of the self-map $\mathscr{A}$, which concludes the proof.

However, if $u_n \neq u_{n+1}$ for each natural number n, we proceed as follows:

Claim 1: The sequence $(u_n)_{n\in\mathbb{N}}$ is asymptotically regular.

Case one: We assume that $\alpha \leq \frac{1}{2}\min\{\gamma, \delta\}$, by substituting $u = u_{n-1}$ and $v = u_n$ into inequality (3.1.4), we obtain

$$\begin{aligned}
\sigma(u_n, u_{n+1}) &\leq \frac{\alpha\sigma(u_{n-1}, u_{n+1})}{\gamma\sigma(u_{n-1}, u_n) + \delta\sigma(u_n, u_{n+1}) + \epsilon}\sigma(u_{n-1}, u_{n+1}); \\
&\leq \frac{\alpha\sigma(u_{n-1}, u_n) + \alpha\sigma(u_n, u_{n+1})}{\gamma\sigma(u_{n-1}, u_n) + \delta\sigma(u_n, u_{n+1}) + \epsilon}[\sigma(u_{n-1}, u_n) + \sigma(u_n, u_{n+1})]; \\
&\leq \frac{\alpha\sigma(u_{n-1}, u_n) + \alpha\sigma(u_n, u_{n+1})}{\min\{\gamma; \delta\}\big(\sigma(u_{n-1}, u_n) + \sigma(u_n, u_{n+1})\big) + \epsilon}[\sigma(u_{n-1}, u_n) + \sigma(u_n, u_{n+1})].
\end{aligned}$$

We define $\Lambda_n = \dfrac{\alpha\sigma(u_{n-1},u_n)+\alpha\sigma(u_n,u_{n+1})}{\min\{\gamma;\delta\}\big(\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1})\big)+\epsilon}$ for each $n\in\mathbb{N}$.
Given that $\alpha \le \frac{1}{2}\min\{\gamma,\delta\}$, it follows that $0\le\Lambda_n<\frac{1}{2}$ for each $n\in\mathbb{N}$.
Furthermore, we establish

$$\sigma(u_n,u_{n+1}) \le \Lambda_n[\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1})],$$

which implies

$$\sigma(u_n,u_{n+1}) \le \frac{\Lambda_n}{1-\Lambda_n}\sigma(u_{n-1},u_n).$$

We denote $\lambda_n = \dfrac{\Lambda_n}{1-\Lambda_n}$ for each $n\in\mathbb{N}$.
Since $0\le\Lambda_n<\frac{1}{2}$ for each n in $\mathbb{N}$, it follows that $\lambda_n\in[0,1)$.
This leads to $\sigma(u_n,u_{n+1})<\sigma(u_{n-1},u_n)$ for each $n\in\mathbb{N}$, which implies $\sigma(u_{n+1},u_{n+2})<\sigma(u_{n-1},u_n)$.

Moreover, the sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ is decreasing due to the following reasoning

$$\begin{aligned}\Lambda_{n+1}-\Lambda_n &= \frac{\alpha\epsilon\big[\sigma(u_{n+1},u_{n+2})-\sigma(u_{n-1},u_n)\big]}{\big[\min\{\gamma;\delta\}(\sigma(u_n,u_{n+1})+\sigma(u_{n+1},u_{n+2}))+\epsilon\big]}\\ &\quad\times\frac{1}{\big[\min\{\gamma;\delta\}(\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1}))+\epsilon\big]},\\ &< 0.\end{aligned}$$

Therefore, the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ is also decreasing. Consequently,

$$\begin{aligned}\sigma(u_n,u_{n+1}) &\le \lambda_n\sigma(u_{n-1},u_n);\\ &\le \lambda_n\lambda_{n-1}\sigma(u_{n-2},u_{n-1});\\ &\vdots\\ &\le \lambda_n\lambda_{n-1}\cdots\lambda_1\sigma(u_0,u_1);\\ &\le \lambda_1^n\sigma(u_0,u_1).\end{aligned}$$

Now, for any pair of natural numbers n and m where $m>n$, we observe

$$\begin{aligned}\sigma(u_n,u_m) &\le \sigma(u_n,u_{n+1})+\sigma(u_{n+1},u_{n+2})+\cdots+\sigma(u_{m-1},u_m);\\ &\le \sum_{k=n}^{m-1}\lambda_1^k\sigma(u_0,u_1);\\ &\le \frac{\lambda_1^n-\lambda_1^m}{1-\lambda_1}\sigma(u_0,u_1).\end{aligned}$$

By executing the limit as n and m approach infinity on both sides of the preceding inequality, we obtain

$$\lim_{n,m\to+\infty}\sigma(u_n,u_m)=0. \tag{3.1.6}$$

As a result, $(u_n)_{n\in\mathbb{N}}$ forms a Cauchy sequence within Υ.

Case two: We assume that $\beta\le\frac{1}{2}\min\{\gamma,\delta\}$. By substituting $u=u_n$ and $v=u_{n-1}$ into inequality (3.1.4), we derive

$$\sigma(u_n,u_{n+1}) \le \frac{\beta\sigma(u_{n-1},u_{n+1})}{\gamma\sigma(u_n,u_{n+1})+\delta\sigma(u_{n-1},u_n)+\epsilon}\sigma(u_{n-1},u_{n+1}).$$

Following a similar argument as in Case One, we can conclude that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence within Υ.

Due to the completeness of the space (Υ,σ), there exists an element $\dot{\xi}$ in Υ such that

$$\lim_{n\to+\infty} u_n = \dot{\xi}.$$

Claim 2: We check that $\dot{\xi}$ is a fixed point of $\mathscr{A}$.

Case one: We assume that $\alpha \le \frac{1}{2}\min\{\gamma,\delta\}$. By substituting $u = u_n$, $v = \dot{\xi}$ into inequality (3.1.4), we obtain

$$\sigma(\mathscr{A}(\dot{\xi}), u_{n+1}) \le \frac{\alpha\sigma(u_n,\mathscr{A}(\dot{\xi})) + \beta\sigma(\dot{\xi}, u_{n+1})}{\gamma\sigma(u_n, u_{n+1}) + \delta\sigma(\dot{\xi},\mathscr{A}(\dot{\xi})) + \epsilon}\max\{\sigma(\dot{\xi}, u_{n+1}), \sigma(u_n,\mathscr{A}(\dot{\xi}))\}. \tag{3.1.7}$$

By executing the limit as n and m approach infinity on both sides of the preceding inequality, we obtain

$$\sigma(\mathscr{A}(\dot{\xi}),\dot{\xi}) \le \frac{\alpha\sigma(\dot{\xi},\mathscr{A}(\dot{\xi}))}{\delta\sigma(\dot{\xi},\mathscr{A}(\dot{\xi})) + \epsilon}\sigma(\dot{\xi},\mathscr{A}(\dot{\xi})). \tag{3.1.8}$$

Hence, $\sigma(\mathscr{A}(\dot{\xi}),\dot{\xi}) = 0$, that means $\mathscr{A}(\dot{\xi}) = \dot{\xi}$. Therefore, $\dot{\xi}$ is a fixed point of $\mathscr{A}$.

Case two: We assume that $\beta \le \frac{1}{2}\min\{\gamma,\delta\}$. By substituting $u = \dot{\xi}$ and $v = u_n$ into inequality (3.1.4), we obtain

$$\sigma(\mathscr{A}(\dot{\xi}), u_{n+1}) \le \frac{\alpha\sigma(\dot{\xi}, u_{n+1}) + \beta\sigma(u_n,\mathscr{A}(\dot{\xi}))}{\gamma\sigma(\dot{\xi},\mathscr{A}(\dot{\xi})) + \delta\sigma(u_n, u_{n+1}) + \epsilon}\max\{\sigma(\dot{\xi}, u_{n+1}), \sigma(u_n,\mathscr{A}(\dot{\xi}))\}.$$

Following a similar argument as in Case One, we can conclude that $\mathscr{A}(\dot{\xi}) = \dot{\xi}$.

Claim 3: We assume that $\mathscr{A}$ has two different fixed points, $\dot{\xi}$ and $\dot{\eta}$, in the metric space Υ. Our objective is to determine the distance between them.
By substituting $u = \dot{\xi}$ and $v = \dot{\eta}$ into inequality (3.1.4), we obtain

$$\begin{aligned}\sigma(\dot{\xi},\dot{\eta}) &\le \frac{\alpha\sigma(\dot{\xi},\dot{\eta}) + \beta\sigma(\dot{\eta},\dot{\xi})}{\gamma\sigma(\dot{\xi},\dot{\xi}) + \delta\sigma(\dot{\eta},\dot{\eta}) + \epsilon}\sigma(\dot{\xi},\dot{\eta});\\ &\le \frac{(\alpha+\beta)\sigma(\dot{\xi},\dot{\eta})}{\epsilon}\sigma(\dot{\xi},\dot{\eta}).\end{aligned}$$

Therefore, we conclude that $\sigma(\dot{\xi},\dot{\eta}) \ge \dfrac{\epsilon}{\alpha+\beta}$. This completes the proof. ■

The following example illustrates and supports Theorem 3.2.

Example 3.3 *Let* $\Upsilon = \{0,1,2\}$ *be a set equipped with a metric* σ *defined as follows:* $\sigma(1,0) = \sigma(0,1) = 0.75$, $\sigma(2,1) = \sigma(1,2) = 0.25$, $\sigma(2,0) = \sigma(0,2) = 1$, *and*

$$\sigma(0,0) = \sigma(1,1) = \sigma(2,2) = 0.$$

Consider a self-map $\mathscr{A}$ *on* Υ *defined by* $\mathscr{A}(0) = \mathscr{A}(2) = 2$ *and* $\mathscr{A}(1) = 1$.

It may be readily inferred that the metric space (Υ,σ) *is complete. Moreover, the inequality* (3.1.4) *holds with constants* $\alpha = \beta = \frac{1}{2}$, $\gamma = \delta = 1$ *and* $\epsilon = \frac{1}{4}$ *for every* $u, v \in \Upsilon$.

By applying Theorem 3.2, we conclude that $\mathscr{A}$ *has at least one fixed point. Specifically,* $\mathscr{A}$ *possesses two distinct fixed points,* 2 *and* 1. *Furthermore, it can be observed that the distance between these fixed points, denoted as* $\sigma(2,1)$, *is greater than or equal to one-fourth.*

The subsequent theorem extends and unifies the two previous findings.

Theorem 3.3 *Let (Υ,σ) represent a complete metric space, and consider the self-map $\mathscr{A}$ on Υ. If there exist non-negative constants $\alpha,\beta,\gamma,\delta$ and ϵ satisfying either $\alpha \leq \frac{1}{2}\min\{\gamma,\delta\}$ or $\beta \leq \frac{1}{2}\min\{\gamma,\delta\}$, and the inequality shown in equation(3.1.9) holds for each $u,v\in\Upsilon$,*

$$\sigma(\mathscr{A}(u),\mathscr{A}(v)) \leq \frac{\alpha\sigma(u,\mathscr{A}(v))+\beta\sigma(v,\mathscr{A}(u))}{\gamma\sigma(u,\mathscr{A}(u))+\delta\sigma(v,\mathscr{A}(v))+\epsilon}\max\{\sigma(u,\mathscr{A}(v)),\sigma(v,\mathscr{A}(u)),\sigma(u,v)\}, \quad (3.1.9)$$

then the subsequent statements are valid:

1. *There exists at minimum one fixed point, denoted by $\dot{\xi}$, within the space Υ for the self-map $\mathscr{A}$;*
2. *Each sequence of the form $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ tends to one of the fixed points;*
3. *If $\mathscr{A}$ has several fixed points, then each pair of fixed points satisfies the inequality (3.1.10).*

$$\sigma(\dot{\xi},\dot{\eta}) \geq \frac{\epsilon}{\alpha+\beta}. \quad (3.1.10)$$

Proof The procedure for proving the above theorem is the same as the proof of Theorem3.2. ■

The following example confirms the validity of Theorem 3.3.

Example 3.4 *Consider the set $\Upsilon = \{m,p,q\}$ equipped with the metric σ defined by the following values: $\sigma(p,q) = 2$, $\sigma(m,p) = \sigma(q,m) = 1$. Furthermore, it holds that $\sigma(u,v) = \sigma(v,u)$ for every pair of elements u and v belonging to the set Υ. Additionally, $\sigma(u,u) = 0$ for each element u in Υ.*

Consider a self-map $\mathscr{A}$ in the set Υ, where $\mathscr{A}(p) = m$, $\mathscr{A}(q) = q$, and $\mathscr{A}(m) = m$.

It can be immediately deduced that (Υ,σ) forms a complete metric space, and the inequality (3.1.9) is satisfied for every $u,v\in\Upsilon$ with the constants $\alpha=\beta=\frac{1}{2}$, $\gamma=\delta=1$, and $\epsilon=1$.

By applying Theorem 3.3, we deduced that the self-map $\mathscr{A}$ possesses no less than one fixed point. Indeed, the given function possesses two distinct fixed points, q and m. Furthermore, it can be observed that the distance between these fixed points is greater than or equal to 1, i.e., $\sigma(m,q)\geq 1$.

We can also provide a generalization of the first result by combining the rational contraction with Kannan's contraction, as stated in the following theorem.

Theorem 3.4 *Let (Υ,σ) represent a complete metric space, and consider the self-map $\mathscr{A}$ on Υ. If there are non-negative constants $\alpha,\beta,\gamma,\delta$, ϵ satisfying either $\alpha\leq\min\{\gamma,\delta\}$ or $\beta\leq\min\{\gamma,\delta\}$ and the inequality shown in equation (3.1.11) holds true for each $u,v\in\Upsilon$,*

$$\sigma(\mathscr{A}(u),\mathscr{A}(v)) \leq \frac{\alpha\sigma(u,\mathscr{A}(v))+\beta\sigma(v,\mathscr{A}(u))}{\gamma\sigma(u,\mathscr{A}(u))+\delta\sigma(v,\mathscr{A}(v))+\epsilon}\max\{\sigma(u,v),\sigma(u,\mathscr{A}(u)),\sigma(v,\mathscr{A}(v))\}. \quad (3.1.11)$$

Under these conditions, the subsequent statements are valid:

1. *There exists, at a minimum, one fixed point denoted as $\dot{\xi}$ within the space Υ for the mapping $\mathscr{A}$;*
2. *Each sequence of the form $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ tends to one of the fixed points;*

3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality (3.1.12)*

$$\sigma(\dot{\xi},\dot{\eta}) \geq \frac{\epsilon}{\alpha+\beta}. \tag{3.1.12}$$

If we consider all the constants in the above theorem to be equal to 1, i.e., $\alpha=\beta=\gamma=\delta=\epsilon=1$, we could get the next corollary.

Corollary 3.5 *Let (Υ,σ) represent a complete metric space, and consider the self-map $\mathscr{A}$ on Υ. If the inequality shown in equation(3.1.13) holds true for each $u,v\in\Upsilon$,*

$$\sigma(\mathscr{A}(u),\mathscr{A}(v)) \leq \frac{\sigma(u,\mathscr{A}(v))+\sigma(v,\mathscr{A}(u))}{\sigma(u,\mathscr{A}(u))+\sigma(v,\mathscr{A}(v))+1}\max\{\sigma(u,v),\sigma(u,\mathscr{A}(u)),\sigma(v,\mathscr{A}(v))\}, \tag{3.1.13}$$

then

1. *There exists, at a minimum, one fixed point, denoted as $\dot{\xi}$, within the space Υ for the self-map $\mathscr{A}$;*
2. *Each sequence of the form $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ tends to one of the fixed points;*
3. *If $\mathscr{A}$ has several fixed points, then each pair of them satisfies the inequality (3.1.14).*

$$\sigma(\dot{\xi},\dot{\eta}) \geq \frac{1}{2}. \tag{3.1.14}$$

The following example illustrates Theorem3.4 and Corollary 3.5.

Example 3.5 *Let $\Upsilon=\{m,p,q\}$ be a set of three distinct elements equipped with a metric σ in the following way $\sigma(p,q)=0.75$, $\sigma(q,m)=0.25$, and $\sigma(p,m)=1$. Additionally, the following standard conditions hold for each $u,v\in\Upsilon$: $\sigma(u,v)=\sigma(v,u)$ and $\sigma(u,u)=0$.*

Consider $\mathscr{A}$ as a self-map on the set Υ, defined by the following equations: $\mathscr{A}(m)=m$, $\mathscr{A}(p)=m$, and $\mathscr{A}(q)=q$.

It can be readily inferred that the metric space (Υ,σ) is complete. Moreover, the inequality (3.1.13) holds for every $u,v\in\Upsilon$.

By Corollary 3.5, we conclude that $\mathscr{A}$ has at least one fixed point (specifically, $\mathscr{A}$ possesses two distinct fixed points, m and q). Furthermore, it can be observed that the distance between these fixed points is greater than or equal to one-half, i.e., $\sigma(m,q)\geq\frac{1}{2}$.

Now, we provide a generalization of the previous theorems.

Theorem 3.6 *Let (Υ,σ) represent a complete metric space, and consider the self-map $\mathscr{A}$ on Υ. If there exist non-negative constants, denoted as $\alpha,\beta,\gamma,\delta$ and ϵ, satisfying either $\alpha\leq\frac{1}{2}\min\{\gamma,\delta\}$ or $\beta\leq\frac{1}{2}\min\{\gamma,\delta\}$, and the inequality shown in equation(3.1.15) holds true for each $u,v\in\Upsilon$,*

$$\sigma(\mathscr{A}(u),\mathscr{A}(v)) \leq \frac{\alpha\sigma(u,\mathscr{A}(v))+\beta\sigma(v,\mathscr{A}(u))}{\gamma\sigma(u,\mathscr{A}(u))+\delta\sigma(v,\mathscr{A}(v))+\epsilon}\max\{\sigma(u,\mathscr{A}(v)),\sigma(v,\mathscr{A}(u)),\sigma(u,\mathscr{A}(u)),\sigma(v,\mathscr{A}(v)),\sigma(u,v)\} \tag{3.1.15}$$

then the subsequent statements are valid:

1. *There exists, at a minimum, one fixed point, denoted as $\dot{\xi}$, within the space Υ for the self-map $\mathscr{A}$;*

2. *Each sequence of the form* $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ *tends to one of the fixed points;*

3. *If* $\mathscr{A}$ *has several fixed points, then each pair of them must satisfy the inequality* (3.1.16).

$$\sigma(\dot{\xi},\dot{\eta}) \geq \frac{\epsilon}{\alpha+\beta}. \tag{3.1.16}$$

Proof The procedure for proving the above theorem is the same as that used for proving Theorem 3.2. ■

The following example confirms the validity of Theorem 3.6.

Example 3.6 *Consider a collection of three elements,* $\Upsilon = \{m, p, q\}$, *equipped with a metric* σ. *The values of* $\sigma(q,p)$, $\sigma(q,m)$, *and* $\sigma(p,m)$ *are* 2, 1, *and* 1, *respectively. Furthermore, it holds that* $\sigma(u,v) = \sigma(v,u)$ *for every pair of elements* u *and* v *belonging to the set* Υ. *Additionally, we have* $\sigma(u,u) = 0$ *for each element* u *in the set* Υ.

Consider a self-map $\mathscr{A}$ *in the set* Υ, *where* $\mathscr{A}(p) = p$, $\mathscr{A}(m) = p$, *and* $\mathscr{A}(q) = q$.

It can be immediately deduced that (Υ,σ) *forms a complete metric space, and the inequality* (3.1.15) *has been established for every* $u, v \in \Upsilon$ *with the constants* $\alpha = \beta = \epsilon = \frac{3}{2}$ *and* $\gamma = \delta = 3$.

From Theorem 3.6, *we deduce that the self-map* $\mathscr{A}$ *possesses at least one fixed point. Indeed, the given function possesses two distinct fixed points,* q *and* p. *Furthermore, it can be observed that the distance separating them is greater than or equal to* $\frac{1}{2}$, *i.e.,* $\sigma(p,q) \geq \frac{1}{2}$.

By ameliorating the previous results, we can provide the following theorems.

Theorem 3.7 *Let* (Υ,σ) *represent a complete metric space, and consider the self-map* $\mathscr{A}$ *on* Υ. *If there exist non-negative constants denoted as* α, β, γ, δ, *and* ϵ *satisfying either* $\alpha \leq \frac{1}{2}\min\{\gamma,\delta\}$ *or* $\beta \leq \frac{1}{2}\min\{\gamma,\delta\}$, *and the inequality shown in equation* (3.1.17) *holds true for each* $u, v \in \Upsilon$,

$$\sigma(\mathscr{A}(u),\mathscr{A}(v)) \leq \frac{\alpha\sigma(u,\mathscr{A}(v)) + \beta\sigma(v,\mathscr{A}(u))}{\gamma\sigma(u,\mathscr{A}(u)) + \delta\sigma(v,\mathscr{A}(v)) + \epsilon} \max\{\sigma(u,\mathscr{A}(v)), \sigma(v,\mathscr{A}(u)), \sigma(u,\mathscr{A}(u)) + \sigma(v,\mathscr{A}(v)), \sigma(u,v) \tag{3.1.17}$$

then

1. *There exists, at a minimum, one fixed point denoted as* $\dot{\xi}$ *within the space* Υ *for the self-map* $\mathscr{A}$.

2. *Each sequence of the form* $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ *converges to one of the fixed points.*

3. *If* $\mathscr{A}$ *has several fixed points, then each pair of them must satisfy the inequality* (3.1.18)*:*

$$\sigma(\dot{\xi},\dot{\eta}) \geq \frac{\epsilon}{\alpha+\beta}. \tag{3.1.18}$$

Proof The procedure for proving the above theorem is analogous to the proof of Theorem 3.2. ■

Theorem 3.8 *Let* (Υ,σ) *represent a complete metric space, and consider the self-map* $\mathscr{A}$ *on* Υ. *If there exist non-negative constants, denoted as* α, β, γ, δ, *and* ϵ, *satisfying either* $\alpha \leq \frac{1}{2}\min\{\gamma,\delta\}$ *or* $\beta \leq \frac{1}{2}\min\{\gamma,\delta\}$, *and the inequality shown in equation* (3.1.19) *holds true for each* $u, v \in \Upsilon$,

$$\sigma(\mathscr{A}(u),\mathscr{A}(v)) \leq \frac{\alpha\sigma(u,\mathscr{A}(v))+\beta\sigma(v,\mathscr{A}(u))}{\gamma\sigma(u,\mathscr{A}(u))+\delta\sigma(v,\mathscr{A}(v))+\epsilon} \max\{\sigma(u,\mathscr{A}(v))+\sigma(v,\mathscr{A}(u)),\sigma(u,\mathscr{A}(u)),\sigma(v,\mathscr{A}(v)),\sigma(u,v)\}, \tag{3.1.19}$$

then

1. *There exists, at a minimum, one fixed point denoted as $\dot{\xi}$ within the space Υ for the self-map $\mathscr{A}$.*
2. *Each sequence of the form $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ converges to one of the fixed points.*
3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality (3.1.20):*

$$\sigma(\dot{\xi},\dot{\eta}) \geq \frac{\epsilon}{2(\alpha+\beta)}. \tag{3.1.20}$$

Proof The procedure for proving the above theorem is analogous to the proof of Theorem 3.2. ∎

Theorem 3.9 *Let (Υ,σ) represent a complete metric space, and consider the self-map $\mathscr{A}$ on Υ. If there exist non-negative constants, denoted as α, β, γ, δ, and ϵ, satisfying either $\alpha \leq \frac{1}{2}\min\{\gamma,\delta\}$ or $\beta \leq \frac{1}{2}\min\{\gamma,\delta\}$, and the inequality shown in equation (3.1.21) holds true for each $u,v\in\Upsilon$:*

$$\sigma(\mathscr{A}(u),\mathscr{A}(v)) \leq \frac{\alpha\sigma(u,\mathscr{A}(v))+\beta\sigma(v,\mathscr{A}(u))}{\gamma\sigma(u,\mathscr{A}(u))+\delta\sigma(v,\mathscr{A}(v))+\epsilon} \max\{\sigma(u,\mathscr{A}(v))+\sigma(v,\mathscr{A}(u)),\sigma(u,\mathscr{A}(u))+\sigma(v,\mathscr{A}(v)),\sigma(u,v)\}, \tag{3.1.21}$$

then

1. *There exists, at a minimum, one fixed point denoted as $\dot{\xi}$ within the space Υ for the self-map $\mathscr{A}$.*
2. *Each sequence of the form $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ converges to one of the fixed points.*
3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality (3.1.22):*

$$\sigma(\dot{\xi},\dot{\eta}) \geq \frac{\epsilon}{2(\alpha+\beta)}. \tag{3.1.22}$$

Proof The procedure for proving the above theorem is analogous to the proof of Theorem 3.2. ∎

Theorem 3.10 *Let (Υ,σ) represent a complete metric space, and consider the self-map $\mathscr{A}$ on Υ. If there exist non-negative constants, denoted as α, β, γ, δ, and ϵ, satisfying either $\alpha \leq \frac{1}{5}\min\{\gamma,\delta\}$ or $\beta \leq \frac{1}{5}\min\{\gamma,\delta\}$, and the inequality shown in equation (3.1.23) holds true for each $u,v\in\Upsilon$:*

$$\sigma(\mathscr{A}(u),\mathscr{A}(v)) \leq \frac{\alpha\sigma(u,\mathscr{A}(v))+\beta\sigma(v,\mathscr{A}(u))}{\gamma\sigma(u,\mathscr{A}(u))+\delta\sigma(v,\mathscr{A}(v))+\epsilon}[\sigma(u,\mathscr{A}(v))+\sigma(v,\mathscr{A}(u))+\sigma(u,\mathscr{A}(u))+\sigma(v,\mathscr{A}(v))+\sigma(u,v)], \tag{3.1.23}$$

then, the following statements are valid:

1. *There exists, at a minimum, one fixed point denoted as $\dot{\xi}$ within the space Υ for the self-map $\mathscr{A}$.*

2. *Each sequence of the form $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ converges to one of the fixed points.*

3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality (3.1.24):*

$$\sigma(\dot{\xi},\dot{\eta}) \geq \frac{\epsilon}{3(\alpha+\beta)}. \tag{3.1.24}$$

Proof The procedure for proving the above theorem is analogous to the proof of Theorem 3.2. ■

This example illustrates the validity of Theorem 3.10.

Example 3.7 *Let $\Upsilon = \{m, p, q\}$ be a set of three distinct elements equipped with a metric σ defined as follows: $\sigma(p,q) = 2$, $\sigma(p,m) = \sigma(q,m) = 1$. Furthermore, it holds that $\sigma(u,v) = \sigma(v,u)$ for every pair of elements u and v belonging to the set Υ. Additionally, we have $\sigma(u,u) = 0$ for each element u in the set Υ.*

Consider a self-map $\mathscr{A}$ on the set Υ, where $\mathscr{A}(p) = p$, $\mathscr{A}(q) = q$, and $\mathscr{A}(m) = p$.

It can be immediately deduced that (Υ,σ) forms a complete metric space, and the inequality (3.1.23) holds for every $u, v \in \Upsilon$ with the constants $\alpha = \beta = \frac{1}{5}$, $\gamma = \delta = 1$, and $\epsilon = \frac{1}{2}$.

From Theorem 3.10, we deduce that the self-map $\mathscr{A}$ has at least one fixed point. Indeed, the given function has two distinct fixed points, p and q. Furthermore, it can be observed that the distance between these fixed points satisfies the inequality $\sigma(p,q) \geq \frac{5}{12}$.

3.2 Application to solvability of equations in $\mathbb{Z}/n\mathbb{Z}$

In this section, let n be a natural number greater than or equal to 2.

We will discuss solving equations in $\mathbb{Z}/n\mathbb{Z}$. Our study will focus on the existence of solutions and the distances between them, without explicitly determining their values.

The problem posed is that $\mathbb{Z}/n\mathbb{Z}$ is not a vector space. However, we can consider it as a metric space, which enables us to apply our theorem.

We begin by presenting our problem in the following form:

$$au \equiv b[n], \tag{3.2.1}$$

where $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$, with the unknown $u \in \mathbb{Z}$.

Next, we reformulate the previous equation to make it applicable to our theorem:

$$\bar{a}\bar{u} = \bar{b}, \tag{3.2.2}$$

where $\bar{a}, \bar{b} \in \mathbb{Z}/n\mathbb{Z}$, with the unknown $\bar{u} \in \mathbb{Z}/n\mathbb{Z}$.

Since the equations (3.2.1) and (3.2.2) are equivalent, it is enough to study the second one.

Define the self-map $\mathscr{A}$ as follows:

$$\mathscr{A} : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$$
$$\bar{u} \longmapsto \mathscr{A}(\bar{u}) = (\overline{a+1})\bar{u} - \bar{b}$$

Theorem 3.11 *Let $(\mathbb{Z}/n\mathbb{Z}, \sigma)$ be a metric space. If the following conditions are satisfied:*

1- rang$(\mathscr{A})$ *is a closed subset of* $\mathbb{Z}/n\mathbb{Z}$,

2- *There exist non-negative constants* $\alpha, \beta, \gamma, \delta$ *and* ϵ *with* $\epsilon > 0$, *satisfying either* $\alpha \leq \frac{1}{2}\min\{\gamma, \delta\}$ *or* $\beta \leq \frac{1}{2}\min\{\gamma, \delta\}$, *and such that inequality* (3.2.3) *holds for each* $u, v \in \text{rang}\mathscr{A}$,

$$\sigma(\mathscr{A}(u), \mathscr{A}(v)) \leq \frac{\alpha\sigma(u, \mathscr{A}(v)) + \beta\sigma(v, \mathscr{A}(u))}{\gamma\sigma(u, \mathscr{A}(u)) + \delta\sigma(v, \mathscr{A}(v)) + \epsilon} \max\{\sigma(u, \mathscr{A}(v)), \sigma(v, \mathscr{A}(u))\}. \tag{3.2.3}$$

Then, equation (3.2.2) admits at least one solution. Moreover, if the equation admits more than one solution, we can provide dynamic information about the solutions, as shown in equation (3.1.5).

Proof Consider the metric space $\Upsilon = (\mathbb{Z}/n\mathbb{Z}, \sigma)$. The space $(\mathbb{Z}/n\mathbb{Z}, \sigma)$ is complete, as the range of the self-map $\mathscr{A}$ is a closed subset of Υ.

Based on the previous result, equation (3.2.3), and by applying Theorem 3.2, we conclude that no less than one fixed point $\bar{u} \in \Upsilon$ exists for $\mathscr{A}$. Since the fixed points of the map $\mathscr{A}$ correspond to the solutions of equation (3.2.2), the desired results follow. ■

Now, we present the following example to confirm the applicability of our theorem to the solvability of algebraic problems in $\mathbb{Z}$.

Example 3.8 *Let* $\mathbb{Z}/6\mathbb{Z}$ *be the set of equivalence classes for congruence modulo* 6, *associated with the metric distance* σ *defined by* $\sigma(\bar{u}, \bar{v}) = |u - v|$, *where* u *and* v *refer to the minimum representations of the classes* $\bar{u}$ *and* $\bar{v}$ *respectively. It is immediately deduced that the metric space* $(\mathbb{Z}/6\mathbb{Z}, \sigma)$ *is complete.*

Consider the following linear equation in $\mathbb{Z}/6\mathbb{Z}$*:*

$$\bar{3}\bar{u} = \bar{3}. \tag{3.2.4}$$

Let $\mathscr{A}$ *be the associated self-map to this equation,*

$$\begin{aligned} \mathscr{A} : \mathbb{Z}/n\mathbb{Z} &\longrightarrow \mathbb{Z}/n\mathbb{Z} \\ \bar{u} &\longmapsto \mathscr{A}(\bar{u}) = \bar{4}\bar{u} + \bar{3}. \end{aligned}$$

Here, $\text{rang}\mathscr{A} = \{\bar{1}, \bar{3}, \bar{5}\}$, *which is a closed subset of* $\mathbb{Z}/6\mathbb{Z}$.

By choosing $\alpha = \beta = 1$, $\gamma = \delta = 2$, *and* $\epsilon = 3$, *the inequality (3.2.3) is satisfied for each* $u, v \in \text{rang}\mathscr{A}$. *Applying Theorem 3.11, we conclude that equation (3.2.4) admits at least one solution.*

We observe that this equation has exactly three solutions $S = \{\bar{1}, \bar{3}, \bar{5}\}$. *Without determining their number or exact values, we can deduce that for each distinct pair of solutions,* $\sigma(\bar{u}, \bar{v}) \geq \frac{3}{2}$.

In order to improve this result, we adjust the constants $\alpha, \beta, \gamma, \delta$, *and* ϵ *to the following values:* $\alpha = \beta = \frac{1}{2}$, $\epsilon = \delta = 2$, *and* $\gamma = 1$. *This modification keeps the validity of the inequality (3.2.3) and upgrades the dynamic information to: for each distinct pair of solutions, we have* $\sigma(\bar{u}, \bar{v}) \geq 2$.

Chapter 4 Dynamic fixed point results for rational contractive maps on Menger PM spaces

The introduction of the general concept of statistical metric spaces is attributed to K. Menger [88], who developed this theory within the framework of probabilistic geometry. Subsequently, many researchers advanced the theory of fundamental probabilistic structures by introducing new derivative spaces, such as fuzzy spaces, Wald spaces, intuitionistic Menger spaces, and others.

The classical metric space is defined by the introduction of the function σ, which represents a positive real number, written as $\sigma(u, v)$ (the distance between u and v), for each pair (u, v) of elements in a non-empty set Υ. However, in certain contexts, the exact value of $\sigma(u, v)$ may not be known, and only the probability of several possible values for this distance can be determined.

In 1942, Menger proposed a probabilistic generalization of a metric space, providing the first definition of a probabilistic metric space (Menger space), which was later refined by Schweizer and Sklar [113]. Menger's idea was to replace the precise distance $\sigma(u, v)$ between two points with a distance distribution function $F_{u,v}$. This function assigns to each positive real number t the probability that the distance between the two points u and v is less than or equal to t, i.e., $F_{u,v}(t) = p(\sigma(u, v) \leq t)$. Additionally, the triangle inequality was replaced by a condition involving triangle functions on these distributions.

Probabilistic metric spaces serve as a significant extension of classical metric spaces. This framework facilitates the analysis of random and probabilistic fluctuations in fields such as biology and physics. These generalizations provide more robust and adaptable tools for addressing complicated nonlinear problems, enabling solutions that are otherwise unattainable within the constraints of traditional metric spaces. For further information on the significance of this field in applied mathematics, the reader is referred to [118].

Additional properties and further information on probabilistic spaces, particularly Menger spaces, can be found in section 1.3 and the references therein.

4.1 Fixed point theorems for contraction mappings in Menger PM spaces

Fixed point theory in probabilistic metric spaces can be considered as part of probabilistic analysis, a highly dynamic area of mathematical research. In 1972, Sehgal and Barucha-Reid

[114] made a significant contribution to this field by presenting the first fixed point theorem for probabilistic metric spaces. In their pioneering work, they introduced the concept of a probabilistic contraction mapping, which generalizes classical notions of contraction mappings to accommodate the inherent uncertainty in probabilistic spaces, as follows.

Definition 4.1 *[114, 115] A mapping $\mathscr{A}$ of a PM-space $(\Upsilon, \mathscr{F})$ into itself is called a contraction mapping if and only if there exists a constant k, with $0 < k < 1$, such that for each $u, v \in \Upsilon$,*

$$F_{\mathscr{A}(u),\mathscr{A}(v)}(kt) \geq F_{u,v}(t) \quad \textit{for all } t > 0. \tag{4.1.1}$$

4.1.1 On the relationship between Banach contractions and PM contractions

Inequality (4.1.1) generalizes the Banach contraction inequality for metric spaces. That is:

Proposition 4.1 *[114, 115] Every Banach contraction mapping on a metric space (Υ, σ) is also a contraction mapping on the PM space induced by the metric σ.*

Proof Let $\mathscr{A} : \Upsilon \to \Upsilon$ be a contraction mapping on the metric space (Υ, σ). To prove that the Banach contraction inequality implies (4.1.1), recall that every metric space (Υ, σ) can be viewed as a Menger space $(\Upsilon, \mathscr{F}, \min)$, where $\mathscr{F}$ is defined as follows:

$$\begin{aligned} F_{u,v}(t) &= H(t - \sigma(u,v)), \\ &= \begin{cases} 1 & \text{if } \sigma(u,v) < t, \\ 0 & \text{if } \sigma(u,v) \geq t \end{cases} \quad (t \in \mathbb{R}). \end{aligned}$$

Suppose that $\mathscr{A} : \Upsilon \to \Upsilon$ is a Banach contraction. We show that inequality (4.1.1) is satisfied, i.e., for every $t > 0$,

$$F_{u,v}(kt) = 1 \implies F_{\mathscr{A}(u),\mathscr{A}(v)}(t) = 1.$$

If $F_{u,v}(kt) = 1$, then $\sigma(u,v) < kt$, and the Banach contraction implies

$$\sigma(\mathscr{A}(u), \mathscr{A}(v)) \leq k^2 \cdot t < t,$$

which means $F_{\mathscr{A}(u),\mathscr{A}(v)}(t) = 1$. ■

4.1.2 Fixed point result for PM contraction mappings

In fact, the previous definition was given by Sehgal [115] years earlier in his thesis (1966), where he proved Theorem 4.2 in the context of metric probability spaces. He showed that every contraction map on a complete PM space that satisfies the strongest form of Menger's triangle inequality has a unique fixed point.

In 1971, Sherwood [117] provided a strong plausibility argument that indicate that this result is the exception rather than the rule for PM spaces. He constructed a complete PM space along with contraction maps in the sense of Sehgal which have no fixed points.

Theorem 4.2 *[115, 117] A contraction map on a PM space has at most one fixed point.*

The following theorem is the first fixed point result within the context of complete Menger spaces, established by Sehgal and Barucha-Reid in 1972 [114]. It generalizes the Banach contraction principle, as noted in the preceding discussion.

Theorem 4.3 *[114] Let $(\Upsilon, \mathscr{F}, *)$ be a complete Menger space, where $*$ is a continuous t-norm satisfying $t * t \geq t$ for each $t \in [0,1]$. If $\mathscr{A}$ is a contraction mapping of Υ into itself, then there exists a unique $\dot{\xi} \in \Upsilon$ such that $\mathscr{A}(\dot{\xi}) = \dot{\xi}$. Moreover, $\mathscr{A}^n(v)$ converges to $\dot{\xi}$ for any $v \in \Upsilon$.*

Proof

Let $\mathscr{A}$ be a contraction mapping on a complete Menger space $(\Upsilon, \mathscr{F}, *)$.

Step 1: We prove that $\mathscr{A}$ has at most one fixed point in Υ.

Assume $\mathscr{A}$ has two distinct fixed points, denoted by u and v, i.e., $\mathscr{A}(u) = u$ and $\mathscr{A}(v) = v$, with $u \neq v$.

From (1.3.2), there exist $t > 0$ and a with $0 \leq a < 1$ such that $F_{u,v}(t) = a$. However, for each positive integer n, it follows from (4.1.1) that

$$a = F_{u,v}(t) = F_{\mathscr{A}(u),\mathscr{A}(v)}(t) \geq F_{u,v}(t/k^n). \tag{4.1.2}$$

Since $F_{u,v}(t/k^n) \to 1$ as $n \to \infty$, it follows that $a = 1$. This contradicts the selection of a. Thus, $\mathscr{A}$ has at most one fixed point in Υ.

Next, we prove the existence of a fixed point. Consider the iterative sequence $u_n = \mathscr{A}^n(v)$, $n \in \mathbb{N}$ based on an arbitrary $v \in \Upsilon$.

Step 2: We show that the sequence $\{u_n\}$ is a Cauchy sequence.

Let ϵ, λ be positive reals. Then, for any pair of natural numbers n and m where $m > n$, we observe

$$\begin{aligned} F_{u_n,u_m}(\epsilon) &\geq F_{u_n,u_{n+1}}(\epsilon - k\epsilon) * F_{u_{n+1},u_m}(k\epsilon), \\ &\geq F_{v,u_1}((\epsilon - k\epsilon)k^{-n}) * F_{u_{n+1},u_m}(k\epsilon). \end{aligned}$$

By Menger's triangle inequality and the monotonicity of the t-norm $*$, we have

$$\begin{aligned} F_{u_n,u_m}(\epsilon) &\geq F_{v,u_1}((\epsilon - k\epsilon)k^{-n}) * F_{u_{n+1},u_{n+2}}(k\epsilon - k^2\epsilon) * F_{u_{n+2},u_m}(k^2\epsilon) \\ &\geq F_{v,u_1}((\epsilon - k\epsilon)k^{-n}) * F_{v,u_1}((\epsilon - k\epsilon)k^{-n}) * F_{u_{n+2},u_m}(k^2\epsilon). \end{aligned}$$

By the associativity of $*$ and the hypothesis $t * t \geq t$, we obtain

$$F_{u_n,u_m}(\epsilon) \geq F_{v,u_1}((\epsilon - k\epsilon)k^{-n}) * F_{u_{n+2},u_m}(k^2\epsilon). \tag{4.1.3}$$

Using the same reasoning repeatedly, we obtain:

$$\begin{aligned} F_{u_n,u_m}(\epsilon) &\geq F_{v,u_1}((\epsilon - k\epsilon)k^{-n}) * F_{u_{m-1},u_m}\left(k^{m-n-1}\epsilon\right) \\ &\geq F_{v,u_1}((\epsilon - k\epsilon)k^{-n}) * F_{v,u_1}\left(k^{-n}\epsilon\right) \\ &\geq F_{v,u_1}((\epsilon - k\epsilon)k^{-n}) * F_{v,u_1}((\epsilon - k\epsilon)k^{-n}) \\ &\geq F_{v,u_1}\left((\epsilon - k\epsilon)k^{-n}\right) \end{aligned}$$

Consequently, if we choose N such that $F_{v,u_1}\left((\epsilon - k\epsilon)k^{-N}\right) > 1-\lambda$, it follows that $F_{u_n,u_m}(\epsilon) > 1 - \lambda$ for all $n \geq N$. Thus, $\{u_n\}$ is a Cauchy sequence.

Since $(\Upsilon, F, *)$ is a complete Menger space, there exists a $\dot{\xi}$ in Υ such that $u_n \to \dot{\xi}$, i.e., $\mathscr{A}^n(v) \to \dot{\xi}$.

Step 3: We show that $\dot{\xi}$ is a fixed point of the mapping $\mathscr{A}$.

Let $N_{\mathscr{A}(\dot{\xi})}(\epsilon, \lambda)$ be any neighborhood of $\mathscr{A}(\dot{\xi})$. Then, $u_n \to \dot{\xi}$ implies the existence of an integer N such that $u_n \in N(\epsilon, \lambda)$ for all $n \geq N$. However,

$$
\begin{aligned}
F_{\mathscr{A}(u_n),\mathscr{A}(\dot{\xi})}(\epsilon) &\geq F_{u_n,\dot{\xi}}\left(\frac{\epsilon}{k}\right) \\
&\geq F_{u_n,\dot{\xi}}(\epsilon) \\
&> 1-\lambda
\end{aligned}
$$

for all $n \geq N$. Therefore, $\mathscr{A}(u_n) \in N(\epsilon,\lambda)$ for all $n \geq N$, i.e., $\mathscr{A}^n(v) \to \mathscr{A}(\dot{\xi})$. Thus, we conclude that $\mathscr{A}(\dot{\xi}) = \dot{\xi}$. This proves the existence part of the theorem. ■

Remark 4.1 *Theorem 4.3 holds for a Menger space equipped with the t-norm* min. *Furthermore, it remains valid for the following t-norms:*

$$
\begin{aligned}
a * b &= \max\{a, b\}, \\
a * b &= a + b - ab, \\
a * b &= \min\{a + b, 1\}.
\end{aligned}
$$

The following example supports and illustrates Theorem 4.3.

Example 4.1 *Let $\Upsilon = [0,1]$ and $F_{u,v}(t) = \dfrac{t}{t+|u-v|}$ for all $u, v \in \Upsilon, t > 0$. Hence, $(\Upsilon, F, \min)$ is a complete Menger space (see section 1.3). Let the mapping $\mathscr{A}$ be defined as:*

$$\mathscr{A}(u) = \frac{u}{4} \quad \text{for all } u \in [0,1].$$

We will show that $\mathscr{A}$ is a $\frac{1}{2}$-contracting mapping:

$$
\begin{aligned}
F_{\mathscr{A}(u)\mathscr{A}(v)}(\frac{1}{2}t) &= \frac{\frac{1}{2}t}{\frac{1}{2}t + \frac{|u-v|}{4}} \\
&= \frac{t}{t+\frac{1}{2}|u-v|}, \\
&\geq F_{u,v}(t).
\end{aligned}
$$

Thus, we have shown that the mapping $\mathscr{A}$ satisfies all the assumptions of Theorem 4.3. Consequently, it has a unique fixed point $\dot{\xi} = 0$.

This foundational result has inspired a wide range of further research (see, for example, [84, 90, 113]). One of the most significant contributions to fixed point theory in probabilistic metric spaces is attributed to Hadzic [53, 54], which led to numerous generalizations and applications across various branches of mathematical analysis. For additional fixed point theorems in this area, we refer the reader to [54] and the references cited therein.

4.2 Dynamic information theorems about fixed points for rational contractive mappings on Menger PM spaces

In this section, we extend the theorems presented in Chapter 2 to the framework of Menger probabilistic metric (PM) spaces. Additionally, we explore the dynamic properties of the fixed-point sets associated with such mappings, highlighting the intricate relationships and behaviors that emerge in this probabilistic context.

Before delving into our original findings, we first introduce the following definition, which plays a crucial role in the theoretical developments presented in this chapter.

Definition 4.2 *Let $(\Upsilon, F, \min)$ be a Menger PM space, and let $\mathscr{A}$ be a self-mapping on Υ.*

The mapping $\mathscr{A}$ is called a rational PM contraction mapping if and only if there exist four non-negative real numbers $\alpha, \beta, \gamma, \delta$ (with at most one of α or β equal to zero) such that

$$\max\left\{\frac{\alpha+\beta}{\gamma+\delta}, \frac{\beta}{\delta}\right\} < 1 \tag{4.2.1}$$

and

$$1 - F_{\mathscr{A}(u),\mathscr{A}(v)}(t) \le \frac{\alpha \min\left\{F_{u,\mathscr{A}(u)}\left(\frac{t}{2}\right), F_{v,\mathscr{A}(v)}\left(\frac{t}{2}\right)\right\} + \beta}{\gamma \min\left\{F_{u,\mathscr{A}(v)}(t), F_{v,\mathscr{A}(u)}(t)\right\} + \delta} \cdot (1 - F_{u,v}(t)), \tag{4.2.2}$$

for all $u, v \in \Upsilon$ and $t > 0$.

4.2.1 Non-uniqueness fixed point result

The following theorem establishes a non-uniqueness fixed point result, guaranteeing the existence of at least one fixed point in the context of a complete Menger PM space under the rational PM contraction condition.

Theorem 4.4 *Any rational PM contraction mapping (in the sens of Definition 4.2) defined on a complete Menger PM space has at least one fixed point. Moreover:*

1. *Every Picard sequence converges to a fixed point;*
2. *If the mapping has two distinct fixed points $\dot{\xi}$ and $\dot{\eta}$, then for all $t > 0$,*

$$F_{\dot{\xi},\dot{\eta}}(t) \le \max\left\{\frac{\alpha+\beta-\delta}{\gamma}, 0\right\} \quad or \quad F_{\dot{\xi},\dot{\eta}}(t) = 1.$$

Proof Let $(\Upsilon, F, \min)$ be a complete Menger PM space, and let $\mathscr{A}$ be a self-mapping on Υ.

Let $(u_n)_{n\in\mathbb{N}}$ be a Picard sequence in Υ with an arbitrary initial value $u_0 \in \Upsilon$ and recurrence relation $u_{n+1} = \mathscr{A}(u_n)$, where $n \in \mathbb{N}$. Let $t > 0$.

Claim 1: The sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

By setting $u = u_{n-1}$ and $v = u_n$ in inequality (4.2.2), we obtain

$$\begin{aligned}
1 - F_{\mathscr{A}(u_{n-1}),\mathscr{A}(u_n)}(t) &\le \frac{\alpha \min\{F_{u_{n-1},u_n}(\frac{t}{2}), F_{u_n,u_{n+1}}(\frac{t}{2})\} + \beta}{\gamma \min\{F_{u_{n-1},u_{n+1}}(t), F_{u_n,\mathscr{A}(u_n)}(t)\} + \delta}(1 - F_{u_{n-1},u_n}(t)) \\
&\le \frac{\alpha F_{u_{n-1},u_{n+1}}(t) + \beta}{\gamma F_{u_{n-1},u_{n+1}}(t) + \delta}(1 - F_{u_{n-1},u_n}(t)) \\
&\le \max\left\{\frac{\alpha+\beta}{\gamma+\delta}, \frac{\beta}{\delta}\right\}(1 - F_{u_{n-1},u_n}(t)) \\
&\vdots \\
&\le \left(\max\left\{\frac{\alpha+\beta}{\gamma+\delta}, \frac{\beta}{\delta}\right\}\right)^n (1 - F_{u_0,u_1}(t)).
\end{aligned}$$

Define $\theta_n(t) = \frac{\alpha F_{u_{n-1},u_{n+1}}(t)+\beta}{\gamma F_{u_{n-1},u_{n+1}}(t)+\delta}$ for each $n \in \mathbb{N}$.

Let $n, m \in \mathbb{N}$ with $m \ge n$. Then, we have

$$F_{u_n,u_m}(t) \ge \min\{F_{u_n,u_{n+1}}(\frac{t}{m-n}), F_{u_{n+1},u_{n+2}}(\frac{t}{m-n}), \ldots, F_{u_{m-1},u_m}(\frac{t}{m-n})\}. \tag{4.2.3}$$

Thus,

$$\begin{aligned} 1-F_{u_n,u_m}(t) &\le \max\{1-F_{u_n,u_{n+1}}(\frac{t}{m-n}),1-F_{u_{n+1},u_{n+2}}(\frac{t}{m-n}),\ldots,1-F_{u_{m-1},u_m}(\frac{t}{m-n})\} \\ &\le \theta_n(\frac{t}{m-n})\theta_{n-1}(\frac{t}{m-n})\cdots\theta_1(\frac{t}{m-n})(1-F_{u_0,u_1}(\frac{t}{m-n})) \\ &\le \max\left\{\frac{\alpha+\beta}{\gamma+\delta},\frac{\beta}{\delta}\right\}^n(1-F_{u_0,u_1}(\frac{t}{m-n})) \\ &\le \max\left\{\frac{\alpha+\beta}{\gamma+\delta},\frac{\beta}{\delta}\right\}^n. \end{aligned}$$

Taking the limit as $n,m\to+\infty$, we deduce

$$\lim_{n,m\to+\infty}F_{u_n,u_m}(t)=1 \quad \forall t>0. \tag{4.2.4}$$

Thus, $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.
As the space Υ is complete, there exists $\dot{\xi}\in\Upsilon$ such that $\lim_{n\to+\infty}u_n=\dot{\xi}$.
Claim 2: $\dot{\xi}$ is a fixed point of $\mathscr{A}$.
By setting $u=\dot{\xi}$ and $v=u_n$ in inequality (4.2.2), we obtain

$$1-F_{\mathscr{A}(\dot{\xi}),\mathscr{A}(u_n)}(t)\le\frac{\alpha\min\{F_{\dot{\xi},\mathscr{A}(\dot{\xi})}(\frac{t}{2});F_{u_n,u_{n+1}}(\frac{t}{2})\}+\beta}{\gamma\min\{F_{\dot{\xi},u_{n+1}}(t);F_{u_n,\dot{\xi}}(t)\}+\delta}(1-F_{\dot{\xi},u_n}(t)).$$

Taking the limit as $n\to+\infty$, we find

$$1-F_{\mathscr{A}(\dot{\xi}),\dot{\xi}}(t)\le\frac{\alpha F_{\dot{\xi},\mathscr{A}(\dot{\xi})}(\frac{t}{2})+\beta}{\gamma F_{\dot{\xi},\dot{\xi}}(t)+\delta}(1-F_{\dot{\xi},\dot{\xi}}(t)).$$

Thus,

$$\lim_{n\to+\infty}F_{\mathscr{A}(\dot{\xi}),\dot{\xi}}(t)=1 \quad \text{for all } t>0.$$

Therefore, $\mathscr{A}(\dot{\xi})=\dot{\xi}$, i.e., $\mathscr{A}$ has at least one fixed point in Υ.

Claim 3: We assume that the mapping $\mathscr{A}$ has two distinct fixed points in Υ. We seek to determine the probabilistic distribution distance between these two fixed points.
Let $\dot{\xi},\dot{\eta}\in\Upsilon$ be two fixed points of $\mathscr{A}$.
By setting $u=\dot{\xi}$ and $v=\dot{\eta}$ in inequality (4.2.2), we obtain

$$1-F_{\mathscr{A}(\dot{\xi}),\mathscr{A}(\dot{\eta})}(t)\le\frac{\alpha+\beta}{\gamma F_{\dot{\xi},\dot{\eta}}(t)+\delta}(1-F_{\dot{\xi},\dot{\eta}}(t)).$$

This implies that

$$\gamma F^2_{\dot{\xi},\dot{\eta}}(t)+(\delta-\gamma-\alpha-\beta)F_{\dot{\xi},\dot{\eta}}(t)+\alpha+\beta-\delta\ge0 \quad \text{for all } t>0,$$

Therefore,

$$F_{\dot{\xi},\dot{\eta}}(t)\le\max\left\{\frac{\alpha+\beta-\delta}{\gamma},0\right\} \quad \text{or} \quad F_{\dot{\xi},\dot{\eta}}(t)=1.$$

This completes the proof.

■

4.2.2 Uniqueness fixed point result

In this section, we derive a uniqueness theorem that refines Theorem 4.4. This result establishes conditions under which rational contraction mappings defined on a complete Menger PM space guarantee the existence of a unique fixed point. The following definition presents a special case of rational contraction mappings that will play a crucial role in the subsequent theorem.

Definition 4.3 *Let $(\Upsilon, F, \min)$ be a Menger PM space, and let $\mathscr{A}$ be a self-mapping on Υ.*

The mapping $\mathscr{A}$ will be called a strict-rational PM contraction mapping if and only if there exist four non-negative real numbers $\alpha, \beta, \gamma, \delta$ (with at most one of α or β equal to zero) such that $\alpha+\beta<\delta$ and satisfying the inequality (4.2.2) for all $u, v \in \Upsilon$ and $t>0$.

Theorem 4.5 *Any strict-rational PM contraction mapping defined on a complete Menger PM space has a unique fixed point on it.*

Proof Let $(\Upsilon, F, \min)$ be a complete Menger PM space, and let $\mathscr{A}$ be a self-mapping on Υ.

We start by proving that the mapping $\mathscr{A}$ has at most one fixed point in Υ.

Let $\dot{\xi}$ and $\dot{\eta} \in \Upsilon$ be two fixed points of $\mathscr{A}$, and let $t>0$.

By setting $u=\dot{\xi}$ and $v=\dot{\eta}$ in inequality (4.2.2), we get

$$1-F_{\mathscr{A}(\dot{\xi}),\mathscr{A}(\dot{\eta})}(t) \leq \frac{\alpha+\beta}{\gamma F_{\dot{\xi},\dot{\eta}}(t)+\delta}(1-F_{\dot{\xi},\dot{\eta}}(t)).$$

Since $\dot{\xi}$ and $\dot{\eta}$ are fixed points, we have $\mathscr{A}(\dot{\xi})=\dot{\xi}$ and $\mathscr{A}(\dot{\eta})=\dot{\eta}$. Substituting these equalities, the inequality becomes

$$\gamma F^2_{\dot{\xi},\dot{\eta}}(t)+(\delta-\gamma-\alpha-\beta)F_{\dot{\xi},\dot{\eta}}(t)+\alpha+\beta-\delta \geq 0 \quad \text{for all } t>0.$$

Since $\alpha+\beta<\delta$, this implies that $F_{\dot{\xi},\dot{\eta}}(t)=1$ for all $t>0$, hence $\dot{\xi}=\dot{\eta}$. Thus, $\mathscr{A}$ possesses at most one fixed point within the space Υ.

For the existence of a fixed point, the proof follows from the argument detailed in Theorem 4.4.

This completes the proof.

■

The following example illustrates and supports Theorem 4.5.

Example 4.2 Let $\Upsilon=\{0,1,2\}$ be a set associated with the following distribution functions:

$$F_{0,1}(t)=F_{1,2}(t)=\begin{cases} 0 & \text{if } t \leq 2, \\ 1 & \text{if } t>2. \end{cases}$$

$$F_{0,2}(t)=\begin{cases} 0 & \text{if } t \leq 0, \\ \frac{1}{2} & \text{if } 0<t \leq 3, \\ 1 & \text{if } t>3. \end{cases}$$

$$F_{u,u}(t)=H(t) \quad \text{for all } u \in \Upsilon \text{ and for all } t \in \mathbb{R},$$

$$F_{u,v}(t)=F_{v,u}(t) \quad \text{for all } u, v \in \Upsilon \quad \text{and for all } t \in \mathbb{R}.$$

The triple $(\Upsilon, \mathscr{F}, \min)$ constitutes a Menger PM space because it satisfies all the conditions outlined in Definitions 1.21 and 1.23. Conditions 1-3 of Definition 1.21 are evidently met. It remains to prove the fourth condition:

- If $F_{0,1}(t) = 1$ and $F_{0,2}(t) = 1$, then $t > 2$ and $s > 3$, leading to $t + s > 5$. Therefore, $F_{1,2}(t + s) = 1$.
- If $F_{0,2}(t) = 1$ and $F_{1,2}(t) = 1$, then $t > 3$ and $s > 2$, leading to $t + s > 5$. Therefore, $F_{0,1}(t + s) = 1$.
- If $F_{0,1}(t) = 1$ and $F_{1,2}(t) = 1$ then, $t > 2$ and $s > 2$, leading to $t + s > 4$. Therefore, $F_{0,2}(t + s) = 1$.

Moreover, Menger's triangle inequality is satisfied, as for all $t, s > 0$

$$F_{0,2}(t+s) \geq \min\{F_{0,1}(t); F_{1,2}(s)\}, \tag{4.2.5}$$

$$F_{0,1}(t+s) \geq \min\{F_{0,2}(t); F_{1,2}(s)\}, \tag{4.2.6}$$

$$F_{1,2}(t+s) \geq \min\{F_{0,1}(t); F_{0,2}(s)\}. \tag{4.2.7}$$

Additionally, $(\Upsilon, \mathscr{F}, \min)$ is a complete Menger PM space. Let $\mathscr{A}$ be a self-mapping defined on Υ as follows:

$$\mathscr{A}(0) = 0 \quad \mathscr{A}(1) = 1 \quad \mathscr{A}(2) = 0.$$

We proceed to confirm the validity of inequality (4.2.2) by setting $\alpha = \gamma = \delta = 1$ and $\beta = \frac{1}{2}$. Given this choice, we observe that $\dfrac{\alpha+\beta}{\gamma+\delta} < 1$ and $\dfrac{\beta}{\delta} < 1$.

We now verify inequality (4.2.2) under each of the possible scenarios.

In the case where $u = v$, inequality (4.2.2) is immediately verified.

For $u = 0$ and $v = 1$, we have

$$1 - F_{\mathscr{A}(0),\mathscr{A}(1)}(t) = 1 - F_{0,1}(t) = \begin{cases} 1 & \text{if } t \leq 2, \\ 0 & \text{if } t > 2. \end{cases}$$

and

$$\begin{aligned} \frac{\min\{F_{0,\mathscr{A}(0)}(\frac{t}{2}); F_{1,\mathscr{A}(1)}(\frac{t}{2})\} + \frac{1}{2}}{\min\{F_{0,\mathscr{A}(1)}(t); F_{1,\mathscr{A}(0)}(t)\} + 1}(1 - F_{0,1}(t)) &= \frac{1.5}{F_{0,1}(t)+1} \\ &= \begin{cases} 1.5 & \text{if } t \leq 2, \\ 0.75 & \text{if } t > 2. \end{cases} \end{aligned}$$

Hence,

$$1 - F_{0,1}(t) \leq \frac{1.5}{F_{0,1}(t)+1} \quad \text{for all } t > 0.$$

For $u = 0$ and $v = 2$, we have

$$\begin{aligned} 1-F_{\mathscr{A}(0),\mathscr{A}(2)}(t) &= 1-F_{0,0}(t) \\ &= 0 \\ &\leq \frac{\min\{F_{0,\mathscr{A}(0)}(\frac{t}{2});F_{2,\mathscr{A}(2)}(\frac{t}{2})\}+\frac{1}{2}}{\min\{F_{0,\mathscr{A}(2)}(t);F_{2,\mathscr{A}(0)}(t)\}+1}(1-F_{0,2}(t)). \end{aligned}$$

For $u=1$ and $v=2$, we have

$$1-F_{\mathscr{A}(1),\mathscr{A}(2)}(t)=1-F_{1,0}(t)=\begin{cases} 1 & \text{if } t\leq 2, \\ 0 & \text{if } t>2. \end{cases}$$

and

$$\begin{aligned} \frac{\min\{F_{1,\mathscr{A}(1)}(\frac{t}{2});F_{2,\mathscr{A}(2)}(\frac{t}{2})\}+\frac{1}{2}}{\min\{F_{1,\mathscr{A}(2)}(t);F_{2,\mathscr{A}(1)}(t)\}+1}(1-F_{1,2}(t)) &= \frac{F_{0,2}(\frac{t}{2})+\frac{1}{2}}{F_{1,2}(t)+1}(1-F_{1,2}(t)) \\ &= \begin{cases} \frac{1}{2} & \text{if } t\leq 0, \\ 1 & \text{if } 0<t\leq 2, \\ 0 & \text{if } t>2. \end{cases} \end{aligned}$$

Hence,

$$1-F_{1,0}(t)\leq \frac{F_{0,2}(\frac{t}{2})+\frac{1}{2}}{F_{1,2}(t)+1}(1-F_{1,2}(t)) \quad \text{for all } t>0.$$

All conditions of Theorem 4.4 have been verified. Therefore, the mapping $\mathscr{A}$ possesses at least one fixed point in Υ (specifically, it has two fixed points: 0 and 1). Moreover, it holds that $F_{0,1}(t)\leq \frac{1}{2}$ or $F_{0,1}(t)=1$.

Additionally, every Picard sequence $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ converges to a fixed point. Specifically:
- If $u_0=0$, then $\mathscr{A}(u_n)=0$ for all $n\in\mathbb{N}$, which converges to 0.
- If $u_0=1$, then $\mathscr{A}(u_n)=1$ for all $n\in\mathbb{N}$, which converges to 1.
- If $u_0=2$, then $\mathscr{A}(u_n)=0$ for all $n\in\mathbb{N}$, which converges to 0.

Remark 4.2 *Very interesting results can be obtained in the same framework of Menger PM space for the case of multivalued operators.*

Chapter 5 Fixed point theorems and dynamic results in b-metric spaces under rational generalized contractive conditions with applications

The concept of a b-metric space generalizes the standard metric space to accommodate a broader class of spaces, where the notion of distance is more flexible. Unlike traditional metric spaces, where the distance function must satisfy the triangular inequality, b-metric spaces relax this requirement by introducing a multiplicative constant. However, this relaxation comes at the cost of losing certain key properties, such as the continuity of the metric function, which can complicate theoretical analysis. Despite these challenges, this generalization has led to the development of new mathematical frameworks and methods, significantly broadening the scope of fixed-point theorems beyond the classical metric setting.

Fixed-point theory, a critical tool in many areas of mathematics, particularly in analysis and topology, has seen substantial advancements within the framework of b-metric spaces. The foundational result in fixed-point theory for metric spaces is Banach's contraction principle, which guarantees the existence and uniqueness of a fixed point under specific contractive conditions. In 1993, Czerwik [32] extended Banach's contraction principle to b-metric spaces, establishing foundational results that have since inspired extensive research in this area. Many authors have subsequently generalized several classical fixed-point theorems to accommodate a variety of contractive conditions and different types of mappings in b-metric spaces, including both single-valued and multi-valued mappings.

Motivated by the extensive literature on linear and nonlinear contractive operators cited in Chapter 2, this chapter presents both uniqueness and non-uniqueness fixed-point results for mappings that satisfy rational-type contractive conditions in b-metric spaces. These results extend the work of Khojasteh [76] and Aouine [9] to the setting of b-metric spaces. Furthermore, compared to Khojasteh's results, this chapter offers more precise dynamic insights. Most of the theorems in this chapter have been published in convenable journals; see [12, 83].

5.1 Banach's contraction principle in the context of b-metric spaces

The following theorem is one of the earliest fixed-point results in b-metric spaces, proven by Czerwik [32]. It serves as an extension of the classical metric case.

Theorem 5.1 *[32] Let (Υ,τ) be a complete b-metric space with constant $s \geq 1$, and suppose that $\mathscr{A}:\Upsilon\to\Upsilon$ satisfies*

$$\tau(\mathscr{A}(u),\mathscr{A}(v)) \leq \varphi(\tau(u,v))$$

for all $u,v\in\Upsilon$, where $\varphi:[0,\infty)\to[0,\infty)$ is an increasing function such that

$$\lim_{n\to\infty}\varphi^n(t)=0$$

for each $t\geq 0$. Then $\mathscr{A}$ has a unique fixed point $\dot{\xi}\in\Upsilon$ and $\lim_{n\to\infty}\mathscr{A}^n(u)=\dot{\xi}$ for each $u\in\Upsilon$.

Proof The proof of this theorem can be found in [32]. Notably, Kajántó and Lukács [66] provided a more refined version of this proof in 2018 for improved accuracy. ■

If we set $\varphi(t)=kt$ for $t\in[0,\infty)$ and some constant $k\in[0,1)$, we get the following corollary, which extends the classical Banach fixed point theorem.

Corollary 5.2 *Let (Υ,τ) be a complete b-metric space with constant $s\geq 1$, and suppose that $\mathscr{A}:\Upsilon\to\Upsilon$ satisfies*

$$\tau(\mathscr{A}(u),\mathscr{A}(v)) \leq k\tau(u,v)$$

for all $u,v\in\Upsilon$, where $k\in[0,1)$. Then, $\mathscr{A}$ has a unique fixed point $\dot{\xi}\in\Upsilon$, and $\lim_{n\to\infty}\mathscr{A}^n(u)=\dot{\xi}$ for each $u\in\Upsilon$.

Remark 5.1 *Note that this corollary was initially proved for some constant $k\in[0,\frac{1}{s})$ (see [77, Theorem 1]). This result was later extended to include the interval $[\frac{1}{s},1)$.*

For additional fixed-point results and examples in b-metric spaces, readers may refer to [3, 8, 18, 31–33, 39, 40, 58, 59, 66, 73, 77, 91, 92, 126, 127].

The results presented in Sections 5.2 and 5.3 are published in [12, 83].

5.2 Uniqueness fixed point results in b-metric spaces

The following theorem provides a fixed point result in the context of a b-metric space under a special contractive condition that combines a rational contraction condition with the Bianchini contraction. This result generalizes and improves the result of Aouine A.C. and Aliouche A. [9] (Theorem 2.33).

Theorem 5.3 *Let (Υ,τ,s) be a complete b-metric space, and let $\mathscr{A}$ be a self-mapping on Υ. If there exist five positive real numbers $\alpha,\beta,\gamma,\delta,\epsilon$ such that $s^2\alpha\leq\min\{\gamma,\delta\}$ or $s^2\beta\leq\min\{\gamma,\delta\}$, and for all $u,v\in\Upsilon$,*

$$\tau(\mathscr{A}(u),\mathscr{A}(v)) \leq \frac{\alpha\tau(u,\mathscr{A}(v))+\beta\tau(v,\mathscr{A}(u))}{\gamma\tau(u,\mathscr{A}(u))+\delta\tau(v,\mathscr{A}(v))+\epsilon}\max\{\tau(u,\mathscr{A}(u)),\tau(v,\mathscr{A}(v))\}. \tag{5.2.1}$$

Then, $\mathscr{A}$ has a unique fixed point $\dot{\xi}\in\Upsilon$.

Proof Let $(u_n)_{n\in\mathbb{N}}$ be a Picard sequence ($u_{n+1} = \mathscr{A}(u_n)$) based on an arbitrary $u_0 \in \Upsilon$. If there exists an $n_0 \in \mathbb{N}$ such that $u_{n_0} = u_{n_0+1}$, then u_{n_0} is the fixed point of $\mathscr{A}$, and the proof is complete. If $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$, we follow the following steps:

Step 1: We show that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case 1: If $s^2\alpha \le \min\{\gamma,\delta\}$, by substituting $u = u_{n-1}$ and $v = u_n$ into inequality (5.2.1), we find:

$$\begin{aligned}
\tau(u_n,u_{n+1}) &\le \frac{\alpha\tau(u_{n-1},u_{n+1})}{\gamma\tau(u_{n-1},u_n)+\delta\tau(u_n,u_{n+1})+\epsilon}\max\{\tau(u_{n-1},u_n),\tau(u_n,u_{n+1})\};\\
&\le \frac{\alpha s\tau(u_{n-1},u_n)+\alpha s\tau(u_n,u_{n+1})}{\gamma\tau(u_{n-1},u_n)+\delta\tau(u_n,u_{n+1})+\epsilon}\max\{\tau(u_{n-1},u_n),\tau(u_n,u_{n+1})\};\\
&\le \frac{\alpha s\tau(u_{n-1},u_n)+\alpha s\tau(u_n,u_{n+1})}{\min\{\gamma,\delta\}\big(\tau(u_{n-1},u_n)+\tau(u_n,u_{n+1})\big)+\epsilon}\tau(u_{n-1},u_n).
\end{aligned}$$

Let $\theta_n = \dfrac{\alpha s\tau(u_{n-1},u_n)+\alpha s\tau(u_n,u_{n+1})}{\min\{\gamma,\delta\}(\tau(u_{n-1},u_n)+\tau(u_n,u_{n+1}))+\epsilon}$ for all $n \in \mathbb{N}$. Since $s^2\alpha \le \min\{\gamma,\delta\}$, then $0 \le \theta_n < \frac{1}{s}$ for all $n \in \mathbb{N}$. Furthermore, the sequence $(\theta_n)_{n\in\mathbb{N}}$ is decreasing.

On the other hand, we have:

$$\begin{aligned}
\tau(u_n,u_{n+1}) &\le \theta_n\tau(u_{n-1},u_n);\\
&\le \theta_n\theta_{n-1}\tau(u_{n-2},u_{n-1});\\
&\ \ \vdots\\
&\le \theta_n\theta_{n-1}\cdots\theta_1\tau(u_0,u_1);\\
&\le \theta_1^n\tau(u_0,u_1).
\end{aligned}$$

Now, for all $n,m \in \mathbb{N}$ such that $m > n$, we have:

$$\begin{aligned}
\tau(u_n,u_m) &\le \sum_{i=n}^{m-1} s^{i-n+1}\tau(u_i,u_{i+1});\\
&\le \sum_{i=n}^{m-1} s^{i-n+1}\theta_1^i\tau(u_0,u_1);\\
&\le \frac{(s\theta_1)^n-(s\theta_1)^m}{1-s\theta_1}\times\frac{1}{s^{n-1}}\tau(u_0,u_1).
\end{aligned}$$

By taking the limit as $n,m \to +\infty$ on both sides of the previous inequality, we get:

$$\lim_{n,m\to+\infty}\tau(u_n,u_m) = 0. \tag{5.2.2}$$

Thus, $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Υ.

Case 2: If $s^2\beta \le \min\{\gamma,\delta\}$, by substituting $u = u_n$ and $v = u_{n+1}$ into inequality (5.2.1), we proceed similarly as in case 1, proving that the sequence $(u_n)_{n\in\mathbb{N}}$ is Cauchy.

Hence, by completeness of the space Υ, the sequence (u_n) converges to a point of Υ, denoted as $\dot{\xi} \in \Upsilon$.

Step 2: We verify that $\dot{\xi}$ is a fixed point of $\mathscr{A}$. Suppose that $\tau(\dot{\xi},\mathscr{A}(\dot{\xi})) > 0$.

Case 1: If $s^2\alpha \le \min\{\gamma,\delta\}$, by substituting $u = u_n$ and $v = \dot{\xi}$ into inequality (5.2.1), we obtain:

$$\tau(\mathscr{A}(\dot{\xi}),u_{n+1}) \le \frac{\alpha\tau(u_n,\mathscr{A}(\dot{\xi}))+\beta\tau(u_{n+1},\dot{\xi})}{\gamma\tau(u_n,u_{n+1})+\delta\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))+\epsilon}\max\{\tau(\dot{\xi},\mathscr{A}(\dot{\xi})),\tau(u_n,u_{n+1})\}.$$

On the other hand, we have:

$$\begin{aligned}\tau(\dot{\xi},\mathscr{A}(\dot{\xi})) &\leq s\tau(\dot{\xi},u_{n+1})+s\tau(u_{n+1},\mathscr{A}(\dot{\xi}));\\ &\leq s\tau(\dot{\xi},u_{n+1})+s\frac{\alpha\tau(u_n,\mathscr{A}(\dot{\xi}))+\beta\tau(u_{n+1},\dot{\xi})}{\gamma\tau(u_n,u_{n+1})+\delta\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))+\epsilon}\max\{\tau(\dot{\xi},\mathscr{A}(\dot{\xi})),\tau(u_n,u_{n+1})\}.\end{aligned} \tag{5.2.3}$$

By taking the limit superior on both sides of (5.2.3), we obtain:

$$\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))\leq\frac{s\alpha\limsup_{n\to+\infty}\tau(u_n,\mathscr{A}(\dot{\xi}))}{\delta\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))+\epsilon}\tau(\dot{\xi},\mathscr{A}(\dot{\xi})).$$

According to Lemma 1.3, we get:

$$\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))\leq\frac{s^2\alpha}{\delta\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))+\epsilon}\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))^2.$$

Then,

$$1\leq\frac{s^2\alpha}{\delta\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))+\epsilon}\tau(\dot{\xi},\mathscr{A}(\dot{\xi})). \tag{5.2.4}$$

Since $s^2\alpha\leq\min\{\gamma,\delta\}$, it follows that $s^2\alpha<\delta$, hence $s^2\alpha\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))<\delta\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))+\epsilon$, which contradicts the inequality (5.2.4). Therefore, $\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))=0$, meaning $\mathscr{A}(\dot{\xi})=\dot{\xi}$.

Case 2: If $s^2\beta\leq\min\{\gamma,\delta\}$, by substituting $u=\dot{\xi}$ and $v=u_n$ into inequality (5.2.1), we obtain:

$$\tau(\mathscr{A}(\dot{\xi}),u_{n+1})\leq\frac{\alpha\tau(\dot{\xi},u_{n+1})+\beta\tau(u_n,\mathscr{A}(\dot{\xi}))}{\gamma\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))+\delta\tau(u_n,u_{n+1})+\epsilon}\max\{\tau(\dot{\xi},\mathscr{A}(\dot{\xi})),\tau(u_n,u_{n+1})\}.$$

Similarly, as in Case 1, we can deduce that $\mathscr{A}(\dot{\xi})=\dot{\xi}$.

Step 3: Suppose that $\mathscr{A}$ has two fixed points $\dot{\xi},\dot{\eta}$ in Υ. By substituting $u=\dot{\xi}$ and $v=\dot{\eta}$ into inequality (5.2.1), we obtain:

$$\tau(\dot{\xi},\dot{\eta})\leq\frac{\alpha\tau(\dot{\xi},\dot{\eta})+\beta\tau(\dot{\eta},\dot{\xi})}{\gamma\tau(\dot{\xi},\dot{\xi})+\delta\tau(\dot{\eta},\dot{\eta})+\epsilon}\max\{\tau(\dot{\xi},\dot{\xi}),\tau(\dot{\eta},\dot{\eta})\}.$$

Thus, $\tau(\dot{\xi},\dot{\eta})=0$, which implies that $\dot{\xi}=\dot{\eta}$. This completes the proof of the theorem.

■

Remark 5.2 *If we take $\alpha=\beta=\gamma=\delta=\epsilon=1$ and $s=1$ in Theorem 5.3, we recover the result of Aouine A.C. and Aliouche A. [9] (Theorem 2.33).*

The following example illustrates and supports Theorem 5.3.

Example 5.1 *Let $\Upsilon=[0,4.5]$ and $\tau:\Upsilon\times\Upsilon\to\mathbb{R}^+$ be defined by $\tau(u,v)=(u-v)^2$ for all $u,v\in\Upsilon$. The pair $(\Upsilon,\tau,2)$ is a complete b-metric space. Let $\mathscr{A}:\Upsilon\to\Upsilon$ be a self-mapping given by*

$$\mathscr{A}(u)=\begin{cases}4.5 & \text{if } u\in[0,2.5[,\\ 4 & \text{if } u\in[2.5,4.5].\end{cases}$$

Let $u,v\in\Upsilon$ and denote

$$m(u,v)=-\tau(\mathscr{A}(u),\mathscr{A}(v))+\frac{\tau(u,\mathscr{A}(v))+\tau(v,\mathscr{A}(u))}{4\tau(u,\mathscr{A}(u))+4\tau(v,\mathscr{A}(v))+1}\max\{\tau(u,\mathscr{A}(u)),\tau(v,\mathscr{A}(v))\}.$$

If $u\in[0,2.5[$ and $v\in[2.5,4.5]$, we plot the curve of the function m over this domain.

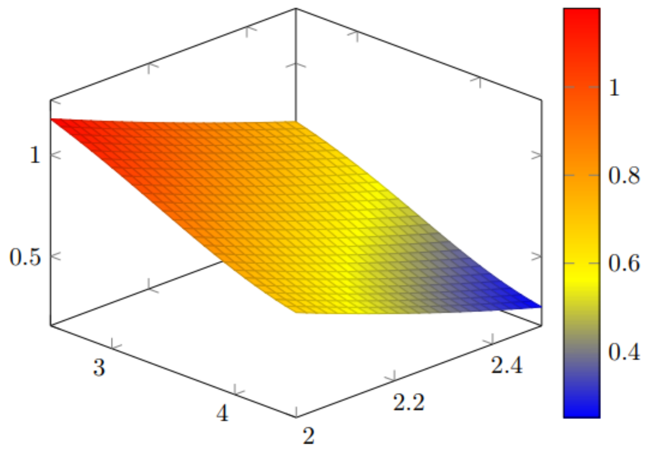

Figure 5.1: Curve of the function m.

We observe that it is positive, which proves the validity of inequality (5.2.1) for all $u \in [0, 2.5[$ *and* $v \in [2.5, 4.5]$. *The other cases are trivial.*
Therefore, by choosing $\alpha = \beta = \epsilon = 1$ *and* $\gamma = \delta = 4$, *all conditions of Theorem 5.3 are satisfied. Hence,* $\mathscr{A}$ *has a unique fixed point* $\dot{\xi}$ *in* Υ *(here* $\dot{\xi} = 4$*).*

If we take $s = 1$ in Theorem 5.3, we obtain the following corollary.

Corollary 5.4 *Let* (Υ, σ) *be a complete metric space and let* $\mathscr{A}$ *be a self-mapping in* Υ. *If there exist five positive real numbers* $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}^+$ *such that* $\alpha \leq \min\{\gamma, \delta\}$ *or* $\beta \leq \min\{\gamma, \delta\}$, *and for all* $u, v \in \Upsilon$,

$$\sigma(\mathscr{A}(u), \mathscr{A}(v)) \leq \frac{\alpha\sigma(u, \mathscr{A}(v)) + \beta\sigma(v, \mathscr{A}(u))}{\gamma\sigma(u, \mathscr{A}(u)) + \delta\sigma(v, \mathscr{A}(v)) + \epsilon} \max\{\sigma(u, \mathscr{A}(u)), \sigma(v, \mathscr{A}(v))\}. \tag{5.2.5}$$

Then, $\mathscr{A}$ *has a unique fixed point* $\dot{\xi} \in \Upsilon$. *Moreover, every Picard sequence converges to* $\dot{\xi}$.

The following example illustrates and supports Corollary 5.4 and Theorem 5.3.

Example 5.2 *Let* $\Upsilon = \{0, 1, 2\}$ *be associated with a metric* σ *such that* $\sigma(0, 1) = 0.6$, $\sigma(0, 2) = 1$, *and* $\sigma(1, 2) = 0.4$. *Also,* $\sigma(u, v) = \sigma(v, u)$ *for all* $u, v \in \Upsilon$, *and* $\sigma(u, u) = 0$ *for all* $u \in \Upsilon$.
Let $\mathscr{A}$ *be a self-mapping in* Υ *such that* $\mathscr{A}(0) = 2$, $\mathscr{A}(1) = 1$, *and* $\mathscr{A}(2) = 1$.
It is easy to conclude that (Υ, σ) *is a complete metric space, and the inequality (5.2.5) was verified for all* $u, v \in \Upsilon$ *with constants* $\alpha = \beta = \gamma = \delta = 3$ *and* $\epsilon = \frac{1}{4}$. *According to Corollary 5.4, we conclude that* $\mathscr{A}$ *has a unique fixed point. Additionally, every Picard sequence converges to* $\dot{\xi}$.

Remark 5.3 *It should be noted that Theorem 2.33 (the result of Aouine A.C. and Aliouche A.[9]) is not applicable in this example, whereas the generalized Corollary 5.4 is applicable, as shown in the example above. This demonstrates the robustness of our results.*

If we take $\gamma = \delta = \beta = 0$ in Theorem 5.3, we obtain the following corollary.

Corollary 5.5 *Let (Υ,τ,s) be a complete b-metric space, and let $\mathscr{A}$ be a self-mapping in Υ. If there exists a nonnegative number k such that for all $u,v\in\Upsilon$,*

$$\tau(\mathscr{A}(u),\mathscr{A}(v)) \leq k\tau(u,\mathscr{A}(v))\tau(u,\mathscr{A}(u)), \tag{5.2.6}$$

then $\mathscr{A}$ has a unique fixed point $\dot{\xi}\in\Upsilon$, and every Picard sequence $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ converges to $\dot{\xi}$.

Proof Since $\tau(u,\mathscr{A}(u)) \leq \max \tau(u,\mathscr{A}(u)),\tau(v,\mathscr{A}(v))$, equation (5.2.1) is satisfied for all $u,v\in\Upsilon$ with the constants $\gamma=\delta=\beta=0$ and $\alpha=k\epsilon$. Thus, all the conditions of Theorem 5.3 are satisfied. Hence, $\mathscr{A}$ has a unique fixed point $\dot{\xi}\in\Upsilon$. ■

5.3 Non-uniqueness fixed point results in b-metric spaces

This section is dedicated to establishing fixed point theorems based on a contraction condition that is more general than (2.5.1) and (2.4.5) within a b-metric space. This condition does not require the continuity of the mapping and enhances the dynamic properties in cases of non-unique fixed points.

Theorem 5.6 *Let (Υ,τ,s) be a complete b-metric space, and let $\mathscr{A}$ be a self mapping on Υ. If there exist five positive real numbers $\alpha,\beta,\gamma,\delta,\epsilon$ such that $s^2\alpha\leq\min\{\gamma,\delta\}$ or $s^2\beta\leq\min\{\gamma,\delta\}$ and for all $u,v\in\Upsilon$ the following inequality holds:*

$$\tau(\mathscr{A}(u),\mathscr{A}(v)) \leq \frac{\alpha\tau(u,\mathscr{A}(v))+\beta\tau(v,\mathscr{A}(u))}{\gamma\tau(u,\mathscr{A}(u))+\delta\tau(v,\mathscr{A}(v))+\epsilon}\tau(u,v), \tag{5.3.1}$$

then

1. *$\mathscr{A}$ has at least one fixed point $\dot{\xi}\in\Upsilon$,*
2. *every Picard sequence $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ converges to a fixed point,*
3. *if $\mathscr{A}$ has two distinct fixed points $\dot{\xi},\dot{\eta}$ in Υ, then $\tau(\dot{\xi},\dot{\eta})\geq\frac{\epsilon}{\alpha+\beta}$.*

Proof Let $(u_n)_{n\in\mathbb{N}}$ be a Picard sequence ($u_{n+1}=\mathscr{A}(u_n)$) based on an arbitrary $u_0\in\Upsilon$. If there exists an $n_0\in\mathbb{N}$ such that $u_{n_0}=u_{n_0+1}$, then u_{n_0} is the fixed point of $\mathscr{A}$, and the proof is complete. If $u_n\neq u_{n+1}$ for all $n\in\mathbb{N}$, we proceed with the following steps:

Step 1: Show that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case 1 If $s^2\alpha\leq\min\{\gamma,\delta\}$, by putting $u=u_{n-1}$ and $v=u_n$ in inequality (5.3.1) we find,

$$\begin{aligned}
\tau(u_n,u_{n+1}) &\leq \frac{\alpha\tau(u_{n-1},u_{n+1})}{\gamma\tau(u_{n-1},u_n)+\delta\tau(u_n,u_{n+1})+\epsilon}\tau(u_{n-1},u_n);\\
&\leq \frac{\alpha s\tau(u_{n-1},u_n)+\alpha s\tau(u_n,u_{n+1})}{\gamma\tau(u_{n-1},u_n)+\delta\tau(u_n,u_{n+1})+\epsilon}\tau(u_{n-1},u_n);\\
&\leq \frac{\alpha s\tau(u_{n-1},u_n)+\alpha s\tau(u_n,u_{n+1})}{\min\{\gamma;\delta\}\big(\tau(u_{n-1},u_n)+\tau(u_n,u_{n+1})\big)+\epsilon}\tau(u_{n-1},u_n).
\end{aligned}$$

We denote that $\theta_n=\dfrac{\alpha s\tau(u_{n-1},u_n)+\alpha s\tau(u_n,u_{n+1})}{\min\{\gamma;\delta\}\big(\tau(u_{n-1},u_n)+\tau(u_n,u_{n+1})\big)+\epsilon}$ for all $n\in\mathbb{N}$.

Since $s^2\alpha\leq\min\{\gamma,\delta\}$, then $0\leq\theta_n<\frac{1}{s}$ for all $n\in\mathbb{N}$, furthermore, the sequence $(\theta_n)_{n\in\mathbb{N}}$ is decreasing because for all $n \in \mathbb{N}$,

$$\begin{aligned} \theta_{n+1}-\theta_n &= \frac{\alpha s\epsilon\left[\tau(u_{n+1},u_{n+2})-\tau(u_{n-1},u_n)\right]}{\left[\min\{\gamma;\delta\}(\tau(u_n,u_{n+1})+\tau(u_{n+1},u_{n+2}))+\epsilon\right]} \\ &\quad \times \frac{1}{\left[\min\{\gamma;\delta\}(\tau(u_{n-1},u_n)+\tau(u_n,u_{n+1}))+\epsilon\right]}, \\ &< 0. \end{aligned}$$

On the other hand we have,

$$\begin{aligned} \tau(u_n,u_{n+1}) &\le \theta_n\tau(u_{n-1},u_n); \\ &\le \theta_n\theta_{n-1}\tau(u_{n-2},u_{n-1}); \\ &\vdots \\ &\le \theta_n\theta_{n-1}\cdots\theta_1\tau(u_0,u_1); \\ &\le \theta_1^n\tau(u_0,u_1). \end{aligned}$$

Now, for all $n, m \in \mathbb{N}$ such that $m > n$, we have

$$\begin{aligned} \tau(u_n,u_m) &\le \sum_{i=n}^{m-1} s^{i-n+1}\tau(u_i,u_{i+1}); \\ &\le \sum_{i=n}^{m-1} s^{i-n+1}\theta_1^i\tau(u_0,u_1); \\ &\le \frac{(s\theta_1)^n-(s\theta_1)^m}{1-s\theta_1}\times\frac{1}{s^{n-1}}\tau(u_0,u_1). \end{aligned}$$

By passing to the limits $n, m \to +\infty$ on both sides of the previous inequality we get,

$$\lim_{n,m\to+\infty}\tau(u_n,u_m)=0. \tag{5.3.2}$$

Then $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Υ.

Case 2 If $s^2\beta \le \min\{\gamma,\delta\}$, by putting $u = u_n$ and $v = u_{n-1}$ in inequality (5.3.1), we find,

$$\tau(u_n,u_{n+1}) \le \frac{\beta\tau(u_{n-1},u_{n+1})}{\gamma\tau(u_n,u_{n+1})+\delta\tau(u_{n-1},u_n)+\epsilon}\tau(u_{n-1},u_n).$$

Similarly, as in Case 1, we can deduce that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Υ.
Since the b-metric space (Υ,τ,s) is complete, there exists $\dot{\xi} \in \Upsilon$ such that,

$$\lim_{n\to+\infty}u_n=\dot{\xi}.$$

Step 2: We check that $\dot{\xi}$ is a fixed point of $\mathscr{A}$.
By putting $u = \dot{\xi}$, $v = u_n$ in inequality (5.3.1), we find,

$$\tau(\mathscr{A}(\dot{\xi}),u_{n+1}) \le \frac{\alpha\tau(\dot{\xi},u_{n+1})+\beta\tau(u_n,\mathscr{A}(\dot{\xi}))}{\gamma\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))+\delta\tau(u_n,u_{n+1})+\epsilon}\tau(\dot{\xi},u_n). \tag{5.3.3}$$

By taking the limit on both sides of (5.3.3), we have $\lim_{n\to+\infty}u_n = \mathscr{A}(\dot{\xi})$. Because of the uniqueness of the limit, we find $\mathscr{A}(\dot{\xi}) = \dot{\xi}$.

Step 3: Suppose that $\mathscr{A}$ has two distinct fixed points $\dot{\xi}, \dot{\eta}$ in Υ and we find the distance between them.
By putting $u = \dot{\xi}$, $v = \dot{\eta}$ in inequality (5.3.1), we find,

$$\begin{aligned} \tau(\dot{\xi}, \dot{\eta}) &\leq \frac{\alpha\tau(\dot{\xi}, \dot{\eta}) + \beta\tau(\dot{\eta}, \dot{\xi})}{\gamma\tau(\dot{\xi}, \dot{\xi}) + \delta\tau(\dot{\eta}, \dot{\eta}) + \epsilon}\tau(\dot{\xi}, \dot{\eta}); \\ &\leq \frac{(\alpha+\beta)\tau(\dot{\xi}, \dot{\eta})}{\epsilon}\tau(\dot{\xi}, \dot{\eta}). \end{aligned}$$

Then, $\tau(\dot{\xi}, \dot{\eta}) \geq \dfrac{\epsilon}{\alpha+\beta}$. This completes the proof of the theorem. ■

Remark 5.4

- If we take $\alpha = \beta = \gamma = \delta = \epsilon = 1$ and $s = 1$ in Theorem 5.6, we return to the results of Khojasteh et al. [76].

- If we take $s = 1$ in Theorem 5.6, we return to Theorem 3.1.

The following example supports our Theorem 5.6.

Example 5.3 *Let $\Upsilon = \{0, 1, 2\}$ and $\tau : \Upsilon \times \Upsilon \to \mathbb{R}^+$ be defined by $\tau(u, v) = (u - v)^2$ for all $u, v \in \Upsilon$. The space $(\Upsilon, \tau, 2)$ is a complete b-metric space. Let $\mathscr{A} : \Upsilon \to \Upsilon$ be a self-mapping given by $\mathscr{A}(0) = 0$, $\mathscr{A}(1) = 0$, and $\mathscr{A}(2) = 2$.*
If $u = v$, the equation is obviously verified. Now, we treat the other cases:

- *If $u = 0$ and $v = 2$, we have*

$$\tau(0,2) \leq (\tau(0,2) + 5\tau(0,2))\tau(0,2).$$

- *If $u = 1$ and $v = 2$, we have*

$$\tau(0,2) \leq \frac{\tau(1,2) + 5\tau(0,2)}{4\tau(1,0) + 1}\tau(1,0).$$

- *If $u = 2$ and $v = 0$, we have*

$$\tau(0,2) \leq (\tau(0,2) + 5\tau(0,2))\tau(0,2).$$

- *If $u = 2$ and $v = 1$, we have*

$$\tau(0,2) \leq \frac{\tau(0,2) + 5\tau(1,2)}{4\tau(1,0) + 1}\tau(1,0).$$

This means that equation (5.3.1) is verified with constants $\alpha = 1$, $\beta = 5$, $\gamma = 4$, $\epsilon = 1$, and $\delta = 4$. Furthermore, all conditions of Theorem 5.6 are satisfied, so $\mathscr{A}$ has at least one fixed point in Υ.
We remark that $\mathscr{A}$ has exactly two fixed points: 0 and 2. Moreover, $\tau(0,2) \geq \dfrac{1}{6}$.

The following theorem confirms the existence of a non-unique fixed point under a modified rational contraction condition combined with a Chatterjea-type condition. Note that this result is a corrected version of [83, Theorem 3].

Theorem 5.7 *Let (Υ,τ,s) be a complete b-metric space and let $\mathscr{A}$ be a self-mapping in Υ. If there exist five positive real numbers $\alpha,\beta,\gamma,\delta,\epsilon\in\mathbb{R}^+$ such that $(s^3+s^2)\alpha\le\min\{\gamma,\delta\}$ or $(s^3+s^2)\beta\le\min\{\gamma,\delta\}$ and for all $u,v\in\Upsilon$,*

$$\tau(\mathscr{A}(u),\mathscr{A}(v))\le\frac{\alpha\tau(u,\mathscr{A}(v))+\beta\tau(v,\mathscr{A}(u))}{\gamma\tau(u,\mathscr{A}(u))+\delta\tau(v,\mathscr{A}(v))+\epsilon}\max\{\tau(u,\mathscr{A}(v)),\tau(v,\mathscr{A}(u))\}. \tag{5.3.4}$$

Then,

1. *$\mathscr{A}$ has at least one fixed point $\dot{\xi}\in\Upsilon$,*
2. *Every Picard sequence $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ converges to a fixed point,*
3. *If $\mathscr{A}$ has two distinct fixed points $\dot{\xi},\dot{\eta}$ in Υ, then $\tau(\dot{\xi},\dot{\eta})\ge\dfrac{\epsilon}{\alpha+\beta}$.*

Proof Let $(u_n)_{n\in\mathbb{N}}$ be a Picard sequence ($u_{n+1}=\mathscr{A}(u_n)$) based on an arbitrary $u_0\in\Upsilon$. If there exists $n_0\in\mathbb{N}$ such that $u_{n_0}=u_{n_0+1}$, then u_{n_0} is the fixed point of $\mathscr{A}$ and the proof is complete. If $u_n\neq u_{n+1}$ for all $n\in\mathbb{N}$, we follow these steps:

Step 1: Show that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case 1 If $(s^3+s^2)\alpha\le\min\{\gamma,\delta\}$, by putting $u=u_{n-1}$ and $v=u_n$ in inequality (5.3.4), we find:

$$\begin{aligned}\tau(u_n,u_{n+1}) &\le \frac{\alpha\tau(u_{n-1},u_{n+1})}{\gamma\tau(u_{n-1},u_n)+\delta\tau(u_n,u_{n+1})+\epsilon}\tau(u_{n-1},u_{n+1});\\ &\le \frac{\alpha s\tau(u_{n-1},u_n)+\alpha s\tau(u_n,u_{n+1})}{\min\{\gamma,\delta\}(\tau(u_{n-1},u_n)+\tau(u_n,u_{n+1}))+\epsilon}[s\tau(u_{n-1},u_n)+s\tau(u_n,u_{n+1})].\end{aligned}$$

Denote $\theta_n=\dfrac{\alpha s\tau(u_{n-1},u_n)+\alpha s\tau(u_n,u_{n+1})}{\min\{\gamma,\delta\}(\tau(u_{n-1},u_n)+\tau(u_n,u_{n+1}))+\epsilon}$ for all $n\in\mathbb{N}$. Since $(s^3+s^2)\alpha\le\min\{\gamma,\delta\}$, then $0\le\theta_n<\frac{1}{s^2+s}$ for all $n\in\mathbb{N}$.

On the other hand, we have:

$$\tau(u_n,u_{n+1})\le\theta_n[s\tau(u_{n-1},u_n)+s\tau(u_n,u_{n+1})].$$

Then,

$$\tau(u_n,u_{n+1})\le\frac{\theta_n s}{1-\theta_n s}\tau(u_{n-1},u_n).$$

Denote $\lambda_n=\dfrac{\theta_n s}{1-\theta_n s}$ for all $n\in\mathbb{N}$. Since $0\le\theta_n<\frac{1}{s^2+s}$ for all $n\in\mathbb{N}$, we have $0\le\lambda_n<\frac{1}{s}$. Thus, $\tau(u_n,u_{n+1})<\tau(u_{n-1},u_n)$ for all $n\in\mathbb{N}$, hence $\tau(u_{n+1},u_{n+2})<\tau(u_{n-1},u_n)$. Furthermore, the sequence $(\theta_n)_{n\in\mathbb{N}}$ is decreasing. consequently, the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is also decreasing, which leads to:

$$\begin{aligned}\tau(u_n,u_{n+1}) &\le \lambda_n\tau(u_{n-1},u_n);\\ &\le \lambda_n\lambda_{n-1}\tau(u_{n-2},u_{n-1});\\ &\vdots\\ &\le \lambda_n\lambda_{n-1}\cdots\lambda_1\tau(u_0,u_1);\\ &\le \lambda_1^n\tau(u_0,u_1).\end{aligned}$$

Now, for all $n, m \in \mathbb{N}$ such that $m > n$, we have:

$$\begin{aligned}\tau(u_n, u_m) &\leq \sum_{i=n}^{m-1} s^{i-n+1}\tau(u_i, u_{i+1});\\ &\leq \sum_{i=n}^{m-1} s^{i-n+1}\lambda_1^i \tau(u_0, u_1);\\ &\leq \frac{(s\lambda_1)^n - (s\lambda_1)^m}{1-s\lambda_1} \times \frac{1}{s^{n-1}}\tau(u_0, u_1).\end{aligned}$$

By taking limits as $n, m \to +\infty$, we obtain:

$$\lim_{n,m\to+\infty} \tau(u_n, u_m) = 0.$$

Thus, $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Υ.

Case 2 If $(s^3+s^2)\beta \leq \min\{\gamma, \delta\}$, by putting $u = u_n$ and $v = u_{n-1}$ in inequality (5.3.4), we find:

$$\tau(u_n, u_{n+1}) \leq \frac{\beta\tau(u_{n-1}, u_{n+1})}{\gamma\tau(u_n, u_{n+1}) + \delta\tau(u_{n-1}, u_n) + \epsilon}\tau(u_{n-1}, u_{n+1}).$$

By following similar steps as in Case 1, we can conclude that $(u_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in Υ. Since the b-metric space (Υ, τ, s) is complete, there exists $\dot{\xi} \in \Upsilon$ such that

$$\lim_{n\to+\infty} u_n = \dot{\xi}.$$

Step 2: Check that $\dot{\xi}$ is a fixed point of $\mathscr{A}$.

Suppose $\tau(\dot{\xi}, \mathscr{A}(\dot{\xi})) > 0$.

Case 1 If $(s^3+s^2)\alpha \leq \min\{\gamma, \delta\}$, by putting $u = u_n$, $v = \dot{\xi}$ in inequality (5.3.4), we find:

$$\tau(\mathscr{A}(\dot{\xi}), u_{n+1}) \leq \frac{\alpha\tau(u_n, \mathscr{A}(\dot{\xi})) + \beta\tau(\dot{\xi}, u_{n+1})}{\gamma\tau(u_n, u_{n+1}) + \delta\tau(\dot{\xi}, \mathscr{A}(\dot{\xi})) + \epsilon}\max\{\tau(\dot{\xi}, u_{n+1}), \tau(u_n, \mathscr{A}(\dot{\xi}))\}. \tag{5.3.5}$$

On the other hand, we have

$$\begin{aligned}\tau(\dot{\xi}, \mathscr{A}(\dot{\xi})) &\leq s\tau(\dot{\xi}, u_{n+1}) + s\tau(u_{n+1}, \mathscr{A}(\dot{\xi}));\\ &\leq s\tau(\dot{\xi}, u_{n+1}) + \frac{s\alpha\tau(u_n, \mathscr{A}(\dot{\xi})) + s\beta\tau(\dot{\xi}, u_{n+1})}{\gamma\tau(u_n, u_{n+1}) + \delta\tau(\dot{\xi}, \mathscr{A}(\dot{\xi})) + \epsilon}\max\{\tau(\dot{\xi}, u_{n+1}), \tau(u_n, \mathscr{A}(\dot{\xi}))\}.\end{aligned} \tag{5.3.6}$$

By taking limit superior on both sides of (5.3.6), we have,

$$\tau(\dot{\xi}, \mathscr{A}(\dot{\xi})) \leq \frac{s\alpha}{\delta\tau(\dot{\xi}, \mathscr{A}(\dot{\xi})) + \epsilon}(\limsup_{n\to+\infty}\tau(u_n, \mathscr{A}(\dot{\xi})))^2. \tag{5.3.7}$$

According to Lemma 1.3, we get,

$$\tau(\dot{\xi}, \mathscr{A}(\dot{\xi})) \leq \frac{s^3\alpha}{\delta\tau(\dot{\xi}, \mathscr{A}(\dot{\xi})) + \epsilon}\tau(\dot{\xi}, \mathscr{A}(\dot{\xi}))^2, \tag{5.3.8}$$

This leads to

$$1 \leq \frac{s^3\alpha}{\delta\tau(\dot{\xi}, \mathscr{A}(\dot{\xi})) + \epsilon}\tau(\dot{\xi}, \mathscr{A}(\dot{\xi})). \tag{5.3.9}$$

Since, $(s^3+s^2)\alpha \le \min\{\gamma,\delta\}$, then $s^3\alpha \le \delta$, then $s^3\alpha\tau(\dot{\xi},\mathscr{A}(\dot{\xi})) < \delta\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))+\epsilon$ which contradict inequality (5.3.9). Therefore, $\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))=0$, that mean $\mathscr{A}(\dot{\xi})=\dot{\xi}$.

Case 2 If $(s^3+s^2)\beta \le \min\{\gamma,\delta\}$, by putting $u=\dot{\xi}$ and $v=u_n$ in inequality (5.3.4), we find:

$$\tau(\mathscr{A}(\dot{\xi}),u_{n+1}) \le \frac{\alpha\tau(\dot{\xi},u_{n+1})+\beta\tau(u_n,\mathscr{A}(\dot{\xi}))}{\gamma\tau(\dot{\xi},\mathscr{A}(\dot{\xi}))+\delta\tau(u_n,u_{n+1})+\epsilon}\max\{\tau(\dot{\xi},u_{n+1}),\tau(u_n,\mathscr{A}(\dot{\xi}))\}.$$

Following similar steps as in Case 1, we can deduce that $\mathscr{A}(\dot{\xi})=\dot{\xi}$.

Step 3: Suppose $\mathscr{A}$ has two distinct fixed points $\dot{\xi},\dot{\eta}$ in Υ and we find the distance between them. By putting $u=\dot{\xi}$ and $v=\dot{\eta}$ in inequality (5.3.4), we find:

$$\begin{aligned}\tau(\dot{\xi},\dot{\eta}) &\le \frac{\alpha\tau(\dot{\xi},\dot{\eta})+\beta\tau(\dot{\eta},\dot{\xi})}{\gamma\tau(\dot{\xi},\dot{\xi})+\delta\tau(\dot{\eta},\dot{\eta})+\epsilon}\tau(\dot{\xi},\dot{\eta});\\ &\le \frac{(\alpha+\beta)\tau(\dot{\xi},\dot{\eta})}{\epsilon}\tau(\dot{\xi},\dot{\eta}).\end{aligned}$$

Then, $\tau(\dot{\xi},\dot{\eta}) \ge \dfrac{\epsilon}{\alpha+\beta}$. ■

If we take $\gamma=\delta=\beta=0$ in theorem 5.6, we get the following corollary

Corollary 5.8 *Let (Υ,τ,s) be a complete b-metric space and let $\mathscr{A}$ be a self mapping on Υ. If there exists a positive number k, for all $u,v\in\Upsilon$*

$$\tau(\mathscr{A}(u),\mathscr{A}(v)) \le k\tau(u,\mathscr{A}(v))\tau(u,v). \tag{5.3.10}$$

Then,

1. *$\mathscr{A}$ has at least one fixed point $\dot{\xi}\in\Upsilon$,*
2. *Every Picard sequence $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ converges to a fixed point,*
3. *If $\mathscr{A}$ has two distinct fixed points $\dot{\xi},\dot{\eta}$ in Υ then, $\tau(\dot{\xi},\dot{\eta}) \ge \dfrac{1}{k}$.*

In the same way, we find

Corollary 5.9 *Let (Υ,τ,s) be a complete b-metric space and let $\mathscr{A}$ be a self mapping on Υ. If there exist nonegative number k, for all $u,v\in\Upsilon$,*

$$\tau(\mathscr{A}(u),\mathscr{A}(v)) \le k\tau(u,\mathscr{A}(v))^2. \tag{5.3.11}$$

Then, $\mathscr{A}$ has at least fixed point $\dot{\xi}\in\Upsilon$, and every Picard sequence $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ converges to a fixed point.

Proof Since $\tau(u,\mathscr{A}(v)) \le \max\tau(u,\mathscr{A}(v)),\tau(v,\mathscr{A}(u))$, equation (5.3.4) is satisfied for all $u,v\in\Upsilon$ with the constants $\gamma=\delta=\beta=0$ and $\alpha=k\epsilon$. Thus, all the conditions of Theorem 5.7 are satisfied. Hence, $\mathscr{A}$ has at least fixed point $\dot{\xi}\in\Upsilon$. ■

The subsequent theorem extends and unifies the two previous theorems.

Theorem 5.10 *Let $\mathscr{A}$ be a self mapping acting on a complete b-metric space denoted by (Υ,τ,s). Assume there are five constants $\alpha,\beta,\gamma,\delta$, and $\epsilon\in\mathbb{R}^+$ that ensure either $s^3\alpha \le \frac{1}{2}\min\{\gamma,\delta\}$ or $s^3\beta \le \frac{1}{2}\min\{\gamma,\delta\}$ and for all $u,v\in\Upsilon$,*

$$\tau(\mathscr{A}(u),\mathscr{A}(v)) \le \frac{\alpha\tau(u,\mathscr{A}(v))+\beta\tau(v,\mathscr{A}(u))}{\gamma\tau(u,\mathscr{A}(u))+\delta\tau(v,\mathscr{A}(v))+\epsilon}\max\{\tau(u,\mathscr{A}(v)),\tau(v,\mathscr{A}(u)),\tau(u,v)\}. \tag{5.3.12}$$

Then,

1. *There is, at a minimum, one fixed point, denoted by $\dot{\xi}$, within the space Υ for the mapping $\mathscr{A}$;*
2. *Each Picard sequence $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ converges to one of the fixed points;*
3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality (5.3.13):*

$$\tau(\dot{\xi},\dot{\eta}) \geq \frac{\epsilon}{\alpha+\beta}. \tag{5.3.13}$$

Proof The procedure for proving the above theorem is omitted because it is similar to the proof of Theorem 5.7.

∎

The following example illustrates and supports Theorem 5.10.

Example 5.4 *Let $\Upsilon = [0;0.2]\cup[1;2]$ associated with a b-metric τ such that $\tau(u,v) = (u-v)^2$ for all $u,v \in \Upsilon$.*

Let $\mathscr{A} : \Upsilon \to \Upsilon$ be a self mapping given by,

$$\mathscr{A}(u) = \begin{cases} 0 & \text{if } u \in [0;0.2], \\ 1 & \text{if } u \in [1;2]. \end{cases}$$

It is easy to conclude that (Υ,τ) is a complete b-metric space with the constant $s = 2$.

Let $u,v \in \Upsilon$ and denote by m the function,

$$m(u,v) = -\tau(\mathscr{A}(u),\mathscr{A}(v)) + \frac{\tau(u,\mathscr{A}(v))+\tau(v,\mathscr{A}(u))}{16\tau(u,\mathscr{A}(u))+16\tau(v,\mathscr{A}(v))+1}\max\{\tau(u,\mathscr{A}(v)),\tau(v,\mathscr{A}(u)),\tau(u,v)\}.$$

If $u = v$, the equation is obviously verified;

If $u,v \in [0;0.2]$ or $u,v \in [1;2]$, the inequality is obviously verified;

If $u \in [0;0.2]$ and $v \in [1;2]$ or $u \in [1;2]$ and $v \in [0;0.2]$, we plot the graph of the function m over this domain.

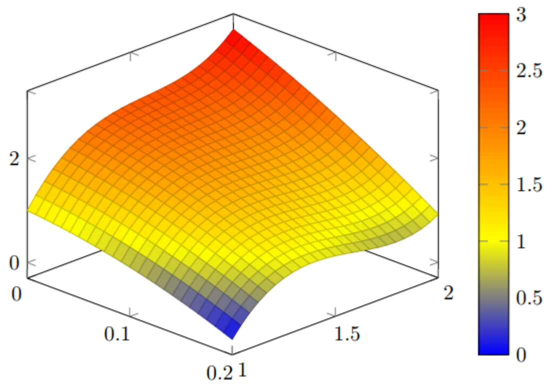

Figure 5.2: Plot of the function m.

It is worth noting that the inequality (5.3.12) holds true on this domain for the constants $\alpha = \beta = 1$, $\gamma = \delta = 16$, and $\epsilon = 1$. By applying Theorem 5.10, we can deduce that $\mathscr{A}$ possesses no less than one fixed point. The function has precisely two fixed points, either 0 and 1. Furthermore, it can be observed that the distance separating each other, denoted as $\tau(0,1)$, is greater than or equal to one-half.

Next, we present the following theorem, which extends and unifies the findings of Khojasteh et al. [76] and A.C. Aouine and A. Aliouche [9].

Theorem 5.11 *Let $\mathscr{A}$ be a self mapping acting on a complete b-metric space denoted by (Υ, τ, s). Assume there are five constants $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}^+$ that ensure either $s^2\alpha \leq \min\{\gamma, \delta\}$ or $s^2\beta \leq \min\{\gamma, \delta\}$. Additionally, the equation (5.3.14) holds for all values of u and v belonging to Υ.*

$$\tau(\mathscr{A}(u), \mathscr{A}(v)) \leq \frac{\alpha\tau(u, \mathscr{A}(v)) + \beta\tau(v, \mathscr{A}(u))}{\gamma\tau(u, \mathscr{A}(u)) + \delta\tau(v, \mathscr{A}(v)) + \epsilon} \max\{\tau(u, v), \tau(u, \mathscr{A}(u)), \tau(v, \mathscr{A}(v))\}. \quad (5.3.14)$$

Then,

1. *There is, at a minimum, one fixed point, denoted by $\dot{\xi}$, within the space Υ for the mapping $\mathscr{A}$;*
2. *Each Picard sequence $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ converges to one of the fixed points;*
3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality (5.3.15):*

$$\tau(\dot{\xi}, \dot{\eta}) \geq \frac{\epsilon}{\alpha + \beta}. \quad (5.3.15)$$

Proof The procedure for proving the above theorem is omitted because it is similar to the proof of Theorem 5.7. ■

The subsequent example explains and affirms Theorem 5.11.

Example 5.5 *Let $\Upsilon = \{0, 1, 2\}$ be a set equipped with a b-metric τ defined as follows: $\tau(0, 1) = 0.2$, $\tau(0, 2) = 1$ and $\tau(1, 2) = 1.5$. Additionally, $\tau(u, v) = \tau(v, u)$ and $\tau(u, u) = 0$ for every $u, v \in \Upsilon$.*

Consider a self-map $\mathscr{A}$ acting on Υ defined by $\mathscr{A}(0) = 0$, $\mathscr{A}(1) = 0$ and $\mathscr{A}(2) = 2$.

It may be readily inferred that the b-metric space (Υ, τ) is a complete b-metric space with the constant $s = \sqrt{2}$. Moreover, the inequality (5.3.14) holds for every $u, v \in \Upsilon$, with constants $\alpha = 2$, $\beta = 1$, $\gamma = \delta = 2$ and $\epsilon = 3$.

Theorem 5.11 leads us to the conclusion that $\mathscr{A}$ has a minimum of one fixed point (precisely, $\mathscr{A}$ possesses two distinct fixed points 2 and 0). Furthermore, it can be observed that the distance separating each other, denoted as $\tau(0, 2)$, is greater than or equal to one.

The following theorem establishes the existence of a non-unique fixed point under a modified rational contraction condition combined with a Ćirić-type condition. It unifies and extends all previously mentioned fixed point results in this chapter.

Theorem 5.12 *Let $\mathscr{A}$ be a self mapping acting on a complete b-metric space noted by (Υ, τ, s). Assume there are five constants $\alpha, \beta, \gamma, \delta$, and $\epsilon \in \mathbb{R}^+$ that ensure either $s^3\alpha \leq \frac{1}{2}\min\{\gamma, \delta\}$ or $s^3\beta \leq \frac{1}{2}\min\{\gamma, \delta\}$ and for all $u, v \in \Upsilon$,*

$$\tau(\mathscr{A}(u), \mathscr{A}(v)) \leq \frac{\alpha\tau(u, \mathscr{A}(v)) + \beta\tau(v, \mathscr{A}(u))}{\gamma\tau(u, \mathscr{A}(u)) + \delta\tau(v, \mathscr{A}(v)) + \epsilon} \max\{\tau(u, \mathscr{A}(v)), \tau(v, \mathscr{A}(u)), \tau(u, \mathscr{A}(u)), \tau(v, \mathscr{A}(v)), \tau(u, v)\}. \quad (5.3.16)$$

Then,

1. *There is, at a minimum, one fixed point denoted as $\dot{\xi}$ within the space Υ for the mapping $\mathscr{A}$;*

2. *Each Picard sequence* $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ *converges to one of the fixed points;*

3. *If* $\mathscr{A}$ *has several fixed points, then each pair of them must satisfy the inequality* (5.3.17)*:*

$$\tau(\dot{\xi},\dot{\eta}) \geq \frac{\epsilon}{\alpha+\beta}. \tag{5.3.17}$$

Proof The procedure for proving the above theorem is omitted because it is similar to the proof of Theorem 5.7. ∎

The following example illustrates and supports Theorem 5.12.

Example 5.6 *Let* $\Upsilon = \{\bigstar, \blacklozenge, \blacktriangle\}$ *be a set equipped with a* b*-metric* τ *defined as follows:* $\tau(\bigstar,\blacklozenge) = 0.2$, $\tau(\bigstar,\blacktriangle) = 1$ *and* $\tau(\blacklozenge,\blacktriangle) = 1.5$. *Additionally,* $\tau(u,v) = \tau(v,u)$ *and* $\tau(u,u) = 0$ *for all* $u, v \in \Upsilon$.

Consider a self-map $\mathscr{A}$ *acting on* Υ *defined by* $\mathscr{A}(\bigstar) = \bigstar$, $\mathscr{A}(\blacklozenge) = \blacklozenge$ *and* $\mathscr{A}(\blacktriangle) = \bigstar$.

It may be readily inferred that the b*-metric space* (Υ,τ) *is a complete* b*-metric space with the constant* $s =^3 \sqrt{2}$. *Moreover, the inequality* (5.3.16) *has been held for every* $u, v \in \Upsilon$, *with constants* $\alpha = 2$, $\beta = 1$, $\gamma = \delta = 4$ *and* $\epsilon = 0.6$.

Theorem 5.12 leads us to the conclusion that $\mathscr{A}$ *has a minimum of one fixed point (precisely,* $\mathscr{A}$ *possesses two distinct fixed points* $\bigstar$ *and* $\blacklozenge$*). Furthermore, it can be observed that the distance separating them, denoted as* $\tau(\bigstar,\blacklozenge)$, *is greater than or equal to* 0.2.

Following the same proof procedure as before, we can introduce additional modifications to these results, yielding the following findings.

Theorem 5.13 *Let* $\mathscr{A}$ *be a self-mapping acting on a complete* b*-metric space noted by* (Υ,τ,s). *Assume there are five constants* $\alpha,\beta,\gamma,\delta,\epsilon \in \mathbb{R}^+$ *that ensure either* $s^3\alpha \leq \frac{1}{2}\min\{\gamma,\delta\}$ *or* $s^3\beta \leq \frac{1}{2}\min\{\gamma,\delta\}$, *and for all* $u, v \in \Upsilon$,

$$\tau(\mathscr{A}(u),\mathscr{A}(v)) \leq \frac{\alpha\tau(u,\mathscr{A}(v)) + \beta\tau(v,\mathscr{A}(u))}{\gamma\tau(u,\mathscr{A}(u)) + \delta\tau(v,\mathscr{A}(v)) + \epsilon}\max\{\tau(u,\mathscr{A}(v)),\tau(v,\mathscr{A}(u)),\tau(u,\mathscr{A}(u))+\tau(v,\mathscr{A}(v)),\tau(u,v)\}. \tag{5.3.18}$$

Then,

1. *There is at least one fixed point denoted as* $\dot{\xi}$ *within the space* Υ *for the mapping* $\mathscr{A}$*;*

2. *Each Picard sequence* $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ *converges to one of the fixed points;*

3. *If* $\mathscr{A}$ *has several fixed points, then each pair of them must satisfy the inequality* (5.3.19)*:*

$$\tau(\dot{\xi},\dot{\eta}) \geq \frac{\epsilon}{\alpha+\beta}. \tag{5.3.19}$$

Proof The procedure for proving the above theorem is omitted because it is similar to the proof of Theorem 5.7. ∎

Theorem 5.14 *Let* $\mathscr{A}$ *be a self-mapping acting on a complete* b*-metric space noted by* (Υ,τ,s). *Assume there are five constants* $\alpha,\beta,\gamma,\delta,\epsilon \in \mathbb{R}^+$ *that ensure either* $s^3\alpha \leq \frac{1}{2}\min\{\gamma,\delta\}$ *or* $s^3\beta \leq \frac{1}{2}\min\{\gamma,\delta\}$, *and for all* $u, v \in \Upsilon$,

$$\tau(\mathscr{A}(u),\mathscr{A}(v)) \leq \frac{\alpha\tau(u,\mathscr{A}(v)) + \beta\tau(v,\mathscr{A}(u))}{\gamma\tau(u,\mathscr{A}(u)) + \delta\tau(v,\mathscr{A}(v)) + \epsilon}\max\{\tau(u,\mathscr{A}(v))+\tau(v,\mathscr{A}(u)),\tau(u,\mathscr{A}(u)),\tau(v,\mathscr{A}(v)),\tau(u,v)\}. \tag{5.3.20}$$

Then,

1. *There is at least one fixed point denoted as $\dot{\xi}$ within the space Υ for the mapping $\mathscr{A}$;*
2. *Each Picard sequence $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ converges to one of the fixed points;*
3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality (5.3.21):*

$$\tau(\dot{\xi},\dot{\eta}) \geq \frac{\epsilon}{2(\alpha+\beta)}. \tag{5.3.21}$$

Proof The procedure for proving the above theorem is omitted because it is similar to the proof of Theorem 5.7. ■

Theorem 5.15 *Let $\mathscr{A}$ be a self-mapping acting on a complete b-metric space noted by (Υ,τ,s). Assume there are five constants $\alpha,\beta,\gamma,\delta,\epsilon \in \mathbb{R}^+$ that ensure either $s^3\alpha \leq \frac{1}{2}\min\{\gamma,\delta\}$ or $s^3\beta \leq \frac{1}{2}\min\{\gamma,\delta\}$, and for all $u,v \in \Upsilon$,*

$$\tau(\mathscr{A}(u),\mathscr{A}(v)) \leq \frac{\alpha\tau(u,\mathscr{A}(v))+\beta\tau(v,\mathscr{A}(u))}{\gamma\tau(u,\mathscr{A}(u))+\delta\tau(v,\mathscr{A}(v))+\epsilon}\max\{\tau(u,\mathscr{A}(v))+\tau(v,\mathscr{A}(u)),\tau(u,\mathscr{A}(u))+\tau(v,\mathscr{A}(v)),\tau(u,v)\}. \tag{5.3.22}$$

Then,

1. *There is at least one fixed point denoted as $\dot{\xi}$ within the space Υ for the mapping $\mathscr{A}$;*
2. *Each Picard sequence $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ converges to one of the fixed points;*
3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality (5.3.23):*

$$\tau(\dot{\xi},\dot{\eta}) \geq \frac{\epsilon}{2(\alpha+\beta)}. \tag{5.3.23}$$

Proof The procedure for proving the above theorem is omitted because it is similar to the proof of Theorem 5.7. ■

Theorem 5.16 *Let $\mathscr{A}$ be a self-mapping acting on a complete b-metric space noted by (Υ,τ,s). Assume there are five constants $\alpha,\beta,\gamma,\delta,\epsilon \in \mathbb{R}^+$ that ensure either $s^2\alpha \leq \frac{1}{5}\min\{\gamma,\delta\}$ or $s^2\beta \leq \frac{1}{5}\min\{\gamma,\delta\}$, and for all $u,v \in \Upsilon$,*

$$\tau(\mathscr{A}(u),\mathscr{A}(v)) \leq \frac{\alpha\tau(u,\mathscr{A}(v))+\beta\tau(v,\mathscr{A}(u))}{\gamma\tau(u,\mathscr{A}(u))+\delta\tau(v,\mathscr{A}(v))+\epsilon}[\tau(u,\mathscr{A}(v))+\tau(v,\mathscr{A}(u))+\tau(u,\mathscr{A}(u))+\tau(v,\mathscr{A}(v))+\tau(u,v)]. \tag{5.3.24}$$

Then,

1. *There is at least one fixed point denoted as $\dot{\xi}$ within the space Υ for the mapping $\mathscr{A}$;*
2. *Each Picard sequence $(\mathscr{A}(u_n))_{n\in\mathbb{N}}$ converges to one of the fixed points;*
3. *If $\mathscr{A}$ has several fixed points, then each pair of them must satisfy the inequality (5.3.25):*

$$\tau(\dot{\xi},\dot{\eta}) \geq \frac{\epsilon}{3(\alpha+\beta)}. \tag{5.3.25}$$

Proof The procedure for proving the above theorem is omitted because it is similar to the proof of Theorem 5.7. ■

5.4 Discussion

- The Theorem 5.6 generalizes the result of Khojasteh et al. [76] and the Corollary 5.4 generalizes the result of Aouine and Aliouche [9].
- We notice the disappearance of the fixed point uniqueness after adding the term $\tau(u,v)$ to the left part of the inequality 5.2.1 as illustrated in Theorem 5.11 and Example 5.5.
- If rang$\mathscr{A}$ is a closed sub-set of Υ, the inequalities (5.3.14),(5.3.12)(5.3.16), (5.3.18), (5.3.20), (5.3.22) and (5.3.24) can be restricted for all $u,v \in$ rang$\mathscr{A}$, and that does not affect the proof and the desired results, which makes it easier for us to verify its validity and become more applicable.
- We note that the choice of constants related to inequalities (5.3.1), (5.2.1), (5.2.5), and (5.3.4) directly affects the dynamic result of Theorems 5.6, 5.3, and Corollaries 5.4 and 5.7, respectively. Example 3.8 illustrates this point precisely.
- Note that the ratio $\dfrac{\alpha\tau(u,\mathscr{A}(v))+\beta\tau(v,\mathscr{A}(u))}{\gamma\tau(u,\mathscr{A}(u))+\delta\tau(v,\mathscr{A}(v))+\epsilon}$ in the inequalities (5.3.1), (5.2.1), (5.2.5), and (5.3.4) may be greater or less than 1; therefore, the corresponding theorems represent special modifications of the Banach contraction principle. Example 5.3 clarifies this point precisely.
- If the range of $\mathscr{A}$ is a closed subset of Υ, the inequalities (5.3.1), (5.2.1), (5.2.5), and (5.3.4) can be restricted to the range of $\mathscr{A}$, which does not affect the proof or the desired results, making it easier to verify its validity and more applicable.
- The above results can be generalized into several generalized metric spaces such as $q_1 - q_2$ b-metric spaces, partial metric spaces, etc.

5.5 Application to integral equations

In this section, we will apply Corollary 5.8 to study the problem of existence of the solution for certain type of integral equation. In this section $(E, \|.\|)$ design a Banach space.

5.5.1 Position of the problem

The integral equation of the form

$$u(t) = h(t) + \int_0^t G(t,s) f(s,u(s))\,ds, \quad t \in [0,T], \tag{5.5.1}$$

is a fundamental type of equation that arises in various fields such as mathematical physics, engineering, and economics. In this equation, the variable $u(t)$ represents the state of the system at time t, while $h(t)$ denotes an initial condition or an external input. The function $G(t,s)$ describes the influence of past states on the current state, and $f(s,u(s))$ models the interaction between the state variable and other inputs at time s.

We consider the following:

$u: [0,T] \to E$ is the unknown, where E: A complete vector space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). It is equipped with a norm $\|\cdot\|$.
$h: [0,T] \to E$, $f: [0,T] \times E \to E$ and $G: [0,T] \times [0,T] \to \mathbb{K}$ be a given functions

5.5.2 Existence and Uniqueness of Solutions

We define the operator $\mathscr{A} : C([0,T],E) \to C([0,T],E)$ as:

$$(\mathscr{A}(u))(t) = h(t) + \int_0^t G(t,s) f(s,u(s))\,ds \quad \text{for all } t \in [0,T],$$

where $C([0,T],E)$: The space of continuous functions from the interval $[0,T]$ to the Banach space E. This space is endowed with the b-metric defined as:

$$\tau(u,v) = \sup_{t\in[0,T]} \|\mathscr{A}(u(t)) - \mathscr{A}(v(t))\|^2$$

for all $u, v \in C([0,T],E)$.

Theorem 5.17 *assuming that the function $f(s,u(s))$ satisfies the condition:*

$$\|f(s,u(s)) - f(s,v(s))\| \le k\|u(s)-v(s)\| \cdot \|u(s) - \mathscr{A}(v(s))\|$$

for all $s \in [0,T]$ and $u, v \in C([0,T],E)$. Then the integral equation (5.5.1) have at least one solution in $C([0,T],E)$.

Proof Let $s \in [0,T]$ and $u, v \in C([0,T],E)$.

$$\tau(\mathscr{A}(u), \mathscr{A}(v)) = \sigma(\mathscr{A}(u), \mathscr{A}(v))^2 = \sup_{t\in[0,T]} \|\mathscr{A}(u(t)) - \mathscr{A}(v(t))\|^2$$

we use the hypothesis and the Cauchy-Schwartz inequality,

$$\begin{aligned}
\|\mathscr{A}(u(t)) - \mathscr{A}(v(t))\| &= \left| \int_0^t f(s,u(s))\,ds - \int_0^t f(s,v(s))\,ds \right| \\
&= \left| \int_0^t \big(f(s,u(s)) - f(s,v(s))\big)\,ds \right| \\
&\le \int_0^t \left| f(s,u(s)) - f(s,v(s)) \right| ds \\
&\le k \int_0^t \|u(s)-v(s)\| \|u(s) - \mathscr{A}(v(s))\| ds \\
&\le k \left(\int_0^t \|u(s)-v(s)\|^2 ds \right)^{\frac{1}{2}} \cdot \left(\int_0^t \|u(s) - \mathscr{A}(v(s))\|^2 ds \right)^{\frac{1}{2}}
\end{aligned}$$

that means,

$$\begin{aligned}
\|\mathscr{A}(u(t)) - \mathscr{A}(v(t))\|^2 &\le k^2 \int_0^t \|u(s)-v(s)\|^2 ds \cdot \int_0^t \|u(s) - \mathscr{A}(v(s))\|^2 ds \\
&\le k^2 \int_0^T \|u(s)-v(s)\|^2 ds \cdot \int_0^T \|u(s) - \mathscr{A}(v(s))\|^2 ds \\
&\le k^2 T^2 \sup_{s\in[0,T]} \|u(s)-v(s)\|^2 \sup_{s\in[0,T]} \|u(s) - \mathscr{A}(v(s))\|^2 \\
&\le k^2 T^2 \tau(u,v) \cdot \tau(u, \mathscr{A}(v))
\end{aligned}$$

implies

$$\sup_{t\in[0,T]} \|\mathscr{A}(u(t)) - \mathscr{A}(v(t))\|^2 \leq k^2 T^2 \tau(u,v)\cdot\tau(u,\mathscr{A}(v))$$

we obtain

$$\tau(\mathscr{A}(u) - \mathscr{A}(v)) \leq k' \tau(u,v)\cdot\tau(u,\mathscr{A}(v))$$

This inequality shows that $\mathscr{A}$ satisfies the condition (5.3.10) and since $(C([0,T],E),\tau)$ is a complete b-metric space with $s=2$. Consequently, according Corollary 5.8 we conclude that the operator $\mathscr{A}$ admits at least one fixed point in $C([0,T],E)$. Thus, the integral equation (5.5.1) has a solution $u \in C([0,T],E)$, which corresponds to the fixed point of the operator $\mathscr{A}$.

■

5.6 Application to coupled fixed point theory

In this section, we apply Theorem 5.11 to examine the existence of coupled fixed points for a certain type of mapping that represents solutions to nonlinear systems of the form:

$$\begin{cases} \mathscr{A}(u,v) &= u, \\ \mathscr{A}(v,u) &= v. \end{cases} \tag{5.6.1}$$

Such problems can be transformed into a fixed point problem by considering the mapping $f : \Upsilon\times\Upsilon \to \Upsilon\times\Upsilon$, defined as $(u,v)\mapsto f(u,v) = (\mathscr{A}(u,v),\mathscr{A}(v,u))$.

Many coupled fixed point theorems are, in fact, direct consequences of well-known fixed point theorems, using the same principle mentioned above. This does not diminish their scientific value, as these theorems have become effective tools for studying problems in the form of system (5.6.1). For example, in [15], Bhaskar and Lakshmikantham established coupled fixed point theorems and used these results to demonstrate the existence and uniqueness of solutions to a class of periodic boundary value problems.

The following is a coupled fixed point result on b-metric space based on our fixed point results. This result was published in [12].

Theorem 5.18 *Let (Υ,τ,s) be a complete b-metric space and let $\mathscr{A} : \Upsilon\times\Upsilon \longrightarrow \Upsilon$ be a mapping. Assume there exist five positive real numbers $\alpha,\beta,\gamma,\delta,\epsilon \in \mathbb{R}^+$, such that $s^2\alpha \leq \min\{\gamma,\delta\}$ or $s^2\beta \leq \min\{\gamma,\delta\}$ and for all $(x_1,y_1),(x_2,y_2) \in \Upsilon\times\Upsilon$,*

$$\tau(\mathscr{A}(x_1,y_1),\mathscr{A}(x_2,y_2)) \leq \frac{\alpha\tau(x_1,\mathscr{A}(x_2,y_2)) + \beta\tau(y_2,\mathscr{A}(y_1,x_1))}{\gamma\tau(x_1,\mathscr{A}(x_1,y_1)) + \gamma\tau(y_1,\mathscr{A}(y_1,x_1)) + \delta\tau(x_2,\mathscr{A}(x_2,y_2)) + \delta\tau(y_2,\mathscr{A}(y_2,x_2)) + \epsilon}$$
$$\max\{\tau(x_1,x_2),\tau(y_1,y_2),\tau(x_1,\mathscr{A}(x_1,y_1)),\tau(y_1,\mathscr{A}(y_1,x_1)),\tau(x_2,\mathscr{A}(x_2,y_2)),\tau(y_2,\mathscr{A}(y_2,x_2))\}. \tag{5.6.2}$$

Then, $\mathscr{A}$ has at least one coupled fixed point $(\dot{x},\dot{y}) \in \Upsilon\times\Upsilon$, moreover, $\tau(\dot{x},\dot{y}) = 0$ or $\tau(\dot{x},\dot{y}) \geq \dfrac{\epsilon}{\alpha+\beta}$.

Proof Let (Υ,τ,s) be a complete b-metric space, then, $(\Upsilon\times\Upsilon,\tau,s)$ is also a complete b-metric space where τ is a b-metric distance defined on $\Upsilon\times\Upsilon$, as follows,

$$\tau((x_1,y_1),(x_2,y_2)) = \max\{\tau(x_1,x_2),\tau(y_1,y_2)\}$$

Let $\mathscr{A} : \Upsilon \times \Upsilon \longrightarrow \Upsilon$ be a mapping. We denote by F the self mapping $F : \Upsilon \times \Upsilon \longrightarrow \Upsilon \times \Upsilon$ defined as follows, $F(x, y) = (\mathscr{A}(x, y), \mathscr{A}(y, x))$ for all couples $(x, y) \in \Upsilon \times \Upsilon$.

According to inequality (5.6.2), we have,

$$\tau(\mathscr{A}(x_1,y_1),\mathscr{A}(x_2,y_2)) \leq \frac{\alpha\max\{\tau(x_1,\mathscr{A}(x_2,y_2)),\tau(y_1,\mathscr{A}(y_2,x_2)\}+\beta\max\{\tau(y_2,\mathscr{A}(y_1,x_1)),\tau(x_2,\mathscr{A}(x_1,y_1))\}}{\gamma\max\{\tau(x_1,\mathscr{A}(x_1,y_1)),\tau(y_1,\mathscr{A}(y_1,x_1))\}+\delta\max\{\tau(x_2,\mathscr{A}(x_2,y_2)),\tau(y_2,\mathscr{A}(y_2,x_2))\}+\epsilon} \max\{\tau(x_1,x_2),\tau(y_1,y_2),\tau(x_1,\mathscr{A}(x_1,y_1)),\tau(y_1,\mathscr{A}(y_1,x_1)),\tau(x_2,\mathscr{A}(x_2,y_2)),\tau(y_2,\mathscr{A}(y_2,x_2))\}. \tag{5.6.3}$$

and

$$\tau(\mathscr{A}(y_1,x_1),\mathscr{A}(y_2,x_2)) \leq \frac{\alpha\max\{\tau(x_1,\mathscr{A}(x_2,y_2)),\tau(y_1,\mathscr{A}(y_2,x_2)\}+\beta\max\{\tau(y_2,\mathscr{A}(y_1,x_1)),\tau(x_2,\mathscr{A}(x_1,y_1))\}}{\gamma\max\{\tau(x_1,\mathscr{A}(x_1,y_1)),\tau(y_1,\mathscr{A}(y_1,x_1))\}+\delta\max\{\tau(x_2,\mathscr{A}(x_2,y_2)),\tau(y_2,\mathscr{A}(y_2,x_2))\}+\epsilon} \max\{\tau(x_1,x_2),\tau(y_1,y_2),\tau(x_1,\mathscr{A}(x_1,y_1)),\tau(y_1,\mathscr{A}(y_1,x_1)),\tau(x_2,\mathscr{A}(x_2,y_2)),\tau(y_2,\mathscr{A}(y_2,x_2))\}. \tag{5.6.4}$$

From inequalities (5.6.3) and (5.6.4), we get,

$$\max\{\tau(\mathscr{A}(x_1,y_1),\mathscr{A}(x_2,y_2)),\tau(\mathscr{A}(y_1,x_1),\mathscr{A}(y_2,x_2)))\} \leq \frac{\alpha\max\{\tau(x_1,\mathscr{A}(x_2,y_2)),\tau(y_1,\mathscr{A}(y_2,x_2)\}+\beta\max\{\tau(y_2,\mathscr{A}(y_1,x_1)),\tau(x_2,\mathscr{A}(x_1,y_1))\}}{\gamma\max\{\tau(x_1,\mathscr{A}(x_1,y_1)),\tau(y_1,\mathscr{A}(y_1,x_1))\}+\delta\max\{\tau(x_2,\mathscr{A}(x_2,y_2)),\tau(y_2,\mathscr{A}(y_2,x_2))\}+\epsilon} \max\{\tau(x_1,x_2),\tau(y_1,y_2),\tau(x_1,\mathscr{A}(x_1,y_1)),\tau(y_1,\mathscr{A}(y_1,x_1)),\tau(x_2,\mathscr{A}(x_2,y_2)),\tau(y_2,\mathscr{A}(y_2,x_2))\}.$$

That means,

$$\tau(F(u),F(v)) \leq \frac{\alpha\tau(u,F(v))+\beta\tau(v,F(u))}{\gamma\tau(u,F(u))+\delta\tau(v,F(v))+\epsilon}\max\{\tau(u,v),\tau(u,F(u)),\tau(v,F(v))\},$$

For all $u, v \in \Upsilon \times \Upsilon$.

According to Theorem 5.11, we conclude that F has at least one fixed point, which means that $\mathscr{A}$ has a coupled fixed point at least noted by $(\dot{x}, \dot{y}) \in \Upsilon \times \Upsilon$.

Since $(\dot{y}, \dot{x})$ is also a coupled fixed point for $\mathscr{A}$, then, by choosing $(x_1, y_1) = (\dot{x}, \dot{y})$ and $(x_2, y_2) = (\dot{y}, \dot{x})$ in inequality (5.6.2), we have,

$$\tau(\dot{x},\dot{y}) \leq \frac{\alpha+\beta}{\epsilon}\tau^2(\dot{x},\dot{y}),$$

That means, $\tau(\dot{x},\dot{y}) = 0$ or $\tau(\dot{x},\dot{y}) \geq \dfrac{\epsilon}{\alpha+\beta}$. This completes the proof of our theorem. ∎

Remark 5.5 *If* rang$\mathscr{A}$ *is a closed subset of* Υ, *the inequality* (5.6.2) *can be restricted to* rang$\mathscr{A} \times$ rang$\mathscr{A}$, *and that does not affect the proof and the desired results, which makes it easier for us to verify its validity and become more applicable.*

In order to prove the possibility of fulfilling the conditions of the previous theorem, we provide the following example.

Example 5.7 *Let $\Upsilon = \{0,1,2\}$ associated with a b-metric τ such that $\tau(x,y) = (x-y)^2$. Let $\mathscr{A}$ be a mapping defined on $\Upsilon \times \Upsilon$ such that*

$$\mathscr{A}(x,y) = \begin{cases} 0 & \text{if } x = 0 \text{ or } x = 2 \text{ or } y = 2, \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to conclude that (Υ, τ, s) is a complete b-metric space with the constant $s = 2$ and the inequality (5.6.2) was verified for all $(x_1, y_1), (x_2, y_2) \in \operatorname{rang}\mathscr{A} \times \operatorname{rang}\mathscr{A}$ with constants $\alpha = \beta = \epsilon = 1$ and $\gamma = \delta = 4$. According to Theorem 5.18, we conclude that $\mathscr{A}$ has at least one coupled fixed point $(\dot{x}, \dot{y}) \in \Upsilon \times \Upsilon$. (In fact, it has four coupled fixed points: $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$). Moreover, the distance between each couple is $\tau(0,0) = \tau(1,1) = 0$ and $\tau(0,1) = \tau(1,0) \geq \frac{1}{2}$.

Conclusion

This thesis has been devoted to the study of fixed points for a new rational-type contraction in metric spaces, with extensions to Menger PM spaces and b-metric spaces. We have developed the dynamic properties associated with the behavior of fixed points, enhancing and generalizing previous results presented by Khojasteh [76], Yildirim [126] and Demma [39]. As well, a brief historical overview was provided to describe the evolution of some prominent fixed point results over the years.

Our original contributions expand on this foundational work by introducing new results related to unique and non-unique fixed point theorems under rational contractive conditions across various frameworks, including complete metric spaces, Menger PM spaces, and b-metric spaces.

As an application, we investigated the existence of solutions to integral equations and congruence problems. Moreover, we established results concerning coupled fixed points in b-metric spaces. Finally, several examples were presented to validate and illustrate the applicability of our theoretical findings.

The fixed point theory in probabilistic metric spaces is still in its formative stages; however, it holds significant promise for both theoretical and applied mathematics. As this field continues to develop, it is expected to have a profound impact, particularly in areas that address uncertainty and probabilistic models. The generalization of classical fixed point theorems to probabilistic settings opens up new avenues for research, offering potential applications across various domains such as optimization, differential equations, and stochastic processes. Furthermore, the results presented in this thesis, while valuable, represent only the beginning of our explorations. There remains considerable room for further investigations, refinements, and expansions of these findings. Future work will focus on enhancing the scope and applicability of these results to real-world problems.

Perspectives

The results obtained in this thesis open several avenues for future research. While we have established key fixed point theorems under rational contractive conditions in metric, Menger, and b-metric spaces, there are numerous unexplored extensions and applications that warrant further investigation:

1. **Generalization to Other Types of Spaces:** Future work could focus on extending the current results to other generalized metric structures, such as partial metric spaces, modular metric spaces, or probabilistic b-metric spaces. These spaces provide a broader framework that may yield new insights into the fixed point theory.

2. **Combining Rational-Type Contractions with Nonlinear Contractions:** Combining rational-type contractions with other nonlinear contractions, such as F-contraction and (F,ψ)-Suzuki-contraction, could lead to more general contraction mappings and new fixed point theorems, enhancing their application in problem-solving across various fields.

3. **Applications to Non-Linear Problems:** The application of fixed point theorems to solving integral equations was touched upon. Future work could extend these applications to non-linear fractional differential evolution problems, optimization problems, and control theory, where fixed points often play a crucial role in proving the existence of mild and classical solutions.

4. **Coupled and tripled Systems:** While coupled fixed point theorems were discussed in the context of b-metric spaces, there remains scope for exploring coupled and tripled fixed points for multi-variable systems. Such studies may lead to significant contributions in systems of non-linear systems.

5. **Computational Algorithms:** Another perspective is to develop efficient numerical algorithms for approximate fixed points in generalized metric spaces. Such algorithms could have practical applications in areas like machine learning, data analysis, and complex systems modeling.

6. **Further Exploration of Non-Uniqueness:** Finally, the phenomenon of non-unique fixed points is not yet fully understood, especially in dynamic systems. Investigating the conditions under which non-uniqueness arises, and its implications in real-world systems, would be an interesting direction for future research.

By pursuing these perspectives, the scope of fixed point theory could be further broadened, offering new applications and deepening our understanding of contraction mappings in various mathematical frameworks.

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