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# On a time-discretization method for a semilinear heat equation with purely integral conditions in a nonclassical function space

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**Abstract**

In this paper, we construct a semidiscrete approximate solution to a semilinear one-dimensional heat equation subject to integral boundary conditions by means of the Rothe discretization in time method. The convergence of the approximation scheme obtained is proved, yielding the well-posedness of the problem considered.


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*Keywords:* Semilinear heat equation; Integral boundary condition; Rothe method; A priori estimate

## 1. Introduction

Since 1930, various classical types of initial–boundary value problems have been investigated by many authors using Rothe time-discretization method, see, for instance, the monographs by Rektorys [12] and Kačur [6] and references cited therein. In particular, for the application of the Rothe method to abstract evolution equations, we refer the reader to the papers by Nečas [9] and Kartsatos and Parrott [7]. In this latter, the authors solved the initial–boundary value problem for the time-dependent functional equation $x' + A(t)x = G(t, x_t)$ with $A(t)$ $m$-accretive and $G$ lipschitzian.

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In the present paper, we consider the problem of determining a function $v = v(x, t)$ satisfying, in a weak sense, the semilinear heat equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \mathrm{f}(x, t, v), \qquad (x, t) \in (0, 1) \times [0, T], \tag{1.1}$$

supplemented by the initial condition

$$v(x, 0) = V_0(x), \qquad 0 \leqslant x \leqslant 1, \tag{1.2}$$

and the integral boundary conditions

$$\int_0^1 v(x, t)\, \mathrm{d}x = E(t), \qquad 0 \leqslant t \leqslant T, \tag{1.3}$$

$$\int_0^1 x v(x, t)\, \mathrm{d}x = G(t), \qquad 0 \leqslant t \leqslant T, \tag{1.4}$$

where f, $V_0$, $E$ and $G$ are sufficiently regular given functions, and $T$ is a positive constant.

The linear version of this problem, i.e. $\mathrm{f} = \mathrm{f}(x, t)$, appears for instance in the modelling of the quasi-static flexure of a thermoelastic rod (see [2]) and has been studied, firstly, by the second author with a more general second-order parabolic equation or a $2m$-parabolic equation in [1, 2,4] by means of the energy-integrals method and, secondly, by the two authors via the Rothe method [11]. The present work, which can be viewed as a continuation of this latter, enrolls in the setting of efforts aiming to develop the Rothe method to certain classes of evolution problems involving nonlocal (integral) conditions over the spatial domain. Indeed, such nonclassical boundary conditions are the source of some great complications when applying the standard Rothe method, and to avoid the difficulties encountered, we make appeal to an appropriate nonclassical function space where we conduct our investigation (see also our earlier works [10] and [5]).

The paper is divided as follows. To begin, in the next section, we transform problem (1.1)–(1.4) to an equivalent one with homogeneous integral conditions, namely problem (2.3)–(2.6). Then, we specify notation and assumptions on data before stating the precise sense of the required solution. In Section 3, using semidiscretization in time, the problem (2.3)–(2.6) is approximated by a sequence of corresponding linearized (time-discretized) problems by means of which an approximate solution of the original problem is constructed. Then, on the basis of some a priori estimates derived in Section 4, it is proved, in the last section, that the approximations converge and the limit turns out to be the unique weak solution of the original problem.

## 2. Preliminaries

Using the transformation

$$u(x, t) = v(x, t) - r(x, t), \qquad (x, t) \in (0, 1) \times [0, T], \tag{2.1}$$

where

$$r(x, t) = 6\,(2G(t) - E(t))\,x - 2\,(3G(t) - 2E(t))\,, \tag{2.2}$$

one easily checks that problem (1.1)–(1.4) with inhomogeneous integral conditions (1.3) and (1.4) is reduced to the following equivalent problem with homogeneous conditions for the new unknown function $u$:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t,u), \qquad (x,t) \in (0,1) \times I, \tag{2.3}$$

$$u(x,0) = U_0(x), \qquad 0 \leqslant x \leqslant 1, \tag{2.4}$$

$$\int_0^1 u(x,t)\,\mathrm{d}x = 0, \qquad t \in I, \tag{2.5}$$

$$\int_0^1 xu(x,t)\,\mathrm{d}x = 0, \qquad t \in I, \tag{2.6}$$

where $I$ stands for the time interval $[0,T]$ and

$$f(x,t,u) := \mathrm{f}(x,t,u+r) - \frac{\partial r(x,t)}{\partial t}, \tag{2.7}$$

and

$$U_0(x) := V_0(x) - r(x,0). \tag{2.8}$$

Hence, instead of looking for the function $v$, we search for the function $u$. The solution of problem (1.1)–(1.4) will be simply given by the formula $v = u + r$.

Following standard notation, we let $H^2(0,1)$ be the (real) second-order Sobolev space on $(0,1)$ with norm $\|\cdot\|_{H^2(0,1)}$. The usual inner product in $L^2(0,1)$ and the corresponding norm will be denoted by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. Observing the nature of the boundary conditions (2.5) and (2.6), it is advisable to consider the space $V$ defined by:

$$V := \left\{ \phi \in L^2(0,1);\ \int_0^1 \phi(x)\,\mathrm{d}x = \int_0^1 x\phi(x)\,\mathrm{d}x = 0 \right\}. \tag{2.9}$$

Being the kernel of the bounded linear mapping $\ell : L^2(0,1) \to \mathbb{R}^2$, $\phi \mapsto \ell(\phi) = \left(\int_0^1 \phi(x)\,\mathrm{d}x,\ \int_0^1 x\phi(x)\,\mathrm{d}x\right)$, $V$ is a Hilbert space for $(\cdot,\cdot)$.

Our analysis requires also the use of the nonclassical function space $B_2^1(0,1)$ introduced by Bouziani (see [1–3], for instance) as the completion of the space $C_0(0,1)$ of real continuous functions with compact support in $(0,1)$ with respect to the inner product

$$(u,v)_{B_2^1} = \int_0^1 \Im_x u \cdot \Im_x v\,\mathrm{d}x, \tag{2.10}$$

where $\Im_x v = \int_0^x v(\xi)\,\mathrm{d}\xi$ for every fixed $x \in (0,1)$. We recall that, if $\|\cdot\|_{B_2^1}$ denotes the corresponding norm, i.e.

$$\|v\|_{B_2^1} = \sqrt{(v,v)_{B_2^1}}, \tag{2.11}$$

then the following inequality

$$\|v\|^2_{B_2^1} \leqslant \frac{1}{2}\|v\|^2 \tag{2.12}$$

holds for every $v \in L^2(0,1)$, and the embedding $L^2(0,1) \to B_2^1(0,1)$ is continuous.

Moreover, we will work in the standard functional spaces of the types $C(I,X)$, $C^{0,1}(I,X)$, $L^2(I,X)$ and $L^\infty(I,X)$ where $X$ is a Banach space, the main properties of which can be found in [8].

The notation $\theta(t)$ is automatically used for the same function $\theta(x,t)$ considered as an abstract function of the variable $t \in I$ into some functional space on (0, 1). Strong and weak convergence are denoted by $\to$ and $\rightharpoonup$ respectively.

In several places, we shall use the Gronwall Lemma in the following continuous and discrete versions:

**Lemma 2.1.** (i) *Let $h(t)$ and $y(t)$ be two real integrable functions on the interval $I$, $h(\tau)$ nondecreasing, and $C$ a positive constant. If*

$$y(t) \leqslant h(t) + C\int_0^t y(\tau)\,\mathrm{d}\tau, \quad \forall t \in I,$$

*then*

$$y(t) \leqslant h(t)\mathrm{e}^{Ct}, \quad \forall t \in I.$$

(ii) *Let $\{a_i\}$ be a sequence of real nonnegative numbers satisfying*

$$\begin{cases} a_1 \leqslant A, \\ a_i \leqslant A + Bh\displaystyle\sum_{k=1}^{i-1} a_k, \quad \forall i = 2, \ldots, \end{cases}$$

*where $A$, $B$ and $h$ are positive constants. Then*

$$a_i \leqslant A\mathrm{e}^{B(i-1)h}, \quad \forall i = 1, 2, \ldots.$$

**Proof.** The proof of assertion (i) is the same as in [6, Lemma 1.3.19]. As for assertion (ii), it suffices to see that from our hypothesis the following estimate follows

$$a_i \leqslant A\,(1+Bh)^{i-1}, \quad \forall i = 1, 2, \ldots.$$

Indeed, we have first $a_1 \leqslant A$ and $a_2 \leqslant A + a_1 Bh \leqslant A\,(1+Bh)$. Next, let us suppose that $a_k \leqslant A\,(1+Bh)^{k-1}$ holds for all $k = 1, \ldots, i-1$, then

$$\begin{aligned} a_i &\leqslant A + Bh\sum_{k=1}^{i-1} A\,(1+Bh)^{k-1} \\ &= A\left[1 + Bh\sum_{k=1}^{i-1}(1+Bh)^{k-1}\right] \\ &= A\left[1 + Bh\frac{1-(1+Bh)^{i-1}}{1-(1+Bh)}\right] = A\,(1+Bh)^{i-1}. \end{aligned}$$

Hence, using the elementary inequality $1 + t \leqslant \mathrm{e}^t$, $\forall t \in \mathbb{R}_+$, we obtain $a_i \leqslant A\mathrm{e}^{B(i-1)h}$ which was to be proved. □

The following assumptions are sufficient for our investigation:

(H$_1$) $f(t,w) \in L^2(0,1)$ for each pair $(t,w) \in I \times L^2(0,1)$ and the following Lipschitz condition

$$\left\| f(t,w) - f(t',w')\right\|_{B_2^1} \leqslant L\left[\left|t-t'\right|\left(1 + \|w\|_{B_2^1} + \left\|w'\right\|_{B_2^1}\right) + \left\|w-w'\right\|_{B_2^1}\right],$$

is satisfied for all $t, t' \in I$ and $w, w' \in V$, where $L$ is some positive constant.
($H_2$) $U_0 \in H^2(0, 1)$.
($H_3$) Compatibility conditions: $U_0 \in V$, i.e. $\int_0^1 U_0(x)\,\mathrm{d}x = \int_0^1 xU_0(x)\,\mathrm{d}x = 0$.

We search for a weak solution in the following sense:

**Definition 2.1.** By a weak solution of problem (2.3)–(2.6), we mean a function $u : I \to L^2(0, 1)$ such that:

(i) $u \in L^\infty(I, V) \cap C^{0,1}\left(I, B_2^1(0, 1)\right)$;
(ii) $u$ has (a.e. in $I$) a strong derivative $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^\infty\left(I, B_2^1(0, 1)\right)$;
(iii) $u(0) = U_0$ in $B_2^1(0, 1)$;
(iv) the identity

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}(t), \phi\right)_{B_2^1} + (u(t), \phi) = (f(t, u(t)), \phi)_{B_2^1}, \tag{2.13}$$

holds for all $\phi \in V$ and a.e. $t \in I$.

Note that since $u \in C^{0,1}\left(I, B_2^1(0, 1)\right) \subset C\left(I, B_2^1(0, 1)\right)$ the condition (iii) makes sense, and in view of (i), (ii) and Assumption ($H_1$) each term in (2.13) is well defined. On the other hand, the fulfillment of the integral conditions (2.5) and (2.6) is included in the fact that $u(t) \in V$ for a.e. $t \in I$.

## 3. Construction of approximate solutions

In order to solve the problem (2.3)–(2.6) by the Rothe method, we divide the time interval $I$ into $n$ subintervals $[t_{j-1}, t_j]$, $j = 1, \ldots, n$, where $t_j = jh$ and $h = T/n$. Then, replacing the first time derivative of $u$ by the corresponding standard difference quotient, the problem (2.3)–(2.6) may be approximated at each point $t = t_j$, $j = 1, \ldots, n$, by the following linearized problem:

*Find function* $u_j : (0, 1) \to \mathbb{R}$, *such that:*

$$\delta u_j - \frac{\mathrm{d}^2 u_j}{\mathrm{d}x^2} = f_j, \quad x \in (0, 1), \tag{3.1$_j$}$$

$$\int_0^1 u_j(x)\,\mathrm{d}x = 0, \tag{3.2$_j$}$$

$$\int_0^1 xu_j(x)\,\mathrm{d}x = 0, \tag{3.3$_j$}$$

where $u_0$ is given by

$$u_0(x) = U_0(x), \quad x \in (0, 1), \tag{3.4}$$

and where $\delta u_j := \frac{u_j - u_{j-1}}{h}$ and $f_j = f(t_j, u_{j-1})$.

Obviously, this is a recurrent system of linear boundary value problems, for the approximates $u_j$, to be solved successively for $j = 1, \ldots, n$, starting from the initial function from (2.4). To prove the existence and uniqueness of such $u_j$, we make use of a slightly modified idea of [13] in the following way: For all $j = 1, \ldots, n$, we associate with problem (3.1$_j$)–(3.3$_j$) the Dirichlet boundary value problem for a second-order linear ordinary differential equation for unknown functions $w_j = w_j(x; \lambda, \mu)$:

$$-\frac{d^2 w_j}{dx^2} + \frac{1}{h} w_j = f_j + \frac{1}{h} w_{j-1}, \quad x \in (0, 1), \tag{3.5$_j$}$$

$$w_j(0) = \lambda, \tag{3.6$_j$}$$

$$w_j(1) = \mu, \tag{3.7$_j$}$$

where $(\lambda, \mu)$ is for the moment an arbitrary, but fixed ordered pair of real numbers, and where $w_0 = U_0$.

Since $f_1 + \frac{1}{h} w_0 := f(t_1, U_0) + \frac{1}{h} U_0 \in L^2(0, 1)$, the Lax–Milgram Lemma implies the existence and uniqueness of a strong solution $w_1 \in H^2(0, 1)$ to the elliptic problem (3.5$_1$)–(3.7$_1$). Let us show that the parameters $\lambda$ and $\mu$ can be chosen in a suitable way such that the corresponding function $w_1(\cdot; \lambda, \mu)$ becomes a solution of problem (3.1$_1$)–(3.3$_1$) provided that $n$ is enough large.

In fact, the function $w_1(\cdot; \lambda, \mu)$ will be a solution to problem (3.1$_1$)–(3.3$_1$) if and only if the pair $(\lambda, \mu)$ is a solution to the following system of equations

$$\begin{cases} \displaystyle\int_0^1 w_1(x; \lambda, \mu)\, dx = 0, \\ \displaystyle\int_0^1 x w_1(x; \lambda, \mu)\, dx = 0, \end{cases} \tag{3.8}$$

and thus, solving (3.8) will provide all the solutions to problem (3.1$_1$)–(3.3$_1$). If, in particular, (3.8) admits a unique solution, so does problem (3.1$_1$)–(3.3$_1$). Since the solvability of system (3.8) requires the explicit expression of $w_1(\cdot; \lambda, \mu)$ in terms of $\lambda$ and $\mu$, we write $w_1(\cdot; \lambda, \mu)$ as the sum of two functions $\widetilde{w}_1$ and $\overline{w}_1(\cdot; \lambda, \mu)$, where $\widetilde{w}_1$ (independent of $\lambda$ and of $\mu$) and $\overline{w}_1$ are the solutions to the following problems respectively:

$$\begin{cases} -\dfrac{d^2 \widetilde{w}_1}{dx^2} + \dfrac{1}{h} \widetilde{w}_1 = f_1 + \dfrac{1}{h} U_0, \quad x \in (0, 1), \\ \widetilde{w}_1(0) = 0, \\ \widetilde{w}_1(1) = 0, \end{cases}$$

and

$$\begin{cases} -\dfrac{d^2 \overline{w}_1}{dx^2} + \dfrac{1}{h} \overline{w}_1 = 0, \quad x \in (0, 1), \\ \overline{w}_1(0) = \lambda, \\ \overline{w}_1(1) = \mu. \end{cases}$$

Similarly to $w_1$, the solution $\widetilde{w}_1$ $\big(\in H^2(0, 1)\big)$ is uniquely determined while an easy computation shows that $\overline{w}_1$ is given by

$$\overline{w}_1(x) = k_1 e^{x/\sqrt{h}} + k_2 e^{-x/\sqrt{h}}, \tag{3.9}$$

where

$$k_1 = \frac{\mu - \lambda e^{-1/\sqrt{h}}}{e^{1/\sqrt{h}} - e^{-1/\sqrt{h}}} \quad \text{and} \quad k_2 = \frac{\lambda e^{1/\sqrt{h}} - \mu}{e^{1/\sqrt{h}} - e^{-1/\sqrt{h}}}. \tag{3.10}$$

Substituting (3.9) in (3.8), we get

$$\begin{cases} \int_0^1 \widetilde{w}_1(x)\,\mathrm{d}x + k_1 \int_0^1 \mathrm{e}^{x/\sqrt{h}}\,\mathrm{d}x + k_2 \int_0^1 \mathrm{e}^{-x/\sqrt{h}}\,\mathrm{d}x = 0, \\ \int_0^1 x\widetilde{w}_1(x)\,\mathrm{d}x + k_1 \int_0^1 x\mathrm{e}^{x/\sqrt{h}}\,\mathrm{d}x + k_2 \int_0^1 x\mathrm{e}^{-x/\sqrt{h}}\,\mathrm{d}x = 0. \end{cases}$$

Computing the integrals and performing some elementary simplifications, we finally obtain the following equivalent linear algebraic system

$$\begin{cases} \lambda + \mu = \dfrac{\sinh\left(1/\sqrt{h}\right)}{\sqrt{h}\left(1 - \cosh\left(1/\sqrt{h}\right)\right)} \displaystyle\int_0^1 \widetilde{w}_1(x)\,\mathrm{d}x, \\ \left(1 - \sqrt{h}\sinh\dfrac{1}{\sqrt{h}}\right)\lambda + \left(\sqrt{h}\sinh\dfrac{1}{\sqrt{h}} - \cosh\dfrac{1}{\sqrt{h}}\right)\mu \\ \quad = \dfrac{\sinh\left(1/\sqrt{h}\right)}{\sqrt{h}} \displaystyle\int_0^1 x\widetilde{w}_1(x)\,\mathrm{d}x \end{cases} \tag{3.11}$$

whose determinant

$$\Phi(h) = 2\sqrt{h}\sinh\frac{1}{\sqrt{h}} - \cosh\frac{1}{\sqrt{h}} - 1$$

vanishes only for the value $h = \overline{h} \simeq 3.448 \times 10^{15}$. Hence, for all $n > n_0$ where $n_0$ denotes the smallest positive integer satisfying $T/n_0 \leqslant h_0 := \min\{\overline{h}, T\}$, the system (3.8) which is equivalent to (3.11) admits a unique solution $(\lambda_1, \mu_1)$ in $\mathbb{R}^2$. Therefore, problem $(3.1_1)$–$(3.3_1)$ admits a unique solution $u_1 = w_1(\cdot; \lambda_1, \mu_1) \in H^2(0,1)$ for $n > n_0$. Next, due to the fact that $f_2 + \frac{1}{h}w_1 := f(t_2, u_1) + \frac{1}{h}u_1 \in L^2(0,1)$, there follows again by the Lax–Milgram Lemma the existence of a unique strong solution $w_2 \in H^2(0,1)$ to problem $(3.5_2)$–$(3.7_2)$. As above, we prove that problem $(3.1_2)$–$(3.3_2)$ has a unique solution $u_2 = w_2(\cdot; \lambda_2, \mu_2) \in H^2(0,1)$ for a certain pair $(\lambda_2, \mu_2) \in \mathbb{R}^2$ provided that $n > n_0$ holds. Thus, proceeding in this way step by step, we arrive at the following result:

**Theorem 3.1.** *For all $n > n_0$ and for all $j = 1, \ldots, n$, the problem $(3.1_j)$–$(3.3_j)$ admits a unique solution $u_j$ in $H^2(0,1)$.*

Now, for all $n > n_0$, we introduce the Rothe function $u^{(n)} : I \to H^2(0,1)$ defined by

$$u^{(n)}(t) = u_{j-1} + \delta u_j \left(t - t_{j-1}\right), \quad t \in \left[t_{j-1}, t_j\right], \quad j = 1, \ldots, n, \tag{3.12}$$

and the corresponding step function $\overline{u}^{(n)} : I \to H^2(0,1)$ defined as follows:

$$\overline{u}^{(n)}(0) = U_0, \quad \overline{u}^{(n)}(t) = u_j \quad \text{for } t \in \left(t_{j-1}, t_j\right], \quad j = 1, \ldots, n. \tag{3.13}$$

It can be expected that for $n \to \infty$ (i.e. $h \to 0$) the limit function (in the sense given later) of the sequence $\{u^{(n)}\}_{n > n_0}$ will be the required solution of our problem (2.3)–(2.6). The establishment of this fact requires some a priori estimates which will be derived in the following section.

## 4. A priori estimates for the approximations

In the rest of the paper, positive constants are denoted by $C$, $C_i$ $(i = 1, \ldots, 9)$.

**Lemma 4.1.** *There exists a* $C > 0$ *such that, for all* $n > n_0$*, the solutions* $u_j$ *of the time-discretized problems* (3.1$_j$)–(3.3$_j$)*,* $j = 1, \ldots, n$*, satisfy the estimates*

$$\left\| u_j \right\| \leqslant C, \tag{i}$$

$$\left\| \delta u_j \right\|_{B_2^1} \leqslant C. \tag{ii}$$

**Proof.** Let us first write the problem (3.1$_j$)–(3.3$_j$) in a weak form.
Suppose $n > n_0$ and let $\phi$ be any function from the space $V$ defined in (2.9). A standard integration by parts yields

$$\int_0^x (x - \xi)\phi(\xi)\,\mathrm{d}\xi = \Im_x^2 \phi, \qquad \text{for all } x \in (0, 1), \tag{4.1}$$

where

$$\Im_x^2 \phi := \Im_x(\Im_\xi \phi) = \int_0^x \mathrm{d}\xi \int_0^\xi \phi(\eta)\,\mathrm{d}\eta.$$

Hence, taking $x = 1$ in (4.1), we get

$$\Im_1^2 \phi = \int_0^1 (1 - \xi)\phi(\xi)\,\mathrm{d}\xi = \int_0^1 \phi(\xi)\,\mathrm{d}\xi - \int_0^1 \xi\phi(\xi)\,\mathrm{d}\xi = 0. \tag{4.2}$$

Next, multiplying for all $j = 1, \ldots, n$, the Eq. (3.1$_j$) by $\Im_x^2 \phi$ and integrating over (0,1), we get

$$\int_0^1 \delta u_j(x)\Im_x^2 \phi\,\mathrm{d}x - \int_0^1 \frac{\mathrm{d}^2 u_j}{\mathrm{d}x^2}(x)\Im_x^2 \phi\,\mathrm{d}x = \int_0^1 f_j(x)\Im_x^2 \phi\,\mathrm{d}x. \tag{4.3}$$

Performing some integrations by parts and invoking (4.2), we obtain for each term in (4.3):

$$\begin{aligned}
\int_0^1 \delta u_j(x)\Im_x^2 \phi\,\mathrm{d}x &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\left(\Im_x(\delta u_j)\right)\Im_x^2 \phi\,\mathrm{d}x \\
&= \Im_x(\delta u_j)\Im_x^2 \phi\Big|_{x=0}^{x=1} - \int_0^1 \Im_x(\delta u_j)\Im_x \phi\,\mathrm{d}x \\
&= -(\delta u_j, \phi)_{B_2^1}, \\
\int_0^1 \frac{\mathrm{d}^2 u_j}{\mathrm{d}x^2}(x)\Im_x^2 \phi \mathrm{d}x &= \frac{\mathrm{d}u_j}{\mathrm{d}x}(x)\Im_x^2 \phi\Big|_{x=0}^{x=1} - \int_0^1 \frac{\mathrm{d}u_j}{\mathrm{d}x}(x)\Im_x \phi\,\mathrm{d}x \\
&= -u_j(x)\Im_x \phi\Big|_{x=0}^{x=1} + \int_0^1 u_j(x)\phi(x)\,\mathrm{d}x \\
&= (u_j, \phi),
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 f_j(x)\Im_x^2 \phi\,\mathrm{d}x &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\left(\Im_x f_j\right)\Im_x^2 \phi\,\mathrm{d}x \\
&= \Im_x f_j \Im_x^2 \phi\Big|_{x=0}^{x=1} - \int_0^1 \Im_x f_j \Im_x \phi\,\mathrm{d}x \\
&= -(f_j, \phi)_{B_2^1},
\end{aligned}$$

so that (4.3) becomes finally:

$$(\delta u_j, \phi)_{B_2^1} + (u_j, \phi) = (f_j, \phi)_{B_2^1}, \quad \forall j = 1, \dots, n. \tag{4.4$_j$}$$

After that, we shall prove the estimate (ii). In particular, from (4.4$_1$), one obtains

$$(\delta u_1, \phi)_{B_2^1} + h(\delta u_1, \phi) = (f_1, \phi)_{B_2^1} - (U_0, \phi). \tag{4.5}$$

Then, integrating by parts the second term in the right-hand side, it follows that

$$\begin{aligned}(U_0, \phi) &= \int_0^1 U_0(x) \frac{\mathrm{d}}{\mathrm{d}x} (\Im_x \phi)\, \mathrm{d}x \\ &= U_0(x) \Im_x \phi|_{x=0}^{x=1} - \int_0^1 \frac{\mathrm{d}U_0}{\mathrm{d}x}(x) \Im_x \phi \, \mathrm{d}x \\ &= -\int_0^1 \frac{\mathrm{d}U_0}{\mathrm{d}x}(x) \Im_x \phi \, \mathrm{d}x,\end{aligned}$$

but,

$$\Im_x \left( \frac{\mathrm{d}^2 U_0}{\mathrm{d}x^2}(x) \right) = \frac{\mathrm{d}U_0}{\mathrm{d}x}(x) - \frac{\mathrm{d}U_0}{\mathrm{d}x}(0), \quad \text{for all } x \in (0,1),$$

whence, due to (4.2)

$$\begin{aligned}(U_0, \phi) &= -\int_0^1 \Im_x \left( \frac{\mathrm{d}^2 U_0}{\mathrm{d}x^2}(x) \right) \Im_x \phi \, \mathrm{d}x - \frac{\mathrm{d}U_0}{\mathrm{d}x}(0) \Im_1^2 \phi \\ &= -\left( \frac{\mathrm{d}^2 U_0}{\mathrm{d}x^2}, \phi \right)_{B_2^1},\end{aligned}$$

in the light of which, (4.5) becomes

$$(\delta u_1, \phi)_{B_2^1} + h(\delta u_1, \phi) = \left( f_1 + \frac{\mathrm{d}^2 U_0}{\mathrm{d}x^2}, \phi \right)_{B_2^1}.$$

Testing this identity with $\phi = \delta u_1 = \frac{u_1 - U_0}{h}$ which is in $V$ thanks to (3.2$_1$)–(3.3$_1$) and condition ($H_3$), we get with the help of the Cauchy–Schwarz inequality

$$\|\delta u_1\|_{B_2^1}^2 + h\, \|\delta u_1\|^2 \leqslant \left[ \|f_1\|_{B_2^1} + \left\| \frac{\mathrm{d}^2 U_0}{\mathrm{d}x^2} \right\|_{B_2^1} \right] \|\delta u_1\|_{B_2^1},$$

consequently

$$\|\delta u_1\|_{B_2^1} \leqslant \|f(t_1, U_0)\|_{B_2^1} + \left\| \frac{\mathrm{d}^2 U_0}{\mathrm{d}x^2} \right\|_{B_2^1},$$

and then

$$\|\delta u_1\|_{B_2^1} \leqslant C_1, \tag{4.6}$$

where $C_1 := \max_{t \in I} \|f(t, U_0)\|_{B_2^1} + \left\| \frac{\mathrm{d}^2 U_0}{\mathrm{d}x^2} \right\|_{B_2^1}$.

On the other hand, taking the difference of the relations (4.4$_j$)–(4.4$_{j-1}$), $j = 2, \ldots, n$, tested with $\phi = \delta u_j$ which belongs to $V$ in view of (3.2$_j$)–(3.3$_j$) and (3.2$_{j-1}$)–(3.3$_{j-1}$), we have

$$\left\|\delta u_j\right\|^2_{B^1_2} + \frac{1}{h}\left\|u_j - u_{j-1}\right\|^2 = (f_j - f_{j-1}, \delta u_j)_{B^1_2} + (\delta u_{j-1}, \delta u_j)_{B^1_2}.$$

Hence, using the Cauchy–Schwarz inequality and omitting the second term in the left-hand side

$$\left\|\delta u_j\right\|_{B^1_2} \leqslant \left\|f_j - f_{j-1}\right\|_{B^1_2} + \left\|\delta u_{j-1}\right\|_{B^1_2},$$

from where we obtain by an iterative procedure

$$\left\|\delta u_j\right\|_{B^1_2} \leqslant \sum_{i=2}^{j} \|f_i - f_{i-1}\|_{B^1_2} + \|\delta u_1\|_{B^1_2}. \tag{4.7}$$

But, by virtue of Assumption (H$_1$), we have for all $i \geqslant 2$:

$$\begin{aligned}\|f_i - f_{i-1}\|_{B^1_2} &= \|f(t_i, u_{i-1}) - f(t_{i-1}, u_{i-2})\|_{B^1_2} \\ &\leqslant L\left[h\left(1 + \|u_{i-1}\|_{B^1_2} + \|u_{i-2}\|_{B^1_2}\right) + \|u_{i-1} - u_{i-2}\|_{B^1_2}\right] \\ &= Lh\left[1 + \|u_{i-1}\|_{B^1_2} + \|u_{i-2}\|_{B^1_2} + \|\delta u_{i-1}\|_{B^1_2}\right],\end{aligned}$$

so that

$$\begin{aligned}&\sum_{i=2}^{j} \|f_i - f_{i-1}\|_{B^1_2} \\ &\quad\leqslant L(j-1)h + Lh\sum_{i=2}^{j}\left(\|u_{i-1}\|_{B^1_2} + \|u_{i-2}\|_{B^1_2}\right) + Lh\sum_{i=2}^{j}\|\delta u_{i-1}\|_{B^1_2} \\ &\quad= L(j-1)h + Lh\left(\sum_{i=1}^{j-1}\|u_i\|_{B^1_2} + \sum_{i=0}^{j-2}\|u_i\|_{B^1_2}\right) + Lh\sum_{i=1}^{j-1}\|\delta u_i\|_{B^1_2} \\ &\quad\leqslant L(j-1)h + Lh\left(\|U_0\|_{B^1_2} + 2\sum_{i=1}^{j-1}\|u_i\|_{B^1_2}\right) + Lh\sum_{i=1}^{j-1}\|\delta u_i\|_{B^1_2}.\end{aligned} \tag{4.8}$$

To continue, we need to estimate the term $\|u_i\|_{B^1_2}$. For this, we insert $\phi = u_i$ in (4.4$_i$), $i = 1, \ldots, n$, and get

$$\frac{1}{h}\|u_i\|^2_{B^1_2} + \|u_i\|^2 \leqslant \left(\|f_i\|_{B^1_2} + \frac{1}{h}\|u_{i-1}\|_{B^1_2}\right)\|u_i\|_{B^1_2},$$

which implies

$$\begin{aligned}\|u_i\|_{B^1_2} &\leqslant h\|f_i\|_{B^1_2} + \|u_{i-1}\|_{B^1_2} \\ &\leqslant h\left(\|f_i\|_{B^1_2} + \|f_{i-1}\|_{B^1_2}\right) + \|u_{i-2}\|_{B^1_2},\end{aligned}$$

then, iterating we may arrive at

$$\|u_i\|_{B^1_2} \leqslant h\sum_{k=1}^{i}\|f_k\|_{B^1_2} + \|U_0\|_{B^1_2}.$$

But, for all $k \geqslant 1$ we have

$$\begin{aligned} \|f_k\|_{B_2^1} &\leqslant \|f(t_k, u_{k-1}) - f(t_k, 0)\|_{B_2^1} + \|f(t_k, 0)\|_{B_2^1} \\ &\leqslant L\, \|u_{k-1}\|_{B_2^1} + M, \end{aligned} \tag{4.9}$$

where $M := \max_{t\in I} \|f(t, 0)\|_{B_2^1}$. Substituting this last inequality in the previous one, we estimate

$$\begin{aligned} \|u_i\|_{B_2^1} &\leqslant h \sum_{k=1}^{i} \left( L\, \|u_{k-1}\|_{B_2^1} + M \right) + \|U_0\|_{B_2^1} \\ &= ihM + (1 + Lh)\, \|U_0\|_{B_2^1} + Lh \sum_{k=2}^{i} \|u_{k-1}\|_{B_2^1} \\ &\leqslant TM + (1 + Lh_0)\, \|U_0\|_{B_2^1} + Lh \sum_{k=1}^{i-1} \|u_k\|_{B_2^1}, \end{aligned}$$

from where we obtain due to the discrete Gronwall Lemma

$$\|u_i\|_{B_2^1} \leqslant \left( TM + (1 + Lh_0)\, \|U_0\|_{B_2^1} \right) \mathrm{e}^{L(i-1)h},$$

or

$$\|u_i\|_{B_2^1} \leqslant C_2, \tag{4.10}$$

with $C_2 := \left( TM + (1 + Lh_0)\, \|U_0\|_{B_2^1} \right) \mathrm{e}^{LT}$. Using (4.10), we dominate the right-hand side in (4.8) as follows

$$\begin{aligned} \sum_{i=2}^{j} \|f_i - f_{i-1}\|_{B_2^1} &\leqslant L(j-1)h + Lh \left( \|U_0\|_{B_2^1} + 2(j-1)C_2 \right) + Lh \sum_{i=1}^{j-1} \|\delta u_i\|_{B_2^1} \\ &\leqslant LT \left( 1 + \|U_0\|_{B_2^1} + 2C_2 \right) + Lh \sum_{i=1}^{j-1} \|\delta u_i\|_{B_2^1}. \end{aligned}$$

Combining (4.6), (4.7) and the last inequality, we write

$$\left\|\delta u_j\right\|_{B_2^1} \leqslant C_1 + LT \left( 1 + \|U_0\|_{B_2^1} + 2C_2 \right) + Lh \sum_{i=1}^{j-1} \|\delta u_i\|_{B_2^1},$$

and hence, applying Gronwall's Lemma in discrete form again we get

$$\left\|\delta u_j\right\|_{B_2^1} \leqslant \left[ C_1 + LT \left( 1 + \|U_0\|_{B_2^1} + 2C_2 \right) \right] \mathrm{e}^{L(j-1)h},$$

for every $j = 1, \ldots, n$. Thus, (ii) is proved with

$$C := \left[ C_1 + LT \left( 1 + \|U_0\|_{B_2^1} + 2C_2 \right) \right] \mathrm{e}^{LT}.$$

Now, to derive the estimate (i), we put $\phi = u_j - u_{j-1}$ in $(4.4_j)$ and apply the identity

$$\left(u_j, u_j - u_{j-1}\right) = \frac{1}{2} \left( \left\|u_j - u_{j-1}\right\|^2 + \left\|u_j\right\|^2 - \left\|u_{j-1}\right\|^2 \right),$$

to get

$$h\left\|\delta u_j\right\|^2_{B_2^1} + \frac{1}{2}\left\|u_j - u_{j-1}\right\|^2 + \frac{1}{2}\left\|u_j\right\|^2 = \left(f_j, u_j - u_{j-1}\right)_{B_2^1} + \frac{1}{2}\left\|u_{j-1}\right\|^2.$$

Ignoring the first two terms in the left-hand side, we have

$$\left\|u_j\right\|^2 \leqslant 2\left\|f_j\right\|_{B_2^1}\left\|u_j - u_{j-1}\right\|_{B_2^1} + \left\|u_{j-1}\right\|^2,$$

whence, using (4.9), (4.10) and the estimation (ii)

$$\left\|u_j\right\|^2 \leqslant 2h\left(LC_2 + M\right)\left[C_1 + LT\left(1 + \|U_0\|_{B_2^1} + 2C_2\right)\right]\mathrm{e}^{LT} + \left\|u_{j-1}\right\|^2.$$

From this recurrent inequality, we successively estimate

$$\left\|u_j\right\|^2 \leqslant 2jh\left(LC_2 + M\right)\left[C_1 + LT\left(1 + \|U_0\|_{B_2^1} + 2C_2\right)\right]\mathrm{e}^{LT} + \|U_0\|^2,$$

from where estimate (i) follows with

$$C := \left\{2T\left(LC_2 + M\right)\left[C_1 + LT\left(1 + \|U_0\|_{B_2^1} + 2C_2\right)\right]\mathrm{e}^{LT} + \|U_0\|^2\right\}^{1/2},$$

and so the proof is complete. □

The following is an immediate consequence of Lemma 4.1.

**Corollary 4.2.** *For all $n > n_0$, the functions $u^{(n)}$ and $\overline{u}^{(n)}$ obey the inequalities*

$$\left\|u^{(n)}(t)\right\| \leqslant C, \qquad \left\|\overline{u}^{(n)}(t)\right\| \leqslant C, \qquad \forall t \in I, \tag{i}$$

$$\left\|\frac{\mathrm{d}u^{(n)}}{\mathrm{d}t}(t)\right\|_{B_2^1} \leqslant C, \qquad a.e.\ in\ I, \tag{ii}$$

$$\left\|\overline{u}^{(n)}(t) - u^{(n)}(t)\right\|_{B_2^1} \leqslant \frac{C}{n}, \qquad \forall t \in I, \tag{iii}$$

$$\left\|\overline{u}^{(n)}(t) - \overline{u}^{(n)}\left(t - \frac{T}{n}\right)\right\|_{B_2^1} \leqslant \frac{C}{n}, \qquad \forall t \in I. \tag{iv}$$

**Proof.** The inequalities (i) follow immediately from the estimate (i) of Lemma 4.1 with the same constant, whereas the inequality (ii) is an easy consequence of the estimate (ii) of Lemma 4.1, also with the same constant, noting that we have

$$\frac{\mathrm{d}u^{(n)}}{\mathrm{d}t}(t) = \delta u_j, \quad \forall t \in (t_{j-1}, t_j], \quad 1 \leqslant j \leqslant n.$$

With regard to the inequalities (iii) and (iv), it suffices to see that we have

$$\overline{u}^{(n)}(t) - u^{(n)}(t) = (t_j - t)\delta u_j, \quad \forall t \in (t_{j-1}, t_j], \quad 1 \leqslant j \leqslant n,$$

and

$$\overline{u}^{(n)}(t) - \overline{u}^{(n)}\left(t - \frac{T}{n}\right) = u_j - u_{j-1}, \quad \forall t \in (t_{j-1}, t_j], \quad 1 \leqslant j \leqslant n,$$

and it follows that

$$\left\|\overline{u}^{(n)}(t) - u^{(n)}(t)\right\|_{B_2^1} \leqslant h \max_{1\leqslant j\leqslant n} \left\|\delta u_j\right\|_{B_2^1} \quad \forall t \in I,$$

and

$$\left\|\overline{u}^{(n)}(t) - \overline{u}^{(n)}\left(t - \frac{T}{n}\right)\right\|_{B_2^1} \leqslant h \max_{1\leqslant j\leqslant n} \left\|\delta u_j\right\|_{B_2^1}, \quad \forall t \in I,$$

hence, applying the estimate (ii) of Lemma 4.1, we get the required inequalities (iii) and (iv) with

$$C := T\left[C_1 + LT\left(1 + \|U_0\|_{B_2^1} + 2C_2\right)\right]\mathrm{e}^{LT}. \quad \square$$

## 5. Existence, uniqueness and convergence of the method

Let us define, for all $n > n_0$, the abstract step function $\overline{f}^{(n)} : I \times V \to L^2(0,1)$ by

$$\overline{f}^{(n)}(t, v) = f(t_j, v), \ \forall t \in (t_{j-1}, t_j], \ \ j = 1, \ldots, n.$$

Then, setting $\overline{u}^{(n)}(t) = U_0$ for $t \in \left(-\frac{T}{n}, 0\right)$, the variational equations (4.4$_j$) may be rewritten in the form:

$$\left(\frac{\mathrm{d}u^{(n)}}{\mathrm{d}t}(t), \phi\right)_{B_2^1} + \left(\overline{u}^{(n)}(t), \phi\right) = \left(\overline{f}^{(n)}\left(t, \overline{u}^{(n)}\left(t - \frac{T}{n}\right)\right), \phi\right)_{B_2^1}, \tag{5.1$^n$}$$

for all $\phi \in V$ and $t \in I$.

To be able to carry out the limiting process in the approximation scheme (5.1$^n$), we have to prove some convergence assertions.

**Theorem 5.1.** *The sequence $\left\{u^{(n)}\right\}_n$ converges in the norm of the space $C(I, B_2^1(0,1))$ to some function $u \in C\left(I, B_2^1(0,1)\right)$ and the error estimate*

$$\left\|u^{(n)} - u\right\|_{C(I, B_2^1(0,1))} \leqslant \frac{C}{n^{1/2}} \tag{5.2}$$

*holds for all $n > n_0$.*

**Proof.** The idea of the proof consists in showing that $\left\{u^{(n)}\right\}_n$ is a Cauchy sequence in the Banach space $C\left(I, B_2^1(0,1)\right)$.

Let $u^{(n)}$ and $u^{(m)}$ be the Rothe approximations corresponding to the step lengths $h_n = \frac{T}{n}$ and $h_m = \frac{T}{m}$ respectively with $m > n > n_0$. Take the (5.1$^n$)–(5.1$^m$) difference tested with $\phi = u^{(n)}(t) - u^{(m)}(t)$ $(\in V)$, this yields for all $t \in I$:

$$\begin{aligned}
&\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(u^{(n)}(t) - u^{(m)}(t)\right), u^{(n)}(t) - u^{(m)}(t)\right)_{B_2^1} \\
&\quad + \left(\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t), u^{(n)}(t) - u^{(m)}(t)\right) \\
&= \left(\overline{f}^{(n)}\left(t, \overline{u}^{(n)}\left(t - \frac{T}{n}\right)\right) - \overline{f}^{(m)}\left(t, \overline{u}^{(m)}\left(t - \frac{T}{m}\right)\right), u^{(n)}(t) - u^{(m)}(t)\right)_{B_2^1},
\end{aligned}$$

or after some rearrangement

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u^{(n)}(t)-u^{(m)}(t)\right\|^2_{B_2^1}+\left\|\overline{u}^{(n)}(t)-\overline{u}^{(m)}(t)\right\|^2$$
$$=\left(\overline{u}^{(n)}(t)-\overline{u}^{(m)}(t),\overline{u}^{(n)}(t)-\overline{u}^{(m)}(t)-u^{(n)}(t)+u^{(m)}(t)\right)$$
$$+\left(\overline{f}^{(n)}\left(t,\overline{u}^{(n)}\left(t-\frac{T}{n}\right)\right)-\overline{f}^{(m)}\left(t,\overline{u}^{(m)}\left(t-\frac{T}{m}\right)\right),u^{(n)}(t)-u^{(m)}(t)\right)_{B_2^1}. \tag{5.3}$$

But, since we have

$$\overline{f}^{(n)}\left(t,\overline{u}^{(n)}\left(t-\frac{T}{n}\right)\right)=f(t_j,u_{j-1}):=f_j,\quad \forall t\in(t_{j-1},t_j],\quad j=1,\ldots,n,$$

it follows in view of (4.9) that

$$\left\|\overline{f}^{(n)}\left(t,\overline{u}^{(n)}\left(t-\frac{T}{n}\right)\right)\right\|_{B_2^1}\leqslant\max_{1\leqslant j\leqslant n}\|f_j\|_{B_2^1}$$
$$\leqslant L\max_{1\leqslant j\leqslant n}\left\|u_{j-1}\right\|_{B_2^1}+M,\quad \forall t\in I,$$

and hence due to (4.10)

$$\left\|\overline{f}^{(n)}\left(t,\overline{u}^{(n)}\left(t-\frac{T}{n}\right)\right)\right\|_{B_2^1}\leqslant LC_2+M,\quad \forall t\in I. \tag{5.4}$$

Thus, estimating the identity

$$\left(\overline{u}^{(n)}(t),\phi\right)=\left(\overline{f}^{(n)}\left(t,\overline{u}^{(n)}\left(t-\frac{T}{n}\right)\right)-\frac{\mathrm{d}u^{(n)}}{\mathrm{d}t}(t),\phi\right)_{B_2^1},\quad \forall t\in I,\ \forall\phi\in V,$$

which follows from (5.1$^n$), we obtain owing to Corollary 4.2/ (ii)

$$\left|\left(\overline{u}^{(n)}(t),\phi\right)\right|\leqslant\left[\left\|\overline{f}^{(n)}\left(t,\overline{u}^{(n)}\left(t-\tfrac{T}{n}\right)\right)\right\|_{B_2^1}+\left\|\tfrac{\mathrm{d}u^{(n)}}{\mathrm{d}t}(t)\right\|_{B_2^1}\right]\|\phi\|_{B_2^1}$$
$$\leqslant C_3\|\phi\|_{B_2^1},\quad \forall t\in I \tag{5.5}$$

with $C_3:=LC_2+M+\left[C_1+LT\left(1+\|U_0\|_{B_2^1}+2C_2\right)\right]\mathrm{e}^{LT}$. This, together with Corollary 4.2 / (iii), allows us to majorize the first term in the right-hand side of (5.3) as follows

$$\left(\overline{u}^{(n)}(t)-\overline{u}^{(m)}(t),\overline{u}^{(n)}(t)-\overline{u}^{(m)}(t)-u^{(n)}(t)+u^{(m)}(t)\right)$$
$$\leqslant 2C_3\left(\left\|\overline{u}^{(n)}(t)-u^{(n)}(t)\right\|_{B_2^1}+\left\|\overline{u}^{(m)}(t)-u^{(m)}(t)\right\|_{B_2^1}\right)$$
$$\leqslant C_4\left(\frac{1}{n}+\frac{1}{m}\right),\quad \forall t\in I, \tag{5.6}$$

with $C_4:=2C_3T\left[C_1+LT\left(1+\|U_0\|_{B_2^1}+2C_2\right)\right]\mathrm{e}^{LT}$.

On the other hand, thanks to the Cauchy inequality

$$\alpha\beta \leqslant \frac{\varepsilon}{2}\alpha^2 + \frac{1}{2\varepsilon}\beta^2, \quad \forall \alpha, \beta \in \mathbb{R}, \quad \forall \varepsilon \in \mathbb{R}_+^*,$$

we can write for every $\varepsilon > 0$ :

$$\begin{aligned}
&\left(\overline{f}^{(n)}\left(t, \overline{u}^{(n)}\left(t - \frac{T}{n}\right)\right) - \overline{f}^{(m)}\left(t, \overline{u}^{(m)}\left(t - \frac{T}{m}\right)\right), u^{(n)}(t) - u^{(m)}(t)\right)_{B_2^1} \\
&\leqslant \left\|\overline{f}^{(n)}\left(t, \overline{u}^{(n)}\left(t - \frac{T}{n}\right)\right) - \overline{f}^{(m)}\left(t, \overline{u}^{(m)}\left(t - \frac{T}{m}\right)\right)\right\|_{B_2^1} \left\|u^{(n)}(t) - u^{(m)}(t)\right\|_{B_2^1} \\
&\leqslant \frac{\varepsilon}{2}\left\|\overline{f}^{(n)}\left(t, \overline{u}^{(n)}\left(t - \frac{T}{n}\right)\right) - \overline{f}^{(m)}\left(t, \overline{u}^{(m)}\left(t - \frac{T}{m}\right)\right)\right\|_{B_2^1}^2 \\
&\quad + \frac{1}{2\varepsilon}\left\|u^{(n)}(t) - u^{(m)}(t)\right\|_{B_2^1}^2, \quad \forall t \in I.
\end{aligned} \tag{5.7}$$

Now, let $t$ be arbitrary but fixed in $(0, T]$, then there exist two integers $k = k(n)$ and $i = i(m)$ such that $t \in (t_{k-1}, t_k] \cap (t_{i-1}, t_i]$, and hence from Assumption ($H_1$), it follows that

$$\begin{aligned}
&\left\|\overline{f}^{(n)}\left(t, \overline{u}^{(n)}\left(t - \frac{T}{n}\right)\right) - \overline{f}^{(m)}\left(t, \overline{u}^{(m)}\left(t - \frac{T}{m}\right)\right)\right\|_{B_2^1}^2 \\
&= \left\|f\left(t_k, \overline{u}^{(n)}\left(t - \frac{T}{n}\right)\right) - f\left(t_i, \overline{u}^{(m)}\left(t - \frac{T}{m}\right)\right)\right\|_{B_2^1}^2 \\
&\leqslant L^2\left[|t_k - t_i|\left\{1 + \left\|\overline{u}^{(n)}\left(t - \frac{T}{n}\right)\right\|_{B_2^1} + \left\|\overline{u}^{(m)}\left(t - \frac{T}{m}\right)\right\|_{B_2^1}\right\}\right. \\
&\quad \left. + \left\|\overline{u}^{(n)}\left(t - \frac{T}{n}\right) - \overline{u}^{(m)}\left(t - \frac{T}{m}\right)\right\|_{B_2^1}\right]^2 \\
&\leqslant L^2\left[(h_n + h_m)\left(1 + \|u_{k-1}\|_{B_2^1} + \|u_{i-1}\|_{B_2^1}\right) + \left\|\overline{u}^{(n)}\left(t - \frac{T}{n}\right) - \overline{u}^{(n)}(t)\right\|_{B_2^1}\right. \\
&\quad \left. + \left\|\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t)\right\|_{B_2^1} + \left\|\overline{u}^{(m)}(t) - \overline{u}^{(m)}\left(t - \frac{T}{m}\right)\right\|_{B_2^1}\right]^2,
\end{aligned}$$

and consequently, with consideration of (4.10) and Corollary 4.2 / (iv) we deduce that

$$\begin{aligned}
&\left\|\overline{f}^{(n)}\left(t, \overline{u}^{(n)}\left(t - \frac{T}{n}\right)\right) - \overline{f}^{(m)}\left(t, \overline{u}^{(m)}\left(t - \frac{T}{m}\right)\right)\right\|_{B_2^1}^2 \\
&\leqslant L^2\left[T\left(\frac{1}{n} + \frac{1}{m}\right)(1 + 2C_2) + C\left(\frac{1}{n} + \frac{1}{m}\right) + \left\|\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t)\right\|_{B_2^1}\right]^2 \\
&= L^2\left[(T(1 + 2C_2) + C)\left(\frac{1}{n} + \frac{1}{m}\right) + \left\|\overline{u}^{(n)}(t) - \overline{u}^{(m)}(t)\right\|_{B_2^1}\right]^2 \\
&\leqslant L^2\left[C_5^2\left(\frac{1}{n} + \frac{1}{m}\right)^2 + 2C_5\left(\frac{1}{n} + \frac{1}{m}\right)\left(\left\|\overline{u}^{(n)}(t)\right\|_{B_2^1} + \left\|\overline{u}^{(m)}(t)\right\|_{B_2^1}\right)\right.
\end{aligned}$$

$$+ \left\| \overline{u}^{(n)}(t) - \overline{u}^{(m)}(t) \right\|^2_{B_2^1} \Big]$$

$$\leqslant (LC_5)^2 \left( \frac{1}{n} + \frac{1}{m} \right)^2 + 4L^2 C_5 C_2 \left( \frac{1}{n} + \frac{1}{m} \right) + L^2 \left\| \overline{u}^{(n)}(t) - \overline{u}^{(m)}(t) \right\|^2_{B_2^1}$$

for all $t \in I$, with $C_5 := T(1 + 2C_2) + C$. Thus, using the notation $C_6 := (LC_5)^2$ and $C_7 := 4L^2 C_5 C_2$, we have

$$\left\| \overline{f}^{(n)} \left( t, \overline{u}^{(n)} \left( t - \frac{T}{n} \right) \right) - \overline{f}^{(m)} \left( t, \overline{u}^{(m)} \left( t - \frac{T}{m} \right) \right) \right\|^2_{B_2^1}$$

$$\leqslant C_6 \left( \frac{1}{n} + \frac{1}{m} \right)^2 + C_7 \left( \frac{1}{n} + \frac{1}{m} \right) + L^2 \left\| \overline{u}^{(n)}(t) - \overline{u}^{(m)}(t) \right\|^2_{B_2^1}, \quad \forall t \in I, \tag{5.8}$$

and hence, going back to (5.7), we have

$$\left( \overline{f}^{(n)} \left( t, \overline{u}^{(n)} \left( t - \frac{T}{n} \right) \right) - \overline{f}^{(m)} \left( t, \overline{u}^{(m)} \left( t - \frac{T}{m} \right) \right), u^{(n)}(t) - u^{(m)}(t) \right)_{B_2^1}$$

$$\leqslant \frac{\varepsilon}{2} C_6 \left( \frac{1}{n} + \frac{1}{m} \right)^2 + \frac{\varepsilon}{2} C_7 \left( \frac{1}{n} + \frac{1}{m} \right) + \frac{\varepsilon}{2} L^2 \left\| \overline{u}^{(n)}(t) - \overline{u}^{(m)}(t) \right\|^2_{B_2^1}$$

$$+ \frac{1}{2\varepsilon} \left\| u^{(n)}(t) - u^{(m)}(t) \right\|^2_{B_2^1}, \quad \forall t \in I. \tag{5.9}$$

Next, combining (5.3), (5.6) and (5.9), we obtain for all $t \in I$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\| u^{(n)}(t) - u^{(m)}(t) \right\|^2_{B_2^1} + 2 \left\| \overline{u}^{(n)}(t) - \overline{u}^{(m)}(t) \right\|^2$$

$$\leqslant \varepsilon C_6 \left( \frac{1}{n} + \frac{1}{m} \right)^2 + (\varepsilon C_7 + 2C_4) \left( \frac{1}{n} + \frac{1}{m} \right) + \varepsilon L^2 \left\| \overline{u}^{(n)}(t) - \overline{u}^{(m)}(t) \right\|^2_{B_2^1}$$

$$+ \frac{1}{\varepsilon} \left\| u^{(n)}(t) - u^{(m)}(t) \right\|^2_{B_2^1},$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\| u^{(n)}(t) - u^{(m)}(t) \right\|^2_{B_2^1} + \left( 2 - \varepsilon L^2 \right) \left\| \overline{u}^{(n)}(t) - \overline{u}^{(m)}(t) \right\|^2$$

$$\leqslant \varepsilon C_6 \left( \frac{1}{n} + \frac{1}{m} \right)^2 + (\varepsilon C_7 + 2C_4) \left( \frac{1}{n} + \frac{1}{m} \right) + \frac{1}{\varepsilon} \left\| u^{(n)}(t) - u^{(m)}(t) \right\|^2_{B_2^1}.$$

Let us choose $\varepsilon > 0$ so that $2 - \varepsilon L^2 = 0$, i.e. $\varepsilon = \frac{2}{L^2}$ and integrate the last inequality over $(0, t)$. Then, invoking the fact that $u^{(n)}(0) = u^{(m)}(0) = U_0$, we obtain for all $t \in I$:

$$\left\| u^{(n)}(t) - u^{(m)}(t) \right\|^2_{B_2^1} \leqslant \frac{2C_6 T}{L^2} \left( \frac{1}{n} + \frac{1}{m} \right)^2 + 2T \left( \frac{C_7}{L^2} + C_4 \right) \left( \frac{1}{n} + \frac{1}{m} \right)$$

$$+ \frac{L^2}{2} \int_0^t \left\| u^{(n)}(\tau) - u^{(m)}(\tau) \right\|^2_{B_2^1} \mathrm{d}\tau,$$

giving by Gronwall's Lemma

$$\left\| u^{(n)}(t) - u^{(m)}(t) \right\|_{B_2^1}^2 \leqslant \left[ C_8 \left( \frac{1}{n} + \frac{1}{m} \right)^2 + C_9 \left( \frac{1}{n} + \frac{1}{m} \right) \right] \mathrm{e}^{\frac{L^2}{2} t} \quad \forall t \in I,$$

with $C_8 := \frac{2C_6 T}{L^2}$ and $C_9 := 2T \left( \frac{C_7}{L^2} + C_4 \right)$. Accordingly

$$\left\| u^{(n)}(t) - u^{(m)}(t) \right\|_{B_2^1} \leqslant \left[ C_8 \left( \frac{1}{n} + \frac{1}{m} \right)^2 + C_9 \left( \frac{1}{n} + \frac{1}{m} \right) \right]^{1/2} \mathrm{e}^{\frac{L^2 T}{4}}, \quad \forall t \in I,$$

and then, taking the supremum with respect to $t$ in the left-hand side of this inequality, we obtain

$$\left\| u^{(n)} - u^{(m)} \right\|_{C(I, B_2^1(0,1))} \leqslant \left[ C_8 \left( \frac{1}{n} + \frac{1}{m} \right)^2 + C_9 \left( \frac{1}{n} + \frac{1}{m} \right) \right]^{1/2} \mathrm{e}^{\frac{L^2 T}{4}}, \quad \forall t \in I, \tag{5.10}$$

which implies the existence of a function $u \in C\left(I, B_2^1(0,1)\right)$ such that $u^{(n)} \to u$ in this space. Moreover, passing to the limit $m \to \infty$ in (5.10) we obtain the error estimate (5.2) with $C := \sqrt{C_8 + C_9} \mathrm{e}^{\frac{L^2 T}{4}}$, which achieves the proof. □

Next, some properties of the function $u$ from Theorem 5.1 are formulated in the following theorem:

**Theorem 5.2.** *For the function $u$ from Theorem 5.1, it holds that:*

(i) $u \in L^\infty(I, V) \cap C^{0,1}\left(I, B_2^1(0,1)\right)$;
(ii) *$u$ is strongly differentiable a.e. in $I$ and* $\frac{\mathrm{d}u}{\mathrm{d}t} \in L^\infty\left(I, B_2^1(0,1)\right)$;
(iii) $u^{(n)}(t), \overline{u}^{(n)}(t) \rightharpoonup u(t)$ *in $V$ for all $t \in I$;*
(iv) $\frac{\mathrm{d}u^{(n)}}{\mathrm{d}t} \rightharpoonup \frac{\mathrm{d}u}{\mathrm{d}t}$ *in* $L^2\left(I, B_2^1(0,1)\right)$.

**Proof.** On the basis of estimates (i) and (ii) from Corollary 4.2, the uniform convergence statement from Theorem 5.1 and the continuous embedding $V \hookrightarrow B_2^1(0,1)$, the assertions of the present theorem are direct consequences of [6, Lemma 1.3.15]. □

Gathering all the results obtained we are in position to state our existence theorem:

**Theorem 5.3.** *The limit function $u$ from Theorem 5.1 is the unique weak solution to the problem (2.3)–(2.6) in the sense of Definition 2.1.*

**Proof.** In the light of the foregoing, the properties (i) and (ii) from Definition 2.1 are already fulfilled. Moreover, since $u^{(n)} \to u$ in $C\left(I, B_2^1(0,1)\right)$ when $n \to \infty$ and, by definition, $u^{(n)}(0) = U_0$, it follows that $u(0) = U_0$ holds in $B_2^1(0,1)$ so the initial condition (2.4) is also fulfilled. It remains to show that $u$ satisfies the integral equation (2.13). Integrating ($5.1^n$) over $(0,t) \subset I$ and invoking the fact that $u^{(n)}(0) = U_0$, we obtain

$$\left( u^{(n)}(t) - U_0, \phi \right)_{B_2^1} + \int_0^t \left( \overline{u}^{(n)}(\tau), \phi \right) \mathrm{d}\tau = \int_0^t \left( \overline{f}^{(n)}\left( \tau, \overline{u}^{(n)}\left( \tau - \frac{T}{n} \right) \right), \phi \right)_{B_2^1} \mathrm{d}\tau. \tag{$5.11^n$}$$

To investigate the behavior of ($5.11^n$) as $n \to \infty$, we prove some convergence statements. Since $u^{(n)}(t) \rightharpoonup u(t)$ in $V$ for all $t \in I$ and since for all fixed $\phi \in V$, the linear functional $v \mapsto (v, \phi)_{B_2^1}$

is bounded on $V$, we have

$$\left(u^{(n)}(t), \phi\right)_{B_2^1} \to (u(t), \phi)_{B_2^1}, \quad \forall t \in I. \tag{5.12}$$

On the other hand, in view of the assumed Lipschitz continuity of $f$, we have

$$\begin{aligned}
&\left\|\overline{f}^{(n)}\left(\tau, \overline{u}^{(n)}\left(\tau - \frac{T}{n}\right)\right) - f(\tau, u(\tau))\right\|_{B_2^1} \\
&\quad = \left\|f\left(t_j, \overline{u}^{(n)}\left(\tau - \frac{T}{n}\right)\right) - f(\tau, u(\tau))\right\|_{B_2^1} \\
&\quad \leqslant L\left[\left|t_j - \tau\right|\left(1 + \|u_{j-1}\|_{B_2^1} + \|u(\tau)\|_{B_2^1}\right) + \left\|\overline{u}^{(n)}\left(\tau - \frac{T}{n}\right) - u(\tau)\right\|_{B_2^1}\right],
\end{aligned}$$

for all $\tau \in (t_{j-1}, t_j]$, $1 \leqslant j \leqslant n$, whence

$$\left\|\overline{f}^{(n)}\left(\tau, \overline{u}^{(n)}\left(\tau - \frac{T}{n}\right)\right) - f(\tau, u(\tau))\right\|_{B_2^1} \leqslant \frac{C}{n} + L\left\|\overline{u}^{(n)}\left(\tau - \frac{T}{n}\right) - u(\tau)\right\|_{B_2^1},$$

$\forall \tau \in I$, where $C := LT\left(1 + C_2 + \|u\|_{C(I, B_2^1(0,1))}\right)$. But, owing to the estimates (iii) and (iv) from Corollary 4.2 and the inequality (5.2), we have

$$\begin{aligned}
\left\|\overline{u}^{(n)}\left(\tau - \frac{T}{n}\right) - u(\tau)\right\|_{B_2^1} &\leqslant \left\|\overline{u}^{(n)}\left(\tau - \frac{T}{n}\right) - \overline{u}^{(n)}(\tau)\right\|_{B_2^1} \\
&\quad + \left\|\overline{u}^{(n)}(\tau) - u^{(n)}(\tau)\right\|_{B_2^1} + \left\|u^{(n)}(\tau) - u(\tau)\right\|_{B_2^1} \\
&\leqslant C\left(\frac{1}{n} + \frac{1}{n^{1/2}}\right), \quad \forall \tau \in I,
\end{aligned}$$

and hence

$$\left\|\overline{f}^{(n)}\left(\tau, \overline{u}^{(n)}\left(\tau - \frac{T}{n}\right)\right) - f(\tau, u(\tau))\right\|_{B_2^1} \leqslant \frac{C}{n^{1/2}}, \quad \forall \tau \in I,$$

therefore

$$\overline{f}^{(n)}\left(\tau, \overline{u}^{(n)}\left(\tau - \frac{T}{n}\right)\right) \underset{n\to\infty}{\longrightarrow} f(\tau, u(\tau)) \quad \text{in } B_2^1(0,1), \quad \forall \tau \in I. \tag{5.13}$$

Now, due to (5.4) and (5.5) the functions $\left|\left(\overline{f}^{(n)}\left(\tau, \overline{u}^{(n)}\left(\tau - \frac{T}{n}\right)\right), \phi\right)_{B_2^1}\right|$ and $|(\overline{u}^{(n)}(\tau), \phi)|$ are uniformly bounded with respect to both $\tau$ and $n$, so the Lebesgue Theorem of dominated convergence may be applied to (5.13) as well as to the convergence statement (iii) from Theorem 5.2 giving

$$\int_0^t \left(\overline{f}^{(n)}\left(\tau, \overline{u}^{(n)}\left(\tau - \frac{T}{n}\right)\right), \phi\right)_{B_2^1} \mathrm{d}\tau \to \int_0^t (f(\tau, u(\tau)), \phi)_{B_2^1} \, \mathrm{d}\tau, \quad \forall t \in I, \tag{5.14}$$

and

$$\int_0^t \left(\overline{u}^{(n)}(\tau), \phi\right) \mathrm{d}\tau \to \int_0^t (u(\tau), \phi) \, \mathrm{d}\tau, \quad \forall t \in I, \tag{5.15}$$

as $n \to \infty$. Then, carrying out a limiting process $n \to \infty$ in (5.11$^n$), we obtain by (5.12), (5.14) and (5.15):

$$(u(t) - U_0, \phi)_{B_2^1} + \int_0^t (u(\tau), \phi)\, \mathrm{d}\tau = \int_0^t (f(\tau, u(\tau)), \phi)_{B_2^1}\, \mathrm{d}\tau,$$

for all $\phi \in V$ and $t \in I$. Finally, differentiating this last identity with respect to $t$ recalling that $u : I \to B_2^1(0,1)$ is strongly differentiable for a.e. $t \in I$, we get the required relation (2.13) thanks to the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t), \phi)_{B_2^1} = \left(\frac{\mathrm{d}u}{\mathrm{d}t}(t), \phi\right)_{B_2^1}, \quad \text{a.e. } t \in I, \ \forall \phi \in V.$$

Thus $u$ weakly solves the problem (2.3)–(2.6).
Regarding the uniqueness, let us consider two weak solutions $\widehat{u}$ and $\widetilde{u}$ of (2.3)–(2.6). Subtracting the identity (2.13) written for $\widetilde{u}$ from the same identity written for $\widehat{u}$ and putting $\phi = \widehat{u}(t) - \widetilde{u}(t)$ in the resulting relation, we get

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}(t), u(t)\right)_{B_2^1} + \|u(t)\|^2 = (f(t, \widehat{u}(t)) - f(t, \widetilde{u}(t)), u(t))_{B_2^1}, \quad \text{a.e. } t \in I,$$

where $u := \widehat{u} - \widetilde{u}$. Then, integrating between 0 and $t$ by taking into account that $\left(\frac{\mathrm{d}u}{\mathrm{d}t}(t), u(t)\right)_{B_2^1} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{B_2^1}$ and $u(0) = 0$, we derive

$$\begin{aligned}
&\|u(t)\|^2_{B_2^1} + 2\int_0^t \|u(\tau)\|^2\, \mathrm{d}\tau \\
&\quad = 2\int_0^t (f(\tau, \widehat{u}(\tau)) - f(\tau, \widetilde{u}(\tau)), u(\tau))_{B_2^1}\, \mathrm{d}\tau \\
&\quad \leqslant 2\int_0^t \|f(\tau, \widehat{u}(\tau)) - f(\tau, \widetilde{u}(\tau))\|_{B_2^1} \|u(\tau)\|_{B_2^1}\, \mathrm{d}\tau \\
&\quad \leqslant 2L\int_0^t \|u(\tau)\|^2_{B_2^1}\, \mathrm{d}\tau, \quad \forall t \in I,
\end{aligned}$$

from where Gronwall's Lemma yields $\|u(t)\|^2_{B_2^1} = 0, \forall t \in I$, which means that $\widehat{u} = \widetilde{u}$. □

We conclude this paper by giving a result of continuous dependence of the solution upon the data, by leaning on an idea due to [6, Theorem 2.2.15].

**Theorem 5.4.** *Let $\widehat{u}$ and $\widetilde{u}$ be two weak solutions of problem* (2.3)–(2.6) *corresponding respectively to $(\widehat{U}_0, \widehat{f})$ and $(\widetilde{U}_0, \widetilde{f})$ satisfying Assumptions* (H$_1$)–(H$_3$). *If the following inequality*

$$\left\|\widehat{f}(t, v) - \widetilde{f}(t, w)\right\|_{B_2^1} \leqslant a(t) + b\,\|v - w\|_{B_2^1}, \quad \forall t \in I, \quad \forall v, w \in V, \tag{5.16}$$

*holds for some continuous nonnegative function $a(t)$ in $I$ and some constant $b \geqslant 0$, then the estimate*

$$\|\widehat{u}(t) - \widetilde{u}(t)\|^2_{B_2^1} \leqslant \left(\left\|\widehat{U}_0 - \widetilde{U}_0\right\|^2_{B_2^1} + \int_0^t a^2(\tau)\, \mathrm{d}\tau\right) \mathrm{e}^{(2b+1)t}, \tag{5.17}$$

*holds for all $t \in I$.*

**Proof.** We take once more the difference of the identities (2.13) corresponding to $\widehat{u}$ and $\widetilde{u}$ tested with $\phi = \widehat{u}(t) - \widetilde{u}(t)$ and integrate the resulting relation over $(0, t)$. We find for all $t \in I$ :

$$\begin{aligned} &\|\widehat{u}(t) - \widetilde{u}(t)\|^2_{B^1_2} + 2\int_0^t \|\widehat{u}(\tau) - \widetilde{u}(\tau)\|^2 \, \mathrm{d}\tau \\ &\leqslant \left\|\widehat{U}_0 - \widetilde{U}_0\right\|^2_{B^1_2} + 2\int_0^t \left\|\widehat{f}(\tau, \widehat{u}(\tau)) - \widetilde{f}(\tau, \widetilde{u}(\tau))\right\|_{B^1_2} \|\widehat{u}(\tau) - \widetilde{u}(\tau)\|_{B^1_2} \, \mathrm{d}\tau, \\ &\leqslant \left\|\widehat{U}_0 - \widetilde{U}_0\right\|^2_{B^1_2} + \int_0^t 2a(\tau) \|\widehat{u}(\tau) - \widetilde{u}(\tau)\|_{B^1_2} \, \mathrm{d}\tau + 2b\int_0^t \|\widehat{u}(\tau) - \widetilde{u}(\tau)\|^2_{B^1_2} \, \mathrm{d}\tau, \end{aligned}$$

by virtue of (5.16). Hence, applying the elementary algebraic inequality $2\alpha\beta \leqslant \alpha^2 + \beta^2$ $(\forall \alpha, \beta \in \mathbb{R})$ to the second term in the right-hand side, we derive

$$\begin{aligned} &\|\widehat{u}(t) - \widetilde{u}(t)\|_{B^1_2} + 2\int_0^t \|\widehat{u}(\tau) - \widetilde{u}(\tau)\|^2 \, \mathrm{d}\tau \\ &\leqslant \left\|\widehat{U}_0 - \widetilde{U}_0\right\|^2_{B^1_2} + \int_0^t a^2(\tau) \, \mathrm{d}\tau + (2b+1)\int_0^t \|\widehat{u}(\tau) - \widetilde{u}(\tau)\|^2_{B^1_2} \, \mathrm{d}\tau, \quad \forall t \in I, \end{aligned}$$

from which the estimate (5.17) follows by means of Gronwall's Lemma. □

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