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Common fixed point theorems of several functions in complex valued metric spaces and applications

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Dedication

To the candle of my life and my source of love, my dear mother,
To my dear father, for his encouragement and his continual prayers,
To my wife who supported and encourage me to finalize this work,
To my dear little girl MIRA.
To my sisters and their husbands, my brothers and their wives and all their children,
To all my friends

To all of mine

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ملخص

ركزنا في هذه الأطروحة على دراسة نظرية النقطة الثابتة والتي تعتبر مفيدة للغاية في مجال التحليل غير الخطي لأنها تساعد على حل العديد من المشكلات في مجالات تطبيقية مختلفة.

برزت أهمية هذه النظرية مع ظهور تقلص Banach واستعمالها في حل المعادلات التكاملية في الفضاء المتري.

إهتممنا في هذا العمل بإيجاد تطبيقات تتضمن تعميم لنظرية النقطة الثابتة في فضاءات مترية خاصة جديدة المنشورة من طرف Azam et al في عام 2011 وهي الفضاءات المترية المركبة.

يحتوي عملنا هذا على فكرتين رئيسيتين:

أولاً، أثبتنا نظرية النقطة الثابتة المشتركة لأربعة تطبيقات ذاتية تلبي تقلص علاقة الترتيب الجزئية في فضاء بي-متري مركب قدمها Rao في عام 2013. تم تحديد النتائج المتحصل عليها عن طريق إثبات كل من وجود ووحدانية حل مشترك لنظام Urysohn للمعادلات التكاملية، كما تم إثبات وجود حل وحيد لنظام المعادلات الخطية.

أثبتنا في فكرتنا الثانية نظرية النقطة الثابتة المشتركة بتحقيق شرط تقلص Pata المنشورة في عام 2011 حيث ان التقلص كان أضعف من تقلص Banach في فضاء متري مركب، كما دعمنا نتائجنا بإثبات كل من وجود ووحدانية حل مشترك لنظام Urysohn للمعادلات التكاملية، وأعطينا بعض الأمثلة.

لإثراء وتدعيم النتائج التي سبق نشرها، تم دعم هاتين الفكرتين بنتائج إضافية في هذه المخطوطة.

الكلمات المفتاحية: النقطة الثابتة، النقطة الثابتة المشتركة، الفضاء المتري المركب، الفضاء بي-متري المركب، تقلص Pata، تقلص علاقة الترتيب الجزئية، المعادلات التكاملية، نظام خطي.

Abstract

The aim of our thesis focused on the study of fixed point theory which is very useful in nonlinear analysis through their contribution to solve a lot of problems in different areas of applications.

Their importance emerged with Banach contraction and its treatment of integral equations problems in metric spaces. We were interested in this work to find some applications involving new generalizations theorems theorems in a new special metric space defined by *Azam et al* in 2011 which was named complex valued metric space.

Our work contains two principal ideas

- Firstly, we have proved a common fixed point theorem for four self-mappings satisfying a rational inequality contraction in complex valued b-metric space introduced by *K. Rao* in 2013. The obtained results were established by proving both existence and uniqueness of a common solution of the system of Urysohn integral equations. In addition, the existence of a unique solution for linear equations system was also proved.
- For our second idea, we have proved a common fixed point theorem in a complex valued metric space under *Pata's* contraction condition developed in 2011 which was weaker compared to renowned Banach contraction. We validated our results on a system of Urysohn integral equations and we gave some examples.

To enrich and consolidate the previous published results, these two ideas were been developed with a better in this manuscript.

Keywords

Fixed point, common fixed point, complex valued metric space, complex valued b-metric space, Pata’s contraction type, rational inequality contraction, integral equations, linear system.

Résumé

Les travaux de cette thèse vise l'étude de la théorie du point fixe, qui est très utile en analyse non linéaire car elle contribue à résoudre de nombreux problèmes dans différents domaines d'applications. L'importance de ce domaine est apparue avec la contraction de Banach et son traitement des problèmes d'équation intégrale dans l'espace métrique.

Nous nous sommes intéressés dans ce travail à trouver des applications impliquant de nouveaux théorèmes des nouvelles généralisations dans le cadre du point fixe commun dans un nouvel espace métrique spécial défini par Azam et al en 2011 et qui a été nommé espace métrique complexe.

Notre travail contient deux idées principales:
- Premièrement, nous avons démontré un théorème du point fixe commun pour quatre auto-applications satisfaisant la contraction d'inégalité rationnelle dans un espace b-métrique complexe introduit par K. Rao en 2013. Les résultats obtenus ont été établis en prouvant à la fois l'existence et l'unicité d'une solution commune du système d'équations intégrales de Urysohn. En outre, l'existence d'une solution unique pour le système d'équations linéaires a été également prouvé.
- Pour notre deuxième idée, nous avons prouvé un théorème du point fixe commun sous la condition de contraction de Pata développée en 2011 et qui était plus faible que celle de la contraction de Banach dans un espace métrique complexe. Nous avons validé nos résultats sur un système d'équations intégrales d'Urysohn et nous avons donné quelques exemples.

Pour consolider les résultats publiés précédemment, ces deux idées ont été développées et enrichir par des résultats additionnels dans ce manuscrit.

Mots-clés

Point fixe, point fixe commun, espace métrique à valeurs complexes, espace b-métrique à valeurs complexes, contraction de Pata, contraction d'inégalité rationnelle, équations intégrales, système linéaire.

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General introduction

Why does this equation, $Sz = z$, make a benefit?

When S is a mapping (*operator*), it can be easy to show the solution of the equation $Sz - z = 0$ where z is the fixed point. For well understanding a fixed point is a point which does not move by a given mapping or transformation, we can see it through a lot of phenomena as a reproducible invariant temperature like boiling point or freezing point, a triple point of a substance such as the liquid used to calibrate a thermometer.

For this reason, we have found numerous research about the existence of fixed point asserts that under certain conditions (on the mapping S and on the space X), a mapping S of X into itself admits one or more fixed points.
There are plenty of results in two areas in framework of fixed point theory. The first area is the topological space (*Brouwer's* fixed point theorem, *Schauder's* fixed point theorem) and the second one is the metric space (Contraction mapping theorem).

The most essential result in the fixed point theorem reported in literature started from theorem of *L. Brouwer* published in 1910, which says that every continuous self-mapping of the closed unit in $\mathbb{R}^n$, the n-dimensional Euclidean space, possesses a fixed point. This result was previously developed by *H. Poincaré* in an equivalent form. In 1886, *Poincaré* proved the following result: If $f : \mathbb{E}^n \to \mathbb{E}^n$ is any continuous function with the property that, for some $r > 0$ and $\alpha > 0, f(z) + \alpha z \neq 0, \|z\| = r$, there exists a point $z_0, \|f(z)\| \leqslant r$ such that $f(z_0) = z_0$.

Actually, it is known that this assertion is equivalent to the *Brouwer* fixed point theorem. In addition, the *Poincaré* theorem was also rediscovered by *P. Bohl* in 1904.

In 1844, *A. Cauchy* was the first mathematician who gave a proof for the existence and uniqueness of the solution of the differential equation $\frac{dz}{dx} = f(x, z); z(x_0) = z_0$ when f is a continuous differentiable function.

In 1877, *R. Lipschitz* simplified the proof made by Cauchy cited above known today as the "*Lipschitz condition*". Latter, a profoundly result, supposing only the continuity of *F. Peano's* approach more related to modern fixed point theorem used to obtain the existence theorem was establish by *G. Peano* in 1890

Also, *E. Sperner* in 1928 proved the combinatorial geometric lemma on the decomposition of a triangle, which is very important in the fixed point theory.

According to its extensive applications, there are nemrous results and still more to come on fixed point theorems. These are the most important tools for proving the existence and the uniqueness of solutions to various mathematical models (integral equations, dynamic programing, ordinary and partial differential equations, variational inequalities). Others are steady state as temperature distribution, chemical reactions, economic theory, flow of fluids, etc. [1]

It is also noted that a lot of fixed point theorems can be approached without any topological background.

One of these theorems generates the conception of a ***contraction*** in complete metric space into itself. This was introduced by *Stefan Banach* in 1922, giving a new field of research in fixed point theorem named contraction mappings, famous as Banach Contraction Principle (BCP).

From that time, there have been many generalizations of (BCP) by weakening its hypothesis while conserving the convergence property of the successive iterates to the unique fixed point of the mapping.
D. Boyd and *J. Wong* in 1969 and *Browder* in 1968 have attempted to generalize BCP by replacing the Lipschitz constant by some real valued functions (their values were less

than 1).

In 1969, *A. Meir* and *E. Keeler*, generalized BCP for a weak and uniformly strict contraction.

Sundry authors studied and generalized the BCP and gave different new type contraction in metric spaces (see [2], [3], [4], [5], [6], [7], [8], [9, 10], [11], [12])

Recently, *Pata* in [7], gave a new theorem with a weaker contraction condition than the renowned Banach contraction principle in metric space. We will give new results and application of this theorem in complex valued metric space which was introduced in 2011 by Azam et al [13]. We note that this generalization is not lonely of metric space. Several researchers before Azam attempted various generalization of this notion in the past such as: rectangular metric spaces, pseudo-metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, 2-metric spaces, D-metric spaces, G-metric spaces, fuzzy metrics and statistical metric spaces, K-metric spaces and cone metric spaces etc. However, there exists a considerable literature on metric spaces generalizations (For more details, one can see [14], [15], [16], [17], [13], [18], [19], [20]. . .).

To be able to turn the problem from the fixed point to the common fixed point, for more than one mapping which have an indispensable interest, the mathematicians needed to originate the notions of the commutative mapping (commutative, weakly commutative, compatibility , weakly compatibility ...)

In 1982, *Sessa* [21], generalized the concept of commutative mappings by introducing the concept of weakly commutative mappings. After that, in 1986, *Jungck* [22], generalized the concept of weakly commutative by introducing the concept of compatible mappings, then in 1996, [23] he introduced the concept of weakly compatible mappings.

We were interested in fixed point theory field especially in a common fixed point in complex valued metric space. We were able to do some remarks and release some results. As part of this thesis, we published papers in two different journals.

Patently, the first chapter, as in all thesis, contains the indispensable elements which will be needed for the following chapters.

Firstly, we gave the notions of complex-valued metric and b-metric spaces with some examples for clarifications. When we operating more than one mapping, it requires some commutativity between these mappings, for that, we will recall some definitions of commutativity used in our research. In particular, we will talk about compatible mappings and weakly compatible mappings.

The remainder of the other chapters will be devoted to the work done and published as part of this thesis. These works will be presented, as follows:

In the second chapter, we presented some common fixed point theorems for mappings satisfying rational inequalities in complex valued b-metric space. The principal result published in [24] entitled *Applications and theorem on common fixed point in complex valued b-metric space*, we have proved a common fixed point theorem for four self-mappings satisfying rational inequalities contraction condition and we gave also in this chapter another results which generalized theorems of Azam and Jaggy in complex valued b-metric space [13, 25, 26]

Our work published in [27] will be detailed in chapter 3. In this publication we proved a common fixed point theorem for two self mappings in complex valued metric space under Pata's contraction condition. Next, we generalized Pata's contraction condition in complex valued b-metric space.

We find in the fourth chapter some applications ensure the existence and the uniqueness of a common solution of the system of *Urysohn* integral equations and the existence of a unique solution for a linear equations system using theorem 2.6 and corolary 2.5 [24].

Thereafter, in [27], we established the existence and the uniqueness solution for the system of *Urysohn* integral equations.

Chapter 1

Notions and preliminaries

1.1 Introduction

Since the appearance of Banach's contraction principle, a number of papers have been consecrated to the improvement and the generalization of this result. Most of these generalizations of the contraction condition were in the metric spaces.

Ghaler [28], generalized the notion of metric space and introduced 2-metric space which was followed by several works dealing with this generalized space. There is also a lot of metric space generalization which were developed such as, rectangular metric spaces, semi-metric spaces, b-metric spaces, D-metric spaces and cone metric spaces (See [13], [14], [15], [16], [17], [18], [19], [20], [29]$\cdots$)

In [13], the authors introduced the notion of a complex valued metric spaces and obtained a common fixed point theorem of two mappings using rational partial inequality contraction condition in the farmwork of a complex valued metric space. The notion of complex-valued metric spaces can be exploited to define complex-valued normed spaces and complex valued Hilbert spaces. Additionally, it offers many research activities in mathematical analysis.

1.2 Complex valued metric spaces

To do so, let us recall a natural relation $\leq$ on $\mathbb{C}$, the set of complex numbers as follows. Let z_1, z_2 in $\mathbb{C}$

$$\begin{aligned} z_1 &\leq z_2 \Leftrightarrow \mathrm{Re}(z_1) \leq \mathrm{Re}(z_2) \text{ and } \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2) \\ z_1 &< z_2 \Leftrightarrow \mathrm{Re}(z_1) < \mathrm{Re}(z_2) \text{ and } \mathrm{Im}(z_1) < \mathrm{Im}(z_2) \end{aligned}$$

In [13], the authors defined a partial order relation $z_1 \precsim z_2$ on $\mathbb{C}$ by

$$z_1 \precsim z_2 \text{ if and only if } \mathrm{Re}(z_1) \leq \mathrm{Re}(z_2) \text{ and } \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2).$$

As a result, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied.

(i) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(ii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,

(iii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,

(iv) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

In (i), (ii) and (iii) we have $|z_1| < |z_2|$. In (iv) we get $|z_1| = |z_2|$, so, $|z_1| \leq |z_2|$. In particular, $z_1 \precsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied. In this case $|z_1| < |z_2|$.
We will write $z_1 \prec z_2$ if only (iii) is satisfied. Further,

$$\begin{aligned} 0 &\precsim z_1 \precnsim z_2 \Rightarrow | z_1 |<| z_2 |, \\ z_1 &\precsim z_2 \textit{ and } z_2 \prec z_3 \Rightarrow z_1 \prec z_3. \end{aligned}$$

In [13], the authors defined the complex valued metric space (X, d) in the following way.

Definition 1.1. *Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{C}$ is called a complex valued metric on X if the following conditions are satisfied.*

(a) $0 \precsim d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0 \Leftrightarrow x = y$,

(b) $d(x, y) = d(y, x)$, *for all* $x, y \in X$,

(c) *$d(x,y) \precsim d(x,z)+d(z,y)$ for all $x,y,z \in X$.*

d is called a complex valued metric in X and (X,d) is called a complex valued metric space.

Example 1.1. *Let $X=\mathbb{C}$. Define the mapping $d : X \times X \longrightarrow \mathbb{C}$ by*

$$d(z_1,z_2)=\mid z_1-z_2\mid +i\mid z_1-z_2\mid .$$

Then (X,d) is a complex valued metric space and d is called usual complex valued metric.

Example 1.2. *Let $X=\mathbb{C}$. Define the mapping $d : X \times X \longrightarrow \mathbb{C}$ by*

$$d(z_1,z_2)=\mid z_1-z_2\mid e^{i\theta},\ \theta \in \left]0,\frac{\pi}{2}\right[.$$

Therefore (X,d) is a complex valued metric space.

In the sequel of this section, (X,d) is a complex valued metric space.

Definition 1.2. *A point $z \in X$ is called the interior point of a set $A \subseteq X$, if there is exists $0 \prec r \in \mathbb{C}$ such that*

$$B(z,r)=\{w \in X : d(z,w) \prec r\} \subseteq A.$$

Definition 1.3. *A point $z \in X$ is called a limit point of a set A whenever for every $0 \prec r \in \mathbb{C}$, we obtain*

$$B(z,r) \cap (A \setminus X) \neq \phi.$$

Definition 1.4. *$A \subseteq X$ is called open if each element of A is an interior point of A.*
A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B.
The family

$$F=\{B(z,r) : z \in X : 0 \prec r\},$$

is a sub-basis for a Hausdorff topology τ on X.

Definition 1.5. *[13] Assume that $\{z_n\}$ is a sequence in X and $z \in X$.*

(i) *We say that $\{z_n\}$ converges to an element $z_0 \in X$ if for every $0 \prec c \in \mathbb{C}$, there exists an integer N such that $d(z_n,z_0) \prec c$ for all $n \geq N$. In this case, we write $z_n \longrightarrow z_0$.*

(ii) *We say that* $\{z_n\}$ *is a Cauchy sequence if for every* $0 \prec c \in \mathbb{C}$*, there exists an integer* N *such that* $d(z_n, z_m) \prec c$ *for all* $n, m \geq N$*.*

(iii) *We say that* (X, d) *is complete, if every Cauchy sequence in* X *converges to a point in* X*.*

Lemma 1.1. *Let* $\{z_n\}$ *be a sequence in* X*. Then* $\{z_n\}$ *converge to* z_0 *if and only if* $|d(z_n, z_0)| \to 0$ *as* $n \to \infty$*.*

Proof. Suppose that $\{z_n\}$ converge to z_0. For a given real number $\epsilon > 0$, let

$$c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}.$$

Then $0 \prec c \in \mathbb{C}$ and there is the natural number N, such that

$$d(z_n, z_0) \prec c \quad \text{for all} \quad n \geq N,$$

therefore,

$$|d(z_n, z_0)| < |c| \qquad \text{for all} \qquad n \geq N.$$

It follows that

$$|d(z_n, z_0)| \to 0 \qquad \text{as} \qquad n \to \infty.$$

Conversely, suppose that $|d(z_n, z_0)| \to 0$ as $n \to \infty$. Where $c \in \mathbb{C}$ with $0 \prec c$, there exists a real number $\delta > 0$, such that for $z \in \mathbb{C}$

$$|z| < \delta \Rightarrow z \prec c,$$

for this δ, there is natural number N such that

$$|d(z_n, z_0)| < \delta \qquad \text{for all} \qquad n \geq N.$$

This means that $d(z_n, z_0) \prec c$ for all $n \geq N$. Hence $\{z_n\}$ converges to z_0 ☐

Lemma 1.2. *Let* (X, d) *be a complex valued metric space and let* $\{z_n\}$ *be a sequence in* X*. Then* $\{z_n\}$ *is a Cauchy sequence if and only if* $|d(z_n, z_{n+m})| \to 0$ *as* $n \to \infty$*.*

Proof. Suppose that $\{z_n\}$ is a Cauchy sequence. For a given real number $\epsilon > 0$, let

$$c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}.$$

Then $0 \prec c \in \mathbb{C}$ and there is a natural number N, such that

$$d(z_n, z_{n+m}) \prec c \qquad \text{for all} \qquad n \geq N,$$

therefore,

$$|d(z_n, z_{n+m})| < |c| = \epsilon \qquad \text{for all} \qquad n \geq N,$$

it follows that

$$|d(z_n, z_{n+m})| \to 0 \qquad \text{as} \qquad n \to \infty.$$

Conversely, suppose that $|d(z_n, z_{n+m})| \to 0$ as $n \to \infty$. Where $c \in \mathbb{C}$ with $0 \prec c$, there exists a real number $\delta > 0$, such that for $z \in \mathbb{C}$

$$|z| < \delta \Rightarrow z \prec c,$$

for this δ, there is a natural number N such that

$$|d(z_n, z_{n+m})| < \delta \qquad \text{for all} \qquad n \geq N.$$

That is $d(z_n, z_{n+m}) \prec c$ for all $n \geq N$ and so $\{z_n\}$ is a Cauchy sequence. □

1.3 Complex valued b-metric spaces

Among many generalizations of the metric space, we have been interested in *b*-metric space, which were initiated from Bakhtin's work in 1989 [29]. Afterwards, Czerwik [30,31], defined the b-metric space as follows.

Definition 1.6. *Let X be a nonempty set and $s \geq 1$ a given real number. A mapping $d : X \times X \to \mathbb{R}_+$ is called b-metric if the following conditions are satisfied for all $x, y, z \in X$.*

(i) $d(x, y) = 0$ *if* $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X,d) formed a b-metric space.

Example 1.3. *Let $X=\mathbb{R}$. Define the mapping $d: X\times X \to \mathbb{R}^+$ by*

$$d(x_1,x_2)=|x_1-x_2|^2+|x_1-x_2|^2 \textit{ for all } \ x_1,x_2\in X.$$

Then (X,d) is a b-metric space with $s=2$.

Remark 1.1. *We note that the definition of the a b-metric space is an extension of a metric space.*

Based on the work of Azam and Czerwik which were given previously, K. Rao [32] introduced the concept of a complex valued b-metric space as follows.

Definition 1.7. *Let X be a nonempty set and $s\geq 1$ a given real number. A mapping $d: X\times X\to\mathbb{C}$ is called a complex b-metric if the following conditions are satisfied.*

(i) $d(x,y)\succsim 0$ *and* $d(x,y)=0$ *if only if* $x=y$ *for all* $x,y\in X$,

(ii) $d(x,y)=d(y,x)$ *for all* $x,y\in X$,

(iii) $d(x,z)\precsim s[d(x,y)+d(y,z)]$ *for all* $x,y,z\in X$.

The pair (X,d) form a complex valued b-metric space.

Example 1.4. *Let $X=\mathbb{C}$. Define the mapping $d: X\times X\to\mathbb{C}$ by*

$$d(z_1,z_2)=|z_1-z_2|^2+i\,|z_1-z_2|^2 \textit{ for all } \ z_1,z_2\in X$$

Therefore (X,d) is a complex valued b-metric space with $s=2$.

Remark 1.2. *We note that the definitions of a complex valued b-metric space is an extension of a complex valued metric space and all properties remain valid.*

1.4 Property of contractive applications

Definition 1.8. *Let (X, d) be a complex valued metric space. A mapping $S : X \to X$ is said to be Lipschitz if there exists a real number $k \geqslant 0$ such as for all $z, w \in X$ we have*

$$d(Sz, Sw) \precsim kd(z, w).$$

The smallest k for which the above inequality holds is the Lipschitz constant of S.
If $k < 1$, we say that S is contraction, if $k = 1$, we say that S is nonexpansive. Finally, S is said contractive if for all $z, w \in X$ and $z \neq w$, so we have

$$d(Sz, Sw) \prec d(z, w),$$

Remark 1.3. *Contraction implies contractive implies non-expansive implies Lipschitz. and all these functions are uniformly continuous.*

Theorem 1.1 (Banach)**.** *Let S be a contraction on a complete complex valued metric space X. Then, S has unique fixed point $u \in X$.*

Proof. Define the iterate sequence $z_{n+1} = Sz_n$. By induction on n,

$$d(z_{n+1}, z_n) \precsim k^n d(z_1, z_0).$$

If $n, m \in \mathbb{N}$ and $m \geqslant 1$

$$\begin{aligned} d(z_{n+m}, z_n) &\precsim d(z_{n+m}, z_{n+m-1}) + \cdots + d(z_{n+1}, z_n) \\ &\precsim (k^{n+m} + \cdots + k^n) d(z_1, z_0) \\ &\precsim \frac{k^n}{1-k} d(z_1, z_0). \end{aligned} \tag{1.1}$$

Therefore,

$$|d(z_{n+m}, z_n)| \leqslant \frac{k^n}{1-k} |d(z_1, z_0)| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, z_n is a Cauchy sequence and admits a limit $u \in X$, since X is complete. When S is continuous, we have $S(u) = \lim_{n\to\infty} S(z_n) = \lim_{n\to\infty} z_{n+1} = u$.
We can prove the uniqueness as fellow.
suppose that $v \in X$ is an other fixed point of S, then

$$d(u, v) = d(S(u), S(v)) \precsim kd(u, v).$$

Further,

$$|d(u,v)| \leq 0.$$

Which is a contradiction, then $d(u,v)=0$. Hence S has a unique fixed point. □

Remark 1.4. *The completeness of X plays here a crucial role. Indeed, a contraction on incomplete complex metric spaces may fail to have fixed points.*

In all the rest of this chapter S and T denote two self-mappings of a complex-valued metric space (X,d).

Definition 1.9. *S and T are said to be commutative if $STz = TSz$ for all $z \in X$.*

Definition 1.10. *S and T are said to be weakly commutative, if for all $z \in X$.*

$$d(STz, TSz) \precsim d(Sz, Tz)$$

Obviously, commutative implies weakly commutative, but the reciprocal is not true in general as it is shown in the example below.

Example 1.5. *Let $X = \{B(z_0,1), \text{where } z_0 = (3,3)\}$ and d is a usual complex valued metric. Define two mappings as follow:*

$$S(z) = \frac{z}{2},\ T(z) = \frac{z}{2+z}, \quad \text{for all } z \in X.$$

We have

$$\begin{aligned} d(STz, TSz) &= |STz - TSz| + i|STz - TSz| \\ &= \left|\frac{z}{4+2z} - \frac{z}{4+z}\right| + i\left|\frac{z}{4+2z} - \frac{z}{4+z}\right| \\ &\precsim \frac{z}{(4+z)(4+2z)} + i\frac{z}{(4+z)(4+2z)} \\ &= \left(\frac{z}{2} - \frac{z}{2+z}\right) + i\left(\frac{z}{2} - \frac{z}{2+z}\right) \\ &\precsim |Sz - Tz| + i|Sz - Tz| = d(Sz, Tz). \end{aligned}$$

Therefore, S and T are weakly commutative, but

$$TSz = \frac{z}{4+z} \succ \frac{z}{4+2z} = STz.$$

So, S and T are not commutative.

Definition 1.11. *[33], [22] S and T are said to be compatibles if*

$$\lim_{n\to\infty} d\left(STz_n, TSz_n\right) = 0, \tag{1.2}$$

for all sequence $\{z_n\}$ in X satisfying

$$\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Tz_n = t \text{ for some } t \in X. \tag{1.3}$$

Example 1.6. *Let $X = \mathbb{C}$ and d is the usual complex valued metric. Define $S, T : X \to X$ as follow: $Sz = \frac{1}{3}z^2$ and $Tz = iz$, for all $z \in X$. Consider the sequence $\{z_n\} = \{\frac{1}{n}\}_{n\geq 1}$ in X. Therefore*

$$\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Tz_n = 0$$

.

Definition 1.12. *Let S and T be self-mappings of a nonempty set X. If $w = Sz = Tz$ for some z in X, then z is called a coincidence point of S and T and w is called a point of coincidence of S and T.*

Definition 1.13. *[23] Let S and T be self-mappings of a nonempty set X. S and T are said to be weakly compatible if they commute at their coincidence points, i.e., $Sz = Tz$, for some z in X implies that $STz = TSz$.*

Definition 1.14. *[34] Let S and T be self-mappings in complex valued metric space (X, d). S and T are said to be weakly compatible if $STz = TSz$, whenever $Sz = Tz$, e.i., they commute at their coincidence points.*

Example 1.7. *Let $X = B(0, 2)$ equipped with the usual complex valued metric Define $S, T : X \to X$ by*

$$S(z) = \ln(z + 1) \quad et \quad S(z) = e^z - 1 \text{ for all } z \in X,$$

We have $S(0) = T(0) = 0$ and $ST(0) = TS(0) = 0$. Thus, the mappings S and T are weakly compatible.

Proposition 1.1. *[35] Let S and T be weakly compatible self-mappings of a nonempty set X. If S and T have a unique point of coincidence $w = Sz = Tz$, therefore w is the unique common fixed point of S and T.*

1.5 Property of comparison in complex valued metric space

Definition 1.15. *[15] Define the* $'\max'$ *function for the partial order relation by*

(i) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$,

(ii) $z_1 \precsim \max\{z_1, z_3\} \Rightarrow z_1 \precsim z_2$, *or* $z_1 \precsim z_3$,

(iii) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$ *or* $|z_1| \leq |z_2|$.

Using the definition (1.14) to get the following Lemma.

Lemma 1.3. *[15] Let* $z_1, z_2, z_{3,\dots} \in \mathbb{C}$ *and the partial order relation* $\precsim$ *defined on* $\mathbb{C}$. *The following statements hold.*

(i) If $z_1 \precsim \max\{z_2, z_3\}$ *then* $z_1 \precsim z_2$ *if* $z_3 \precsim z_2$,

(ii) If $z_1 \precsim \max\{z_2, z_3, z_4\}$ *then* $z_1 \precsim z_2$ *if* $\max\{z_3, z_4\} \precsim z_2$,

(iii) If $z_1 \precsim \max\{z_2, z_3, z_4, z_5\}$ *then* $z_1 \precsim z_2$ *if* $\max\{z_3, z_4, z_5\} \precsim z_2$, *and soon.*

1.6 Integral equations

An equation in which the unknown function of one or more variables appears under the integral sign is said an *integral equation*. This general definition takes into account many naturally occurring forms of modeling different problems of mechanics and mathematical physics or by reworking an important class of problems previously formulated by differential operators, especially the boundary problems and those of Cauchy.

1.6.1 Linear integral equations

The ordinary form of a linear integral equation is given by

$$\alpha(t)z(t) = f(t) + \lambda \int K(t, s)z(s)ds, \tag{1.4}$$

where α, f and K are given functions, the function z which appears inside and outside of the integral sign is the unknown to be determined, λ is a real or complex parameter different than zero. The function K is called the kernel of the integral equation.

1.6.1.1 Fredholm's integral equations

An equation of the form (1.4), whose integration terminals are fixed is called Fredholm's linear integral equation.
i) If $\alpha(t) = 0$, the equation is written

$$f(t) + \lambda \int_a^b K(t,s)z(s)ds = 0 \tag{1.5}$$

It is said of the first kind.
ii) If $\alpha(t) = 1$, the equation is written

$$z(t) = f(t) + \lambda \int_a^b K(t,s)z(s)ds \tag{1.6}$$

It is said of the second kind.
iii) If $\alpha(t)$ is continuous and vanishes at some points, but not at all points of $[a,b]$, it is said of the first kind.
iv) If $f(t) = 0$, the equation is written

$$z(t) = \lambda \int_a^b K(t,s)z(s)ds \tag{1.7}$$

It is said homogeneous.
v) The equation

$$\alpha(t)z(t) = f(t) + \lambda \int_a^b K(t,s)z(s)ds, \tag{1.8}$$

It is said of the third kind.

1.6.1.2 Volterra's integral equations

The integral equations of Volterra of the first, second and third kind or homogeneous are defined in the same previous way except that the upper bound of integration is variable, i.e., $b = t$

1.6.2 Nonlinear integral equations

The ordinary form of a nonlinear integral equation is given by

$$\alpha(t)z(t) = f(t) + \lambda \int K(t, s, z(s))ds, \tag{1.9}$$

where α, f and K are given functions. The function z which appears inside and outside of the integral sign is the unknown to be determined, λ is a real or complex parameter different to zero. The function K is called kernel of the integral equation.

1.6.2.1 Fredholm's integral equations

An equation of the form (1.8) whose integration terminals are fixed is called Fredholm's nonlinear integral equation.

i) If $\alpha(t) = 0$, the equation is written

$$f(t) + \lambda \int_a^b K(t, s, z(s))ds = 0, \tag{1.10}$$

it is said of the first kind.

ii) If $\alpha(t) = 1$ constant, the equation is written

$$z(t) = f(t) + \lambda \int_a^b K(t, s, z(s))ds, \tag{1.11}$$

it is said of the second kind.

iii) If $\alpha(t)$ is continuous and vanishes at some points, but not at all points of $[a, b]$, it is said of the first kind.

iv) If $f(t) = 0$, the equation is written

$$z(t) = \lambda \int_a^b K(t, s, z(s))ds, \tag{1.12}$$

it is said homogeneous.

v) The equation

$$\alpha(t)z(t) = f(t) + \lambda \int_a^b K(t, s, z(s))ds, \tag{1.13}$$

it is said of the third kind.

1.6.2.2 Volterra's integral equations

The integral equations of Volterra of the first, second and third kind or homogeneous are defined in the same previous way except that the upper bound of integration is variable, i.e., $b = t$

1.6.2.3 Urysohn's integral equations

An equation of the form (1.8) whose integration terminals are fixed is called Urysohn nonlinear integral equation.

i) If $\alpha(t) = 0$ and $\lambda = 1$, the equation is written

$$f(t) + \int_a^b K(t, s, z(s))ds, \tag{1.14}$$

it is said of the first kind.

ii) If $\alpha(t) = 1$ and $\lambda = 1$, the equation is written

$$z(t) = f(t) + \int_a^b K(t, s, z(s))ds, \tag{1.15}$$

it is said of the second kind.

iii) If $\lambda = 1$ and $\alpha(t)$ is continuous and vanishes at some points, but not at all points of $[a, b]$, it is called of the first kind.

iv) If $f(t) = 0$, the equation is written

$$z(t) = \int_a^b K(t, s, z(s))ds, \tag{1.16}$$

it is said homogeneous.

v) The equation

$$\alpha(t)z(t) = f(t) + \int_a^b K(t, s, z(s))ds, \tag{1.17}$$

it is said of the third kind.

Chapter 2

Common fixed point theorems for self-mappings satisfying rational contraction

2.1 Introduction

In 1989, Bakhtin [29], initiated of b-metric space, after that Czerwik in [30, 31], defined it such as current structure which is considered generalization of metric spaces. The complex valued b-metric spaces concept was introduced in 2013 by Rao et al. [32], which was more general than the well-known complex valued metric spaces that were introduced in 2011 by Azam et al. [13], which proved some common fixed point theorems for mappings satisfying rational inequalities which are not worthwhile in cone metric spaces [36–39]. Sundry authors have studied and proved fixed point results for mappings satisfying different type of contraction conditions in the framework of complex valued metric and b-metric spaces(see [26, 27, 32, 40, 41]). In this chapter we present common fixed point theorems for two and four self-mappings verifying rational inequalities in complex valued b-metric spaces.

Theorem 2.1. *Let (X,d) be a complete complex valued metric space and $S,T : X \to X$ be two mappings satisfying*

$$d(Sz,Tw) \precsim \lambda d(z,w) + \frac{\mu d(z,Sz)d(w,Tw)}{1+d(z,w)} \tag{2.1}$$

for all $z,w \in X$, where λ, μ are nonnegative reals with $\lambda + \mu < 1$. Then S and T have a unique common fixed point.

Proof. see [13] ☐

2.2 Common fixed point for two mappings

In this section, we given some results in complex valued b-metric space using different inequalities for two self-mappings. In the first result, we generalize theorem 2.1.

Theorem 2.2. *Let (X, d) be a complete complex valued b-metric space ($s \geq 1$) and $S, T : X \to X$ be two mappings satisfying*

$$d(Sz, Tw) \precsim \frac{\lambda}{s} d(z, w) + \frac{\mu d(z, Sz) d(w, Tw) + \gamma d(w, Sz) d(z, Tw)}{1 + d(z, w)} \tag{2.2}$$

for all $z, w \in X$, where λ, μ, γ are nonnegative reals with $\lambda + \mu + \gamma < 1$. Then S and T have a unique common fixed point.

Proof. Let z_0 be an arbitrary point in X. Define a sequence $\{z_n\}$ such that

$$z_{2n+1} = Sz_{2n} \quad \text{and} \quad z_{2n+2} = Tz_{2n+1}. \quad n = 0, 1, 2, \cdots \tag{2.3}$$

Using (2.3) in (2.2) we have

$$\begin{aligned} d(z_{2n+1}, z_{2n+2}) &= d(Sz_{2n}, Tz_{2n+1}) \\ &\precsim \frac{\lambda}{s} d(z_{2n}, z_{2n+1}) \\ &\quad + \frac{\mu d(z_{2n}, Sz_{2n}) d(z_{2n+1}, Tz_{2n+1}) + \gamma d(z_{2n+1}, Sz_{2n}) d(z_{2n}, Tz_{2n+1})}{1 + d(z_{2n}, z_{2n+1})} \\ &\precsim \frac{\lambda}{s} d(z_{2n}, z_{2n+1}) \\ &\quad + \frac{\mu d(z_{2n}, z_{2n+1}) d(z_{2n+1}, z_{2n+2}) + \gamma d(z_{2n+1}, z_{2n+1}) d(z_{2n}, z_{2n+2})}{1 + d(z_{2n}, z_{2n+1})}. \end{aligned}$$

We have

$$1 + d(z_{2n}, z_{2n+1}) \succsim d(z_{2n}, z_{2n+1}) \Rightarrow \frac{d(z_{2n}, z_{2n+1})}{1 + d(z_{2n}, z_{2n+1})} \leqslant 1.$$

Therefore,

$$d(z_{2n+1}, z_{2n+2}) \precsim \frac{\lambda}{s(1-\mu)} d(z_{2n}, z_{2n+1}).$$

Similarly,

$$\begin{aligned}
d(z_{2n}, z_{2n+1}) &= d(Sz_{2n-1}, Tz_{2n}) \\
&\precsim \frac{\lambda}{s} d(z_{2n-1}, z_{2n}) \\
&\quad + \frac{\mu d(z_{2n-1}, Sz_{2n-1}) d(z_{2n}, Tz_{2n}) + \gamma d(z_{2n}, Sz_{2n-1}) d(z_{2n-1}, Tz_{2n})}{1 + d(z_{2n-1}, z_{2n})} \\
&\precsim \frac{\lambda}{s} d(z_{2n-1}, z_{2n}) \\
&\quad + \frac{\mu d(z_{2n-1}, z_{2n}) d(z_{2n}, z_{2n+1}) + \gamma d(z_{2n}, z_{2n}) d(z_{2n-1}, z_{2n+1})}{1 + d(z_{2n-1}, z_{2n})} \\
&\precsim \frac{\lambda}{s} d(z_{2n-1}, z_{2n}) + \mu d(z_{2n}, z_{2n+1}) \\
&\precsim \frac{\lambda}{s(1-\mu)} d(z_{2n-1}, z_{2n}).
\end{aligned}$$

Put $k = \frac{\lambda}{1-\mu}$, we have

$$\begin{aligned}
d(z_{n+1}, z_{n+2}) &\precsim \frac{k}{s} d(z_n, z_{n+1}) \\
&\precsim \cdots \precsim \left(\frac{k}{s}\right)^{n+1} d(z_0, z_1).
\end{aligned}$$

So for any $m > n$ where $n, m \in \mathbb{N}$,

$$\begin{aligned}
|d(z_n, z_m)| &\leq s \left(\frac{k}{s}\right)^{n} |d(z_0, z_1)| + s^2 \left(\frac{k}{s}\right)^{n+1} |d(z_0, z_1)| + s^3 \left(\frac{k}{s}\right)^{n+2} |d(z_0, z_1)| + \\
&\quad \cdots + s^{m-n-1} \left(\frac{k}{s}\right)^{m-2} |d(w_0, w_1)| + s^{m-n} \left(\frac{k}{s}\right)^{m-1} |d(z_0, z_1)| \\
&= \sum_{i=1}^{m-n} s^i \left(\frac{k}{s}\right)^{i+n-1} |d(z_0, z_1)|.
\end{aligned}$$

Therefore,

$$\begin{aligned}|d(z_n, z_m)| &\leq \sum_{i=1}^{m-n} s^{i+n-1}\left(\frac{k}{s}\right)^{i+n-1}|d(z_0, z_1)| = \sum_{t=n}^{m-1} s^t\left(\frac{k}{s}\right)^t|d(z_0, z_1)| \\ &\leq \sum_{i=1}^{\infty}(k)^t|d(z_0, z_1)| = \frac{(k)^n}{(1-k)}|d(z_0, z_1)|.\end{aligned}$$

Hence,

$$|d(z_n, z_m)| \leq \frac{(k)^n}{(1-k)}|d(z_0, z_1)| \to 0 \,\text{as}\, n \to \infty.$$

Thus, $\{z_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $z_n \to u$. For its sub-sequences we also have $Tz_{2n+1} \to u, Sz_{2n} \to u$. It follows that $u = Su$, otherwise $d(u, Su) = v > 0$ and we would then have

$$\begin{aligned}v &\precsim sd(u, z_{2n+2}) + sd(z_{2n+2}, Su) \\ &\precsim sd(u, z_{2n+2}) + sd(Tz_{2n+1}, Su) \\ &\precsim sd(u, z_{2n+2}) + \lambda d(u, z_{2n+1}) \\ &\quad + \frac{\mu d(u, Su)d(z_{2n+1}, Tz_{2n+1}) + \gamma d(z_{2n+1}, Su)d(u, Tz_{2n+1})}{1 + d(u, z_{2n+1})}\end{aligned}$$

Passing to the limit as $n \to \infty$

$$\begin{aligned}v &\precsim sd(u, z_{2n+2}) + \lambda d(u, z_{2n+1}) \\ &\quad + \frac{\mu v d(z_{2n+1}, z_{2n+2}) + \gamma d(z_{2n+1}, Su)d(u, u)}{1 + d(u, z_{2n+1})}.\end{aligned}$$

This implies that

$$|v| \leqslant s|d(u, z_{2n+2})| + \lambda|d(u, z_{2n+1})| + \frac{\mu|d(z_{2n+1}, z_{2n+2})||v|}{|1 + d(u, z_{2n+1})|}.$$

Nevertheless $|v| = 0$, is a contradiction. Hence, $u = Su$, it follows similarly that $u = Tu$. We now show that S and T have unique common fixed point. For that supposing u^* is a second common fixed point of S and T. Using (2.2)

$$\begin{aligned} d(u,u^*) &= d(Su,Tu^*) \\ &\precsim \frac{\lambda}{s}d(u,u^*)+\frac{\mu d(u,Su)d(u^*,Tu^*)+\gamma d(u^*,Su)d(u,Tu^*)}{1+d(u,u^*)} \\ &\precsim (\frac{\lambda}{s}+\gamma)d(u,u^*). \end{aligned}$$

This implies that $u=u^*$, completing the proof of the theorem. □

Corollary 2.1. *Let* (X,d) *be a complete complex valued b-metric space* ($s\geq 1$) *and* $S:X\to X$ *be a mapping satisfying*

$$d(Sz,Sw)\precsim \frac{\lambda}{s}d(z,w)+\frac{\mu d(z,Sz)d(w,Sw)+\gamma d(w,Sz)d(z,Sw)}{1+d(z,w)} \tag{2.4}$$

for all $z,w\in X$*, where* λ,μ,γ *are nonnegative reals with* $\lambda+\mu+\gamma<1$*. So* S *has a unique fixed point in* X*.*

Corollary 2.2. *Let* (X,d) *be a complete complex valued b-metric space* ($s\geq 1$) *and* $S:X\to X$ *be a mappings satisfying*

$$d(S^nz,S^nw)\precsim \frac{\lambda}{s}d(z,w)+\frac{\mu d(z,S^nz)d(w,S^nw)+\gamma d(w,S^nz)d(z,S^nw)}{1+d(z,w)} \tag{2.5}$$

for all $z,w\in X$*, where* λ,μ,γ *are nonnegative reals with* $\lambda+\mu+\gamma<1$*. Therefore* S *has a unique fixed point in* X*.*

Proof. By Corollary 2.1 we obtain $v\in X$ such that

$$S^nv=v.$$

Using (2.5) we get

$$\begin{aligned} d(Sv,v) &= d(SS^nv,S^nv)=d(S^nSv,S^nv) \\ &\precsim \frac{\lambda}{s}d(Sv,v)+\frac{\mu d(Sv,S^nSv)d(v,S^nv)+\gamma d(v,S^nSv)d(Sv,S^nv)}{1+d(Sv,v)} \\ &= \frac{\lambda}{s}d(Sv,v) \end{aligned}$$

□

Definition 2.1. *[25] Let (X, d) be a metric space. A self mapping S in X is called an almost Jaggi contraction if it satisfies the following condition*

$$d(Sz, Sw) \leqslant \frac{\alpha d(z, Sz)d(w, Sw)}{d(z, w)} + \beta d(z, w) + L\min\{d(z, Sw), d(w, Sz)\}, \tag{2.6}$$

for all $z, w \in X$ with $z \neq w$, where $L \geq 0$ and $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$.

Theorem 2.3. *Let (X, d) be a complete complex valued b-metric space $(s \geq 1)$ and $S : X \to X$ be an almost Jaggi contraction with $s(\alpha + \beta) < 1$. Then S has a unique fixed point in X.*

Proof. See [25] □

Theorem 2.4. *Let (X, d) be a complete complex valued b-metric space $(s \geq 1)$. Suppose that the mappings $S, T : X \to X$ verify the following almost Jaggi contraction*

$$d(Sz, Tw) \precsim \frac{\alpha d(z, Sz)d(w, Tw)}{d(z, w)} + \beta d(z, w) + L\min\{d(z, Tw), d(w, Sz)\}, \tag{2.7}$$

for all $z, w \in X$ with $z \neq w$, where $L \geq 0$ and α, β are nonnegative reals with $s(\alpha+\beta) < 1$. Therefore S and T have a unique common fixed point in X.

Proof. See [26] □

We propose this theorem similar to the previous result.

Theorem 2.5. *Let (X, d) be a complete complex valued b-metric space $(s \geq 1)$. Assume that the mappings $S, T : X \to X$ verify the following almost Jaggi contraction*

$$\begin{aligned} d(Sz, Tw) \quad &\precsim \quad \frac{\alpha d(z, Sz)d(w, Tw) + \gamma d(w, Sz)d(z, Tw)}{d(z, w)} \\ &\qquad + \beta d(z, w) + L\min\{d(z, Tw), d(w, Sz)\}, \end{aligned} \tag{2.8}$$

for all $z, w \in X$ with $z \neq w$, where $L \geq 0$ and α, β, γ are nonnegative reals with $s(\alpha + \beta + \gamma) < 1$. So, S and T have a unique common fixed point in X.

Proof. Let z_0 be an arbitrary point in X. Define a sequence $\{z_n\}$ such that

$$z_{2n+1} = Sz_{2n} \quad \text{and} \quad z_{2n+2} = Tz_{2n+1}. \quad n = 0, 1, 2, \cdots \tag{2.9}$$

Using (2.9) in (2.8) we have

$$\begin{aligned}
d(z_{2n+1}, z_{2n+2}) &= d(Sz_{2n}, Tz_{2n+1}) \\
&\precsim \frac{\alpha d(z_{2n}, Sz_{2n}) d(z_{2n+1}, Tz_{2n+1}) + \gamma d(z_{2n+1}, Sz_{2n}) d(z_{2n}, Tz_{2n+1})}{d(z_{2n}, z_{2n+1})} \\
&\quad + \beta d(z_{2n}, z_{2n+1}) + L \min\{d(z_{2n}, Tz_{2n+1}), d(z_{2n+1}, Sz_{2n})\} \\
&= \frac{\alpha d(z_{2n}, z_{2n+1}) d(z_{2n+1}, z_{2n+2})}{d(z_{2n}, z_{2n+1})} \\
&\quad + \beta d(z_{2n}, z_{2n+1}) + L \min\{d(z_{2n}, z_{2n+2}), d(z_{2n+1}, z_{2n+1})\} \\
d(z_{2n+1}, z_{2n+2}) &\precsim \frac{\alpha d(z_{2n}, z_{2n+1}) d(z_{2n+1}, z_{2n+2})}{d(z_{2n}, z_{2n+1})} + \beta d(z_{2n}, z_{2n+1}).
\end{aligned}$$

So that

$$|d(z_{2n+1}, z_{2n+2})| \leqslant \alpha |d(z_{2n+1}, z_{2n+2})| + \beta |d(z_{2n}, z_{2n+1})|,$$

hence,

$$|d(z_{2n+1}, z_{2n+2})| \leqslant \frac{\beta}{1-\alpha} |d(z_{2n}, z_{2n+1})|.$$

Similarly, we obtain

$$|d(z_{2n}, z_{2n+1})| \leqslant \frac{\beta}{1-\alpha} |d(z_{2n-1}, z_{2n})|.$$

Putting $k = \frac{\beta}{1-\alpha}$, we have

$$\begin{aligned}
d(z_{n+1}, z_{n+2}) &\precsim k d(z_n, z_{n+1}) \\
&\precsim \cdots \precsim k^{n+1} d(z_0, z_1).
\end{aligned}$$

So for any $m > n$, where, $n, m \in \mathbb{N}$ and since $sk < 1$ we get

$$\begin{aligned}
|d(z_n, z_m)| &\leq sk^n |d(z_0, z_1)| + s^2 k^{n+1} |d(z_0, z_1)| + s^3 k^{n+2} |d(z_0, z_1)| \\
&\quad + \cdots + s^{m-n-1} k^{m-2} |d(w_0, w_1)| + s^{m-n} k^{m-1} |d(z_0, z_1)| \\
&= \sum_{i=1}^{m-n} s^i k^{i+n-1} |d(z_0, z_1)|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|d(z_n, z_m)| &\leq \sum_{i=1}^{m-n} s^{i+n-1}k^{i+n-1}|d(z_0, z_1)| = \sum_{t=n}^{m-1} s^t k^t |d(z_0, z_1)| \\
&\leq \sum_{i=1}^{\infty} (sk)^t \, |d(z_0, z_1)| = \frac{(sk)^n}{(1-sk)}|d(z_0, z_1)|,
\end{aligned}$$

hence,

$$|d(z_n, z_m)| \leq \frac{(sk)^n}{(1-sk)}|d(z_0, z_1)| \to 0 \text{ as } n \to \infty.$$

Thus, $\{z_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $z_n \to u$. For its sub-sequences we also have $Tz_{2n+1} \to u, Sz_{2n} \to u$, it follows that $u = Su$, otherwise $d(u, Su) = v > 0$ and we would then have

$$\begin{aligned}
v &\precsim sd(u, z_{2n+2}) + sd(z_{2n+2}, Su) \\
&\precsim sd(u, z_{2n+2}) + sd(Su, Tz_{2n+1}) \\
&\precsim sd(u, z_{2n+2}) + s\frac{\alpha d(u, Su)d(z_{2n+1}, Tz_{2n+1}) + \gamma d(z_{2n+1}, Su)d(u, Tz_{2n+1})}{d(u, z_{2n+1})} \\
&\quad +\beta d(u, z_{2n+1}) + L\min\{d(u, Tz_{2n+1}), d(z_{2n+1}, Su)\}.
\end{aligned}$$

Passing to the limit as $n \to \infty$

$$\begin{aligned}
v &\precsim sd(u, z_{2n+2}) + s\frac{\alpha d(u, Su)d(z_{2n+1}, u) + \gamma d(z_{2n+1}, Su)d(u, u)}{d(u, z_{2n+1})} \\
&\quad +\beta d(u, z_{2n+1}) + L\min\{d(u, u), d(z_{2n+1}, Su)\} \\
&= sd(u, z_{2n+2}) + s\frac{\alpha v d(z_{2n+1}, u)}{d(u, z_{2n+1})} + \beta d(u, z_{2n+1})
\end{aligned}$$

$$|v| \leqslant s|d(u, z_{2n+2})| + s\frac{\alpha |v||d(z_{2n+1}, u)|}{|d(u, z_{2n+1})|} + \beta |d(u, z_{2n+1})|.$$

Nevertheless, $|v| = 0$, is a contradiction, hence, $u = Su$. It follows similarly that $u = Tu$. We now show that S and T have a unique common fixed point.

For that supposing u^* is a second common fixed point of S and T. Using (2.8)

$$\begin{aligned} d(u,u^*) &= d(Su,Tu^*) \\ &\precsim \frac{\alpha d(u,Su)d(u^*,Tu^*)+\gamma d(u^*,Su)d(u,Tu^*)}{d(u,u^*)} \\ &\quad +\beta d(u,u^*)+L\min\{d(u,Tu^*),d(u^*,Su)\}, \end{aligned}$$

further,

$$|d(u,u^*)| \leqslant (\gamma+\beta+L)|d(u,u^*)|.$$

Which is a contradiction, so that $u = u^*$. □

Taking $T = I_X$ in the previous theorem, we get the following corollaries

Corollary 2.3. *Let (X,d) be a complete complex valued b-metric space ($s \geq 1$). Suppose that the mapping $S : X \to X$ verify the following contraction*

$$\begin{aligned} d(Sz,Sw) &\precsim \frac{\alpha d(z,Sz)d(w,Sw)+\gamma d(w,Sz)d(z,Sw)}{d(z,w)} \\ &\quad +\beta d(z,w)+L\min\{d(z,Sw),d(w,Sz)\}, \end{aligned}$$

for all $z,w \in X$ with $z \neq w$, where $L \geq 0$ and α,β,γ are nonnegative reals with $s(\alpha+\beta+\gamma) < 1$. Then S has a unique fixed point in X.

Corollary 2.4. *Let (X,d) be a complete complex valued b-metric space ($s \geq 1$). Assume that the mapping $S : X \to X$ verify the following contraction*

$$\begin{aligned} d(S^nz,S^nw) &\precsim \frac{\alpha d(z,S^nz)d(w,S^nw)+\gamma d(w,S^nz)d(z,S^nw)}{d(z,w)} \\ &\quad +\beta d(z,w)+L\min\{d(z,S^nw),d(w,S^nz)\}, \end{aligned}$$

for all $z,w \in X$ with $z \neq w$, where $L \geq 0$ and α,β,γ are nonnegative reals with $s(\alpha+\beta+\gamma) < 1$. Therefore, S has a unique fixed point in X.

Proof. By Corollary 2.3 we obtain $v \in X$ such that

$$S^nv = v$$

From the previous condition, we get

$$\begin{aligned}
d(Sv,v) &= d(SS^nv,S^nv)=d(S^nSv,S^nv)\\
&\precsim \frac{\alpha d(Sv,S^nSv)d(v,S^nv)+\gamma d(v,S^nSv)d(Sv,S^nv)}{d(Sv,v)}\\
&\quad +\beta d(Sv,v)+L\min\{d(Sv,S^nv),d(v,S^nSv)\}\\
&= \beta d(Sv,v).
\end{aligned}$$

□

2.3 Common fixed point for four self-mappings in complex valued b-metric space

In this section, we itemize our paper published in [24] in complex valued b-metric spaces for four mappings using contraction inequality and the notions of compatibility and weakly compatibility. We illustrate our main theorem by two examples.

Theorem 2.6. *Let (X,d) be a complete complex valued b-metric space and $S,T,P,Q:X\to X$ be a mappings satisfying the conditions*

(C_1) *$S(X)\subset Q(X)$ and $T(X)\subset P(X)$.*

(C_2) *$d(Sz,Tw)\precsim \frac{\lambda}{s^2}R(z,w)$, for all $z,w\in X$ where $s\geq 1$, $\lambda\in(0,1)$ and*

$$\begin{aligned}
R(z,w) &= \max\Big\{(Pz,Qw),d(Pz,Sz),d(Qw,Tw)\\
&\quad ,\frac{1}{2}[d(Qw,Sz)+d(Pz,Tw)],\frac{d(Pz,Sz)d(Qw,Tw)}{1+d(Pz,Qw)}\Big\}.
\end{aligned}$$

(C_3) *The pair (S,P) is compatible and the pair (T,Q) is weakly compatible.*

(C_4) *Either P or S is continuous.*
Therefore S,T,P and Q have a unique common fixed point in X.

Proof. Let $z_0\in X$ be arbitrary. From the condition (C_1), there exist z_1,z_2 such that $w_0=Qz_1=Sz_0$ and $w_1=Pz_2=Tz_1$. We can construct successively the sequences $\{w_n\}$ and $\{z_n\}$ in X as follows

$$w_{2n}=Qz_{2n+1}=Sz_{2n}\ \text{ and }\ w_{2n+1}=Pz_{2n+2}=Tz_{2n+1}. \tag{2.10}$$

Using (2.10) in (C_2) we get

$$d(w_{2n}, w_{2n+1}) = d(Sz_{2n}, Tz_{2n+1}) \precsim \frac{\lambda}{s^2} R(z_{2n}, z_{2n+1}),$$

where,

$$\begin{aligned} R(z_{2n}, z_{2n+1}) &= \max\left\{d(Pz_{2n}, Qz_{2n+1}), d(Pz_{2n}, Sz_{2n}), d(Qz_{2n+1}, Tz_{2n+1})\right. \\ &\quad , \frac{1}{2}[d(Qz_{2n+1}, Sz_{2n}) + d(Pz_{2n}, Tz_{2n+1})] \\ &\quad \left. , \frac{d(Pz_{2n}, Sz_{2n}) d(Qz_{2n+1}, Tz_{2n+1})}{1 + d(Pz_{2n}, Qz_{2n+1})}\right\} \\ &= \max\left\{d(w_{2n-1}, w_{2n}), d(w_{2n-1}, w_{2n}), d(w_{2n}, w_{2n+1})\right. \\ &\quad , \frac{1}{2}[d(w_{2n}, w_{2n}) + d(w_{2n-1}, w_{2n+1})] \\ &\quad \left. , \frac{d(w_{2n-1}, w_{2n}) d(w_{2n}, w_{2n+1})}{1 + d(w_{2n-1}, w_{2n})}\right\}. \end{aligned}$$

We have

$$\begin{aligned} \frac{1}{2} d(w_{2n-1}, w_{2n+1}) &\precsim \frac{1}{2}[d(w_{2n-1}, w_{2n}) + d(w_{2n}, w_{2n+1})] \\ &\precsim \max\{d(w_{2n-1}, w_{2n}), d(w_{2n}, w_{2n+1})\}, \end{aligned} \tag{2.11}$$

and we have

$$d(w_{2n-1}, w_{2n}) \precsim 1 + d(w_{2n-1}, w_{2n}),$$

which implies

$$\frac{d(w_{2n-1}, w_{2n}) d(w_{2n}, w_{2n+1})}{1 + d(w_{2n-1}, w_{2n})} \precsim d(w_{2n}, w_{2n+1}). \tag{2.12}$$

From (2.11) and (2.12) we get

$$R(z_{2n}, z_{2n+1}) = \max\{d(w_{2n-1}, w_{2n}), d(w_{2n}, w_{2n+1})\}$$

with

$$d(w_{2n}, w_{2n+1}) = d(Sz_{2n}, Tz_{2n+1}) \precsim \frac{\lambda}{s^2} R(z_{2n}, z_{2n+1}).$$

If

$$R(z_{2n}, z_{2n+1}) = d(w_{2n}, w_{2n+1}).$$

Then,

$$d(w_{2n}, w_{2n+1}) \precsim \frac{\lambda}{s^2} d(w_{2n}, w_{2n+1}), \text{ therefore } \left(1 - \frac{\lambda}{s^2}\right) d(w_{2n}, w_{2n+1}) \precsim 0,$$

which is a contradiction, since $\lambda \in (0,1), s \geq 1$. We conclude that

$$d(w_{2n}, w_{2n+1}) \precsim \frac{\lambda}{s^2} d(w_{2n-1}, w_{2n})$$

Similarly we get

$$d(w_{2n+1}, w_{2n+2}) \precsim \frac{\lambda}{s^2} d(w_{2n}, w_{2n+1}).$$

It follows that

$$d(w_n, w_{n+1}) \precsim \frac{\lambda}{s^2} d(w_{n-1}, w_n) \precsim \cdots \precsim \left(\frac{\lambda}{s^2}\right)^n d(w_0, w_1),$$

which implies

$$|d(w_n, w_{n+1})| \leq \frac{\lambda}{s^2} |d(w_{n-1}, w_n)| \leq \cdots \leq \left(\frac{\lambda}{s^2}\right)^n |d(w_0, w_1)|,$$

For $m > n \quad (n, m \in \mathbb{N})$ we have

$$\begin{aligned} |d(w_n, w_m)| &\leq s \left(\frac{\lambda}{s^2}\right)^n |d(w_0, w_1)| + s^2 \left(\frac{\lambda}{s^2}\right)^{n+1} |d(w_0, w_1)| + s^3 \left(\frac{\lambda}{s^2}\right)^{n+2} |d(w_0, w_1)| + \\ &\quad \cdots + s^{m-n-1} \left(\frac{\lambda}{s^2}\right)^{m-2} |d(w_0, w_1)| + s^{m-n} \left(\frac{\lambda}{s^2}\right)^{m-1} |d(w_0, w_1)| \\ &= \sum_{i=1}^{m-n} s^i \left(\frac{\lambda}{s^2}\right)^{i+n-1} |d(w_0, w_1)|. \end{aligned}$$

Therefore,

$$\begin{aligned}|d(w_n, w_m)| &\leq \sum_{i=1}^{m-n} s^{i+n-1}\left(\frac{\lambda}{s^2}\right)^{i+n-1} |d(w_0, w_1)| = \sum_{t=n}^{m-1} s^t \left(\frac{\lambda}{s^2}\right)^t |d(w_0, w_1)| \\ &\leq \sum_{i=1}^{\infty} \left(\frac{\lambda}{s}\right)^t |d(w_0, w_1)| = \frac{\left(\frac{\lambda}{s}\right)^n}{\left(1-\frac{\lambda}{s}\right)} |d(w_0, w_1)|.\end{aligned}$$

Hence,

$$|d(w_n, w_m)| \leq \frac{\left(\frac{\lambda}{s}\right)^n}{\left(1-\frac{\lambda}{s}\right)} |d(w_0, w_1)| \to 0 \text{ as } n \to \infty.$$

Thus, $\{w_n\}$ is a Cauchy sequence in X. Since X is complete, so there exists some $u \in X$ such that $w_n \to u$ as $n \to \infty$. For its sub-sequences we also have $Qz_{2n+1} \to u, Sz_{2n} \to u, Pz_{2n+1} \to u$ and $Tz_{2n} \to u$.

From (C_4) if P is continuous, then $PPz_{2n} \to Pu$ and $PSz_{2n} \to Pu$, as $n \to \infty$.

Also, since the pair (S, P) is compatible, this implies that $SPz_{2n} \to Pu$. Indeed,

$$d(SPz_{2n}, Pu) \precsim s[d(SPz_{2n}, PSz_{2n}) + d(PSz_{2n}, Pu)].$$

So,

$$|d(SPz_{2n}, Pu)| \leq s|d(SPz_{2n}, PSz_{2n})| + s|d(PSz_{2n}, Pu)| \to 0 \quad \text{as} \quad n \to \infty.$$

We prove $Pu = u$. On the contrary we suppose that $Pu \neq u$

$$d(Pu, u) \precsim sd(Pu, SPz_{2n}) + s^2 d(SPz_{2n}, Tz_{2n+1}) + s^2 d(Tz_{2n+1}, u).$$

Using (C_2) with $z = Pz_{2n}, w = z_{2n+1}$, we get

$$d(SPz_{2n}, Tz_{2n+1}) \precsim \lambda R(Pz_{2n}, z_{2n+1}),$$

where,

$$\begin{aligned}R(Pz_{2n}, z_{2n+1}) &= \max\{d(PPz_{2n}, Qz_{2n+1}), d(PPz_{2n}, SPz_{2n}), d(Qz_{2n+1}, Tz_{2n+1}) \\ &\quad , \frac{1}{2}[d(Pz_{2n+1}, SPz_{2n}) + d(QPz_{2n}, Tz_{2n+1})] \\ &\quad \left., \frac{d(PPz_{2n}, SPz_{2n}) d(Qz_{2n+1}, Tz_{2n+1})}{1 + d(PPz_{2n}, Qz_{2n+1})}\right\}.\end{aligned}$$

Passing to the limit as $n \to \infty$ we get

$$\begin{aligned} R(Pu,u) &= \max\{d(Pu,u), d(Pu,Pu), d(u,Pu) \\ &\quad , \frac{1}{2}[d(Pu,Pu)+d(Pu,u)], \frac{d(Pu,Pu)d(u,u)}{1+d(Pu,u)}\bigg\} = d(Pu,u). \end{aligned}$$

Further,

$$|d(Pu,u)| \leq \frac{\lambda}{s^2}|d(Pu,u)|.$$

So, $(1-\frac{\lambda}{s^2})|d(Pu,u)| \leq 0$, which is a contradiction. Then, $|d(Pu,u)| = 0$, hence, $Pu = u$.
We prove $Su = u$. On the contrary we suppose that $Su \neq u$

$$d(Su,u) \precsim sd(Pu,Tz_{2n+1}) + sd(Tz_{2n+1},u).$$

Using (C_2) with $z = u, w = z_{2n+1}$, we get $d(Su,Tz_{2n+1}) \precsim \frac{\lambda}{s^2}R(u,z_{2n+1})$ where

$$\begin{aligned} R(u,z_{2n+1}) &= \max\{d(Pu,Qz_{2n+1}), d(Pu,Su), d(Qz_{2n+1},Tz_{2n+1}) \\ &\quad , \frac{1}{2}[d(Qz_{2n+1},Su)+d(Pu,Tz_{2n+1})] \\ &\quad , \frac{d(Pu,Su)d(Qz_{2n+1},Tz_{2n+1})}{1+d(Pu,Qz_{2n+1})}\bigg\}. \end{aligned}$$

Passing to the limit as $n \to \infty$ we get

$$\begin{aligned} R(u,u) &= \max\{d(u,u), d(u,Su), d(u,u) \\ &\quad , \frac{1}{2}[d(u,Su)+d(u,u)], \frac{d(u,Su)d(u,u)}{1+d(u,u)}\bigg\} = d(Su,u). \end{aligned}$$

So, $d(Su,u) \precsim \frac{\lambda}{s^2}d(Su,u)$, further, $|d(Su,u)| \leq \frac{\lambda}{s^2}|d(Su,u)|$, which is a contradiction. Then $|d(Su,u)| = 0$, hence, $Su = u$.
We prove $Qu = Tu$, as $S(X) \subset Q(X)$, so there exists $v \in X$ such that $u = Su = Qv$. Firstly, we shall show that $Qv = Tv$ for this we get

$$d(Qv,Tv) = d(Su,Tv) \precsim \frac{\lambda}{s^2}R(u,v),$$

where,

$$\begin{aligned} R(u,v) &= \max\left\{d(Pu,Qv), d(Pu,Su), d(Qv,Tv)\right. \\ &\quad \left., \frac{1}{2}[d(Qv,Su)+d(Pu,Tv)], \frac{d(Pu,Su)d(Qv,Tv)}{1+d(Pu,Qv)}\right\}, \end{aligned}$$

then,

$$\begin{aligned} R(u,v) &= \max\left\{d(Qv,Qv), d(u,u), d(Qv,Tv)\right. \\ &\quad \left., \frac{1}{2}[d(Qv,Qv)+d(Qv,Tv)], \frac{d(u,u)d(Qv,Tv)}{1+d(Qv,Qv)}\right\}. \end{aligned}$$

So, $d(Qv,Tv) \precsim \frac{\lambda}{s^2}d(Qv,Tv)$, further, $|d(Qv,Tv)| \leq \frac{\lambda}{s^2}|d(Qv,Tv)|$, which is a contradiction. Then $|d(Qv,Tv)| = 0$, hence, $Qv = Tv = u$. As the pair (T,Q) is weakly compatible, so we have $TQv = QTv$, therefore $Qu = Tu$.
We prove $u = Tu$, On the contrary we suppose that $Tu \neq u$,

$$d(u,Tu) = d(Su,Tu) \precsim \frac{\lambda}{s^2}R(u,u),$$

where,

$$\begin{aligned} R(u,u) &= \max\left\{d(Pu,Qu), d(Pu,Su), d(Qu,Tu)\right. \\ &\quad \left., \frac{1}{2}[d(Qu,Su)+d(Pu,Tu)], \frac{d(Pu,Su)d(Qu,Tu)}{1+d(Pu,Qu)}\right\}, \end{aligned}$$

then,

$$\begin{aligned} R(u,v) &= \max\left\{d(u,Tu), d(u,u), d(Tu,Tu)\right. \\ &\quad \left., \frac{1}{2}[d(Tu,u)+d(Tu,Tu)], \frac{d(u,u)d(Tu,Tu)}{1+d(u,Tu)}\right\}. \end{aligned}$$

So, $d(u,u) \precsim \frac{\lambda}{s^2}d(u,u)$, further, $|d(u,Tu)| \leq \frac{\lambda}{s^2}|d(u,Tu)|$, which is a contradiction. Then $|d(u,Tu)| = 0$, hence, $u = Tu$.

Now we prove that $Qu = u$, On the contrary we suppose that $Qu \neq u$,, we have

$$d(u, Qu) = d(Su, QTu) = d(Su, TQu).$$

From (C_2) we get

$$d(u, Qu) = d(Su, TQu) \precsim \frac{\lambda}{s^2} R(u, Qu),$$

where,

$$\begin{aligned} R(u, Qu) &= \max\left\{d(Pu, QQu), d(Pu, Su), d(QQu, TQu)\right. \\ &\quad \left., \frac{1}{2}[d(QQu, Su) + d(Pu, TQu)], \frac{d(Pu, Su)d(QQu, TQu)}{1 + d(Pu, QQu)}\right\} \\ &= \max\left\{d(u, Qu), d(u, u), d(Qu, Qu)\right. \\ &\quad \left., \frac{1}{2}[d(Qu, u) + d(u, Qu)], \frac{d(u, u)d(Qu, Qu)}{1 + d(u, Qu)}\right\} = d(u, Qu). \end{aligned}$$

So, $|d(u, Qu)| \leq \frac{\lambda}{s^2}|d(u, Qu)|$, which is contradiction. Then $|d(u, Qu)| = 0$, hence, $u = Qu$. On conclude $Su = Tu = Pu = Qu = u$ when P is continuous, we get the same results when S is continuous.

Now we prove the uniqueness, Let u^* be another common fixed point of S, T, P and Q, then

$$Su^* = Tu^* = Pu^* = Qu^* = u^*$$

Putting $z = u, w = u^*$ in (C_2), we get $d(u, u^*) = d(Su, Tu^*) \precsim \frac{\lambda}{s^2} R(u, u^*)$,
where,

$$\begin{aligned} R(u, u^*) &= \max\left\{d(Pu, Qu^*), d(Pu, Su), d(Qu^*, Tu^*)\right. \\ &\quad \left., \frac{1}{2}[d(Qu^*, Su) + d(Pu, Tu^*)], \frac{d(Pu, Su)d(Qu^*, Tu^*)}{1 + d(Pu, Qu^*)}\right\} \\ &= \max\left\{d(u, u^*), d(u, u), d(u^*, u^*)\right. \\ &\quad \left., \frac{1}{2}[d(u^*, u) + d(u, u^*)], \frac{d(u, u)d(u^*, u^*)}{1 + d(u, u^*)}\right\}. \end{aligned}$$

Further, $|d(u,u^*)| \leq \frac{\lambda}{s^2}|d(u,u^*)|$, which is a contradiction, hence, $|d(u,u^*)| = 0$, which implies that $u = u^*$. Thus, u is the unique common fixed point of S, T, P and Q in X. $\square$

If $S = T$ and $P = Q = I_X$ in Theorem 2.6, we get the following Corollary.

Corollary 2.5. *Let (X, d) be a complete complex valued b-metric space and $S : X \to X$ be a mapping verifying the inequality*

$$d(Sz, Sw) \precsim \frac{\lambda}{s^2} d(z, w)$$

for all $z, w \in X$, where $s \geq 1$ and $\lambda \in (0, 1)$ Then, T has a unique fixed point in X.

Example 2.1. *Let $X = [0.1]$. For all $z, w \in X$, define $d :\ X \times X \to \mathbb{C}$ by*

$$d(z, w) = |z - w|^2 + i\,|z - w|^2 .$$

Now, define the mappings $S, T, P, Q : X \to X$ by

$$Sz = \frac{z}{32},\ Tz = \frac{z^2}{48},\ Pz = \frac{z}{2},\quad and\quad Qz = \frac{z^2}{3}.$$

We have

$$\begin{aligned} d(Sz, Tw) &= \left[\left|\frac{z}{32} - \frac{w^2}{48}\right|^2 + i\left|\frac{z}{32} - \frac{w^2}{48}\right|^2\right] = \frac{1}{256}\left[\left|\frac{z}{2} - \frac{w^2}{3}\right|^2 + i\left|\frac{z}{2} - \frac{w^2}{3}\right|^2\right], \\ d(Pz, Qw) &= \left[\left|\frac{z}{2} - \frac{w^2}{3}\right|^2 + i\left|\frac{z}{2} - \frac{w^2}{3}\right|^2\right], \\ d(Sz, Tw) &= \frac{1}{256} d(Pz, Qw). \end{aligned}$$

Thus, all the conditions of theorem 2.6 are satisfied, where $\lambda = \frac{1}{64}$ and $s = 2$. Then legibly $'0'$ is the unique common fixed point of the mappings S, T, P and Q.

Example 2.2. *Let $X = B(0, r), r > 1$. For all $z, w \in X$, define $d : X \times X \to \mathbb{C}$ by*

$$d(z(u), w(u)) = \frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u} du - \int_\Gamma \frac{w(u)}{u} du\right|^2 ,$$

a complex valued b-metric where Γ *is a closed path in* X *containing a zero.*
We prove that d *is a complex valued b-metric with* $s = 2$

$$\begin{aligned}
d(z(u), w(u)) &= \frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u}du - \int_\Gamma \frac{w(u)}{u}du\right|^2 \\
&= \frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u}du - \int_\Gamma \frac{x(u)}{u}du + \int_\Gamma \frac{x(u)}{u}du - \int_\Gamma \frac{w(u)}{u}du\right|^2 \\
&\precsim \frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u}du - \int_\Gamma \frac{x(u)}{u}du\right|^2 + \frac{i}{2\pi}\left|\int_\Gamma \frac{x(u)}{u}du - \int_\Gamma \frac{w(u)}{u}du\right|^2 + \\
&\quad 2\frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u}du - \int_\Gamma \frac{x(u)}{u}du\right|\left|\int_\Gamma \frac{x(u)}{u}du - \int_\Gamma \frac{w(u)}{u}du\right| \\
&\precsim \frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u} - \int_\Gamma \frac{x(u)}{u}\right|^2 + \frac{i}{2\pi}\left|\int_\Gamma \frac{x(u)}{u}du - \int_\Gamma \frac{w(u)}{u}du\right|^2 + \\
&\quad \frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u}du - \int_\Gamma \frac{x(u)}{u}du\right|^2 + \frac{i}{2\pi}\left|\int_\Gamma \frac{x(u)}{u}du - \int_\Gamma \frac{w(u)}{u}du\right|^2 \\
&\precsim 2\left\{\frac{i}{2\pi}\left|\int_\Gamma \frac{z(u)}{u}du - \int_\Gamma \frac{x(u)}{u}du\right|^2 + \frac{i}{2\pi}\left|\int_\Gamma \frac{x(u)}{u}du - \int_\Gamma \frac{w(u)}{u}du\right|^2\right\} \\
d(z(u), w(u)) &\precsim 2\left\{d(z(u), x(u)) + d(x(u), w(u))\right\}.
\end{aligned}$$

Now we define the mappings $S, T, P, Q : X \to X$ *by*

$$Sz(u) = u, Tz(u) = e^{\frac{u}{2}}, Pz(u) = e^u - 1 \quad \text{and} \quad Qz(u) = u^2 + \frac{1}{2}u.$$

Using the Cauchy formula when the mappings S, T, P *and* Q *are analytics we get*

$$\begin{aligned}
d(Sz(u), Tw(u)) &= \frac{i}{2\pi}\left|\int_\Gamma \frac{u}{u}du - \int_\Gamma \frac{e^u - 1}{u}du\right|^2 = 0, \\
d(Pz(u), Qw(u)) &= \frac{i}{2\pi}\left|\int_\Gamma \frac{e^{\frac{u}{2}}}{u}du - \int_\Gamma \frac{u^2 + \frac{1}{2}u}{u}du\right|^2 = \frac{(2\pi)^2 i}{2\pi}, \\
d(Pz(u), Sz(u)) &= \frac{i}{2\pi}\left|\int_\Gamma \frac{e^{\frac{u}{2}}}{u}du - \int_\Gamma \frac{u}{u}du\right|^2 = 0,
\end{aligned}$$

$$
\begin{aligned}
d(Qw(u),Tw(u)) &= \frac{i}{2\pi}\left|\int_\Gamma \frac{u^2+\frac{1}{2}u}{u}du - \int_\Gamma \frac{e^u-1}{u}du\right|^2 = 0,\\
d(Qw(u),Sz(u)) &= \frac{i}{2\pi}\left|\int_\Gamma \frac{u^2+\frac{1}{2}u}{u}du - \int_\Gamma \frac{u}{u}du\right|^2 = 0,\\
d(Pz(u),Tw(u)) &= \frac{i}{2\pi}\left|\int_\Gamma \frac{e^{\frac{u}{2}}}{u}du - \int_\Gamma \frac{e^u-1}{u}du\right|^2 = \frac{(2\pi)^2 i}{2\pi},\\
R(z(u),w(u)) &= \max\{2\pi i, 0\} = 2\pi i.
\end{aligned}
$$

Further,

$$0 = d(Sz(u),Tw(u)) \precsim \frac{\pi\lambda i}{2}.$$

As all the conditions of theorem 2.6 are satisfied, the mappings S, T, P and Q have a unique common fixed point in X.

Chapter 3

Common fixed point theorems under Pata's contraction type

3.1 Introduction

Pata [7] gave a new weak contraction compared to renowned Banach contraction [42], in metric space. In addition Kadelburg and Radenovic in [43–45], developed common fixed point results of Pata contraction condition for two self-mappings in metric spaces. Based on these results we have showed common fixed point theorems of Pata contraction type for two self-mappings in complex valued metric spaces [27] and we generalized Kadelburg and Radenovic's theorem which is under Pata's contraction in complex valued b-metric spaces.

3.2 Common fixed point theorems in complex metric space

In this section, we give the main results published in [27], in complex valued metric spaces using Pata contraction condition for two self-mappings using the notion of weak compatibility. Two examples involving this theorem are given.

We started this section by giving some theorems about Pata's contraction condition. Select an arbitrary point $z_0 \in X$, and $\|z\| = d(z, z_0)$, for all $z \in X$.

In the sequel of this section, $\Gamma : [0, 1] \to [0, \infty)$ be a nondecreasing function, continuous at zero, with $\Gamma(0) = 0$.

Pata in [7] obtained the following refinement of the classical Banach contraction principle.

Theorem 3.1. *Let S be a self-mapping in a complete metric space (X, d). Assume that*

$\tau \geqslant 0$, $\alpha \geqslant 1$ and $\beta \in [0, \alpha]$ are fixed constants such that the following inequality

$$d(Sz, Sw) \leqslant (1-\epsilon)d(z, w) + \tau\epsilon^{\alpha}\Gamma(\epsilon)[1 + \|z\| + \|w\|]^{\beta}, \tag{3.1}$$

is satisfied for each $\epsilon \in [0, 1]$ and for all $z, w \in X$. Then S has a unique fixed point $u \in X$. Furthermore, the sequence $\{S^n z_0\}$ converges to u for all $z_0 \in X$..

Theorem 3.2 (Boyd-Wong)**.** *Let $\varrho : \mathbb{R}^+ \to [0, \infty)$ be a continuous function satisfying the inequality $\varrho(r) < r$ for every $r > 0$. If*

$$d(S(z), S(w)) \leqslant \varrho(d(z, w)) \quad \forall\ z \neq w. \tag{3.2}$$

Then, the mapping S has a unique fixed point $u \in X$ and $d(u, S^n(z_0)) \to 0$ for every $z_0 \in X$.

Proposition 3.1. *If X is a bounded metric space, Theorems 3.1 and 3.2 implies each other.*

Proof. [7] ☐

Chakraborty and Samanta extended in [46] the result of Pata to the case of Kannan-type contraction condition.
Kadelburg and Radenovic in [47] showed the following Chatterjea type fixed point theorem.

Theorem 3.3. *Let S be a self-mapping in a complete metric space (X, d). Assume that $\tau \geqslant 0$, $\alpha \geqslant 1$ and $\beta \in [0, \alpha]$ are fixed constants such that the following inequality*

$$d(Sz, Sw) \leqslant \frac{1-\epsilon}{2}(d(z, Sw) + d(w, Sz)) + \tau\epsilon^{\alpha}\Gamma(\epsilon)[1 + \|z\| + \|w\| + \|Sz\| + \|Sw\|]^{\beta}, \tag{3.3}$$

is verified for each $\epsilon \in [0, 1]$ and for all $z, w \in X$. Then S has a unique fixed point in $u \in X$.

Kadelburg and Radenovic in [47] proved the following common fixed point theorem under Pata contraction condition

Theorem 3.4. *Let S and T be two self mappings of the given metric space (X, d), such that $SX \subset TX$ and at least one of these sub-spaces of X is complete. Assume that $\tau \geqslant 0$, $\alpha \geqslant 1$ and $\beta \in [0, \alpha]$ are fixed constants such that the following inequality*

$$d(Sz, Sw) \leqslant (1-\epsilon)d(Tz, Tw) + \tau\epsilon^{\alpha}\Gamma(\epsilon)[1 + \|Tz\| + \|Tw\|]^{\beta}, \tag{3.4}$$

is satisfied for each $\epsilon \in [0, 1]$ and for all $z, w \in X$. Then the pair (S, T) has a unique point of coincidence in X. Moreover, if the pair (S, T) is weakly compatible, then S and T have a unique common fixed point in X.

To prove our main theorem, we start with the following lemma

Lemma 3.1. *Let (X, d) be a complex valued metric space and $\{w_n\}$ be a sequence in X such that the sequence $\{d(w_{n+1}, w_n)\}$ is nonincreasing and*

$$\lim_{n\to\infty} d(w_{n+1}, w_n) = 0. \tag{3.5}$$

If $\{w_{2n}\}$ is not a Cauchy sequence, then there exists $\epsilon > 0$ and two sequences $\{m_k\}$, $\{n_k\}$ of positive integers such that the following four sequences tend to ϵ as $k \to \infty$

$$d(w_{2m_k}, w_{2n_k}), d(w_{2m_k}, w_{2n_k+1}), d(w_{2m_k-1}, w_{2n_k}), d(w_{2m_k-1}, w_{2n_k+1}). \tag{3.6}$$

Proof. If $\{w_{2n}\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and two sequences $\{m_k\}$, $\{n_k\}$ of positive integers such that

$$n_k > m_k,\ d(w_{2m_k}, w_{2n_k-2}) \prec \epsilon,\ d(w_{2m_k}, w_{2n_k}) \succsim \epsilon,$$

for all positive integer k.
Therefore

$$\begin{aligned} \epsilon \precsim d(w_{2m_k}, w_{2n_k}) &\precsim d(w_{2m_k}, w_{2n_k-2}) + d(w_{2n_k-2}, w_{2n_k-1}) + d(w_{2n_k-1}, w_{2n_k}) \\ &\prec \epsilon + d(w_{2n_k-2}, w_{2n_k-1}) + d(w_{2n_k-1}, w_{2n_k}). \end{aligned}$$

Passing to the limit as $k \to \infty$ and using (3.6), we get

$$\epsilon \precsim \lim_{k\to\infty} d(w_{2m_k}, w_{2n_k}) \precsim \epsilon.$$

So,

$$\begin{aligned} \mid \epsilon \mid \leq \lim_{n\to\infty} \mid d(w_{2m_k}, w_{2n_k}) \mid \ &\leq \ \mid \epsilon \mid \\ \lim_{n\to\infty} \mid d(w_{2m_k}, w_{2n_k}) \mid \ &= \ \mid \epsilon \mid, \end{aligned}$$

hence,

$$\lim_{n\to\infty} d(w_{2m_k}, w_{2n_k}) = \epsilon. \tag{3.7}$$

Further,

$$d(w_{2m_k}, w_{2n_k+1}) \precsim d(w_{2m_k}, w_{2n_k}) + d(w_{2n_k}, w_{2n_k+1}).$$

Passing to the limit as $k \to \infty$ and using (3.5) and (3.7), we obtain

$$\lim_{n\to\infty} d(w_{2m_k}, w_{2n_k+1}) = \epsilon.$$

Similarly we can prove that the remaining two sequences in (3.6) tend to ϵ. □

Now we present our main theorem
Let $\Gamma : [0,1] \to [0,\infty)$ be a nondecreasing function, continuous at zero, with $\Gamma(0) = 0$.

Theorem 3.5. *Let S and T be two self-mappings in a complex valued metric space (X, d), such that $S(X) \subset T(X)$, $T(X)$ is a complete subspace of X. Assume that $\tau \geqslant 0$, $\alpha \geqslant 1$ and $\beta \in [0, \alpha]$ be fixed constants such that the following inequality*

$$d(Sz, Sw) \precsim (1-\epsilon)d(Tz, Tw) + \tau\epsilon^{\alpha}\Gamma(\epsilon)[1 + \|Tz\| + \|Tw\|]^{\beta}, \tag{3.8}$$

is verified for each $\epsilon \in [0,1]$ and for all $z, w \in X$. Then, the pair (S,T) has a unique point of coincidence in X. Moreover, if the pair (S,T) is weakly compatible. Then, S and T have a unique common fixed point in X.

Proof. Let $z_0 \in X$ be an arbitrary initial point. When $w_0 = Sz_0$, putting $\|z\| = d(z, w_0)$. Using $S(X) \subset T(X)$ we can define a sequence $\{w_n\}$ such that $w_n = Sz_n = Tz_{n+1}$, $n = 0, 1, 2, \cdots$

1. The uniqueness

Let v_1 and v_2 two points of coincidence of the pair (S,T), then $v_1 = Su_1 = Tu_1$ and $v_2 = Su_2 = Tu_2$. Applying (3.8) we have

$$\begin{aligned} d(v_1, v_2) &= d(Su_1, Su_2) \precsim (1-\epsilon)d(Tu_1, Tu_2) + \tau\epsilon^{\alpha}\Gamma(\epsilon)[1 + \|Tu_1\| + \|Tu_2\|]^{\beta} \\ &= (1-\epsilon)d(v_1, v_2) + C\epsilon^{\alpha}\Gamma(\epsilon), \end{aligned}$$

where, $C = \tau[1 + \|Tu_1\| + \|Tu_2\|]^{\beta} \succ 0$. Therefore

$$d(v_1, v_2) \precsim C\epsilon^{\alpha-1}\Gamma(\epsilon),$$

hence,

$$|d(v_1, v_2)| \leq |C|\epsilon^{\alpha-1}\Gamma(\epsilon).$$

Letting $\epsilon \to 0$ and using the properties of the function Γ, it follows that $v_1 = v_2$.

2. The existence

A. We prove that the sequence $\{d(w_{n+1}, w_n)\}$ is nonincreasing.

Putting $\epsilon = 0, z = z_{n+1}, w = z_n$ in (3.8) we get

$$\begin{aligned} d(Sz_{n+1}, Sz_n) &\precsim d(Tz_{n+1}, Tz_n), \ \textit{i.e.}, \\ d(w_{n+1}, w_n) &\precsim d(w_n, w_{n-1}). \end{aligned}$$

So, we deduce by induction that, for all $n \in \mathbb{N}$

$$d(w_{n+1}, w_n) \precsim d(w_n, w_{n-1}) \precsim d(w_{n-1}, w_{n-2}) \cdots \precsim d(w_1, w_0). \tag{3.9}$$

Then, the sequence $\{d(w_{n+1}, w_n)\}$ is nonincreasing.

B. We prove that the sequence $\{c_n\}$ where $c_n = d(w_n, w_0)$ is bounded.

Using (3.9), we obtain the following estimation

$$\begin{aligned} c_n = d(w_n, w_0) &\precsim d(w_n, w_{n+1}) + d(w_{n+1}, w_1) + d(w_1, w_0) \\ &= d(w_n, w_{n+1}) + d(Sz_{n+1}, Sz_1) + d(w_1, w_0). \end{aligned}$$

Using (3.8) we obtain

$$\begin{aligned} c_n &\precsim 2c_1 + (1-\epsilon)d(Tz_{n+1}, Tz_1) + \tau\epsilon^{\alpha}\Gamma(\epsilon)[1 + \|Tz_{n+1}\| + \|Tz_1\|]^{\beta} \\ &= 2c_1 + (1-\epsilon)d(w_n, w_0) + \tau\epsilon^{\alpha}\Gamma(\epsilon)[1 + \|w_n\| + \|w_0\|]^{\beta} \\ &\precsim 2c_1 + (1-\epsilon)c_n + \tau\epsilon^{\alpha}\Gamma(\epsilon)[1 + c_n]^{\alpha} \\ &\precsim (1-\epsilon)c_n + a\epsilon^{\alpha}\Gamma(\epsilon)c_n^{\alpha} + b, \end{aligned}$$

where $a \succ 0$ and $b \succ 0$. Hence,

$$\epsilon c_n \precsim a\epsilon^{\alpha}\Gamma(\epsilon)c_n^{\alpha} + b.$$

Therefore,

$$\epsilon|c_n| \leq |a|\epsilon^{\alpha}\Gamma(\epsilon)|c_n^{\alpha}| + |b|.$$

If there is a subsequence $\{c_{n_k}\}$ such that $c_{n_k} \to \infty$, the choice $\epsilon = \epsilon_k = \frac{|1+b|}{|c_{n_k}|}$ leads to the contradiction $1 \leq |a||1+b|^{\alpha}\Gamma(\epsilon_k) \to 0$. Then the sequence $\{c_n\}$ is bounded.

C. We show that $\delta = \lim_{n\to\infty} d(w_{n+1}, w_n) = 0$.

We have

$$\begin{aligned} d(w_{n+1}, w_n) &= d(Sz_{n+1}, Sz_n) \\ &\precsim (1-\epsilon)d(Tz_{n+1}, Tz_n) + \tau\epsilon^{\alpha}\Gamma(\epsilon)[1 + \|Tz_{n+1}\| + \|Tz_n\|]^{\beta}. \end{aligned}$$

Using the fact that $\{c_n\}$ is bounded and modifying the constant C, we get

$$d(w_{n+1}, w_n) \precsim (1-\epsilon)d(w_n, w_{n-1}) + C\epsilon^{\alpha}\Gamma(\epsilon).$$

Further,

$$|d(w_{n+1}, w_n)| \leq (1-\epsilon)|d(w_n, w_{n-1})| + |C|\epsilon^{\alpha}\Gamma(\epsilon).$$

Passing to the limit as $n \to \infty$ we obtain

$$\begin{aligned} |\delta| &\leq (1-\epsilon)|\delta| + \epsilon^{\alpha}|C|\Gamma(\epsilon) \\ |\delta| &\leq \epsilon^{\alpha-1}|C|\Gamma(\epsilon). \end{aligned}$$

Using the properties of the function Γ, we get $|\delta| \leq 0$ which is a contradiction, hence $\delta = 0$.

D. We prove that the sequence $\{w_n\}_{n\in\mathbb{N}}$ is a Cauchy.

By Lemma 3.1, choosing $\epsilon > 0$ and $\{m_k\}$, $\{n_k\}$ two sequences of positive integers. Putting $z = z_{2m_k-1}, w = z_{2n_k}$ in (3.8) we get

$$d(w_{2m_k-1}, w_{2n_k}) \precsim (1-\epsilon)d(w_{2n_k-2}, w_{2n_k-1}) + C\epsilon^{\alpha}\Gamma(\epsilon).$$

Further,

$$|d(w_{2m_k-1}, w_{2n_k})| \leq (1-\epsilon)|d(w_{2n_k-2}, w_{2n_k-1})| + |C|\epsilon^{\alpha}\Gamma(\epsilon).$$

Passing to the limit as $k \to \infty$ we obtain

$$\begin{aligned} |\delta| &\leq (1-\epsilon)|\delta| + |C|\epsilon^{\alpha}\Gamma(\epsilon) \\ |\delta| &\leq \epsilon^{\alpha-1}|C|\Gamma(\epsilon). \end{aligned}$$

Using the properties of the function Γ, we get $|\delta| \leq 0$ which is a contradiction, then $\delta = 0$. Hence $\{w_n\}$ is a Cauchy sequence.

E. We prove that ν is a common fixed point of S and T.

Assume that, $T(X)$ is a complete subspace of X, therefore $w_n = Sz_n \to \nu = T\xi$ for some $\xi \in X$. Putting $\epsilon = 0, z = z_n, w = \xi$ in (3.8) we get

$$d(Sz_n, S\xi) \precsim d(Tz_n, T\xi).$$

Hence,

$$|d(Sz_n, S\xi)| \leq |d(Tz_n, T\xi)|.$$

Taking the limit as $n \to \infty$, we obtain $S\xi = T\xi = \nu$, which is the unique point of coincidence of S and T (by the uniqueness). Since (S, T) is weakly compatible by proposition 1.1 of Abbas and Jungck [48] ν is the unique common fixed point of S and T. ☐

Taking $T = I_X$ (the identity mapping in X) in theorem 3.5, we get the following corollary which is an extension of the theorem 3.1 of Pata [7] from metric space to the framework of complex valued metric spaces.

Corollary 3.1. *Let S be a self-mapping in a complete complex valued metric space (X, d). Assume that $\tau \geqslant 0$, $\alpha \geqslant 1$ and $\beta \in [0, \alpha]$ are fixed constants such that the inequality*

$$d(Sz, Sw) \precsim (1-\epsilon)d(z, w) + \tau\epsilon^{\alpha}\Gamma(\epsilon)[1 + \|z\| + \|w\|]^{\beta},$$

holds for each $\epsilon \in [0, 1]$ and for all $z, w \in X$. So, S has a unique fixed point $u \in X$. Furthermore, the sequence $\{S^n(z_0)\}$ converges to u for all $z_0 \in X$.

In a similar manner, we can prove the following theorem of Pata-Kannan type for two mappings.

Theorem 3.6. *Let S and T be two self-mappings in a complex valued metric space (X,d) such that $S(X)\subset T(X)$ and $T(X)$ is a complete subspace of X. Suppose that $\tau\geqslant 0,\alpha\geqslant 1$ and $\beta\in[0,\alpha]$ are fixed constants such that the inequality*

$$\begin{aligned}d(Sz,Sw) &\precsim \frac{(1-\epsilon)}{2}(d(Sz,Tz)+d(Sw,Tw))+\\ &\quad \tau\epsilon^{\alpha}\Gamma(\epsilon)[1+\|Sz\|+\|Sw\|+\|Tz\|+\|Tw\|]^{\beta},\end{aligned}$$

is verified for each $\epsilon\in[0,1]$ and for all $z,w\in X$. So, the pair (S,T) has a unique point of coincidence in X. Moreover, if the pair (S,T) is weakly compatible. Then, S and T have a unique common fixed point in X.

Example 3.1. *Let $X=C([1,3],\mathbb{R}),a>0$ and $d:X\times X\to\mathbb{C}$ defined as follows*

$$d(z,w)=\max_{t\in[1,3]}|z(t)-w(t)|\sqrt{1+a^2}e^{itan^{-1}a}.$$

Define $S,T:X\to X$ by

$$Sz(t)=4+\int_1^t(z(s)+s^2)e^{s-1}ds,t\in[1,3],$$

$$Tz(t)=2+\int_1^t(z(s)+s^2+s)e^{2s-1}ds,t\in[1,3].$$

For every $z,w\in X$, we have

$$\begin{aligned}d(Sz,Sw) &= \max_{t\in[1,3]}|Sz(t)-Sw(t)|\sqrt{1+a^2}e^{i\tan^{-1}a}\\ &= \int_1^3 e^2ds\ \max_{t\in[1,3]}|z(t)-w(t)|\sqrt{1+a^2}e^{i\tan^{-1}a}\\ &= 2e^2d(z,w),\\ d(Tz,Tw) &= \max_{t\in[1,3]}|Tz(t)-Tw(t)|\sqrt{1+a^2}e^{i\tan^{-1}a}\\ &= \int_1^3 e^5ds\ \max_{t\in[1,3]}|z(t)-w(t)|\sqrt{1+a^2}e^{i\tan^{-1}a}\\ &= 2e^5d(z,w).\end{aligned}$$

Therefore,

$$d(Sz,Sw) \precsim \frac{1}{e^4} d(Tz,Tw),$$

which implies that

$$d(Sz,Sw) \precsim \frac{1}{e^4} d(Tz,Tw) + \tau\epsilon^{\alpha}\Gamma(\epsilon)[1+\|Tz\|+\|Tw\|]^{\beta}.$$

So for $\epsilon = 1 - \frac{1}{e^4}$ *and for every* τ, α. *all the conditions of theorem 3.5 are satisfied. Consequently,* S *and* T *have a unique fixed point in* X*, which is the unique common solution of the integral equations*

$$\begin{cases} z(t) = 4 + \int_1^t (z(s)+s^2)e^{s-1}ds, & t \in [1,3], \\ z(t) = 2 + \int_1^t (z(s)+s^2+s)e^{2s-1}ds, & t \in [1,3]. \end{cases}$$

Or the system of differential equations

$$\begin{cases} z^{'}(t) = (z(t)+t^2)e^{t-1}, & t \in [1,3], \quad z(1) = 4, \\ \\ z^{'}(t) = (z(t)+t^2+t)e^{2t-1}, & t \in [1,3], \quad z(1) = 2. \end{cases}$$

Example 3.2. *Let* $X = B(0,r), r > 1$ *for every* $z, w \in X$*. Define* $d: X \times X \to \mathbb{C}$ *by*

$$d(z(u), w(u)) = \frac{i}{2\pi}\left| \int_\Gamma \frac{z(u)}{u}du - \int_\Gamma \frac{w(u)}{u}du \right|,$$

where Γ *is a closed path in* X *containing zero. Define the mappings* $S, T : X \to X$ *by*

$$Sz(u) = e^{\frac{u}{2}},\ Tz(u) = u^2 + \frac{1}{2}u.$$

Using the Cauchy formula when the mappings S *and* T *are analytics we get*

$$\begin{aligned} d(Sz(u), Sw(u)) &= \frac{i}{2\pi}\left| \int_\Gamma \frac{e^{\frac{u}{2}}}{u}du - \int_\Gamma \frac{e^{\frac{u}{2}}}{u}du \right| du = 0, \\ d(Sz(u), Tz(u)) &= \frac{i}{2\pi}\left| \int_\Gamma \frac{e^{\frac{u}{2}}}{u}du - \int_\Gamma \frac{u^2+\frac{1}{2}u}{u}du \right| = \frac{2\pi i}{2\pi} = i, \\ d(Sw(u), Tw(u)) &= \frac{i}{2\pi}\left| \int_\Gamma \frac{e^{\frac{u}{2}}}{u}du - \int_\Gamma \frac{u^2+\frac{1}{2}u}{u}du \right| = \frac{2\pi i}{2\pi} = i, \end{aligned}$$

$$0 = d(Sz(u), Tw(u)) \precsim \frac{(1-\epsilon)}{2}(2i) + C\epsilon^{\alpha}\Gamma(\epsilon)$$

where

$$C = \tau[1 + \|Sz\| + \|Sw\| + \|Tz\| + \|Tw\|]^{\beta} \succ 0$$

. Hence, all the conditions of theorem 3.6 are satisfied and the mappings S and T have unique common fixed point in X.

3.3 Common fixed point theorems in complex b-metric spaces

In this section, we give new results in complex valued b-metric spaces using Pata contraction condition for two mappings using the notion weak compatibility.
We begin this section by proving the following lemma which is used to prove our new theorem

Lemma 3.2. *Let (X, d) be a complex valued b-metric space and $\{w_n\}$ be a sequence in X such that the sequence*

$$\lim_{n\to\infty} d(w_n, w_{n+1}) = 0 \tag{3.10}$$

If $\{w_n\}$ is not a Cauchy sequence, there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences

$$d(w_{m(k)}, w_{n(k)}),\ d(w_{m(k)}, w_{n(k)+1}),\ d(w_{m(k)+1}, w_{n(k)}) \text{ and } d(w_{m(k)+1}, w_{n(k)+1}).$$

verify

$$\begin{aligned}
\epsilon &\precsim \lim_{k\to\infty} \inf d(w_{m(k)}, w_{n(k)}) \precsim \lim_{k\to\infty} \sup d(w_{m(k)}, w_{n(k)}) \precsim s\epsilon,\\
\frac{\epsilon}{s} &\precsim \lim_{k\to\infty} \inf d(w_{m(k)}, w_{n(k)+1}) \precsim \lim_{k\to\infty} \sup d(w_{m(k)}, w_{n(k)+1}) \precsim s^2\epsilon,\\
\frac{\epsilon}{s} &\precsim \lim_{k\to\infty} \inf d(w_{m(k)+1}, w_{n(k)}) \precsim \lim_{k\to\infty} \sup d(w_{m(k)+1}, w_{n(k)}) \precsim s^2\epsilon,\\
\frac{\epsilon}{s^2} &\precsim \lim_{k\to\infty} \inf d(w_{m(k)+1}, w_{n(k)+1}) \precsim \lim_{k\to\infty} \sup d(w_{m(k)+1}, w_{n(k)+1}) \precsim s^3\epsilon.
\end{aligned}$$

If $s = 1$, the above four sequences tend to ϵ as $k \to \infty$

Remark 3.1. *If s=1 in lemma 3.2, we find lemma 3.1.*

Proof. If w_n is not a Cauchy sequence, there exist $\epsilon > 0$ and sequences $m(k)$, $n(k)$ of positive integers such that

$$m(k) > n(k) > k, \quad d(w_{m(k)}, w_{n(k)-1}) \prec \epsilon, \; d(w_{m(k)}, w_{n(k)-1}) \succsim \epsilon \tag{3.11}$$

for all positive integer k. Now, from (3.11) and using the triangle inequality we have

$$\epsilon \precsim d(w_{m(k)}, w_{n(k)}) \precsim s[d(w_{m(k)}, w_{n(k)-1}) + d(w_{n(k)-1}, w_{m(k)})] \prec s\epsilon + sd(w_{n(k)-1}, w_{n(k)}). \tag{3.12}$$

Taking the upper and lower and lower limits as $k \to \infty$ in (3.12) and using (3.10) we obtain that

$$\epsilon \precsim \lim_{k\to\infty} \inf d(w_{m(k)}, w_{n(k)}) \precsim \lim_{k\to\infty} \inf d(w_{m(k)}, w_{n(k)}) \precsim s\epsilon. \tag{3.13}$$

Using the triangle inequality again we have

$$\begin{aligned} d(w_{m(k)}, w_{n(k)}) &\precsim s[d(w_{m(k)}, w_{n(k)+1}) + d(w_{n(k)+1}, w_{n(k)})] \\ &\precsim s^2[d(w_{m(k)}, w_{n(k)}) + d(w_{n(k)+1}, w_{n(k)})] + sd(w_{n(k)+1}, w_{n(k)}). \end{aligned}$$

Taking the upper and lower limits as $k \to \infty$ in (3.14) and using (3.10) and (3.13), we have

$$\epsilon \precsim s \lim_{k\to\infty} \sup d(w_{m(k)}, w_{n(k)+1}) \precsim s^3\epsilon,$$

or equivalently,

$$\frac{\epsilon}{s} \precsim \lim_{k\to\infty} \sup d(w_{m(k)}, w_{n(k)+1}) \precsim s^2\epsilon.$$

The remaining two conditions of the lemma can be proved in a similar way ☐

Kadelburg and Radenovic in [47] present a common fixed point theorem under Pata contraction condition

Theorem 3.7. *Let (X, d) be a complete b-metric space with the a parameter $(s \geq 1)$ and $S, T : X \to X$ be two self-mappings such that $S(X) \subseteq T(X)$. Assume that for some fixed constants $\tau \geqslant 0$, $\alpha \geqslant 1$ and $\beta \in [0, \alpha]$ the inequality*

$$\begin{aligned} d(Sz, Sw) \;\leqslant\; & \frac{(1-\epsilon)}{s} \max\left\{\frac{d(Tz, Tw)}{2s}, d(Tz, Sz), d(Tw, Sw)\right\} \\ & + \tau\epsilon^{\alpha}\Gamma(\epsilon)[1 + \|Sz\| + \|Sw\| + \|Tz\| + \|Tw\|]^{\beta}, \end{aligned} \tag{3.14}$$

is satisfied for each $\epsilon \in [0,1]$ and for all $z, w \in X$. Then, S and T have a unique point of coincidence in X. Moreover, if the pair (S,T) is weakly compatible. Then, S and T have a unique common fixed point in X.

Now we establish a common fixed point theorem under Pata contraction condition in complex valued b-metric spaces.

Theorem 3.8. *Let (X,d) be a complete complex valued b-metric space with a parameter $(s \geq 1)$ and $S,T : X \to X$ be two mappings such that $S(X) \subseteq T(X)$. Suppose that $\tau \geqslant 0, \alpha \geqslant 1$ and $\beta \in [0,\alpha]$ are fixed constants such that the inequality*

$$\begin{aligned} d(Sz,Sw) \quad &\precsim \quad \frac{(1-\epsilon)}{s}\max\left\{d(Tz,Sz), d(Tw,Sw), \frac{d(Tz,Tw)}{2s}, \frac{d(Tz,Sz)d(Sw,Tw)}{1+d(Sz,Tz)}\right\} \\ &\quad +\tau\epsilon^{\alpha}\Gamma(\epsilon)[1+\|Sz\|+\|Sw\|+\|Tz\|+\|Tw\|]^{\beta}, \end{aligned} \tag{3.15}$$

is verified for each $\epsilon \in [0,1]$ and for all $z, w \in X$. So, S and T have a unique point of coincidence in X. Moreover, if the pair (S,T) is weakly compatible. Then S and T have a unique common fixed point in X.

Proof. Let $z_0 \in X$ be an arbitrary initial point, when $w_0 = Sz_0$ putting $\|z\| = d(z, w_0)$. Using $S(X) \subset T(X)$ we can define a sequence $\{w_n\}$ such that $w_n = Sz_n = Tz_{n+1}$, $n = 0, 1, 2, \cdots$

1. The uniqueness

Let v_1 and v_2 be two points of coincidence of the pair (S,T), then $v_1 = Su_1 = Tu_1$ and $v_2 = Su_2 = Tu_2$. Applying (3.15) we have

$$\begin{aligned} d(v_1,v_2) \quad &= \quad d(Su_1,Su_2) \\ &\precsim \quad \frac{(1-\epsilon)}{s}\max\left\{d(Tu_1,Su_1), d(Tu_2,Su_2), \frac{d(Tu_1,Tu_2)}{2s}, \frac{d(Tu_1,Su_1)d(Su_2,Tu_2)}{1+d(Su_1,Tu_1)}\right\} \\ &\quad +\tau\epsilon^{\alpha}\Gamma(\epsilon)[1+\|Su_1\|+\|Su_2\|+\|Tu_1\|+\|Tu_2\|]^{\beta}, \end{aligned}$$

where, $C = \tau[1+\|Su_1\|+\|Su_2\|+\|Tu_1\|+\|Tu_2\|]^{\beta} \succ 0$. Therefore

$$d(v_1,v_2) \precsim C\epsilon^{\alpha-1}\Gamma(\epsilon),$$

hence,

$$|d(v_1, v_2)| \leq 2s^2 C\epsilon^{\alpha-1}\Gamma(\epsilon).$$

Letting $\epsilon \to 0$ and using the properties of the function Γ, it follows that $v_1 = v_2$.

2. The existence

A. We prove that the sequence $\{d(w_{n+1}, w_n)\}$ is nonincreasing.

Putting $\epsilon = 0, z = z_{n+1}, w = z_n$ in (3.15) we get

$$\begin{aligned} d(Sz_{n+1}, Sz_n) &\precsim \frac{1}{s}\max\{d(Tz_{n+1}, Sz_{n+1}) \\ &\quad , d(Tz_n, Sz_n), \frac{d(Tz_{n+1}, Tz_n)}{2s}, \frac{d(Tz_{n+1}, Sz_{n+1})d(Sz_n, Tz_n)}{1 + d(Sz_{n+1}, Tz_{n+1})}\Bigg\}, \\ d(w_{n+1}, w_n) &\precsim \frac{1}{s}\max\{d(w_n, w_{n+1}) \\ &\quad , d(w_{n-1}, w_n), \frac{d(w_n, w_{n-1})}{2s}, \frac{d(w_n, w_{n+1})d(w_n, w_{n-1})}{1 + d(w_{n+1}, w_n)}\Bigg\}, \end{aligned}$$

we have

$$\frac{d(w_n, w_{n+1})d(w_n, w_{n-1})}{1 + d(w_{n+1}, w_n)} \precsim d(w_n, w_{n-1}).$$

Then,

$$d(w_{n+1}, w_n) \precsim \frac{1}{s}d(w_n, w_{n-1}).$$

So, we deduce by induction that, for all $n \in \mathbb{N}$

$$d(w_{n+1}, w_n) \precsim \frac{1}{s}d(w_n, w_{n-1}) \precsim \frac{1}{s^2}d(w_{n-1}, w_{n-2}) \cdots \precsim \frac{1}{s^n}d(w_1, w_0). \tag{3.16}$$

Therefore, the sequence $\{d(w_{n+1}, w_n)\}$ is nonincreasing.

B. We prove that the sequence $\{c_n\}$ where $c_n = d(w_n, w_0)$ is bounded.

Using (3.15), we obtain the following estimation

$$\begin{aligned} c_{n+1} = d(w_{n+1}, w_0) &\precsim s(d(w_{n+1}, w_1) + d(w_1, w_0)) \\ &= s(d(Sz_{n+1}, Sz_1) + d(w_1, w_0)). \end{aligned}$$

Then, again taking $\epsilon = 0$, we obtain that

$$\begin{aligned}
c_{n+1} &\precsim sd(w_1, w_0) \\
&\quad + s\frac{1}{s}\max\left\{d(Tz_{n+1}, Sz_{n+1}), d(Tz_1, Sz_1), \frac{d(Tz_{n+1}, Tz_1)}{2s}, \frac{d(Tz_{n+1}, Sz_{n+1})d(Sz_1, Tz_1)}{1 + d(Sz_{n+1}, Tz_{n+1})}\right\} \\
&= \max\left\{d(w_n, w_{n+1}), d(w_0, w_1), \frac{d(w_n, w_0)}{2s}, \frac{d(w_n, w_{n+1})d(w_1, w_0)}{1 + d(w_{n+1}, w_n)}\right\} + sd(w_1, w_0) \\
&\precsim d(w_1, w_0) + sd(w_1, w_0) \\
&\precsim (1 + s)c_1 \precsim 2sc_1.
\end{aligned}$$

Since both $\frac{d(w_n, w_0)}{2s}$ and $d(w_n, w_{n+1})$ are not greater than $d(w_1, w_0)$. This finishes the inductive proof.

C. In order to prove that $\{w_n\}$ is a Cauchy sequence, supposing the contrary. Using Lemma 3.2 with ϵ replaced by $\delta > 0$), there exists two sequences $n(k)$, $m(k)$ of positive integers such that $n(k) > m(k) > k$

$$d(w_{m(k)}, w_{n(k)}) \succsim \delta \quad d(w_{m(k)}, w_{n(k)}) \prec \delta,$$

and

$$\frac{\delta}{s} \precsim \lim_{n\to\infty} \sup d(w_{m(k)+1}, w_{n(k)}). \tag{3.17}$$

Replacing $z = z_{m(k)+1}, w = z_{n(k)}$ in the condition (3.15) we get

$$\begin{aligned}
d(Sz_{m(k)+1}, Sz_{n(k)}) &\precsim \frac{(1-\epsilon)}{s}\max\Big\{d(Tz_{m(k)+1}, Sz_{m(k)+1}), d(Tz_{n(k)}, Sz_{n(k)}) \\
&\quad , \frac{d(Tz_{m(k)+1}, Tw)}{2s}, \frac{d(Tz_{m(k)+1}, Sz_{m(k)+1})d(Sz_{n(k)}, Tz_{n(k)})}{1+d(Sz_{m(k)+1}, Tz_{m(k)+1})}\Big\} \\
&\quad +\tau\epsilon^{\alpha}\Gamma(\epsilon)[1+\|Sz_{m(k)+1}\|+\|Sz_{n(k)}\|+\|Tz_{m(k)+1}\|+\|Tz_{n(k)}\|]^{\beta} \\
&= \frac{(1-\epsilon)}{s}\max\Big\{d(w_{m(k)}, w_{m(k)+1}), d(w_{n(k)-1}, w_{n(k)}) \\
&\quad , \frac{d(w_{m(k)}, w_{n(k)-1})}{2s}, \frac{d(w_{m(k)}, w_{m(k)+1})d(w_{n(k)}, w_{n(k)-1})}{1+d(w_{m(k)+1}, w_{m(k)})}\Big\} \\
&\quad +\tau\epsilon^{\alpha}\Gamma(\epsilon)[1+\|w_{m(k)+1}\|+\|w_{n(k)}\|+\|w_{m(k)}\|+\|w_{n(k)-1}\|]^{\beta} \\
d(w_{m(k)+1}, w_{n(k)}) &\precsim \frac{(1-\epsilon)}{s}\max\Big\{d(w_{m(k)}, w_{m(k)+1}), d(w_{n(k)-1}, w_{n(k)}) \\
&\quad , \frac{d(w_{m(k)}, w_{n(k)-1})}{2s}, \frac{d(w_{m(k)}, w_{m(k)+1})d(w_{n(k)}, w_{n(k)-1})}{1+d(w_{m(k)+1}, w_{m(k)})}\Big\} \\
&\quad +K\tau\epsilon^{\alpha}\Gamma(\epsilon).
\end{aligned}$$

For some constant K, since the sequence $\{c_n\}$ is bounded. Passing to the upper limit, and using (3.17), we get

$$\begin{aligned}
\frac{\delta}{s} &\precsim \lim_{n\to\infty}\sup d(w_{m(k)+1}, w_{n(k)}) \\
&\precsim \frac{(1-\epsilon)}{s}\max\left\{0, 0, \lim_{n\to\infty}\sup\frac{d(w_{m(k)}, w_{n(k)-1})}{2s}, 0\right\} + K\tau\epsilon^{\alpha}\Gamma(\epsilon) \\
&= \frac{(1-\epsilon)}{s}\frac{\delta}{2s} + K\tau\epsilon^{\alpha}\Gamma(\epsilon),
\end{aligned}$$

hence,

$$\frac{\delta}{s} \precsim \frac{(1-\epsilon)}{s}\frac{\delta}{2s} + K\tau\epsilon^{\alpha}\Gamma(\epsilon) \precsim \frac{(1-\epsilon)\delta}{s} + K\tau\epsilon^{\alpha}\Gamma(\epsilon).$$

Further,

$$|\delta| \leq s|K|\tau\epsilon^{\alpha-1}\Gamma(\epsilon)$$

Putting $\epsilon = 0$, we get $\delta = 0$, which is a contradiction. Hence, $w_n = Sz_n = Tz_{n+1}$ is Cauchy sequence.
We will show that $Su = Tu$. Assume that $T(X)$ is a complete subspace of X, therefore $w_n = Sz_n = Tz_{n+1} \to u$ as $n \to \infty$, for some $u \in X$. We will show that $Su = Tu$, we have

$$\begin{aligned}
\frac{1}{s}d(Su, Tu) &\precsim d(Su, Sz_n) + d(Sz_n, Tu) \\
&\precsim \frac{1-\epsilon}{s}\max\left\{d(Tu, Su), d(Tz_n, Sz_n), \frac{d(Tu, Tz_n)}{2s}, \frac{d(Tu, Su)d(Sz_n, Tz_n)}{1 + d(Su, Tu)}\right\} \\
&\quad + K\tau\epsilon^{\alpha}\Gamma(\epsilon) + d(Sz_n, Tu).
\end{aligned}$$

Passing to the limit as $n \to \infty$

$$\begin{aligned}
\frac{1}{s}d(Su, Tu) &\precsim \frac{1-\epsilon}{s}\max\left\{d(Tu, Su), d(Tu, Su), \frac{d(Tu, Tu)}{2s}, \frac{d(Tu, Su)d(Su, Tu)}{1 + d(Su, Tu)}\right\} \\
&\quad + K\tau\epsilon^{\alpha}\Gamma(\epsilon) + d(Su, Tu) \\
\frac{1}{s}d(Su, Tu) &\precsim \frac{1-\epsilon}{s}d(Su, Tu) + K\tau\epsilon^{\alpha}\Gamma(\epsilon).
\end{aligned}$$

Further,

$$\begin{aligned}
|d(Su, Tu)| &\leqslant (1-\epsilon)|d(Su, Tu)| + s|K|\tau\epsilon^{\alpha}\Gamma(\epsilon) \\
|d(Su, Tu)| &\leqslant s|K|\tau\epsilon^{\alpha-1}\Gamma(\epsilon),
\end{aligned}$$

Consequently, $Su = Tu$ which is the unique point of coincidence of S and T. Since (S, T) is weakly compatible by the proposition 1.1 of Abbas and Jungck [48]. Then u is the unique common fixed point of S and T. □

Putting $T = I_X$ in the previous theorem, we obtain

Corollary 3.2. *Let (X, d) be a complete complex valued b-metric space with a parameter $(s \geq 1)$ and $S : X \to X$ be a mapping. Assume that $\tau \geqslant 0, \alpha \geqslant 1$ and $\beta \in [0, \alpha]$ are fixed constants such the inequality*

$$\begin{aligned}
d(Sz, Sw) &\precsim \frac{(1-\epsilon)}{s}\max\left\{d(z, Sz), d(w, Sw), \frac{d(z, w)}{2s}, \frac{d(z, Sz)d(Sw, w)}{1 + d(Sz, z)}\right\} \\
&\quad + \tau\epsilon^{\alpha}\Gamma(\epsilon)[1 + \|Sz\| + \|Sw\| + \|z\| + \|w\|]^{\beta},
\end{aligned} \tag{3.18}$$

holds for each $\epsilon \in [0,1]$ *and for all* $z, w \in X$*. Therefore* S *has a unique fixed point.*

Remark 3.2. *The theorem 3.7 is a generalization of theorem 3.6 and all results extracted of it.*

Chapter 4

Applications

In this chapter we present some applications upon both a rational inequality and Pata's contraction condition in complex valued metric, b-metric spaces. In the first section, we apply theorem 2.6 to a system of Urysohn integral equations and apply corollary 2.5 to linear system. As a result of section 2, using theorem 3.5, we establish the existence and the uniqueness of a common solution of Urysohn integral equations.

4.1 Application to integral equations upon rational inequality contraction

4.1.1 Application to Urysohn integral equations

Our first new result in this section is the following

Theorem 4.1. *Let* $X = C([a,b], \mathbb{R}^n), a > 0$ *and* $d : X \times X \to \mathbb{C}$ *defined as follows.*

$$d(z, w) = \max_{u \in [a,b]} \|z(u) - w(u)\|_\infty \sqrt{1 + a^2} e^{itan^{-1}a}.$$

Consider the Urysohn integral equations

$$z(u) = \int_a^b K_1(t, s, z(u))ds + g(u), \tag{1}$$

$$z(u) = \int_a^b K_2(t, s, z(u))ds + h(u), \tag{2}$$

where, $u \in [a,b] \subset \mathbb{R}$ *and* $z, g, h \in X$.
Assume that $K_1, K_2 : [a,b] \times [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ *such that* $F_z, G_z \in X$, *for each* $z \in X$, *where*

$$F_z(u) = \int_a^b K_1(t,s,z(u))ds, \; G_z(u) = \int_a^b K_2(t,s,z(u))ds \quad \textit{for all} \;\; u \in [a,b].$$

If there exist $s \geq 1, \; \lambda \in (0,1)$ *such that the inequality*

$$A(z,w)(u) \precsim \frac{\lambda}{s^2} R(z,w)(u), \tag{4.1}$$

holds for all $z, w \in X$ *where*

$$\begin{aligned} R(z,w) &= \max\{D(z,w)(u), B(z,w)(u), C(z,w)(u) \\ &\quad , \frac{1}{2}[B(z,w)(u) + C(z,w)(u)], \frac{B(z,w)(u)C(z,w)(u)}{1 + D(z,w)(u)}\Big\} \end{aligned}$$

and

$$\begin{aligned} A(z,w)(u) &= \|F_z(u) - G_w(u) + g(u) - h(u)\| \sqrt{1+a^2} e^{itan^{-1}a}, \\ B(z,w)(u) &= \|z(u) - F_z(u) - g(u)\| \sqrt{1+a^2} e^{itan^{-1}a}, \\ C(z,w)(u) &= \|w(u) - G_w(u) - h(u)\| \sqrt{1+a^2} e^{itan^{-1}a}, \\ D(z,w)(u) &= \|z(u) - w(u)\| \sqrt{1+a^2} e^{itan^{-1}a}, \end{aligned}$$

Then, the system of Urysohn integral equations has a unique common solution in X.

Proof. Define $S, T : X \to X$ by

$$Sz = F_z + g, Tz = G_z + h.$$

Then,

$$\begin{aligned} d(Sz, Tw) &= \max_{u \in [a,b]} \|F_z(u) - G_w(u) + g(u) - h(u)\|_\infty \sqrt{1+a^2} e^{itan^{-1}a}, \\ d(z, Sz) &= \max_{u \in [a,b]} \|z(u) - F_z(u) - g(u)\|_\infty \sqrt{1+a^2} e^{itan^{-1}a}, \\ d(w, Tw) &= \max_{u \in [a,b]} \|w(u) - G_w(u) - h(u)\|_\infty \sqrt{1+a^2} e^{itan^{-1}a}, \\ d(z, w) &= \max_{u \in [a,b]} \|z(u) - w(u)\|_\infty \sqrt{1+a^2} e^{itan^{-1}a}. \end{aligned}$$

From assumption 3.1, for each $u \in [a,b]$ we have

$$\begin{aligned} A(z,w)(u) &\precsim \frac{\lambda}{s^2} R(z,w)(u) \\ &\precsim \frac{\lambda}{s^2} \max \{D(z,w)(u), B(z,w)(u), C(z,w)(u) \\ &\quad , \frac{1}{2}\left[B(z,w)(u) + C(z,w)(u)\right], \frac{B(z,w)(u)C(z,w)(u)}{1+D(z,w)(u)} \Bigg\}, \end{aligned}$$

which implies that

$$\begin{aligned} \max_{u\in[a,b]} A(z,w)(u) &\precsim \frac{\lambda}{s^2} \max_{u\in[a,b]} \max \{D(z,w)(u), B(z,w)(u), C(z,w)(u) \\ &\quad , \frac{1}{2}\left[B(z,w)(u) + C(z,w)(u)\right], \frac{B(z,w)(u)C(z,w)(u)}{1+D(z,w)(u)} \Bigg\} \\ &\precsim \frac{\lambda}{s^2} \max \Bigg\{ \max_{u\in[a,b]} D(z,w)(u), \max_{u\in[a,b]} B(z,w)(u) \\ &\quad , \max_{u\in[a,b]} C(z,w)(u), \frac{1}{2}[\max_{u\in[a,b]} B(z,w)(u) + \max_{u\in[a,b]} C(z,w)(u)] \\ &\quad , \frac{\max_{u\in[a,b]} B(z,w)(u) \max_{u\in[a,b]} C(z,w)(u)}{1+\max_{u\in[a,b]} D(z,w)(u)} \Bigg\}. \end{aligned}$$

Therefore,

$$\begin{aligned} d(Sz,Tw) &\precsim \frac{\lambda}{s^2} \max\{d(z,w), d(z,Sz), d(w,Tw), \\ &\quad \frac{1}{2}\left[d(w,Sz) + d(z,Tw)\right], \frac{d(z,Sz)d(w,Tw)}{1+d(z,w)}\}. \end{aligned}$$

Thus all the conditions of theorem 2.6 with $P = Q = I_X$ are satisfied. Therefore, the system of Urysohn integral equations has a unique common solution in X. □

4.1.2 Application to linear system

In this section we give an application using the Corollary 2.5 in complete complex valued b-metric space $(X = \mathbb{C}^n, d_2)$ the where,

$$d_2(z,w) = \left[\sum_{i=1}^{n} \left(|z_i - w_i|^2 + i|z_i - w_i|^2\right)\right]^{\frac{1}{2}}.$$

Theorem 4.2. *Let $(X = \mathbb{C}^n, d_2)$ where, $z = (z_1, \ldots, z_n)^t \in X$ and $w = (w_1, \ldots, w_n)^t \in X$, if $\beta < \frac{1}{n}$ where,*

$$\beta_{ij} = \begin{cases} a_{ij} & \text{if, } i \neq j \\ a_{ii} + 1 & \text{if, } i = j \end{cases} \qquad \text{and} \qquad \beta = \max\{\beta_{ij}\}, \forall 1 \leq i,j \leq n.$$

So, the following linear system of n equations and n unknowns $AZ = B$ has a unique solution.

$$\begin{cases} a_{11}z_1 + a_{12}z_2 + \ldots + & a_{1n}z_n = b_1 \\ a_{21}z_1 + a_{22}z_2 + \ldots + & a_{2n}z_n = b_2 \\ \vdots & \\ a_{n1}z_1 + a_{n2}z_2 + \ldots + & a_{nn}z_n = b_n \end{cases} \Leftrightarrow \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Where, $z = (z_1, \ldots, z_n)^t \in X$ and $a_{ij} \in \mathbb{C}$, where, $1 \leq i,j \leq n$ and $b_1, b_2, b_n \in \mathbb{C}$

Proof. Define $T : X \to X$ by $Tz = (A + I)Z - B$. to prove that a linear system $AZ = B$ has a unique solution, its enough to prove that T is a contraction.
Since

$$\begin{aligned} d_2(Tz, Tw) &= \left[\sum_{i=1}^{n} \left(|(Tz)_i - (Tw)_i|^2 + i|(Tz)_i - (Tw)_i|^2\right)\right]^{\frac{1}{2}} \\ &= \left[\sum_{i=1}^{n} \left(\left|\sum_{j=1}^{n} \beta_{ij}(z_j - w_j)\right|^2 + i\left|\sum_{j=1}^{n} \beta_{ij}(z_j - w_j)\right|^2\right)\right]^{\frac{1}{2}}, \end{aligned}$$

where

$$\beta_{ij} = \begin{cases} a_{ij} & \text{if } i \neq j \\ a_{ii} + 1 & \text{if } i = j \end{cases} \qquad \text{and } \beta = \max\{\beta_{ij}\}, \forall 1 \leq i,j \leq n.$$

Then,

$$\begin{aligned} d_2(Tz,Tw) &\precsim \left[\left(\sum_{i=1}^{n}\max_{1\leq i,j\leq n}\beta_{ij}^2\right)\left(\left|\sum_{j=1}^{n}(z_j-w_j)\right|^2+i\left|\sum_{j=1}^{n}(z_j-w_j)\right|^2\right)\right]^{\frac{1}{2}} \\ &\precsim \left(n\beta^2\right)^{\frac{1}{2}}\left[n\left(\left|\sum_{j=1}^{n}(z_j-w_j)\right|^2+i\left|\sum_{j=1}^{n}(z_j-w_j)\right|^2\right)\right]^{\frac{1}{2}} \\ &\precsim n\beta\left[\left(\left|\sum_{j=1}^{n}(z_j-w_j)\right|^2+i\left|\sum_{j=1}^{n}(z_j-w_j)\right|^2\right)\right]^{\frac{1}{2}} \\ &= n\beta d_2(z,w). \end{aligned}$$

So, we get finally $d_2(Tz,Tw)\precsim n\beta d_2(z,w)$ where $\beta=\max\{|a_{ij}|,|a_{ii}+1|\ \forall\ 1\leq i,j\leq n\}$.

We conclude that T is a contraction mapping. By applying corollary 2.5, the linear system has a unique solution. □

4.2 Application to integral equations upon Pata's contraction condition

Theorem 4.3. *Let* $X=C([a,b],\mathbb{R}^n)$ *and* $d:X\times X\to\mathbb{C}$ *be defined by*

$$d(z,w)=max_{t\in[a,b]}\|z(t)-w(t)\|_\infty e^{i\theta},\ \theta\in]0,\frac{\pi}{2}[.$$

Consider the Urysohn integral equations

$$z(t)=\int_a^b K_1(t,s,z(t))ds+g(t), \tag{1}$$

$$z(t)=\int_a^b K_2(t,s,z(t))ds+h(t), \tag{2}$$

where $t\in[a,b]\subset\mathbb{R}$ *and* $z,g,h\in X$. *Assume that* $K_1,K_2:[a,b]\times[a,b]\times\mathbb{R}^n\to\mathbb{R}^n$ *such that* $F_z,G_z\in X$ *for each* $z\in X$, *where*

$$F_z(t)=\int_a^b K_1(t,s,z(t))ds,\ G_z(t)=\int_a^b K_2(t,s,z(t))ds\ \ \textit{for all}\ \ t\in[a,b].$$

If there exists $\tau \geqslant 0$, $\alpha \geqslant 1$ *and* $\beta \in [0,\alpha]$ *are fixed constants such that the inequality*

$$\|F_z(t) - F_w(t)\|_\infty e^{i\theta} \precsim (1-\epsilon)V(z,w)(t) + \tau\epsilon^\alpha\Gamma(\epsilon)R(z,w)(t),$$

holds for each $\epsilon \in [0,1]$ *and for all* $z, w \in X$*, where*

$$\begin{aligned} R(z,w)(t) &= [1 + \|G_z(t) + h(t)\|e^{i\theta} + \|G_w(t) + h(t)\|e^{i\theta}]^\beta, \\ V(z,w)(t) &= \|G_z(t) - G_z(t)\|e^{i\theta}, \end{aligned}$$

Therefore, the system of Urysohn integral equations 1 *and* 2 *has a unique common solution in* X*.*

Proof. Define $S, T : X \to X$ by

$$Sz = F_z + g, Tz = G_z + h.$$

Then,

$$\begin{aligned} d(Sz,Sw) &= \max_{t\in[a,b]} \|F_z(t) - F_w(t)\|_\infty e^{i\theta}, \\ d(Tz,Tw) &= \max_{t\in[a,b]} \|G_z(t) - G_w(t)\|_\infty e^{i\theta}, \end{aligned}$$

and

$$\begin{aligned} d(Tz,w_0) &= \|Tz\| = \max_{t\in[a,b]} \|G_z(t) + h(t)\|_\infty e^{i\theta}, \\ d(Tw,w_0) &= \|Tw\| = \max_{t\in[a,b]} \|G_w(t) + h(t)\|_\infty e^{i\theta}. \end{aligned}$$

We can show easily that for all $z, w \in X$,

$$d(Sz,Sw) \precsim (1-\epsilon)d(Tz,Tw) + \tau\epsilon^\alpha\Gamma(\epsilon)[1 + \|Tz\| + \|Tw\|]^\beta.$$

By applying theorem 3.6, therefore, the system of Urysohn integral equations (1) and (2) has a unique common solution in X. ☐

Conclusion

This thesis presents a systematic and comprehensive study of common fixed point theorems in complex valued metric spaces defined by Azam et al. We have used complex valued b-metric spaces as defined by K. Rao, to deduce a number of corollaries and to present examples to illustrate our results.

Our studies originated with two ideas.

- Firstly, we have proved common fixed point theorems under Pata's contraction condition in complex metric space for two mappings using weakly compatibility property.

- Secondly, we have proved common fixed point theorems verifying rational inequalities in complex b-metric space for four mappings using compatibility and weak compatibility properties.

As perspectives, there are furthermore ideas in this topic which we can do. In particular, one may study common fixed point theorems with more than two mappings under Pata's contraction condition with feeble properties in complex valued metric and b-metric spaces or may be fruitfully utilized in establishing some common fixed point theorems in complex valued 2-metric spaces.
Also, one may try to use the results of this thesis in numerical analysis to find numerical solution of Urysohn integral equations.

To find applications of this research subject in the real life seems to be very difficult, the reason why we are looking forward to work with these aspects in our future investigations.

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