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Présentée par

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# ON THE THEORY AND APPLICATIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS

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# Preface

In recent years a considerable interest has been shown in the so-called fractional calculus, which allows us to consider integration and differentiation of any order, not necessarily integer. To a large extent this is due to the applications of the fractional calculus to problems in different areas of physics and engineering. The fractional calculus can be considered an old and yet novel topic. Starting from some speculations of Leibniz and Euler, followed by the works of other eminent mathematicians including Laplace, Fourier, Abel, Liouville and Riemann, it has undergone a rapid development especially during the past two decades. One of the emerging branches of this study is the theory of fractional quasilinear equations, i.e. quasilinear equations where the integer derivative with respect to time is replaced by a derivative of fractional order. The increasing interest in this class of equations is motivated both by their application to problems from viscoelasticity, heat conduction in materials with memory, electrodynamics with memory, and also because they can be employed to approach nonlinear conservation laws

The purpose of this thesis is to extend the classical results (existence, uniqueness, regularity of solution and controllability) of ordinary abstract evolution equations to fractional abstract quasilinear equations. For completeness, these results are derived using resolvent operators, family resolvent operators, the properties of characteristic solution operators, fixed point theorem as well as fractional calculus.

This thesis is an outcome of the author's research during his Dr.Sc. study at the ENS Vieux-Kouba (October 2007 - October 2011) and Larbi Ben M'hidi University (November 2011 - December 2012). Most of the material in the thesis is based on the following articles from this period:

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- [1]. M. Bragdi and M. Hazi, Controllability for systems governed by semilinear fractional differential inclusion in a Banach spaces, *Adv. Dyn. Sys. Appl.*, Vol.7, No.2 (2012), 163-170.
- [2]. M. Bragdi and M. Hazi, Existence and uniqueness of solutions of fractional quasilinear mixed integrodifferential equations with nonlocal condition in Banach spaces, *E. J. Qualitative Theory of Diff. Equ.*, No. 51 (2012), 1-16.
- [3]. M. Hazi and M. Bragdi, Controllability of fractional integrodifferential systems via semigroup theory in Banach space, *Math. J. Okayama Univ.* 54 (2012), 133-143.
- [4]. M. Bragdi and M. Hazi, Existence and controllability result for an evolution fractional integrodifferential systems, *Int. J. Contemp. Math. Sciences*, Vol. 5, No. 19 (2010), 901-910.

The rest of this thesis is organized as follows: In Chapter 1 we recall briefly some basic definitions of fractional derivative. Chapter 2 contains some background material about resolvent and family operators which are used throughout this work. In Chapter 3 applying resolvent operators and fixed point theorem, the solvability of some fractional quasilinear equations are investigated. Chapter 4 is devoted to the controllability of some fractional quasilinear equations applying family resolvent operators. Chapter 5 deals with to the controllability of fractional Integrodifferential equations with almost sectorial operators, after the description of this system, we use the properties of characteristic solution operators to obtain sufficient conditions for the controllability results. Chapter 6 is devoted to the investigation of controllability for systems governed by nonlocal quasilinear differential inclusion in a Banach spaces by using resolvent family and a fixed point theorem. In Chapter 7 the problem of existence of solutions for fractional differential inclusions is tackled in Banach spaces settings by the fixed point theorem introduced by Covitz and Nadler for contraction multi-valued maps.

*“God exists since mathematics is consistent, and the Devil exists since we cannot prove it.”*

André Weil

# *Abstract*

In this work we consider two type of fractional abstract differential equations. The purpose of the first part is to extend the classical results (existence, uniqueness, regularity of solution and controllability) of ordinary abstract evolution equations to fractional abstract quasilinear equations. In the second part we study existence, uniqueness and controllability of nonlinear control system governed by a semilinear differential inclusion. For completeness, these results are derived using resolvent operators, family resolvent operators, almost sectorial operators theory, fixed point theorems as well as fractional calculus.

**Keywords:** Controllability, Mild solution, Resolvent Operators, Family Resolvent Operators, Almost Sectorial Operators, Fixed Point Theorems, Fractional Calculus.

# Résumé

## Résumé

Dans le présent travail nous nous sommes intéressés à deux types d'équations différentielles fractionnaires abstraites. Nous l'avons structuré en deux parties. La première partie est consacrée à étendre des résultats classiques (existence, unicité et régularité des solutions, et contrôlabilité, ) des équations d'évolution ordinaires abstraites aux équations fractionnaires quasi-linéaires abstraites. Nous nous sommes attelés dans la deuxième partie, à l'étude de l'existence, l'unicité et la contrôlabilité des systèmes de contrôle non linéaires régis par une inclusion différentielle semi-linéaire. Il y a lieu de signaler que ces résultats sont obtenus en utilisant la théorie des semi-groupes, les opérateurs résolvents, le théorème du point fixe, ainsi que le calcul différentiel fractionnaire.

**Mots Clés:** Opérateurs résolvents, théorie des semi-groupes, théorème du point fixe, calcul différentiel fractionnaire.

## ملخص

يدور فحوى العمل الحاضر حول نوعين من المعادلات التفاضلية ذات الرتب الكسرية. جننا به في قسمين. توخينا في الأول منهما تسليط الضوء على بعض الجوانب والنتائج الكلاسيكية الدائرة في فلك هذا الصنف من المعادلات، من وجود الحلّ ووحدايته وكذا قابلية التحكم وانتظام الحلول وغيرها من الخواص المتداولة والمعهودة لدي المعادلات التفاضلية التطورية المجردة العادية، وتعميمها إلى المعادلات التفاضلية المجردة شبه الخطية ذات الرتب الكسرية، وهذا في فضاءات بناخية. تطرقنا في القسم الثاني إلى دراسة هذه الخصائص واستعرضناها في إطار الجمل المدبرة باحتواءات تفاضلية كسرية شبه خطية. لقد تمّ التوصل إلى هذه النتائج عبر استخدام المؤثرات الحالة وعائلة المؤثرات الحالة ونظرية المؤثرات شبه المقطعية ونظريات النقطة الصامدة إلى جانب الحساب التفاضلي الكسري.

## كلمات مفتاحية:

مؤثرات حالة، عائلة مؤثرات حالة، نظرية أنصاف الزمر، نظرية النقطة الصامدة، مشتق كسري.

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Oum El Bouaghi,  
December 1, 2012.

Mabrouk Bragdi

# Contents

<b>Preface</b>	<b>i</b>
<b>Abstract</b>	<b>iv</b>
<b>Résumé</b>	<b>v</b>
<b>Acknowledgements</b>	<b>vi</b>
<b>Abbreviations</b>	<b>x</b>
<b>Symbols</b>	<b>xi</b>
<b>1 Twofold Introduction to Fractional Derivatives</b>	<b>1</b>
1.1 Historical Introduction . . . . .	1
1.2 Mathematical Introduction . . . . .	2
1.2.1 Fractional Integrals . . . . .	2
1.2.1.1 Iterated Integrals . . . . .	3
1.2.1.2 Riemann–Liouville Fractional Integrals . . . . .	3
1.2.1.3 Basic Properties of RL Fractional Integrals . . . . .	4
1.2.1.4 Weyl Integral . . . . .	4
1.2.1.5 Basic Properties of Weyl integral . . . . .	5
1.2.1.6 Riesz Fractional Integrals . . . . .	5
1.2.1.7 Riesz–Feller fractional Integral . . . . .	6
1.2.2 Fractional Derivatives . . . . .	6
1.2.2.1 Riemann–Liouville Fractional Derivative . . . . .	7
1.2.2.2 Caputo Fractional Derivative . . . . .	7
1.2.2.3 Basic Properties of RL and Caputo Fractional Derivatives . . . . .	8
1.2.2.4 The generalized Riemann–Liouville fractional derivatives . . . . .	9
1.2.2.5 Marchaud–Hadamard fractional derivatives . . . . .	10

---

1.2.2.6	Weyl–Liouville fractional derivatives . . . . .	10
1.2.2.7	Grünwald–Letnikov fractional derivatives . . . . .	10
1.2.2.8	Local fractional derivatives . . . . .	11
<b>2</b>	<b>Abstract Fractional Differential Equations in Banach Spaces</b>	<b>12</b>
2.1	Preliminaries . . . . .	12
2.2	Operators in Banach spaces . . . . .	14
2.3	Solution Operators . . . . .	17
2.3.1	Resolvent Operators . . . . .	17
2.3.2	$(\alpha, x)$ -Resolvent Family . . . . .	19
2.3.3	$\alpha$ -Mildly Solvability . . . . .	20
<b>3</b>	<b>Existence and Uniqueness of Solutions of Fractional Quasilinear Mixed Integrodifferential Equations with Nonlocal Condition in Banach Spaces</b>	<b>23</b>
3.1	Introduction . . . . .	23
3.2	Preliminaries . . . . .	24
3.3	Main Results . . . . .	25
3.4	Application . . . . .	38
<b>4</b>	<b>Existence and Controllability Result for an Evolution Fractional Integrodifferential Systems</b>	<b>40</b>
4.1	Introduction . . . . .	40
4.2	Preliminaries . . . . .	41
4.3	Existence result . . . . .	42
4.4	Controllability result . . . . .	43
4.5	Application . . . . .	46
<b>5</b>	<b>Controllability of Abstract Fractional Differential Equations with Almost Sectorial Operators</b>	<b>48</b>
5.1	Introduction . . . . .	48
5.2	Main results . . . . .	49
5.3	Application . . . . .	56
<b>6</b>	<b>Controllability for Systems Governed by Nonlocal Quasilinear Differential Inclusion in a Banach Spaces</b>	<b>58</b>
6.1	Introduction . . . . .	58
6.2	Preliminaries . . . . .	59
6.3	Main results . . . . .	60
6.4	Application . . . . .	65
<b>7</b>	<b>Existence of Solutions for Fractional Differential Inclusions in Banach Spaces</b>	<b>67</b>
7.1	Introduction . . . . .	67
7.2	Preliminaries . . . . .	68
7.3	Main results . . . . .	70

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Step 1: $T$ has non-empty closed-values. . . . .	72
Step 2: $T$ is a contraction. . . . .	72
<b>Conclusion and Direction for Further Research</b>	<b>75</b>
<b>A Special Functions of the Fractional Calculus</b>	<b>76</b>
A.1 Euler’s Gamma Function . . . . .	76
A.2 Beta Function . . . . .	77
A.3 The Error Function . . . . .	78
A.4 Mittag-Leffler’s Function . . . . .	78
A.5 Wright–type Function . . . . .	80
A.6 Mellin-Ross Function . . . . .	80
A.7 Robotov’s Function . . . . .	81
<b>B Fixed Point Theorems</b>	<b>82</b>
<b>Bibliography</b>	<b>84</b>

# Abbreviations

<b>FQMIDE</b>	<b>F</b> ractional <b>Q</b> uasilinear <b>M</b> ixed <b>I</b> ntegro <b>D</b> ifferential <b>E</b> quations
<b>FIDS</b>	<b>F</b> ractional <b>I</b> ntegro <b>D</b> ifferential <b>S</b> ystems
<b>QDI</b>	<b>Q</b> uasilinear <b>D</b> ifferential <b>I</b> nclusion
<b>FDI</b>	<b>F</b> ractional <b>D</b> ifferential <b>I</b> nclusion
<b>AFDE</b>	<b>A</b> bstract <b>F</b> ractional <b>D</b> ifferential <b>E</b> quations
<b>FIO</b>	<b>F</b> ractional <b>I</b> ntegral <b>O</b> perator
<b>FDO</b>	<b>F</b> ractional <b>D</b> ifferential <b>O</b> perator
<b>LFD</b>	<b>L</b> ocal <b>F</b> ractional <b>D</b> erivative
<b>RL</b>	<b>R</b> iemann– <b>L</b> iouville definition
<b>MH</b>	<b>M</b> archaud– <b>H</b> adamard definition
<b>WL</b>	<b>W</b> eyl– <b>L</b> iouville definition
<b>GL</b>	<b>G</b> rünwald– <b>L</b> etnikov definition
<b>KG–LFD</b>	<b>K</b> olwankar– <b>G</b> angal <b>L</b> ocal <b>F</b> ractional <b>D</b> erivative

# Symbols

$\mathbb{N}$	The set of natural numbers
$\mathbb{N}_0$	The set of natural numbers with zero included
$\mathbb{Z}$	The set of integer numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\Re(z)$	A real part of a complex number $z$
$\delta(z)$	The Dirac delta function
$\Gamma(z)$	The Euler's gamma function
$\Gamma^*(\nu, t), \gamma(\nu, t)$	The incomplete Gamma function
$B(z, w)$	The Beta function
$E_{\alpha, \beta}(z)$	The Mittag-Leffler function
$\Psi_\alpha(z)$	The Wright-type function
$E_t(\nu, a)$	The Mellin-Ross function
$R_\alpha(\beta, t)$	The Robotov function
$f * g$	The convolution product for functions on $\mathbb{R}$
$I_{a+}^\alpha, {}_aI_x^\alpha$	The RL FIO of order $\alpha$ with lower limit $a$
$I_{b-}^\alpha, {}_xI_b^\alpha$	The RL FIO of order $\alpha$ with upper limit $b$
${}_xW_\infty^\alpha, I_-^\alpha$	The Weyl FIO of order $\alpha$ with upper limit $\infty$
${}_{-\infty}W_x^\alpha, I_+^\alpha$	The Weyl FIO of order $\alpha$ with lower limit $-\infty$
$I^\alpha$	The Riesz FIO of order $\alpha$
$I^{\alpha, \beta}$	The Riesz-Feller FIO of order $\alpha$ and $\beta$
$K_\pm^\alpha$	The Weyl kernel with parameter $\alpha$
$K^\alpha$	The Riesz kernel with parameter $\alpha$
$K^{\alpha, \beta}$	The Riesz-Feller kernel with parameters $\alpha$ and $\beta$

$D_{a+}^{\alpha}, {}_a D_x^{\alpha}$	The RL FDO of order $\alpha$ with lower limit $a$
$D_{b-}^{\alpha}, {}_x D_b^{\alpha}$	The RL FDO of order $\alpha$ with upper limit $b$
${}^C D_x^{\alpha}, {}^C D_{a+}^{\alpha}$	The Caputo FDO of order $\alpha$ with lower limit $a$
${}^C D_b^{\alpha}, {}^C D_{b-}^{\alpha}$	The Caputo FDO of order $\alpha$ with upper limit $b$
${}^C D_x^{\alpha}, \frac{d^{\alpha}}{dx^{\alpha}}$	The Caputo FDO of order $\alpha$ with lower limit 0
$D_{a\pm}^{\alpha,\beta}$	The generalized RL FDO of order $\alpha$ and type $\beta$
$M_{a\pm}^{\alpha}$	The Marchaud–Hadamard FDO of order $\alpha$
$D_{\pm}^{\alpha}$	The Weyl–Liouville FDO of order $\alpha$
${}^{GL} D_{a\pm}^{\alpha}$	The Grünwald-Letnikov FDO of order $\alpha$
${}^L D_{\pm}^{\alpha}$	The right (left) KG LFD operator of order $\alpha$
$X, Y$	Banach spaces
$\ \cdot\ _X$	The norm in the space $X$
$B(X, Y)$	The space of all bounded linear operators from $X$ to $Y$
$B(X)$	The space of all bounded linear operators from $X$ to $X$
$C(J; X)$	The Banach space of $X$ -valued continuous functions on $J$
$A$	A linear operator in $X$
$D(A)$	The domain space of $A$
$R(A)$	The range space of $A$
$N(A)$	The null space of $A$
$\sigma(A)$	The spectrum set of $A$
$\rho(A)$	The resolvent set of $A$
$T(t)$	A semigroup of bounded linear operators on $X$
$R(\lambda, A)$	The resolvent operator of $A$
$R_z(t, s)$	A family resolvent operators for an equations
$R_{\alpha,x}(t, s)$	$(\alpha, x)$ -resolvent family
$\mathcal{S}_{\alpha}, \mathcal{P}_{\alpha}$	The characteristic solution operators

*Dedicated to my Parents...*

# Chapter 1

## Twofold Introduction to Fractional Derivatives

### 1.1 Historical Introduction

The term fractional calculus is more than 300 years old. It is a generalization of the ordinary differentiation and integration to non-integer order. The subject is as old as the calculus of differentiation and goes back to times when Leibniz, Gauss, and Newton invented this kind of calculation.

As to the history of fractional calculus, already in 1695 L'Hospital raised the question as to the meaning of  $d^n y/dx^n$  if  $n = 1/2$ , that is "what if  $n$  is fractional?". "This is an apparent paradox from which, one day, useful consequences will be drawn", Leibniz replied, together with " $d^{1/2}x$  will be equal to  $x\sqrt{dx : x}$ ". In 1730 the subject of fractional calculus did not escape Eulers attention. J. L. Lagrange in 1772 contributed to fractional calculus indirectly, when he developed the law of exponents for differential operators. In 1812, P. S. Laplace defined the fractional derivative by means of integral and in 1819 S. F. Lacroix mentioned a derivative of arbitrary order in his 700-page long text, followed by J. B. J. Fourier in 1822, who mentioned the derivative of arbitrary order. The first use of fractional operations was made by N. H. Abel in 1823 in the solution of tautochrone problem. J. Liouville made the first major study of fractional calculus in 1832, where he applied his definitions to problems in theory. In 1867, A. K. Grunwald worked on the fractional operations. G. F. B. Riemann developed the theory of fractional

integration during his school days and published his paper in 1892. A. V. Letnikov wrote several papers on this topic from 1868 to 1872. Oliver Heaviside published a collection of papers in 1892, where he showed the so-called Heaviside operational calculus concerned with linear generalized operators. In the period of 1900 to 1970 the principal contributors to the subject of fractional calculus were, for example, H. H. Hardy, S. Samko, H. Weyl, M. Riesz, S. Blair, etc. From 1970 to the present, they are for instance J. Spanier, K. B. Oldham, B. Ross, K. Nishimoto, O. Marichev, A. Kilbas, H. M. Srivastava, R. Bagley, K. S. Miller, M. Caputo, I. Podlubny, and many others.

Nowadays, the theory of derivatives and integrals of fractional order has played an important role in engineering, physics, chemistry, and other sciences. Some of the most prominent examples are given in a book by Oldham and Spanier [63] (diffusion processes) and the classic papers of Bagley and Torvik [6], and Caputo and Mainardi [24] (these two papers dealing with the modeling of viscoelastic materials). More recent results are described, for example, in the works of Gaul et al. [41] (description of mechanical systems subject to damping) and Podlubny [67] (control theory). We refer to [36] and [66] for more details about the history of the development of fractional calculus.

## 1.2 Mathematical Introduction

The brief historical introduction has shown that fractional derivatives may be defined in numerous ways. In this Section, we recall several definitions on fractional integral and fractional derivatives from [48] with a slight change.

### 1.2.1 Fractional Integrals

This section deals with the definition and properties of various operators of fractional integration of arbitrary order. Among the various operators studied, it involves the Riemann–Liouville fractional integral operators, Weyl operators.

### 1.2.1.1 Iterated Integrals

Consider a locally integrable<sup>1</sup> real valued function  $f : J \rightarrow \mathbb{R}$  whose domain of definition  $J = [a, b] \subset \mathbb{R}$  is an interval with  $-\infty \leq a < b \leq \infty$ . Integrating  $n$  times gives the fundamental formula

$$\begin{aligned} (I_{a+}^n f)(x) &= \int_a^x \int_a^{x_1} \dots \int_a^{x_{n-1}} f(x_n) dx_n \dots dx_2 dx_1 \\ &= \frac{1}{(n-1)!} \int_a^x (x-y)^{n-1} f(y) dy \end{aligned} \quad (1.1)$$

where  $a < x < b$  and  $n \in \mathbb{N}$ . This formula may be proved by induction. It reduces  $n$ -fold integration to a single convolution integral. The subscript  $a+$  indicates that the integration has  $a$  as its lower limit. An analogous formula holds with lower limit  $x$  and upper limit  $a$ . In that case the subscript  $a-$  will be used.

### 1.2.1.2 Riemann–Liouville Fractional Integrals

Equation (1.1) for  $n$ -fold integration can be generalized to noninteger values of  $n$  using the relation  $(n-1)! = \Gamma(n)$  where  $\Gamma(z)$  is Euler's  $\Gamma$ -function defined for all  $z \in \mathbb{C}$  by A.1 (see Appendix A).

**Definition 1.1.** Let  $-\infty \leq a < x < b \leq \infty$ . The Riemann–Liouville fractional integral of order  $\alpha > 0$  with lower limit  $a$  is defined for locally integrable functions  $f : [a, b] \rightarrow \mathbb{R}$  as

$$(I_{a+}^\alpha f)(x) = ({}_a I_x^\alpha f)(x) = ({}_a D_x^{-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt. \quad (1.2)$$

The Riemann–Liouville fractional integral of order  $\alpha > 0$  with upper limit  $b$  is defined as

$$(I_{b-}^\alpha f)(x) = ({}_x I_b^\alpha f)(x) = ({}_x D_b^{-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt. \quad (1.3)$$

---

<sup>1</sup>A function  $f : J = [a, b] \subset \mathbb{R}$  is called locally integrable if it is integrable on all compact subsets  $K \subset J$

### 1.2.1.3 Basic Properties of RL Fractional Integrals

Fractional integrals obey the following semigroup property:

$$I_{a+}^{\alpha} I_{a+}^{\beta} \phi = I_{a+}^{\alpha+\beta} \phi = I_{a+}^{\beta} I_{a+}^{\alpha} \phi \quad (1.4)$$

$$I_{b-}^{\alpha} I_{b-}^{\beta} \phi = I_{b-}^{\alpha+\beta} \phi = I_{b-}^{\beta} I_{b-}^{\alpha} \phi \quad (1.5)$$

The following results

$$\int_a^b f(x) (I_{a+}^{\alpha} g(x)) dx = \int_a^b g(x) (I_{b-}^{\alpha} f(x)) dx, \quad f, g \in L^1(a, b), \quad (1.6)$$

is called the property of "integration by parts" for fractional integrals.

### 1.2.1.4 Weyl Integral

The Weyl fractional integral is defined as

**Definition 1.2.** Let  $-\infty < x < \infty$ . The Weyl fractional integral of order  $\alpha > 0$  with upper limit  $\infty$  is defined for locally integrable functions  $f : (-\infty, \infty) \rightarrow \mathbb{R}$  as

$$({}_x W_{\infty}^{\alpha} f)(x) = (I_{-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t) dt \quad (1.7)$$

The Weyl fractional integral of order  $\alpha > 0$  with lower limit  $-\infty$  is defined as

$$({}_{-\infty} W_x^{\alpha} f)(x) = (I_{+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-t)^{\alpha-1} f(t) dt \quad (1.8)$$

It can be shown that the Weyl definition coincides with the Riemann-Liouville definition  $I_{\infty-}^{\alpha} f$  respectively  $I_{(-\infty)+}^{\alpha} f$ . For this reason the Riemann-Liouville fractional integrals with limits  $\pm\infty$ ,  $I_{+}^{\alpha} f = I_{(-\infty)+}^{\alpha} f$  and  $I_{-}^{\alpha} f = I_{\infty-}^{\alpha} f$  are often called Weyl fractional integrals.

The Weyl fractional integral may be rewritten as a convolution

$$(I_{\pm}^{\alpha} f)(x) = (K_{\pm}^{\alpha} * f)(x), \quad (1.9)$$

where the convolution product for functions on  $\mathbb{R}$  is defined as

$$(K * f)(x) = \int_{-\infty}^{\infty} K(x-t)f(t)dt \quad (1.10)$$

and the convolution kernels are defined as

$$K_{\pm}^{\alpha}(x) = \frac{(\pm x)_{+}^{\alpha-1}}{\Gamma(\alpha)}, \quad (1.11)$$

for  $\alpha > 0$ . Here the suffix  $+$  in  $(x)_{+}^{\alpha-1}$  is just denoting that the function is vanishing for  $x < 0$ . Being  $\alpha > 0$ , this function turns out to be locally absolutely integrable in  $\mathbb{R}^{+}$ . For  $\alpha = 0$  the kernel  $K_{+}^0(x) = K_{-}^0(x) = \delta(x)$  is the Dirac  $\delta$ -function.

### 1.2.1.5 Basic Properties of Weyl integral

Weyl fractional integral obeys the semigroup property. That is,

$${}_x W_{\infty x}^{\alpha} W_{\infty}^{\beta} f = {}_x W_{\infty}^{\alpha+\beta} f = {}_x W_{\infty x}^{\beta} W_{\infty}^{\alpha} f \quad (1.12)$$

Also the following relation holds

$$\int_0^{\infty} \phi(x)({}_0 I_x^{\alpha} \psi)(x) dx = \int_0^{\infty} ({}_0 W_x^{\alpha} \phi)(x) \psi(x) dx \quad (1.13)$$

Equality 1.13 is called Parseval equality. It can be established by interchanging the order of integration.

Riemann–Liouville and Weyl fractional integrals have upper or lower limits of integration are sometimes called left-sided resp. right-sided integrals.

### 1.2.1.6 Riesz Fractional Integrals

**Definition 1.3.** Let  $f \in L_{loc}^1(\mathbb{R})$  be locally integrable. The Riesz fractional integral or Riesz potential of order  $\alpha > 0$  is defined as the linear combination of right– and left–sided Weyl fractional integrals.

$$(I^{\alpha} f)(x) = \frac{(I_{+}^{\alpha} f)(x) + (I_{-}^{\alpha} f)(x)}{2 \cos(\alpha\pi/2)} = \frac{1}{2\Gamma(\alpha) \cos(\alpha\pi/2)} \int_{-\infty}^{\infty} \frac{f(t)}{|x-t|^{1-\alpha}} dt. \quad (1.14)$$

The definition is again completed with  $(I^0 f)(x) = f(x)$  for  $\alpha = 0$ .

Riesz fractional integration may be written as a convolution

$$(I^a f)(x) = (K^a * f)(x), \quad (1.15)$$

with the (one-dimensional) Riesz kernels

$$K^\alpha(x) = \frac{K_+^\alpha(x) + K_-^\alpha(x)}{2 \cos(\alpha\pi/2)} = \frac{|x|^{\alpha-1}}{2\Gamma(\alpha) \cos(\alpha\pi/2)}, \quad (1.16)$$

for  $\alpha \neq 2k + 1$ ,  $k \in \mathbb{Z}$ .

### 1.2.1.7 Riesz–Feller fractional Integral

The generalized Riesz–Feller fractional integral of order  $\alpha > 0$  and type  $\beta \in \mathbb{R}$  is defined as

$$(I^{\alpha,\beta} f)(x) = (K^{\alpha,\beta} * f)(x), \quad (1.17)$$

where

$$K^{\alpha,\beta}(x) = \frac{|x|^{\alpha-1} \sin[\alpha(\pi/2 + \beta \operatorname{sgn} x)]}{2\Gamma(\alpha) \sin(\alpha\pi/2)}$$

is the generalized Riesz–Feller kernels with parameter  $\beta$  and  $f \in L_{loc}^1(\mathbb{R})$ .

## 1.2.2 Fractional Derivatives

The notation for fractional derivatives is not standardized [48]. Leibniz and Euler used  $d^\alpha$ , Riemann wrote  $\partial_x^\alpha$ , Liouville preferred  $d^\alpha/dx^\alpha$ , Grünwald used  $\{d^\alpha f/dx^\alpha\}_{x=a}$  or  $D^\alpha[f]_{x=a}$ , Marchaud wrote  $D_a^{(\alpha)}$ , and Hardy–Littlewood used an index  $f^\alpha$ . Modern authors also use  $I^{-\alpha}$ ,  $I_x^{-\alpha}$ ,  ${}_a D_x^\alpha$ ,  $d^\alpha/dx^\alpha$ ,  ${}_a D_x^\alpha$ ,  $d^\alpha/d(x-a)^\alpha$  instead of  $D_{a+}^\alpha$ .

The fractional derivatives are defined using integrals, so they are non-local operators. The fractional derivative in time contains information about the function at earlier points, so it possess a memory effect. Because of their non–local property fractional derivatives can be used to construct simple material models and unified principles. They have important applications in astrophysics, economics, fusion plasmas, mechanics and viscoelasticity. (see [29]).

Here we recall the following known definition, for more details see [42, 60, 68].

### 1.2.2.1 Riemann–Liouville Fractional Derivative

**Definition 1.4.** Let  $-\infty \leq a < x < b \leq \infty$ . The Riemann–Liouville fractional derivative of order  $\alpha > 0$  with lower limit  $a$  (resp. upper limit  $b$ ) is defined for functions such that  $f \in L^1([a, b])$  and  $f * K^{n-\alpha} \in W^{n,1}([a, b])$  as

$$(D_{a\pm}^\alpha f)(x) = (\pm 1)^n \frac{d^n}{dx^n} (I_{a\pm}^{n-\alpha} f)(x), \quad (1.18)$$

and  $(D_{a\pm}^0 f)(x) = f(x)$  for  $\alpha = 0$ .

Here  $n = [\alpha] + 1$  is smallest integer larger than  $\alpha$ , and  $W^{k,p}(G) = \{f \in L^p(G) : f^{(k)} \in L^p(G)\}$  denotes a Sobolev space. For  $k = p = 1$  the space  $W^{1,1}([a, b]) = AC^0([a, b])$  coincides with the space of absolutely continuous functions<sup>2</sup>.

### 1.2.2.2 Caputo Fractional Derivative

Let  $f(x)$  be absolutely continuous on the finite interval  $[a, b]$ . Then, its derivative  $f'$  exists almost everywhere on  $[a, b]$  with  $f' \in L^1([a, b])$ , and the function  $f$  can be written as

$$f(x) = \int_a^x f'(y)dy + f(a) = (I_{a+}^1 f')(x) + f(a),$$

Substituting this into  $I_{a+}^\alpha f$  gives

$$(I_{a+}^\alpha f)(x) = (I_{a+}^1 I_{a+}^\alpha f')(x) + \frac{f(a)}{\Gamma(\alpha + 1)}(x - a)^\alpha,$$

where commutativity of  $I_{a+}^1 f$  and  $I_{a+}^\alpha f$  was used. It follows that

$$(DI_{a+}^\alpha f)(x) - (I_{a+}^\alpha Df)(x) + \frac{f(a)}{\Gamma(\alpha)}(x - a)^{\alpha-1},$$

for  $0 < \alpha < 1$ . Above, the notations  $(Df)(x) = \frac{df(x)}{dx} = f'(x)$  were used for the first order derivative.

<sup>2</sup>A function  $f : I \rightarrow \mathbb{R}$  is said to be absolutely continuous on  $I$  if for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\sum_{k=1}^l |f(b_k) - f(a_k)| \leq \varepsilon$  for every finite number of nonoverlapping intervals

$(a_k, b_k)$ ,  $k = 1, \dots, l$  with  $[a_k, b_k] \subset I$  and  $\sum_{k=1}^l (b_k - a_k) \leq \delta$ .

This observation suggests to introduce a modified Riemann–Liouville fractional derivative through

$$({}^C D_{a+}^\alpha f)(x) := I_{a+}^{n-\alpha} f^{(n)}(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(y)}{(x-y)^{\alpha-n+1}} dy, \quad (1.19)$$

where  $n = [\alpha] + 1$ . Note, that  $f$  must be at least  $n$ -times differentiable. Formula 1.19 is due to Liouville, but nowadays named after Caputo. We should point out that we write from now on,  $\frac{d^\alpha}{dx^\alpha}$  instead  ${}^C D_{0+}^\alpha$  for the Caputo derivative of order  $\alpha$ . If  $f$  is an abstract function with values in Banach space  $X$ , then the integrals and derivatives which appear in (1.19) are taken in Bochner's sense.

### 1.2.2.3 Basic Properties of RL and Caputo Fractional Derivatives

Let  $\lambda, \mu, \alpha, a$  be real constants and  $f, g$  arbitrary functions for which the needed operations are defined. Then the linearity can be express by the formula:

$$D_{a\pm}^\alpha(\lambda f + \mu g) = \lambda D_{a\pm}^\alpha(f) + \mu D_{a\pm}^\alpha(g), \quad (1.20)$$

$${}^C D_{a\pm}^\alpha(\lambda f + \mu g) = \lambda {}^C D_{a\pm}^\alpha(f) + \mu {}^C D_{a\pm}^\alpha(g). \quad (1.21)$$

Similarly we can derive a general formula for an infinite series, where the fundamental condition is uniform convergence of original series. The idea is that we have to deal with uniformly convergent series all the time

$$\left( D_{a\pm}^\alpha \sum_{k=0}^{\infty} f_k \right) (x) = \sum_{k=0}^{\infty} (D_{a\pm}^\alpha f_k) (x), \quad (1.22)$$

$$\left( {}^C D_{a\pm}^\alpha \sum_{k=0}^{\infty} f_k \right) (x) = \sum_{k=0}^{\infty} ({}^C D_{a\pm}^\alpha f_k) (x). \quad (1.23)$$

Especially for finite lower bound  $a$  we have to care only about the uniform convergence connected with derivatives, because integrals stay uniformly convergent automatically.

The Caputo differential operator is a commute operator

$$\left({}^C D_{x_0}^\alpha {}^C D_x^\beta f\right)(x) = \left({}^C D_{t_0}^\beta {}^C D_x^\alpha f\right)(x) = \left({}^C D_x^{\alpha+\beta} f\right)(x), \quad \forall \alpha, \beta \in \mathbb{R}_+, \quad (1.24)$$

if  $f$  is sufficiently smooth, and it possesses the desirable propriety that

$${}^C D_x^\alpha c = 0 \quad \text{for any constant } c. \quad (1.25)$$

The more common RL fractional derivative need not commute [40]; furthermore,

$${}_0 D_x^\alpha c = \frac{cx^{-\alpha}}{\Gamma(\alpha-1)}.$$

The relation between (1.18) and (1.19) is given by

$$\left(D_{a+}^\alpha f\right)(x) = \left({}^C D_{a+}^\alpha f\right)(x) + \sum_{k=0}^{n-1} \frac{(x-a)^{k-a}}{\Gamma(k-\alpha+1)} f^{(k)}(a), \quad \alpha \geq 0. \quad (1.26)$$

The RL integral operator and the RL differential operator are inverse operators in the sense that

$$D_{a+}^\alpha I_{a+}^\alpha f(x) = f(x); \quad (1.27)$$

$$I_{a+}^\alpha D_{a+}^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \left(D^{n-k-1} I_{a+}^{n-\alpha} f\right)(a), \quad (1.28)$$

where

$$I_{a+}^{n-\alpha} f \in AC^n([a, b]) := \{f \in C^n([a, b]) \mid f^{(n)} \text{ is absolutely continuous}\},$$

$$f \in L^1([a, b]), \quad \alpha > 0 \text{ and } n = [\alpha] + 1.$$

#### 1.2.2.4 The generalized Riemann–Liouville fractional derivatives

**Definition 1.5.** The generalized Riemann–Liouville fractional derivative of order  $0 < \alpha < 1$  and type  $0 \leq \beta \leq 1$  with lower (resp. upper) limit  $a$  is defined as

$$\left(D_{a\pm}^{\alpha, \beta} f\right)(x) = \left(\pm I_{a\pm}^{\beta(1-\alpha)} \frac{d}{dx} \left(I_{a\pm}^{(1-\beta)(1-\alpha)} f\right)\right)(x), \quad (1.29)$$

for functions such that the expression on the right hand side exists.

The type  $\beta$  of a fractional derivative allows to interpolate continuously from  $D_{a\pm}^\alpha = D_{a\pm}^{\alpha,0}$  to  ${}^C D_{a\pm}^\alpha = D_{a\pm}^{\alpha,1}$ .

### 1.2.2.5 Marchaud–Hadamard fractional derivatives

**Definition 1.6.** Let  $-\infty < a < b < \infty$  and  $0 < \alpha < 1$ . The Marchaud–Hadamard fractional derivative of order  $\alpha$  with lower limit  $a$  is defined as

$$(M_{a+}^\alpha f)(x) = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy, \quad (1.30)$$

and the Marchaud–Hadamard fractional derivative of order  $\alpha$  with upper limit  $b$  is defined as

$$(M_{b-}^\alpha f)(x) = \frac{f(x)}{\Gamma(1-\alpha)(b-x)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_x^b \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy, \quad (1.31)$$

For  $a = -\infty$  (resp.  $b = \infty$ ) the definition is

$$(M_{\pm}^\alpha f)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - f(x \mp y)}{y^{\alpha+1}} dy. \quad (1.32)$$

The definition is completed with  $M^0 f = f$  for all variants.

### 1.2.2.6 Weyl–Liouville fractional derivatives

**Definition 1.7.** The Weyl–Liouville fractional derivative is defined as

$$(D_{\pm}^\alpha f)(x) = \pm \frac{d}{dx} (I_{\pm}^{1-\alpha} f)(x), \quad (1.33)$$

for  $0 < \alpha < 1$ , where the Weyl integral  $\pm I_{\pm}^{1-\alpha} f$  was defined in (1.8).

### 1.2.2.7 Grünwald–Letnikov fractional derivatives

The basic idea of the Grünwald approach is to generalize finite difference quotients to noninteger order, and then take the limit to obtain a differential quotient.

**Definition 1.8.** The Grünwald–Letnikov fractional derivative of order  $\alpha > 0$  with lower limit  $a$  is defined as

$$({}^{GL}D_{a+}^{\alpha} f)(x) = \lim_{h \rightarrow 0^+} \frac{1}{h^{\alpha}} \sum_{k=0}^{[(x-a)/h]} (-1)^k \binom{\alpha}{k} f(x - kh), \quad (1.34)$$

and the Grünwald–Letnikov fractional derivative of order  $\alpha$  with upper limit  $b$  is defined as the limit

$$({}^{GL}D_{b-}^{\alpha} f)(x) = \lim_{h \rightarrow 0^+} \frac{1}{h^{\alpha}} \sum_{k=0}^{[(b-x)/h]} (-1)^k \binom{\alpha}{k} f(x + kh), \quad (1.35)$$

of fractional difference quotients whenever the limit exists, where  $\binom{\alpha}{k}$  is the generalized binomial coefficient.

### 1.2.2.8 Local fractional derivatives

All fractional derivatives defined above are nonlocal operators. A local fractional derivative (LFD) operators have been introduced in the recent literature [52]. Being local in nature these derivatives have proven useful in studying fractional differentiability properties of highly irregular and nowhere differentiable functions.

**Definition 1.9.** For  $-\infty < a < \infty$  the right (left) Kolwankar–Gangal local fractional derivative (KG–LFD) of order  $0 < \alpha < 1$  at  $x = a$  is defined by

$$({}^L D_{\pm}^{\alpha} f)(a) = f^{(\alpha)}(a \pm 0) = \lim_{x \rightarrow a_{\pm}} (D_{a \pm}^{\alpha} f)(x), \quad (1.36)$$

The function  $f$  is called KG fractionally differentiable at the limit  $a$  whenever the two limits (1.36) exist and are equal.

# Chapter 2

## Abstract Fractional Differential Equations in Banach Spaces

This chapter serves as an introduction to the other chapters. It contains the essential background materials required throughout this thesis. In this section, we recall some materials about differential operators, resolvent operator and family resolvent operator and its applications to abstract fractional differential equations. We will only state results without proofs. See the references [5], [16]–[18], [30], [45] and [65] for more details and summary of this theory.

### 2.1 Preliminaries

Throughout this thesis, most notations used are standard. So  $\mathbb{N}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$  denote the sets of natural, real and complex numbers, respectively, and  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ ,  $\mathbb{R}_+ := [0, \infty)$ ,  $\Sigma_\theta(\omega) := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z - \omega)| < \theta\}$ ,  $\theta \in (0, \pi)$ ,  $\omega \in \mathbb{R}$ ,  $\Sigma_0 = (0, \infty)$  and  $\Sigma_\theta = \Sigma_\theta(0)$  for short, also,  $\partial\Sigma_\theta$  and  $\overline{\Sigma_\theta}$  denote its boundary and closure, respectively.

Let  $X$ ,  $Y$  be Banach spaces with norms  $\|\cdot\|_X$ ,  $\|\cdot\|_Y$ ; the subscripts will be dropped when there is no danger of confusion. By  $B(X, Y)$  we denote the space of all bounded linear operators from  $X$  to  $Y$ ,  $B(X) := B(X, X)$  for short. Unless otherwise specified, we denote by  $E$  the Banach space  $C(J; X)$  of  $X$ -valued continuous functions on  $J$  equipped with the sup norm. If  $A$  is a linear operator in  $X$  then  $D(A)$ ,  $R(A)$ ,  $N(A)$  denote domain, range and null space of  $A$ ,

respectively, while  $\sigma(A)$  and  $\rho(A)$  mean spectrum and resolvent set of  $A$  and  $R(\lambda, A) := (\lambda I - A)^{-1}$  stands for the resolvent operator of  $A$ . We say that the Banach space  $X$  is continuously embedded in the Banach space  $Y$  if  $X \subset Y$  and  $\|\cdot\|_X \leq C\|\cdot\|_Y$ .

Let  $I = (a, b)$ , where  $-\infty \leq a < b \leq \infty$ , and  $1 \leq p < \infty$ . Then  $L^p(I; X)$  denotes the space of all (equivalent classes of) Bochner-measurable functions  $f : I \rightarrow X$ , such that  $\|f(t)\|_X^p$  is integrable for  $t \in I$ . It is a Banach space when normed by

$$\|f\|_{L^p(I; X)} := \left( \int_I \|f(s)\|_X^p ds \right)^{1/p}.$$

If  $p = \infty$  the space  $L^p(I; X)$  consists of all measurable functions with a finite norm

$$\|f\|_{L^\infty(I; X)} := \operatorname{ess\,sup}_{t \in I} \|f(t)\|_X.$$

Let us recall the following definition:

**Definition 2.1** ([65]). A two parameter family of bounded linear operators  $U(t, s)$ ,  $0 \leq s \leq t \leq T$ , on  $X$  is called an evolution system if the following two conditions are satisfied.

- (i)  $U(t, t) = I$ ,  $U(t, r)U(r, s) = U(t, s)$  for  $0 \leq s \leq r \leq t \leq T$ ,
- (ii)  $(t, s) \rightarrow U(t, s)$  is strongly continuous for  $0 \leq s \leq t \leq T$ .

**Definition 2.2** ([65]). A one parameter family  $T(t)$ ,  $0 \leq t < \infty$ , of bounded linear operators from  $X$  into  $X$  is a semigroup of bounded linear operators on  $X$  if

- (i)  $T(0) = I$  (the identity operator in  $X$ ),
- (ii)  $T(t)T(s) = T(t + s)$ , for  $t, s \geq 0$  (the semigroup property).

A semigroup of bounded linear operators  $T(t)$  is uniformly continuous if

$$\lim_{t \rightarrow 0} \|T(t) - I\| = 0.$$

The linear operator  $A$  defined by

$$D(A) = \{x \in X \mid \lim_{h \rightarrow 0} \frac{T(h)(x) - x}{h} \text{ exists}\}$$

and

$$A(x) = \lim_{h \rightarrow 0} \frac{T(h)(x) - x}{h}, \quad \text{for } x \in D(A),$$

is the infinitesimal generator of the semigroup  $T(t)$ ,  $D(A)$  is the domain of  $A$ .

**Definition 2.3** ([65]). A one parameter family  $T(t)$ ,  $0 \leq t < \infty$ , of bounded linear operators on  $X$  is a strongly continuous semigroup of bounded linear operators if

$$\lim_{t \rightarrow 0} T(t)x = x \quad \text{for every } x \in X.$$

A strongly continuous semigroup of bounded linear operators on  $X$  will be called a semigroup of class  $C_0$  or simply  $C_0$  semigroup.

**Theorem 2.4** ([65]). Let  $T(t)$  be a  $C_0$  semigroup. There exists  $\omega \geq 0$  and  $M \geq 1$  such that

$$\|T(t)\| \leq Me^{\omega t}, \quad \text{for } 0 \leq t < \infty.$$

If  $\omega = 0$ ,  $T(t)$  is called uniformly bounded and if moreover  $M = 1$  it is called a  $C_0$  semigroup of contractions.

For more details on  $C_0$  semigroups, we refer the reader to the book of A. Pazy [65].

## 2.2 Operators in Banach spaces

We now turn to a short introduction to sectorial operators

Let  $X$  be a complex Banach space, and let  $A : D(A) \subset X \rightarrow X$  be a closed linear densely defined operator in  $X$ . In the sequel we suppose that  $D(A)$  is equipped with the graph norm of  $A$ , i.e.  $\|x\|_{D(A)} := \|x\|_X + \|Ax\|_X$ ; since  $A$  is closed,  $D(A)$  is a Banach space, continuously and densely embedded into  $X$ .

**Definition 2.5** ([7]). We call an operator  $A : D(A) \subset X \rightarrow X$  nonnegative iff the following two conditions are satisfied:

- (i) there exists  $K \geq 0$  such that for all  $\lambda > 0$  and all  $x \in D(A)$ ,  $\lambda\|x\|_X = K\|\lambda x + Ax\|_X$  holds;
- (ii)  $R(\lambda I + A) = X$  for all  $\lambda > 0$ .

Observe that if  $A$  satisfies (i) and (ii), it is closed. Moreover, any nonnegative operator in a reflexive Banach space is densely defined.

**Definition 2.6** ([7]). An operator  $A$  satisfying (i) of Definition 2.5 with  $K = 1$  is called accretive. If  $A$  moreover satisfies (ii) then  $A$  is called  $m$ -accretive.

Note that if  $A$  is accretive and (ii) holds for some  $\lambda_0 > 0$  then it holds for all  $\lambda > 0$ , so it is  $m$ -accretive.

It is well known that if  $X$  is a Hilbert space with inner product  $(\cdot, \cdot)$  then  $A$  is accretive iff  $\Re(Ax, x) \geq 0$  for all  $x \in D(A)$ .

**Definition 2.7** ([7]). An operator  $A$  is said to be positive if it is nonnegative and  $0 \in \rho(A)$ .

There are many examples of positive operators. For instance, any positive-definite selfadjoint operator acting in Hilbert space is a positive operator. If  $A$  generates a  $C_0$ -semigroup of negative type then  $A$  is a positive operator. The reverse statement, however, is untrue, since there exist positive operators which are not generators of suitable semigroups.

**Definition 2.8** ([28]). The operator  $A$  is called sectorial of angle  $\theta \in [0, \pi)$  ( $A \in \text{Sect}(\theta)$ , in short) if:

(i)  $\sigma(A)$  is contained in the closure of the sector

$$\Sigma_\theta := \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg z| < \theta\};$$

(ii) For every  $\theta' \in (\theta, \pi)$ ,  $\sup\{\|zR(z, A)\| : z \in \mathbb{C} \setminus \overline{\Sigma_{\theta'}}\} < \infty$ .

Obviously, any positive operator is sectorial. Examples of sectorial, but not positive operators are some differential operators on unbounded regions, like the Laplace operator or the Stokes operator on exterior domains.

**Theorem 2.9** ([7]). *An operator  $A$  is sectorial of angle  $\theta \in [0, \pi)$  (sectorial, in short) if it is nonnegative,  $N(A) = \{0\}$  and  $\overline{R(A)} = X$ .*

**Definition 2.10** ([28]). A family of operators  $(A_\tau)_{\tau \in \Lambda}$  is called uniformly sectorial of angle  $\theta \in [0, \pi)$  if  $A_\tau \in \text{Sect}(\theta)$  for each  $\tau$ , and  $\sup\{\|zR(z, A_\tau)\| : \tau \in \Lambda, z \in \mathbb{C} \setminus \overline{\Sigma_{\theta'}}\} < \infty$ .

If  $0 \in \rho(A)$  for a sectorial operator  $A$ , then we can define its fractional powers as follows. For  $b > 0$ , define  $A^{-b}$  by

$$A^{-b} := -\frac{1}{2\pi i} \int_{\Gamma(\zeta)} \lambda^{-b} R(\lambda, A) d\lambda,$$

where  $\zeta \in (\theta, \pi)$  and the path  $\Gamma(\zeta)$  runs in the resolvent set of  $A$  from  $\infty e^{-i\zeta}$  to  $\infty e^{i\zeta}$ , avoiding the negative real axis and the origin, and  $\lambda^{-b}$  is taken as the principle branch (see [28]).

**Definition 2.11** ([76]). Let  $-1 < \gamma < 0$  and  $0 < \theta < \pi/2$ . By  $\Theta_\theta^\gamma(X)$  we denote the family of all linear closed operators  $A : D(A) \subset X \rightarrow X$  which satisfy

(i)  $\Sigma(A) \subset S_\theta = \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg z| \leq \theta\} \cup \{0\}$  and

(ii) For every  $\mu \in (\theta, \pi)$  there exists a constant  $C_\mu$  such that

$$\|R(z, A)\| \leq C_\mu |z|^\gamma \text{ for all } z \in \mathbb{C} \setminus S_\mu.$$

A linear operator  $A$  will be called an almost sectorial operator on  $X$  if  $A \in \Theta_\theta^\gamma(X)$ . Note in particular that if  $A \in \Theta_\theta^\gamma(X)$ , then  $0 \in \rho(A)$  and  $A$  generates a semigroup  $T(t)$  with a singular behavior at  $t = 0$  in a sense, called semigroup of growth  $1 + \gamma$ . Moreover, the semigroup  $T(t)$  is analytic in an open sector of the complex plane  $\mathbb{C}$ , but the strong continuity fails at  $t = 0$  for data which are not sufficiently smooth (see [76]).

## 2.3 Solution Operators

In this section, we introduce preliminary facts about solution operators in Banach spaces which are used throughout this thesis.

### 2.3.1 Resolvent Operators

We consider the quasilinear fractional integrodifferential equation

$$\frac{d^\alpha x(t)}{dt^\alpha} + A(t, x(t))x(t) = f(t, x(t), \int_0^t k(t, s, x(s))ds, \int_0^T h(t, s, x(s))ds), \quad t \in J, \quad (2.1)$$

$$x(0) + g(t_1, t_2, \dots, t_p, x(\cdot)) = x_0, \quad x_0 \in X, \quad (2.2)$$

where  $J = [0, T]$ ,  $0 < \alpha \leq 1$  and  $0 \leq t_1 < t_2 < \dots < t_p \leq T$ ,  $-A(t, \cdot)$  is a closed linear operator defined on a dense domain  $D(A)$  in  $X$  into  $X$  such that  $D(A)$  is independent of  $t$ . It is assumed also that  $-A(t, \cdot)$  generates resolvent operator in the Banach space  $X$ . The nonlinear functions  $f : J \times X \times X \times X \rightarrow X$ ,  $g : J^p \times X \rightarrow X$ ,  $k, h : J \times J \times X \rightarrow X$  are given. The operator  $\frac{d^\alpha}{dt^\alpha}$  denotes the Caputo fractional derivative of order  $\alpha$ .

We now give the definition of resolvent operators

**Definition 2.12** ([17], [72]). A family of bounded linear operators  $R_z(t, s) \in B(X)$ ,  $0 \leq s \leq t \leq T$  is called resolvent operator for equations (2.1) and (2.2) if the following conditions hold:

- (a)  $R_z(t, s)$  is strongly continuous in  $t$  and  $s$ ,  $R_z(t, t) = I$ ,  $t \in J$ .
- (b) For each  $x \in X$ ,  $R_z(t, s)x$  is a continuously differentiable function in  $t$  and  $s$  such that

$$\frac{\partial^\alpha R_z}{\partial t^\alpha}(t, s)x = -A(t, z(t))R_z(t, s)x.$$

With  $\frac{\partial^\alpha R_z}{\partial t^\alpha}(t, s)x$  strongly continuous on  $0 \leq s \leq t \leq b$ . Here  $R_z(t, s)$  can be extracted from the evolution operator of the generator  $-A(t, \cdot)$ . The resolvent operator is similar to the evolution operator for nonautonomous differential equations in a Banach space.

Next we introduce the so-called "Mild Solution" and "Classical Solution" for (2.1)–(2.2). Let us point out that the definition of mild solution will depend on whether the operator solution was considered.

**Definition 2.13** ([17, 72]). A continuous solution  $x$  of the integral equation

$$\begin{aligned} x(t) &= R_x(t, 0)x_0 - R_x(t, 0)g(t_1, t_2, \dots, t_p, x(\cdot)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} R_x(t, s) f(s, x(s), \int_0^s k(s, \tau, x(\tau)) d\tau, \int_0^T h(s, \tau, x(\tau)) d\tau) ds, \end{aligned} \quad (2.3)$$

with  $t \in J$ , is said to be a mild solution of (2.1)–(2.2) on  $J$ .

**Definition 2.14** ([30, 32]). By a classical solution of (2.1)–(2.2) on  $J$ , we mean a function  $x$  with values in  $X$  such that:

- (i)  $x$  is continuous function on  $J$  and  $x(t) \in D(A)$ ,
- (ii)  $\frac{d^\alpha x}{dt^\alpha}$  exists and is continuous on  $(0, T)$ , and satisfying (2.1)–(2.2) on  $J$ .

**Definition 2.15** ([31, 43]). The system (2.1)–(2.2) is said to be controllable on the interval  $J$ , if for every  $x_0, x_1 \in X$  there is exists a control  $u \in L^2(J, U)$  such that the mild solution  $x(t)$  of (2.1)–(2.2) satisfies  $x(0) + g(t_1, t_2, \dots, t_p, x) = x_0$  and  $x(b) = x_1$ .

Also, we need the following lemma

**Lemma 2.16.** [30, Lemma 3.1] *Let  $\Omega \subset X$ ,  $Y$  be a densely and continuously imbedded Banach space in  $X$  and let  $R_z(t, s)$  be the resolvent operator for the problem (2.1)–(2.2), there exists a constant  $C_0 > 0$  such that*

$$\|R_{z_1}(t, s)\omega - R_{z_2}(t, s)\omega\| \leq C_0\|\omega\|_Y \int_s^t \|z_1(\tau) - z_2(\tau)\| d\tau,$$

for every  $z_1, z_2 \in E$  with values in  $\Omega$  and every  $\omega \in Y$ .

### 2.3.2 $(\alpha, x)$ -Resolvent Family

Let  $E$  be the Banach space formed from  $D(A)$  with the graph norm. Since  $-A(t, \cdot)$  is a closed operator, it follows that  $-A(t, \cdot)$  is in the set of bounded operators from  $E$  to  $X$ .

In order to give another definition to an  $(\alpha, x)$ -resolvent family solutions we need the following theoretical approach to our system.

**Definition 2.17** ([31]). Let  $A(t, x)$  be a closed and linear operator with domain  $D(A)$  defined on a Banach space  $X$  and  $\alpha > 0$ . Let  $\rho(A(t, x))$  be the resolvent set of  $A(t, x)$ . We call  $A(t, x)$  the generator of an  $(\alpha, x)$ -resolvent family if there exist  $\omega \geq 0$  and a strongly continuous function  $R_{(\alpha, x)} : \mathbb{R}_+^2 \rightarrow L(X)$  such that  $\{\lambda^\alpha : \Re(\lambda) > \omega\} \subset \rho(A(t, x))$  and for  $0 \leq s \leq t < \infty$ ,

$$[\lambda^\alpha I - A(s, x)]^{-1}x = \int_0^\infty e^{-\lambda(t-s)} R_{(\alpha, x)}(t, s)y dt, \quad \Re(\lambda) > \omega, \quad (x, y) \in X^2.$$

In this case,  $R_{(\alpha, x)}(t, s)$  is called the  $(\alpha, x)$ -resolvent family generated by  $A(t, x)$ .

- (i) We can deduce that (2.1)–(2.2) is well posed if and only if  $-A(t, x)$  is the generator of  $(\alpha, x)$ -resolvent family. ,
- (ii) Here,  $R_{(\alpha, x)}(t, s)$  can be extracted from the evolution operator of the generator  $-A(t, x)$ ,

- (iii) The  $(\alpha, x)$ -resolvent family is similar to the evolution operator for nonautonomous differential equations in a Banach space.

In the case where  $-A(t, x)$  is not dependent on  $x$ , a 1-resolvent family is the same as a  $C_0$ -semigroup whereas that a 2-resolvent family corresponds to the concept of sine family, see [5].

Here we give another definition of mild solution (see [31])

**Definition 2.18** ([17, 31]). Let the linear part of the system (2.1)–(2.2) generates an  $(\alpha, x)$ -resolvent family  $R_{(\alpha, x)}(t, s)$ . A continuous solution  $x$  of the integral equation

$$\begin{aligned} x(t) &= R_{(\alpha, x)}(t, 0)x_0 - R_{(\alpha, x)}(t, 0)g(t_1, t_2, \dots, t_p, x(\cdot)) \\ &\quad + \int_0^t R_{(\alpha, x)}(t, 0)(t, s)f(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau, \int_0^T h(s, \tau, x(\tau))d\tau)ds, \end{aligned} \quad (2.4)$$

with  $t \in J$ , is said to be a mild solution of (2.1)–(2.2) on  $J$ .

### 2.3.3 $\alpha$ -Mildly Solvability

E. Hernández et al. in [47] showed that it is inappropriate to define the mild solution of problem

$$\begin{cases} {}^C D_+^\alpha x(t) + Ax(t) = f(t), & t > 0, \\ x(0) = 1, & 0 < \alpha < 1. \end{cases} \quad (2.5)$$

as follows

$$x(t) = T(t)x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s)f(s)ds,$$

where  $T(t)$  is the semigroup generated by  $A$ , though this fashion was used in some situations of previous research (see, e.g., [50]).

We should point out that in the treatment of problem 2.5 and related problems, one of the difficult points is to give reasonable concept of solutions. Another is

that even though the operator  $A$  generates a semigroup  $T(t)$  in  $X$ , it will not be continuous at  $t = 0$  for nonsmooth initial data  $x_0$  (see [76]).

Let us now give definition of  $\alpha$ -mild solution in a way close to that given by R.-N. Wang et al. and J. Wang et al.

**Definition 2.19** ([17, 31]). Let the linear part  $A$  of the system (2.1)–(2.2) be an operator in the class  $\Theta_\theta^\gamma$ . For each  $u \in L^2(J, X)$ , an  $\alpha$ -mild solution of the system (2.1)–(2.2), we mean a function  $x \in C(J, X)$  which satisfies

$$\begin{aligned} x(t) &= \mathcal{S}_\alpha(t)x_0 - \mathcal{S}_\alpha(t)g(t_1, t_2, \dots, t_p, x(\cdot)) \\ &\quad + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s)(t, s)f(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau, \int_0^T h(s, \tau, x(\tau))d\tau)ds, \end{aligned} \quad (2.6)$$

where the operator families  $\{\mathcal{S}(t)\}_{t \in S_{\pi/2-\theta}^0}$  and  $\{\mathcal{P}(t)\}_{t \in S_{\pi/2-\theta}^0}$  are called characteristic solution operators and given by

$$\mathcal{S}_\alpha(t)x := E_\alpha(-zt^\alpha)(A)x = \int_0^\infty \Psi_\alpha(s)T(st^\alpha)x ds,$$

$$\mathcal{P}_\alpha(t)x := E_{\alpha, \alpha}(-zt^\alpha)(A)x = \int_0^\infty \alpha s \Psi_\alpha(s)T(st^\alpha)x ds,$$

where  $E_{\alpha, \alpha}$ ,  $E_\alpha$  and  $\Psi_\alpha$  are the generalized Mittag–Leffler function, Mittag–Leffler function and Wright–type function respectively (see Appendix A).

We need some basic properties of these families which are used further in this thesis.

**Lemma 2.20.** [76, Theorem 3.1]

(i) For each  $t \in S_{\pi/2-\theta}^0$ ,  $\mathcal{S}_\alpha(t)$  and  $\mathcal{P}_\alpha(t)$  are linear and bounded operators on  $X$ , i.e., for any  $x \in X$ ,

$$\|\mathcal{S}_\alpha(t)x\| \leq M \|x\| \quad \text{and} \quad \|\mathcal{P}_\alpha(t)x\| \leq \frac{M}{\Gamma(\alpha)} \|x\|, \quad (2.7)$$

(ii) There exist constants  $C_s = C(\alpha, \gamma) > 0$ ,  $C_p = C(\alpha, \gamma) > 0$  such that for all  $t > 0$ ,

$$\|\mathcal{S}_\alpha(t)x\| \leq C_s t^{-\alpha(1+\gamma)} \quad \text{and} \quad \|\mathcal{P}_\alpha(t)x\| \leq C_s t^{-\alpha(1+\gamma)}. \quad (2.8)$$

**Lemma 2.21.** [76, Theorem 3.2] For  $t > 0$ ,  $\mathcal{S}_\alpha(t)$  and  $\mathcal{P}_\alpha(t)$  are continuous in the uniform operator topology. Moreover, for every  $r > 0$ , the continuity is uniform on  $[r, \infty)$ .

**Lemma 2.22.** [76, Theorem 3.4] The following properties hold.

(i) Let  $\beta > 1 + \gamma$  for all  $x \in D(A^\beta)$ ,  $\lim_{t \rightarrow 0^+} \mathcal{S}_\alpha(t)x = x$ ;

(ii) For all  $x \in D(A)$ ,  $(\mathcal{S}_\alpha(t) - I)x = \int_0^t -s^{\alpha-1} A \mathcal{P}_\alpha(s)x ds$ ;

## Chapter 3

# Existence and Uniqueness of Solutions of Fractional Quasilinear Mixed Integrodifferential Equations with Nonlocal Condition in Banach Spaces

### 3.1 Introduction

This chapter have been extracted from the research paper [17] .

The existence results to evolution equations with nonlocal conditions in Banach space was studied first by [21], subsequently, many authors were pointed in the same field, see [4, 11, 12, 22, 23, 34]. [34] indicated that, using the nonlocal condition  $x(0) + \sum_{i=1}^p c_i x(t_i) = x_0$  where  $c_i, (i = 1, 2, \dots, p)$  are given constants and  $t = 0 < t_1 < \dots < t_p \leq b$  to describe for instance, the diffusion phenomenon

of a small amount of gas in a transparent tube can give better result than using the usual local Cauchy problem  $x(0) = x_0$ .

Recently, the existence of solutions of fractional abstract differential equations with nonlocal initial condition was investigated by [61]. Much attention has been paid to existence results for the nonlinear mixed integrodifferential equations with nonlocal condition in Banach spaces, see Dhakne et al. [35]. Several authors have studied the existence of solutions of abstract nonlocal problems by using different techniques, see [8, 20, 37, 49, 57, 73, 74] and the references given therein.

Consider the quasilinear fractional integrodifferential equation as in chapter

$$\frac{d^\alpha x(t)}{dt^\alpha} + A(t, x(t))x(t) = f(t, x(t), \int_0^t k(t, s, x(s))ds, \int_0^T h(t, s, x(s))ds), \quad t \in J, \quad (3.1)$$

$$x(0) + g(t_1, t_2, \dots, t_p, x(\cdot)) = x_0, \quad x_0 \in X, \quad (3.2)$$

where  $J = [0, T]$ ,  $0 < \alpha \leq 1$ ,  $0 \leq t_1 < t_2 < \dots < t_p \leq T$ ,  $-A(t, \cdot)$  and the nonlinear functions  $f : J \times X \times X \times X \rightarrow X$ ,  $g : J^p \times X \rightarrow X$ ,  $k, h : J \times J \times X \rightarrow X$  are given as in Chapter 2.

In this chapter our aims is to study the existence, uniqueness and other properties of solutions of the problem (3.1)–(3.2). The main tool employed in our analysis is based on the Banach fixed point theorem, resolvent operators and fractional calculus. Our results generalizes the correspondence results in [35] to nonlocal quasilinear mixed integrodifferential equations of arbitrary orders. We indicate that the method used in this work is different from that in [59].

The rest of this chapter is organized as follows: In section 2 we give some necessarily conditions which ensuring the main results. The existence and uniqueness theorems for the problem (3.1)–(3.2) and their proofs are arranged in section 3. Finally in section 4 we give example to illustrate the application of our results.

## 3.2 Preliminaries

Now, we list the following hypotheses for our convenience. For the rest of this chapter, let  $Z$  be taken as both  $X$  and  $Y$ .

(H1) There exists a constant  $G > 0$  such that

$$\|g(t_1, t_2, \dots, t_p, x_1(\cdot)) - g(t_1, t_2, \dots, t_p, x_2(\cdot))\| \leq G \|x_1 - x_2\|_E$$

for  $x_1, x_2 \in E$ .

(H2) There are constants  $L_1, K_1, H_1, G_1$  and  $M_0$  such that

$$\begin{aligned} L_1 &= \max_{0 \leq t \leq T} \|f(t, 0, 0, 0)\|_Z, \\ K_1 &= \max_{0 \leq s \leq t \leq T} \|k(t, s, 0)\|, \\ H_1 &= \max_{0 \leq s, t \leq T} \|h(t, s, 0)\|, \\ G_1 &= \max_{x \in E} \|g(t_1, t_2, \dots, t_p, x(\cdot))\|_Z, \\ M_0 &= \max_{0 \leq s \leq t \leq T} \|R_z(t, s)\|. \end{aligned}$$

(H3) The constants  $\|x_0\|, M, G_1, L, K, K_1, H, H_1, T$  and  $r$  satisfy the following two inequalities:

$$\begin{aligned} &[C_0 \|x_0\|_Y T + M_0 G + C_0 G T] + \frac{T^{\alpha+1} C_0}{\Gamma(\alpha+1)} [Lr + LT(Kr + K_1) + LT(Hr + H_1) + \\ &L_1] + \frac{T^\alpha M_0 L}{\Gamma(\alpha+1)} [1 + KT + HT] < 1, \end{aligned}$$

and

$$M_0 \|x_0\| + M_0 G_1 + \frac{M_0 T^\alpha}{\Gamma(\alpha+1)} [Lr + TLKr + TLK_1 + TLHr + TLH_1 + L_1] \leq r.$$

With these preparations we are now in a position to state our main results to be proved in the present chapter.

### 3.3 Main Results

**Theorem 3.1.** *Assume that*

(i) *hypotheses (H1)–(H3) hold,*

(ii)  $f : J \times X \times X \times X \rightarrow Z$  is continuous in  $t$  on  $J$  and there exists a constant  $L > 0$  such that

$$\|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)\|_Z \leq L(\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|),$$

for  $x_i, y_i, z_i \in B_r$ ,  $i = 1, 2$ , where  $B_r = \{x \in X : \|x\| \leq r\}$ .

(iii)  $k, h : J \times J \times X \rightarrow X$  are continuous in  $s, t$  on  $J$  and there exist positive constants  $K, H$  such that

$$\|k(t, s, x_1) - k(t, s, x_2)\| \leq K(\|x_1 - x_2\|),$$

$$\|h(t, s, x_1) - h(t, s, x_2)\| \leq H(\|x_1 - x_2\|),$$

for  $x_i, y_i \in B_r$ ,  $i = 1, 2$ .

Then problem (3.1)–(3.2) has a unique mild solution on  $J$ .

*Proof of Theorem 3.1.* We shall use the notions and notations introduced in the preceding section. We define an operator  $F : E \rightarrow E$  by

$$\begin{aligned} (Fz)(t) &= R_z(t, 0)x_0 - R_z(t, 0)g(t_1, t_2, \dots, t_p, z(\cdot)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} R_z(t, s) f(s, z(s), \int_0^s k(s, \tau, z(\tau)) d\tau, \int_0^T h(s, \tau, z(\tau)) d\tau) ds, \end{aligned} \quad (3.3)$$

for  $t \in J$ . It follows from assumption on the functions  $f$ ,  $h$  and  $k$  that  $F : E \rightarrow E$  and for every  $z \in E$ ,  $Fz(0) = x_0 - g(t_1, t_2, \dots, t_p, z(\cdot))$ .

Let  $S$  be the nonempty closed and bounded set given by

$$S = \{z \in E : z(0) = x_0 - g(t_1, t_2, \dots, t_p, z(\cdot)), \|z(t)\| \leq r\}. \quad (3.4)$$

Then for  $z \in S$  we have

$$\begin{aligned}
& \|(Fz)(t)\| \\
& \leq \|R_z(t, 0)x_0\| - \|R_z(t, 0)g(t_1, t_2, \dots, t_p, z(\cdot))\| \\
& \quad + \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} R_z(t, s) f(s, z(s), \int_0^s k(s, \tau, z(\tau)) d\tau, \int_0^T h(s, \tau, z(\tau)) d\tau) ds \right\| \\
& \leq M_0 \|x_0\| + M_0 G_1 + \frac{M_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\|f(s, z(s), \int_0^s k(s, \tau, z(\tau)) d\tau, \\
& \quad \int_0^T h(s, \tau, z(\tau)) d\tau) - f(s, 0, 0, 0)\| + \|f(s, 0, 0, 0)\|] ds \\
& \leq M_0 \|x_0\| + M_0 G_1 + \frac{M_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [L(\|z(s) - 0\| + \|\int_0^s k(s, \tau, z(\tau)) d\tau - 0\| \\
& \quad + \|\int_0^T h(s, \tau, z(\tau)) d\tau - 0\|) + \|f(s, 0, 0, 0)\|] ds \\
& \leq M_0 \|x_0\| + M_0 G_1 \\
& \quad + \frac{M_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Lr + L \int_0^s \|k(s, \tau, z(\tau)) - k(s, \tau, 0) + k(s, \tau, 0)\| d\tau \\
& \quad + L \int_0^T \|h(s, \tau, z(\tau)) - h(s, \tau, 0) + h(s, \tau, 0)\| d\tau + L_1] ds \\
& \leq M_0 \|x_0\| + M_0 G_1 \\
& \quad + \frac{M_0}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Lr + LT(Kr + K_1) + LT(Hr + H_1) + L_1] ds \\
& \leq M_0 \|x_0\| + M_0 G_1 \\
& \quad + \frac{M_0 T^\alpha}{\Gamma(\alpha + 1)} [Lr + TLKr + TLK_1 + TLHr + TLH_1 + L_1] \leq r.
\end{aligned}$$

Thus, we have  $F : S \rightarrow S$ .

Now, for every  $z_1, z_2 \in S$  and  $t \in J$ , we have

$$\begin{aligned}
& \|(Fz_1)(t) - (Fz_2)(t)\| \\
& \leq \|R_{z_1}(t, 0)x_0 - R_{z_2}(t, 0)x_0\| \\
& \quad + \|R_{z_1}(t, 0)g(t_1, t_2, \dots, t_p, z_1(\cdot)) - R_{z_2}(t, 0)g(t_1, t_2, \dots, t_p, z_2(\cdot))\|
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
& \times \left[ \|R_{z_1}(t,0)f(s, z_1(s), \int_0^s k(s, \tau, z_1(\tau))d\tau, \int_0^T h(s, \tau, z_1(\tau))d\tau \right. \\
& \left. - R_{z_2}(t,0)f(s, z_2(s), \int_0^s k(s, \tau, z_2(\tau))d\tau, \int_0^T h(s, \tau, z_2(\tau))d\tau) \| \right] ds \\
& \leq \|R_{z_1}(t,0)x_0 - R_{z_2}(t,0)x_0\| \\
& + \|R_{z_1}(t,0)g(t_1, t_2, \dots, t_p, z_1(\cdot)) - R_{z_2}(t,0)g(t_1, t_2, \dots, t_p, z_2(\cdot))\| \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
& \times \left[ \|R_{z_1}(t,0)f(s, z_1(s), \int_0^s k(s, \tau, z_1(\tau))d\tau, \int_0^T h(s, \tau, z_1(\tau))d\tau \right. \\
& \left. - R_{z_2}(t,0)f(s, z_1(s), \int_0^s k(s, \tau, z_1(\tau))d\tau, \int_0^T h(s, \tau, z_1(\tau))d\tau) \| \right. \\
& \left. + \|R_{z_2}(t,0)f(s, z_1(s), \int_0^s k(s, \tau, z_1(\tau))d\tau, \int_0^T h(s, \tau, z_1(\tau))d\tau \right. \\
& \left. - R_{z_2}(t,0)f(s, z_2(s), \int_0^s k(s, \tau, z_2(\tau))d\tau, \int_0^T h(s, \tau, z_2(\tau))d\tau) \| \right] ds \\
& \leq I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 & = \|R_{z_1}(t,0)x_0 - R_{z_2}(t,0)x_0\| \\
& + \|R_{z_1}(t,0)g(t_1, t_2, \dots, t_p, z_1(\cdot)) - R_{z_2}(t,0)g(t_1, t_2, \dots, t_p, z_2(\cdot))\|,
\end{aligned}$$

$$\begin{aligned}
I_2 & = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
& \times \left[ \|R_{z_1}(t,0)f(s, z_1(s), \int_0^s k(s, \tau, z_1(\tau))d\tau, \int_0^T h(s, \tau, z_1(\tau))d\tau \right. \\
& \left. - R_{z_2}(t,0)f(s, z_1(s), \int_0^s k(s, \tau, z_1(\tau))d\tau, \int_0^T h(s, \tau, z_1(\tau))d\tau) \| \right] ds,
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\quad + \left[ \|R_{z_2}(t,0)f(s, z_1(s), \int_0^s k(s, \tau, z_1(\tau))d\tau, \int_0^T h(s, \tau, z_1(\tau))d\tau) \right. \\
&\quad \left. - R_{z_2}(t,0)f(s, z_2(s), \int_0^s k(s, \tau, z_2(\tau))d\tau, \int_0^T h(s, \tau, z_2(\tau))d\tau) \right] ds.
\end{aligned}$$

Using Lemma (2.16) and hypotheses (H1), (H2), we obtain

$$\begin{aligned}
I_1 &\leq \|R_{z_1}(t,0)x_0 - R_{z_2}(t,0)x_0\| \\
&\quad + \|R_{z_1}(t,0)g(t_1, t_2, \dots, t_p, z_1(\cdot)) - R_{z_1}(t,0)g(t_1, t_2, \dots, t_p, z_2(\cdot))\| \\
&\quad + \|R_{z_1}(t,0)g(t_1, t_2, \dots, t_p, z_2(\cdot)) - R_{z_2}(t,0)g(t_1, t_2, \dots, t_p, z_2(\cdot))\| \\
&\leq C_0 \|x_0\|_Y \int_0^t \|z_1(\tau) - z_2(\tau)\| d\tau \\
&\quad + \|R_{z_1}(t,0)\| \|g(t_1, t_2, \dots, t_p, z_1(\cdot)) - g(t_1, t_2, \dots, t_p, z_2(\cdot))\| \\
&\quad + C_0 \|g(t_1, t_2, \dots, t_p, z_2(\cdot))\|_Y \int_0^t \|z_1(\tau) - z_2(\tau)\| d\tau \\
&\leq C_0 \|x_0\|_Y \|z_1 - z_2\|_E \int_0^t d\tau + M_0 G \|z_1 - z_2\|_E \\
&\quad + C_0 \|g(t_1, t_2, \dots, t_p, z_2(\cdot))\|_Y \|z_1 - z_2\|_E \int_0^t d\tau
\end{aligned}$$

Thus

$$I_1 \leq [C_0 \|x_0\|_Y T + M_0 G + C_0 G T] \|z_1 - z_2\|_E. \quad (3.5)$$

Applying Lemma (2.16), hypotheses (H2), and assumptions (ii), (iii), we get

$$\begin{aligned}
I_2 &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} C_0 \int_0^t \|z_1(\tau) - z_2(\tau)\| d\tau \\
&\quad \times \left[ \|f(s, z_1(s), \int_0^s k(s, \tau, z_1(\tau))d\tau, \int_0^T h(s, \tau, z_1(\tau))d\tau) - f(s, 0, 0, 0)\|_Y \right. \\
&\quad \left. + \|f(s, 0, 0, 0)\|_Y \right] ds
\end{aligned}$$

$$\begin{aligned}
I_2 &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} C_0 \|z_1 - z_2\|_E \int_0^t d\tau \\
&\quad \times [L(\|z(s) - 0\| + \|\int_0^s k(s, \tau, z_1(\tau))d\tau - 0\| \\
&\quad + \|\int_0^T h(s, \tau, z_1(\tau))d\tau - 0\|) + \|f(s, 0, 0, 0)\|_Y] ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} C_0 \|z_1 - z_2\|_E T \\
&\quad \times [Lr + L \int_0^s \|k(s, \tau, z_1(\tau)) - k(s, \tau, 0) + k(s, \tau, 0)\| d\tau \\
&\quad + L \int_0^T \|h(s, \tau, z_1(\tau)) - h(s, \tau, 0) + h(s, \tau, 0)\| d\tau + L_1] ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} C_0 \|z_1 - z_2\|_E T \\
&\quad \times [Lr + LT(Kr + K_1) + LT(Hr + H_1) + L_1] ds \\
&\leq \frac{T^{\alpha+1} C_0}{\Gamma(\alpha + 1)} [Lr + LT(Kr + K_1) + LT(Hr + H_1) + L_1] \|z_1 - z_2\|_E, \\
I_2 &\leq \frac{T^{\alpha+1} C_0}{\Gamma(\alpha + 1)} [Lr + LT(Kr + K_1) + LT(Hr + H_1) + L_1] \|z_1 - z_2\|_E. \quad (3.6)
\end{aligned}$$

Again by using Lemma (2.16), hypotheses (H2), and assumptions (ii), (iii), we obtain

$$\begin{aligned}
I_3 &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|R_{z_2}(t, 0)\| \\
&\quad \times [\|f(s, z_1(s), \int_0^s k(s, \tau, z_1(\tau))d\tau, \int_0^T h(s, \tau, z_1(\tau))d\tau) \\
&\quad - f(s, z_2(s), \int_0^s k(s, \tau, z_2(\tau))d\tau, \int_0^T h(s, \tau, z_2(\tau))d\tau)\|] ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_0 L [\|z_1(s) - z_2(s)\| \\
&\quad + \int_0^s \|k(s, \tau, z_1(\tau)) - k(s, \tau, z_2(\tau))\| d\tau \\
&\quad + \int_0^T \|h(s, \tau, z_1(\tau)) - h(s, \tau, z_2(\tau))\| d\tau] ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_0 L \|z_1 - z_2\|_E [1 + K \int_0^s d\tau + H \int_0^T d\tau] ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M_0 L \|z_1 - z_2\|_E [1 + KT + HT] ds, \\
I_3 &\leq \frac{T^\alpha M_0 L}{\Gamma(\alpha+1)} [1 + KT + HT] \|z_1 - z_2\|_E. \tag{3.7}
\end{aligned}$$

Hence from (3.5)–(3.7), we have

$$\|Fz_1 - Fz_2\|_E \leq q \|z_1 - z_2\|_E,$$

where  $q = [C_0 \|x_0\|_Y T + M_0 G + C_0 GT] + \frac{T^{\alpha+1} C_0}{\Gamma(\alpha+1)} [Lr + LT(Kr + K_1) + LT(Hr + H_1) + L_1] + \frac{T^\alpha M_0 L}{\Gamma(\alpha+1)} [1 + KT + HT]$ , with  $0 < q < 1$ . Thus  $F$  is a strict contraction map from  $S$  into  $S$  and therefore by Banach contraction principle there exists unique fixed point  $x$  of  $F$  in  $S$  and this point is the mild solution of problem (3.1)–(3.2) on  $J$ . This completes the proof of the Theorem 3.1.  $\square$

To establish the existence of unique classical solution to (3.1)–(3.2), we shall require the following lemma.

**Lemma 3.2.** *Assume that  $|\tilde{t}_2 - \tilde{t}_1| \leq 1$  and  $0 < \alpha \leq 1$ . Then, there exists a constant  $N_0 > 0$  such that*

$$\|[R_z(\tilde{t}_2, s) - R_z(\tilde{t}_1, s)]x\| \leq (N_0 M_0 \|x\| + \epsilon) |\tilde{t}_2 - \tilde{t}_1|^\alpha \quad \text{for every } x, z \in D(A). \tag{3.8}$$

*Proof.* It is not difficult to show that  $D(A)$  with the norm  $|x|_A := \|x\| + \|Ax\|$  is a Banach space which we denote by  $X_A$ . Using the closed graph theorem and closeness of  $-A(t, \cdot)$ , we obtain  $-A(t, \cdot)$  is in the Banach space  $B(X_A, X)$  of bounded linear operators from  $X_A$  to  $X$ . Therefore, there exists a constant  $N_0 > 0$  such that

$$\|A(t, z)x\| \leq N_0 \|x\| \quad \text{for every } x, z \in D(A) \quad \text{and every } t \in J.$$

Now, using (iv) of Definition 2.12, we obtain

$$\begin{aligned}
\| [R_z(\tilde{t}_2, s) - R_z(\tilde{t}_1, s)]x \| &\leq \| A(\tilde{t}_1, z)R_z(\tilde{t}_1, s)x \| |\tilde{t}_2 - \tilde{t}_1| + o(\tilde{t}_2 - \tilde{t}_1) \\
&\leq N_0 M_0 \|x\| |\tilde{t}_2 - \tilde{t}_1| + o(\tilde{t}_2 - \tilde{t}_1) \\
&= (N_0 M_0 \|x\| + \epsilon) |\tilde{t}_2 - \tilde{t}_1| \\
&\leq (N_0 M_0 \|x\| + \epsilon) |\tilde{t}_2 - \tilde{t}_1|^\alpha.
\end{aligned} \tag{3.9}$$

□

**Theorem 3.3.** *Assume that*

- (i) *hypotheses (H1)–(H3) hold,*
- (ii) *X is a reflexive Banach space with norm  $\|\cdot\|$  and  $x_0 \in D(A)$ , the domain of  $A(t, \cdot)$ ,*
- (iii)  *$g(t_1, t_2, \dots, t_p, x(\cdot)) \in D(A)$ ,*
- (iv) *There exists a constant  $L > 0$  such that*

$$\begin{aligned}
\| f(t_1, x_1, y_1, z_1) - f(t_2, x_2, y_2, z_2) \|_Z &\leq L(|t_1 - t_2| + \|x_1 - x_2\| + \|y_1 - y_2\| \\
&\quad + \|z_1 - z_2\|),
\end{aligned}$$

- (v) *There exist constants  $K, H > 0$  such that*

$$\| k(t_1, s, x_1) - k(t_2, s, x_2) \| \leq K(|t_1 - t_2| + \|x_1 - x_2\|),$$

$$\| h(t_1, s, x_1) - h(t_2, s, x_2) \| \leq H(|t_1 - t_2| + \|x_1 - x_2\|),$$

*Then  $x$  is a unique classical solution of (3.1)–(3.2) on  $J$ .*

*Proof of Theorem 3.3.* All the assumptions of Theorem 3.1 are being satisfied, then problem (3.1)–(3.2) has a unique mild solution belonging to  $S$  and given by

$$\begin{aligned} x(t) &= R_x(t, 0)x_0 - R_x(t, 0)g(t_1, t_2, \dots, t_p, x(\cdot)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} R_x(t, s) f(s, x(s), \int_0^s k(s, \tau, x(\tau)) d\tau, \int_0^T h(s, \tau, x(\tau)) d\tau) ds. \end{aligned} \quad (3.10)$$

Since  $J$  is compact it is easy to check that  $x$  is Hölder continuous on  $J$  if it is locally Hölder continuous. Now we will show that  $x$  is locally Hölder continuous.

For simplification, set

$$\tilde{f}(t) = f(t, x(t), \int_0^t k(t, s, x(s)) ds, \int_0^T h(t, s, x(s)) ds). \quad (3.11)$$

Then (3.10) can be written as

$$x(t) = R_x(t, 0)x_0 - R_x(t, 0)g(t_1, t_2, \dots, t_p, x(\cdot)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} R_x(t, s) \tilde{f}(s) ds. \quad (3.12)$$

Since  $x$  is continuous on  $J$  and the map  $f$  satisfy the assumptions (iv) and (v), it follows that  $\tilde{f}$  is continuous, and therefore bounded on  $J$ . Next, let  $t \in J$  be fixed and let  $\tilde{t}_1, \tilde{t}_2$  be in  $(t - \delta, t + \delta)$  with  $\tilde{t}_1 \leq \tilde{t}_2$  and  $\delta > 0$ , we have

$$\begin{aligned} x(\tilde{t}_2) - x(\tilde{t}_1) &= [R_x(\tilde{t}_2, 0) - R_x(\tilde{t}_1, 0)]x_0 - [R_x(\tilde{t}_2, 0) - R_x(\tilde{t}_1, 0)]g(t_1, t_2, \dots, t_p, x(\cdot)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{\tilde{t}_1}^{\tilde{t}_2} (\tilde{t}_2 - s)^{\alpha-1} R_x(\tilde{t}_2, s) \tilde{f}(s) ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^{\tilde{t}_1} [(\tilde{t}_1 - s)^{\alpha-1} - (\tilde{t}_2 - s)^{\alpha-1}] R_x(\tilde{t}_2, s) \tilde{f}(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^{\tilde{t}_1} (\tilde{t}_1 - s)^{\alpha-1} [R_x(\tilde{t}_2, s) - R_x(\tilde{t}_1, s)] \tilde{f}(s) ds. \end{aligned} \quad (3.13)$$

$$\|x(\tilde{t}_2) - x(\tilde{t}_1)\| \leq \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4. \quad (3.14)$$

Using Lemma (3.2) for a small enough  $\delta > 0$ , we get

$$\begin{aligned} \tilde{I}_1 &\leq \| [R_x(\tilde{t}_2, 0) - R_x(\tilde{t}_1, 0)]x_0 - [R_x(\tilde{t}_2, 0) - R_x(\tilde{t}_1, 0)]g(t_1, t_2, \dots, t_p, x(\cdot)) \| \\ &\leq [N_0M_0\|x\| + N_0M_0G_1 + 2\epsilon]|\tilde{t}_2 - \tilde{t}_1|^\alpha, \end{aligned} \quad (3.15)$$

for  $\tilde{I}_2$ , we have

$$\begin{aligned} \tilde{I}_2 &\leq \frac{1}{\Gamma(\alpha)} \int_{\tilde{t}_1}^{\tilde{t}_2} (\tilde{t}_2 - s)^{\alpha-1} \|R_x(\tilde{t}_2, s)\tilde{f}(s)\| ds \\ &\leq \frac{M_0N_1}{\Gamma(\alpha + 1)} |\tilde{t}_2 - \tilde{t}_1|^\alpha, \end{aligned} \quad (3.16)$$

and for  $\tilde{I}_3$ , we have

$$\begin{aligned} \tilde{I}_3 &\leq \frac{1}{\Gamma(\alpha)} \int_0^{\tilde{t}_1} |(\tilde{t}_1 - s)^{\alpha-1} - (\tilde{t}_2 - s)^{\alpha-1}| \|R_x(\tilde{t}_2, s)\tilde{f}(s)\| ds \\ &\leq \frac{M_0N_1}{\Gamma(\alpha)} \int_0^{\tilde{t}_1} |(\tilde{t}_1 - s)^{-\mu} - (\tilde{t}_2 - s)^{-\mu}| ds, \end{aligned} \quad (3.17)$$

with  $\mu = 1 - \alpha$ . Here we can use the calculation presented in [64, Theorem 3.2] to find the upper bound of integral and thus we get

$$\tilde{I}_3 \leq \frac{M_0N_1}{\Gamma(\alpha)} \mu \delta_1^{\mu-1} (1-c)^{(\mu-1)-1} |\tilde{t}_2 - \tilde{t}_1|^{1-\mu}, \quad (3.18)$$

where  $c = (1 - \mu)^{\frac{1}{\mu}}$ ,  $0 < \delta_1 \leq 1$  and  $N_1 = \sup_{t \in J} \|\tilde{f}(t)\|$ .

Using again (3.8), we may calculate the bound of  $\tilde{I}_4$  as

$$\begin{aligned} \tilde{I}_4 &\leq \frac{1}{\Gamma(\alpha)} \int_0^{\tilde{t}_1} (\tilde{t}_1 - s)^{\alpha-1} \| [R_x(\tilde{t}_2, s) - R_x(\tilde{t}_1, s)]\tilde{f}(s)\| ds \\ &\leq \frac{(N_0M_0N_1 + \epsilon)}{\Gamma(\alpha)} \int_0^{\tilde{t}_1} (\tilde{t}_1 - s)^{\alpha-1} |\tilde{t}_2 - \tilde{t}_1|^\alpha ds \\ &\leq \frac{T^\alpha}{\Gamma(\alpha + 1)} (N_0M_0N_1 + \epsilon) |\tilde{t}_2 - \tilde{t}_1|^\alpha, \end{aligned} \quad (3.19)$$

Hence from (3.15)–(3.19), locally Hölder continuity of  $x(t)$  follows.

As pointed out earlier in this proof, we may deduce that  $x(t)$  is Hölder continuous on  $J$ . The Hölder continuity of  $x(t)$  on  $J$  combined with (iv) and (v) of Theorem (3.3) implies  $\tilde{f}(t)$  is Hölder continuous on  $J$ . According to [30, Theorem 3.4] and [65], we observe that the equation

$$\begin{aligned} \frac{d^\alpha y(t)}{dt^\alpha} + A(t, y(t))y(t) &= f(t, x(t), \int_0^t k(t, s, x(s))ds, \int_0^T h(t, s, x(s))ds), \quad t \in J \\ y(0) &= x_0 - g(t_1, t_2, \dots, t_p, x(\cdot)) \end{aligned}$$

has a unique classical solution  $y(t)$  on  $J$  satisfying the equation

$$\begin{aligned} y(t) &= R_x(t, 0)x_0 - R_x(t, 0)g(t_1, t_2, \dots, t_p, x(\cdot)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} R_x(t, s) f(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau, \int_0^T h(s, \tau, x(\tau))d\tau) ds \\ &= x(t), \quad t \in J. \end{aligned}$$

Consequently,  $x(t)$  is the classical solution of initial value problem (3.1)–(3.2) on  $J$ . This completes the proof of Theorem 3.3.  $\square$

**Theorem 3.4.** *Suppose that the functions  $f, g, k$  and  $h$  satisfy hypotheses (H1)–(H4) and assumptions (ii), (iii) of Theorem 3.1. Then, for each pair of elements  $x_0^*, x_0^{**} \in X$ , and for the corresponding mild solutions  $x_1, x_2$  of problem (3.1) with  $x_1(t_0) + g(t_1, t_2, \dots, t_p, x_1(\cdot)) = x_0^*$  and  $x_2(t_0) + g(t_1, t_2, \dots, t_p, x_2(\cdot)) = x_0^{**}$ , the inequality*

$$\|x_1 - x_2\|_E \leq \frac{M_0}{(1-p_1)} \|x_0^* - x_0^{**}\| E_\alpha \left( \frac{p_2}{(1-p_1)} \Gamma(\alpha) t^\alpha \right),$$

is true, whenever

$$\begin{aligned} p_1 &= \frac{T^{\alpha+1}C_0}{\Gamma(\alpha+1)}[Lr + LT(Kr + K_1) + LT(Hr + H_1) + L_1] \\ &\quad + [C_0\|x_0\|_Y T + M_0G + C_0GT] \\ &< 1 \end{aligned}$$

and  $p_2 = M_0L[1 + KT + HT]$ .

*Proof of Theorem 3.4.* Suppose that  $x_1(t)$  and  $x_2(t)$  satisfy (3.1) on  $J$  with  $x_1(t_0) + g(t_1, t_2, \dots, t_p, x_1(\cdot)) = x_0^*$  and  $x_2(t_0) + g(t_1, t_2, \dots, t_p, x_2(\cdot)) = x_0^{**}$ , respectively and  $x_1, x_2 \in E$ . Using the equation (2.6), hypotheses (H1)–(H4) and assumptions (ii), (iii), we obtain

$$\begin{aligned} &\|x_1(t) - x_2(t)\| \\ &\leq \|R_{x_1}(t, 0)x_0^* - R_{x_2}(t, 0)x_0^{**}\| \\ &\quad + \|R_{x_1}(t, 0)g(t_1, t_2, \dots, t_p, x_1(\cdot)) - R_{x_2}(t, 0)g(t_1, t_2, \dots, t_p, x_2(\cdot))\| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times \left[ \|R_{x_1}(t, 0)f(s, x_1(s), \int_0^s k(s, \tau, x_1(\tau))d\tau, \int_0^T h(s, \tau, x_1(\tau))d\tau) \right. \\ &\quad \left. - R_{x_2}(t, 0)f(s, x_2(s), \int_0^s k(s, \tau, x_2(\tau))d\tau, \int_0^T h(s, \tau, x_2(\tau))d\tau) \| ds \right] \\ &\leq \|R_{x_1}(t, 0)x_0^* - R_{x_2}(t, 0)x_0^*\| + \|R_{x_2}(t, 0)x_0^* - R_{x_2}(t, 0)x_0^{**}\| \\ &\quad + \|R_{x_1}(t, 0)g(t_1, t_2, \dots, t_p, x_1(\cdot)) - R_{x_1}(t, 0)g(t_1, t_2, \dots, t_p, x_2(\cdot))\| \\ &\quad + \|R_{x_1}(t, 0)g(t_1, t_2, \dots, t_p, x_2(\cdot)) - R_{x_2}(t, 0)g(t_1, t_2, \dots, t_p, x_2(\cdot))\| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times \left[ \|R_{x_1}(t, 0)f(s, x_1(s), \int_0^s k(s, \tau, x_1(\tau))d\tau, \int_0^T h(s, \tau, x_1(\tau))d\tau) \right. \\ &\quad \left. - R_{x_2}(t, 0)f(s, x_1(s), \int_0^s k(s, \tau, x_1(\tau))d\tau, \int_0^T h(s, \tau, x_1(\tau))d\tau) \| ds \right] \end{aligned}$$

$$\begin{aligned}
& + \|R_{x_2}(t, 0)f(s, x_1(s), \int_0^s k(s, \tau, x_1(\tau))d\tau, \int_0^T h(s, \tau, x_1(\tau))d\tau) \\
& - R_{x_2}(t, 0)f(s, x_2(s), \int_0^s k(s, \tau, x_2(\tau))d\tau, \int_0^T h(s, \tau, x_2(\tau))d\tau)\| \Big] ds
\end{aligned}$$

Now, we can use the same calculation presented in proof of Theorem (3.1) to find

$$\begin{aligned}
\|x_1(t) - x_2(t)\| & \leq M_0\|x_0^* - x_0^{**}\| + p_1\|x_1 - x_2\|_E \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_2 \|x_1 - x_2\|_E ds,
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\|x_1 - x_2\|_E & \leq \frac{M_0}{(1-p_1)} \|x_0^* - x_0^{**}\| \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{p_2}{(1-p_1)} \|x_1 - x_2\|_E ds.
\end{aligned}$$

Using Lemma (A.7), we get

$$\|x_1 - x_2\|_E \leq \frac{M_0}{(1-p_1)} \|x_0^* - x_0^{**}\| E_\alpha\left(\frac{p_2}{(1-p_1)} \Gamma(\alpha) t^\alpha\right),$$

provided that  $p_1 < 1$ . From this inequality, it follows that the continuous dependence of solutions depends upon the initial data. This completes the proof of the Theorem 3.4.  $\square$

### 3.4 Application

In this section we present an example to illustrate the applications of some of our main results, we consider the fractional mixed Volterra–Fredholm partial integrodifferential equation

$$\begin{aligned} & \frac{\partial^\alpha w(u, t)}{\partial t^\alpha} + a(u, t, w(u, t)) \frac{\partial^2 w(u, t)}{\partial u^2} \\ &= P(t, w(u, t), \int_0^t k_1(t, s, w(u, s)) ds, \int_0^T h_1(t, s, w(u, s)) ds) \quad (3.20) \\ & 0 < u < 1, \quad 0 \leq t \leq T \end{aligned}$$

with initial and boundary conditions

$$w(0, t) = w(1, t) = 0, \quad 0 \leq t \leq T, \quad (3.21)$$

$$w(u, 0) + \sum_{i=1}^p w(u, t_i) = w_0(u), \quad 0 < t_1 < t_2 < \dots < t_p \leq T. \quad (3.22)$$

where  $a : (0, 1) \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $P : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $k_1, h_1 : [0, T] \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous functions.

First, we reduce the equations (3.20)–(3.22) into (3.1)–(3.2) by making suitable choices of  $A, f, g, k$  and  $h$ .

Let  $X = L^2[0, 1]$  be the space of square integrable functions. Define the operator  $A(t, \cdot) : X \rightarrow X$  by  $(A(t, \cdot)z)(u) = a(u, t, \cdot)z''$  with dense domain  $D(A(t, \cdot)) = \{z \in X : z, z' \text{ are absolutely continuous, } z'' \in X \text{ and } z(0) = z(1) = 0\}$ , generates an evolution system and  $R_x(t, s)$  can be extracted from evolution system, such that  $\|R_x(t, s)\| \leq M_0$ ,  $M_0 > 0$  for  $s < t$  and  $x \in \Omega \subset X$  (see [30, 31, 70]).

Define the functions  $f : [0, T] \times X \times X \times X \rightarrow X$ ,  $k : [0, T] \times [0, T] \times X \rightarrow X$ ,  $h : [0, T] \times [0, T] \times X \rightarrow X$  and  $g : [0, T]^p \times X \rightarrow X$  as follows

$$\begin{aligned} f(t, x, y, z)(u) &= P(t, x(u), y(u), z(u)), \\ k(t, s, x)(u) &= k_1(t, s, x(u)), \\ h(t, s, x)(u) &= h_1(t, s, x(u)), \\ g(t_1, t_2, \dots, t_p, x(\cdot))u &= \sum_{i=1}^p w(u, t_i) \end{aligned}$$

for  $t \in [0, T]$ ,  $x, y, z \in X$  and  $0 < u < 1$ . We assume that the functions  $P$ ,  $k_1$  and  $h_1$  in (3.20) satisfy all the hypotheses of the Theorem 3.1. Also we suppose that

$$\left| \sum_{i=1}^p w(u, t_i) - \sum_{i=1}^p w(v, t_i) \right| \leq G^* \sup_{t \in [0, T]} |u(t) - v(t)|$$

for  $u, v \in E_1 = C([0, T]; \mathbb{R})$  and some constant  $G^* > 0$ . Then the above problem (3.20)–(3.22) can be formulated abstractly as quasilinear mixed integrodifferential equation in Banach space  $X$ :

$$\frac{\partial^\alpha x(t)}{\partial t^\alpha} + A(t, x(t))x(t) = f(t, x(t), \int_0^t k(t, s, x(s))ds, \int_0^T h(t, s, x(s))ds), \quad t \in J \quad (3.23)$$

$$x(t_0) + g(t_1, t_2, \dots, t_p, x(\cdot)) = x_0. \quad (3.24)$$

Since all the hypotheses of the Theorem 3.1 are satisfied, the Theorem 3.1 can be applied to guarantee the mild solution of the fractional mixed Volterra–Fredholm partial integrodifferential equations (3.20)–(3.22).

# Chapter 4

## Existence and Controllability

### Result for an Evolution Fractional Integrodifferential Systems

#### 4.1 Introduction

This Chapter essentially corresponds to the paper [16].

In the recent years, the studies of fractional differential equations and control problems have attracted the attention of many mathematician and physicists.

A large amount of literature developed concerning fractional differential equations [15, 33, 53, 54], and their controllability [15] to investigated various scientific models. Motivated by the fact that many partial fractional differential and integrodifferential equations can be converted into fractional equations in some Banach spaces [11, 39], we feel that there is a real need to discuss the controllability problem of fractional order systems in abstract spaces [72]. In this Chapter, we study the existence and controllability result with the help of resolvent operators.

This Chapter is organized as follows: Section 4.2 gives the necessary preliminaries. In Sections 4.3, the existence result of the fractional integrodifferential systems with the help of resolvent operators is studied. In Sections 4.4 the controllability

of the fractional integrodifferential control systems is investigated via the Banach fixed point theorem. The last Section is devoted to an illustrated example.

## 4.2 Preliminaries

In the Section 4.3, we consider the nonautonomous semilinear evolution fractional integrodifferential system of the form

$$\frac{d^\alpha x(t)}{dt^\alpha} = A(t)x(t) + f\left(t, x(t), \int_0^t h(t, s, x(s))ds\right), \quad (4.1)$$

$$x(0) = x_0, \quad (4.2)$$

where  $t \in J := [0, b]$ ,  $0 < \alpha < 1$ , the state  $x(\cdot)$  takes values in the Banach space  $X$ ,  $x_0 \in X$ . Here the family  $\{A(t)\}_{t \in J}$  is a closed linear operator on  $X$  with dense domain  $D(A)$  which is independent of  $t$  and generates an evolution operators in the Banach space  $X$ . The nonlinear operators  $h : \Delta \times X \rightarrow X$  and  $f : J \times X \times X \rightarrow X$  are given function and  $\Delta = \{(t, s) \mid 0 \leq s \leq t \leq b\}$ .

In the Section 4.4, we consider the semilinear evolution fractional integrodifferential control system of the form

$$\frac{d^\alpha x(t)}{dt^\alpha} = A(t)x(t) + f\left(t, x(t), \int_0^t h(t, s, x(s))ds\right) + Bu(t), \quad (4.3)$$

$$x(0) = x_0, \quad (4.4)$$

where the control function  $u(\cdot)$  is given in  $L^2(J, U)$ , a Banach space of admissible control functions with  $U$  as a Banach space and  $B$  is a bounded linear operator from  $U$  to  $X$ .

Now, we shall make the following hypotheses

(H1)  $f : J \times X \times X \rightarrow X$  is continuous and there exists constants  $M_1, M_2 > 0$ , such that for all  $x_i, y_i \in X, i = 1, 2$  we have

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq M_1 [\|x_1 - x_2\| + \|y_1 - y_2\|],$$

and

$$M_2 = \max_{t \in J} \|f(t, 0, 0)\|.$$

(H2)  $h : \Delta \times X \rightarrow X$  is continuous and there exists constants  $L_1, L_2 > 0$ , such that for all  $x_1, x_2 \in X$  we have

$$\|h(t, s, x_1) - h(t, s, x_2)\| \leq L_1 \|x_1 - x_2\|,$$

and

$$L_2 = \max_{t, s \in \Delta} \|h(t, s, 0)\|.$$

(H3) Denote  $M = \max_{t, s \in J} \|R(t, s)\|$  for  $s < t$ .

(H4) Let  $M \|x_0\| + \gamma M N_1 \leq n$  for some  $n > 0$ , where  $N_1 = M_1 (n + nL_1b + L_2b) + M_2$ ,  $\gamma = \frac{b^\alpha}{\Gamma(\alpha+1)}$  and  $p_1 = \gamma M [M_1 + M_1 L_1 b]$  be such that  $0 \leq p_1 < 1$ .

Under these assumptions, we can prove the existence of mild solutions.

### 4.3 Existence result

**Theorem 4.1.** *Let assumptions (H1)–(H4) be satisfied. Then for every  $x_0 \in X$ , problem 4.1–4.2 has a mild solution on  $J$ .*

**Proof** As in [70], we let  $x_0 \in X$  be fixed and let  $Z = C(J, X)$  be the Banach space of all continuous functions from  $J$  into  $X$  with sup-norm and  $Z_n = \{x \mid x \in Z, x(0) = x_0, \|x(t)\| \leq n, \text{ for } t \in J\}$ . Define an operator  $\Phi : Z_n \rightarrow Z_n$  by

$$\Phi x(t) = R(t, 0)x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} R(t, s) f(s, x(s), Hx(s)) ds,$$

where  $Hx(s) = \int_0^s h(s, \tau, x(\tau)) d\tau$ .

First we show that  $\Phi$  maps  $Z_n$  into it self. For  $x \in Z_n$

$$\begin{aligned}
\| \Phi x(t) \| &\leq \| R(t, 0) x_0 \| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left\| \int_0^t (t-s)^{\alpha-1} R(t, s) f(s, x(s), Hx(s)) ds \right\| \\
&\leq \| R(t, 0) x_0 \| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| R(t, s) \| \\
&\quad \times \left[ \| f(s, x(s), Hx(s)) - f(s, 0, 0) \| + \| f(s, 0, 0) \| \right] ds \\
&\leq M \|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)} M [M_1 (n + nL_1b + L_2b) + M_2] \\
&\leq M \|x_0\| + \gamma MN_1 \\
&\leq n.
\end{aligned}$$

Therefore  $\Phi$  maps  $Z_n$  into it self. Now for  $x, y \in Z_n$  we have

$$\begin{aligned}
\| \Phi x(t) - \Phi y(t) \| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| R(t, s) \| \| f(s, x(s), Hx(s)) \\
&\quad - f(s, y(s), Hy(s)) \| ds \\
&\leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds [M_1 + M_1L_1b] \|x - y\| \\
&\leq \frac{Mb^\alpha}{\Gamma(\alpha+1)} [M_1 + M_1L_1b] \|x - y\|,
\end{aligned}$$

Thus

$$\| \Phi x - \Phi y \| \leq \gamma M [M_1 + M_1L_1b] \|x - y\| = p_1 \|x - y\|.$$

which implies that  $\Phi$  is a contraction on  $Z_n$  by assumption (H4).

## 4.4 Controllability result

For all  $x_0 \in X$  and admissible control  $u \in L^2(J, U)$ , system 4.3–4.4 admits a mild solution give by

$$x_u(t) = R(t, 0)x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} R(t, s) \left[ Bu(s) + f(s, x(s), Hx(s)) \right] ds.$$

To prove the controllability result, we further consider the following additional conditions.

(H5) The bounded linear operator  $W : L^2(J, U) \rightarrow X$  defined by

$$Wu = \frac{1}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} R(b, s) Bu(s) ds,$$

has an induced inverse operator  $\tilde{W}^{-1}$  which takes values in  $L^2(J, U)/\ker W$  and there exists constants  $K_1, K_2 > 0$ , such that  $\|B\| \leq K_1$  and  $\|\tilde{W}^{-1}\| \leq K_2$

(H6) Let  $M\|x_0\| + \gamma MK_1 K_2 [\|x_1\| + M\|x_0\| + \gamma MN_2] + \gamma MN_2 \leq k$  for some  $k > 0$ , where  $N_2 = M_1(k + kL_1b + L_2b) + M_2$  and  $\gamma = \frac{b^\alpha}{\Gamma(\alpha+1)}$ .

(H7)  $p_2 = \gamma MM_1 [MK_1 K_2 \gamma + 1] [1 + L_1 b]$  be such that  $0 \leq p_2 < 1$ .

**Theorem 4.2.** *If the hypotheses (H1)–(H3) and (H5)–(H7) are satisfied, then the control system 4.3–4.4 is controllable on  $J$ .*

**Proof** Let  $Z_k = \{x \mid x \in Z, x(0) = x_0, \|x(t)\| \leq k, \text{ for } t \in J\}$ . Define an operator  $\Omega : Z_k \rightarrow Z_k$  by

$$\begin{aligned} \Omega x_u(t) &= R(t, 0)x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1} R(t, \eta) B\tilde{W}^{-1} \left[ x_1 - R(b, 0)x_0 \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} R(b, s) f(s, x(s), Hx(s)) ds \right] (\eta) d\eta \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} R(t, s) f(s, x(s), Hx(s)) ds. \end{aligned}$$

Using the hypothesis (H5), for an arbitrary function  $x(\cdot)$  choose the control

$$u(t) = \tilde{W}^{-1} \left[ x_1 - R(b, 0)x_0 - \frac{1}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} R(b, s) f(s, x(s), Hx(s)) ds \right] (t).$$

Using this control we shall show that the operator  $\Omega$  has a fixed point. This fixed point is then a solution of the control problem 4.3–4.4. Clearly  $\Omega x(b) = x_1$ , which means that the control  $u$  steers the system 4.3–4.4 from initial state  $x_0$  to  $x_1$  in time  $b$  provided we can obtain a fixed point of the operator  $\Omega$ . As in [11, 70] we first show that  $\Omega$  maps  $Z_k$  into itself.

$$\begin{aligned} \|\Omega x_u(t)\| &\leq \|R(t, 0)x_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1} \|R(t, \eta)\| \|B\| \|\tilde{W}^{-1}\| \\ &\quad \times \left[ \|x_1\| + \|R(b, 0)x_0\| + \frac{1}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} \|R(b, s)\| \right. \\ &\quad \times \left. \left[ \|f(s, x(s), Hx(s)) - f(s, 0, 0)\| + \|f(s, 0, 0)\| \right] ds \right] d\eta \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|R(t, s)\| \left[ \|f(s, x(s), Hx(s)) \right. \\ &\quad \left. - f(s, 0, 0)\| + \|f(s, 0, 0)\| \right] ds \\ &\leq M \|x_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1} MK_1K_2 \left[ \|x_1\| + M \|x_0\| \right. \\ &\quad \left. + \frac{b^\alpha}{\Gamma(\alpha+1)} M [M_1(k + kL_1b + L_2b) + M_2] \right] d\eta \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M [M_1(k + kL_1b + L_2b) + M_2] ds \\ &\leq M \|x_0\| + \gamma MK_1K_2 [\|x_1\| + M \|x_0\| + \gamma MN_2] + \gamma MN_2 \\ &\leq k. \end{aligned}$$

Thus,  $\Omega$  maps  $Z_k$  into itself. Now, for  $x, y \in Z_k$ , we have

$$\begin{aligned} \|\Omega x(t) - \Omega y(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1} \|R(t, \eta) B \tilde{W}^{-1}\| \\ &\quad \times \frac{1}{\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \|R(b, s)\| \left[ \|f(s, x(s), Hx(s)) \right. \\ &\quad \left. - f(s, y(s), Hy(s)) \right] \|ds d\eta \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \| R(t,s) \left[ f(s, x(s), Hx(s)) \right. \\
& \left. - f(s, y(s), Hy(s)) \right] \| ds
\end{aligned}$$

Therefore

$$\begin{aligned}
\| \Omega x(t) - \Omega y(t) \| & \leq \frac{b^\alpha}{\Gamma(\alpha+1)} MK_1 K_2 \frac{b^\alpha}{\Gamma(\alpha+1)} M [M_1 + M_1 L_1 b] \|x - y\| \\
& + \frac{b^\alpha}{\Gamma(\alpha+1)} M [M_1 + M_1 L_1 b] \|x - y\| \\
& \leq \gamma M M_1 [MK_1 K_2 \gamma + 1] [1 + L_1 b] \|x - y\|,
\end{aligned}$$

Thus

$$\| \Omega x - \Omega y \| \leq p_2 \|x - y\|.$$

From assumption (H7), we find that  $\Omega$  is a contraction mapping on  $Z_k$  and hence there exists a unique fixed point  $x \in Z_k$ . Any fixed point of  $\Phi$  is a Mild solution of 4.3–4.4 which satisfies  $x(b) = x_1$ . Thus, system 4.3–4.4 is controllable on  $J$ .

## 4.5 Application

In this section we present an example to illustrate our main results. Let us consider the following nonlinear partial integrodifferential equation of the form

$$\frac{\partial^\alpha}{\partial t^\alpha} z(t, y) = a(t, y) \frac{\partial^2}{\partial y^2} z(t, y) + \mu(t, y) + \mu_1 \left( t, z(t, y), \int_0^t \mu_2(t, s, z(t, y)) ds \right), \tag{4.5}$$

$$z(0, y) = z_0(y), \quad 0 < y < 1, \tag{4.6}$$

$$z(t, 0) = z(t, 1), \quad t \in J = [0, 1], \tag{4.7}$$

where  $0 < \alpha < 1$ ,  $a$  and  $\mu : J \times [0, 1] \rightarrow [0, 1]$  are continuous functions.

Assume that nonlinear functions  $\mu_1, \mu_2$  satisfy the following Lipschitz conditions

$$\|\mu_1(t, v_1, w_1) - \mu_1(t, v_2, w_2)\| \leq W_1 [\|v_1 - v_2\| + \|w_1 - w_2\|],$$

$$\|\mu_2(t, s, w_1) - \mu_2(t, s, w_2)\| \leq W_2 \|w_1 - w_2\|.$$

where  $W_1, W_2 > 0$ ,  $v_i, w_i \in X, i = 1, 2$ .

Now, let us take  $X = U = L^2[0, 1]$ , and define  $A(t) : X \rightarrow X$  by

$$(A(t)w)(y) = a(t, y)w'',$$

with domain  $D(A) = \{w \in X \mid w, w'' \text{ are absolutely continuous, } w'' \in X, w(0) = w(1) = 0\}$ , generates an evolution system (see [70]), such that  $R(t, s) \leq k$ ,  $k > 0$  for  $s < t$ .

Let  $Bu : J \rightarrow X$  be defined by

$$(Bu)(t)(y) = \mu(t, y), \quad y \in (0, 1).$$

Assume that the linear operator  $W$  is given by

$$(Wu)(y) = \frac{1}{\Gamma(\alpha)} \int_0^b (t-s)^{\alpha-1} R(b, s) \mu(s, y) ds, \quad y \in (0, 1),$$

has a bonded invertible operator  $\tilde{W}^{-1}$  in  $L^2(J, U)/\ker W$ .

With the choice of  $A, B, h$  and  $f$ , we see that Eqs. 4.3–4.4 is the abstract formulation of Eqs. 4.5–4.6.

Further, all the conditions states in Theorem 4.2 are satisfied. Hence, system 4.5–4.6 is controllable on  $J$ .

# Chapter 5

## Controllability of Abstract Fractional Differential Equations with Almost Sectorial Operators

### 5.1 Introduction

The present chapter is devoted to some controllability results for nonlinear fractional integrodifferential systems in infinite dimensional spaces. without imposing severe compactness condition on the characteristic solution operators, we obtain sufficient conditions for the controllability results by using fractional calculus, almost sectorial operators theory and Schauder's second fixed point theorem.

Surely there are works dealing with controllability of solutions to this type of equations ( cf., for instance, [2, 11, 27, 71]). However, most existing results concern the case where the order is integer and the extension to the case of non-integer order looks quite natural. Some of our results are an extension of L Gorniewicz [43] to the situation where the order is non integer. Techniques employed in [43], have been generalized to the study of this type of equation. That is, under different appropriate assumptions, controllability of solutions are obtained, using a fixed point argument. The main tool in the approach followed in our proves is

the properties of characteristic solution operators of the theory of almost sectorial operators.

Now consider the following system represented by the fractional integrodifferential equation of the form

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + f\left(t, x(t), \int_0^t h(t, s, x(s)) ds\right) + Bu(t), \quad t \in J := [0, b], \quad (5.1)$$

$$x(0) = x_0, \quad (5.2)$$

where  $0 < \alpha < 1$ ,  $h : \Delta \times X \rightarrow X$ ,  $\Delta = \{(t, s) \mid 0 \leq s \leq t \leq b\}$ ,  $f : J \times X \times X \rightarrow X$  are given function,  $x_0 \in X$ ,  $A$  is in the class  $\Theta_\theta^\gamma$  and  $-1 < \gamma < 0$ ,  $0 < \theta < \pi/2$ , and the state  $x(\cdot)$  takes values in the real Banach space  $X$ . Also the control function  $u(\cdot)$  is given in  $L^2(J, U)$ , a Banach space of admissible control functions with  $U$  as a Banach space. Finally,  $B$  is a bounded linear operator from  $U$  to  $X$ .

To begin, we quickly recall some definitions from Chapter 2.

**Definition 5.1.** A continuous solution of the integral equation

$$x(t) = \mathcal{S}_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) \left[ Bu(s) + f\left(s, x(s), \int_0^s h(s, \tau, x(\tau)) d\tau\right) \right] ds, \quad t \in J,$$

is said to be a mild solution of the problem (5.1)–(5.2) on  $J$ .

**Definition 5.2.** The system (5.1)–(5.2) is said to be controllable on the interval  $J$ , if for every  $x_0, x_1 \in X$  there is exists a control  $u \in L^2(J, U)$  such that the mild solution  $x(t)$  of (5.1)–(5.2) satisfies  $x(b) = x_1$ .

## 5.2 Main results

To investigate the controllability of the system (5.1)–(5.2), we assume the following conditions:

(H1) The linear operator  $B : L^2(J, U) \rightarrow L^1(J, U)$  is bounded,  $W : L^2(J, U) \rightarrow X$  defined by

$$Wu = \int_0^b (b-s)^{\alpha-1} \mathcal{S}_\alpha(b-s) Bu(s) ds,$$

has an inverse operator  $W^{-1}$  which takes values in  $L^2(J, U)/\ker W$  and there exist positive constants  $M_1, M_2$  such that

$$\|B\|_{B(U, X)} \leq M_1 \text{ and } \|W^{-1}\|_{B(X, L^2(J, U)/\ker W)} \leq M_2;$$

(H2)  $h$  satisfies Caratheodory condition, i.e.

- for each  $(t, s) \in \Delta$ , the function  $h(t, s, \cdot) : X \rightarrow X$  is continuous, and
- for each  $x \in X$ ,  $h(\cdot, \cdot, x) : \Delta \rightarrow X$  is strongly measurable;

(H3)  $f$  satisfies Caratheodory condition, i.e.

- for each  $t \in J$ , the function  $f(t, \cdot, \cdot) : X \times X \rightarrow X$  is continuous, and
- for each  $x, y \in X$ ,  $f(\cdot, x, y) : J \rightarrow X$  is strongly measurable;

(H4) For every positive integer  $k$ , there exists  $\varphi_k \in L^p((0, b))$  such that for a.e.  $t \in J$  and  $x \in C(J, X)$

$$\sup_{\|x\| \leq k} \left\| f \left( t, x(t), \int_0^t h(t, s, x(s)) ds \right) \right\| \leq \varphi_k(t),$$

where  $p > -\frac{1}{\alpha\gamma}$ ,  $q = \frac{p}{(p-1)}$  and  $\|x\| = \sup_{t \in J} \|x(t)\|$ ;

(H5) There exists  $q \in L^1(J, \mathbb{R}_+)$  such that

$$\|h(t, s, u)\| \leq q(t) \Psi(\|u\|), \quad (t, s) \in \Delta, \quad u \in X,$$

where  $\Psi : [0, \infty) \rightarrow (0, \infty)$  is a continuous nondecreasing function;

(H6) There exists  $p \in L^1(J, \mathbb{R}_+)$  such that

$$\|f(t, u, v)\| \leq p(t) \Psi(\|u\|) + \|v\| \quad \text{for each } (t, u, v) \in J \times X \times X$$

and there exists a constant  $M_* > 0$  with

$$\frac{M_*}{C_1 + \left(C_2 + \frac{b^{\alpha-1}}{\Gamma(\alpha)}M\right) \Psi(M_*) \int_0^b [p(s) + q(s)] ds} > 1$$

where

$$C_1 = M \|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)} M M_1 M_2 (\|x_1\| + M \|x_0\|)$$

and

$$C_2 = \frac{b^{2\alpha-1}}{\Gamma(\alpha)\Gamma(\alpha+1)} M^2 M_1 M_2;$$

(H7) For all bounded subsets  $B$ , the set

$$Y_{h,\delta}(t) = \left\{ \int_0^{t-h} \int_\delta^\infty \alpha \tau (t-s)^{\alpha-1} \Psi_\alpha(\tau) T((t-s)^\alpha \tau) \right. \\ \left. \times f(s, x(s), \int_0^s h(s, \varsigma, x(\varsigma)) d\varsigma) d\tau ds, x \in B \right\},$$

is relatively compact in  $X$  for arbitrary  $h \in (0, t)$  and  $\delta > 0$ .

**Theorem 5.3.** *If the hypotheses (H1)–(H6) are satisfied, system (5.1)–(5.2) is controllable on  $J$ .*

*Proof.* In view of the hypothesis (H1) for an arbitrary function  $x(t)$ , the control is defined as follows

$$u_x(t) = W^{-1} \left[ x_1 - \mathcal{S}_\alpha(b)x_0 \right. \\ \left. - \int_0^b (b-s)^{\alpha-1} \mathcal{P}_\alpha(b-s) f \left( s, x(s), \int_0^s h(s, \tau, x(\tau)) d\tau \right) ds \right] (t).$$

In what follows, it suffices to show that when using this control the operator  $\Phi : C(J, X) \rightarrow C(J, X)$  defined by

$$\Phi x(t) = \mathcal{S}_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) \left[ Bu_x(s) \right. \\ \left. + f \left( s, x(s), \int_0^s h(s, \tau, x(\tau)) d\tau \right) \right] ds, t \in J,$$

has a fixed point  $x(\cdot)$  from which it follows that this fixed point is a mild solution of the system 5.1–5.2. Clearly  $\Phi x(b) = x_1$ , from which we conclude that the system is controllable.

As in ([43]), we first prove that  $\Phi$  is continuous and completely continuous. For convenience we let

$$G(\eta) = BW^{-1} \left[ x_1 - \mathcal{S}_\alpha(b)x_0 - \int_0^b (b-s)^{\alpha-1} \mathcal{P}_\alpha(b-s) \hat{f}(s) ds \right] (\eta),$$

where  $\hat{f}(s) = f(s, x(s), \int_0^s h(s, \tau, x(\tau)) d\tau)$ .

Let  $B_k = \{x \in C(J, X) \mid \|x\| \leq k\}$  for some  $k \geq 1$ . It is clear for  $x \in B_k$  that

$$\|G(\eta)\| \leq M_1 M_2 \left[ \|x_1\| + M \|x_0\| + \frac{b^{\alpha-1}}{\Gamma(\alpha)} M \int_0^b \varphi_k(s) ds \right] := G_0$$

and

$$\|\Phi x(t)\| \leq M \|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)} M G_0 + \frac{b^{\alpha-1}}{\Gamma(\alpha)} M \|\varphi_k\|_{L^1};$$

where we have used the Lemma 2.20 and the hypotheses (H1), (H4). Let  $\tau_1, \tau_2 \in J$  with  $\tau_2 > \tau_1 > 0$ , and  $\varepsilon > 0$  small enough, we have

$$\|\Phi x(\tau_2) - \Phi x(\tau_1)\| \leq I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$\begin{aligned} I_1 &= \|\mathcal{S}_\alpha(\tau_2)x_0 - \mathcal{S}_\alpha(\tau_1)x_0\|, \\ I_2 &= \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} \|\mathcal{P}_\alpha(\tau_2 - s) \hat{f}(s)\| ds, \\ I_3 &= \int_0^{\tau_1 - \varepsilon} (\tau_1 - s)^{\alpha-1} \|\mathcal{P}_\alpha(\tau_2 - s) - \mathcal{P}_\alpha(\tau_1 - s)\| \|\hat{f}(s)\| ds, \\ I_4 &= \int_{\tau_1 - \varepsilon}^{\tau_1} (\tau_1 - s)^{\alpha-1} \|\mathcal{P}_\alpha(\tau_2 - s) - \mathcal{P}_\alpha(\tau_1 - s)\| \|\hat{f}(s)\| ds, \\ I_5 &= \int_0^{\tau_1} |(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}| \cdot \|\mathcal{P}_\alpha(\tau_2 - s)\| \|\hat{f}(s)\| ds, \end{aligned}$$

From Lemma 2.21 and Lemma 2.22(i) it is easy to see that  $I_1 \rightarrow 0$  when  $t_1 \rightarrow t_2$ . Moreover, using (H4) and Lemma 2.20 we get

$$\begin{aligned} I_2 &\leq \frac{b^{\alpha-1}}{\Gamma(\alpha)} M \int_{\tau_1}^{\tau_2} \varphi_k(s) ds, \\ I_3 &\leq \sup_{s \in [0, \tau_1 - \varepsilon]} \|\mathcal{P}_\alpha(\tau_2 - s) - \mathcal{P}_\alpha(\tau_1 - s)\| \int_0^{\tau_1 - \varepsilon} (\tau_1 - s)^{\alpha-1} \varphi_k(s) ds, \\ I_4 &\leq \sup_{s \in [0, \tau_1 - \varepsilon]} \|\mathcal{P}_\alpha(\tau_2 - s) - \mathcal{P}_\alpha(\tau_1 - s)\| \int_{\tau_1 - \varepsilon}^{\tau_1} (\tau_1 - s)^{\alpha-1} \varphi_k(s) ds. \end{aligned}$$

It follows from Lemma 2.21 that  $I_i$  ( $i = 2, 3, 4, 5$ ) tends to zero independent of  $x \in B_k$  as  $t_2 - t_1 \rightarrow 0$ ,  $\varepsilon \rightarrow 0$ . Hence, we can conclude that  $\|\Phi x(\tau_2) - \Phi x(\tau_1)\| \rightarrow 0$  as  $t_2 - t_1 \rightarrow 0$ , and the limit is independent of  $x \in B_k$ . For the case when  $0 = t_1 < t_2 \leq T$ , since  $I_2 \leq \frac{b^{\alpha-1}}{\Gamma(\alpha)} M \int_{\tau_1}^{\tau_2} \varphi_k(s) ds$ , in view of (H4) and Lemma 2.20,  $\|\Phi x(\tau_2)\|$  can be made small when  $t_2$  is small independently of  $x \in B_k$ . Thus, the equicontinuity of  $\Phi$  holds.

Define  $\Pi = \Phi B_k$  and  $\Pi(t) = \{\Phi x(t) : x \in B_k\}$  for  $t \in J$ . Clearly,  $\Pi(0) = \{x_0\}$  is compact, and hence, it is only necessary to consider  $t > 0$ . For each  $h \in (0, t)$ ,  $t \in (0, T]$ , arbitrary  $\delta > 0$ , define  $\Pi_{h,\delta}(t) = \{\Phi_{h,\delta} x(t) : x \in B_k\}$  where

$$\Phi_{h,\delta} x(t) = \mathcal{S}_\alpha(t)x_0 + \int_0^{t-h} \int_\delta^\infty \alpha \tau (t-s)^{\alpha-1} \Psi_\alpha(\tau) T((t-s)^\alpha \tau) \hat{f}(s) d\tau ds.$$

Using Lemma 2.21, [H7] and the same method used in [75, Theorem 3.1], one can verify that the set  $\Pi(t)$  can be arbitrary approximated by the relatively compact sets  $\Pi_{h,\delta}(t)$ . So,  $\Pi_{h,\delta}(t)$  is relatively compact in  $X$  for every  $h \in (0, t)$  and  $\delta > 0$ . Moreover, for every  $x \in B_k$  we have

$$\begin{aligned} &\|\Phi x(t) - \Phi_{h,\delta} x(t)\| \\ &\leq \left\| \int_0^t \int_0^\delta \alpha \tau (t-s)^{\alpha-1} \Psi_\alpha(\tau) T((t-s)^\alpha \tau) \hat{f}(s) d\tau ds \right\| \\ &\quad + \left\| \int_{t-h}^t \int_\delta^\infty \alpha \tau (t-s)^{\alpha-1} \Psi_\alpha(\tau) T((t-s)^\alpha \tau) \hat{f}(s) d\tau ds \right\| \\ &\leq \int_0^t C_p(t-s)^{-1-\alpha\gamma} \varphi_k(s) ds \int_0^\delta \tau^{-\gamma} \Psi_\alpha(\tau) d\tau \\ &\quad + \int_{t-h}^t C_p(t-s)^{-1-\alpha\gamma} \varphi_k(s) ds \int_\delta^\infty \tau^{-\gamma} \Psi_\alpha(\tau) d\tau \end{aligned}$$

$$\begin{aligned}
& \|\Phi x(t) - \Phi_{h,\delta} x(t)\| \\
& \leq C_p \left( \int_0^t s^{-(1+\alpha\gamma)q} ds \right)^{\frac{1}{q}} \left( \int_0^t \varphi_k^p(s) ds \right)^{\frac{1}{p}} \int_0^\delta \tau^{-\gamma} \Psi_\alpha(\tau) d\tau \\
& \quad + C_p \frac{\Gamma(1-\delta)}{\Gamma(1-\delta\alpha)} \left( \int_0^h s^{-(1+\alpha\gamma)q} ds \right)^{\frac{1}{q}} \left( \int_{t-h}^t \varphi_k^p(s) ds \right)^{\frac{1}{p}} \\
& \leq C_p \left( \frac{b^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q} \right)^{\frac{1}{q}} \|\varphi_k\|_{L^p(0,b)} \int_0^\delta \tau^{-\gamma} \Psi_\alpha(\tau) d\tau \\
& \quad + C_p \frac{\Gamma(1-\delta)}{\Gamma(1-\delta\alpha)} \left( \frac{\varepsilon^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q} \right)^{\frac{1}{q}} \|\varphi_k\|_{L^p(0,b)}.
\end{aligned}$$

Therefore, the set  $\Pi(t) = \{\Phi x(t) \mid x \in B_k\}$  is totally bounded. Hence  $\Pi(t)$  is relatively compact in  $X$  and as a consequence from the Arzila–Ascoli theorem we can conclude that  $\Phi : C(J, X) \rightarrow C(J, X)$  is completely continuous.

Now let  $\{x_n\}_0^\infty \subseteq C(J, X)$  with  $x_n \rightarrow x$  in  $C(J, X)$ . Then, there exists an integer  $k$  such that  $\|x_n(t)\| \leq k$  for all  $n \in \mathbb{N}$  and  $t \in J$ , so  $x_n \in B_k$  and  $x \in B_k$ . For brevity we let

$$G_n(\eta) = BW^{-1} \left[ x_1 - \mathcal{S}(b)x_0 - \int_0^b (b-s)^{\alpha-1} \mathcal{P}(b-s) \hat{f}_n(s) ds \right](\eta),$$

where  $\hat{f}_n(s) = f(s, x_n(s), \int_0^s h(s, \tau, x_n(\tau)) d\tau)$ . By (H3) and (H4),

$$\hat{f}_n(t) \rightarrow \hat{f}(t) \quad \text{as } n \rightarrow \infty,$$

for each  $t \in J$ , since

$$\|\hat{f}_n(t) - \hat{f}(t)\| \leq 2\varphi_k(t)$$

and

$$\begin{aligned}
\|\Phi x_n(t) - \Phi x(t)\| & \leq \frac{b^{\alpha-1}}{\Gamma(\alpha)} M \int_0^t \|G_n(s) - G(s)\| ds \\
& \quad + \frac{b^{\alpha-1}}{\Gamma(\alpha)} M \int_0^t \|\hat{f}_n(s) - \hat{f}(s)\| ds,
\end{aligned}$$

where

$$\|G_n(s) - G(s)\| \leq M_1 M_2 \frac{b^{\alpha-1}}{\Gamma(\alpha)} M \int_0^b \|\hat{f}_n(s) - \hat{f}(s)\| ds,$$

by using the dominated convergence theorem, we get

$$\|\Phi x_n - \Phi x\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus,  $\Phi$  is continuous and completely continuous.

Now let  $\lambda \in (0, 1)$  and let  $x = \lambda\Phi x$ . Then for  $t \in J$

$$\begin{aligned} x(t) &= \lambda \mathcal{S}(t) x_0 + \lambda \int_0^t (t-s)^{\alpha-1} \mathcal{P}(t-s) B u_x(s) ds \\ &\quad + \lambda \int_0^t (t-s)^{\alpha-1} \mathcal{P}(t-s) \hat{f}(s) ds, \end{aligned}$$

where

$$u_x(s) = W^{-1} \left[ x_1 - \mathcal{S}(b) x_0 - \int_0^b (b-s)^{\alpha-1} \mathcal{P}(b-s) \hat{f}(s) ds \right] (s)$$

and we have

$$\begin{aligned} \|x(t)\| &\leq M \|x_0\| + \frac{b^\alpha}{\Gamma(\alpha+1)} M M_1 M_2 \\ &\quad \times \left[ \|x_1\| + M \|x_0\| + \frac{b^{\alpha-1}}{\Gamma(\alpha)} M \int_0^b [p(s) + q(s)] \Psi(\|x(s)\|) ds \right] \\ &\quad + \frac{b^{\alpha-1}}{\Gamma(\alpha)} M \int_0^t [p(s) + q(s)] \Psi(\|x(s)\|) ds \\ &\leq C_1 + C_2 \Psi(\|x\|) \int_0^b [p(s) + q(s)] ds \\ &\quad + \frac{b^{\alpha-1}}{\Gamma(\alpha)} M \Psi(\|x\|) \int_0^b [p(s) + q(s)] ds \end{aligned}$$

Consequently

$$\frac{\|x\|}{C_1 + \left( C_2 + \frac{b^{\alpha-1}}{\Gamma(\alpha)} M \right) \Psi(\|x\|) \int_0^b [p(s) + q(s)] ds} \leq 1$$

Then by (H7) there exists  $M_*$  such that  $\|x\| \neq M_*$ .

Finally set

$$V \in \{x \in C(J, X) \mid \|x\| < M_*\}.$$

From the choice of  $V$  there is no  $x \in \partial V$  such that  $x = \lambda \Phi x$  for some  $\lambda \in (0, 1)$ . As a consequence of Theorem B.4 we deduce that  $\Phi$  has fixed point  $x$  in  $\bar{V}$  and hence system 5.1–5.2 is controllable on  $J$ . This completes the proof.

### 5.3 Application

In this section we present an example to illustrate our main results. Let us consider the following nonlinear partial integro-differential equation of the form [76, Exemple 6.3.]

$$\begin{aligned} & \frac{\partial^\alpha}{\partial t^\alpha} \omega(t, y) + (-i\Delta + \lambda)^{\frac{1}{2}} \omega(t, y) \\ &= \mu(t, y) + P(t, \omega(t, y), \int_0^t a(t, s, \omega(s, y)) ds), \\ & t > 0, \quad y \in \mathbb{R}^2, \\ & \omega(0, y) = \omega_0(y), \quad y \in \mathbb{R}^2, \end{aligned} \tag{5.3}$$

where  $\lambda > 0$  is a suitable constant,  $i\Delta$  is the Schrödinger operator,  $0 < \alpha < 1$ ,  $\mu : J \times (0, \frac{\pi}{2}) \rightarrow (0, \frac{\pi}{2})$  is continuous. Let us take  $X = U = L^3([0, \frac{\pi}{2}]^2)$ ,  $u(t) = \mu(t, \cdot)$  and let

$$\hat{A} = (-i\Delta + \sigma)^{\frac{1}{2}}, \quad D(\hat{A}) = W^{1,3}(\mathbb{R}^2)$$

Then  $i\Delta$  generates a  $\beta$ -times integrated semigroup  $S^\beta(t)$  with  $\beta = \frac{5}{12}$  on  $L^3(\mathbb{R}^2)$  such that  $\|S^\beta(t)\|_{B(L^3(\mathbb{R}^2))} \leq \hat{M}t^\beta$ , for all  $t \geq 0$  and some constant  $\hat{M} > 0$ . We can deduce that the operator  $i\Delta + \lambda$  belongs to  $\Theta_{\pi/2}^{\beta-1}(L^3(\mathbb{R}^2))$ , which denotes the family of all linear closed operators  $A : D(A) \subset L^3(\mathbb{R}^2) \rightarrow L^3(\mathbb{R}^2)$  satisfying  $\sigma(A) \subset S_{\pi/2} = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \pi/2\} \cup \{0\}$ , and for every  $\pi/2 < p < \pi$  there exists a constant  $C_p$  such that  $\|R(z, A)\| \leq C_p |z|^{\beta-1}$  for all  $z \in \mathbb{C} \setminus S_p$ . Thus, it follows that  $\hat{A} \in \Theta_\theta^{-1+2\beta}(L^3(\mathbb{R}^2))$  for some  $0 < \theta < \pi/2$ . Here  $a : \Delta \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $\Delta = \{(t, s) \mid 0 \leq s \leq t \leq 1\}$  and  $P : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous and we assume that these functions  $a, P$  satisfy (H2)–(H7).

Define the function  $f : J \times X \times X \rightarrow X$  and  $h : \Delta \times X \rightarrow X$  as follows

$$f(t, x, z)(y) = P(t, x(y), z(y))$$

and

$$h(t, s, x)(y) = a(t, s, x(y)).$$

for  $t \in J$ ,  $x, z \in X$  and  $0 < y < \pi/2$ .

With the choice of  $\hat{A}, B, h, f$  and  $B = I$ , the identity operator, we see that Eqs. 5.1–5.2 is the abstract formulation of system 5.3.

Now assume that the operator  $W : L^2[J, U] \rightarrow X$  defined by

$$Wu = \int_0^{\pi/2} (\pi/2 - s)^{\alpha-1} E_\alpha(-z(\pi/2 - s)^\alpha) (\hat{A})Bu(s) ds,$$

has an inverse operator and satisfies the condition (H1).

Thus all the conditions of the above Theorem 5.3 are satisfied. Hence system 5.3 is controllable on  $J$ .

# Chapter 6

## Controllability for Systems

## Governed by Nonlocal

## Quasilinear Differential Inclusion

## in a Banach Spaces

### 6.1 Introduction

This Chapter presents some results of the paper [18].

The investigation of controllability problems for nonlinear systems by the methods of fixed point theory has a long history [62]. In recent years, the study of differential inclusions and control problems have attracted the attention of many mathematicians and physicists. A large amount of articles developed concerning the investigation of controllability problems for systems governed by semilinear differential inclusion in Banach spaces [10, 13, 14, 26, 43]. In the literature there are a few papers dealing with the controllability for systems governed by nonlocal quasilinear differential inclusion. Motivated by our work [18], Górniewicz et al. [43] and Debbouche [31], our main purpose, in this work, is to study the controllability of nonlocal quasilinear fractional differential inclusions by using resolvent

family and a fixed point theorem. We indicate that the method used in this work is different from that in [43].

This chapter is organized as follows. In Section 6.2 we give the necessary preliminaries for the chapter. In Section 6.3 we give the outline of the corresponding results for the case when the multivalued map is lower semicontinuous. The last Section is devoted to an illustrated example.

## 6.2 Preliminaries

In this section, we give some basic definition and proprieties of resolvent family.

Let  $X$  be a separable real Banach space with the norm  $\|\cdot\|$ . We denote by  $E$  the Banach space  $C(J; X)$  of  $X$ -valued continuous functions on  $J$  equipped with the sup-norm and by  $B(X)$  the space of all bounded linear operators from  $X$  to  $X$ . For  $x \in X$  and for nonempty sets  $A, B$  of  $X$  we denote  $d(x, A) = \inf\{\|x - y\|; y \in A\}$ ,  $e(A, B) := \sup\{d(x, B); x \in A\}$  and  $\chi(A, B) := \max\{e(A, B), e(B, A)\}$  for a Hausdorff metric. For more detail, we refer the reader to [25, 62].

Motivated by the work in [18, 31, 43], we consider the following nonlinear control system governed by a semilinear differential inclusion of the form:

$$\frac{d^\alpha x(t)}{dt^\alpha} \in A(t, x(t))x(t) + F(t, x(t)) + Bu(t), \quad t \in J := [0, b], \quad (6.1)$$

$$x(0) + h(x) = x_0, \quad (6.2)$$

where  $0 < \alpha < 1$ ,  $F : J \times X \rightarrow \mathcal{P}(X)$  is a bounded, closed, multivalued map,  $\mathcal{P}(X)$  the set of all nonempty subsets of  $X$ ,  $x_0 \in X$  and  $h : C(J, X) \rightarrow X$ . We assume that  $A(t, \cdot)$  is a closed linear operator defined on a dense domain  $D(A)$  in  $X$  into  $X$  such that  $D(A)$  is independent of  $t$ . It is assumed also that  $A(t, \cdot)$  generates an  $(\alpha, x)$ -resolvent family  $R_{(\alpha, x)}(t, s)$  in the Banach space  $X$  and the control function  $u(\cdot)$  is given in  $L^2(J, U)$ , a Banach space of admissible control functions with  $U$  as a Banach space. Finally  $B$  is a bounded linear operator from  $U$  to  $X$ . The following definition is similar to the concept defined in [5, 31, 43].

**Definition 6.1.** A continuous function  $x(\cdot)$  is said to be a mild solution of the problem (6.1)–(6.2), if  $x(0) + h(x) = x_0$  and there exists a function  $v \in L^1(J, X)$

such that  $v(t) \in F(t, x(t))$  a.e. on  $J$  and

$$x(t) = R_{(\alpha, x)}(t, 0)x_0 - R_{(\alpha, x)}(t, 0)h(x) + \int_0^t R_{(\alpha, x)}(t, s) [Bu(s) + v(s)] ds.$$

**Definition 6.2** ([31, 43]). The system (6.1)–(6.2) is said to be controllable on the interval  $J$ , if for every  $x_0, x_1 \in X$  there is exists a control  $u \in L^2(J, U)$  such that the mild solution  $x(t)$  of (6.1)–(6.2) satisfies  $x(0) + h(x) = x_0$  and  $x(b) = x_1$ .

We give now a suitable forms of the Lemma 2.16.

**Lemma 6.3.** [31, Lemma 3.1] *Let  $\Omega \subset X, Y$  be a densely and continuously imbedded Banach space in  $X$  and let  $R_{(\alpha, x)}(t, s)$  be the resolvent operator for the problem (6.1)–(6.2), there exists a constant  $K > 0$  such that*

$$\|R_{(\alpha, y)}(t, s)\omega - R_{(\alpha, x)}(t, s)\omega\| \leq K\|\omega\|_Y \int_s^t \|y(\tau) - x(\tau)\| d\tau,$$

for every  $z_1, z_2 \in E$  with values in  $\Omega$  and every  $\omega \in Y$ .

### 6.3 Main results

To investigate the controllability of the system (6.1)–(6.2), we assume the following conditions:

(H1) The bounded linear operator  $W : L^2(J, U) \rightarrow X$ , defined by

$$Wu = \int_0^b R_{(\alpha, x)}(b, s)Bu(s)ds,$$

has an induced inverse operator  $\tilde{W}^{-1}$  which takes values in  $L^2(J, U)/\ker W$  and there exist positive constants  $M_1, M_2$  such that  $\|B\| \leq M_1$  and  $\|\tilde{W}^{-1}\| \leq M_2$ ;

(H2)  $F : J \times X \rightarrow \mathcal{P}(X)$  is a nonempty, compact-valued, multivalued map such that:

- (i)  $(t, x) \mapsto F(t, x)$  is  $\mathcal{L} \otimes \mathcal{B}$  measurable,
- (ii)  $x \mapsto F(t, x)$  is lower semi-continuous for a.e.  $t \in J$ ;

(H3) For each  $k > 0$ , there exists  $\varphi_k(\cdot) \in L^1(J, \mathbb{R}_+)$  such that

$$\|F(t, x)\| = \sup \{\|v\| \mid v \in F(t, x)\} \leq \varphi_k(t),$$

for all  $\|x\| \leq k$  and for a.e.  $t \in J$ ;

(H4) There exists a function  $\psi(\cdot) \in L^1(J, \mathbb{R}_+)$  such that  $F$  is  $\psi(t)$ -Lipschitz in the sense that

$$\chi(F(t, y), F(t, x)) \leq \psi(t)\|y - x\|, \text{ for a.e. } t \in J,$$

and for each  $x, y \in X$ , where  $\chi$  is the Hausdorff metric on  $\mathcal{P}(X)$  (see[25, 62]);

(H5)  $h : E \rightarrow Y$  is Lipschitz continuous in  $X$  and bounded in  $Y$ , that is, there exist positive constants  $H_1$  and  $H_2$  such that

$$\|h(x)\|_Y \leq H_1,$$

$$\|h(y) - h(x)\|_Y \leq H_2\|y - x\|_E, \quad x, y \in E;$$

(H6) Denote  $M = \max \|R_{(\alpha, x)}(t, s)\|_{B(X)}$ ,  $0 \leq s \leq t \leq b$ ,  $x \in X$ ;

(H7) There exist positive constants  $k_1$  and  $k_2$  such that

$$M\|x_0\| + MM_1M_2 \left[ \|x_1\| + M \left( \|x_0\| + H_1 + \int_0^b \varphi_{k_1}(s) ds \right) \right] + M\|\varphi_{k_1}\|_{L^1} \leq k_1,$$

and

$$\begin{aligned} k_2 = & MM_1M_2b \left[ (Kb^2\|f\|_{L^1} + M \int_0^b \psi(s) ds) + (Kb\|x_0\| + MH_2 + KbH_1) \right] \\ & + KbM_1M_2 \left[ \|x_1\| + M\|x_0\| + MH_1 + M \int_0^b \varphi_{k_1}(s) ds \right] < 1. \end{aligned}$$

**Theorem 6.4.** *If the hypotheses (H1)–(H7) are satisfied, system (6.1)–(6.2) is controllable on  $J$ .*

*Proof.* As in [43], the assumption (H2) and (H3) imply that  $F$  is of lower semi-continuous type (i.e., it has nonempty closed and decomposable values (see[19])), then there exists a continuous function  $f : E \rightarrow L^1(J, X)$  such that  $f(x) \in \mathcal{F}(x)$  for all  $x \in E$ , where  $\mathcal{F}$  is the Nemitsky operator defined by

$$\mathcal{F}(x) = \{ w \in L^1(J, X) \mid w(t) \in F(t, x(t)) \text{ for a.e. } t \in J \}.$$

Consider the problem

$$\frac{d^\alpha x(t)}{dt^\alpha} - A(t, x)x(t) - Bu(t) = f(x)(t), \quad t \in J, \quad (6.3)$$

$$x(0) + h(x) = x_0. \quad (6.4)$$

It is clear that if  $x \in E$  is a solution of the problem (6.3)–(6.4), then it is also solution to the problem (6.1)–(6.2).

In view of the hypothesis (H1) for an arbitrary function  $x(t)$ , the control is defined as follows (compar with [31])

$$u_x(t) = \tilde{W}^{-1} \left[ x_1 - R_{(\alpha, x)}(b, 0)x_0 + R_{(\alpha, x)}(b, 0)h(x) - \int_0^b R_{(\alpha, x)}(b, s)f(x)(s) ds \right] (t).$$

In what follows, it suffices to show that when using this control the operator  $\Phi : E \rightarrow E$  defined by

$$\Phi x(t) = R_{(\alpha, x)}(t, 0)(x_0 - h(x)) + \int_0^t R_{(\alpha, x)}(t-s)[Bu_x(s) + f(x)(s)] ds, \quad t \in J,$$

has a fixed point  $x(\cdot)$  from which it follows that this fixed point is a mild solution of the system (6.1)–(6.2). Clearly  $\Phi x(b) = x_1$ , from which we conclude that the system is controllable.

Let  $B_{k_1}$  be the nonempty closed and bounded set given by

$$B_{k_1} = \{z \in E : z(0) = x_0, \|z\| \leq k_1\}. \quad (6.5)$$

We first prove that  $\Phi(B_{k_1}) \subset B_{k_1}$ . For convenience we let

$$G_x(\eta) = B\tilde{W}^{-1} \left[ x_1 - R_{(\alpha,x)}(b,0)(x_0 - h(x)) - \int_0^b R_{(\alpha,x)}(b,s)f(x)(s) ds \right] (\eta),$$

It is clear for  $x \in B_{k_1}$  that

$$\|G_x(\eta)\| \leq M_1 M_2 \left[ \|x_1\| + M \|x_0\| + MH_1 + M \int_0^b \varphi_{k_1}(s) ds \right] := G_0$$

and

$$\|\Phi x(t)\| \leq M \|x_0\| + MG_0 + M \|\varphi_{k_1}\|_{L^1},$$

where the last two inequalities follow from (H1), (H3), (H5) and (H6).

From assumption (H7), one gets  $\|\Phi x\| \leq k_1$ . Thus,  $\Phi$  maps  $B_{k_1}$  into it self.

Now prove that  $\Phi$  is a contraction. Then for  $x, y \in B_k$

$$\|\Phi y(t) - \Phi x(t)\| \leq I_1(t) + I_2(t) + I_3(t),$$

where

$$I_1(t) = \|R_{(\alpha,x)}(t,0)x_0 - R_{(\alpha,y)}(t,0)x_0\| + \|R_{(\alpha,x)}(t,0)h(x) - R_{(\alpha,y)}(t,0)h(y)\|,$$

$$I_2(t) = \int_0^t \|R_{(\alpha,y)}(t,s)G_y(s) - R_{(\alpha,x)}(t,s)G_x(s)\| ds,$$

and

$$I_3(t) = \int_0^t \|R_{(\alpha,y)}(t,s)f(y)(s) - R_{(\alpha,x)}(t,s)f(x)(s)\| ds.$$

Applying Lemma 6.3, (H5) and (H6), we get

$$\begin{aligned} I_1(t) &\leq \|R_{(\alpha,x)}(t,0)x_0 - R_{(\alpha,y)}(t,0)x_0\| + \|R_{(\alpha,x)}(t,0)h(x) - R_{(\alpha,x)}(t,0)h(y)\| \\ &\quad + \|R_{(\alpha,x)}(t,0)h(y) - R_{(\alpha,y)}(t,0)h(y)\| \\ &\leq (Kb\|x_0\| + MH_2 + KbH_1)\|x - y\|_E. \end{aligned}$$

Also, we apply Lemma 6.3, and (H4)–(H6), we obtain

$$\begin{aligned} I_3(t) &\leq \int_0^t [\|R_{(\alpha,y)}(t,s)f(y)(s) - R_{(\alpha,y)}(t,s)f(x)(s)\| \\ &\quad + \|R_{(\alpha,y)}(t,s)f(x)(s) - R_{(\alpha,x)}(t,s)f(x)(s)\|] ds \\ &\leq \int_0^t M[\|f(y)(s) - f(x)(s)\| + Kb\|f\|_{L^1}] ds \\ &\leq (Kb^2\|f\|_{L^1} + M \int_0^b \psi(s) ds)\|y - x\|_E. \end{aligned}$$

Again, Lemma 6.3 and (H4)–(H6), imply that

$$\begin{aligned} I_2(t) &\leq \int_0^t [\|R_{(\alpha,y)}(t,s)G_y(s) - R_{(\alpha,y)}(t,s)G_x(s)\| \\ &\quad + \|R_{(\alpha,y)}(t,s)G_x(s) - R_{(\alpha,x)}(t,s)G_x(s)\|] ds \\ &\leq \int_0^t [M\|G_y(s) - G_x(s)\| + Kb\|G_x(s)\|\|y - x\|_E] ds \\ &\leq \int_0^t [MM_1M_2(I_1(b) + I_3(b)) + KbG_0\|y - x\|_E] ds \\ &\leq \{MM_1M_2b[(Kb^2\|f\|_{L^1} + M \int_0^b \psi(s) ds) + (Kb\|x_0\| + MH_2 + KbH_1)] \\ &\quad + KbG_0\}\|y - x\|_E. \end{aligned}$$

Now, from (H5), we have

$$\|\Phi y(t) - \Phi x(t)\| \leq k_2 \|y - x\|_E.$$

Therefore,  $\Phi$  is a contraction mapping and hence there exists a unique fixed point  $x \in X$ , which is a solution to problem 6.3–6.4. This completes the proof.

□

## 6.4 Application

In this section we present an example to illustrate our main results. Let us consider the following nonlinear partial differential equation of the form

$$\frac{\partial^\alpha w(y, t)}{\partial t^\alpha} = a(y, t, w(y, t)) \frac{\partial^2 w(y, t)}{\partial y^2} + \mathcal{Q}(t, w(y, t)) + \mu(y, t), \quad (6.6)$$

$$w(0, t) = w(\pi, t) = 0, \quad t \in J := [0, b], \quad (6.7)$$

$$w(y, 0) + \sum_{i=1}^p c_i w(y, t_i) = w_0(y), \quad 0 < t_1 < t_2 < \cdots < t_p \leq b, \quad c_i \in \mathbb{R}, \quad y \in [0, \pi], \quad (6.8)$$

where  $0 < \alpha < 1$ ,  $a : (0, \pi) \times [0, b] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $\mu : [0, \pi] \times J \rightarrow [0, \pi]$  is continuous and  $\mathcal{Q} : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$  is a multivalued map.

Let us take  $X = U = L^2[0, \pi]$ ,  $u(t) = \mu(t, \cdot)$ ,  $x(t) = w(t, \cdot)$ ,  $h(w(\cdot, t)) = \sum_{i=1}^p c_i w(\cdot, t_i)$  and

$$F(t, x(t))(y) = \mathcal{Q}(t, w(t, y)), \quad t \in J, \quad y \in [0, \pi].$$

such that the hypotheses (H2)–(H4) are satisfied.

We define  $A(t, \cdot) : X \rightarrow X$  by  $(A(t, \cdot)z)(y) = a(y, t, \cdot)z''$  with

- (i) The domain  $D(A(t, \cdot)) = \{z \in X : z, z'$  are absolutely continuous,  $z'' \in X$  and  $z(0) = z(\pi) = 0\}$  is dense in the Banach space  $X$  and independent of  $t$ .

Then  $A$  can be written as

$$A(t, x)z = - \sum_{n=1}^{\infty} n^2 (z, z_n) z_n, \quad z \in D(A),$$

where  $(\cdot, \cdot)$  is the inner product in  $L^2[0, \pi]$  and  $z_n = Z_n \circ x$  is the orthogonal set of eigenvectors in  $A(t, x)$ , and  $Z_n(t, s) = \sqrt{2/\pi} \sin n(t-s)^\alpha$ ,  $0 \leq s \leq t \leq b$ ,  $n = 1, 2, \dots$ ;

(ii) The operator  $[A(t, \cdot) + \lambda^\alpha I]^{-1}$  exists in  $L(X)$  for any  $\lambda$  with  $\Re(\lambda) \leq 0$  and

$$\|[A(t, \cdot) + \lambda^\alpha I]^{-1}\| \leq \frac{C_\alpha}{|\lambda| + 1}, \quad t \in J;$$

(iii) There exist constants  $\eta \in (0, 1]$  and  $C_\alpha$  such that

$$\|[A(t_1, \cdot) - A(t_2, \cdot)]A^{-1}(s, \cdot)\| \leq C_\alpha |t_1 - t_2|^\eta, \quad t_1, t_2, s \in J.$$

Under these conditions each operator  $A(s, \cdot)$ ,  $s \in J$  generates an evolution operator which is an  $(\alpha, x)$ -resolvent family and has the form (see[31, Example]) :

$$R_{\alpha, x}(t, s)z = \sum_{n=1}^{\infty} \exp[-n^2(t-s)^\alpha](z, z_n)z_n, \quad z \in X.$$

With the choice of  $A, F$  and  $(Bu)(y, t) = \mu(t, y)$ , we see that Eqs. 6.1–6.2 is the abstract formulation of Eqs. 6.6–6.8. Now assume that the linear operator  $W$  is given by

$$Wu(y) = \sum_{n=1}^{\infty} \int_0^b \exp[-n^2(b-t)^\alpha](\mu(y, s), z_n)z_n ds, \quad y \in [0, \pi].$$

has a bounded invertible operator  $\tilde{W}$  in  $L^2(J, U)/\ker W$ . Thus all the conditions of the Theorem 6.4 are satisfied. Hence system 6.6–6.8 is controllable on  $J$ .

# Chapter 7

## Existence of Solutions for Fractional Differential Inclusions in Banach Spaces

### 7.1 Introduction

Delbosco and Rodino [33] considered the existence of a solution for the nonlinear fractional differential equation  $\frac{d^\alpha x(t)}{dt^\alpha} = f(t, x)$ , where  $0 < \alpha < 1$ , and  $f : [0, a] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $0 < a \leq +\infty$  is a given function, continuous in  $(0, a) \times \mathbb{R}$ . They obtained results for solutions by using the Schauder fixed point theorem and the Banach contraction principle. Qiu and Bai [69] considered the existence of positive solution for equation

$$\begin{aligned} \frac{d^\alpha x(t)}{dt^\alpha} + f(t, x(t)) &= 0, \quad 0 < t < 1, \\ x(0) = x'(1) = x''(0) &= 0, \end{aligned} \tag{7.1}$$

where  $2 < \alpha \leq 3$ , and  $f : (0, 1] \times [0, \infty) \rightarrow [0, \infty)$  with  $f$  is singular at  $t = 0$  (that is  $\lim_{t \rightarrow 0^+} f(t, \cdot) = +\infty$ ), by using Krasnoselskii's fixed point theorem and nonlinear alternative of Leray-Schauder type in a cone. Recently, Aitaliobrahim

[3] considered the existence of solutions to the boundary-value problem

$$\begin{aligned} x''(t) &\in F(t, x(t), x'(t)), \quad \text{a.e. on } [0, 1] \\ x'(0) &= r, \quad x'(1) = s, \end{aligned} \tag{7.2}$$

where  $F$  is a closed multifunction, measurable in the first argument and Lipschitz continuous in the second argument, by using the fixed point theorem introduced by Covitz and Nadler for contraction multi-valued maps. In this Chapter we establish the existence of solutions to the boundary-value problem

$$\begin{aligned} \frac{d^\alpha x(t)}{dt^\alpha} &\in F(t, x(t), x'(t), x''(t)), \quad \text{a.e. on } [0, 1], \\ x(0) &= 0, \quad x'(1) = s, \quad x''(0) = r, \end{aligned} \tag{7.3}$$

where  $2 < \alpha \leq 3$  and  $F$  is nonconvex, closed multifunction, measurable in the first argument and Lipschitz continuous in the second argument; and  $r, s$  are in a Banach space  $X$ . We should point out that the technique used in this chapter is the same technique of [3]

The Chapter is organized as follows. In Section 7.2 we recall some preliminary facts that we need in the sequel while in Section 7.2, we give the main result.

## 7.2 Preliminaries

Let  $X$  be a real separable Banach space with the norm  $\|\cdot\|$ . We denote by  $\mathcal{C}([0, 1], X)$  the Banach space of continuous functions from  $[0, 1]$  to  $E$  equipped with the norm  $\|x(\cdot)\|_\infty := \sup\{\|x(t)\|; t \in [0, 1]\}$ . For  $x \in X$  and for nonempty sets  $A, B$  of  $X$  we denote  $d(x, A) = \inf\{d(x, y); y \in A\}$ ,  $e(A, B) := \sup\{d(x, B); x \in A\}$  and  $H(A, B) := \max\{e(A, B), e(B, A)\}$ . A multifunction is said to be measurable if its graph is measurable.

Also, we recall the following results that will be used in this chapter.

**Definition 7.1** ([3]). Let  $T : X \rightarrow 2^X$  be a multi function with closed values

- (i)  $T$  is  $k$ -Lipschitz if  $H(T(x), T(y)) \leq k\|x - y\|$  for each  $x, y \in X$ .

(ii)  $T$  is a contraction if it is  $k$ -Lipschitz with  $k < 1$ .

(iii)  $T$  has a fixed point if there exists  $x \in X$  such that  $x \in T(x)$ .

**Lemma 7.2** ([3]). *If  $T : X \rightarrow 2^X$  is a contraction with nonempty closed values, then it has a fixed point.*

**Lemma 7.3** ([3]). *Assume that  $F : [a, b] \times X \rightarrow 2^X$  is a multi function with nonempty closed values satisfying:*

(i) *For every  $x \in X$ ,  $F(\cdot, x)$  is measurable on  $[a, b]$ ;*

(ii) *For every  $t \in [a, b]$ ,  $F(t, \cdot)$  is (Hausdorff) continuous on  $E$ .*

*Then for any measurable function  $x(\cdot) : [a, b] \rightarrow X$ , the multi function  $F(\cdot, x(\cdot))$  is measurable on  $[a, b]$ .*

**Definition 7.4** ([3]). A measurable multi-valued function  $F : [0, 1] \rightarrow 2^X$  is said to be integrably bounded if there exists a function  $h \in L^1([0, 1], X)$  such that for all  $v \in F(t)$ ,  $\|v\| \leq h(t)$  for almost every  $t \in [0, 1]$ .

**Definition 7.5** ([3]). A function  $x(\cdot) : [0, 1] \rightarrow X$  is said to be a solution of (7.3) if  $x(\cdot)$  is absolutely continuous on  $[0, 1]$  and satisfies (7.3).

**Lemma 7.6** ([69]). *Given  $\varphi \in C[0, 1]$ , and  $2 < \alpha \leq 3$ , the unique solution of*

$$\begin{aligned} \frac{d^\alpha y(t)}{dt^\alpha} + \varphi(t) &= 0, & 0 < t < 1, \\ y(0) = y'(1) = y''(0) &= 0. \end{aligned} \tag{7.4}$$

is

$$y(t) = \int_0^1 G(t, s)\varphi(s)ds$$

where

$$G(t, s) = \begin{cases} \frac{(\alpha-1)t(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{7.5}$$

Obviously,  $G(t, s)$  is continuous on  $[0, 1] \times [0, 1]$  and  $0 < G(t, s) \leq \lambda$ , for each  $t, s \in [0, 1]$  and some  $\lambda$ .

### 7.3 Main results

The following is the main result of this section.

**Theorem 7.7.** *Let  $F : [0, 1] \times X \times E \rightarrow 2^X$  be a set-valued map with nonempty closed values satisfying*

- (i) *For each  $(x, y, z) \in X \times X \times X$ ,  $t \mapsto F(t, x, y, z)$  is measurable and integrably bounded;*
- (ii) *There exists a function  $m(\cdot) \in L^1([0, 1], \mathbb{R}^+)$  such that for all  $t \in [0, 1]$  and for all  $x_1, x_2, y, z \in X$*

$$H(F(t, x_1, y, z), F(t, x_2, y, z)) \leq m(t)\|x_1 - x_2\|.$$

*Then, if  $\int_0^1 (1 + m(s))ds < 1/\lambda$ , for all  $r, s \in X$ , Problem (7.3) has at least one solution on  $[0, 1]$ .*

*Proof.* The proof of this theorem is similar to [3, Theorem 2.6] with a slight change.

Let  $r, s$  be in  $X$ . We introduce first the function  $\rho : [0, 1] \rightarrow X$  defined by

$$\rho(t) = (r + s)t - \frac{1}{2}rt^2, \quad \forall t \in [0, 1],$$

and the multifunction  $H : [0, 1] \times \mathcal{C}([0, 1], X) \rightarrow 2^X$  defined by

$$H(t, y(\cdot)) = \varphi(t) - F(t, y(t) + \rho(t), y'(t) + \rho'(t), y''(t) + \rho''(t)), \quad (7.6)$$

for all  $(t, y(\cdot)) \in [0, 1] \times \mathcal{C}([0, 1], X)$ . Consider the problem:

$$\begin{aligned} -\frac{d^\alpha x(t)}{dt^\alpha} + \varphi(t) &\in H(t, y(\cdot)), \quad \text{a.e. on } [0, 1] \\ y(0) = y'(1) = y''(0) &= 0. \end{aligned} \tag{7.7}$$

We should note that the function  $y(\cdot)$  is a solution of (7.7), if and only if the function  $x(t) = y(t) + \rho(t)$  is a solution of (7.3), for all  $t \in [0, 1]$ .

Next, by Lemma 7.3, for  $y(\cdot) \in \mathcal{C}([0, 1], X)$ ,  $F(\cdot, y(\cdot), y'(\cdot), y''(\cdot))$  is closed and measurable, then it has a measurable selection which, by hypothesis (i), belongs to  $L^1([0, 1], X)$ . Thus the set

$$S_{F, y(\cdot)} := \left\{ f \in L^1([0, 1], X) : f(t) \in F(t, y(t), y'(t), y''(t)) \text{ for a.e. } t \in [0, 1] \right\}$$

is nonempty. Let us transform problem (7.7) into a fixed point problem. Consider the multi-valued map,

$$T : \mathcal{C}([0, 1], X) \rightarrow 2^{\mathcal{C}([0, 1], X)}$$

defined as follows, for  $y(\cdot) \in \mathcal{C}([0, 1], X)$ ,

$$T(y(\cdot)) = \left\{ z(\cdot) \in \mathcal{C}([0, 1], X) : z(t) = \int_0^1 G(t, s)h(s)ds, \forall t \in [0, 1], h \in S_{H, y(\cdot)} \right\},$$

where

$$S_{H, y(\cdot)} := \left\{ h \in L^1([0, 1], X) : h(t) \in H(t, y(\cdot)) \text{ for a.e. } t \in [0, 1] \right\}.$$

We shall show that  $T$  satisfies the assumptions of Lemma 7.2. The proof will be given in two steps:

**Step 1:  $T$  has non-empty closed-values.** Indeed, let  $(y_p(\cdot))_{p \geq 0} \in T(y(\cdot))$  such that  $(y_p(\cdot))_{p \geq 0}$  converges to  $\bar{y}(\cdot)$  in  $\mathcal{C}([0, 1], X)$ . Then  $\bar{y}(\cdot) \in \mathcal{C}([0, 1], X)$  and for each  $t \in [0, 1]$ ,

$$y_p(t) \in \int_0^1 G(t, s)H(s, y(\cdot))ds,$$

where  $\int_0^1 G(t, s)H(s, y(\cdot))ds$  is the Aumann's integral of  $G(t, \cdot)H(\cdot, y)$ , which is defined as

$$\int_0^1 G(t, s)H(s, y(\cdot))ds = \left\{ \int_0^1 G(t, s)h(s)ds, h \in S_{H, y(\cdot)} \right\}.$$

Using the fact that the set-valued map  $F$  is closed and from (7.6), we conclude that the set

$$\int_0^1 G(t, s)H(s, y(\cdot))ds$$

is closed for all  $t \in [0, 1]$ . Then

$$\bar{y}(t) \in \int_0^1 G(t, s)H(s, y(\cdot))ds.$$

So, there exists  $h \in S_{H, y(\cdot)}$  such that

$$\bar{y}(t) = \int_0^1 G(t, s)h(s)ds.$$

Hence  $\bar{y}(\cdot) \in T(y(\cdot))$ . So  $T(y(\cdot))$  is closed for each  $y(\cdot) \in \mathcal{C}([0, 1], X)$ .

**Step 2:  $T$  is a contraction.** Indeed, let  $y_1(\cdot), y_2(\cdot) \in \mathcal{C}([0, 1], X)$  and consider  $z_1(\cdot) \in T(y_1(\cdot))$ . Then there exists  $h_1 \in S_{H, y_1(\cdot)}$  such that

$$z_1(t) = \int_0^1 G(t, s)h_1(s)ds, \quad \forall t \in [0, 1].$$

Using (7.6), there exists  $f_1 \in S_{F, y_1(\cdot)}$  such that

$$h_1(t) = y_1(t) - f_1(t), \quad \forall t \in [0, 1].$$

On the other hand, let  $\varepsilon > 0$  and consider the valued map  $U_\varepsilon : [0, 1] \rightarrow 2^X$ , given by

$$U_\varepsilon(t) = \{x \in X : \|f_1(t) - x\| \leq m(t)\|y_1(t) - y_2(t)\| + \varepsilon\}.$$

We claim that  $U_\varepsilon(t)$  is nonempty, for each  $t \in [0, 1]$ . Indeed, let  $t \in [0, 1]$ , we have

$$H(F(t, y_1(t), y'_1(t), y''_1(t)), F(t, y_2(t), y'_2(t), y''_2(t))) \leq m(t)\|y_1(t) - y_2(t)\|.$$

Hence, there exists  $x \in F(t, y_2(t), y'_2(t), y''_2(t))$ , such that

$$\|f_1(t) - x\| \leq m(t)\|y_1(t) - y_2(t)\| + \varepsilon.$$

The multifunction

$$V : t \rightarrow U_\varepsilon(t) \cap F(t, y_2(t), y'_2(t), y''_2(t)) \quad \text{is measurable.} \quad (7.8)$$

Then there exists a measurable selection for  $V$  denoted  $f_2$  such that, for all  $t \in [0, 1]$ ,

$$f_2(t) \in F(t, y_2(t), y'_2(t), y''_2(t))$$

and

$$\|f_1(t) - f_2(t)\| \leq m(t)\|y_1(t) - y_2(t)\| + \varepsilon.$$

Now, for all  $t \in [0, 1]$ , set  $h_2(t) = y_2(t) - f_2(t)$  and

$$z_2(t) = \int_0^1 G(t, s)h_2(s)ds.$$

We have

$$\begin{aligned}
\|z_1(t) - z_2(t)\| &\leq \int_0^1 \|G(t, s)\| \|h_1(s) - h_2(s)\| ds \\
&\leq \lambda \int_0^1 \|y_1(s) - y_2(s)\| ds + \lambda \int_0^1 \|f_1(s) - f_2(s)\| ds \\
&\leq \lambda \int_0^1 \|y_1(s) - y_2(s)\| ds + \lambda \int_0^1 m(s) \|y_1(s) - y_2(s)\| ds + \lambda \varepsilon \\
&\leq \lambda \|y_1(\cdot) - y_2(\cdot)\|_\infty \int_0^1 (1 + m(s)) ds + \lambda \varepsilon.
\end{aligned}$$

So, we conclude that

$$\|z_1(\cdot) - z_2(\cdot)\|_\infty \leq \lambda \|y_1(\cdot) - y_2(\cdot)\|_\infty \int_0^1 (1 + m(s)) ds + \lambda \varepsilon.$$

By an analogous relation, obtained by interchanging the roles of  $y_1(\cdot)$  and  $y_2(\cdot)$ , it follows that

$$H(T(y_1(\cdot)), T(y_2(\cdot))) \leq \lambda \|y_1(\cdot) - y_2(\cdot)\|_\infty \int_0^1 (1 + m(s)) ds + \lambda \varepsilon.$$

By letting  $\varepsilon \rightarrow 0$ , we obtain

$$H(T(y_1(\cdot)), T(y_2(\cdot))) \leq \lambda \|y_1(\cdot) - y_2(\cdot)\|_\infty \int_0^1 (1 + m(s)) ds.$$

Consequently, if  $\int_0^1 (1 + m(s)) ds < \frac{1}{\lambda}$ ,  $T$  is a contraction. By Lemma 7.2,  $T$  has a fixed point which is a solution of (7.7). The proof is complete.

□

# *Conclusion and Direction for Further Research*

## **Conclusion**

The major objective of this thesis was to develop efficient methods and supporting analysis for solving abstract fractional differential equations of fractional order. Since the classical semigroup operator solutions are not adequate to handle the fractional case, which naturally leads to the necessity of deriving new techniques for fractional derivatives. A series of four published papers and three manuscript in preparation has been presented on the controllability for systems governed by semilinear fractional differential inclusion, solutions of fractional quasilinear mixed integrodifferential equations with nonlocal condition, controllability of fractional integrodifferential systems via almost operators, existence and controllability result for an evolution fractional integrodifferential systems, controllability for systems governed by nonlocal quasilinear differential inclusion, existence of solutions for fractional differential inclusions with boundary conditions in a Banach Spaces. these papers comprise the bulk of Chapters 3 to 7 and form the backbone of the theory associated with solution operators methods.

## **Direction for Further Research**

There are a number of extensions and applications to the methods presented in this thesis that could be pursued in the future. In Particular, the following areas could lead to fruitful research

- Existence and controllability for Riemann–Liouville fractional integrodifferential equations.
- Classical and global solutions to fractional quasilinear differential equations with infinite delay in Banach spaces.
- Mittag-Leffler stability of fractional quasilinear differential equations.
- Time–Optimal control of systems displaying fractional dynamics in the sense of the Riemann-Liouville and Caputo.

# Appendix A

## Special Functions of the Fractional Calculus

In this Appendix we shall consider the Mittag–Leffler function and some of the related functions which are relevant for their connection with fractional calculus.

### A.1 Euler’s Gamma Function

We recall the definition

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad (\text{A.1})$$

for  $x > 0$ . Elementary considerations from the theory of improper integrals reveal that the integral exists.

Now we have the all important relation between the Gamma function and the factorial:

**Theorem A.1.** *If  $x > 0$  then  $\Gamma(x + 1) = x\Gamma(x)$ .*

As a consequence of these considerations, we find another important identity involving the Gamma function:

**Theorem A.2.** *Let  $n \notin \mathbb{Z}$  and  $k \in \mathbb{N}_0$ . Then,*

$$(-1)^{k+1} \Gamma(n - k) \Gamma(k + 1 - n) = \Gamma(-n) \Gamma(n + 1)$$

Another useful identity in this context is the Reflection Formula

**Theorem A.3.** *Let  $0 < x < 1$ . Then,*

$$\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

It is also possible to find an alternative representation, due to Gauss, for the Gamma function.

**Theorem A.4.** *Let  $x \in \mathbb{R}$  and  $-x \in \mathbb{N}_0$ . Then,*

$$\Gamma(x) = \lim_{n \rightarrow \infty} \frac{n! n^x}{x(x+1)(x+2) \cdots (x+n)}$$

The asymptotic behaviour of  $\Gamma(x)$  as  $x \rightarrow \infty$  is sometimes important; it can be described by the Stirlings Formula

**Theorem A.5.** *For  $x \rightarrow \infty$ . Then,*

$$\Gamma(x+1) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x} (1 + o(1))$$

One last result that we shall mention explicitly is the following integral identity

**Theorem A.6.** *Let  $\alpha, \beta \in \mathbb{R}_+$ . Then,*

$$\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

and hence

$$\int_0^x t^{\alpha-1} (1-t)^{\beta-1} dt = x^{\alpha+\beta-1} \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

The incomplete Gamma function is a closely related function defined as

$$\Gamma^*(\nu, t) = \frac{2}{\Gamma(\nu) t^\nu} \int_0^t e^{-x} x^{\nu-1} dx, \quad \Re(\nu) > 0. \quad (\text{A.2})$$

## A.2 Beta Function

Like the Gamma function, the Beta function (also known as Euler's integral of the first kind) is defined by a definite integral. Its definition is given by the integral in the first equation of Theorem [A.6](#)

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt \quad (\text{A.3})$$

### A.3 The Error Function

The definition of the Error function is given by

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad x \in \mathbb{R}. \quad (\text{A.4})$$

The complementary Error function (*Erfc*) is a closely related function that can be written in terms of the Error function as

$$\text{Erfc}(x) = 1 - \text{Erf}(x) \quad (\text{A.5})$$

### A.4 Mittag-Leffler's Function

The exponential function  $e^z$ , plays a very important role in the theory of integer-order differential equations. Its one-parameter generalization, the function which is now denoted by  $E$  was introduced by G. M. Mittag-Leffler

The two-parameter function of the Mittag-Leffler type, which plays a very important role in the fractional calculus was in fact introduced by Agarwal. This function could have been called the Agarwal function. However, Humbert and Agarwal generously left the same notation as for the one-parameter Mittag-leffler function and that is the reason that now the two-parameter function is called the Mittag-Leffler function.

A two-parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \Re(\alpha) > 0, \quad \Re(\beta) > 0. \quad (\text{A.6})$$

For  $\beta = 1$ , we obtain the Mittag-Leffler function in one parameter:

$$E_{\alpha, 1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \equiv E_{\alpha}(z) \quad (\text{A.7})$$

As a result of the definition A.6, the following relations hold:

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z), \quad (\text{A.8})$$

$$E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}(z), \quad (\text{A.9})$$

and

$$\left(\frac{d}{dz}\right)^m \left[ z^{\beta-1} E_{\alpha,\beta}(z^\alpha) \right] = z^{\beta-m-1} E_{\alpha,\beta-m}(z^\alpha), \quad \Re(\beta - m) > 0, \quad m = 0, 1, \dots \quad (\text{A.10})$$

For some specific values of  $\alpha$  and  $\beta$ , the Mittag–Leffler function reduces to some familiar functions. For example  $E_{1,1}(x) = e^x$ ,  $E_{1,3}(z) = \frac{e^z - 1 - z}{z^2}$ ,  $E_{1,2}(x) = \frac{e^x - 1}{x}$ ,  $E_{2,1}(z^2) = \cosh(z)$ ,  $E_{2,2}(z^2) = \frac{\sinh(z)}{z}$ .

The following generalized Gronwall’s inequality is essential to prove continuous dependence of mild solutions of equations (3.1)–(3.2) in chapter three.

**Lemma A.7** ([46]). *Suppose  $b \geq 0$ ,  $\beta > 0$  and  $a(t)$  is a nonnegative function locally integrable on  $0 \leq t < T$  (some  $T \leq \infty$ ), and suppose  $u(t)$  is nonnegative and locally integrable on  $0 \leq t < T$  with*

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1} u(s) ds, \quad 0 \leq t < T.$$

Then

$$u(t) \leq a(t) + \int_0^t \sum_{j=1}^{\infty} \frac{(b\Gamma(\beta))^j}{\Gamma(j\beta)} (t-s)^{j\beta-1} a(s) ds, \quad 0 \leq t < T. \quad (\text{A.11})$$

If  $a(t) \equiv a$ , constant on  $0 \leq t < T$ , then the inequality (A.11) is reduced to

$$u(t) \leq aE_\beta(b\Gamma(\beta)t^\beta),$$

where  $E_\beta$  is the Mittag–Leffler function of order  $\beta$ .

## A.5 Wright–type Function

The definition of the Wright–type Function is given by

$$\Psi_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(-\alpha n + 1 - \alpha)} = \sum_{n=1}^{\infty} \frac{(-z)^n}{(n-1)!} \Gamma(n\alpha) \sin(n\pi\alpha), \quad z \in \mathbb{C}, \quad (\text{A.12})$$

with  $0 < \alpha < 1$ . For  $-1 < r < \infty$ ,  $\lambda > 0$ , the following results hold.

$$\Psi_\alpha(t) \geq 0, \quad t > 0; \quad (\text{A.13})$$

$$\int_0^\infty \frac{\alpha}{t^{\alpha+1}} \Psi_\alpha\left(\frac{1}{t^\alpha}\right) e^{-\lambda t} dt = e^{-\lambda^\alpha}; \quad (\text{A.14})$$

$$\int_0^\infty \Psi_\alpha(t) t^r dt = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}; \quad (\text{A.15})$$

$$\int_0^\infty \Psi_\alpha(t) e^{-zt} dt = E_\alpha(-z), \quad z \in \mathbb{C}. \quad (\text{A.16})$$

## A.6 Mellin-Ross Function

The Mellin-Ross function  $E_t(\nu, a)$  arises when finding the fractional integral of an exponential  $e^{at}$ . The function is closely related to both the incomplete Gamma and Mittag-Leffler functions. Its definition is given by

$$E_t(\nu, a) = t^\nu e^{at} \Gamma^*(\nu, t). \quad (\text{A.17})$$

We can also write,

$$E_t(\nu, a) = t^\nu \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k + \nu + 1)} = t^\nu E_{1, \nu+1}(at). \quad (\text{A.18})$$

## A.7 Robotov's Function

The definition of the Robotov's function is given by

$$R_\alpha(\beta, t) = t^\alpha \sum_{k=0}^{\infty} \frac{\beta^k t^{k(\alpha+1)}}{\Gamma((\alpha+1)(k+1))} = t^\alpha E_{\alpha+1, \alpha+1}(\beta t^{\alpha+1}). \quad (\text{A.19})$$

# Appendix B

## Fixed Point Theorems

In this Appendix we collect some very basic point fixed theorems.

The proofs of various existence and uniqueness theorems throughout this text have been based on classical theorems asserting existence or uniqueness of fixed points of certain operators.

The first of these theorems is the following generalization of Banach's fixed point theorem that we take from.

**Theorem B.1 (Weissinger's Fixed Point Theorem).** *Let  $Y$  be a Banach space and  $\alpha_k$  a sequence of positive numbers such that  $\sum_{k=0}^{\infty} \alpha_k$  converges. Let  $F \subset Y$  be closed and  $A : F \rightarrow F$  be an operator such that*

$$\|A^k \varphi - A^k \psi\| \leq \alpha_k \|\varphi - \psi\|, \quad \forall k \in \mathbb{N} \quad \text{and} \quad \forall \varphi, \psi \in F. \quad (\text{B.1})$$

*Then  $A$  possesses a unique fixed point. Furthermore the fixed point can be obtained as the limit of the sequence defined by  $\psi_0 = \Phi \in F$ , and  $\psi_{k+1} = A\psi_k$ .*

An immediate consequence is

**Theorem B.2 (Banach's Fixed Point Theorem).** *Assume  $Y$  be a Banach space and  $0 \leq \alpha < 1$ . Let  $F \subset Y$  be closed and  $A : F \rightarrow F$  be an operator such that*

$$\|A\varphi - A\psi\| \leq \alpha \|\varphi - \psi\|, \quad \forall \varphi, \psi \in F. \quad (\text{B.2})$$

Then,  $A$  has a uniquely determined fixed point  $\psi^*$ . Furthermore, for any  $\psi_0 = \Phi \in F$ , the sequence defined by  $\psi_{k+1} = A\psi_k$  converges to this fixed point  $\psi^*$ .

Moreover we also used a slightly different result that asserts only the existence but not the uniqueness of a fixed point. Here we may work with weaker assumptions on the operator in question.

**Theorem B.3 (Schauder's Fixed Point Theorem).** *Let  $Y$  be a Banach space, let  $F$  be a closed convex subset of  $Y$ , and let  $A : F \rightarrow F$  be a mapping such that the set  $\{A\varphi : \varphi \in F\}$  is relatively compact in  $Y$ . Then  $A$  has at least one fixed point.*

As a key tool for developing the controllability in the chapter five, the Leray–Schauder fixed point theorem will be introduced as follows.

**Theorem B.4** ([43, 44]). *Let  $E$  be a Banach space,  $C$  a closed convex subset of  $E$ ,  $V$  an open subset of  $C$  and  $0 \in V$ . Suppose that  $F : \overline{V} \rightarrow C$  is a continuous, compact (that is,  $F(\overline{V})$  is a relatively compact subset of  $C$ ) map. Then either*

- (i)  $F$  has a fixed point in  $\overline{V}$ , or
- (ii) there is a  $x \in \partial V$  (the boundary of  $V$  in  $C$ ) and  $\lambda \in (0, 1)$  with  $x = \lambda F(x)$ .

The Krasnoselskii fixed-point theorem is a well-known principle that generalizes the Schauder fixed-point theorem and the Banach contraction principle as follows.

**Theorem B.5 (Krasnoselskii's Fixed Point Theorem).** *Let  $M$  be a nonempty closed convex subset of a Banach space  $Y$ ,  $S : F \rightarrow Y$  and  $T : F \rightarrow Y$ . Suppose that*

- (a)  $S$  is compact and continuous;
- (b)  $T$  is a  $k$ -contraction;
- (c)  $Sx + Tx \in F$  for every  $x, y \in F$ .

Then there exists  $x^* \in F$  such that  $Sx^* + Tx^* = x^*$ .

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كلية العلوم الدقيقة وعلوم الطبيعة والحياة  
جامعة العربي بن مهدي  
أم البواقي

أطروحة لنيل شهادة الدكتوراه في الرياضيات  
تحت عنوان:

## حول نظرية وتطبيقات المعادلات التفاضلية الكسرية

من تقديم:  
مبروك براقدي

### ملخص مفيض

لقد عرف حساب التفاضل والتكامل الكسري رواجاً ملحوظاً في أوساط الرياضياتيين والفيزيائيين وأثبت أهمية كبرى في السنوات الأخيرة من خلال تطبيقاته المتنوعة والمفيدة. يتميز باعتناؤه بالاشتقاق والمكاملة وتوسيعهما من الرتب الصحيحة التقليدية إلى الرتب الكسرية؛ ويعدّ إطاراً مفضلاً لحلّ العديد من المسائل المختلفة في الفيزياء والهندسة.

يمكن اعتبار حساب التفاضل والتكامل الكسري موضوعاً قديماً وجديداً في الوقت ذاته. إذ أنّه ظهر بدءاً من بعض تكهّنات العالمين لايبنتز وأولر، ومن بعض الأعمال البارزة الأخرى لعلماء رياضياتيين باريزين، من أمثال لابلاس وفورييه وآبل وريمان وليوفيل. خضع حساب التفاضل والتكامل الكسري لتطور سريع وخاصة خلال العقدين الماضيين. تعتبر نظريتنا المعادلات نصف الخطية وشبه الخطية ذات الرتب الكسرية، أي المعادلات التفاضلية التي عوض فيها المشتقّ الصحيح بالنسبة للزمن بمشتقّ كسريّ، أحد الفروع الناشئة من هذه الدراسة. لعلّ الدافع وراء الاهتمام المتزايد بهذه الفئة من المعادلات يكمن في تطبيقاتها الكثيرة على المسائل الهندسية ذات اللزوجة والمطاطية، ومسائل توصيل الحرارة في المواد مع الذاكرة، الديناميكا الكهربائية مع الذاكرة؛ كما يمكن أيضاً أن تكون بسبب استخدامها لنهج قوانين الحفظ غير الخطية.

يدور فحوى الأطروحة الحاضرة حول نوعين من المعادلات التفاضليّة ذات الرتب الكسريّة. جننا به في قسمين. توخينا في الأوّل منهما تسليط الضوء على بعض الجوانب والنتائج الكلاسيكيّة الدائرة في فلك هذا الصنف من المعادلات، من وجود الحلّ ووحديّته وكذا قابليّة التحكّم وانتظام الحلول وغيرها من الخواص المتداولة والمعهودة لدي المعادلات التفاضليّة التطوريّة المجرّدة العادية، وتعميمها إلى المعادلات التفاضليّة المجرّدة شبه الخطيّة ذات الرتب الكسريّة، وهذا في فضاءات بناخيّة. تطرّقنا في القسم الثّاني إلى دراسة هذه الخصائص واستعرضناها في إطار الجمل المدبّرة باحتواءات تفاضليّة كسريّة شبه خطيّة. لقد تمّ التوصل إلى هذه النتائج عبر استخدام المؤثّرات الحالّة وعائلة المؤثّرات الحالّة ونظريّة المؤثّرات شبه المقطعيّة ونظريات النقطة الصامدة إلى جانب الحساب التفاضليّ الكسريّ.

هيكنا هذه الرسالة وفق سبعة فصول.

قمنا في الفصل الأوّل بتقديم التعاريف الأساسيّة التي تخصّ التكامل الكسريّ وخصائصه وأنواعه، وكذا الاشتقاق الكسريّ خصائصه وأنواعه.

يحتوي الفصل الثّاني على بعض المعلومات والمواد الأساسيّة عن نظرية أنصاف الزمر والمؤثّرات الحالّة وأسرة المؤثّرات الحالّة والمؤثّرات المقطعيّة تقريبا والتي تستخدم في بقية الفصول اللاحقة.

اعتبرنا في الفصل الثّالث دراسة وجود ووحديّة وانتظاميّة حلول المعادلات التفاضليّة-التكامليّة الكسريّة شبه الخطيّة المجرّدة. قمنا بفرض بعض الشروط الأساسيّة لضمان تطبيق خواص المؤثّرات الحالّة وتطبيق نظرية النقطة الصامدة. تعدّ النتائج المحصّل عليها امتدادا لأعمال سابقة، حيث قمنا في هذا الفصل بتعميم هذه النتائج من أجل المعادلات التفاضليّة-التكامليّة شبه الخطيّة برتب كسريّة. حريّ بنا أن نشير إلى أنّ الطريقة المستخدمة هنا تختلف عن الطرق المستخدمة في الأعمال السابقة.

خصّصنا الفصل الرابع لدراسة وجود ووحديّة الحلّ وقابليّة المراقبة لبعض المعادلات نصف الخطيّة ذات الرتب الكسريّة في الفضاءات اللانهائيّة. لقد قمنا بإعطاء الشروط الأساسيّة لضمان ذلك وهذا بدون اعتبار شرط التراص على المؤثّرات الحالّة لضمان أن المراقبة تكون في الفضاءات اللانهائيّة. تعتبر النتائج المحصّل عليها تعميما لنتائج سابقة حيث استخدمنا هنا، ولأوّل مرّة، المؤثّرات الحالّة الخاصّة بحساب التفاضل والتكامل الكسريّ.

يتعامل الفصل الخامس مع قابلية المراقبة لبعض المعادلات التفاضليّة-التكامليّة الكسريّة المجرّدة، حيث قمنا بإعطاء الشروط الأساسيّة لضمان تطبيق خواص الحلّ المؤثّر المميّز لنظريّة المؤثّرات المقطعيّة تقريبا ونظريّة النقطة الصامدة لشودر. لوري. تعدّ النتائج المحصّل عليها امتدادا لأعمال سابقة، حيث قمنا بتعميم هذه

النتائج من أجل المعادلات التفاضلية-التكاملية ذات رتب كسرية. علينا أن نشير هنا أيضا إلى أن الوسائل المستخدمة تختلف عن تلك المستخدمة في الأعمال السابقة.

كرّسنا الفصل السادس لدراسة قابلية المراقبة لأنظمة شبه الخطية المنمذجة بمعادلات تفاضلية كسرية احتوائية في فضاءات بناخية. لقد حصلنا على النتائج المرجوة باستخدام نظرية أسرة المؤثرات الحالة ونظرية النقطة الصامدة. النتائج المحصل عليها امتداد لأعمال سابقة. لقد قمنا بتعميم هذه النتائج من أجل المعادلات التفاضلية الاحتوائية وذات رتب كسرية، مستخدمين وسائل تختلف عن تلك المستخدمة في الأعمال السابقة.

قمنا في الفصل السابع والأخير بدراسة وجود ووحدانية الحل لمعادلات تفاضلية احتوائية ذات رتب كسرية مع قيم حدية. لجأنا إلى إعطاء الشروط الأساسية لضمان ذلك، وهذا بدون اعتبار شرط التحدّب. استخدمنا أيضا حساب التفاضل والتكامل الكسريّ نظرية فريين ونظرية النقطة الصامدة للتطبيقات المتعددة القيم لإثبات الوجود والوحدانية. النتائج المحصل عليها تمديد وتعميم لنتائج سابقة.