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**Solvability of Solution and  
Controllability of Ordinary and Partial  
Fractional Differential Problem**

By

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## Solvability of Solution and Controllability of Ordinary and Partial Fractional Differential Problems

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# Dedication



I dedicate this thesis to :

My great parents, whose unconditional love, sacrifice, and unwavering belief in me have been my greatest source of strength and motivation throughout my life.

My dearest brothers, "Ammar, Ilyes " for their constant support, words of encouragement, and for always being there when I needed them most.

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# Abstract

This thesis aims to accomplish three essential goals. First, we study the solvability of the solution and controllability of fractional singular and degenerate problem. We then use the energy inequality method to demonstrate the solution's existence and uniqueness. On other hand, Tikhonov regularization examines the weak controllability of the problem. Secondly, we aims to identify the unknown source term in an inverse problem by converting this problem into an optimal control problem with missing data. The proposed method to solve the optimal control problem is no-regret control concept, the source term will be characterized by an optimality system. Finally, the thesis explores optimal control in bilinear system with incomplete data using an iterative sequence we linearize the bilinear problem. The optimality system of the bilinear problem obtained us limit of the optimality system of the linearized problem.

**Keywords.** Age-structured population, Bilinear systems, Energy inequality method, Fractional problem, Identification problem, Incomplete data, No-regret control, Optimality system, Solvability, Weak controllability, Tikhonov regularization.

**2020 Mathematics Subject Classification 2020.** 35A01, 65L10, 65L12, 65L20, 65L70, 35R11, 35K58, 35K67.

## Résumé

Cette thèse vise à atteindre trois objectifs essentiels. Premièrement, nous étudions la solvabilité de la solution et la contrôlabilité du problème fractionnaire singulière et dégénéré. On utilise la méthode d'inégalité énergétique pour démontrer l'existence et l'unicité de la solution. D'un autre côté, la régularisation de Tikhonov examine la contrôlabilité approchée du problème étudié. Deuxièmement, nous visons à identifier le terme source inconnu dans un problème inverse en transformé ce problème en un problème de contrôle optimal avec des données manquantes.

La méthode proposée pour résoudre le problème de contrôle optimal est le concept de contrôle sans regret, le terme source sera caractérisé par un système d'optimalité. Enfin, la thèse explore le contrôle optimal dans un système bilinéaire avec données incomplètes en utilisant une séquence itérative pour linéariser le problème bilinéaire. Le système d'optimalité du problème bilinéaire est une limite du système d'optimalité du problème linéarisé.

**Mots clés.** Population structurée par âge, Problème bilinéaire, Méthode des inégalités d'énergétique, Problème fractionnaire, Problème d'identification, Données incomplètes, Contrôle sans regret, Système d'optimalité, solvabilité, Contrôlabilité approchée, Régularisation de Tikhonov.

## المخلص

تهدف هذه الأطروحة الى تحقيق ثلاث أهداف أساسية. أولاً، ندرس قابلية الحل وإمكانية التحكم في مسألة كسرية مفردة. ثم نستخدم طريقة عدم المساواة في الطاقة لإثبات وجود الحل ووحده. من ناحية أخرى، تسمح طريقة تعديل تيخونوف بدراسة التحكم الضعيف للمسألة. ثانياً، نهدف الى تحديد المصدر المجهول لمشكلة عكسية عن طريق تحويل هذه المشكلة الى مشكلة التحكم الأمثل مع بيانات ناقصة.

الطريقة المقترحة لحل المشكلة الأخيرة هي طريقة التحكم بلا ندم. سيميز المصدر المجهول بنظام استمثالي. أخيراً، تستكشف الأطروحة التحكم الأمثل في الأنظمة الثنائية مع بيانات ناقصة باستخدام سلسلة تكرارية المشكلة تصبح خطية. النظام الإستمثالي للمشكلة الثنائية هو عبارة عن نهاية للنظام الإستمثالي للمشكلة الخطية.

**الكلمات المفتاحية:** السكان المنظمون حسب العمر، المشكلة الثنائية، طريقة متراجحة الطاقة، المشكلة الكسرية، مشكلة التحديد، البيانات غير المكتملة، التحكم دون ندم، نظام استمثالي، القدرة على الحل، التحكم التقريبي، تعديل تيخونوف.

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# Notations and Abbreviations

## Notations

$\mathbb{R}$	The set of all real numbers.
$C([0, T], E)$	The space of continuous functions assuming values in $E$ .
$\ \cdot\ _H$	A scalar product in Hilbert space $H$ .
$(\cdot, \cdot)_H$	A scalar product in Hilbert space $H$ .
$ \cdot _H$	A semi-norm in $H$ .
$\frac{\partial y}{\partial \nu} = \nabla y \cdot \nu$	The conormal derivative.
$\Delta = \sum_{i=1}^n \frac{\partial}{\partial x_i}$	The laplacien operator.
$\nabla = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)^T$	The gradient operator.
$div$	Divergence.
$\mathcal{A}^*$	The adjoint operator of $\mathcal{A}$ .
$d\Gamma$	Lebesgue measure on boundary $\Gamma$ .
$\chi_\omega$	Characteristic function of the set $\omega$ .
$\mathcal{L}(Y, Z)$	The space of linear bounded operators from $Y$ to $Z$ .
$\mathcal{D}(Q)$	The space of functions in $C^\infty$ with a compact support in $Q$ . $\mathcal{D}'(Q)$ The dual space of
$L^2(0, T, H)$	The bounded linear operator space.
$Im$	The image.
$\Re$	The real part.
$:=$	Is defined as.

## Abbreviations

- i.e.* Namely or that is.
- ODE Ordinary differential equation.
- PDE Partial differential equation.
- FDE Fractiona differential equation.
- $\rightharpoonup$  Symbol of weak convergence.
- iff If and only if.
- resp. respectively.

# Introduction

Mathematical modelling of natural phenomena serves as a powerful tool in understanding the complexities of the world around us and is applied in a wide range of disciplines, including physics, biology, chemistry, ecology, meteorology, etc. These models help scientists predict future behaviours and outcomes based on current data, understand the underlying mechanisms and relationships between variables in natural systems, develop strategies for intervention or optimization, and model the spread of infectious diseases. These models can be more efficient and cost-effective than real-world experiments, which might be impractical, expensive, or ethically questionable. Moreover, they provide a means for analysis and exploration, contributing to our understanding of the fundamental principles that govern the world.

To ensure that systems management is achieved and directed towards specific objectives, control theory must be intervened, which also helps stabilize these systems through designing appropriate control strategies that make the systems able to withstand disturbances and shocks this is important in many fields for example in aeronautics fields which preserving their stability is essential for safety. Controllability allows to design and improve the strategies to control the spread of infectious diseases through vaccination or other interventions. It also enables us to make the right decisions to achieve the desired results. Moreover, control systems greatly improve performance. In robotics, control ensures that robots can perform complex tasks accurately and efficiently.

Controllability is an important area in control theory that examines the possibility of directing systems from any initial state to a desired state using a set of admissible controls. The theory of controllability was initiated by the famous Kalman in 1962-1963 [25] for finite dimensional systems, in 1978 Russell [42] extended the study of controllability for linear PDE's. This is followed by these papers [24, 26] addressing the controllability of semi-linear PDE's, for the controllability of the

non-linear PDE's investigated by Balachandran and Dauer in 2002 [6].

Among the various types of controllability : exact controllability, null controllability and approximate controllability are essential and done in many studies for example in [33, 41, 31, 15]. The concept of exact controllability consists of driving a system from initial state to the final state in finite time, this concept is characterized by Kalman's algebraic rank condition. A null controllability is a specific case from the exact controllability that drives a system to the zero states from any arbitrary state in finite time. In practice, exact controllability is not necessarily achieved, therefore we can reach arbitrarily close to a desired state within a finite time, This is known as approximate controllability.

While controllability theory ensures the principles and methods to influencing the behaviour of dynamical systems, it does not interest in the quality or efficiency of the control. Unlike optimal control theory is concerned with designing controllers that not only achieve desired outcomes but also do so in the best manner respecting certain criteria.

The theory of optimal control is a branch of control theory that seeks to determine a control law that will guide the system from any initial state to a desired state while minimizing or maximizing an objective function subject to the system's dynamic constraints and control limitations. To formulate an optimal control problem we necessitate essential components :

- The control input : represents the variable that can change the behaviour of the system.
- The system state : is the set of variables describing the current condition of a dynamical system at any given moment.
- The observation function : is used to measure the system's performance.
- The objective function : establish constraints on the control input or system states to ensure the validity of solutions.

To solve an optimal control problem, using techniques from calculus of variation, differential equation and numerical analysis. Calculus of variation helps to obtain the necessary condition for optimality, differential equation models the dynamic of the system being controlled, and numerical analysis employs computational tools to find approximate solutions to the optimization problem.

Their origins can be traced back to the 18th and 20th centuries by the work of Leonhard Euler, Joseph-Louis Lagrange, and Carl Gustav Jacob Jacobi, who provided the basic of the calculus of variation to find the extrema of functionals. In the 1950s, Richard Bellman developed the fundamental

technique for optimal control theory in ODE's, known as dynamic programming. Later, in 1956s Lev Pontryagin formulated the Pontryagin Maximum Principle which established the necessary conditions for optimality. Meanwhile, the development of optimal control of PDEs occurred in the middle of the 20th century via the Linear-Quadratic Regulator problem, the method of characteristics, and several numerical methods. During the 1970s and 1980s, Jacques Louis Lions made a major contribution to this field where his theory allowed to control of systems described by PDEs on spatial and temporal domains.

Often real-world applications frequently place us in situations where the available data on the phenomenon under study is insufficient or noisy. Here the traditional approach of optimal control is unable to overcome the difficulties brought on by missing data. To ensure that systems with incomplete data can be controlled and effectively optimized, the techniques of optimal control theory must be developed. In this publication [28, 29] Jacques Louis Lions is credited as being the first who study this kind of problem and establish the concept of Pareto control. The methods of no-regret control and low-regret control were later introduced in 1992 [30], they changing the study of optimal control with missing data to the standard optimal control theory. The equivalence between the concepts of no-regret control and Pareto control was demonstrated by Nakoulima, Omrane & Velin in 2003 [34]. Hafdallah and Ayadi in 2020 [20] recently explored the notion of low-regret control in the electromagnetic wave equation with missing data.

In this thesis, we investigate the fields of controllability theory and optimal control theory, to explore and analyze the weak controllability of fractional singular degenerate problem using the Tikhonov regularization method. Additionally, to identify the unknown source term in the coupled partial differential equation using both concepts of no-regret control and low-regret control. In order to accomplish these goals, this work is divided into four chapters. The first two chapters contain the foundation and an outline of controllability and classical optimal control theory, as well as the theory of optimal control with incomplete data. The most significant part of this dissertation is the third chapter, where in the first by use of Tikhonov regularization we achieve the weak controllability of the fractional singular degenerate problem which describes the heat conduction in relation to the body geometric shape. Next, we attempt to identify the unknown local changes in the age-structured population with missing initial population age. Our approach to solving this identification problem is

to transform it into an optimal control problem with missing data, and by using the no-regret control the local changes will be characterized by a coupled optimality system. Also, in [22] the author used the no regret control to identify the diffusion coefficient in a semi-linear parabolic equation. The last chapter deals with studying the optimal control of a bilinear system with incomplete data. We think that by an iterative sequence, we can linearise the bilinear problem.

### **Thesis' overview**

This thesis is organized into four chapters :

The first chapter is divided into three sections. The first section introduces the fundamental concepts of Semi-groups theory. The second section, focuses on the controllability of localized systems. The final section of this chapter extends the controllability of localized systems to distributed systems.

The second chapter contains four sections. We start with the essential techniques of optimization theory. Then, we delve into the standard optimal control theory. Moreover, we concentrated on the optimal control problem with incomplete data. We end this chapter with the presentation of the sentinel method.

The third chapter is divided into two sections. The first section discusses the solvability and controllability of the fractional heat equation. In the second section, we aim to identify the unknown source term in a coupled partial differential equation.

The last chapter is devoted to three sections. The first section deals with setting the bilinear problem. The second section explores the methods to linearize the bilinear problem. The last section presents the full characterization of the no-regret control bilinear problem.

We finish this dissertation with a conclusion and perspectives summarizing the main results obtained and suggestions for future research related to this topic.

# Chapitre 1

## Basics on Controllability

In this first chapter, we lay the foundational groundwork necessary for understanding the concept of controllability in dynamic systems. We begin with some preliminaries on semi-group theory. The second section is concerned with the controllability concept, we explore it in two distinct settings : finite-dimensional and infinite-dimensional systems. We discuss the techniques and the highlighting tools that will be used to tackle the controllability of the systems.

## 1.1 Basic Semigroups theory

In this section, we give some major definitions and theorems about Semigroups theory underpins our subsequent discussions.

Proofs of theorems, corollary and properties presented in this section are taken from [36].

### 1.1.1 Preface

Let  $E$  be a functional space. We consider the following Cauchy Problem : find  $y \in E$  solution of :

$$\begin{cases} y' = ay, \\ y(0) = y_0, \end{cases} \quad (1.1.1)$$

where  $a$  is a given constant, the solution of the problem 1.1.1 given by  $y(t) = \exp(ta)y_0$ , such that  $\exp(ta) = \sum_{i=0}^{\infty} \frac{(at)^i}{i!}$ .

Now, we consider the following problem

$$\begin{cases} y' = Ay, \\ y(0) = y_0. \end{cases} \quad (1.1.2)$$

If  $\dim E = n$ ,  $A$  is a matrix  $n \times n$  (its evident that  $A$  is a bounded operator). The solution of the previous problem is  $y(t) = \exp(tA)y_0$  we noted by  $y(t) = \mathcal{S}_1(t)y_0$ . When  $t \in \mathbb{R}$ , the set  $\{\mathcal{S}_1(t) : t \in \mathbb{R}\}$  has properties of Group.

If  $A$  is unbounded linear operator the solution of problem 1.1.2 given by  $y(t) = \mathcal{S}_2(t)y_0$  ( $t \geq 0$  sometimes  $t \in \mathbb{R}$ ). Where  $\mathcal{S}_2$  is a bounded operator in  $E$  verifying certain properties of  $\exp(tA)$ , precisely

$$\mathcal{S}_2(0) = I, \text{ (} I \text{ is the identity operator on } E\text{)}.$$

$$\mathcal{S}_2(t+s) = \mathcal{S}_2(t)\mathcal{S}_2(s) \text{ for every } t, s \geq 0.$$

$$\lim_{t \rightarrow 0} \mathcal{S}_2(t)y_0 = y_0 \text{ in } E.$$

**Definition 1.1** Let  $E$  be a Banach space, a one parameter family  $\mathcal{S}(t)$ ,  $0 \leq t < +\infty$ , of bounded linear operators from  $E$  into  $E$  is a semigroup of bounded linear operator on  $E$  if

$$\text{— } \mathcal{S}(0) = I, \text{ (} I \text{ is the identity operator on } E\text{)}.$$

—  $\mathcal{S}(t+s) = \mathcal{S}(t)\mathcal{S}(s)$  for every  $t, s \geq 0$  (the semigroup property).

A semigroup of bounded linear operator,  $\mathcal{S}(t)$ , is uniformly continuous if

$$\lim_{t \rightarrow 0} \|\mathcal{S}(t) - I\| = 0.$$

The linear operator  $A$  defined by

$$D(A) = \left\{ y \in E : \lim_{t \rightarrow 0} \frac{\mathcal{S}(t)y - y}{t} \text{ exists} \right\},$$

and

$$Ay = \lim_{s \rightarrow t} \frac{\mathcal{S}(t)y - y}{t} = \frac{d^+ \mathcal{S}(t)y}{dt} \Big|_{t=0} \text{ for } y \in D(A),$$

is the infinitesimal generator of the semigroup  $\mathcal{S}(t)$ ,  $D(A)$  is the domain of  $A$ .

From the definition, it is clear that if  $\mathcal{S}(t)$  is a uniformly continuous semigroup of bounded linear operator, then

$$\lim_{t \rightarrow 0} \|\mathcal{S}(s) - \mathcal{S}(t)\| = 0.$$

**Theorem 1.1** A linear operator  $A$  is the infinitesimal generator of a uniformly continuous semigroup if and only if  $A$  is a bounded linear operator.

**Theorem 1.2** Let  $\mathcal{S}(t)$  and  $\mathcal{T}(t)$  be uniformly continuous semigroups of bounded linear operators. If

$$\lim_{t \rightarrow 0} \frac{\mathcal{S}(t) - I}{t} = \lim_{t \rightarrow 0} \frac{\mathcal{T}(t) - I}{t} = A,$$

then  $\mathcal{S}(t) = \mathcal{T}(t)$  for  $t \geq 0$ .

**Corollary 1.1** Let  $\mathcal{S}(t)$  be a uniformly continuous semigroup of bounded linear operators. Then

- There exists a constant  $\omega \geq 0$  such that  $\|\mathcal{S}(t)\| \leq e^{\omega t}$ .
- There exists a unique bounded linear operator  $A$  such that  $\mathcal{S}(t) = e^{tA}$ .
- The previous operator  $A$  is the infinitesimal generator of  $\mathcal{S}(t)$ .
- $t \rightarrow \mathcal{S}(t)$  is differentiable in norm and

$$\frac{d\mathcal{S}(t)}{dt} = A\mathcal{S}(t) = \mathcal{S}(t)A.$$

### 1.1.2 Strongly Continuous Semigroups of Bounded Linear Operators

**Definition 1.2** A semigroup  $\mathcal{S}(t)$ ,  $0 \leq t < \infty$ , of bounded linear operators on  $E$  is a strongly continuous semigroup of bounded linear operators if

$$\lim_{t \rightarrow 0} \mathcal{S}(t)y = y \quad \text{for every } y \in E.$$

A strongly continuous semigroup of bounded linear operators on  $E$  will be called a semigroup of class  $\mathcal{C}_0$  or simply a  $\mathcal{C}_0$ -semigroup.

**Theorem 1.3** Let  $\mathcal{S}(t)$  be a  $\mathcal{C}_0$ -semigroup. There exist constants  $w \geq 0$  and  $M \geq 1$  such that

$$\|\mathcal{S}(t)\| \leq Me^{\omega t}, \quad \text{for } 0 \leq t < \infty.$$

**Corollary 1.2** If  $\mathcal{S}(t)$  is a  $\mathcal{C}_0$ -semigroup then for every  $y \in E$ ,  $t \rightarrow \mathcal{S}(t)y$  is a continuous function from  $\mathbb{R}_0^+$  (the nonnegative real line) into  $E$ .

**Theorem 1.4** Let  $\mathcal{S}(t)$  be a  $\mathcal{C}_0$ -semigroup and let  $A$  be its infinitesimal generator. Then

— For  $y \in E$ ,

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} \mathcal{S}(s)y ds = \mathcal{S}(t)y.$$

— For  $y \in E$ ,  $\int_0^t \mathcal{S}(s)y ds \in D(A)$  and

$$A \left( \int_0^t \mathcal{S}(s)y ds \right) = \mathcal{S}(t)y - y.$$

— For  $y \in D(A)$ ,  $\mathcal{S}(t)y \in D(A)$  and

$$\frac{d}{dt} \mathcal{S}(t)y = A\mathcal{S}(t)y = \mathcal{S}(t)Ay.$$

— For  $y \in D(A)$ ,

$$\mathcal{S}(t)y - \mathcal{S}(s)y = \int_s^t \mathcal{S}(\tau)Ay d\tau = \int_s^t A\mathcal{S}(\tau)y d\tau.$$

**Corollary 1.3** If  $A$  is the infinitesimal generator of  $\mathcal{C}_0$ -semigroup  $\mathcal{S}(t)$ , then  $D(A)$  the domain of  $A$ , is dense in  $E$  and  $A$  is a closed linear operator.

**Theorem 1.5** *Let  $\mathcal{S}(t)$  and  $\mathcal{T}(t)$  be  $\mathcal{C}_0$ -semigroups of bounded linear operators with infinitesimal generators  $A$  and  $B$  resp. If  $A = B$  then  $\mathcal{S}(t) = \mathcal{T}(t)$  for  $t \geq 0$ .*

If  $A$  is the infinitesimal generator of a  $\mathcal{C}_0$ -semigroups then by Corollary 1.3,  $\overline{D(A)} = E$ . Actually, a much stronger result is true. Indeed we have.

**Theorem 1.6** *Let  $A$  be the infinitesimal generator of the  $\mathcal{C}_0$ -semigroup  $\mathcal{S}(t)$ . If  $\mathcal{D}(A^n)$  is the domain of  $A^n$ , then  $\bigcap_{n=1}^{\infty} \mathcal{D}(A^n)$  is dense in  $E$ .*

We finish this section with a useful result for establishing that a given operator  $A$  satisfies the sufficient conditions of the Hille-Yosida theorem is the infinitesimal generator of a  $\mathcal{C}_0$ -semigroup of contractions.

### The Hille-Yosida Theorem

Let  $\mathcal{S}(t)$  be a  $\mathcal{C}_0$ -semigroup. From Theorem 1.3 it follows that there are constants  $\omega \geq 0$  and  $M \geq 1$  such that  $\|\mathcal{S}(t)\| \leq Me^{\omega t}$  for  $t \geq 0$ . If  $\omega = 0$ ,  $\mathcal{S}(t)$  is called uniformly bounded and if moreover  $M = 1$  it is called a  $\mathcal{C}_0$ -semigroup of contractions. This section is devoted to the characterization of the infinitesimal generators of  $\mathcal{C}_0$ -semigroups of contractions. Conditions on the behaviour of the resolvent of an operator  $A$ , which are necessary and sufficient for  $A$  to be the infinitesimal generator of a  $\mathcal{C}_0$ -semigroup of contractions, are given.

Recall that if  $A$  is a linear, not necessarily bounded operator in  $E$ , the resolvent set  $\rho(A)$  of  $A$  is the set of all complex numbers  $\lambda$  for which  $(\lambda I - A)$  is invertible, i.e.,  $(\lambda I - A)^{-1}$  is a bounded linear operator in  $E$ . The family  $R(\lambda, A) = (\lambda I - A)^{-1}$ ,  $\lambda \in \rho(A)$  of bounded linear operators is called the resolvent of  $A$ .

**Theorem 1.7 Hille-Yosida in general case** *A linear operator  $A$  is the infinitesimal generator of a  $\mathcal{C}_0$ -semigroup satisfying  $\|\mathcal{S}(t)\| \leq Me^{\omega t}$ ,  $t \geq 0$ , with  $\omega \geq 0$  and  $M \geq 0$ , iff :*

- $A$  is closed and  $\overline{D(A)} = E$ .
- For all  $\lambda \in \mathbb{C}$  such that  $\text{Re}\lambda > \omega$ , we have  $\lambda \in \rho(A)$  and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\Re(\lambda) - \omega)^n}, \forall n \in \mathbb{N}.$$

## 1.2 Controllability of localized and distributed Systems

The problem of controllability is the possibility to change the behaviour of a dynamics systems from a given state to our wishes or guiding it to a specific state in a finite time. We give some basic ideas on the controllability in finite and infinite dimensions.

Proofs of theorems, corollary and properties presented in this section are taken from [21, 39].

### 1.2.1 Controllability of localized Systems

This subsection explores the essential principles underlying controllability within the context of localized systems, shedding light on the Kalman criterion, and elucidating its role in determining the controllability in finite dimension.

#### Description of the systems

Let  $T > 0$ ,  $\Omega$  be an open bounded domain with smooth boundary  $\Gamma$ . We consider the following localized controllability system :

$$\begin{cases} y' &= Ay + Bv, \\ y(0) &= y_0 \in D(A), \end{cases} \quad (1.2.1)$$

where :

- $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $B : \mathbb{R}^m \rightarrow \mathbb{R}^n$  are bounded linear matrices with  $m, n \in \mathbb{N}^* \times \mathbb{N}^*$  and  $m \leq n$ .
- $v \in L^2(0, T, U)$  is the control function ( $U = \mathbb{R}^m$ ).
- $y(\cdot) \in E = \mathbb{R}^n$  is the state with  $\dim E = n$ .
- $y_0$  the initial state.
- The solution  $y \in L^2(0, T, E)$  characterized by the following variation constant formula

$$y(t, y_0, v) = \mathcal{S}(t)y_0 + \int_0^t \mathcal{S}(t-s)Bv(s)ds, \quad \forall t \in [0, T],$$

where  $\mathcal{S}(t) = e^{tA}$  is the semi group.

When  $y_0 = 0$ , the solution can be written as follows

$$y(T, v) = L_T v,$$

where  $L_T$  is a linear bounded operator defined by :

$$\begin{aligned} L_T & : L^2(0, T, U) \rightarrow E \\ v & \longmapsto \int_0^T \mathcal{S}(T-s)Bv(s)ds. \end{aligned}$$

**Definition 1.3** We say that the system 1.2.1 is controllable in a finite time  $T$ , if for any initial state  $y_0 \in E$  and a desired state  $y_d \in E$ , it exist a control function  $v \in L^2(0, T, U)$  such that :

$$y_d = y(t, y_0, v),$$

then .

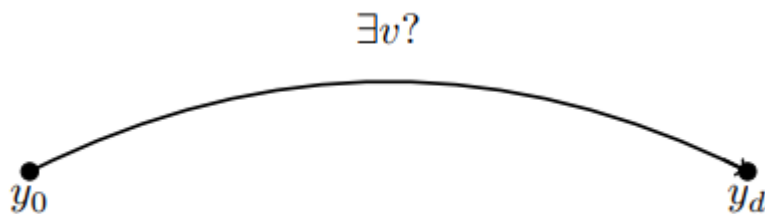


FIGURE 1.1 – Notion of controllability.

**Proposition 1.1** We say that the system 1.2.1 is controllable on  $T > 0$  iff  $L_T$  is surjective.

## The Gramian of controllability

### CNS of Controllability

Since the dimension of the space is finite the operator  $L_T$  will be surjective if and only if its adjoint is injective.

The adjoint operator of  $L_T$  define by the following expression

$$L_T^* \cdot = B^* \mathcal{S}^*(T - \cdot),$$

where  $B^*$  (resp  $\mathcal{S}^*(T - \cdot)$ ) are the adjoint matrix of  $B$  (resp  $\mathcal{S}(T - \cdot)$ ).

### Controllability matrix - Gramian of controllability

Let the operator  $Q_T \in \mathcal{L}(\mathbb{R}^n)$  given by

$$Q_T = L_T L_T^* = \int_0^T \mathcal{S}(T-s) B B^* \mathcal{S}^*(T-s) ds, \quad T > 0,$$

and

$$\langle Q_T y, y \rangle = \int_0^T |B^* \mathcal{S}^*(T-s)y|^2 ds = \|L_T^* y\|^2 \geq 0, \quad \forall y \in \mathbb{R}^n.$$

We called  $Q_T = L_T L_T^*$  the controllability matrix or Gramian of controllability.

**Theorem 1.8** *The following proprieties are equivalent :*

- *The pair  $(A, B)$  is controllable in  $T > 0$ .*
- *The operator  $L_T$  is surjective.*
- *The operator  $L_T^*$  is injective.*
- *The operator  $Q_T$  is invertible.*

### Kalman's criterion

The previous theorem 1.8 is not easy to applied it. For this, we present the famous Kalman controllability condition which is facilitated to applicable because it is an algebraic criterion which provides a powerful tool for assessing controllability by examining the rank of the matrix by block of  $A$  and  $B$  given by :

$$[B, AB, \dots, A^{n-1}B],$$

this matrix is called by Kalman controllability matrix. We distinct two different systems autonomous systems and non-autonomous systems we shall gives the Kalman's creterion in the two cases.

### Kalman's criterion of autonomous systems

**Theorem 1.9** (*Kalman's criterion*) *The pair  $(A, B)$  is controllable iff*

$$rg [B, AB, A^2B, \dots, A^{n-1}B] = n, \text{ with } \dim A = n.$$

**Remark 1.1** (Regular controllability) If  $\text{rank}[B] = j, j < n$ , then the pair  $(A, B)$  is controllable iff

$$\text{rg}[B; AB; \dots; A^{n-j}B] = n.$$

**Example 1.1** Consider the following dynamic system

$$y' = Ay + Bv,$$

where

$$A = \begin{pmatrix} -3 & 0 \\ 0 & -5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & \alpha \\ 1 & 2 \end{pmatrix} = (b_1 | b_2),$$

with  $\dim A = n = 2$  and  $\alpha$  is a constant, we examine the controllability and the regular controllability of the system.

We have

$$\text{rg}[B, AB] = \text{rg} \begin{bmatrix} 1 & \alpha & -3 & -3\alpha \\ 1 & 2 & 5 & -10 \end{bmatrix} = 2,$$

then, the system is controllable.

### Regular controllability

We have

$$\text{rg}[b_1, Ab_1] = \text{rg} \begin{bmatrix} 1 & -1 \\ 1 & 5 \end{bmatrix} = 2,$$

and

$$\text{rg}[b_2, Ab_2] = \text{rg} \begin{bmatrix} \alpha & -3\alpha \\ 2 & -10 \end{bmatrix} = -4\alpha,$$

then the system is regular controllable if  $\alpha \neq 0$ .

### Kalman's criterion of non-autonomous systems

The Kalman criterion to the case of systems for which the matrices  $A$  and  $B$  depend on time.

Suppose that  $A \in C^{n-2}(0, t, M_n(\mathbb{R}))$  and  $B \in C^{n-1}(0, t, M_{n,m}(\mathbb{R}))$ , the system given by

$$\begin{cases} y'(t) = A(t)y(t) + B(t)v(t), \\ y(0) = y_0. \end{cases} \quad (1.2.2)$$

Consider the following sequence  $(N_k(t))_{0 \leq k \leq n-1}$

$$\begin{cases} N_0(t) = B(t), \\ N_{k+1}(t) = A(t)N_k(t) - \frac{d}{dt}N_k(t), \quad k = \overline{0, n-2}. \end{cases}$$

If the  $\text{rank}[N_0(t), N_1(t), \dots, N_{n-1}(t)] = n$ , then the system 1.2.2 is controllable.

**Example 1.2** We study the controllability of the following system :

$$y'(t) = A(t)y(t) + B(t)v(t), \quad t > 0.$$

Where

$$A(t) = \begin{pmatrix} t & 1 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^2 \end{pmatrix} \quad \text{and} \quad B(t) = B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

with  $\dim A = n = 3$ , we shall calculate the matrix  $N_0(t), N_1(t), N_2(t)$ ,

$$\begin{aligned} N_0(t) &= B(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\ N_1(t) &= -A(t)B(t) + \frac{d}{dt}N_0(t) = -\begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}, \\ N_2(t) &= -A(t)N_1(t) + \frac{d}{dt}N_1(t) = -\begin{pmatrix} 2t \\ t^2 - 1 \\ t^4 - 2t \end{pmatrix}, \end{aligned}$$

then

$$rg[N_0(t), N_1(t), N_2(t)] = rg \begin{bmatrix} 0 & -1 & 2t \\ 1 & -t & t^2 - 1 \\ 1 & -t^2 & t^4 - 2t \end{bmatrix} = 3.$$

We conclude that our system is controllable.

## 1.2.2 Controllability of infinite dimensional Systems

The study of controllability of distributed systems presents challenges and opportunities unlike localized systems. We aim in this section to explore the fundamental condition for controllability of distributed system and we discuss the different types of controllability.

### Description of the system

We keep the same theoretical which we mentioned in the previous subsection the difference that the space  $E, U$  are Hilbert spaces. Let  $\Omega$  be an open bounded domain of  $\mathbb{R}^n$  ( $n = 1, 2, 3$ ),  $T > 0$  and  $\Gamma = \partial\Omega$ . Let us consider a controlled abstract equation :

$$\begin{cases} y'(t) = Ay + Bv & \text{in } (0, T), \\ y(0) = y_0. \end{cases} \quad (1.2.3)$$

The state  $y$  is given by :

$$y(t, y_0, v) = \mathcal{S}(t)y_0 + L_T v,$$

where  $\mathcal{S}(t)$  is a semi group and  $L_T$  is a linear bounded operator defined by

$$\begin{aligned} L_T & : L^2(0, T, U) \rightarrow E \\ v & \longmapsto \int_0^T \mathcal{S}(T-s)Bv(s)ds. \end{aligned}$$

## Different concepts of controllability

We show the different types of controllability on the following picture :

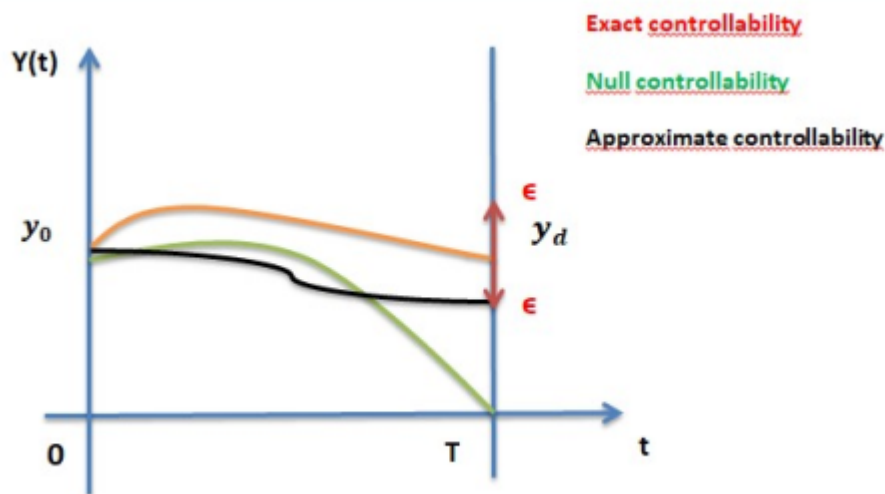


FIGURE 1.2 – Different types of controllability.

### Exact controllability

**Definition 1.4** *The system 1.2.3 is said exactly controllable in time  $T$  if for any  $(y_0, y_d) \in E^2$  there exist a control  $v \in L^2(0, T, U)$  such that the solution of 1.2.3 satisfying*

$$y(T; y_0, v) = y_d.$$

The previous definition equivalent to

$$ImL_T = E.$$

### Null controllability

**Definition 1.5** *The system 1.2.3 is said null controllable in time  $T$  if for any initial state  $y_0$  there exist a control  $v \in L^2(0, T, U)$  such that*

$$y(T; y_0, v) = 0.$$

### Approximate controllability

**Definition 1.6** *The system 1.2.3 is said approximately controllable in time  $T$  if, for any  $(y_0, y_d) \in E^2$  there exist a control  $v \in L^2(0, T, U)$  such that the solution of 1.2.3 satisfying*

$$\text{for all } \epsilon > 0, \|y(T; y_0, v) - y_d\|_E \leq \epsilon.$$

*The previous definition equivalent to*

$$\overline{\text{Im}(L_T)} = E.$$

**Theorem 1.10** *The pair  $(A, B)$  is approximately controllable iff  $\ker L_T^* = \{0\}$  and  $\text{Im}L_T^*$  is closed.*

**Theorem 1.11** *We define the Grammian of controllability by*

$$Q_T = L_T L_T^*,$$

*we have the following conditions to check the exact and approximate controllability of the problem 1.2.3*

**a)** *The problem 1.2.3 is exactly controllable iff the following conditions are satisfying :*

- $\exists \gamma > 0, \forall y \in E : \langle Q_T y, y \rangle_E \geq \gamma \|y\|_Y^2,$
- $\exists \gamma > 0, \forall y \in E : \|L_T^* y\|_{L^2(0, T; U)}^2 \geq \gamma \|y\|_E^2,$
- $\exists \gamma > 0, \forall y \in E : \|B^* \mathcal{S}^*(t) y\|_{L^2(0, T; U)}^2 \geq \gamma \|y\|_E^2,$
- $\ker L_T^* = \{0\}$  and  $\text{Im}L_T^*$  is closed.

**b)** *The problem 1.2.3 is weakly controllable iff the following conditions are satisfying :*

- $Q_T$  is positive,
- $\ker L_T^* = \{0\},$
- $B^* \mathcal{S}^*(t) y = 0$  on  $[0, T] \Rightarrow y = 0,$
- $\overline{\cup_{n \geq 0} \text{Im}(A^n \mathcal{S}(t) B)} = E.$

**c)** *The problem 1.2.3 is null controllable iff the following conditions are satisfying :*

- $\exists \gamma > 0, \forall y \in E : \|B^* \mathcal{S}^*(t) y\|_{L^2(0, T; U)}^2 \geq \gamma \|\mathcal{S}^*(T) y\|_E^2,$
- $\text{Im} \mathcal{S}(t) \subset \text{Im} L_T$  is closed.

# Chapitre 2

## Theory of the optimal control of distributed systems

In this chapter, we will delve into the optimal control theory as applied to distributed systems, examining it through two distinct cases : complete and incomplete data. We'll start by the classical case when we have all the data we need laying down fundamental principles before transitioning to scenarios where data may be missing. Here, we introduce the concepts of no-regret control and low regret control strategies which are designed to tackle challenge effectively of systems with missing information on source term, boundary or initial condition. We finish this chapter by the sentinel method which is the key tools using to identify the missing infected people.

## 2.2 Optimization

Throughout this section, we aim to provide a clear and comprehensive introduction to optimisation theory.

Proofs of theorems, corollary and properties presented in this section are taken from [8].

### 2.2.1 Basic

Let  $U$  a Hilbert space (on  $\mathbb{R}^n$ ),  $U_{ad}$  a subset of  $U$ . Let  $J$  be a function defined from  $U_{ad}$  to  $\mathbb{R}$ .

#### Coercivity

**Definition 2.2.1** *Let  $U \subset \mathbb{R}^n$  an unbounded set and  $J : U \rightarrow \mathbb{R}$ , we say that  $J$  is coercive on  $U$  if we have*

$$\lim_{\|v\| \rightarrow +\infty} J(v) = +\infty, \quad v \in U.$$

#### Convexity

**Definition 2.2.2** *We say that  $U_{ad} \subset U$  is convex if*

$$\forall (u, v) \in U_{ad} \times U_{ad}, \forall t \in [0, 1] \quad tu + (1 - t)v \in U_{ad}.$$

*A geometric object  $U_{ad}$  is convex when, each time we take two points  $u$  and  $v$  of  $U_{ad}$ , the segment  $[u, v]$  which joins them is entirely contained in  $U_{ad}$ .*

**Definition 2.2.3** *We say that  $J : U_{ad} \subset U \rightarrow \mathbb{R} \cup \{+\infty\}$  is convex if  $U_{ad}$  is convex and if*

$$\forall (u, v) \in U_{ad} \times U_{ad}, \forall t \in [0, 1] \quad J(tu + (1 - t)v) \leq tJ(u) + (1 - t)J(v).$$

**Definition 2.2.4** *We say that  $J : U_{ad} \subset U \rightarrow \mathbb{R} \cup \{+\infty\}$  is strictly convex if  $U_{ad}$  is convex and if*

$$\forall (u, v) \in U_{ad} \times U_{ad}; u \neq v \quad \forall t \in [0, 1] \quad J(tu + (1 - t)v) < tJ(u) + (1 - t)J(v).$$

#### Differentiability of function

**Definition 2.2.5**  $J$  is differentiable (Frechet sense) on  $u \in U$  it exist  $l_u(v)$  on  $U'$  such that

$$\forall v \in U, \quad J(u + v) = J(u) + l_u(v) + \epsilon(v)\|v\|, \quad \lim_{\epsilon \rightarrow 0} \epsilon(v) = 0,$$

where  $l_u(v)$  is the derivative of  $J$ , we noted  $J'(u)$  and we write  $J'(u).v = l_u(v)$ . By Riesz theorem

$$J'(u).v = (\nabla J(u), v).$$

**Proposition 2.2.1** Let  $J$  a function of  $U$  into  $\mathbb{R} \cup \{+\infty\}$ . We say that  $J$  is differentiable-gateaux in  $u \in D(J)$  if directional derivative

$$J'(u, v) = \lim_{t \rightarrow 0^+} \frac{J(u + tv) - J(u)}{t} = \langle \nabla J(x), h \rangle,$$

exist in the all of direction  $v$  of  $U$  if the application  $v \rightarrow J'(u, v)$  is linear and continuous.

Where  $D(J)$  is the domain of  $J$  is the next set  $D(J) = \{v \in U ; J(v) < +\infty\}$  which is convex.

**Remark 2.2.1** If  $J$  is Frechet differentiable then  $J$  is Gateaux differentiable. The inverse is false.

## 2.2.2 Optimization problem

The optimization problem is written as follows

$$\min_{v \in U} J(v), \tag{2.2.1}$$

or

$$\max_{v \in U} J(v),$$

where the function  $J$  called : the objective function, cost function or optimization criterion.

The optimization problem is called unconstraints if  $U = U_{ad}$ , with constraints if  $U \neq U_{ad}$ .

**Definition 2.2.6** *Local extrema*

We say that  $u$  is a local minimum (resp local maximum ) of  $J$  if it exist  $r > 0$  such that :

$$\forall u \in B(v, r) \cap U, \quad J(u) \leq J(v) \text{ (resp } J(v) \leq J(u)).$$

**Definition 2.2.7** *Global extrema*

We say that  $u$  is a global minimum (resp global maximum ) of  $J$  if :

$$\forall u \in U, J(u) \leq J(v) \text{ (resp } J(v) \leq J(u)).$$

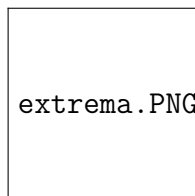


FIGURE 2.1 – Local and Global extrema.

**Remark 2.2.2** If  $u$  is a global or local maximum of  $J$  it satisfies a global or local minimum of  $(-J)$ , i.e.

$$\max \{J(v), v \in U\} = - \min \{-J(v), v \in U\}.$$

All results will be determined on the minimization problem.

## Unconstraints minimization problem

### Existence and uniqueness result

**Theorem 2.2.1** Let  $J : U \rightarrow \mathbb{R}$  propre, continuous and coercive function, then the minimization problem 2.2.1 admits a solution. Moreover, if  $J$  is strictly convex these solution is unique.

### First order optimality condition

**Theorem 2.2.2** (Necessary condition) Let  $J$  a Gateaux-differentiable function at point  $u \in U$ , if  $u$  is a local minimum of  $J$ , then

$$\nabla J(u) = 0.$$

**Remark 2.2.3** —  $u$  is said a stationary point of  $J(v)$ .

— The relation  $\nabla J(u) = 0$  is said a Euler equation.

**Theorem 2.2.3** (Sufficient condition) Let  $J$  be a convex and differentiable function, if  $\nabla J(u) = 0$  then  $u$  is a global minimum.

## Second order optimality condition

**Theorem 2.2.4** (Necessary condition) *Let  $J$  a second Gateaux-differentiable function at point  $u \in U$ , if  $u$  is a local minimum of  $J$ , then  $\nabla J(u) = 0$  and  $\nabla^2 J(u) \geq 0$ .*

**Theorem 2.2.5** (Sufficient condition) *Let  $J$  a second Gateaux differentiable function at point  $u \in U$ , if  $\nabla J(u) = 0$  and  $\nabla^2 J(u) \geq 0$  then  $u$  is a local strict minimum of  $J$ .*

**Theorem 2.2.6** *If  $J$  is convex then all local minimum is also global. Furthermore, if  $J$  is differentiable then every stationary point of  $J$  is a global minimum of  $J$ .*

## Constraints minimization problem

General minimisation problem with constraints written as :

$$\begin{cases} \min_{v \in U_{ad}} J(v), \\ g_i(v) \leq 0, i = 1 \dots p \\ h_i(v) = 0, i = 1 \dots q. \end{cases}$$

where  $g_i, h_i : \mathbb{R}^n \rightarrow \mathbb{R}$ , are inequality and equality constraints function resp. The feasible set is  $U_{ad} = \{v \in \mathbb{R}^n : g_i \leq 0, i = 1 \dots p \text{ and } h_i = 0, i = 1 \dots q\}$ .

**Theorem 2.2.7** *Let  $U_{ad}$  be a closed, convex and nonempty subset,  $J$  a continue convex function, we supposed that  $J$  is coercive or  $U_{ad}$  is bounded. Then the minimization problem admit a solution, if  $J$  is strictly convex the minimizer is unique.*

## First order optimality condition

**Theorem 2.2.8** *If  $J$  is Gateaux differentiable and  $U_{ad}$  is closed convex then all solution  $u$  of 2.2.2 satisfy the following necessary condition*

$$\forall u \in U_{ad} \quad (\nabla J(u), v - u) \geq 0.$$

## Lagrange multipliers problem

The Lagrange multipliers method is a powerful technique in optimization problems which returns optimization problems with constrained into optimization problems without constrained.

Lagrange problem given by

$$L(v, \lambda, \gamma) = J(v) + \sum_{i=1}^p \lambda_i g_i(v) + \sum_{j=1}^q \gamma_j h_j(v),$$

where  $\lambda_i$  and  $\gamma_j$  called Lagrange multipliers.

### Optimality condition

#### Inequality constraints

We consider the following problem :

$$\begin{cases} \min_{v \in U_{ad}} J(v), \\ g_i(v) \leq 0, i = 1 \dots p, \end{cases}$$

where  $U_{ad} = \{v \in \mathbb{R}^n : g_i(v) \leq 0, i = 1 \dots p\}$ .

**Theorem 2.2.9** *We supposed that  $\nabla g_i(v) \neq 0, i = 1 \dots p$ , if  $J(v)$  and  $g(v) = (g_1(v), g_2(v), \dots, g_p(v))^T$  are differentiable functions from  $\mathbb{R}^n$  to  $\mathbb{R}$  and  $u$  is a local minimum of  $J$  on  $U_{ad}$  then it exist  $\gamma \geq 0$  such that*

$$\begin{cases} \nabla J(u) + \gamma \nabla g_i(u) = 0, i = 1 \dots p, \\ \gamma g_i(u) = 0, i = 1 \dots p. \end{cases}$$

*also, if  $J(v)$  and  $g(v)$  are convex then the previous equality are sufficient condition to ensure that  $u$  is a local minimum.*

#### Equality constraints

We consider the following problem :

$$\begin{cases} \min_{v \in U_{ad}} J(v), \\ h_i(v) = 0 \end{cases}$$

where  $U_{ad} = \{v \in \mathbb{R}^n : h_i(v) = 0, i = 1 \dots q\}$ .

**Theorem 2.2.10** *We supposed that  $\nabla h_i(v) \neq 0, i = 1 \dots q$ , if  $J(v)$  is differentiable functions from  $\mathbb{R}^n$  to*

$\mathbb{R}$  and  $u$  is a local minimum of  $J$  on  $U_{ad}$  then it exist  $m$  real numbers such that

$$\nabla L(u) = 0$$

where

$$L(v) = J(v) + \sum_{i=1}^q \lambda_i h_i(v)$$

$(\lambda_1, \lambda_2, \dots, \lambda_n)$  are Lagrange multipliers.

## 2.3 Classical optimal control theory of distributed systems

We start by elucidating the fundamental principle to study the optimal control of distributed systems. Our studies commence with a presentation of the optimal control problem, followed by an examination of crucial aspects such as existence, uniqueness and characterization of the optimum. We conclude this segment by illuminating these concepts through illustrative examples, offering practical insights into their application and significance.

### 2.3.1 Description of the optimal control problem

Let the following abstract PDE equation related to the control  $v \in U_{ad}$

$$Ay = f + Bv, \tag{2.3.1}$$

associated by the following quadratic cost function

$$J(v) = \|Cy(v) - y_d\|_Z^2 + N \|v\|_U^2 \quad \forall v \in U_{ad}, \tag{2.3.2}$$

where

- $E$ ,  $U$  and  $Z$  be infinite dimensional Hilbert spaces of state, control and observation resp.
- $U_{ad} \subset U$  is a subset of admissible controls supposed non empty, closed and convex.
- $A \in \mathcal{L}(E)$  is a partial differential operator makes an isomorphism on  $E$ ,  $B \in \mathcal{L}(U, E)$  is the control operator and  $C \in \mathcal{L}(E, Z)$  is the observation operator.

—  $J$  is convex function from  $U_{ad} \subset U$  to  $\mathbb{R} \cup \{+\infty\}$ , and  $N > 0$ ,  $y_d$  is the fixed observation in  $Z$ .

The problem of optimal control consists of determining a control function  $u \in U_{ad}$  which is solution of

$$J(u) = \inf_{v \in U_{ad}} J(v). \quad (2.3.3)$$

The pair  $(u, y(u))$  which satisfy 2.3.3 is called the optimal pair.

### 2.3.2 Existence, Uniqueness and Characterization of the optimal control

Let  $U_{ad} \subset U$  closed and nonempty,  $J$  is lower semicontinuous, bounded from below, coercive on  $U_{ad}$  and strictly convex. Then there exists a unique minimizer for  $J$  on  $U_{ad}$  which characterized by the following optimality system

$$\begin{cases} Ay(u) = f + Bu, \\ A^*p(u) = C^*(Cy(u) - y_d), \\ (B^*p + Nu, v - u)_U \geq 0, \quad \forall v \in U_{ad}. \end{cases}$$

**Proof.** The proof is divided into two steps. In the first step, we prove the existence and the uniqueness of the minimiser. The second step deals with the characterisation of the optimal control.

#### Step1 : Existence and Uniqueness

Since  $J$  is bounded from below, there exists  $m$  positive such that :

$$m = \inf_{v \in U_{ad}} J(v),$$

Let  $(v_n)$  be a minimizing sequence<sup>1</sup> in  $U_{ad}$  since  $J$  is coercive,  $(v_n)$  is converge to  $u \in U_{ad}$ , and  $J$  is lower semicontinuous, then

$$J(u) \leq \lim_{n \rightarrow +\infty} \inf_{v \in U_{ad}} J(v_n) = \inf_{v \in U_{ad}} J(v) = m,$$

then  $u$  is a minimizer of  $J$  on  $U_{ad}$ .

---

1. a minimizing sequence of the criterion  $J$  on the set  $U_{ad}$  is a sequence  $(v_n)$  such that

$$\lim_{n \rightarrow +\infty} J(v_n) = \inf_{v \in U_{ad}} J(v)$$

We suppose that the problem 2.3.3 admits two distinct solutions  $u_1, u_2$ . We set  $u = \frac{u_1+u_2}{2}$ , due to strict convexity of  $J$  we get

$$J(u) < \frac{1}{2}J(u_1) + \frac{1}{2}J(u_2) = m,$$

we obtain a contradiction with the assumption that  $u_1, u_2$  are two solutions of 2.3.3. Thus 2.3.3 admits a unique solution.

**Step2 : Characterization of the optimal control**

The cost function  $J$  is a Gateaux-differentiable function, then the first order optimality condition gives :

$$J'(u)(v-u) = 2(C^*(Cy(u) - y_d), y(v-u))_E + 2N(u, v-u)_U \geq 0, \quad \forall v \in U_{ad}.$$

Let's introduce the adjoint state  $p = p(u)$  given by

$$A^*p(u) = C^*(Cy(u) - y_d),$$

where  $A^*$  and  $C^*$  are the adjoint operators of  $A$  and  $C$  resp, we have that

$$\begin{aligned} (C^*(Cy(u) - y_d), y(v-u))_E &= (A^*p(u), y(v-u))_E = (p(u), Ay(v-u))_E \\ &= (p(u), B(v-u))_E = (B^*p(u), (v-u))_U, \end{aligned}$$

Finally, the optimality condition becomes

$$J'(u)(v-u) = (B^*p(u) + Nu, v-u)_U \geq 0, \quad \forall v \in U_{ad}.$$

■

**Remark 2.3.1** *In the case of no constraint on the control i.e.  $U_{ad} = U$ , we get*

$$J'(u)(v-u) = 0, \quad \forall v \in U.$$

Therefore the optimality system becomes as following

$$\begin{cases} Ay(u) = f + Bu, \\ A^*p(u) = C^*(Cy(u) - y_d), \\ u = \frac{-1}{N}B^*p(u), \quad \forall v \in U. \end{cases}$$

## 2.4 Optimal control of distributed system with incomplete data

Now, we transition to scenarios where some information is unavailable, known as an optimal control with incomplete data. To address this challenge, we introduce two key methods : no-regret control and low-regret control. Initially, we define the no-regret control and then proceed to relax the problem, leading to the formulation of the low-regret control. We delve into the existence, uniqueness and characterization of low-regret control, also we show the convergence of low-regret control to the no-regret control which facilitates characterizing the no-regret control.

### 2.4.1 Statement of the problem

Let  $G$  be a nonempty closed vector subspace of the Hilbert space of uncertainties  $F$ , and we will denote by  $\beta \in \mathcal{L}(F, E)$  the operator of the missing data. For  $f \in E$  the abstract equation related to the control  $v \in U_{ad}$  and the uncertainty  $g \in G$  is given by

$$Ay(v, g) = f + Bv + \beta g. \quad (2.4.1)$$

The equation 2.4.1 is well posed in  $E$  and her solution  $y(v, g)$ , which associate to her the following cost function :

$$J(v, g) = \| Cy(v, g) - y_d \|_Z^2 + N \| v \|_U^2, \quad \forall v \in U_{ad}, \forall g \in G, \quad (2.4.2)$$

we are concerned with the optimal control of 2.4.1 and 2.4.2 is to search  $u$  solution of :

$$\inf_{v \in U_{ad}} J(v, g), \quad \forall g \in G, \quad (2.4.3)$$

we remark that

- If  $G = \{0\}$  then  $J(v, g) = J(v, 0)$ . Therefore, the problem 2.4.3 becomes a classical problem of optimal control

$$\inf_{v \in U_{ad}} J(v, 0).$$

- If  $G \neq \{0\}$ , the problem 2.4.3 has no sense, because  $G$  is an infinite dimensional space. The idea is to look for a solution for the following inf sup problem

$$\inf_{v \in U_{ad}} \left( \sup_{g \in G} J(v, g) \right), \quad (2.4.4)$$

but we can get  $\sup_{g \in G} J(v, g) = +\infty$ , to avoid this difficulty the famous mathematician J.L. Lions presented the notion of no-regret control.

## 2.4.2 The no-regret control notion

**Definition 2.4.1** [34, 18] We say that  $u \in U_{ad}$  is a no-regret control for 2.4.1 and 2.4.2 if  $u$  solves

$$\inf_{v \in U_{ad}} \left( \sup_{g \in G} (J(v, g) - J(0, g)) \right).$$

**Lemma 2.4.1** For every  $u \in U_{ad}$  and  $g \in G$  we have :

$$J(v, g) - J(0, g) = J(v, 0) - J(0, 0) + 2(S(v), g)_G, \quad (2.4.5)$$

where  $S(v) = \beta^* \xi(v)$  and  $\xi(v)$  defined for  $v \in U_{ad}$  by

$$A^* \xi(v) = C^* C(y(v, 0) - y(0, 0)).$$

**Proof.** Due to linearity of the operator  $A$  in  $E$ , so :

$$y(v, g) = y(v, 0) + y(0, g) - y(0, 0),$$

with  $y(v, 0)$  and  $y(0, g)$  are a solution of 2.4.1 when  $g = 0$  and  $v = 0$  resp.

By a simple calculus, we obtain

$$J(v, g) - J(0, g) = J(v, 0) - J(0, 0) + 2(C^*C(y(v, 0) - y(0, 0)), y(0, g) - y(0, 0))_E.$$

Introduce an adjoint state  $\xi(v)$  given by  $A^*\xi(v) = C^*C(y(v, 0) - y(0, 0))$  to write

$$\begin{aligned} J(v, g) - J(0, g) &= J(v, 0) - J(0, 0) + 2(A^*\xi(v), y(0, g) - y(0, 0))_E \\ &= J(v, 0) - J(0, 0) + 2(\xi(v), A(y(0, g) - y(0, 0)))_E \\ &= J(v, 0) - J(0, 0) + 2(\xi(v), \beta g)_E = J(v, 0) - J(0, 0) + 2(\beta^*\xi(v), g)_G \\ &= J(v, 0) - J(0, 0) + 2(S(v), g)_G \text{ where } S(v) = \beta^*\xi(v). \end{aligned}$$

■ We remark that the no-regret control is realized only for  $v \in K$ , where  $K$  is a closed subspace of  $U_{ad}$  given by

$$K = \{v \in U_{ad}, (S(v), g) = 0, \forall g \in G\}.$$

### 2.4.3 The low-regret control

We relax the problem by adding a quadratic perturbation as following

$$J(v, g) \leq J(0, g) + \gamma \|g\|_G^2, \quad \gamma > 0,$$

using Legendre transform (see appendix Definition 4.3.1) [44], we obtain

$$\begin{aligned} \sup_{g \in G} (J(v, g) - J(0, g) - \gamma \|g\|_G^2) &= J(v, 0) - J(0, 0) + \sup_{g \in G} (2(S(v), g)_G - \gamma \|g\|_G^2) \\ &= J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|S(v)\|_G^2, \end{aligned}$$

we are concerned with the following standard optimal control problem

$$\inf_{v \in U_{ad}} J^\gamma(v), \tag{2.4.6}$$

where  $J^\gamma$  is given by

$$J^\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|S(v)\|_G^2. \quad (2.4.7)$$

Now, we can define the low-regret control by

**Definition 2.4.2** [34, 18] *We say that  $u_\gamma \in U_{ad}$  is a low-regret control for 2.4.2 and 2.4.3 if  $u$  solves*

$$\inf_{v \in U_{ad}} \sup_{g \in G} (J(v, g) - J(0, g) - \gamma \|g\|_G^2), \quad \gamma > 0.$$

**Theorem 2.4.1** (*Low-regret control existence and uniqueness*)

*The problem 2.4.1 and 2.4.6 with 2.4.7 has a unique solution  $u_\gamma$  which is converge weakly to the unique no-regret control  $u$  in  $U_{ad}$  when  $\gamma$  tends to 0.*

**Proof.** The proof is divided into two steps :

**Step1 : Existence and uniqueness**

We know that :

$$J^\gamma(v) \geq -J(0, 0) = \text{constant},$$

which implies that  $d_\gamma = \inf_{v \in U_{ad}} J^\gamma(v)$  exists. Let a minimizing sequence  $(v_n^\gamma)$  verifying  $\lim_{n \rightarrow \infty} J^\gamma(v_n^\gamma) = d_\gamma$ , we have that :

$$-J(0, 0) \leq J^\gamma(v_n^\gamma) = J(v_n^\gamma, 0) - J(0, 0) + \frac{1}{\gamma} \|\beta^* \zeta(v_n^\gamma)\|_G^2 \leq d_\gamma + 1,$$

which implies that

$$\|Cy(v_n^\gamma, 0) - y_d\|_Z^2 + N \|v_n^\gamma\|_U^2 + \frac{1}{\gamma} \|\beta^* \zeta(v_n^\gamma)\|_G^2 \leq d_\gamma + J(0, 0) + 1 = C_\gamma.$$

we deduce that

$$\|v_n^\gamma\|_U \leq C_\gamma, \quad (2.4.8)$$

$$\|Cy(v_n^\gamma, 0) - y_d\|_Z \leq C_\gamma, \text{ implies } \|Cy(v_n^\gamma, 0)\|_Z \leq C_\gamma, \quad (2.4.9)$$

$$\|\beta^* \zeta(v_n^\gamma)\|_G \leq C_\gamma \sqrt{\gamma}, \quad (2.4.10)$$

where  $C_\gamma$  is a constant independent of  $n$ . we deduce that  $(v_n^\gamma)$  is bounded in compact space  $U_{ad}$  then we can extracting a subsequence still denoting by  $(v_n^\gamma)$  converges weakly to  $u_\gamma$  in  $U_{ad}$ , due to isomorphism of  $A$  we deduce that  $y(v_n^\gamma, 0)$  converge weakly to  $y(u_\gamma, 0)$  in  $E$ . The cost function  $J^\gamma(v)$  is a lower semi continuous

$$J^\gamma(u_\gamma) \leq \liminf_{n \rightarrow \infty} \inf_{v \in U_{ad}} J^\gamma(v_n^\gamma) = \inf_{v \in U_{ad}} J^\gamma(v) = d_\gamma,$$

which means that

$$J^\gamma(u_\gamma) = \inf_{v \in U_{ad}} J^\gamma(v).$$

Since  $J^\gamma$  is strict convexity, then  $u_\gamma$  is unique.

### Step2. Converges of low-regret control to the no-regret control

Let  $u_\gamma$  be a low-regret control in  $U_{ad}$  then for all  $v \in U_{ad}$  :

$$J^\gamma(u_\gamma) \leq J^\gamma(v),$$

from the definition of  $J^\gamma(v)$ , we find

$$J(u_\gamma, 0) - J(0, 0) + \frac{1}{\gamma} \|\beta^* \zeta(u_\gamma)\|_G^2 \leq J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|\beta^* \zeta(v)\|_G^2, \quad \forall v \in U_{ad},$$

which implies

$$J(u_\gamma, 0) + \frac{1}{\gamma} \|\beta^* \zeta(u_\gamma)\|_G^2 \leq J(v, 0) + \frac{1}{\gamma} \|\beta^* \zeta(v)\|_G^2, \quad \forall v \in U_{ad},$$

we choose  $v = 0$  to find :

$$J(u_\gamma, 0) + \frac{1}{\gamma} \|\beta^* \zeta(u_\gamma)\|_G^2 = \|Cy(u_\gamma, 0) - y_d\|_Z^2 + N \|u_\gamma\|_U^2 + \frac{1}{\gamma} \|\beta^* \zeta(u_\gamma)\|_G^2 \leq J(0, 0) = \text{constant},$$

where  $C$  is a constant independent of  $\gamma$ .

We deduce from 2.4.8 that  $(u_\gamma)$  is bounded in  $U_{ad}$  then we can extract a subsequence still denoting  $(u_\gamma)$  converges weakly to  $u \in U_{ad}$ .

It's clear that for every  $v \in U_{ad}$

$$J(v, g) - J(0, g) - \gamma \|g\|_G^2 \leq J(v, g) - J(0, g), \quad \forall g \in G,$$

i.e,

$$J(v, g) - J(0, g) - \gamma \|g\|_G^2 \leq \sup_{g \in G} (J(v, g) - J(0, g)), \quad \forall g \in G,$$

On another side we have

$$J(u_\gamma, g) - J(0, g) - \gamma \|g\|_G^2 \leq J(v, g) - J(0, g) - \gamma \|g\|_G^2,$$

so

$$J(u_\gamma, g) - J(0, g) - \gamma \|g\|_G^2 \leq \sup_{g \in G} (J(v, g) - J(0, g)), \quad \forall g \in G,$$

when  $\gamma$  tend to 0 we obtain :

$$J(u, g) - J(0, g) \leq \sup_{g \in G} (J(v, g) - J(0, g)), \quad \forall g \in G,$$

which means that

$$\sup_{g \in G} (J(u, g) - J(0, g)) = \inf_{v \in U_{ad}} \left\{ \sup_{g \in G} (J(v, g) - J(0, g)) \right\}.$$

In conclusion,  $u$  is a no-regret control. ■

## Characterization of the low-regret control

The next proposition gives the optimality system characterizing the low-regret control.

**Proposition 2.4.1** *The low-regret control is characterized by the following optimality system :*

$$\left\{ \begin{array}{l} Ay_\gamma = f + Bu_\gamma, \\ A^* \xi_\gamma = C^* C(y_\gamma - y(0, 0)), \\ A\rho_\gamma = \frac{1}{\gamma} \beta \beta^* \xi_\gamma, \\ A^* p_\gamma = C^* (Cy_\gamma - y_d) + C^* C\rho_\gamma, \end{array} \right. \quad (2.4.11)$$

with the next optimality condition

$$(B^*p_\gamma + Nu_\gamma, v - u_\gamma)_U \geq 0, \quad \forall v \in U_{ad},$$

where  $y(u_\gamma, 0) = y_\gamma$ ,  $\xi(u_\gamma) = \xi_\gamma$ .

**Proof.** A first order optimality condition gives

$$J'(u_\gamma)(v - u_\gamma) \geq 0, \quad \forall v \in U_{ad},$$

where  $J'(u_\gamma)(v - u_\gamma)$  given by

$$J'(u_\gamma)(v - u_\gamma) = 2(Cy(u_\gamma, 0) - y_d, Cy(v - u_\gamma, 0))_Z + 2N(u_\gamma, v - u_\gamma)_U + \frac{2}{\gamma}(S(u_\gamma), S(v - u_\gamma))_G.$$

Thanks to linearity, we get

$$\begin{aligned} J'(u_\gamma)(v - u_\gamma) &= 2(C^*(Cy(u_\gamma, 0) - y_d), y(v - u_\gamma, 0) - y(0, 0))_E \\ &\quad + 2N(u_\gamma, v - u_\gamma)_U + \frac{2}{\gamma}(S(u_\gamma), S(v - u_\gamma))_G. \end{aligned}$$

Let the adjoint state defined previously by  $A^*\xi(u_\gamma) = C^*C(y(u_\gamma, 0) - y(0, 0))$ , then

$$(S(u_\gamma), S(v - u_\gamma))_G = (\beta^*\xi(u_\gamma), \beta^*\xi(v - u_\gamma))_G = (\beta\beta^*\xi(u_\gamma), \xi(v - u_\gamma))_E.$$

Also, let  $\rho_\gamma = \rho(u_\gamma)$  solution of

$$A\rho_\gamma = \frac{1}{\gamma}\beta\beta^*\xi(u_\gamma),$$

this leads to the following equality

$$\begin{aligned} (A\rho_\gamma, \xi(v - u_\gamma))_E &= (\rho_\gamma, A^*\xi(v - u_\gamma))_E = (\rho_\gamma, C^*C(y(v - u_\gamma, 0) - y(0, 0)))_E \\ &= (C^*C\rho_\gamma, y(v - u_\gamma, 0) - y(0, 0))_E, \end{aligned}$$

Let the new adjoint state  $p_\gamma = p(u_\gamma)$  defined by

$$A^*p_\gamma = C^*(Cy_\gamma - y_d) + C^*C\rho_\gamma,$$

to find

$$\begin{aligned} (A^*p_\gamma, y(v - u_\gamma, 0) - y(0, 0))_E &= (p_\gamma, A(y(v - u_\gamma, 0) - y(0, 0)))_E \\ &= (p_\gamma, B(v - u_\gamma))_E \\ &= (B^*p_\gamma, v - u_\gamma)_U, \end{aligned}$$

Finally, we conclude that

$$J^{\gamma'}(u_\gamma)(v - u_\gamma) = (B^*p_\gamma + Nu_\gamma, v - u_\gamma)_U \geq 0, \quad \forall v \in U_{ad}.$$

■

### Characterization of the no-regret control

**Theorem 2.4.2** *The no-regret control is characterized by the following optimality system :*

$$\left\{ \begin{array}{l} Ay = f + Bu, \\ A^*\zeta = C^*Cy(u, 0) - y_d, \\ A\rho = \kappa, \kappa \in \widehat{G}, \\ A^*p = C^*(Cy(u, 0) - y_d) + C^*C\rho, \\ (B^*p + Nu, v - u)_U \geq 0, \quad \forall v \in U_{ad}, \end{array} \right. \quad (2.11)$$

where  $y(u, 0) = y$ ,  $\xi(u) = \xi$  and  $\widehat{G}$  is the completion of  $G$  in  $F$ .

**Proof.** Let's solve the following state

$$A\rho = \beta g, \quad g \in G, \quad \rho \in E,$$

then

$$A^*\sigma = C^*C\rho, \quad \sigma \in E,$$

We define an operator  $\mathcal{R}$  as follows  $\mathcal{R}g = B^*\sigma$ . We suppose that

$$\|\mathcal{R}g\|_{\widehat{G}} \geq c\|g\|_G, \quad c > 0, \text{ for any } g \in G. \quad (2.4.12)$$

where  $\widehat{G}$  is the completion of  $G$  in  $F$ , containing the elements  $\mathcal{R}g$ . The sequence  $\{u_\gamma\}$  is weakly converges in  $U$  to  $u$  the unique no-regret control. The operator  $B$  being continuous from  $U$  to  $E$ ,  $\{Bu_\gamma\}$  weakly converges in  $E$  to  $Bu$ . Now, from the above optimality system of Proposition 2.4.1 the sequence  $\{Ay_\gamma\}$  is bounded in  $E$  and, as  $A$  is an isomorphism, weakly converges to  $Ay$  in  $E$ . Passing to the limit in the first equation we obtain  $Ay = f + Bu$ . We also deduce from Proposition 2.4.1 that  $B^*p_\gamma = -Nu_\gamma$  is bounded in  $E$ . According to the hypothesis 2.4.12, let  $\mathcal{R}$  be the operator such that  $\mathcal{R}\left(\frac{1}{\gamma}\beta^*\xi_\gamma\right) = B^*p_\gamma$ . We deduce under 2.4.12 that  $\left\{\frac{1}{\gamma}\beta^*\xi_\gamma\right\}$  is bounded in  $G$  subset of the Hilbert space  $F$ . Then it converges to  $\kappa \in \widehat{G} \subset F$ . Hence,  $A\rho_\gamma = \frac{1}{\gamma}\beta^*\xi_\gamma$  is bounded, and then  $\{\rho_\gamma\}$  also bounded thanks to the isomorphism of  $A$  weakly converges to  $\rho \in E$ . Consequently,  $A\rho_\gamma \rightharpoonup A\rho$ . From the boundness of  $\{\rho_\gamma\}$  and  $\{y_\gamma\}$  we obtain that  $A^*p_\gamma$  is bounded. Then  $\{p_\gamma\}$  converges to  $p$ . The optimality system follows. ■

## 2.5 Sentinel Method

Throughout this section, we will discuss the theoretical foundations of the Sentinel method. The effectiveness of the Sentinel method in our context lies in identifying the unknown infected individuals to implement targeted measures to isolate them and prevent further transmission also, to minimize the infection number of people.

### 2.5.1 An overview about the sentinel method

We begin by introducing the basic principles of the Sentinel method.

## Position of the problem

Let  $Q = \Omega \times (0, T)$  and  $\Sigma = \Gamma \times (0, T)$ , we consider the following problem

$$\begin{cases} \frac{\partial y}{\partial t} + Ay = \xi + \lambda \hat{\xi} & \text{in } \Omega \times (0, T), \\ y(x, 0) = y_0 + \tau \hat{y}_0 & \text{in } \Omega, \\ y(x, t) = 0 & \text{on } \Sigma. \end{cases} \quad (2.5.1)$$

Where

- $A$  is elliptic differential operator of the 2nd order .
- $\xi$  is given in a suitable space  $\mathcal{L}^2(Q)$ ,  $\lambda \hat{\xi} \in \mathcal{L}^2(Q)$  is unknown (is called the pollution term) and with  $\lambda \in \mathcal{R}$  small.
- $y_0$  is given,  $\hat{y}_0$  remains in the unit ball of  $\mathcal{L}^2(\Omega)$  and with  $\tau$  small. We supposed that the boundary condition is given.
- The problem 2.5.1 admit a weak solution in  $\mathcal{L}^2(Q)$  denoted by

$$y(\lambda, \tau) = y(x, t; \lambda, \tau) \quad \text{on } O \times (0, T), \quad (2.5.2)$$

where  $O \subset \Omega$ .

The sentinel method is used to obtain some information on the important term  $\lambda \hat{\xi}$  without caring about the unimportant term  $\tau \hat{y}_0$ .

**-State observation** Let  $O \subset \Omega$ , is arbitrarily small is called the observatory, All observations are made within the time interval  $(0, T)$ . We observed the state  $E$  on  $O$ ,

there are two types of observations :

1. Noise free observation : We define

$$y(x, t; \lambda, \tau) = y_{obs} \quad \text{on } O \times (0, T), \quad (2.5.3)$$

where  $y_{obs}$  is a given measurement. We distinguish the following cases :

- Distributed observation :  $O \subset \Omega$ ,
- Boundary observation :  $O \subset \Gamma$ ,

— Observation dependent on time :  $O = O(t)$ ,

2. Noisy observation : The measurements can be affected by some perturbation is known as noise.

The noise arises from the errors in the measurement instruments or inaccuracies in the equation approximations, this situation is complicated to account it, the observation operator is defined as follows

$$y_{obs} = m_0 + \sum_{i=1}^n \beta_i m_i, \quad (2.5.4)$$

where the functions  $(m_0, m_1, \dots, m_n)$  are given, and  $\beta_i$  are the noise parameters which not known.

**-The sentinel functional** Let  $h_0$  be a function given on  $O \times (0, T)$ , we consider the following function

$$S(\lambda, \tau) = \int_0^T \int_O (h_0 + u) y(\lambda, \tau) dx dt, \quad (2.5.5)$$

where  $u = u(x, t)$  is a function to be determined such that :

$$\left. \frac{\partial S}{\partial \tau}(\lambda, \tau) \right|_{\lambda=0, \tau=0} = 0, \quad (2.5.6)$$

and

$$\|u\|_{L^2(O \times (0, T))} = \text{minimum}. \quad (2.5.7)$$

The functional defined by (2.5.5) - (2.5.7) is called the sentinel.

**Remark 2.5.1** -Condition (2.5.6), expresses the insensitivity of the functional  $S$  with respect to  $\tau$  (to the first order), and condition (2.5.7) expresses that we move away from the mean "as little as possible".

-The choice  $u = -h_0$  gives rise to (2.5.6). Therefore, under very general assumptions, the problem (2.5.6) - (2.5.7) admits a unique solution. But it will be necessary to ensure that under suitable conditions  $u \neq -h_0$ , the functional  $S(\lambda, \tau)$  not being likely to bring us much information.

### Equivalence to controllability problem

The insensitivity condition (2.5.6) gives

$$\int_0^T \int_O (h_0 + u) y_\tau dx dt = 0,$$

where  $y_\tau = \frac{\partial y}{\partial \tau}$  solution of

$$\left\{ \begin{array}{ll} \frac{\partial y_\tau}{\partial t} + Ay_\tau + f'(y^0)y_\tau = 0 & \text{in } \Omega \times (0, T), \\ y_\tau(x, 0) = \hat{y}_0 & \text{in } \Omega, \\ y_\tau = 0 & \text{on } \Sigma, \end{array} \right.$$

where  $y^0$  is solution of 2.5.1 when  $\lambda = \tau = 0$ .

We define the adjoint state  $p = p(x, t)$  solution of

$$\left\{ \begin{array}{ll} -\frac{\partial p}{\partial t} + A^*p + f'(y^0)p = (h_0 + u)\chi_O & \text{in } \Omega \times (0, T), \\ p(x, T) = 0 & \text{in } \Omega, \\ p = 0 & \text{on } \Sigma. \end{array} \right. \quad (2.5.8)$$

**Proposition 2.5.1** *The existence of the sentinel (2.5.5) is equivalent to solving the optimal control problem (2.5.8) with (2.5.7) which satisfies*

$$p(x, 0) = 0. \quad (2.5.9)$$

**Proof.** [14, 18, 17] It's easy to see, by multiplying the first equation of the system (2.5.8) by  $y_\tau$  and then integrating by parts with some calculations ■

## Resolution of the controllability problem

We proved that the existence of a sentinel is equivalent to solving a controllability problem. To tackle this, we employ several key tools :

**Careleman estimates :** These are powerful inequalities used to establish the controllability problems that involve establishing bounds on the observability of system states and designing control strategies that leverage these insights.

**Penalisation method :** This technique transforms the optimization problem with constraint to an optimization problem without constraint.

**HUM method :** This method is used to establish the exact controllability of PDEs by transforming the controllability problem into an equivalent optimization problem.

### Information given by the sentinel

If we assume that the state  $y(\lambda, \tau)$  depends differently on  $\lambda$  and  $\tau$ , we can write formally

$$S(\lambda, \tau) \simeq S(0, 0) + \lambda \frac{\partial S}{\partial \lambda}(0, 0) + \tau \frac{\partial S}{\partial \tau}(0, 0),$$

(since by definition of the sentinel  $\frac{\partial S}{\partial \tau}(0, 0) = 0$ ). Using 2.5.3, we can therefore write

$$\lambda \frac{\partial S}{\partial \lambda}(0, 0) \simeq \int_0^T \int_O (h_0 + u) (y_{obs} - y^0) dx dt.$$

### 2.5.2 Identification of the missing infected people using sentinel method

This subsection is devoted to estimating the number of missing infected individuals within a population. Identifying these individuals, who may not have been detected through regular testing or reporting, is crucial for effective disease control and prevention.

#### Formulation of problem

Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^3$  with smooth boundary  $\Gamma$ , we denoted by  $Q = \Omega \times ]0, T[$  the space-time cylinder. We are interested in modelling the problem of a diseases. We start by divide our population of the world  $N(t)$  into three categories  $S(t)$ ,  $I(t)$  and  $R(t)$  where

$S(t)$  : represent the number of susceptible people,

$I(t)$  : represent the number of infected people,

$R(t)$  : represent the number of recovered people. i.e ;

$$N(t) = S(t) + I(t) + R(t).$$

Our problem is given by the following nonlinear system of differential equation [23] :

$$\begin{cases} S' = (\alpha - \mu S - bcSI); & S(0) \geq 0, \\ I' = I(bcS - \mu - k - \alpha - \rho); & I(0) \geq 0, \\ R' = kI - \mu R; & R(0) \geq 0, \end{cases} \quad (2.5.10)$$

where  $\alpha$  is the rate of newborn and migrated individuals.

$b$  is the transmission rate from susceptible to infected.

$c$  is the contact rate of susceptible with infected.

$\mu$  is the natural death rate.

$k$  is the recovery rate.

$\rho$  is the death rate of the infected class due to virus.

We must note that some information related to this disease is missing for example the total number of infected people  $I(t)$  is not all known the results of the first infected people were not mentioned by health organizations, also the causes of this virus which make us in problem of incomplete data.

The motivation of the problem is to take

$$\begin{aligned} I(t) &= \xi + \lambda \tilde{\xi}, \\ I(0) &= I_0 + \tau \tilde{I}_0. \end{aligned}$$

Where  $\xi$  represent the known infected people,  $I_0$  represent the first infected people announced by The Ministry of Health in "wuhan",  $\tilde{\xi}$  represent the missing infected people and  $\tilde{I}_0$  represent the number of first infected people unannounced.

We aim to minimize as much as possible the infected people  $I$  without caring about the missing information, for this reason the sentinel method is the most suitable to solve this kind of problem.

We start by linear the system 2.5.10

**Theorem 2.5.1** *System 2.5.10 became to linear system presented by*

$$\begin{cases} y' = Ay + \lambda \hat{\xi}^*, \\ y(0) = y_0, \end{cases} \quad (2.5.11)$$

$$\text{where } A = \begin{pmatrix} -\mu & -bc_{\mu}^{\alpha} & 0 \\ 0 & bc_{\mu}^{\alpha} - \mu - k & 0 \\ 0 & k & -\mu \end{pmatrix}, \quad v = \begin{pmatrix} -bc_{\mu}^{\alpha} \\ (bc_{\mu}^{\alpha} - \mu - k) \\ k \end{pmatrix}.$$

According to the data, we know that system 2.5.11 admits a unique solution

$$y = \begin{pmatrix} S^* \\ \xi^* \\ R^* \end{pmatrix} \in L^2(Q), \quad y(0) = \begin{pmatrix} S^*(0) \\ I_0 + \tau \tilde{I}_0 \\ R^*(0) \end{pmatrix} \in L^2(\Omega).$$

**Remark 2.5.2** The problem 2.5.11 admit the following equilibrium point are  $P_1 = \left(\frac{\alpha}{\mu}, 0, 0\right)$ .

**Proof.** Supposed that

$$\begin{cases} S^* = S - \frac{\alpha}{\mu}, \\ I^* = I, \\ R^* = R, \end{cases}$$

we get

$$\begin{cases} S^{*'} = \alpha - \mu \left(S^* + \frac{\alpha}{\mu}\right) - bc \left(S^* + \frac{\alpha}{\mu}\right) I^* \text{ in } Q ; S^*(0) \geq -\frac{\alpha}{\mu}, \\ I^{*'} = I^* \left(bc \left(S^* + \frac{\alpha}{\mu}\right) - \mu - k - \alpha - \rho\right) \text{ in } Q ; I^*(0) \geq 0, \\ R^{*'} = kI^* - \mu R^* \text{ in } Q ; R^*(0) \geq 0, \end{cases}$$

the Jacobian matrix is

$$J(0, 0, 0) = \begin{pmatrix} -\mu & -bc\frac{\alpha}{\mu} & 0 \\ 0 & bc\frac{\alpha}{\mu} - \mu - k & 0 \\ 0 & k & -\mu \end{pmatrix},$$

then

$$\begin{cases} S^{*'} = -\mu S^* - bc\frac{\alpha}{\mu} I^* \text{ in } Q ; S^*(0) \geq -\frac{\alpha}{\mu} \text{ in } \Omega, \\ I^{*'} = I^* \left(bc\frac{\alpha}{\mu} - \mu - k\right) \text{ in } Q ; I^*(0) \geq 0 \text{ in } \Omega, \\ R^{*'} = kI^* - \mu R^* \text{ in } Q ; R^*(0) \geq 0 \text{ in } \Omega, \end{cases}$$

where

$$\begin{aligned} I^* &= \xi^* + \lambda \hat{\xi}^*, \\ I^*(0) &= I_0^* + \tau \tilde{I}_0^*. \end{aligned}$$

Consequently

$$\left\{ \begin{pmatrix} S^{*'} \\ \xi^{*'} \\ R^{*'} \end{pmatrix} \right. = \begin{pmatrix} -\mu & -bc_{\mu}^{\alpha} & 0 \\ 0 & bc_{\mu}^{\alpha} - \mu - k & 0 \\ 0 & k & -\mu \end{pmatrix} \begin{pmatrix} S^* \\ \xi^* \\ R^* \end{pmatrix} + \lambda \begin{pmatrix} -bc_{\mu}^{\alpha} \\ (bc_{\mu}^{\alpha} - \mu - k) \\ k \end{pmatrix} \hat{\xi}^*.$$

■

### Application the Sentinel method

We devoted to applied the Sentinel method to estimate the missing infected people introduced the first time by the famous mathematician J.L. Lions (1992) and developed by many authors in this papers ([7, 9] [19][43] [37, 38] [32] ). We observed that the existence of Sentinel is equivalent to problem of null controllability. This last one developed by Nakoulima, O. (2004) [35], Y. J. Lin Guo, Littman. W [27]. In 2023 ( [40]) Imad, R., & Fatma, A. used the sentinel method to identify the pollution term in parabolic evolution equation.

**Definition 2.5.1** [32] *We define the functional  $S$  as following*

$$S(\lambda, \tau) = \int_Q (h_0 \chi_O + u \chi_{\omega}) y(\lambda, \tau) dxdt,$$

where  $O \subset \Omega$ ,  $h_0$  is a function on  $O \times [0, T]$ . The observed state given by

$$y_{obs} = m_0 \in L^2(O \times [0; T]),$$

$\omega$  is an open empty subset of  $\Omega$  is the control space,  $w \in L^2(\omega \times [0, T])$ .

**Definition 2.5.2** *We say that the functional  $S(\lambda, \tau)$  is define a sentinel if the following conditions are satisfied*

$$\left| \frac{dS}{d\tau}(0, 0) \right| \leq \epsilon, \forall y_0. \quad (2.5.12)$$

$$\|u\|_{L^2(\omega \times (0, T))} = \min_{v \in \omega} \|v\|_{L^2(\omega \times (0, T))}. \quad (2.5.13)$$

**Lemma 2.5.1** *Let  $m \in \mathcal{M}$ . Then there is no  $\varphi \in L^2(Q)$ ,  $\varphi \neq 0$  such that  $\varphi$  satisfies*

$$\begin{cases} \varphi' - A^t\varphi = 0, & \text{in } Q, \\ \varphi(0) = m\chi_\omega. \end{cases} \quad (2.5.14)$$

**Proof.** If the problem 2.5.14 admits a solution, then it is given by

$$\varphi(x, t) = \sum_{j \geq 1} \alpha_j(t) u_j(x), \quad (2.5.15)$$

where  $u_j$  are eigenfunctions of

$$\begin{cases} A^t u = \nu u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (2.5.16)$$

Differentiate the solution 2.5.15 once with respect to  $t$  and twice with respect to  $x$  and substitute these derivatives into the first equation of (2.5.14). We then obtain

$$\sum_{j \geq 1} (\alpha_j'(t) - \nu_j \alpha_j(t)) u_j(x) = 0.$$

Thus,

$$\alpha_j'(t) - \nu_j \alpha_j(t) = 0,$$

because  $(u_j)$  form an orthonormal base of  $L^2(\Omega)$ . Furthermore, the function  $\varphi$  satisfies the boundary conditions if and only if

$$\sum_{j=1}^{\infty} \alpha_j(0) u_j(x) = m\chi_\omega.$$

As  $m\chi_\omega \in L^2(\Omega)$  then

$$m\chi_\omega = \sum_{j=1}^{\infty} \langle m\chi_\omega, u_j \rangle_{L^2(\Omega)} u_j(x).$$

Consequently

$$\alpha_j(0) = \langle m\chi_\omega, u_j \rangle_{L^2(\Omega)}.$$

Finally, we have

$$\begin{cases} \alpha_j'(t) - \nu_j \alpha_j(t) = 0, & \text{in } (0, T), \\ \alpha_j(0) = \langle m\chi_\omega, u_j \rangle_{L^2(\Omega)}. \end{cases}$$

Then the solution of the first order linear is given by

$$\alpha_j(t) = \langle m\chi_\omega, u_j \rangle_{L^2(\Omega)} e^{\nu_j t}.$$

Consequently, if the problem 2.5.14 admits a solution, it is necessarily in the form

$$\varphi(x, t) = \sum_{j=1}^{\infty} \langle m\chi_\omega, u_j \rangle_{L^2(\Omega)} e^{\nu_j t} u_j(x).$$

We prove now that  $\varphi \notin L^2(\Omega)$ . Indeed,

$$\begin{aligned} \int_0^t |\alpha_j(t)|^2 dt &= \left| \langle m\chi_\omega, u_j \rangle_{L^2(\Omega)} \right|^2 \int_0^t e^{2\nu_j t} dt \\ &= \left| \langle m\chi_\omega, u_j \rangle_{L^2(\Omega)} \right|^2 \left[ \frac{-1}{2\nu_j} + \frac{1}{2\nu_j} e^{2\nu_j t} \right] \end{aligned}$$

But,  $\nu_j$  is the eigenvalue of problem (2.5.16), then  $\nu_j \xrightarrow{j \rightarrow \infty} \infty$ . Consequently,

$$\int_0^T |\alpha_j(t)|^2 dt \xrightarrow{j \rightarrow \infty} \infty. \quad (2.5.17)$$

Which means that the series whose general term  $\alpha_j(t)$  is not normally convergent. So, problem 2.5.14 admits no solution. ■

**Theorem 2.5.2** *For  $\epsilon > 0, h_0 \in L^2(Q)$ , there exist some control  $u$  and some state  $p$  such that*

$$\begin{cases} \left(-\frac{d}{dt} - A^T\right) p = h_0 \chi_O + u \chi_\omega, & \text{in } Q, \\ p(T) = 0, \end{cases} \quad (2.5.18)$$

and

$$\left\{ \begin{array}{l} \mathcal{M} \text{ a real closed vector subspace of } L^2(\omega), \\ \mathcal{M}^\perp \text{ the orthogonal subspace of } \mathcal{M} \text{ in } L^2(\omega), \\ \text{we look for a control variable } u \in L^2(Q) \text{ such that } u \in \mathcal{M}^\perp, \end{array} \right. \quad (2.5.19)$$

and

$$\|p(0)\| \leq \epsilon \text{ in } \Omega, \quad (2.5.20)$$

hold. Moreover, there exists a unique pair  $(\widehat{u}_\epsilon, \widehat{p}_\epsilon)$  with  $\widehat{u}_\epsilon$  of minimal norm in  $L^2(Q)$ , i.e. such that 2.5.18, 2.5.19, 2.5.20 and 2.5.13 hold.

**Proof.** Let  $p$  be a solution of the system 2.5.18 and  $p_1$  a solution of the following system :

$$\left\{ \begin{array}{l} \left(-\frac{d}{dt} - A^t\right) p_1 = h_0 \chi_O, \text{ in } Q, \\ p_1(T) = 0, \end{array} \right.$$

We put  $p = z + p_1$ . Then,  $z$  is the solution of the following problem :

$$\left\{ \begin{array}{l} \left(-\frac{d}{dt} - A^t\right) z = u \chi_\omega, \text{ in } Q, \\ z(T) = 0. \end{array} \right. \quad (2.5.21)$$

We now introduce the set of states reachable at time 0 defined by :

$$R(0) = \{z(u, 0), u \in \mathcal{M}^\perp\}. \quad (2.5.22)$$

We give now the proof of Theorem 2.5.2.

It is clear that  $R(0)$  is a vector subspace of  $L^2(\omega)$ . According to the Hahn-Banach theorem, it will be dense in  $L^2(\omega)$  if and only if its orthogonal in  $L^2(\omega)$  is reduced to zero. As  $\{0\} \subset R^\perp(0)$ , it remains to show that  $R^\perp(0) \subset \{0\}$ . Let  $\varphi^0 \in R^\perp(0)$ , then

$$\langle \varphi^0, z(0) \rangle_{L^2(\omega)} = \int_\omega \varphi^0 z(0) dx = 0,$$

where  $z$  is solution of 2.5.21. It is therefore natural to define the adjoint  $\varphi$  of  $z$ , this is the solution of

the following problem :

$$\begin{cases} \left(\frac{d}{dt} - A^t\right) \varphi = 0, & \text{in } Q, \\ \varphi(0) = \varphi^0. \end{cases} \quad (2.5.23)$$

Now multiply the first equation of system 2.5.23 by  $z$ . After integration by parts on  $Q$ , it comes

$$\begin{aligned} 0 &= \int_Q \left(\frac{d}{dt} \varphi - A^t \varphi\right) z dx dt \\ &= \int_{\Omega} \varphi(T) z(T) dx - \int_{\Omega} \varphi(0) z(0) dx - \int_Q u \chi_{\omega} \varphi dx dt. \end{aligned} \quad (2.5.24)$$

Since  $z$  and  $\varphi$  are solutions of 2.5.21 and 2.5.23 resp, 2.5.24 becomes

$$\int_{\Omega} \varphi^0 z(0) dx + \int_Q u \chi_{\omega} \varphi dx dt = 0.$$

This is equivalent to

$$\int_Q u \chi_{\omega} \varphi dx dt = 0, \quad \forall u \in \mathcal{M}^{\perp},$$

because,  $\varphi^0 \in R^{\perp}(0)$  and  $z(0) \in R(0)$ . Finally, we have

$$\varphi \chi_{\omega} \in \mathcal{M}. \quad (2.5.25)$$

Therefore,  $\varphi$  satisfies 2.5.23 and 2.5.25 and by applying Lemma 2.5.1, we deduce that

$$\varphi = 0 \quad \text{in } Q.$$

As a consequence,  $\varphi^0 = 0$  which shows that  $R^{\perp}(0) = \{0\}$ . ■

## Resolution of approximate controllability problem

**Theorem 2.5.3** [32] *Supposed that*

$$a(w, v) = \int_Q LwLvdxdt + \int_0^T \int_\omega wvdxdt,$$

$$l(v) = \int_Q h_0\chi_Ovdxdt,$$

due to Lax-Milgram theorem

$$a(w, v) = l(v), \quad \forall v \in L^2(Q), \quad (2.5.26)$$

**Theorem 2.5.4** *We pose  $h_0 \in L^2(Q)$  and  $w$  is a unique solution of problem 2.5.26 let*

$$u = -w\chi_\omega,$$

$$p = Lw,$$

then the pair  $(u, p)$  is solution of 2.5.18 and 2.5.20.

**Proof.** We have that  $w$  is solution of 2.5.26, then

$$\int_Q LwLvdxdt + \int_0^T \int_\omega wvdxdt = \int_Q h_0\chi_Ovdxdt, \quad \forall v \in L^2(Q).$$

Replace  $w$  and  $Lw$  by it's value, we obtain

$$\int_Q pLvdxdt = \int_Q (h_0\chi_O + u\chi_\omega)vdxdt, \quad \forall v \in L^2(Q), \quad (2.5.27)$$

we take  $v \in D(Q)$

$$\langle p, Lv \rangle_{L^2(Q)} = \langle h_0\chi_O + u\chi_\omega, v \rangle_{L^2(Q)}, \quad \forall v \in D(Q),$$

implies that

$$\langle L^*p, v \rangle_{L^2(Q)} = \langle h_0\chi_O + u\chi_\omega, v \rangle_{L^2(Q)}, \quad \forall v \in D(Q),$$

we deduce,

$$L^*p = h_0\chi_O + u\chi_\omega, \quad (2.5.28)$$

multiply 2.5.28 by  $\Phi \in C^\infty(Q)$  and integrate by parts, we obtain

$$\int_Q L^*p\Phi dxdt = \int_Q (h_0\chi_O + u\chi_\omega) \Phi dxdt,$$

implies that

$$\int_Q (h_0\chi_O + u\chi_\omega) \Phi dxdt = \int_Q pL\Phi dxdt - \int_\Omega p(0) \Phi(0) dx + \int_\Omega p(T) \Phi(T) dx,$$

replace in 2.5.18  $v$  by  $\Phi$ , we get

$$\int_Q pL\Phi dxdt = \int_Q (h_0\chi_O + u\chi_\omega) \Phi dxdt,$$

Finally for  $\epsilon > 0$ , and to obtain the approximate controllability, we must put

$$\left| - \int_\Omega p(0) \Phi^0 dx + \int_\Omega p(T) \Phi(T) dx \right| \leq \epsilon, \forall \Phi^0 (\Phi^0 \in L^2(\Omega)).$$

Consequently,

$$\|p(0)\| \leq \epsilon.$$

■

## Identification of the unknown infected people

Using Taylor develop

$$\begin{aligned} S(\lambda, \tau) &\simeq S(0, 0) + \lambda \frac{\partial S}{\partial \lambda}(0, 0) + \tau \frac{\partial S}{\partial \tau}(0, 0), \\ &\simeq S(0, 0) + \lambda \frac{\partial S}{\partial \lambda}(0, 0), \end{aligned}$$

we have :

$$y_{obs} = m_0.$$

$S(\lambda, \tau)$  is observed

$$S(\lambda, \tau) = \int_Q (h_O \chi_O + u \chi_\omega) m_0 dx dt \simeq \int_Q (h_O \chi_O + u \chi_\omega) y_0 dx dt + \lambda \frac{\partial S}{\partial \lambda}(0, 0).$$

$$\lambda \frac{\partial S}{\partial \lambda}(0, 0) \simeq \int_Q (h_O \chi_O + u \chi_\omega) (m_0 - y_0) dx dt$$

in other side

$$\frac{\partial S}{\partial \lambda}(0, 0) = \int_Q (h_O \chi_O + u \chi_\omega) y_\lambda dx dt, \quad y_\lambda = \frac{\partial y}{\partial \lambda}(0, 0),$$

where  $y_\lambda$  is solution of the following system

$$\begin{cases} \frac{d}{dt} y_\lambda - A y_\lambda = \hat{\xi}^*, \\ y_\lambda(0) = 0. \end{cases}$$

Multiply the first equation in the last problem by  $p$  and integrate over  $Q$  we get

$$\int_Q \left( -\frac{d}{dt} - A^t \right) y_\lambda p dx dt = \int_Q \hat{\xi}^* p dx dt$$

$$\int_Q (h_O \chi_O + u \chi_\omega) y_\lambda dx dt - \int_\Omega y_\lambda(0) p(0) dx = \int_Q \hat{\xi}^* p dx dt$$

Hence

$$\int_Q \hat{\xi}^* p dx dt = \int_Q (h_O \chi_O + w \chi_\omega) (m_0 - y_0) dx dt.$$

To get information about the missing  $\hat{\xi}^*$ , it is enough to find the optimal control  $u$ .

## Chapitre 3

# Solvability and controllability of fractional differential problems

In this essential chapter, we present two of our main research works, both of them deals with the field of control theory and its application in complex systems, these works have been recognized by their publication in Scopus-indexed journals. The first work [2] was already published in a Scopus journal, delves into the existence, uniqueness, and controllability of a fractional singular degenerate problem. This study clarified the obstacles to realised the weak controllability of fractional differential equations. The second work [1] is published in a in another Scopus journal, concentrate on identifying the unknown source term in a coupled inverse problem using an optimal control framework.

## 3.2 Solvability and weak controllability of fractional parabolique problem

In this section, we study the solvability and the weak controllability of a singular degenerate fractional parabolic controller problem, when the control fact in the second member of the equation. Our studies devoted into two part, the first part deals with studying the existence and the uniqueness of the strong solution to our problem. In the second part, we try to achieve the weak controllability of the problem.

### 3.2.1 Fractional differentiation and integration

It is important to present some concepts of fractional differentiation and integration, as these are fundamental to our studies.

**Definition 3.2.1** [5, 10, 2] For  $\alpha \in \mathbb{R}_+$  we define the fractional integral of a function  $f \in L^1([0, T], E)$  of order  $\alpha$  as follows

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad (3.2.1)$$

where  $\Gamma$  is the Gamma function.

**Definition 3.2.2** [5, 2] Let  $0 < \alpha < 1$  and  $f \in L^1([0, T], E)$  we define :

1. The left Caputo derivatives :

$${}^C D_t^\alpha f = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds,$$

2. The right Caputo derivatives :

$${}^C D_t^\alpha f = \frac{-1}{\Gamma(1-\alpha)} \int_t^T (s-t)^{-\alpha} f'(s) ds. \quad (3.2.2)$$

3. The left Riemann-Liouville derivatives :

$${}^R D_t^\alpha f = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} f(s) ds. \quad (3.2.3)$$

4. The right Riemann-Liouville derivatives :

$${}^R D_t^\alpha f = \frac{-1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_t^T (s-t)^{-\alpha} f(s) ds. \quad (3.2.4)$$

The right Caputo and Riemann-Liouville derivatives are connected by the following relationship

$${}^R D_t^\alpha f = {}^C D_t^\alpha f + \frac{f(0)}{\Gamma(1-\alpha) t^\alpha}, \quad (3.2.5)$$

if  $f(0) = 0$  then Riemann-Liouville derivative and Caputo derivative are coincides

$${}^R D_t^\alpha f = {}^C D_t^\alpha f. \quad (3.2.6)$$

**Definition 3.2.3** [5, 2] For any  $\sigma > 0$  we define the semi-norms

$$|f|_{{}^L H^\sigma(\Omega)}^2 : = \|{}^R D_t^\sigma f\|_{L^2(\Omega)}^2, \quad (3.2.7)$$

$$|f|_{{}^r H^\sigma(\Omega)}^2 : = \|{}^R D_t^\sigma f\|_{L^2(\Omega)}^2, \quad (3.2.8)$$

$$|f|_{{}^c H^\sigma(\Omega)}^2 : = \left| \frac{({}^R D_t^\sigma f, {}^R D_t^\sigma f)_{L^2(\Omega)}}{\cos(\sigma\pi)} \right|^{\frac{1}{2}}, \quad (3.2.9)$$

and the norms

$$\|f\|_{{}^L H^\sigma(\Omega)}^2 = \left( \|f\|_{L^2(\Omega)}^2 + |f|_{{}^L H^\sigma(\Omega)}^2 \right)^{\frac{1}{2}}, \quad (3.2.10)$$

$$\|f\|_{{}^r H^\sigma(\Omega)}^2 : = \left( \|f\|_{L^2(\Omega)}^2 + |f|_{{}^r H^\sigma(\Omega)}^2 \right)^{\frac{1}{2}}, \quad (3.2.11)$$

$$\|f\|_{{}^c H^\sigma(\Omega)}^2 : = \left( \|f\|_{L^2(\Omega)}^2 + |f|_{{}^c H^\sigma(\Omega)}^2 \right)^{\frac{1}{2}}, \quad (3.2.12)$$

where the spaces  ${}^L H_0^\sigma(\Omega)$  and  ${}^r H_0^\sigma(\Omega)$  are the closure space of  $C_0^\infty(\Omega)$  with respect to the norms

$\|\cdot\|_{L^2 H^\sigma(\Omega)}^2$  and  $\|\cdot\|_{r H^\sigma(\Omega)}^2$  resp.

**Lemma 3.2.1** [5, 2] For any real  $\sigma \in \mathbb{R}_+$ , if  $f \in {}^L H^\sigma(\Omega)$  and  $g \in C_0^\infty(\Omega)$  then

$$({}^R D_t^\sigma f, g)_{L^2(\Omega)} = (f, {}^R D_t^\sigma g)_{L^2(\Omega)}.$$

**Lemma 3.2.2** [5, 2] For  $0 < \sigma < 2$ ,  $\sigma \neq 1$   $f \in H_0^{\frac{\sigma}{2}}(\Omega)$ , we have

$${}^R D_t^\sigma f = {}^R D_t^{\frac{\sigma}{2}} {}^R D_t^{\frac{\sigma}{2}} f. \quad (3.2.13)$$

**Lemma 3.2.3** [5, 2] For  $\sigma \in \mathbb{R}_+$ ,  $\sigma \neq n + \frac{1}{2}$ , the semi-norms  $|\cdot|_{L^2 H^\sigma(\Omega)}$ ,  $|\cdot|_{r H^\sigma(\Omega)}$  and  $|\cdot|_{c H^\sigma(\Omega)}$  are equivalent. Then we pose

$$|\cdot|_{L^2 H^\sigma(\Omega)} \cong |\cdot|_{r H^\sigma(\Omega)} \cong |\cdot|_{c H^\sigma(\Omega)}.$$

**Lemma 3.2.4** For  $\sigma > 0$ , the space  ${}^r H^\sigma(\Omega)$  with respect to the norm  $\|\cdot\|_{r H^\sigma(\Omega)}$  is complete.

**Theorem 3.2.1** [?] The mild solution  $z = z(x, t, v) \in C([0, T], E)$  of the problem 3.2.15 is given by

$$z(t) = S_\alpha(t)z_0 + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s)(f(s) - v(s)) ds, t \in [0, T], \text{ for every } v \in L^2([0, T], U)$$

Let  $\Phi_\alpha$  be a Mainardi function defined as

$$\Phi_\alpha(z) = \sum_{n=0}^{+\infty} \frac{(-z)^n}{n! \Gamma(-\alpha n + 1 - \alpha)},$$

We set

$$\begin{aligned} S_\alpha(t) &= \int_0^\infty \Phi_\alpha(\theta) R(\theta t^\alpha) d\theta, \\ S_\alpha^*(t) &= \int_0^\infty \Phi_\alpha(\theta) R^*(\theta t^\alpha) d\theta, \\ P_\alpha(t) &= \int_0^\infty \alpha \theta \Phi_\alpha(\theta) R(t^\alpha \theta) d\theta, \end{aligned}$$

where  $S_\alpha$  and  $P_\alpha$  verifying the assumptions in [?].

### 3.2.2 Solvability of the fractional controller system

In this subsection, we use the energy estimate method to prove the strong solution of the fractional problem.

Proofs of theorems, proposition and corollary taken from [5, 15, 2].

Let  $\Omega$  be an open bounded domain with smooth boundary  $\Gamma$  we denoted by  $Q = \Omega \times (0, T)$  and  $\Sigma = \Gamma \times (0, T)$  where  $\Omega = (0, l)$ . We consider the following fractional controller degenerate and singular problem

$$\begin{cases} {}_0^C D_t^\alpha y(x, t) - \frac{\partial}{\partial x} (x^\beta \frac{\partial}{\partial x} y(x, t)) + b(x, t)y(x, t) = \tilde{f} + v, & \text{in } Q \\ y(x, 0) = \varphi(x), & \text{on } \Omega, \\ y(0, t) = y(l, t) = 0, & \text{in } \Sigma, \end{cases} \quad (3.2.14)$$

where  $0 < \alpha < 1$ ,  $0 \leq \beta < 1$ ,  ${}^C D^\alpha$  is the Caputo fractional derivative. The functions  $b$ ,  $\tilde{f}$  and  $\varphi$  are known functions,  $v$  is the control function belongs to the control space  $U$ , the function  $\varphi$  satisfies the compatibility conditions

$$\varphi(0) = \varphi(l) = 0,$$

and the function  $b$  verify :

$$0 < b_0 \leq b(x, t) \leq b_1.$$

Now, we assume that

$$z(x, t) = y(x, t) - \varphi(x),$$

implies that

$$y(x, t) = z(x, t) + \varphi(x),$$

the problem 3.2.14 becomes

$$\begin{cases} {}_0^C D_t^\alpha z(x, t) - \frac{\partial}{\partial x} (x^\alpha \frac{\partial}{\partial x} z(x, t)) + b(x, t)z(x, t) = f + v, & \text{in } Q \\ z(x, 0) = 0, & \text{on } \Omega, \\ z(0, t) = z(l, t) = 0, & \text{in } \Sigma, \end{cases} \quad (3.2.15)$$

where

$$f = \tilde{f} - {}_0^C D_t^\alpha \varphi(x) + \frac{\partial}{\partial x} (x^\alpha \frac{\partial}{\partial x} \varphi(x)) - b(x, t)\varphi(x),$$

the problem 3.2.15 can be written as follows

$$Lz = \mathcal{F},$$

where  $L = (\mathcal{L}, l)$  with the domain of definition  $E$ ,  $\mathcal{F} = (f + v, 0)$  belongs to  $F = L^2(Q)$  such that

$$\mathcal{L}z = {}^C D_t^\alpha z(x, t) - \frac{\partial}{\partial x} \left( x^\alpha \frac{\partial}{\partial x} z(x, t) \right) + b(x, t)z(x, t) = f + v,$$

with the initial condition

$$lz = z(x, 0) = 0, \quad \forall x \in (0, l),$$

we shall study the existence and the uniqueness of strong solution to the controller problem 3.2.15.

### Estimation a priori

We choose the following multiplier  $Mv(x, t) = v(x, t)$  to establish an a priori estimate in a specific space.

We consider

$$Lz = \mathcal{F},$$

where  $L = (\mathcal{L}, l)$ , with the domain of definition  $E$  such that  $v, {}^C D_t^{\frac{\alpha}{2}}, \frac{\partial v}{\partial x} \in L^2(Q)$ ,  $\mathcal{F} = (f + v, 0)$ .

**Theorem 3.2.2** *For any function  $z \in E$ , we have the inequality*

$$\|z\|_E \leq c \|Lz\|_{L^2(Q)}, \quad (3.2.16)$$

where  $c$  is a positive constant independent of  $z$ . The precedent estimation proves the uniqueness of the solution.

**Proposition 3.2.1** *The operator  $L$  with the domain of definition  $D(L)$  has a closure  $\bar{L}$ .*

**Definition 3.2.4** *We say that  $z$  is a strong solution to the problem 3.2.15 iff  $z$  verify the following equation*

$$\bar{L}z = \mathcal{F},$$

where  $\bar{L}$  is the closure of  $L$  with the domain of definition  $D(\bar{L})$ . The a priori estimate stays valid for

the strong solution, i.e.

$$\|z\|_E \leq c \left\| \bar{L}z \right\|_{L^2(Q)}.$$

We deduce from the last estimation that the strong solution of 3.2.15 if it exists is unique.

**Corollary 3.2.1** *The range  $R(\bar{L})$  is equal to the closure of  $R(L)$  it means that  $R(\bar{L}) = \overline{R(L)}$ .*

### Existence of solution

In this step, we prove the existence of a strong solution to the problem 3.2.15 is sufficient to check the density of the operator  $R(L)$ .

**Theorem 3.2.3** *Let the estimation (3.2.16) satisfied then for all  $\mathcal{F} = (f + v, 0) \in F = L^2(Q)$ , the problem 3.2.15 admits a unique strong solution.*

### 3.2.3 Weak controllability of fractional problem

Studying the weak controllability of fractional problems is essential due to the prevalence of such systems in real-world applications. By incorporating Tikhonov regularization, we can prove the controllability of our degenerate and singular fractional parabolic problem and characterizing this controller.

The method of Tikhonov regularization for the control problem consists to solve the following problem

$$\inf_{v \in L^2(Q)} J(v), \quad (3.2.17)$$

where

$$J(v) = \|I^{1-\alpha}z(T) - h\|_{L^2(Q)}^2 + \theta \|v\|_{L^2(Q)}^2, \quad (3.2.18)$$

$z \in E = L^2(Q)$ ,  $h$  is desired state belongs to  $L^2(Q)$ ,  $\theta$  is regularized parameter.

**Definition 3.2.5** *We say that the problem of control 3.2.15 is approximate controllable on  $[0, T]$  if*

$$\overline{C(T, z_0, v)} = E,$$

where

$$C(T, z_0, v) = \{I^{1-\alpha}z(T, z_0, v), v \in L^2(Q)\}.$$

**Theorem 3.2.4** For  $\frac{1}{2} < \alpha < 1$ , the problem (3.2.17) with (3.2.18) admit a unique optimal control.

**Proof.** Let a minimizing sequences  $(v_n)$  such that

$$\inf_{v \in L^2(Q)} J(v) = \lim_{n \rightarrow +\infty} J(v_n) = m.$$

Moreover, we have

$$J(v_n) \leq m,$$

this implies that exist a constant  $c$  such that

$$\begin{aligned} \|I^{1-\alpha}z_n(T) - h\|_{L^2(Q)} &\leq c, \\ \|v_n\|_{L^2(Q)} &\leq \theta^{\frac{-1}{2}} c, \end{aligned}$$

so there exist a subsequences denoted by  $(v_n)$  and  $(I^{1-\alpha}z_n(T))$  such that

$$\begin{aligned} I^{1-\alpha}z_n(T) &\rightharpoonup \Theta \text{ in } L^2(Q), \\ v_n &\rightharpoonup u \text{ in } L^2(Q), \end{aligned}$$

we have that

$$z_n(t) = S_\alpha(t)z_0 + \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) (g(s) + v_n(s)) ds, t \in [0, T].$$

for any function  $\phi \in L^2(Q)$ , we have

$$\begin{aligned} \langle z_n, \phi \rangle_{L^2(Q)} &= \langle S_\alpha(t)z_0, \phi \rangle_{L^2(Q)} + \int_0^t (t-s)^{\alpha-1} \langle P_\alpha(t-s)g(s), \phi \rangle_{L^2(Q)} ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} \langle v_n(s), P_\alpha^*(t-s)\phi \rangle_{L^2(Q)} ds, \end{aligned}$$

passing to the limit in the previous expression

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle z_n, \phi \rangle_{L^2(Q)} &= \langle S_\alpha(t)z_0, \phi \rangle_{L^2(Q)} + \int_0^t (t-s)^{\alpha-1} \langle P_\alpha(t-s)g(s), \phi \rangle_{L^2(Q)} ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} \langle P_\alpha(t-s)u(s), \phi \rangle_{L^2(Q)} ds \\ &= \langle z, \phi \rangle_{L^2(Q)}, \end{aligned}$$

which means that

$$z_n \rightarrow z \text{ in } L^2(Q),$$

According to the proposition 3.5 in [?] we get

$$z_n \rightharpoonup z \text{ in } C([0, T], E),$$

Also  $z_n \in C([0, T], E)$  then  $I^{1-\alpha}z_n \in C([0, T], E)$ , and we have

$$\begin{aligned} \langle I^{1-\alpha}z_n(T), \phi \rangle_{L^2(Q)} &= \frac{1}{\Gamma(1-\alpha)} \int_0^T (T-s)^{-\alpha} \langle S_\alpha(s)z_0, \phi \rangle_{L^2(Q)} \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^s \frac{(T-s)^{-\alpha}}{(s-\tau)^{1-\alpha}} \langle P_\alpha(s-\tau)g(\tau), \phi \rangle_{L^2(Q)} d\tau ds \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^s \frac{(T-s)^{-\alpha}}{(s-\tau)^{1-\alpha}} \langle v_n(\tau), P_\alpha^*(s-\tau)\phi \rangle_{L^2(Q)} d\tau ds. \end{aligned}$$

By passing to limit, we get

$$\begin{aligned} \langle \theta, \phi \rangle_{L^2(Q)} &= \frac{1}{\Gamma(1-\alpha)} \int_0^T (T-s)^{-\alpha} \langle S_\alpha(s)z_0, \phi \rangle_{L^2(Q)} \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^s \frac{(T-s)^{-\alpha}}{(s-\tau)^{1-\alpha}} \langle P_\alpha(s-\tau)g(\tau), \phi \rangle_{L^2(Q)} d\tau ds \\ &\quad + \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^s \frac{(T-s)^{-\alpha}}{(s-\tau)^{1-\alpha}} \langle P_\alpha(s-\tau)u(\tau), \phi \rangle_{L^2(Q)} d\tau ds \\ &= \langle I^{1-\alpha}z(T), \phi \rangle_{L^2(Q)}. \end{aligned}$$

We deduce

$$I^{1-\alpha} z_n(T) \rightharpoonup I^{1-\alpha} z(T) \text{ in } L^2(Q),$$

since  $J$  is coercive and lower semi continuous, we get

$$J(u) \leq \lim_{n \rightarrow +\infty} \inf_{v \in L^2(Q)} J(v_n) = \inf_{v \in L^2(Q)} J(v) = m,$$

then  $u$  is the optimal control corresponding to the associate optimal state  $z$ . The uniqueness of  $u$  result from the strictly convexity of  $J$ . ■

### Characterization of the optimal control

This part is devoted to characterizing the optimal control solution of the regularization problem via an optimality systems, also to give some necessary conditions to achieve the weak controllability. A first order optimality condition for  $J$  gives :

$$J'(u)(v - u) = 0, \quad \forall v \in L^2(Q),$$

by a simple calculation, we obtain

$$J'(u)(v - u) = \langle I^{1-\alpha} z(T) - h, I^{1-\alpha} \psi(T) \rangle_{L^2(Q)} + \theta \langle u, v - u \rangle_{L^2(Q)} = 0, \quad \forall v \in L^2(Q),$$

where  $\psi(t)$  is solution of the following problem

$$\left\{ \begin{array}{l} {}^C D_t^\alpha \psi(x, t) - \frac{\partial}{\partial x} (x^\beta \frac{\partial}{\partial x} \psi(x, t)) + b(x, t) \psi(x, t) = v - u, \text{ in } Q \\ \psi(x, 0) = 0, \text{ on } \Omega, \\ \psi(0, t) = \psi(l, t) = 0, \text{ in } \Sigma. \end{array} \right. \quad (3.2.19)$$

We define an adjoint state  $p = p(t, x, v)$  solution of

$$\left\{ \begin{array}{l} -D^\alpha p(x, t) - \frac{\partial}{\partial x} (x^\beta \frac{\partial}{\partial x} p) + b(x, t) p = 0, \text{ in } Q \\ p(T) = I^{1-\alpha} z(T) - h, \text{ on } \Omega, \\ p(0, t) = p(l, t) = 0, \text{ in } \Sigma. \end{array} \right. \quad (3.2.20)$$

The solution of the problem (3.2.20) given by

$$p(t) = S_\alpha^*(T-t) [I^{1-\alpha}z(T) - h], t \in [0, T].$$

Multiplying the first equation in (3.2.19) by  $p$  solution of (3.2.20), we get

$$\langle I^{1-\alpha}z(T) - h, I^{1-\alpha}\psi(T) \rangle_{L^2(Q)} = \langle p(T), I^{1-\alpha}\psi(T) \rangle_{L^2(Q)} = \langle p, v - u \rangle_{L^2(Q)},$$

hence

$$J'(u)(v - u) = \langle p, v - u \rangle_{L^2(Q)} + \theta \langle u, u - v \rangle_{L^2(Q)} = 0, \forall v \in L^2(Q),$$

consequently

$$u = -\theta^{-1}q = -\theta^{-1}S_\alpha^*(T-t) [I^{1-\alpha}z(T) - h] \text{ a.e in } Q.$$

Then the optimal control is characterized by the following optimality system

$$\begin{cases} {}_0^C D_t^\alpha z(x, t) - \frac{\partial}{\partial x}(x^\beta \frac{\partial}{\partial x} z(x, t)) + b(x, t)z(x, t) = f + u, \text{ in } Q \\ z(x, 0) = 0, \text{ on } \Omega, \\ z(0, t) = z(l, t) = 0, \text{ in } \Sigma. \\ -D^\alpha p(x, t) - \frac{\partial}{\partial x}(x^\alpha \frac{\partial}{\partial x} p(x, t)) + b(x, t)p(x, t) = 0, \text{ in } Q \\ p(x, T) = I^{1-\alpha}z(T) - h, \text{ on } \Omega, \\ p(0, t) = p(l, t) = 0, \text{ in } \Sigma. \end{cases}$$

with

$$u = -\theta^{-1}p = -\theta^{-1}S_\alpha^*(T-t) [I^{1-\alpha}z(T) - h] \text{ a.e in } Q.$$

**Theorem 3.2.5** *The problem 3.2.15 is weak controllable iff the operator  $\theta R(\theta, -\Lambda_T)$  converge to zero when  $\theta \rightarrow 0$ . where  $\Lambda_T$  is an operator defined from  $L^2(Q)$  to  $L^2(Q)$  by*

$$\Lambda_T = \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^s \frac{(T-s)^{-\alpha}}{(s-\tau)^{1-\alpha}} P_\alpha(s-\tau) S(s-\tau) d\tau ds$$

and

$$R(\theta, -\Lambda_T) = (\theta I + \Lambda_T)^{-1}$$

such that  $(\theta I + \Lambda_T)$  is invertible.

**Proof.** [16, 2] We have that

$$I^{1-\alpha} z(T) = \frac{1}{\Gamma(1-\alpha)} \int_0^T (T-s)^{-\alpha} [S_\alpha(s)z_0 + r(s)] ds - \theta^{-1} \Lambda_T [I^{1-\alpha} z(T) - h],$$

where

$$r(s) = \int_0^s (s-\tau)^{\alpha-1} P_\alpha(s-\tau) g(\tau) d\tau,$$

it's easy to get

$$I^{1-\alpha} z(T) - h = \theta R(\theta, -\Lambda_T) \left( \frac{1}{\Gamma(1-\alpha)} \int_0^T (T-s)^{-\alpha} [S_\alpha(s)z_0 + r(s)] ds - h, \right)$$

by using the theorem(2) in [16] if  $\theta R(\theta, -\Lambda_T) \rightarrow 0$  when  $\theta$  tends to 0, then

$$\overline{C(T, z_0, v)} = E,$$

which prove the approximate controllability. ■

### 3.3 Identification of the local changes in age-structured population dynamics with missing initial condition

In this section we focus on an inverse problem of a dynamic population structured by age with missing initial conditions when we aims to identifying the source term from the knowledge of some measurement on the state. Our way to solve this problem is different, we seems that the study of the inverse problem transformed into an optimal control problem with incomplete data. This leads us to use the no-regret control approximated by a sequence of low-regret control, we show the existence and uniqueness of the sequence of low-regret control, which converges weakly to the unique no-regret

control. The source term will be characterized by a coupled optimality system.

### 3.3.1 Transformation of the inverse problem to an optimal control problem with incomplete data

We begin this subsection by the presentation of our inverse problem of identification an unknown couple of source terms in a dynamic population structured by age with discontinuous diffusion coefficients and missing initial conditions from some kind of measurement. Then, we transformed it into an optimal control problem with incomplete data. By using the no-regret and low-regret controls we find the characterization of the unknown source term.

Let  $\Omega$  be an open-bounded domain of  $\mathbb{R}^n$  ( $n \leq 3$ ) with boundary  $\Gamma$  of class  $C^2$ ,  $T$  be a positive constant and  $A$  be the natural life of an individual. We denoted by  $Q = \Omega \times (0, T) \times (0, A)$ ,  $Q_T = \Omega \times (0, T)$ ,  $Q_A = \Omega \times (0, A)$  and  $\Sigma = \Gamma \times (0, T) \times (0, A)$ , with  $Dy = y_t + y_a$ ,  $Dz = z_t + z_a$

$$\left\{ \begin{array}{l} Dy = \text{div}(r(x)\nabla y) + a(x)y + b(x)z + f_1(x, t, a), (x, t, a) \in Q, \\ Dz = \text{div}(\tilde{r}(x)\nabla z) + \tilde{a}(x)z + \tilde{b}(x)z + f_2(x, t, a), (x, t, a) \in Q, \\ y(x, 0, a) = y_0(x, a), z(x, 0, a) = z_0(x, a), (x, a) \in Q_A, \\ y(x, t, 0) = y_0(x, t), z(x, t, 0) = z_0(x, t), (x, t) \in Q_t, \\ y(x, t, a) = z(x, t, a) = 0 \text{ on } \Sigma. \end{array} \right. \quad (3.3.1)$$

Where

- $f_1(t, a, x), f_2(t, a, x) \in L^2(Q) \times L^2(Q)$  are the unknown source terms representing the local change of population density.
- $(y_0(x, a), z_0(x, a)) \in L^2(Q_A) \times L^2(Q_A)$  represents the initial population age distribution is supposed to be unknown.
- The measurement  $(y(x, t, a), z(x, t, a)) \in L^2(0, T, H_0^1(\Omega) \cap H^2(\Omega)) \times L^2(0, T, H_0^1(\Omega) \cap H^2(\Omega))$  represent the age-density at time  $t$  and a point  $x$  [13].

Mathematically, our question is : Determine  $(f_1, f_2)$  for a given observation

$$y(t, f_1, y_0), z(t, f_2, z_0) = (y_{obs}, z_{obs})?$$

In order to solve the previous question we replace the source couple  $(f_1, f_2)$  by the control pair

$(v_1, v_2)$  and the couple initial condition  $(y_0, z_0)$  by the couple function  $(g_1, g_2)$ , we obtain the following problem

$$\left\{ \begin{array}{l} Dy = \operatorname{div}(r(x)\nabla y) + a(x)y + b(x)z + v_1(x, t, a), (x, t, a) \in Q, \\ Dz = \operatorname{div}(\tilde{r}(x)\nabla z) + \tilde{a}(x)z + \tilde{b}(x)z + v_2(x, t, a), (x, t, a) \in Q \\ y(x, 0, a) = g_1(x, a), \quad z(x, 0, a) = g_2(x, a), (x, a) \in Q_A, \\ y(x, t, 0) = y_0(x, t), \quad z(x, t, 0) = z_0(x, t), (x, t) \in Q_t, \\ y(x, t, a) = z(x, t, a) = 0 \text{ on } \Sigma, \end{array} \right. \quad (3.3.2)$$

where  $(g_1, g_2)$  are unknowns functions. Our goal is to identify the control function  $(v_1, v_2)$  without carrying to the missing initial condition using the no-regret control method.

Given an observation

$$(y(t, v_1, g_1), z(t, v_2, g_2)) = (y_{obs}, z_{obs}),$$

where  $(y_{obs}, z_{obs})$  are given observation in  $L^2(Q)$  and  $y(t, v_1, g_1) = y(t, a, x, v_1, g_1)$  and  $z(t, v_2, g_2) = z(t, a, x, v_2, g_2)$ .

with the following cost function

$$\begin{aligned} J(v_1, v_2, g_1, g_2) &= \|y(t, v_1, y_0) - y_{obs}\|_{L^2(Q)}^2 + \|z(t, v_2, z_0) - z_{obs}\|_{L^2(Q)}^2 \\ &\quad + N(\|v_1\|_{L^2(Q)}^2 + \|v_2\|_{L^2(Q)}^2), \end{aligned} \quad (3.3.3)$$

where  $N > 0$ . We aim to solve the following optimal control problem with missing data

$$\left\{ \begin{array}{l} \text{Find the couple } (u_1, u_2) \in (L^2(Q))^2 \text{ such that :} \\ J(u_1, u_2, g_1, g_2) = \inf_{(v_1, v_2) \in (L^2(Q) \times L^2(Q))} J(v_1, v_2, g_1, g_2), \end{array} \right. \quad (3.3.4)$$

### 3.3.2 No-regret control method for age-structured population with missing data

We will now be concerned with the optimal control of the coupled system with missing initial condition in order to identify the unknown source pair. Recently this paper [12] was published to study the optimal control of the coupled system with missing data; In this situation, we can't use the classical approach of optimal control, as we said before the no-regret control method is full to return the problem to a classical case.

In general, the problem 3.3.4 has no sense so, we try to solve the following problem

$$\inf_{(v_1, v_2) \in (L^2(Q) \times L^2(Q))} \sup_{(g_1, g_2) \in L^2(Q_A) \times L^2(Q_A)} (J(v_1, v_2, g_1, g_2)).$$

But here  $J(v_1, v_2, g_1, g_2)$  is not upper-bounded, another idea proposed by Lions on [30] is to look only for the controls such that  $(v_1, v_2) \in L^2(Q) \times L^2(Q)$

$$J(v_1, v_2, g_1, g_2) \leq J(0, 0, g_1, g_2), \quad \forall (g_1, g_2) \in L^2(Q_A) \times L^2(Q_A). \quad (3.3.5)$$

thus makes  $\sup_{(g_1, g_2) \in L^2(Q_A) \times L^2(Q_A)} (J(v_1, v_2, g_1, g_2))$  exists. In this case, we can solve the problem below :

$$\inf_{(v_1, v_2) \in (L^2(Q) \times L^2(Q))} \sup_{(g_1, g_2) \in L^2(Q_A) \times L^2(Q_A)} (J(v_1, v_2, g_1, g_2) - J(0, 0, g_1, g_2)). \quad (3.3.6)$$

**Definition 3.3.1** We called that the couple  $(u_1, u_2)$  is no-regret control pair for the problem 3.3.2 and 3.3.3 iff it solves the problem 3.3.6

**Lemma 3.3.1** For every  $(u_1, u_2) \in (L^2(Q) \times L^2(Q))$  and  $(g_1, g_2) \in L^2(Q) \times L^2(Q)$  the problem 3.3.6 is equivalent to

$$\inf_{(v_1, v_2) \in (L^2(Q) \times L^2(Q))} \left( \begin{array}{c} J(v_1, v_2, 0, 0) - J(0, 0, 0, 0) \\ + 2 \sup_{(g_1, g_2) \in L^2(Q_A) \times L^2(Q_A)} \left( \int_0^A \int_{\Omega} \xi(0, x, a) g_1 dx da \right. \\ \left. + \int_0^A \int_{\Omega} \varsigma(0, x, a) g_2(a, x) dx da \right) \end{array} \right)$$

where  $(\xi, \varsigma) = (\xi(t, x, a), \varsigma(t, x, a))$  is a solution of the following backward coupled equations

$$\left\{ \begin{array}{l} -D\xi - \operatorname{div}(r(x)\nabla\xi) - a(x) - b(x)\varsigma = y(t, v_1, 0) - y(t, 0, 0), (x, t, a) \in Q, \\ -D\varsigma - \operatorname{div}(\tilde{r}(x)\nabla\varsigma) - \tilde{b}(x)\xi - \tilde{b}(x)\varsigma = z(t, v_2, 0) - z(t, 0, 0), (x, t, a) \in Q, \\ \xi(x, t, a) = 0, \varsigma(x, t, a) = 0, (x, a) \in Q_A, \\ \xi(x, t, A) = 0, \varsigma(x, t, A) = 0, (x, t) \in Q_t, \\ \xi(x, t, a) = \varsigma(x, t, a) = 0 \text{ on } \Sigma. \end{array} \right. \quad (3.3.7)$$

**Proof.** Due to linearity of  $(y, z)$ , we can write

$$\begin{aligned} y(t, v_1, g_1) &= y(t, 0, g_1) + y(t, v_1, 0) - y(t, 0, 0), \\ z(t, v_2, g_2) &= z(t, 0, g_2) + z(t, v_2, 0) - z(t, 0, 0). \end{aligned}$$

where  $(y(t, 0, g_1), z(t, 0, g_2))$ ,  $(y(t, v_1, 0), z(t, v_2, 0))$  and  $(y(t, 0, 0), z(t, 0, 0))$  are solutions of the following systems resp.

$$\left\{ \begin{array}{l} Dy(t, 0, g_1) = \operatorname{div}(r(x)\nabla y(t, 0, y_0)) + a(x)y(t, 0, y_0) + b(x)y(t, 0, y_0), (x, t, a) \in Q, \\ Dz(t, 0, g_2) = \operatorname{div}(\tilde{r}(x)\nabla z(t, 0, z_0)) + a(x)z(t, 0, z_0) + b(x)z(t, 0, z_0), (x, t, a) \in Q, \\ y(x, 0, a) = g_1(x, a), \quad z(x, 0, a) = g_2(x, a), (x, a) \in Q_A, \\ y(x, t, 0) = y(x, t), \quad z(x, t, 0) = z(x, t), (x, t) \in Q_t, \\ y(x, t, a) = z(x, t, a) = 0 \text{ on } \Sigma, \end{array} \right.$$

$$\left\{ \begin{array}{l} Dy(t, v_1, 0) = \operatorname{div}(r(x)\nabla y(t, v_1, 0)) + a(x)y(t, v_1, 0) + b(x)z(t, v_1, 0) + v_1, (x, t, a) \in Q, \\ Dz(t, v_2, 0) = \operatorname{div}(\tilde{r}(x)\nabla z(t, v_2, 0)) + \tilde{a}(x)y(t, v_1, 0) + \tilde{b}(x)z(t, v_2, 0) + v_2, (x, t, a) \in Q, \\ y(x, 0, a) = 0, \quad z(x, 0, a) = 0, (x, a) \in Q_A, \\ y(x, t, 0) = y(t, x), \quad z(x, t, 0) = z(t, x), (x, t) \in Q_t, \\ y(x, t, a) = z(x, t, a) = 0 \text{ on } \Sigma, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} Dy(t, 0, 0) = \operatorname{div}(k(x)\nabla y(t, 0, 0)) + \mu(x)y(t, 0, 0) + \alpha(x)z(t, 0, 0), (t, a, x) \in Q, \\ Dz(t, 0, 0) = \operatorname{div}(\tilde{k}(x)\nabla z(t, 0, 0)) + \tilde{\mu}(x)y(t, 0, 0) + \tilde{\alpha}(x)z(t, 0, 0), (t, a, x) \in Q, \\ y(x, 0, a) = 0, \quad z(x, 0, a) = 0, (x, a) \in Q_A, \\ y(x, t, 0) = y(x, t), \quad z(x, t, 0) = z(x, t), (x, t) \in Q_t, \\ y(x, t, a) = z(x, t, a) = 0 \text{ on } \Sigma. \end{array} \right.$$

By a simple calculus

$$\begin{aligned} J(v_1, v_2, g_1, g_2) - J(0, 0, g_1, g_2) &= J(v_1, v_2, 0, 0) - J(0, 0, 0, 0) \\ &\quad + 2(y(t, v_1, 0) - y(t, 0, 0), y(t, 0, g_1) - y(t, 0, 0))_{L^2(Q_A)} \\ &\quad + 2(z(t, v_2, 0) - z(t, 0, 0), z(t, 0, g_2) - z(t, 0, 0))_{L^2(Q_A)}. \end{aligned}$$

We introduce the adjoint coupled state  $(\xi, \varsigma)$  solution of 3.3.7 and we use integration by parts and Green formula, we obtain

$$\begin{aligned} 0 &= \iiint_Q (-D\xi - \operatorname{div}(r(x)\nabla\xi) - a(x)\xi - b(x)\varsigma) (y(t, 0, g_1) - y(t, 0, 0)) dxda dt \\ &= \int_0^A \int_\Omega \xi(x, 0, a)g_1(x, a)dxda - \int_Q (y(t, v_1, 0) - y(t, 0, 0)) (y(t, 0, g_1) - y(t, 0, 0)) dxda, \end{aligned}$$

and

$$\begin{aligned} 0 &= \int_Q (-D\xi - \operatorname{div}(\tilde{r}(x)\nabla\xi) - \tilde{a}(x)\xi - \tilde{b}(x)\varsigma) (z(t, 0, g_2) - z(t, 0, 0)) dxda dt \\ &= \int_0^A \int_\Omega \varsigma(x, 0, a)g_2(x, a)dxda - \int_Q (z(t, v_2, 0) - z(t, 0, 0)) ((z(t, 0, g_2) - z(t, 0, 0))) dxda. \end{aligned}$$

Then

$$\begin{aligned} J(v_1, v_2, g_1, g_2) - J(0, 0, g_1, g_2) &= J(v_1, v_2, 0, 0) - J(0, 0, 0, 0) \\ &+ 2 \int_0^A \int_\Omega \xi(x, 0, a)g_1(x, a)dxda + 2 \int_0^A \int_\Omega \varsigma(x, 0, a)g_2(x, a)dxda, \end{aligned}$$

■

The no-regret control exists only if  $g_1(x, a)$  and  $g_2(x, a)$  are elements of the kernel of the  $(\xi(x, 0, a), \varsigma(x, 0, a))$  in  $L^2(Q_A)$  resp. Because it is difficult to test this assumption, we relax our problem by introducing a quadratic perturbation to the inequality 3.3.5

$$\begin{aligned} J(v_1, v_2, g_1, g_2) &\leq J(0, 0, g_1, g_2) + \gamma \left( \|g_1(x, a)\|_{L^2(Q_A)}^2 + \|g_2(x, a)\|_{L^2(Q_A)}^2 \right), \\ \text{where } \gamma &> 0, \forall (g_1, g_2) \in (L^2(Q_A))^2. \end{aligned}$$

Then the previous inf sup problem becomes

$$\inf_{(v_1, v_2) \in (L^2(Q) \times L^2(Q))} \left( \begin{array}{l} J(v_1, v_2, 0, 0) - J(0, 0, 0, 0) + 2 \sup_{(g_1, g_2) \in L^2(Q_A) \times L^2(Q_A)} \left( \int_0^A \int_\Omega \xi(x, 0, a) \right. \\ \left. g_1(x, a) dxda + \int_0^A \int_\Omega \varsigma(x, 0, a) g_2(x, a) dxda \right. \\ \left. - \gamma (\|g_1(x, a)\|_{L^2(Q_A)}^2 + \|g_2(x, a)\|_{L^2(Q_A)}^2) \right) \end{array} \right)$$

by Legendre transform [44], we get

$$\begin{aligned} & \sup_{(g_1, g_2) \in L^2(Q_A) \times L^2(Q_A)} \left( \int_0^A \int_{\Omega} \xi(x, 0, a) g_1(x, a) dx da + \int_0^A \int_{\Omega} \varsigma(x, 0, a) g_2(x, a) dx da \right. \\ & \left. - \gamma (\|g_1(x, a)\|_{L^2(Q_A)}^2 + \|g_2(x, a)\|_{L^2(Q_A)}^2) \right), \\ & = \frac{1}{2\gamma} \left( \|\xi(x, 0, a)\|_{L^2(Q_A)}^2 + \|\varsigma(x, 0, a)\|_{L^2(Q_A)}^2 \right), \end{aligned}$$

by replacing this result in the last inf sup problem, we obtain

$$\inf_{(v_1, v_2) \in (L^2(Q)) \times L^2(Q)} J_{\gamma}(v_1, v_2)$$

where

$$J_{\gamma}(v_1, v_2) = J(v_1, v_2, 0, 0) - J(0, 0, 0, 0) + \frac{1}{\gamma} \left( \|\xi(0, x, a)\|_{L^2(Q_A)}^2 + \|\varsigma(0, x, a)\|_{L^2(Q_A)}^2 \right). \quad (3.3.8)$$

Hence, the previous problem is equivalent to the following standard optimal control problem

$$\begin{cases} \forall \gamma > 0, \text{ find } (u_1^{\gamma}, u_2^{\gamma}) \in (L^2(Q))^2 \text{ such that} \\ J_{\gamma}(u_1^{\gamma}, u_2^{\gamma}) = \inf_{(v_1, v_2) \in (L^2(Q))^2} J_{\gamma}(v_1, v_2). \end{cases} \quad (3.3.9)$$

**Definition 3.3.2** We called that the couple  $(u_1^{\gamma}, u_2^{\gamma}) \in (L^2(Q) \times L^2(Q))$  is low-regret control pair for the problem 3.3.2 and 3.3.8 iff it solves the problem 3.3.9 .

**Theorem 3.3.1** There exists a unique coupled of low-regret control denoted by  $(u_1^{\gamma}, u_2^{\gamma}) \in (L^2(Q))^2$  solution of the problem 3.3.9 and 3.3.8. Moreover, when  $\gamma$  tends to 0 the couple of low-regret control converges weakly to the unique coupled of the no-regret control .

**Proof.** From the definition of  $J_{\gamma}$ , we have :

$$J_{\gamma}(v_1, v_2) \geq -J(0, 0, 0, 0) = \text{constant},$$

then, there exists  $d_{\gamma} = \inf_{(v_1, v_2) \in (L^2(Q)) \times L^2(Q)} J_{\gamma}(v_1, v_2)$ . Let a couple minimizing sequence  $(v_{n1}, v_{n2})$

verifying

$$\lim_{n \rightarrow \infty} J_\gamma(y, z, v_{n1}, v_{n2}) = \inf_{(v_1, v_2) \in (L^2(Q) \times L^2(Q))} J_\gamma(v_1, v_2) = d_\gamma,$$

again, we have

$$\begin{aligned} -J(0, 0, 0, 0) &\leq J_\gamma(v_{n1}, v_{n2}) = J(v_{n1}, v_{n2}, 0, 0) - J(0, 0, 0, 0) + \\ &\frac{1}{\gamma} \left( \|\xi_n(0, x, a)\|_{L^2(Q_A)}^2 + \|\varsigma_n(0, x, a)\|_{L^2(Q_A)}^2 \right) \leq d_\gamma + 1 = C_\gamma \end{aligned}$$

from which we deduce the following estimates

$$\left\{ \begin{array}{l} \|v_{n1}\|_{L^2(Q)} \leq C_\gamma, \|v_{n2}\|_{L^2(Q)} \leq C_\gamma, \\ \|\xi_n(0, x, a)\|_{L^2(Q_A)} \leq C_\gamma \sqrt{\gamma}, \|\varsigma_n(0, x, a)\|_{L^2(Q_A)} \leq C_\gamma \sqrt{\gamma}, \end{array} \right. \quad (3.3.10)$$

where  $C_\gamma$  is a constant independent of  $n$ . From 3.3.10, we can extract a subsequence denoting by  $(v_{n1}, v_{n2})$ , such that when  $n$  tends to  $\infty$  we have  $(v_{n1}, v_{n2}) \rightharpoonup (u_1^\gamma, u_2^\gamma)$  weakly in  $L^2(Q)$ . Firstly, we know that  $(y_n, z_n) = (y(t, v_{n1}, 0), z(t, v_{n2}, 0))$  solves the following coupled systems

$$\left\{ \begin{array}{l} Dy_n - \operatorname{div}(r(x)\nabla y_n) - a(x)y_n - b(x)z_n = v_{n1}, (x, t, a) \in Q, \\ Dz_n - \operatorname{div}(\tilde{r}(x)\nabla z_n) - \tilde{a}(x)y_n - \tilde{b}(x)z_n = v_{n2}, (x, t, a) \in Q, \\ y_n(x, 0, a) = 0, z_n(x, 0, a) = 0, (x, a) \in Q_A, \\ y_n(x, t, 0) = y_n(x, t), z_n(x, t, 0) = z_n(x, t), (x, t) \in Q_t, \\ y_n(x, t, a) = z_n(x, t, a) = 0 \text{ on } \Sigma. \end{array} \right. \quad (3.3.11)$$

Let's start by multiplying the first equation of 3.3.11 by  $y_n$  and we integrate over  $\Omega$ , we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\Omega} |y_n|^2 dx + \frac{1}{2} \frac{d}{da} \int_{\Omega} |y_n|^2 dx + \int_{\Omega} r(x) |\nabla y_n|^2 dx \\ &= \int_{\Omega} v_{n1} y_n dx + \int_{\Omega} a(x) |y_n|^2 dx + \int_{\Omega} b(x) z_n y_n dx, \end{aligned}$$

by the same way with the second equation

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\Omega} |z_n|^2 dx + \frac{1}{2} \frac{d}{da} \int_{\Omega} |z_n|^2 dx + \int_{\Omega} \tilde{k}(x) |\nabla z_n|^2 dx \\ &= \int_{\Omega} v_{n2} z_n dx + \int_{\Omega} \tilde{a}(x) |z_n|^2 dx + \int_{\Omega} \tilde{\mu}(x) y_n z_n dx, \end{aligned}$$

by summation of the two last equalities, and we integrate over  $(0, t) \times (0, a)$  then applying

Cauchy-Schwarz inequality and Poincar's inequality, we found

$$\begin{aligned} & \int_0^a \int_{\Omega} (|y_n|^2 + |z_n|^2) dx d\alpha + \int_0^t \int_{\Omega} (|y_n|^2 + |z_n|^2) dx ds \leq \iint_Q (|v_{n1}|^2 + |v_{n2}|^2) dx d\alpha ds \\ & + \int_0^t \int_{\Omega} (|y_n(x, t)|^2 + |z_n(x, t)|^2) dx ds + c_1 \iint_Q (|y_n|^2 + |z_n|^2) dx d\alpha ds, \end{aligned}$$

where

$$c_1 = |b_1 + \tilde{a}_1 + \beta - 1|, \quad \beta = 2 \max(a_1, \tilde{b}_1).$$

Moreover, we use the Gronwall lemma we obtain

$$\|y_n\|_{L^\infty(0,t,L^2(Q_A))} \leq C_\gamma, \quad \|z_n\|_{L^\infty(0,A,L^2(Q_A))} \leq C_\gamma.$$

From 3.3.10 and we integrate over  $Q$  then we deduce that

$$\begin{aligned} \|Dy_n - \operatorname{div}(k(x)\nabla y_n) - \mu(x)y_n - \alpha(x)z_n\|_{L^2(Q)} &\leq C_\gamma, \\ \|Dz_n - \operatorname{div}(\tilde{k}(x)\nabla z_n) - \tilde{\mu}(x)y_n - \tilde{\alpha}(x)z_n\|_{L^2(Q)} &\leq C_\gamma, \end{aligned}$$

then, we deduce

$$\begin{aligned} y_n &\rightharpoonup y^\gamma \text{ weakly in } L^\infty(Q), \quad z_n \rightharpoonup z^\gamma \text{ weakly in } L^\infty(Q), \\ Dy_n - \operatorname{div}(k(x)\nabla y_n) - \mu(x)y_n - \alpha(x)z_n &\rightharpoonup h_1 \text{ weakly in } L^2(Q), \\ Dz_n - \operatorname{div}(\tilde{k}(x)\nabla z_n) - \tilde{\mu}(x)y_n - \tilde{\alpha}(x)z_n &\rightharpoonup h_2 \text{ weakly in } L^2(Q), \end{aligned}$$

according to continuity of the embedding between  $L^\infty(Q)$  and  $L^2(Q)$  we have

$$y_n \rightharpoonup y^\gamma \text{ weakly in } L^2(Q) \text{ and } z_n \rightharpoonup z^\gamma \text{ weakly in } L^2(Q), \quad (3.3.12)$$

also

$$Dy_n - \operatorname{div}(r(x)\nabla y_n) - a(x)y_n - b(x)z_n \rightharpoonup Dy^\gamma - \operatorname{div}(r(x)\nabla y^\gamma) - a(x)y^\gamma - b(x)z^\gamma$$

$$\text{in } D'(Q),$$

$$Dz_n - \operatorname{div}(\tilde{r}(x)\nabla z_n) - \tilde{a}(x)y_n - \tilde{b}(x)z_n \rightharpoonup Dz^\gamma - \operatorname{div}(\tilde{r}(x)\nabla z^\gamma) - \tilde{a}(x)y^\gamma - \tilde{b}(x)z^\gamma$$

$$\text{in } D'(Q),$$

where  $D'(Q)$  is the dual space of  $D(Q) = C_0^\infty(Q)$ .

From limit uniqueness we get :

$$Dy^\gamma - \operatorname{div}(r(x)\nabla y^\gamma) - a(x)y^\gamma - b(x)z^\gamma = u_1^\gamma \text{ in } Q,$$

$$Dz^\gamma - \operatorname{div}(\tilde{r}(x)\nabla z^\gamma) - \tilde{a}(x)y^\gamma - \tilde{b}(x)z^\gamma = u_2^\gamma \text{ in } Q.$$

Also, from 3.3.12 and the initial conditions in 3.3.11, we deduce

$$y^\gamma(x, 0, a) = 0, \quad z^\gamma(x, 0, a) = 0, \quad (x, a) \in Q_A,$$

$$y^\gamma(x, t, 0) = y^\gamma(x, t), \quad z^\gamma(x, t, 0) = z^\gamma(x, t), \quad (x, t) \in Q_t.$$

Now, we multiply the two equalities in 3.3.11 by  $(\phi, \varphi) \in D(Q) \times D(Q)$  where  $\phi(x, t, a) = 0$ ,  $\varphi(x, t, a) = 0$ ,  $\phi(x, t, A) = 0$ ,  $\varphi(x, t, A) = 0$  in  $\Omega$ ,  $\phi(x, t, a) = \varphi(x, t, a) = 0$  on  $\Sigma$  and integrate over  $Q$  to get

$$y^\gamma = z^\gamma = 0 \text{ on } \Sigma.$$

Secondly, we have that  $(\xi_n, \varsigma_n) = (\xi(v_{n1}^\gamma), \varsigma(v_{n2}^\gamma))$  is a solution of the coupled adjoint system below

$$\left\{ \begin{array}{l} -D\xi_n - \operatorname{div}(r(x)\nabla \xi_n) - a(x)\xi_n - b(x)\varsigma_n = y(t, v_{n1}, 0) - y(t, 0, 0), \quad (x, t, a) \in Q, \\ -D\varsigma_n - \operatorname{div}(\tilde{r}(x)\nabla \varsigma_n) - \tilde{a}(x)\xi_n - \tilde{b}(x)\varsigma_n = z(t, v_{n2}, 0) - z(t, 0, 0), \quad (t, a, x) \in Q, \\ \xi_n(x, t, a) = 0, \quad \varsigma_n(x, t, a) = 0, \quad (x, a) \in Q_A, \\ \xi_n(x, t, A) = 0, \quad \varsigma_n(x, t, A) = 0, \quad (x, t) \in Q_t, \\ \xi_n(x, t, a) = \varsigma_n(x, t, a) = 0 \text{ on } \Sigma. \end{array} \right.$$

We want to prove that the couple  $(\xi_n, \varsigma_n)$  converges to  $(\xi^\gamma, \varsigma^\gamma)$ . We follow the same way in the previous coupled state, we get

$$\|\xi_n\|_{L^\infty(0,t,H^1(Q_A))} \leq C_\gamma, \quad \|\varsigma_n\|_{L^\infty(0,t,H^1(Q_A))} \leq C_\gamma,$$

we deduce

$$\xi_n \rightharpoonup \xi^\gamma \text{ weakly in } L^2(Q), \quad \varsigma_n \rightharpoonup \varsigma^\gamma \text{ weakly in } L^2(Q),$$

and

$$\begin{aligned} D\xi^\gamma - \operatorname{div}(r(x)\nabla\xi^\gamma) - a(x)\xi^\gamma - b(x)\varsigma^\gamma &= y(t, v_1^\gamma, 0) - y(t, 0, 0) \text{ in } Q, \\ D\varsigma^\gamma - \operatorname{div}(\tilde{r}(x)\nabla\varsigma^\gamma) - \tilde{a}(x)\xi^\gamma - \tilde{b}(x)\varsigma^\gamma &= z(t, u_2^\gamma, 0) - z(t, 0, 0) \text{ in } Q. \end{aligned}$$

With the following initial conditions

$$\begin{aligned} \xi^\gamma(x, 0, a) &= 0, \quad \varsigma^\gamma(x, 0, a) = 0, \quad (x, a) \in Q_A, \\ \xi^\gamma(x, t, 0) &= \xi(x, t), \quad \varsigma^\gamma(x, t, 0) = \varsigma^\gamma(x, t), \quad (x, t) \in Q_t, \end{aligned}$$

and the following boundary conditions

$$\xi^\gamma = \varsigma^\gamma = 0 \text{ on } \Sigma.$$

Moreover, the cost function  $J_\gamma$  is a lower semi-continuous

$$J_\gamma(u_1^\gamma, u_2^\gamma) \leq \lim_{n \rightarrow \infty} \inf_{(v_{n1}, v_{n2}) \in (L^2(Q))^2} J_\gamma(v_{n1}, v_{n2}) = \inf_{(v_1, v_2) \in (L^2(Q))^2} J_\gamma(v_1, v_2) = d_\gamma,$$

we deduce that

$$J_\gamma(u_1^\gamma, u_2^\gamma) = \inf_{(v_1, v_2) \in (L^2(Q))^2} J_\gamma(v_1, v_2),$$

As a result, from strictly convexity of  $J_\gamma$  we deduce that  $(u_1^\gamma, u_2^\gamma)$  is a unique minimizer of  $J_\gamma$ . ■

### 3.3.3 Full characterization of the coupled low-regret control

In this step, we will see how using Euler's first-order optimality criterion can be used to characterize the low-regret control.

Here is an essential property is the first order optimality condition for  $J_\gamma$  gives :

$$J'_\gamma(u_1^\gamma, u_2^\gamma)(v_1 - u_1^\gamma, v_2 - u_2^\gamma) = 0, \quad \forall (v_1, v_2) \in L^2(Q) \times L^2(Q).$$

This is equivalent to

$$\begin{aligned}
J'_\gamma(u_1^\gamma, u_2^\gamma) (v_1 - u_1^\gamma, v_2 - u_2^\gamma) &= (y(t, u_1^\gamma, 0) - y_{obs}, y(t, v_1 - u_1^\gamma, 0))_{L^2(Q)} \\
&+ (z(t, u_2^\gamma, 0) - z_{obs}, z(t, v_2 - u_2^\gamma, 0))_{L^2(Q)} + \\
&N \left( (u_1^\gamma, v_1 - u_1^\gamma)_{L^2(Q)} + (u_2^\gamma, v_2 - u_2^\gamma)_{L^2(Q)} \right) + \\
&\frac{1}{\gamma} ((\xi(x, 0, a) (u_1^\gamma), \xi(x, 0, a) (v_1 - u_1^\gamma))_{L^2(Q_A)} + \\
&(\varsigma(x, 0, a) (u_2^\gamma), \varsigma(x, 0, a) (v_2 - u_2^\gamma))_{L^2(Q_A)}) = 0.
\end{aligned}$$

Let's introduce new coupled state  $(\rho_\gamma, \mu_\gamma) = (\rho(u_1^\gamma, 0)(x, t, a), \mu_\gamma(u_2^\gamma, 0)(x, t, a))$  given by

$$\left\{ \begin{array}{l}
D\rho_\gamma - \text{div}(r(x)\nabla\rho_\gamma) - a(x)\rho_\gamma - b(x)\mu_\gamma = 0, (x, t, a) \in Q, \\
D\mu_\gamma - \text{div}(\tilde{r}(x)\nabla\mu_\gamma) - \tilde{a}(x)\rho_\gamma - \tilde{b}(x)\mu_\gamma = 0, (x, t, a) \in Q, \\
\rho_\gamma(x, 0, a) = \frac{1}{\gamma}\xi(x, 0, a) (u_1^\gamma), \mu_\gamma(x, 0, a) = \frac{1}{\gamma}\varsigma(x, 0, a) (u_2^\gamma), (x, a) \in Q_A, \\
\rho_\gamma(x, t, 0) = 0, \mu_\gamma(x, t, 0) = 0, (x, t) \in Q_t, \\
\rho_\gamma(x, t, a) = \mu_\gamma(x, t, a) = 0 \text{ on } \Sigma.
\end{array} \right.$$

Multiply the two equalities of the last one by  $\xi(v_1 - u_1^\gamma)$  and  $\varsigma(v_2 - u_2^\gamma)$  resp, apply the second Green formula and integration by parts, we obtain

$$\begin{aligned}
0 &= \int_Q (D\rho_\gamma - \text{div}(r(x)\nabla\rho_\gamma) - a(x)\rho_\gamma - b(x)\mu_\gamma) \xi(v_1 - u_1^\gamma) dxdadt \\
&= \int_Q \rho_\gamma(t) y(t, v_1 - u_1^\gamma, 0) dxdadt - \frac{1}{\gamma} \iint_{Q_A} \xi(x, 0, a) (u_1^\gamma) \xi(x, 0, a) (v_1 - u_1^\gamma) dxda,
\end{aligned}$$

and

$$\begin{aligned}
0 &= \int_Q (D\mu_\gamma - \text{div}(\tilde{r}(x)\nabla\mu_\gamma) - \tilde{a}(x)\rho_\gamma - \tilde{b}(x)\mu_\gamma) \varsigma(v_2 - u_2^\gamma) dxdadt \\
&= \int_Q \mu_\gamma(t) z(t, v_2 - u_2^\gamma, 0) dxdadt - \frac{1}{\gamma} \iint_{Q_A} \varsigma(x, 0, a) (u_2^\gamma) \varsigma(x, 0, a) (v_2 - u_2^\gamma) dxda.
\end{aligned}$$

Hence,

$$\begin{aligned} J'_\gamma(u_1^\gamma, u_2^\gamma)(v_1 - u_1^\gamma, v_2 - u_2^\gamma) &= (y(t, u_1^\gamma, 0) - y_{obs} + \rho_\gamma(t), y(t, v_1 - u_1^\gamma, 0))_{L^2(Q)} \\ &\quad + (z(t, u_2^\gamma, 0) - z_{obs} + \varsigma_\gamma(t), z(t, v_2 - u_2^\gamma, 0))_{L^2(Q)} \\ &\quad + N \left( (u_1^\gamma, v_1 - u_1^\gamma)_{L^2(Q)} + (u_2^\gamma, v_2 - u_2^\gamma)_{L^2(Q)} \right)_{L^2(Q)}. \end{aligned}$$

Also, for the second time, we introduce the coupled adjoint states  $(p_\gamma, q_\gamma) = (p(u_1^\gamma), q(u_2^\gamma))$

$$\left\{ \begin{array}{l} -Dp_\gamma - \operatorname{div}(r(x)\nabla p_\gamma) - a(x)p_\gamma - b(x)q_\gamma = y(t, u_1^\gamma, 0) - y_d + \rho_\gamma(t), (x, t, a) \in Q, \\ -Dq_\gamma - \operatorname{div}(\tilde{r}(x)\nabla q_\gamma) - \tilde{a}(x)p_\gamma - \tilde{b}(x)q_\gamma = z(t, u_2^\gamma, 0) - z_d + \varsigma_\gamma(t), (x, t, a) \in Q, \\ p_\gamma(x, t, a) = 0, q_\gamma(x, t, a) = 0, (x, a) \in Q_A, \\ p_\gamma(x, t, A) = 0, q_\gamma(x, t, A) = 0, (x, t) \in Q_t, \\ p_\gamma(x, t, a) = q_\gamma(x, t, a) = 0 \text{ on } \Sigma. \end{array} \right.$$

Again it's easy to get

$$\begin{aligned} 0 &= \iiint_Q (-Dp_\gamma - \operatorname{div}(r(x)\nabla p_\gamma) - a(x)p_\gamma - b(x)q_\gamma) y(t, v_1 - u_1^\gamma, 0) dx dadt \\ &= \iiint_Q p_\gamma(v_1 - u_1^\gamma) dx dadt - \int_Q (y(t, u_1^\gamma, 0) - y_d + \rho_\gamma(t)) y(t, v_1 - u_1^\gamma, 0) dx dadt, \end{aligned}$$

and

$$\begin{aligned} 0 &= \iiint_Q (-Dq_\gamma - \operatorname{div}(\tilde{r}(x)\nabla q_\gamma) - \tilde{a}(x)p_\gamma - \tilde{b}(x)q_\gamma) z(t, v_2 - u_2^\gamma, 0) dx dadt \\ &= \iiint_Q q_\gamma(v_2 - u_2^\gamma) dx dadt - \int_Q (z(t, u_2^\gamma, 0) - z_d + \varsigma_\gamma(t)) z(t, v_2 - u_2^\gamma, 0), \end{aligned}$$

then

$$\begin{aligned} J'_\gamma(u_1^\gamma, u_2^\gamma)(v_1 - u_1^\gamma, v_2 - u_2^\gamma) &= (p_\gamma + Nu_1^\gamma, v_1 - u_1^\gamma)_{L^2(Q)} + (q_\gamma + Nu_2^\gamma, v_2 - u_2^\gamma)_{L^2(Q)} \\ &= 0 \quad \forall (v_1, v_2) \in L^2(Q) \times L^2(Q). \end{aligned}$$

Finally, the coupled optimality system, which characterized the coupled low-regret control is

$$\left\{ \begin{array}{l} Dy_\gamma - \operatorname{div}(r(x)\nabla y_\gamma) - a(x)y_\gamma - b(x)z_\gamma = u_1^\gamma, (x, t, a) \in Q, \\ Dz_\gamma - \operatorname{div}(\tilde{r}(x)\nabla z_\gamma) - \tilde{a}(x)y_\gamma - \tilde{b}(x)z_\gamma = u_2^\gamma, (x, t, a) \in Q, \\ y_\gamma(x, 0, a) = 0, z_\gamma(x, 0, a) = 0, (x, a) \in Q_A, \\ y_\gamma(x, t, 0) = y_\gamma(x, t), z_\gamma(x, t, 0) = z_\gamma(x, t), (x, t) \in Q_t, \\ y_\gamma(x, t, a) = z_\gamma(x, t, a) = 0 \text{ on } \Sigma, \end{array} \right.$$

$$\left\{ \begin{array}{l} -D\xi_\gamma - \operatorname{div}(r(x)\nabla \xi_\gamma) - a(x)\xi_\gamma - b(x)\varsigma_\gamma = y(t, u_1^\gamma, 0) - y(t, 0, 0), (x, t, a) \in Q, \\ -D\varsigma_\gamma - \operatorname{div}(\tilde{r}(x)\nabla \varsigma_\gamma) - \tilde{a}(x)\xi_\gamma - \tilde{b}(x)\varsigma_\gamma = z(t, u_2^\gamma, 0) - z(t, 0, 0), (x, t, a) \in Q, \\ \xi_\gamma(x, t, a) = 0, \varsigma_\gamma(x, t, a) = 0, (x, a) \in Q_A, \\ \xi_\gamma(x, t, A) = 0, \varsigma_\gamma(x, t, A) = 0, (x, t) \in Q_t, \\ \xi_\gamma(x, t, a) = \varsigma_\gamma(x, t, a) = 0 \text{ on } \Sigma, \end{array} \right.$$

$$\left\{ \begin{array}{l} D\rho_\gamma - \operatorname{div}(r(x)\nabla \rho_\gamma) - a(x)\rho_\gamma - b(x)\mu_\gamma = 0, (x, t, a) \in Q, \\ D\mu_\gamma - \operatorname{div}(\tilde{r}(x)\nabla \mu_\gamma) - \tilde{a}(x)\rho_\gamma - \tilde{b}(x)\mu_\gamma = 0, (x, t, a) \in Q, \\ \rho_\gamma(x, 0, a) = \frac{1}{\gamma}\xi(x, 0, a) (u_1^\gamma), \mu_\gamma(x, 0, a) = \frac{1}{\gamma}\varsigma(x, 0, a) (u_2^\gamma), (x, a) \in Q_A, \\ \rho_\gamma(x, t, 0) = 0, \mu_\gamma(x, t, 0) = 0, (x, t) \in Q_t, \\ \rho_\gamma(x, t, a) = \mu_\gamma(x, t, a) = 0 \text{ on } \Sigma, \end{array} \right.$$

$$\left\{ \begin{array}{l} -Dp_\gamma - \operatorname{div}(r(x)\nabla p_\gamma) - a(x)p_\gamma - b(x)q_\gamma = y(t, u_1^\gamma, 0) - y_{obs} + \rho_\gamma, (x, t, a) \in Q, \\ -Dq_\gamma - \operatorname{div}(\tilde{r}(x)\nabla q_\gamma) - \tilde{a}(x)p_\gamma - \tilde{b}(x)q_\gamma = z(t, u_2^\gamma, 0) - z_{obs} + \varsigma_\gamma(t), (x, t, a) \in Q, \\ p_\gamma(x, t, a) = 0(t), q_\gamma(x, t, a) = 0, (x, a) \in Q_A, \\ p_\gamma(x, t, a) = 0, q_\gamma(x, t, A) = 0, (x, t) \in Q_t, \\ p_\gamma(x, t, a) = q_\gamma(x, t, a) = 0 \text{ on } \Sigma, \end{array} \right.$$

with

$$(p_\gamma + Nu_1^\gamma, v_1 - u_1^\gamma)_{L^2(Q)} + (q_\gamma + Nu_2^\gamma, v_2 - u_2^\gamma)_{L^2(Q)} = 0, \forall (v_1, v_2) \in (L^2(Q) \times L^2(Q)).$$

### 3.3.4 Full characterization of the coupled no-regret control

Now let's show that the coupled low-regret control sequence converges to the unique coupled of no-regret control, based on this assumption we can construct the optimality system of the coupled no-regret control.

**Theorem 3.3.2** *The coupled sequences of low-regret control  $(u_1^\gamma, u_2^\gamma)$  converges to the unique coupled of no-regret control  $(u_1, u_2)$  which characterized by the following coupled optimality system :*

$$\left\{ \begin{array}{l} Dy - \operatorname{div}(r(x)\nabla y) - a(x)y - b(x)z = u_1, (x, t, a) \in Q, \\ Dz - \operatorname{div}(\tilde{r}(x)\nabla z) - \tilde{a}(x)y - \tilde{b}(x)z = u_2, (t, a, x) \in Q, \\ y(x, 0, t) = 0, z(x, 0, t) = 0, (x, t) \in Q_A, \\ y(x, t, 0) = y(x, t), z(x, t, 0) = z(x, t), (x, t) \in Q_t, \\ y(x, t, a) = z(x, t, a) = 0 \text{ on } \Sigma, \end{array} \right.$$

$$\left\{ \begin{array}{l} -D\xi - \operatorname{div}(r(x)\nabla\xi) - a(x)\xi - b(x)\varsigma = y(t, u_1, 0) - y(t, 0, 0), (x, t, a) \in Q, \\ -D\varsigma - \operatorname{div}(\tilde{r}(x)\nabla\varsigma) - \tilde{a}(x)\xi - \tilde{b}(x)\varsigma = z(t, u_2, 0) - z(t, 0, 0), (x, t, a) \in Q, \\ \xi(x, t, a) = 0, \varsigma(x, t, a) = 0, (x, a) \in Q_A, \\ \xi(x, t, A) = 0, \varsigma(x, t, A) = 0, (x, t) \in Q_t, \\ \xi(x, t, a) = \varsigma(x, t, a) = 0 \text{ on } \Sigma, \end{array} \right.$$

$$\left\{ \begin{array}{l} D\rho - \operatorname{div}(r(x)\nabla\rho) - a(x)\rho - b(x)\mu = 0, (x, t, a) \in Q, \\ D\mu - \operatorname{div}(\tilde{k}(x)\nabla\mu) - \tilde{a}(x)\rho - \tilde{b}(x)\mu = 0, (x, t, a) \in Q, \\ \rho(x, 0, a) = \lambda, \mu(x, 0, a) = \delta, (x, a) \in Q_A, \\ \rho(x, t, 0) = 0, \mu(x, t, 0) = 0, (x, t) \in Q_t, \\ \rho(x, t, a) = \mu(x, t, a) = 0 \\ \text{on } \Sigma, \end{array} \right.$$

$$\left\{ \begin{array}{l} -Dp - \operatorname{div}(r(x)\nabla p) - a(x)p - b(x)q = y(t, u_1, 0) - y_{obs} + \rho(t), (x, t, a) \in Q, \\ -Dq - \operatorname{div}(\tilde{r}(x)\nabla q) - \tilde{a}(x)p - \tilde{b}(x)q = z(t, u_2) - z_{obs} + \mu(t), (x, t, a) \in Q, \\ p(x, t, a) = 0, q(x, t, a) = 0, (x, a) \in Q_A, \\ p(x, t, A) = 0, q(x, t, A) = 0, (x, t) \in Q_t, \\ p(x, t, a) = q(x, t, a) = 0 \text{ on } \Sigma, \end{array} \right.$$

with

$$(p + Nu_1, v_1 - u_1)_{L^2(Q)} + (q + Nu_2, v_2 - u_2)_{L^2(Q)} = 0 \quad \forall (v_1, v_2) \in (L^2(Q) \times L^2(Q)).$$

Where  $(u_1, u_2)$  is the best approximate estimation of the unknown pair source term, i.e;

$$(u_1, u_2) \simeq (f_1, f_2) .$$

**Proof.** With the assumption that  $(u_1^\gamma, u_2^\gamma)$  is the minimum of  $J_\gamma$ , so

$$J_\gamma(y, z, u_1^\gamma, u_2^\gamma) \leq J_\gamma(y, z, 0, 0) = 0,$$

which implies

$$\begin{aligned} & \|y(t, u_1^\gamma, 0) - y_{obs}\|_{L^2(Q)}^2 + \|z(t, u_2^\gamma, 0) - z_{obs}\|_{L^2(Q)}^2 + N(\|u_1^\gamma\|_{L^2(Q)}^2 + \|u_2^\gamma\|_{L^2(Q)}^2) \\ & + \frac{1}{\gamma} \left( \|\xi_\gamma(x, 0, a)\|_{L^2(Q_A)}^2 + \|\varsigma_\gamma(x, 0, a)\|_{L^2(Q_A)}^2 \right) \\ & \leq J(y, z, 0, 0, 0, 0) = \text{constant}, \end{aligned}$$

this gives the following bounds

$$\begin{aligned} \|u_1^\gamma\|_{L^2(Q)} & \leq c, \quad \|u_2^\gamma\|_{L^2(Q)} \leq c, \\ \|y(t, u_1^\gamma, 0)\|_{L^2(Q)} & \leq c, \quad \|z(t, u_2^\gamma, 0)\|_{L^2(Q)} \leq c, \\ \|\xi_\gamma(0, x, a)\|_{L^2(Q_A)} & \leq c\sqrt{\gamma}, \quad \|\varsigma_\gamma(0, x, a)\|_{L^2(Q_A)} \leq c\sqrt{\gamma}, \end{aligned}$$

where  $c$  is a constant independent of  $\gamma$ . We have that

$$\left\{ \begin{array}{l} Dy_\gamma - \operatorname{div}(r(x)\nabla y_\gamma) - a(x)y_\gamma - b(x)z_\gamma = u_1^\gamma, (x, t, a) \in Q, \\ Dz_\gamma - \operatorname{div}(\tilde{r}(x)\nabla z_\gamma) - \tilde{a}(x)y_\gamma - \tilde{b}(x)z_\gamma = u_2^\gamma, (x, t, a) \in Q, \\ y_\gamma(x, 0, a) = 0, z_\gamma(x, 0, a) = 0, (x, a) \in Q_A, \\ y_\gamma(x, t, 0) = y_\gamma(x, t), z_\gamma(x, t, 0) = z_\gamma(x, t), (x, t) \in Q_t, \\ y_\gamma(x, t, a) = z_\gamma(x, t, a) = 0 \text{ on } \Sigma. \end{array} \right.$$

Multiply the previous coupled system by the following coupled state  $(y_\gamma, z_\gamma)$  and integrate over  $Q$ , apply Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|y_\gamma\|_{L^2(Q_A)}^2 + \|z_\gamma\|_{L^2(Q_A)}^2 + \|y_\gamma\|_{L^2(Q_t)}^2 + \|z_\gamma\|_{L^2(Q_t)}^2 &\leq \|u_1^\gamma\|_{L^2(Q)}^2 + \|u_2^\gamma\|_{L^2(Q)}^2 + \|y_\gamma\|_{L^2(Q_t)}^2 + \\ &\|z_\gamma\|_{L^2(Q_t)}^2 + c_1(\|y_\gamma\|_{L^2(Q)}^2 + c_2\|z_\gamma\|_{L^2(Q)}^2), \end{aligned}$$

then, by Gronwall lemma

$$\|y_\gamma\|_{L^\infty(0,t,H^1(Q_A))} \leq c, \quad \|z_\gamma\|_{L^\infty(0,t,H^1(Q_A))} \leq c,$$

also, we have

$$\begin{aligned} \|Dy_\gamma - \operatorname{div}(k(x)\nabla y_\gamma) - \mu(x)y_\gamma - \alpha(x)z_\gamma\|_{L^2(Q)} &\leq c, \\ \|Dz_\gamma - \operatorname{div}(\tilde{k}(x)\nabla z_\gamma) - \tilde{\mu}(x)y_\gamma - \tilde{\alpha}(x)z_\gamma\|_{L(Q)} &\leq c. \end{aligned}$$

From the previous estimation and by the same way in the previous section, we deduce the following weak convergences

$$\begin{aligned} u_1^\gamma &\rightharpoonup u_1 \text{ in } L^2(Q), \quad u_2^\gamma \rightharpoonup u_2 \text{ in } L^2(Q), \\ y_\gamma &\rightharpoonup y \text{ in } L^2(Q), \quad z_\gamma \rightharpoonup z \text{ in } L^2(Q), \end{aligned}$$

and

$$\begin{aligned} Dy_\gamma - \operatorname{div}(r(x)\nabla y_\gamma) - a(x)y_\gamma - b(x)z_\gamma &\rightharpoonup Dy - \operatorname{div}(r(x)\nabla y) - a(x)y - b(x)z \text{ in } Q, \\ Dz_\gamma - \operatorname{div}(\tilde{r}(x)\nabla z_\gamma) - \tilde{a}(x)y_\gamma - \tilde{b}(x)z_\gamma &\rightharpoonup Dz - \operatorname{div}(\tilde{r}(x)\nabla z) - \tilde{a}(x)y - \tilde{b}(x)z \text{ in } Q, \\ y(x, 0, a) = 0, z(x, 0, a) = 0, (x, a) &\in Q_A, \\ y(x, t, 0) = y(x, t), z(x, t, 0) = z(x, t), (x, t) &\in Q_t, \\ y(x, t, a) = z(x, t, a) = 0 \text{ on } \Sigma. \end{aligned}$$

Also, we have the following bounds

$$\begin{aligned} \|\xi_\gamma\|_{L^2(Q)} &\leq c, \quad \|\varsigma_\gamma\|_{L^2(Q)} \leq c, \\ \|-D\xi_\gamma - \operatorname{div}(k(x)\nabla\xi_\gamma) - \mu(x)\xi_\gamma - \alpha(x)\varsigma_\gamma\|_{L^2(Q)} &\leq c, \\ \|-D\varsigma_\gamma - \operatorname{div}(\tilde{k}(x)\nabla\varsigma_\gamma) - \tilde{\mu}(x)\xi_\gamma - \tilde{\alpha}(x)\varsigma_\gamma\|_{L^2(Q)} &\leq c, \end{aligned}$$

we deduce that :

$$\begin{aligned} \xi_\gamma &\rightharpoonup \xi \text{ weakly in } L^2(Q), \quad \varsigma_\gamma \rightharpoonup \varsigma \text{ weakly in } L^2(Q), \\ -D\xi_\gamma - \operatorname{div}(r(x)\nabla\xi_\gamma) - a(x)\xi_\gamma - b(x)\varsigma_\gamma &\rightharpoonup -D\xi - (r(x)\nabla\xi) - a(x)\xi - b(x)\varsigma \text{ in } Q \\ -D\varsigma_\gamma - \operatorname{div}(\tilde{r}(x)\nabla\varsigma_\gamma) - \tilde{a}(x)\xi_\gamma - \tilde{b}(x)\varsigma_\gamma &\rightharpoonup -D\varsigma - \operatorname{div}(r(x)\nabla\varsigma) - a(x)\xi - b(x)\varsigma \text{ in } Q, \\ \xi(x, t, a) = 0, \quad \varsigma(x, t, a) = 0, \quad (x, a) &\in Q_A, \\ \xi(x, t, 0) = 0, \quad \varsigma(x, t, 0) = 0, \quad (x, t) &\in Q_t, \\ \xi(x, t, a) = \varsigma(x, t, a) = 0 \text{ on } \Sigma, \\ \xi_\gamma(x, 0, a) &\rightarrow 0 \text{ strongly in } L^2(Q_A), \quad \varsigma_\gamma(x, 0, a) \rightarrow 0 \text{ strongly in } L^2(Q_A). \end{aligned}$$

This means that the couple  $(u_1, u_2)$  is a no-regret control. The remainder of this proof will be devised into two steps :

**Step 1 :** We have the following important estimation we obtained like the previous states

$$\begin{aligned} \|\rho_\gamma\|_{L^2(Q)} &\leq c, \quad \|\mu_\gamma\|_{L^2(Q)} \leq c, \\ \|D\rho_\gamma - \operatorname{div}(r(x)\nabla\rho_\gamma) - a(x)\rho_\gamma - b(x)\mu_\gamma\|_{L^2(Q)} &\leq c, \\ \|D\mu_\gamma - \operatorname{div}(\tilde{r}(x)\nabla\mu_\gamma) - \tilde{a}(x)\rho_\gamma - \tilde{b}(x)\mu_\gamma\|_{L^2(Q)} &\leq c, \\ \frac{1}{\gamma} \|\xi_\gamma(x, 0, a)\|_{L^2(Q_A)} &\leq \frac{1}{\gamma} \|\xi_\gamma(x, 0, a)\|_{L^2(Q_A)}^2 \leq c, \\ \frac{1}{\gamma} \|\varsigma_\gamma(x, 0, a)\|_{L^2(Q_A)} &\leq \frac{1}{\gamma} \|\varsigma_\gamma(x, 0, a)\|_{L^2(Q_A)}^2 \leq c, \end{aligned}$$

which implies the following convergence

$$\begin{aligned} \rho_\gamma &\rightharpoonup \rho \text{ weakly in } L^2(Q), \quad \mu_\gamma \rightharpoonup \mu \text{ weakly in } L^2(Q), \\ D\rho_\gamma - \operatorname{div}(r(x)\nabla\rho_\gamma) - a(x)\rho_\gamma - b(x)\mu_\gamma &\rightharpoonup D\rho - \operatorname{div}(r(x)\nabla\rho) - a(x)\rho - b(x)\mu \text{ in } Q, \\ D\mu_\gamma - \operatorname{div}(\tilde{r}(x)\nabla\mu_\gamma) - \tilde{a}(x)\rho_\gamma - \tilde{b}(x)\mu_\gamma &\rightharpoonup D\mu - \operatorname{div}(\tilde{r}(x)\nabla\mu) - \tilde{a}(x)\rho - \tilde{b}(x)\mu \text{ in } Q, \\ \frac{1}{\gamma} \xi_\gamma(x, 0, a) &\rightharpoonup \lambda \text{ weakly in } Q_A, \quad \frac{1}{\gamma} \varsigma_\gamma(x, 0, a) \rightharpoonup \delta \text{ weakly in } Q_A, \end{aligned}$$

we get the following coupled system governed  $(\rho, \mu)$

$$\left\{ \begin{array}{l} D\rho - \operatorname{div}(r(x)\nabla\rho) - a(x)\rho - b(x)\mu = 0, (x, t, a) \in Q, \\ D\mu - \operatorname{div}(\tilde{r}(x)\nabla\mu) - \tilde{a}(x)\rho - \tilde{b}(x)\mu = 0, (x, t, a) \in Q, \\ \rho(x, 0, a) = \lambda, \mu(x, 0, a) = \delta, (x, a) \in Q_A, \\ \rho(x, t, 0) = 0, \mu(x, t, 0) = 0, (x, t) \in Q_t, \\ \rho(t, a, x) = \mu(t, a, x) = 0 \text{ on } \Sigma. \end{array} \right.$$

**Step 2 :** We know that  $(\rho, \mu) \in (L^2(Q))^2$  and  $(y(t, u_1^\gamma, 0) - y_{obs}, z(t, u_2^\gamma, 0) - z_{obs}) \in (L^2(Q))^2$  requires to  $(y(t, u_1^\gamma, 0) - y_{obs} + \rho_\gamma, z(t, u_2, 0) - z_{obs} + \mu_\gamma) \in (L^2(Q))^2$ . It is easy to get

$$\begin{aligned} \|p_\gamma\|_{L^2(Q)} &\leq c, \|q_\gamma\|_{L^2(Q)} \leq c, \\ \|Dp_\gamma - \operatorname{div}(r(x)\nabla p_\gamma) - a(x)p_\gamma - b(x)q_\gamma\|_{L^2(Q)} &\leq c, \\ \|Dq_\gamma - \operatorname{div}(\tilde{r}(x)\nabla q_\gamma) - \tilde{a}(x)p_\gamma - \tilde{b}(x)q_\gamma\|_{L^2(Q)} &\leq c. \end{aligned}$$

As usual, we deduce the following convergence

$$\begin{aligned} p_\gamma &\rightharpoonup p \text{ weakly in } L^2(Q), \quad q_\gamma \rightharpoonup q \text{ weakly in } L^2(Q), \\ Dp_\gamma - \operatorname{div}(r(x)\nabla p_\gamma) - a(x)p_\gamma - b(x)q_\gamma &\rightharpoonup Dp - \operatorname{div}(r(x)\nabla p) - a(x)p - b(x)q \text{ weakly in } Q, \\ Dq_\gamma - \operatorname{div}(\tilde{r}(x)\nabla q_\gamma) - \tilde{a}(x)p_\gamma - \tilde{b}(x)q_\gamma &\rightharpoonup Dq - \operatorname{div}(\tilde{r}(x)\nabla q) - \tilde{a}(x)p - \tilde{b}(x)q \text{ weakly in } Q, \end{aligned}$$

we have that

$$\|y(t, u_1^\gamma, 0) - y_{obs}\|_{L^2(Q)} \leq c, \quad \|z(t, u_2^\gamma, 0) - z_{obs}\|_{L^2(Q)} \leq c,$$

which implies that

$$\begin{aligned} y(t, u_1^\gamma, 0) - y_{obs} + \rho_\gamma(t) &\rightharpoonup y - y_{obs} + \rho(t) \text{ weakly in } Q, \\ z(t, u_2^\gamma, 0) - z_{obs} + \mu_\gamma(t) &\rightharpoonup z - z_{obs} + \mu(t) \text{ weakly in } Q. \end{aligned}$$

Hence, using the same way as in the previous coupled states, and we pass to limit we get the coupled

system governed  $(p, q)$

$$\left\{ \begin{array}{l} -Dp - \operatorname{div}(r(x)\nabla p) - a(x)p - b(x)q = y - y_{obs} + \rho(t), (x, t, a) \in Q, \\ -Dq - \operatorname{div}(\tilde{r}(x)\nabla q) - \tilde{a}(x)p - \tilde{b}(x)q = z - z_{obs} + \varsigma(t), (x, t, a) \in Q, \\ p(x, t, a) = 0, q(x, t, a) = 0, (x, t) \in Q_A, \\ p(x, t, A) = 0, q(x, t, A) = 0, (x, t) \in Q_t, \\ p(x, t, a) = q(x, t, a) = 0 \text{ on } \Sigma, \end{array} \right.$$

with the optimality condition is given by

$$(p + Nu_1, v_1 - u_1)_{L^2(Q)} + (q + Nu_2, v_2 - u_2)_{L^2(Q)} = 0 \quad \forall (v_1, v_2) \in (L^2(Q) \times L^2(Q)).$$

■

**Remark 3.3.1** *In the case when we take the final measurement  $(y(T, v_1, g_1), z(T, v_2, g_2)) = (y_{obs}, z_{obs})$  we shall assume that the initial condition  $(g_1, g_2)$  belongs to  $H_0^1(\Omega) \times H_0^1(\Omega)$  to accurate that  $(y(t, v_1, g_1), y(t, v_2, g_2))$  is continuous and belongs to  $L^2(0, T, H^2(\Omega)) \times L^2(0, T, H^2(\Omega))$ .*

## Chapitre 4

# Optimal control of bilinear systems with incomplete data

This chapter introduces new research that may be published in the future, addressing the optimal control of a bilinear problem with missing data, various kinds of literature published done on optimal control of bilinear systems for example [3, 4, 11]. Our approach involves linearizing the bilinear system using an iterative sequence that converges to a linear problem.

## 4.1 Statement of the problem

Let  $V$  be a separable Hilbert space and  $E$  a subspace of  $V$  which is a reflexive Banach space continuously and densely embedded in  $V$ . Identifying  $V$  and  $E$  with their duals  $V'$  and  $E'$  resp.  $Z$  and  $G$  be an observations and uncertainties spaces resp, and  $A \in L(E)$  be a linear partially differential operator makes an isomorphism on  $E$  and  $A^{-1}$  is a linear bounded operators,  $\mathcal{B} : E \rightarrow V$  is a linear and continuous,  $C \in L(Z, E)$  and  $\beta \in L(E, G)$  are observation and uncertainties operators. We introduce the spaces  $L^2(H) = L^2(0, T; H)$ ,  $L^2(E) = L^2(0, T; E)$ , The control  $v(t) \in L^\infty(0, T)$ ,  $f \in E$  is a source function, and  $g$  is the uncertainty in  $G$ . The corresponding state  $y = y(v, g; x, t)$  satisfies the bilinear equation with incomplete data is given by

$$\begin{cases} Ay = f + v\mathcal{B}y + \beta g, \\ y(0) = y_0, \end{cases} \quad (4.1.1)$$

such that  $Ay = y' + \mathcal{A}y$  and  $y' = \frac{\partial y}{\partial t}$ .

**Theorem 4.1.1** *The equation 4.1.1 has a unique continuous solution  $y$  satisfying*

$$y \in L^\infty(V) \cap L^2(E).$$

**Proof.** see [?] ■

We associate with the state equation 4.1.1 by the following cost function

$$j(v, g) = \|Cy(v, g) - y_d\|_Z^2 + N\|v\|_{L^2(0, T)}^2, \quad (4.1.2)$$

where  $y_d \in Z$  is a fixed observation,  $N > 0$ .

We want to study the optimal control of the problem 4.1.1 and 4.1.2. Since 4.1.1 is a bilinear problem. We try to linearise using an iterative coupled sequences, and we supposed that the characterisation of the linear optimal control problem converges to the bilinear optimal control problem.

**Theorem 4.1.2** *Given  $y_0 \in E$  we define the sequence  $(v_n, y_n)$  solution of the following linearized*

problem

$$\begin{cases} Ay_n = f + v_n \mathcal{B}y_{n-1} + \beta g, \\ y_n(0) = y_n^0, \end{cases} \quad (4.1.3)$$

Where  $(v_n, y_n)$  converges weakly to  $(v, y)$  in  $L^\infty(0, T) \times L^2(E)$  is the solution of the bilinear problem.

**Proof.** Let  $(v_n, y_n)$  be a bounded sequences in  $L^\infty(0, T) \times L^2(E)$ , then it exist a constant  $M$  such that  $\|v_n\|_{L^\infty(0, T)} < M$ , there exist a subsequence still denoted by  $(v_n)$  such that

$$v_n \rightharpoonup v^* \text{ weakly in } L^\infty(0, T),$$

we have that

$$y_n \rightharpoonup y^* \text{ weakly in } E, \quad (4.1.4)$$

From the continuity of the operators  $A$  and  $\mathcal{B}$ , we deduce that  $\mathcal{B}y_n \rightharpoonup \mathcal{B}y^*$  weakly in  $V$   $Ay_n \rightharpoonup Ay^*$  weakly in  $V$ .

From another side, we have

$$\begin{aligned} \|v_n \mathcal{B}y_{n-1} - v^* \mathcal{B}y^*\|_V &= \|v_n \mathcal{B}(y_{n-1} - y^*) + (v_n - v^*) \mathcal{B}y^*\|_V \leq \|v_n \mathcal{B}(y_{n-1} - y^*)\|_V + \|(v_n - v^*) \mathcal{B}y^*\|_V \\ &\leq \|v_n\|_{L^\infty(0, T)} \|\mathcal{B}(y_{n-1} - y^*)\|_V + \|v_n - v^*\|_{L^\infty(0, T)} \|\mathcal{B}y^*\|_V \end{aligned}$$

which gives that  $v_n \mathcal{B}y_{n-1} \rightharpoonup v^* \mathcal{B}y^*$  weakly in  $V$ .

We pass to limit in problem 4.1.3, we get

$$Ay^* = f + v^* \mathcal{B}y^* + \beta g(t),$$

Let  $\phi \in D'(Q)$  be a test function such that  $\phi(x, T) = 0$  on  $V$ .

We multiply the first equation in 4.1.3 by  $\phi$ , we get

$$\langle Ay_n, \phi \rangle_{L^2(E)} = \langle f + v_n \mathcal{B}y_{n-1} + \beta g, \phi \rangle_{L^2(E)}$$

this last one becomes

$$\langle y_n, A^* \phi \rangle_{L^2(E)} - \langle y_n^0, \phi \rangle_V = \langle f + v_n \mathcal{B} y_{n-1} + \beta g, \phi \rangle_{L^2(E)},$$

by passing to limit and integrate another time, we get  $y_n^0 \rightharpoonup y^*(0) = y_0^*$  weakly in  $E$ . which means that  $(y^*, v^*)$  is the solution of

$$\begin{cases} Ay^* = f + v^* \mathcal{B} y^* + \beta g, \\ y^*(0) = y_0^*. \end{cases}$$

by the uniqueness of the solution we deduce that  $(v^*, y^*) = (v, y)$  ■

## 4.2 Optimal control of the linearized problem

We consider the following optimal control problem

$$\inf_{v_n \in L^2(0,T)} j_n(v_n, g), \quad (4.2.1)$$

where  $j_n(v_n, g)$  given by

$$j_n(v_n, g) = \|C y_n(v_n, g) - z\|_Z^2 + N \|v_n\|_{L^2(0,T)}^2, \quad (4.2.2)$$

with the following linear equation

$$\begin{cases} Ay_n = f + B v_n + \beta g, \\ y_n(0) = y_n^0, \end{cases} \quad (4.2.3)$$

where

$$B v_n = v_n \mathcal{B} y_{n-1}.$$

### The no-regret control of linearized problem

**Definition 4.2.1** We say that  $u_n \in L^2(0, T)$  is a no-regret control for 4.2.1 and 4.2.3 with 4.2.2 if  $u_n$  solution of

$$\inf_{v_n \in L^2(0,T)} (\sup_{g \in G} (j_n(v_n, g) - j_n(0, g))).$$

We know that the no-regret control belongs only to a set that is impractical to characterize. To address this difficulty, we introduce the concept of low-regret control.

**Definition 4.2.2** We say that  $u_n^\gamma \in L^2(0, T)$  is a low-regret control for 4.2.1 and 4.2.3 with 4.2.2 if  $u_n^\gamma$  is a solution of

$$\inf_{v_n \in L^2(0, T)} (\sup_{g \in G} (j_n(v_n, g) - j_n(0, g) - \gamma \|g\|_G^2)).$$

**Proposition 4.2.1** The characteristics of low-regret control of linear problem are determined by the following optimality system

$$\left\{ \begin{array}{l} Ay_n^\gamma = f + Bu_n^\gamma, y_n^\gamma(0) = y_{0n}^\gamma, \\ A^* \xi_n^\gamma = C^* C(y_n^\gamma - y_n^\gamma(0, 0)), \xi_n^\gamma(T) = 0, \\ A\rho_n^\gamma = \lambda_n^\gamma, \rho_n^\gamma(0) = 0, \\ A^* p_n^\gamma = C^*(Cy_n^\gamma - y_d^\gamma) + C^* C\rho_n^\gamma, p_n^\gamma(T) = 0, \\ (p_n^\gamma, B(v_n^\gamma - u_n^\gamma))_E + (Nu_n^\gamma, v_n^\gamma - u_n^\gamma)_{L^2(0, T)} = 0, \end{array} \right. \quad (4.2.4)$$

where

$$B(v_n^\gamma - u_n^\gamma) = (v_n^\gamma - u_n^\gamma) \mathcal{B} y_{n-1}^\gamma (v_n^\gamma - u_n^\gamma, 0),$$

and  $y_n^\gamma(u_n^\gamma, 0) = y_n^\gamma$ ,  $\xi_n^\gamma(u_n^\gamma) = \xi_n^\gamma$ .

**Proof.** see [18, 34]. ■

**Proposition 4.2.2** The characteristic of no-regret control of linear problem is established by taking the limit in the optimal system 4.2.4, resulting in the following optimality system.

$$\left\{ \begin{array}{l} Ay_n = f + Bu_n, y_n(0) = y_{n0}, \\ A^* \xi_n = C^* C(y_n - y_n(0, 0)), \xi_n(T) = 0, \\ A\rho_n = \lambda_n, \rho_n(0) = 0, \\ A^* p_n = C^*(Cy_n - y_d) + C^* C\rho_n, p_n(T) = 0, \\ (p_n, \mathcal{B}(v_n - u_n))_E + (Nu_n, v_n - u_n)_{L^2(0, T)} = 0 \end{array} \right. \quad (4.2.5)$$

where

$$B(v_n - u_n) = (v_n - u_n) \mathcal{B} y_{n-1} (v_n - u_n, 0),$$

and  $y_n(u_n, 0) = y_n$ ,  $\xi_n(u_n) = \xi_n$ .

**Proof.** see [18, 34]. ■

### 4.3 Optimality system of the bilinear system

Here, we shall give an optimality system characterizing the no-regret control of the bilinear problem. From theorem 4.1.2 we know that the sequence  $(v_n, y_n)$  converges to  $(v, y)$ , then the optimality system of the bilinear system is obtained as a limit of the optimality system of the linearized problem.

**Theorem 4.3.1** *The optimal control of the bilinear system 4.1.1 with 4.1.2 is characterized by the following optimality system*

$$\left\{ \begin{array}{l} Ay = f + Bu, \quad y(0) = y_0, \\ A^*\xi = C^*C(y - y(0, 0)), \quad \xi(T) = 0, \\ A\rho = \lambda, \quad \rho(0) = 0, \\ A^*p = C^*(Cy - y_d) + C^*C\rho, \quad p(T) = 0, \\ (p, B(v - u))_E + (Nu, v - u)_{L^2(0, T)} = 0, \end{array} \right. \quad (4.3.1)$$

where  $Bu = u\mathcal{B}y$  and  $B(v - u) = (v - u)\mathcal{B}y(v - u, 0)$ .

**Proof.** We pass to limit in the first equation in 4.2.4 and we follow the same steps in theorem 4.1.2, we get

$$\left\{ \begin{array}{l} Ay = f + u\mathcal{B}y, \\ y(0) = y_0. \end{array} \right.$$

By the same way, we get the following convergences

$$\xi_n \rightharpoonup \xi \text{ weakly in } E,$$

$$\rho_n \rightharpoonup \rho \text{ weakly in } E,$$

$$p_n \rightharpoonup p \text{ weakly in } E,$$

Since  $A$  and  $A^*$  are a continuous maps

$$A^*\xi_n \rightharpoonup A^*\xi \text{ weakly in } E, \quad (4.3.2)$$

$$A\rho_n \rightharpoonup A\rho \text{ weakly in } E, \quad (4.3.3)$$

$$A^*p_n \rightharpoonup A^*p \text{ weakly in } E, \quad (4.3.4)$$

Then

$$\lambda_n \rightharpoonup \lambda \text{ weakly in } E,$$

Also, according to continuity of the operators  $C$  and  $C^*$ , we deduce

$$\begin{aligned} C^*C(y_n - y_n(0,0)) &\rightharpoonup C^*C(y - y(0,0)) \text{ weakly in } E, \\ C^*(Cy_n - y_d) + C^*C\rho_n &\rightharpoonup C^*(Cy - y_d) + C^*C\rho \text{ weakly in } E. \end{aligned}$$

According to the unicity of limit, we get :

$$\left\{ \begin{array}{l} Ay = f + Bu, y(0) = y_0, \\ A^*\xi = C^*C(y - y(0,0)), \xi(T) = 0, \\ A\rho = \lambda, \rho(0) = 0, \\ A^*p = C^*(Cy - y_d) + C^*C\rho, p(T) = 0, \\ (p, B(v - u))_E + (Nu, v - u)_{L^2(0,T)} = 0, \end{array} \right. \quad (4.3.5)$$

where  $Bu = u\mathcal{B}y$  and  $B(v - u) = (v - u)\mathcal{B}y(v - u, 0)$ .

for the initial condition, we choose a test function and we follow the same way in theorem 4.1.2 we get  $\xi(T) = 0, \rho(0) = 0, p(T) = 0$  ■

# Conclusion

In this dissertation, we have thoroughly investigated the unique solvability and weak controllability of a singular degenerate fractional problem, we have proved the existence and uniqueness of the strong solution using the energy inequality method. By leveraging the Tikhonov regularization method, we established weak controllability. This method has provided significant insights into controllability problem, enabling us to use it in complicated mathematical models. Additionally, we addressed identification problems, which we studied in two different ways. First, we introduced the sentinel method and applied it to diseases problems, where we identify the unknown infected people. Then, a new idea that we adopted was to transform the identification problem into an optimal control problem with missing data to determine the source term in a coupled population densities problem. This idea provides a better estimation than the sentinel method. The results obtained from this allow us to address numerous challenges across different fields to gain accurate information.

For future research, the optimal control of bilinear problems with incomplete data is a really interesting field to explore and develop techniques for solving . Advancing this field will significantly enhance optimal control theory, opening new pathways for improving practical applications across a variety of disciplines.

# Appendices

**Theorem 4.3.2 (Green Formulas)** Let  $\Omega \subset \mathbb{R}^n$  a bounded and regular domain, and the normal vector unit to the outside  $\Gamma = \partial\Omega$ . So, for  $u \in H^1(\Omega)$  and  $v \in H^2(\Omega)$  we have : the first Green's formula

$$\int_{\Omega} u \Delta v dx = - \int_{\Omega} \nabla u \nabla v dx + \int_{\Gamma} u \frac{\partial v}{\partial \nu} d\Gamma,$$

for  $u, v \in H^2(\Omega)$  we have the second formula of Green's :

$$\int_{\Omega} (u \Delta v - v \Delta u) dx = \int_{\Gamma} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} d\Gamma.$$

**Definition 4.3.1** Let  $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$  is a proper function, so Legendre transform  $f^*$  of  $f$  is a function of  $E \rightarrow \mathbb{R} \cup \{+\infty\}$  defined by

$$f^*(p) = \sup_{x \in E} ((p, x) - f(x))$$

**Lemma 4.3.1 (Gronwall)** Let  $\Psi, G$  be continuous in  $[0, T]$ , with  $G$  nondecreasing and  $\gamma > 0$ . If

$$\Psi(t) \leq G(t) + \gamma \int_0^t \Psi(s) ds \quad \text{for all } t \in [0, T],$$

then

$$\Psi(t) \leq G(t) \exp(\gamma t), \quad \text{for all } t \in [0, T].$$

**Proposition 4.3.1 (Cauchy inequality)**

Let  $a, b$  are any real numbers and  $p, q$  are real numbers connected by the relationship  $\frac{1}{p} + \frac{1}{q} = 1$ .

Then we have the Cauchy inequality

$$ab \leq \frac{1}{2} (a^2 + b^2).$$

**Proposition 4.3.2** (*Poincare inequality*)

For  $1 \leq p < \infty$ , and  $\Omega$  a subset bounded at least in one direction. there exists a constant  $C$  depending only on  $\Omega$  and  $p$ , so that, for every function  $u \in W_0^{1,p}(\Omega)$  such that

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

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