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# Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type

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**Abstract**

We prove a common fixed point theorem of Gregus type for four mappings satisfying a contractive condition of integral type in metric spaces using the concept of weak compatibility which generalizes Theorem 2 of [A. Djoudi, L. Nisse, Gregus type fixed points for weakly compatible mappings, Bull. Belg. Math. Soc. 10 (2003) 369–378] and other papers. We prove also common fixed point theorems of Gregus type using a strict contractive condition of integral type, a property (E.A) and a common property (E.A) introduced by [M. Aamri, D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002) 181–188] and [W. Liu, J. Wu, Z. Li, Common fixed points of single-valued and multi-valued maps, Int. J. Math. Math. Sci. 19 (2005) 3045–3055], respectively.


*Keywords:* Weakly compatible mappings; Common fixed point; Metric space; Property (E.A); Common property (E.A)

## 1. Introduction

Let $S$ and $T$ be self-mappings of a metric space $(X, d)$. $S$ and $T$ are commuting if $STx = TSx$ for all $x \in X$. Sessa [17] defined $S$ and $T$ to be weakly commuting if for all $x \in X$,

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$$d(STx, TSx) \leqslant d(Tx, Sx). \tag{1.1}$$

Jungck [5] defined $S$ and $T$ to be compatible as a generalization of weakly commuting if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0, \tag{1.2}$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

It is easy to show that commuting implies weakly commuting implies compatible and there are examples in the literature verifying that the inclusions are proper. See [5] and [17].

Jungck et al. [6] defined $S$ and $T$ to be compatible mappings of type (A) if

$$\lim_{n\to\infty} d\big(STx_n, T^2x_n\big) = 0 \quad \text{and} \quad \lim_{n\to\infty} d\big(TSx_n, S^2x_n\big) = 0, \tag{1.3}$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly, weakly commuting implies compatible of type (A). By [6], the converse is not true. Examples are given to show that the two concepts of compatibility are independent. See [6].

Recently, Pathak and Khan [12] defined $S$ and $T$ to be compatible mappings of type (B) as a generalization of compatible mappings of type (A) if

$$\lim_{n\to\infty} d\big(TSx_n, S^2x_n\big) \leqslant \frac{1}{2}\Big[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d\big(Tt, T^2x_n\big)\Big] \quad \text{and}$$
$$\lim_{n\to\infty} d\big(STx_n, T^2x_n\big) \leqslant \frac{1}{2}\Big[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d\big(St, S^2x_n\big)\Big], \tag{1.4}$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly, compatible mappings of type (A) are compatible mappings of type (B), but the converse is not true. See [11]. However, compatibility, compatibility of type (A) and compatibility of type (B) are equivalent if $S$ and $T$ are continuous. See [12].

Pathak et al. [13] defined $S$ and $T$ to be compatible mappings of type (P) if

$$\lim_{n\to\infty} d\big(S^2x_n, T^2x_n\big) = 0, \tag{1.5}$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

However, compatibility, compatibility of type (A) and compatibility of type (P) are equivalent if $S$ and $T$ are continuous. See [13].

Pathak et al. [14] defined $S$ and $T$ to be compatible mappings of type (C) as a generalization of compatible mappings of type (A) if

$$\lim_{n\to\infty} d\big(TSx_n, S^2x_n\big) \leqslant \frac{1}{3}\Big[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d\big(Tt, S^2x_n\big) + \lim_{n\to\infty} d\big(Tt, T^2x_n\big)\Big],$$
$$\lim_{n\to\infty} d\big(STx_n, T^2x_n\big) \leqslant \frac{1}{3}\Big[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d\big(St, T^2x_n\big) + \lim_{n\to\infty} d\big(St, S^2x_n\big)\Big], \tag{1.6}$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Compatibility, compatibility of type (A) and compatibility of type (C) are equivalent if $S$ and $T$ are continuous. See [14].

## 2. Preliminaries

**Definition 1.** Jungck and Rhoades [7] defined $S$ and $T$ to be weakly compatible if they commute at their coincidence points; i.e., if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

**Lemma 1.** [5,6,12–14] *If $S$ and $T$ are compatible, or compatible of type* (A)*, or compatible of type* (P)*, or compatible of type* (B)*, or compatible of type* (C)*, then they are weakly compatible.*

The following example shows that the converse is not true in general.

**Example.** Let $(X,d) = ([0,10], |.|)$. Define $S$ and $T$ by

$$Sx = \begin{cases} 3 & \text{if } x \in (0,2], \\ 0 & \text{if } x \in \{0\} \cup (2,10], \end{cases} \qquad Tx = \begin{cases} 0 & \text{if } x = 0, \\ x+8 & \text{if } x \in (0,2], \\ x-2 & \text{if } x \in (2,10]. \end{cases}$$

We have $Sx = Tx$ iff $x = 0$. $ST(0) = T(0) = 0$, $TS(0) = S(0) = 0$. Then, $\{S,T\}$ is weakly compatible. Let $\{x_n\}$ be a sequence in $X$ defined by: $x_n = 2 + \frac{1}{n}$, $n \geqslant 1$. $Sx_n = S(2 + \frac{1}{n}) = 0$, $Tx_n = T(2 + \frac{1}{n}) = \frac{1}{n}$. $Sx_n, Tx_n \to t = 0$ as $n \to \infty$. $STx_n = S(\frac{1}{n}) = 3$, $TSx_n = T(0) = 0$, $|STx_n - TSx_n| \to 3 \neq 0$. So, $\{S,T\}$ is not compatible.

$S^2x_n = S(0) = 0$, $T^2x_n = T(\frac{1}{n}) = 8 + \frac{1}{n}$. Therefore, $|TSx_n - S^2x_n| = 0$ and $|STx_n - T^2x_n| = 5 + \frac{1}{n} \to 5 \neq 0$. Then, $\{S,T\}$ is not compatible of type (A).

$$\lim_{n\to\infty} \left|STx_n - T^2x_n\right| = 5 > \frac{1}{2}\Big[\lim_{n\to\infty} |STx_n - St| + \lim_{n\to\infty} \left|St - S^2x_n\right|\Big] = \frac{3}{2}.$$

Hence, $\{S,T\}$ is not compatible of type (B). $\lim_{n\to\infty} |S^2x_n - T^2x_n| = 8 \neq 0$. Therefore, $\{S,T\}$ is not compatible of type (P).

$$\lim_{n\to\infty} \left|STx_n - T^2x_n\right| = 5 > \frac{1}{3}\Big[\lim_{n\to\infty} |STx_n - St| + \lim_{n\to\infty} \left|St - T^2x_n\right| + \lim_{n\to\infty} \left|St - S^2x_n\right|\Big] = \frac{11}{3}.$$

So, $\{S,T\}$ is not compatible of type (C).

**Definition 2.** Pant [10] defined $S$ and $T$ to be $R$-weakly commuting at a point $x \in X$ if for some $R > 0$,

$$d(STx, TSx) \leqslant Rd(Tx, Sx). \tag{2.1}$$

**Definition 3.** $S$ and $T$ are pointwise $R$-weakly commuting on $X$ if given $x \in X$, there exists $R > 0$ such that (2.1) holds.

It is proved in [11] that $R$-weak commutativity is equivalent to commutativity at coincidence points; i.e., $S$ and $T$ are pointwise $R$-weakly commuting if and only if they are weakly compatible.

M. Aamri and D. El Moutawakil [1] defined property (E.A) as follows.

**Definition 4.** Let $S, T : X \to X$. The pair $(S,T)$ satisfies property (E.A) if there exists a sequence $\{x_n\}$ in $X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \in X. \tag{2.2}$$

It is clear from the definition of compatibility that the pair $(S,T)$ of a metric space $(X,d)$ is noncompatible if there exists at least one sequence $\{x_n\}$ in $X$ such that (2.2) holds but,

$\lim_{n\to\infty} d(STx_n, TSx_n)$ is either nonzero or does not exist. Therefore, two noncompatible mappings of a metric space $(X,d)$ satisfy property (E.A).

Recently, Y. Liu et al. [9] defined a common property (E.A) as follows.

**Definition 5.** Let $A, S, B, T : X \to X$. The pairs $(A,S)$ and $(B,T)$ satisfy a common property (E.A) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t \in X. \tag{2.3}$$

If $B = A$ and $T = S$ in (2.3), we obtain the definition of property (E.A).

Gregus [4] proved the following theorem.

**Theorem 1.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$ and $T$ be a mapping of $C$ into itself satisfying the inequality*

$$\|Tx - Ty\| \leqslant a\|x - y\| + b\|x - Tx\| + c\|y - Ty\|$$

*for all $x, y$ in $C$, where $a > 0$, $b, c \geqslant 0$, $a + b + c = 1$. Then, $T$ has a unique fixed point.*

Several authors have generalized Theorem 1. See [3,8,12,14].

Let $\mathbb{R}_+$ be the set of nonnegative real numbers and $F$ the family of mappings $\varphi$ from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that each $\varphi$ is upper semicontinuous, nondecreasing and $\varphi(t) < t$ for all $t > 0$.

The following theorem has been proved in [3].

**Theorem 2.** *Let $A$, $B$, $S$ and $T$ be mappings from a Banach space $X$ into itself satisfying*

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X),$$

$$\begin{aligned} \|Ax - By\|^p \leqslant \varphi\biggl(a\|Sx - Ty\|^p + (1-a)\max\biggl\{&\alpha\|Ax - Sx\|^p, \beta\|By - Ty\|^p, \\ &\|Ax - Sx\|^{\frac{p}{2}} \cdot \|Ax - Ty\|^{\frac{p}{2}}, \|Ax - Ty\|^{\frac{p}{2}} \cdot \|Sx - By\|^{\frac{p}{2}}, \\ &\frac{1}{2}\bigl(\|Ax - Sx\|^p + \|By - Ty\|^p\bigr)\biggr\}\biggr) \end{aligned} \tag{2.4}$$

*for all $x, y$ in $X$, where $0 < a \leqslant 1$, $0 < \alpha$, $\beta \leqslant 1$, $p \geqslant 1$ and $\varphi \in F$. If $A(X)$ or $B(X)$ is closed and the pairs $(A,S)$ and $(B,T)$ are weakly compatible, then $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$.*

Since for any two real nonnegative numbers $b$ and $c$, $\frac{b+c}{2} \leqslant \max\{b, c\}$ and $\varphi$ is nondecreasing, therefore the right-hand side of (2.4) implies the following inequality:

$$\begin{aligned} \|Ax - By\|^p \leqslant \varphi\bigl(a\|Sx - Ty\|^p + (1-a)\max\bigl\{&\|Ax - Sx\|, \|By - Ty\|, \\ &\|Ax - Sx\|^{\frac{1}{2}} \cdot \|Ax - Ty\|^{\frac{1}{2}}, \|Sx - By\|^{\frac{1}{2}} \cdot \|Ax - Ty\|^{\frac{1}{2}}\bigr\}^p\bigr). \end{aligned} \tag{2.5}$$

**Lemma 2.** [18] *For any $t > 0$, $\varphi(t) < t$ iff $\lim_{n\to+\infty} \varphi^n(t) = 0$, where $\varphi^n$ denotes the $n$-times repeated composition of $\varphi$ with itself.*

## 3. Main results

Let $A$, $B$, $S$ and $T$ be mappings from a metric space $(X,d)$ into itself such that

$$A(X)\subset T(X) \quad\text{and}\quad B(X)\subset S(X), \tag{3.1}$$

$$\begin{aligned}\left(\int_0^{d(Ax,By)}\psi(t)\,dt\right)^p \leqslant \varphi\Bigg[a\left(\int_0^{d(Sx,Ty)}\psi(t)\,dt\right)^p &+ (1-a)\max\Bigg\{\int_0^{d(Ax,Sx)}\psi(t)\,dt,\\ &\int_0^{d(By,Ty)}\psi(t)\,dt, \left(\int_0^{d(Ax,Sx)}\psi(t)\,dt\right)^{\frac12}\cdot\left(\int_0^{d(Ax,Ty)}\psi(t)\,dt\right)^{\frac12},\\ &\left(\int_0^{d(Sx,By)}\psi(t)\,dt\right)^{\frac12}\cdot\left(\int_0^{d(Ax,Ty)}\psi(t)\,dt\right)^{\frac12}\Bigg\}^p\Bigg]\end{aligned} \tag{3.2}$$

for all $x, y$ in $X$, where $0 < a \leqslant 1$, $p \geqslant 1$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue integrable mapping which is summable nonnegative and such that

$$\int_0^{\epsilon}\psi(t)\,dt > 0 \quad \text{for each } \epsilon > 0. \tag{3.3}$$

By (3.1), for an arbitrary $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$, for this point $x_1$ we can choose a point $x_2$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in $X$ such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \quad\text{and}\quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \tag{3.4}$$

for all $n = 0, 1, 2, \ldots$.

**Lemma 3.** *Let $A$, $B$, $S$ and $T$ be mappings from a metric space $(X,d)$ into itself satisfying* (3.1) *and* (3.2) *for all $x, y$ in $X$, where $0 < a \leqslant 1$, $p \geqslant 1$ and $\psi$ satisfies* (3.3). *Then, the sequence $\{y_n\}$ defined by* (3.4) *is a Cauchy sequence in $X$.*

**Proof.** First of all, assume that $y_n \neq y_{n+1}$ for all $n$.

By (3.2) and (3.4) we have

$$\begin{aligned}\left(\int_0^{d(y_{2n},y_{2n+1})}\psi(t)\,dt\right)^p &= \left(\int_0^{d(Ax_{2n},B_{2n+1})}\psi(t)\,dt\right)^p\\ &\leqslant \varphi\Bigg[a\left(\int_0^{d(y_{2n-1},y_{2n})}\psi(t)\,dt\right)^p\\ &\quad + (1-a)\max\Bigg\{\int_0^{d(y_{2n-1},y_{2n})}\psi(t)\,dt, \int_0^{d(y_{2n+1},y_{2n})}\psi(t)\,dt\Bigg\}^p\Bigg].\end{aligned}$$

If

$$\int_0^{d(y_{2n-1},y_{2n})} \psi(t)\,dt \leqslant \int_0^{d(y_{2n+1},y_{2n})} \psi(t)\,dt$$

in the above inequality, then

$$\left(\int_0^{d(y_{2n+1},y_{2n})} \psi(t)\,dt\right)^p \leqslant \varphi\left(\left(\int_0^{d(y_{2n+1},y_{2n})} \psi(t)\,dt\right)^p\right) < \left(\int_0^{d(y_{2n+1},y_{2n})} \psi(t)\,dt\right)^p$$

which is a contradiction. Therefore

$$\left(\int_0^{d(y_{2n},y_{2n+1})} \psi(t)\,dt\right) \leqslant \varphi\left(\left(\int_0^{d(y_{2n-1},y_{2n})} \psi(t)\,dt\right)^p\right).$$

Similarly, we get

$$\left(\int_0^{d(y_{2n+1},y_{2n+2})} \psi(t)\,dt\right)^p \leqslant \varphi\left(\left(\int_0^{d(y_{2n},y_{2n+1})} \psi(t)\,dt\right)^p\right).$$

By induction we obtain

$$\left(\int_0^{d(y_n,y_{n+1})} \psi(t)\,dt\right)^p \leqslant \varphi\left(\left(\int_0^{d(y_{n-1},y_n)} \psi(t)\,dt\right)^p\right) \leqslant \cdots \leqslant \varphi^n\left(\left(\int_0^{d(y_0,y_1)} \psi(t)\,dt\right)^p\right). \tag{3.5}$$

By Lemma 2, it follows that

$$\lim_{n\to\infty} \int_0^{d(y_n,y_{n+1})} \psi(t)\,dt = 0 \tag{3.6}$$

and (3.3) implies that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{3.7}$$

Now, we show that $\{y_n\}$ is a Cauchy sequence in $X$. By (3.7), it suffices to show that the subsequence $\{y_{2n}\}$ of $\{y_n\}$ is a Cauchy sequence in $X$. Suppose not. As in [3] we have

$$d(y_{2n(k)}, y_{2m(k)}) \to \varepsilon \quad \text{as } k \to \infty, \tag{3.8}$$

$$d(y_{2n(k)}, y_{2m(k)-1}) \to \varepsilon \quad \text{and} \quad d(y_{2n(k)+1}, y_{2m(k)-1}) \to \varepsilon \quad \text{as } k \to \infty. \tag{3.9}$$

Using (3.4) we have

$$d(y_{2n(k)}, y_{2m(k)}) \leqslant d(y_{2n(k)}, y_{2n(k)+1}) + d(Ax_{2m(k)}, Bx_{2n(k)+1}).$$

By (3.6) and (3.2) we get

$$\lim_{k\to\infty}\int_0^{d(y_{2n(k)},y_{2m(k)})}\psi(t)\,dt$$
$$\leqslant \lim_{k\to\infty}\int_0^{d(Ax_{2m(k)},Bx_{2n(k)+1})}\psi(t)\,dt$$
$$\leqslant \lim_{k\to\infty}\left(\varphi\left[a\left(\int_0^{d(Sx_{2m(k)},Tx_{2n(k)+1})}\psi(t)\,dt\right)^p\right.\right.$$
$$+(1-a)\max\left\{\int_0^{d(Ax_{2m(k)},Sx_{2m(k)})}\psi(t)\,dt,\ \int_0^{d(Bx_{2n(k)+1},Tx_{2n(k)+1})}\psi(t)\,dt,\right.$$
$$\left(\int_0^{d(Ax_{2m(k)},Sx_{2m(k)})}\psi(t)\,dt\right)^{\frac{1}{2}}\cdot\left(\int_0^{d(Ax_{2m(k)},Tx_{2n(k)+1})}\psi(t)\,dt\right)^{\frac{1}{2}},$$
$$\left.\left.\left.\left(\int_0^{d(Sx_{2m(k)},Bx_{2n(k)+1})}\psi(t)\,dt\right)^{\frac{1}{2}}\cdot\left(\int_0^{d(Ax_{2m(k)},Tx_{2n(k)+1})}\psi(t)\,dt\right)^{\frac{1}{2}}\right\}^p\right]\right)^{\frac{1}{p}}$$
$$=\lim_{k\to\infty}\left(\varphi\left[a\left(\int_0^{d(y_{2m(k)-1},y_{2n(k)})}\psi(t)\,dt\right)^p+(1-a)\max\left\{\int_0^{d(y_{2m(k)},y_{2m(k)-1})}\psi(t)\,dt,\right.\right.\right.$$
$$\int_0^{d(y_{2n(k)+1},y_{2n(k)})}\psi(t)\,dt,\left(\int_0^{(y_{2m(k)},y_{2m(k)-1})}\psi(t)\,dt\right)^{\frac{1}{2}}\cdot\left(\int_0^{d(y_{2m(k)},y_{2n(k)})}\psi(t)\,dt\right)^{\frac{1}{2}},$$
$$\left.\left.\left.\left(\int_0^{d(y_{2m(k)-1},y_{2n(k)+1})}\psi(t)\,dt\right)^{\frac{1}{2}}\cdot\left(\int_0^{d(y_{2m(k)},y_{2n(k)})}\psi(t)\,dt\right)^{\frac{1}{2}}\right\}^p\right]\right)^{\frac{1}{p}}.$$

Using (3.6), (3.8) and (3.9) we have as $k\to\infty$

$$\int_0^{\epsilon}\psi(t)\,dt\leqslant\left[\varphi\left(a\left(\int_0^{\epsilon}\psi(t)\,dt\right)^p+(1-a)\left(\int_0^{\epsilon}\psi(t)\,dt\right)^p\right)\right]^{\frac{1}{p}}<\int_0^{\epsilon}\psi(t)\,dt$$

which is a contradiction. Therefore, $\{y_n\}$ is a Cauchy sequence in $X$. □

**Theorem 3.** *Let $A$, $B$, $S$ and $T$ be mappings from a metric space $(X,d)$ into itself satisfying* (3.1), (3.2) *for all $x$, $y$ in $X$, where $0<a\leqslant 1$, $p\geqslant 1$ and $\psi$ satisfies* (3.3). *Suppose that one of $S(X)$ or $T(X)$ is complete and the pairs $(A,S)$ and $(B,T)$ are weakly compatible. Then, $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$.*

**Proof.** By Lemma 3, the sequence $\{y_{2n+1}\} = \{Sx_{2n+2}\} \subset S(X)$ is a Cauchy sequence in $S(X)$. Since $S(X)$ is complete, it converges to a point $z = Su$ for some $u \in X$. Hence, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Tx_{2n+1}\}$ also converge to $z$.

If $Au \neq z$, using (3.2) we get

$$\left(\int_0^{(Au,Bx_{2n+1})} \psi(t)\,dt\right)^p \leqslant \varphi\Bigg[a\left(\int_0^{d(Su,Tx_{2n+1})} \psi(t)\,dt\right)^p + (1-a)\max\Bigg\{\int_0^{d(Au,Su)} \psi(t)\,dt, \int_0^{d(Bx_{2n+1},Tx_{2n+1})} \psi(t)\,dt, \left(\int_0^{d(Au,Su)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au,Tx_{2n+1})} \psi(t)\,dt\right)^{\frac{1}{2}}, \left(\int_0^{d(Su,Bx_{2n+1})} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au,Tx_{2n+1})} \psi(t)\,dt\right)^{\frac{1}{2}}\Bigg\}^p\Bigg].$$

Letting $n \to \infty$ we obtain

$$\left(\int_0^{d(Au,z)} \psi(t)\,dt\right)^p \leqslant \varphi\left[(1-a)\left(\int_0^{d(Au,z)} \psi(t)\,dt\right)^p\right] < \left(\int_0^{d(Au,z)} \psi(t)\,dt\right)^p$$

which is a contradiction. Then

$$\int_0^{d(Au,z)} \psi(t)\,dt = 0,$$

and (3.3) implies that $z = Au = Su$.

Since $A(X) \subset T(X)$, there exists $v \in X$ such that $z = Tv$.

If $z \neq Bv$, using (3.2) we have

$$\left(\int_0^{d(z,Bv)} \psi(t)\,dt\right)^p = \left(\int_0^{d(Au,Bv)} \psi(t)\,dt\right)^p \leqslant \varphi\Bigg[a\left(\int_0^{d(Su,Tv)} \psi(t)\,dt\right)^p + (1-a)\max\Bigg\{\int_0^{d(Au,Su)} \psi(t)\,dt, \int_0^{d(Bv,Tv)} \psi(t)\,dt, \left(\int_0^{d(Au,Su)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au,Tv)} \psi(t)\,dt\right)^{\frac{1}{2}}, \left(\int_0^{d(Su,Bv)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au,Tv)} \psi(t)\,dt\right)^{\frac{1}{2}}\Bigg\}^p\Bigg]$$

$$= \varphi\left[(1-a)\left(\int_0^{d(z,Bv)} \psi(t)\,dt\right)^p\right]$$
$$< \left(\int_0^{d(z,Bv)} \psi(t)\,dt\right)^p$$

which is a contradiction. Therefore, $z = Bv = Tv$.

Since the pair $(A, S)$ is weakly compatible, we have $SAu = ASu$; i.e., $Az = Sz$.

If $Az \neq z$, using (3.2) we obtain

$$\left(\int_0^{d(Az,z)} \psi(t)\,dt\right)^p = \left(\int_0^{d(Az,Bv)} \psi(t)\,dt\right)^p$$
$$\leqslant \varphi\left[a\left(\int_0^{d(Sz,Tv)} \psi(t)\,dt\right)^p + (1-a)\max\left\{\int_0^{d(Az,Sz)} \psi(t)\,dt,\right.\right.$$
$$\int_0^{d(Bv,Tv)} \psi(t)\,dt, \left(\int_0^{d(Az,Sz)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Az,Tv)} \psi(t)\,dt\right)^{\frac{1}{2}},$$
$$\left.\left.\left(\int_0^{d(Sz,Bv)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Az,Tv)} \psi(t)\,dt\right)^{\frac{1}{2}}\right\}^p\right]$$
$$< \left(\int_0^{d(Az,z)} \psi(t)\,dt\right)^p$$

which is a contradiction. Hence, $z = Az = Sz$.

Similarly, we can prove that $z = Bz = Tz$.

Suppose there exists $n$ such that $y_n = y_{n+1}$. Therefore, $y_n = y_{n+k}$ for $k \geqslant 1$. So, there exist $u, v \in X$ such that $Au = Su$ and $Bv = Tv$. As in Theorem 3, we can prove that $z = Az = Bz = Tz$.

The uniqueness of $z$ follows from (3.2). □

If $B = A$ and $T = S$ in Theorem 3, we get the following corollary.

**Corollary 1.** *Let $A$ and $S$ be mappings from a metric space $(X, d)$ into itself satisfying*

$$A(X) \subset S(X), \tag{3.10}$$

$$\left(\int_0^{d(Ax,Ay)} \psi(t)\,dt\right)^p \leqslant \varphi\left[a\left(\int_0^{d(Sx,Sy)} \psi(t)\,dt\right)^p + (1-a)\max\left\{\int_0^{d(Ax,Sx)} \psi(t)\,dt,\right.\right.$$
$$\int_0^{d(Ay,Sy)} \psi(t)\,dt, \left(\int_0^{d(Ax,Sx)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax,Sy)} \psi(t)\,dt\right)^{\frac{1}{2}},$$

$$\left(\int_0^{d(Sx,Ay)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax,Sy)} \psi(t)\,dt\right)^{\frac{1}{2}}\Bigg\}^{p}\Bigg]$$

*for all $x, y$ in $X$, where $0 < a \leqslant 1$, $p \geqslant 1$ and $\psi$ satisfies* (3.3). *Suppose that $S(X)$ is complete and the pair $(A, S)$ is weakly compatible. Then, $A$ and $S$ have a unique common fixed point in $X$.*

If $S = I_X$ in Corollary 1, where $I_X$ is the identity mapping in $X$, we get the following corollary.

**Corollary 2.** *Let $A$ be a mapping from a Banach space $(X, d)$ into itself satisfying*

$$\left(\int_0^{d(Ax,Ay)} \psi(t)\,dt\right)^{p} \leqslant \varphi\Bigg[a\left(\int_0^{d(x,y)} \psi(t)\,dt\right)^{p} + (1-a)\max\Bigg\{\int_0^{d(x,Ax)} \psi(t)\,dt,$$
$$\int_0^{d(y,Ay)} \psi(t)\,dt, \left(\int_0^{d(x,Ax)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(y,Ax)} \psi(t)\,dt\right)^{\frac{1}{2}},$$
$$\left(\int_0^{d(x,Ay)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(y,Ax)} \psi(t)\,dt\right)^{\frac{1}{2}}\Bigg\}^{p}\Bigg]$$

*for all $x, y$ in $X$, where $0 < a \leqslant 1$, $p \geqslant 1$ and $\psi$ satisfies* (3.3). *Then, $A$ has a unique fixed point in $X$.*

If $\psi(t) = 1$ in Theorem 3, we obtain Theorem 2 of [3].

If $p = a = 1$ and $\varphi(t) = kt$, $0 < k < 1$ in Corollary 2, we obtain Theorem 2.1 of [2].

Now, we prove a common fixed point Theorem of Gregus type using a strict contraction of integral type and property (E.A).

**Theorem 4.** *Let $A$, $B$, $S$ and $T$ be mappings from a metric space $(X, d)$ into itself satisfying* (3.1) *and*

$$\int_0^{d(Ax,By)} \psi(t)\,dt < a\int_0^{d(Sx,Ty)} \psi(t)\,dt + (1-a)\max\Bigg\{\int_0^{d(Ax,Sx)} \psi(t)\,dt,$$
$$\int_0^{d(By,Ty)} \psi(t)\,dt, \left(\int_0^{d(Ax,Sx)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax,Ty)} \psi(t)\,dt\right)^{\frac{1}{2}},$$
$$\left(\int_0^{d(Sx,By)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax,Ty)} \psi(t)\,dt\right)^{\frac{1}{2}}\Bigg\} \tag{3.11}$$

*for all $x, y$ in $X$ for which the right-hand side of* (3.11) *is positive, where $0 < a < 1$ and $\psi$ satisfies* (3.3). *Suppose that $(A, S)$ or $(B, T)$ satisfies property* (E.A)*, one of $A(X)$, $B(X)$, $S(X)$, $T(X)$ is a closed subspace of $X$ and the pairs $(A, S)$ and $(B, T)$ are weakly compatible. Then, $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$.*

**Proof.** Suppose that $(B,T)$ satisfies property (E.A). Then there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Therefore, $\lim_{n\to\infty} d(Bx_n, Tx_n) = 0$. Since $B(X) \subset S(X)$, there exists in $X$ a sequence $\{y_n\}$ such that $Bx_n = Sy_n$.

Hence, $\lim_{n\to\infty} Sy_n = z$. Let us show that $\lim_{n\to\infty} Ay_n = z$.

Suppose that $\limsup_{n\to\infty} d(Ay_n, z) = \varepsilon > 0$. Using (3.11) we get

$$\int_0^{d(Ay_n,Bx_n)} \psi(t)\,dt < a \int_0^{d(Sy_n,Tx_n)} \psi(t)\,dt + (1-a)\max\Biggl\{ \int_0^{d(Ay_n,Sy_n)} \psi(t)\,dt, \int_0^{d(Bx_n,Tx_n)} \psi(t)\,dt, \left(\int_0^{d(Ay_n,Sy_n)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ay_n,Tx_n)} \psi(t)\,dt\right)^{\frac{1}{2}}, \left(\int_0^{d(Sy_n,Bx_n)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ay_n,Tx_n)} \psi(t)\,dt\right)^{\frac{1}{2}} \Biggr\}$$

$$= a \int_0^{d(Bx_n,Tx_n)} \psi(t)\,dt + (1-a)\max\Biggl\{ \int_0^{d(Ay_n,Bx_n)} \psi(t)\,dt, \int_0^{d(Bx_n,Tx_n)} \psi(t)\,dt, \left(\int_0^{d(Ay_n,Bx_n)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ay_n,Tx_n)} \psi(t)\,dt\right)^{\frac{1}{2}} \Biggr\}.$$

Taking the limit as $n \to \infty$, we obtain

$$\int_0^{\varepsilon} \psi(t)\,dt \leqslant (1-a)\int_0^{\varepsilon} \psi(t)\,dt < \int_0^{\varepsilon} \psi(t)\,dt$$

which is a contradiction. Hence

$$\int_0^{\varepsilon} \psi(t)\,dt = 0$$

and (3.3) implies that $\varepsilon = 0$; i.e., $\lim_{n\to\infty} Ay_n = z$.

Suppose that $S(X)$ is a closed subspace of $X$. Then, $z = Su$ for some $u \in X$.

If $Au \neq z$, using (3.11) we get

$$\int_0^{d(Au,Bx_{2n+1})} \psi(t)\,dt < a \int_0^{d(Su,Tx_{2n+1})} \psi(t)\,dt + (1-a)\max\Biggl\{ \int_0^{d(Au,Su)} \psi(t)\,dt, \int_0^{d(Bx_{2n+1},Tx_{2n+1})} \psi(t)\,dt, \left(\int_0^{d(Au,Su)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au,Tx_{2n+1})} \psi(t)\,dt\right)^{\frac{1}{2}},$$

$$\left(\int_0^{d(Su,Bx_{2n+1})}\psi(t)\,dt\right)^{\frac{1}{2}}\cdot\left(\int_0^{d(Au,Tx_{2n+1})}\psi(t)\,dt\right)^{\frac{1}{2}}\Bigg\}.$$

Letting $n\to\infty$ we obtain

$$\int_0^{d(Au,z)}\psi(t)\,dt\leqslant(1-a)\int_0^{d(Au,z)}\psi(t)\,dt<\int_0^{d(Au,z)}\psi(t)\,dt$$

which is a contradiction. Then, $z=Au=Su$.

Since $A(X)\subset T(X)$, there exists $v\in X$ such that $z=Tv$.

If $z\neq Bv$, using (3.11) we have

$$\begin{aligned}\int_0^{d(z,Bv)}\psi(t)\,dt&=\int_0^{d(Au,Bv)}\psi(t)\,dt\\&<a\int_0^{d(Su,Tv)}\psi(t)\,dt+(1-a)\max\Bigg\{\int_0^{d(Au,Su)}\psi(t)\,dt,\\&\quad\int_0^{d(Bv,Tv)}\psi(t)\,dt,\left(\int_0^{d(Au,Su)}\psi(t)\,dt\right)^{\frac{1}{2}}\cdot\left(\int_0^{d(Au,Tv)}\psi(t)\,dt\right)^{\frac{1}{2}},\\&\quad\left(\int_0^{d(Su,Bv)}\psi(t)\,dt\right)^{\frac{1}{2}}\cdot\left(\int_0^{d(Au,Tv)}\psi(t)\,dt\right)^{\frac{1}{2}}\Bigg\}\\&=(1-a)\int_0^{d(z,Bv)}\psi(t)\,dt\\&<\int_0^{d(z-Bv)}\psi(t)\,dt\end{aligned}$$

which is a contradiction. Therefore, $z=Bv=Tv$.

Since the pair $(A,S)$ is weakly compatible, we have $SAu=ASu$; i.e., $Az=Sz$.

If $Az\neq z$, using (3.11) we obtain

$$\begin{aligned}\int_0^{d(Az,z)}\psi(t)\,dt&=\int_0^{d(Az,Bv)}\psi(t)\,dt\\&<a\int_0^{d(Sz,Tv)}\psi(t)\,dt+(1-a)\max\Bigg\{\int_0^{d(Az,Sz)}\psi(t)\,dt,\\&\quad\int_0^{d(Bv,Tv)}\psi(t)\,dt,\left(\int_0^{d(Az,Sz)}\psi(t)\,dt\right)^{\frac{1}{2}}\cdot\left(\int_0^{d(Az,Tv)}\psi(t)\,dt\right)^{\frac{1}{2}},\end{aligned}$$

$$\left(\int_0^{d(Sz,Bv)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Az,Tv)} \psi(t)\,dt\right)^{\frac{1}{2}}\Bigg\}$$

$$= \int_0^{d(Az,z)} \psi(t)\,dt$$

which is a contradiction. Hence, $z = Az = Sz$.

Similarly, we can prove that $z = Bz = Tz$. The uniqueness of $z$ follows from (3.11). □

If we let $B = A$ and $T = S$ in Theorem 4, we get the following corollary.

**Corollary 3.** *Let $A$ and $S$ be mappings from a metric space $(X, d)$ into itself satisfying* (3.10) *and*

$$\int_0^{d(Ax,Ay)} \psi(t)\,dt < a \int_0^{d(Sx,Sy)} \psi(t)\,dt + (1-a)\max\Bigg\{\int_0^{d(Ax,Sx)} \psi(t)\,dt,$$
$$\int_0^{d(Ay,Sy)} \psi(t)\,dt, \left(\int_0^{d(Ax,Sx)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax,Sy)} \psi(t)\,dt\right)^{\frac{1}{2}},$$
$$\left(\int_0^{d(Sx,Ay)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax,Sy)} \psi(t)\,dt\right)^{\frac{1}{2}}\Bigg\} \tag{3.12}$$

*for all $x, y$ in $X$ for which the right-hand side of* (3.12) *is positive, where $0 < a < 1$ and $\psi$ satisfies* (3.3)*. Suppose that $(A, S)$ satisfies property* (E.A)*, $A(X)$ or $S(X)$ is a closed subspace of $X$ and the pair $(A, S)$ is weakly compatible. Then, $A$ and $S$ have a unique common fixed point in $X$.*

If $\psi(t) = 1$ in Theorem 4, we get the following corollary.

**Corollary 4.** *Let $A$, $B$, $S$ and $T$ be mappings from a metric space $(X, d)$ into itself satisfying* (3.1) *and*

$$d(Ax, By) < ad(Sx, Ty) + (1-a)\max\big\{d(Ax, Sx), d(By, Ty),$$
$$d(Ax, Sx)^{\frac{1}{2}} \cdot d(Ax, Ty)^{\frac{1}{2}}, d(Ax, Ty)^{\frac{1}{2}} \cdot d(Sx, By)^{\frac{1}{2}}\big\} \tag{3.13}$$

*for all $x, y$ in $X$ for which the right-hand side of* (3.13) *is positive. Suppose that $(A, S)$ or $(B, T)$ satisfies property* (E.A)*, one of $A(X)$, $B(X)$, $S(X)$, $T(X)$ is a closed subspace of $X$ and the pairs $(A, S)$ and $(B, T)$ are weakly compatible. Then, $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$.*

Now, we prove a common fixed point Theorem of Gregus type using a strict contraction of integral type and a common property (E.A).

**Theorem 5.** *Let $A$, $B$, $S$ and $T$ be mappings from a metric space $(X, d)$ into itself satisfying* (3.11) *for all $x, y$ in $X$ for which the right-hand side of* (3.11) *is positive, where $0 < a < 1$ and $\psi$ satisfies* (3.3). *Suppose that $(A, S)$ and $(B, T)$ satisfy a common property* (E.A)*, $S(X)$ and $T(X)$ are closed subspaces of $X$ and the pairs $(A, S)$ and $(B, T)$ are weakly compatible. Then, $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$.*

**Proof.** Suppose that $(A, S)$ and $(B, T)$ satisfy a common property (E.A). Then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$ for some $z \in X$. Assume that $S(X)$ and $T(X)$ are closed subspaces of $X$. Then, $z = Su = Tv$ for some $u, v \in X$. If $Au \neq z$, using (3.11) we get

$$\int_0^{d(Au,By_n)} \psi(t)\,dt < a \int_0^{d(Su,Ty_n)} \psi(t)\,dt + (1-a)\max\Bigg\{ \int_0^{d(Au,Su)} \psi(t)\,dt, \int_0^{d(By_n,Ty_n)} \psi(t)\,dt, \left(\int_0^{d(Au,Su)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au,Ty_n)} \psi(t)\,dt\right)^{\frac{1}{2}}, \left(\int_0^{d(Su,By_n)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Au,Ty_n)} \psi(t)\,dt\right)^{\frac{1}{2}} \Bigg\}.$$

Letting $n \to \infty$ we obtain

$$\int_0^{d(Au,z)} \psi(t)\,dt \leqslant (1-a) \int_0^{d(Au,z)} \psi(t)\,dt < \int_0^{d(Au,z)} \psi(t)\,dt$$

which is a contradiction. Then, $z = Au = Su$.

The rest of the proof follows as in Theorem 4. □

If $B = A$ and $T = S$ in Theorem 5, we get the following corollary.

**Corollary 5.** *Let $A$ and $S$ be mappings from a metric space $(X, d)$ into itself satisfying* (3.12) *for all $x, y$ in $X$ for which the right-hand side of* (3.12) *is positive, where $0 < a < 1$ and $\psi$ satisfies* (3.3). *Suppose that $(A, S)$ satisfies property* (E.A)*, $S(X)$ is a closed subspace of $X$ and the pair $(A, S)$ is weakly compatible. Then, $A$ and $S$ have a unique common fixed point in $X$.*

If $\psi(t) = 1$ in Theorem 5, we get the following corollary.

**Corollary 6.** *Let $A$, $B$, $S$ and $T$ be mappings from a metric space $(X, d)$ into itself satisfying* (3.13) *for all $x, y$ in $X$ for which the right-hand side of* (3.13) *is positive, where $0 < a < 1$. Suppose that $(A, S)$ and $(B, T)$ satisfy common property* (E.A)*, $S(X)$ and $T(X)$ are closed subspaces of $X$ and the pairs $(A, S)$ and $(B, T)$ are weakly compatible. Then, $A$, $B$, $S$ and $T$ have a unique common fixed point in $X$.*

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