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Journal of Number Theory 127 (2007) 103–117

JOURNAL OF
Number
Theory

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# Approximation by polynomials with bounded coefficients

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Received 24 June 2006; revised 2 December 2006

Available online 30 March 2007

Communicated by David Goss

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**Abstract**

Let $\theta$ be a real number satisfying $1 < \theta < 2$, and let $A(\theta)$ be the set of polynomials with coefficients in $\{0, 1\}$, evaluated at $\theta$. Using a result of Bugeaud, we prove by elementary methods that $\theta$ is a Pisot number when the set $(A(\theta) - A(\theta) - A(\theta))$ is discrete; the problem whether Pisot numbers are the only numbers $\theta$ such that 0 is not a limit point of $(A(\theta) - A(\theta))$ is still unsolved. We also determine the three greatest limit points of the quantities $\inf\{c,\ c > 0,\ c \in C(\theta)\}$, where $C(\theta)$ is the set of polynomials with coefficients in $\{-1, 1\}$, evaluated at $\theta$, and we find in particular infinitely many Perron numbers $\theta$ such that the sets $C(\theta)$ are discrete.

*MSC:* 11R06; 11Y60; 11C08

*Keywords:* Polynomial approximation; Beta-expansion; Pisot numbers

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## 1. Introduction

For a real number $\theta > 1$ and a rational integer $m \geqslant 1$, let

$$A_m = A_m(\theta) = \{\varepsilon_0 + \varepsilon_1\theta + \cdots + \varepsilon_n\theta^n,\ n \in \mathbb{N},\ \varepsilon_k \in \{0, 1, \ldots, m\}\},$$

where $\mathbb{N}$ is the set of positive rational integers, and let

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0022-314X/$ – see front matter
doi:10.1016/j.jnt.2007.01.014

$$B_m = B_m(\theta) = A_m - A_m = \{\varepsilon_0 + \varepsilon_1\theta + \cdots + \varepsilon_n\theta^n,\ n \in \mathbb{N},\ \varepsilon_k \in \{-m, -m+1, \ldots, m\}\}.$$

Several authors have studied the distribution of the elements of the sets above in the real line $\mathbb{R}$ (see for instance [2–7,15]), and considered in this context the quantities

$$\beta_m = \beta_m(\theta) = \inf\{b,\ b \in B_m,\ b > 0\}.$$

It is clear that $A_m$ is uniformly discrete if and only if $\beta_m > 0$, or equivalently if and only if 0 is not a limit point of $B_m$. Recall that a subset $X$ of $\mathbb{R}$ is uniformly discrete if the usual distance between two distinct elements of $X$ is greater than a positive constant depending only on $X$; a uniformly discrete set is a discrete set, that is a set with no finite limit point. Notice also by the relation $B_{2m} = B_m - B_m$, that the two propositions: *the sets $B_m$ are uniformly discrete for all $m$*, and *the sets $B_m$ are discrete for all $m$*, are equivalent, since we have $\beta_{2m} > 0$ when $B_{2m}$ is discrete, and $B_m$ is uniformly discrete if and only if $\beta_{2m} > 0$. As usual, Pisot numbers were the first algebraic integers $\theta$ which had been considered in such a problem. A Pisot number is a real algebraic integer greater than 1 whose other conjugates over the field of the rationals $\mathbb{Q}$ are of modulus less than 1. By the Pigeon-hole principle, it is easy to check (see also [3,8] and [16]) that the sets $B_m$ are discrete for all $m$ when $\theta$ is a Pisot number. Bugeaud [3] was the first to show that the converse of the last sentence is also true:

**Theorem A.** *If $B_m$ is discrete for each $m$, then $\theta$ is a Pisot number.*

In fact Bugeaud proved that $\theta$ is a Pisot number, when the equivalent condition: $\beta_m > 0$ *for all $m$*, holds. The proof of Theorem A uses a result of Frougny [9] from automata theory, and does not provide any estimation for $m$. Generalizing some former results of Frougny [9] and Erdős, Joó and Schnitzer [7], Erdős and Komornik [8] obtained an improvement of Theorem A:

**Theorem B.** *Let $m$ be the smallest positive rational integer satisfying $m \geqslant \theta - \frac{1}{\theta}$. If $B_m$ is discrete, then $\theta$ is a Pisot number.*

In these pages we denote by $\lfloor x \rfloor$ and $\{x\}$ the integer and the fractional parts of a real number $x$, respectively ($\lfloor x \rfloor$ is the greatest rational integer less than or equal to $x$ and $\{x\} = x - \lfloor x \rfloor$). The first aim of this paper is to show:

**Theorem 1.** *If $(B_{\lfloor\theta\rfloor} - A_{\lfloor\theta\rfloor})$ is discrete, then $B_m$ is discrete for each $m$.*

Clearly, we can infer by Theorem 1 and Theorem A that $\theta$ is a Pisot number when the set $(B_{\lfloor\theta\rfloor} - A_{\lfloor\theta\rfloor})$ is discrete; in particular, if $\theta < 2$ and $(A(\theta) - A(\theta) - A(\theta))$ is discrete, where $A(\theta)$ is the set of polynomials with coefficients in $\{0, 1\}$, evaluated at $\theta$, then $\theta$ is a Pisot number. It is worth noting that Theorem 1 (together with Theorem A) improves Theorem B for $\theta \in ]\frac{1+\sqrt{5}}{2}, 2[$, as the inclusion $B_1 - A_1 \subset B_2$ is strict for a.e. $\theta \in ]\frac{1+\sqrt{5}}{2}, 2[$ (for instance we have that $2 \in B_2$, and if $2 \in B_1 - A_1$ then $\theta$ is a root of a non-zero polynomial with rational integers coefficients of modulus at most 4). We prefer to state Theorem 1 in the general case for the simplicity of its proof. In Section 2, we shall also show the following weaker form of Theorem A without using automata theory: *If $B_m$ is discrete for each $m$, then $\theta$ is a Pisot or a Salem number.* Recall that a Salem number is an algebraic integer greater than 1, whose other conjugates over $\mathbb{Q}$ are of modulus at most 1 and with a conjugate of modulus 1.

Now, assume $\theta < 2$ and consider the set

$$C = C(\theta) = \{\varepsilon_0 + \varepsilon_1\theta + \cdots + \varepsilon_n\theta^n,\ n \in \mathbb{N},\ \varepsilon_k \in \{-1, 1\}\}.$$

Then, we have $C = -C \subset B_1$ and the set $C$ is uniformly discrete when $\theta$ is a Pisot number. It has been proved in [12] that $C$ is dense in $\mathbb{R}$ for a.e. $\theta \in ]\sqrt{2}, 2[$. Further, if $\theta \in ]1, \sqrt{2}]$ and $\theta^2$ is not a root of a polynomial with coefficients in $\{-1, 0, 1\}$, then the set $C$ is also dense in $\mathbb{R}$. In somewhat the opposite direction, Borwein and Hare [2] found a family of Salem numbers $\theta$ such that the corresponding sets $C$ are discrete. They also exhibit a finite set of Perron numbers that are not Pisot nor Salem numbers, with the same property. A Perron number is a real algebraic integer $\theta > 1$ whose other conjugates over $\mathbb{Q}$ are of modulus less than $\theta$. In their proof Borwein and Hare used a simple algorithm to determine the elements of the set $C \cap ]0, \frac{1}{\theta-1}[$ (the same algorithm was used in [2] and in [16] to determine the elements of $B_m \cap ]0, \frac{m}{\theta-1}[$ for some Pisot numbers $\theta$), and the following discreteness test:

**Theorem C.** *Let $\theta$ be a real number satisfying $1 < \theta < 2$. Then, the set $C$ (respectively, $B_m$) is discrete if and only if $C \cap [0, \frac{1}{\theta-1}]$ (respectively, $B_m \cap [0, \frac{m}{\theta-1}]$) is finite.*

We shall use the same arguments in the proof of the next result to determine whether the sets $C(\theta)$ are discrete for some classes of Perron numbers $\theta$, and also to compute the quantities

$$\gamma = \gamma(\theta) = \inf\{c,\ c \in C(\theta),\ c > 0\},$$

when the corresponding sets $C(\theta)$ are discrete.

**Theorem 2.** *Let $\theta$ be a real number satisfying $1 < \theta < 2$. Then, the possible values of $\gamma$ greater than*

$$\frac{1}{\theta}\prod_{0\leqslant i}\left(1 - \frac{1}{\theta^{2^i}}\right),$$

*are*

$$\frac{1}{\theta},\quad \frac{1}{\theta+1},\quad \frac{1}{\theta^{2^{n+1}}}\prod_{0\leqslant i\leqslant n}\left(\theta^{2^i} - 1\right) \quad \text{and} \quad \frac{1}{\theta^{2^{n+1}}+1}\prod_{0\leqslant i\leqslant n}\left(\theta^{2^i} - 1\right),$$

*where $n$ is a non-negative rational integer. Moreover, each of the equalities $\gamma(\theta) = \frac{1}{\theta}$, $\gamma(\theta) = \frac{1}{\theta+1}$, $\gamma(\theta) = \frac{\theta-1}{\theta^2}$ and $\gamma(\theta) = \frac{\theta-1}{\theta^2+1}$ hold for infinitely many Perron numbers $\theta$ for which the sets $C(\theta)$ are discrete.*

From the proof of Theorem 2 we easily deduce:

**Corollary 1.** *The three greatest limit points of the set $\{\gamma(\theta),\ \theta \in ]1, 2[\}$ are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ ($\frac{1}{2}$ and $\frac{1}{3}$ are both right hand limit points and $\frac{1}{4}$ is a left hand limit point).*

The solutions $\theta \in ]1, 2[$ of the inequality $\beta_1(\theta) > \frac{1}{2}$ have been determined in Theorem 3 of [15]. The same result asserts also that the implication $\beta_1(\theta) \leqslant \frac{1}{2} \Rightarrow \beta_1(\theta) < \frac{2}{5}$, is true. By Theorem 2 we have:

**Corollary 2.** *If* $\beta_1(\theta) \leqslant \frac{1}{2}$, *then* $\beta_1(\theta) \leqslant \frac{\theta-1}{\theta^2}$.

The proofs of Theorems 1 and 2 appear in Sections 2 and 3, respectively. The proof of Theorem 1 uses elementary properties of the beta-expansion of a real number, and the proof of Theorem 2 is inductive. We also show in Section 4:

**Theorem 3.** *The set* $B_m$ *is discrete if and only if* $B_m \cap [0, \frac{1}{\theta+1}]$ *is finite.*

It is clear when $\theta < 2$ that Theorem 3 improves Theorem C for the sets $B_m$. The proof of Theorem 3 follows essentially from the ones of Theorems 1 and 2. All the computations in the paper were performed using the computer system Pari [1].

## 2. Proof of Theorem 1

It is clear that $B_m$ is discrete when $\theta \in \mathbb{N}$, since a subset of a discrete set is discrete and $B_m \subset \mathbb{Z}$, the ring of the rational integers. So, assume that $B_{\lfloor\theta\rfloor} - A_{\lfloor\theta\rfloor}$ is discrete and $\theta \notin \mathbb{N}$. Notice also by the relations $B_{\lfloor\theta\rfloor} = B_{\lfloor\theta\rfloor} - \{0\} \subset B_{\lfloor\theta\rfloor} - A_{\lfloor\theta\rfloor}$ that $B_m$ is discrete when $m \leqslant \lfloor\theta\rfloor$, as $B_m \subset B_{\lfloor\theta\rfloor}$. To prove the result for $m > \lfloor\theta\rfloor$, we shall only use the relations

$$B_{k\lfloor\theta\rfloor} \subset B_{\lfloor\theta\rfloor} + F_k, \tag{1}$$

where $k$ is rational integer greater than 1, and $F_k$ is a finite subset of $\mathbb{R}$ depending on $k$ and $\theta$. To show that the inclusion (1) is true for $k = 2$, we first recall the definition of the beta-expansion of a real number.

Following [10], let $x$ be a positive real number, and let $p = p(x) \in \mathbb{Z}$ be such that $\theta^p \leqslant x < \theta^{p+1}$. Then, the beta-expansion of $x$ in base $\theta$, or simply the beta-expansion of $x$, is the sequence $(\varepsilon_k)_{k\leqslant p} = (\varepsilon_k(x))_{k\leqslant p}$ defined by the relations $\varepsilon_p = \lfloor\frac{x}{\theta^p}\rfloor$, $r_p = r_p(x) = \{\frac{x}{\theta^p}\}$, and $\varepsilon_k = \lfloor\theta r_{k+1}\rfloor$ and $r_k = r_k(x) = \{\theta r_{k+1}\}$ for $k$ running through the set of the rational integers less than $p$. In this case we have

$$x = \varepsilon_p\theta^p + \varepsilon_{p-1}\theta^{p-1} + \cdots + \varepsilon_0 + \varepsilon_{-1}\theta^{-1} + \varepsilon_{-2}\theta^{-2} + \cdots,$$

$\varepsilon_k \in \{0, 1, \ldots, \lfloor\theta\rfloor\}$ and $r_k \in [0, 1[$.

Now, let $b \in B_{\lfloor\theta\rfloor} \cap [1, \infty[$ and let $p = p(b)$. If the beta-expansion of $b$ is the sequence $(\varepsilon_k)_{k\leqslant p}$, then

$$b = \varepsilon_p\theta^p + \varepsilon_{p-1}\theta^{p-1} + \cdots + \varepsilon_0 + \varepsilon_{-1}\theta^{-1} + \varepsilon_{-2}\theta^{-2} + \cdots,$$
$$\varepsilon_p\theta^p + \varepsilon_{p-1}\theta^{p-1} + \cdots + \varepsilon_0 \in A_{\lfloor\theta\rfloor}$$

and so the number $b - (\varepsilon_p\theta^p + \varepsilon_{p-1}\theta^{p-1} + \cdots + \varepsilon_0)$ belongs to the finite set

$$E := (B_{\lfloor\theta\rfloor} - A_{\lfloor\theta\rfloor}) \cap [0, 1[,$$

since $(B_{\lfloor\theta\rfloor} - A_{\lfloor\theta\rfloor})$ is discrete and $b - (\varepsilon_p\theta^p + \varepsilon_{p-1}\theta^{p-1} + \cdots + \varepsilon_0) = r_0(b)$. Consider an element $d \in B_{2\lfloor\theta\rfloor}$. It is clear that $d$ can be written as

$$d = b - b',$$

for some elements $b$ and $b'$ of the set $B_{\lfloor\theta\rfloor}$. Let $N$ be a sufficiently large rational integer so that $\theta^N + b$ and $\theta^N + b'$ belong to $B_{\lfloor\theta\rfloor} \cap [1, \infty[$. By the above, there are $a \in A_{\lfloor\theta\rfloor}$, $r \in E$, $a' \in A_{\lfloor\theta\rfloor}$ and $r' \in E$ such that $b + \theta^N = a + r$ and $b' + \theta^N = a' + r'$. It follows that

$$d = b + \theta^N - \left(b' + \theta^N\right) = \left(a - a'\right) + \left(r - r'\right) \in (A_{\lfloor\theta\rfloor} - A_{\lfloor\theta\rfloor}) + (E - E) = B_{\lfloor\theta\rfloor} + (E - E)$$

and so (1) is satisfied with

$$F_2 := E - E.$$

Assume that (1) is true for some $k \geqslant 2$. Then, by the relations

$$B_{(k+1)\lfloor\theta\rfloor} = B_{k\lfloor\theta\rfloor} + B_{\lfloor\theta\rfloor} \subset B_{\lfloor\theta\rfloor} + F_k + B_{\lfloor\theta\rfloor} = B_{2\lfloor\theta\rfloor} + F_k \subset B_{\lfloor\theta\rfloor} + (F_k + F_2),$$

we see that the inclusion (1) is true for $k + 1$ with $F_{k+1} := F_k + F_2$. Now, by (1) we have immediately that $B_{k\lfloor\theta\rfloor}$ is discrete for each $k \geqslant 2$. Indeed, otherwise there is a convergent sequence of distinct elements of $B_{k\lfloor\theta\rfloor}$, say $(d_n)_{n\in\mathbb{N}}$. Since $F_k$ is finite and each term of $(d_n)_{n\in\mathbb{N}}$ can be written as $d_n = b_n + f_n$, where $b_n \in B_{\lfloor\theta\rfloor}$ and $f_n \in F_k$, we can extract from $(d_n)_{n\in\mathbb{N}}$ a subsequence of the form $(b_n + f)_{n\in I}$, where $f \in \{f_n, n \in \mathbb{N}\}$ and $I$ is an infinite subset of $\mathbb{N}$, and so we obtain a convergent sequence of distinct elements of $B_{\lfloor\theta\rfloor}$, namely the sequence $(b_n)_{n\in I}$; this is absurd because $B_{\lfloor\theta\rfloor}$ is discrete. Hence, $B_{k\lfloor\theta\rfloor}$ is discrete and so are all the sets $B_m$, since for each $m$ there is $k \in \mathbb{N}$ such that $m \leqslant k\lfloor\theta\rfloor$. □

**Remark 1.** Next we give two simple proofs of the following weaker form of Theorem A: *If $B_m$ is discrete for each $m$, then $\theta$ is a Pisot or a Salem number.* The first proof uses a result due to Schmidt [14] and the second one a theorem of Parry's [11]. Recall that if the beta-expansion (to base $\theta$) of a positive real number $x$ is the sequence $(\varepsilon_k)_{k\leqslant p}$, then $(-\varepsilon_k)_{k\leqslant p}$ is the beta-expansion of $-x$, and we say that the real $x$ has a periodical expansion if its beta-expansion is eventually periodic. Let $Per(\theta)$ be the set of numbers having periodical expansions. Then, $Per(\theta) \subset \mathbb{Q}(\theta)$, and by Theorem 2.4 of [14] we have that if $\mathbb{Q}(\theta) = Per(\theta)$ then $\theta$ is a Pisot or a Salem number. Recall also that the number $\theta$ is said to be a beta-number when $\{\theta\} \in Per(\theta)$. In [11] it has been shown that a beta-number is an algebraic integer and the other conjugates of a beta-number over $\mathbb{Q}$ are of modulus less than 2. To make the notation easier (as in [11] and [13]), we let the beta-expansion of a real number $\alpha$ satisfying $\frac{1}{\theta} \leqslant \alpha < 1$ to be the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, where $\varepsilon_n = [\theta r_{n-1}]$ and $r_n = \{\theta r_{n-1}\}$ when $n \in \mathbb{N}$, and $r_0 = \alpha$. Then, $r_n = \varepsilon_{n+1}\theta^{-1} + \varepsilon_{n+2}\theta^{-2} + \varepsilon_{n+3}\theta^{-3} + \cdots$,

$$\alpha = \varepsilon_1\theta^{-1} + \varepsilon_2\theta^{-2} + \cdots + \varepsilon_n\theta^{-n} + r_n\theta^{-n} \tag{2}$$

and the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ is eventually periodic if and only if the set $\{r_n, n \geqslant 0\}$ is finite. Finally, suppose that $\theta \notin \mathbb{N}$, since $\theta$ is a Pisot number when it is a rational integer.

**The first proof.** Let $k \in \mathbb{N}$, $k \geqslant 2$, and let $l$ be the greatest rational integer such that $\theta^l < k$. Then, $\frac{\theta^l}{k} \in [\frac{1}{\theta}, 1[$, and by (2) we have for $\alpha := \frac{\theta^l}{k}$,

$$kr_n = \theta^{l+n} - k\bigl(\varepsilon_1\theta^{n-1} + \varepsilon_2\theta^{n-2} + \cdots + \varepsilon_n\bigr).$$

Hence, there is an $m \in \mathbb{N}$ independent of $n$ (we may choose $m = k\lfloor\theta\rfloor$) such that

$$kr_n \in B_m(\theta)$$

for all $n$, and so $\{kr_n, n \geqslant 0\}$ is a subset of the finite set $B_m(\theta) \cap [0, k[$. It follows that $\{r_n, n \geqslant 0\}$ is finite and $\alpha \in Per(\theta)$. Moreover, there are $u$ and $v$ such that $u > v$ and $kr_u = kr_v$, and so $\theta$ is a root of the polynomial

$$x^{l+u} - k\bigl(\varepsilon_1 x^{u-1} + \cdots + \varepsilon_u\bigr) - x^{l+v} + k\bigl(\varepsilon_1 x^{v-1} + \cdots + \varepsilon_v\bigr).$$

Thus, $\theta$ is an algebraic integer of degree, say $d$, over $\mathbb{Q}$ and $\mathbb{Q}(\theta) = \{P(\theta), P \in \mathbb{Q}[X], \deg(P) \leqslant d-1\}$. To prove the inclusion $\mathbb{Q}(\theta) \subset Per(\theta)$, it suffices to show that $\mathbb{Q}(\theta) \cap [\frac{1}{\theta}, 1[ \subset Per(\theta)$, since by definition $0 \in Per(\theta)$, $-x \in Per(\theta)$ when $x \in Per(\theta)$, and if $\theta^p \leqslant x < \theta^{p+1}$ for some $p \in \mathbb{Z}$ then $\theta^{-1} \leqslant \theta^{-p-1}x < 1$, $\theta^{-p-1}x \in \mathbb{Q}(\theta)$ when $x \in \mathbb{Q}(\theta)$ and the beta-expansion of $x$ and $\theta^{-p-1}x$ are identical. Let $\alpha \in \mathbb{Q}(\theta) \cap [\frac{1}{\theta}, 1[$. Then, $\alpha$ can be written as

$$\alpha = \frac{n_0 + n_1\theta + \cdots + n_{d-1}\theta^{d-1}}{k}$$

for some $n_0, n_1, \ldots, n_{d-1} \in \mathbb{Z}$ and $k \in \mathbb{N}$, and so by (2) we have

$$kr_n = \bigl(n_0 + n_1\theta + \cdots + n_{d-1}\theta^{d-1}\bigr)\theta^n - k\bigl(\varepsilon_1\theta^{n-1} + \varepsilon_2\theta^{n-2} + \cdots + \varepsilon_n\bigr);$$

thus $kr_n \in B_m(\theta)$, where $m = \max\{|n_0|, |n_1|, \ldots, |n_{d-1}|, k\lfloor\theta\rfloor\}$, and similarly as for the case where $\alpha = \frac{\theta^a}{k}$ we easily obtain that $\alpha \in Per(\theta)$. After this we use Schmidt's result to infer that $\theta$ is a Pisot or a Salem number. Finally, recall that the question whether Pisot numbers are the only numbers $\theta$ satisfying the relation $Per(\theta) = \mathbb{Q}(\theta)$, remains open. □

**The second proof.** Let $k \in \mathbb{N}$. Then, the set $B_{\lfloor\theta^k\rfloor}(\theta^k)$ is discrete, since $B_{\lfloor\theta^k\rfloor}(\theta^k) \subset B_{\lfloor\theta^k\rfloor}(\theta)$ and $B_{\lfloor\theta^k\rfloor}(\theta)$ is discrete. Considering the beta-expansion of $\{\theta^k\}$ to base $\theta^k$, we obtain identically as in the first proof that $\theta^k$ is a beta number. Hence, $\theta$ is an algebraic integer with no other conjugate over $\mathbb{Q}$ of modulus greater than 1, since otherwise we obtain a contradiction with Parry's result when $k$ is large. □

## 3. Proof of Theorem 2 and its corollaries

**Proof of Theorem 2.** To make the proof clear we consider the cases corresponding to the greatest limit points of the set $\{\gamma(\theta), \theta \in ]1, 2[\}$ separately.

**Step 1.** *We show that $C \cap ]0, \frac{1}{\theta}] \neq \emptyset$, and solve the equation $\gamma(\theta) = \frac{1}{\theta}$.*

Consider the function

$$T_0(x) = \theta x - 1$$

in the real variable $x$. It is clear that

$$T_0(C) \subset C$$

and

$$T_0\left(\left]\sum_{k=1}^{n}\frac{1}{\theta^k}, \sum_{k=1}^{n+1}\frac{1}{\theta^k}\right]\right) \subset \left]\sum_{k=1}^{n-1}\frac{1}{\theta^k}, \sum_{k=1}^{n}\frac{1}{\theta^k}\right],$$

where $n \in \mathbb{N}$. Since

$$1 \in C \cap \left]\frac{1}{\theta}, \frac{1}{\theta - 1}\right[ \quad \text{and} \quad \frac{1}{\theta - 1} = \sum_{k \geqslant 1}\frac{1}{\theta^k},$$

by iterating the map $T_0$ we obtain that there exists $n \in \mathbb{N}$ such that $T_0^{(n)}(1) \in C \cap ]0, \frac{1}{\theta}]$, and so $C \cap ]0, \frac{1}{\theta}] \neq \emptyset$. To find the numbers $\theta$ satisfying the equation $\gamma(\theta) = \frac{1}{\theta}$, we shall use the polynomials

$$f_n(x) = x^n - x^{n-1} - \cdots - x - 1,$$

where $n \geqslant 2$. It is known (see [5] and [15]) that $f_n$ is the minimal polynomial of a Pisot number, say $q_n$, the sequence $(q_n)_{n \geqslant 2}$ is increasing towards 2 with $q_2 = 1.618\ldots$, and the real function $f_n(x)$ is increasing on the interval $[\frac{1}{q_{n-1}}, \infty[$; by convention $q_1 := 1$ and $f_1(x) := x - 1$. Let $\theta$ be such that $\gamma(\theta) = \frac{1}{\theta}$. Then, $C \cap ]0, \frac{1}{\theta}[ = \emptyset$ and so $\theta \geqslant q_2$, since $\theta - 1 \in C$ (and $\theta - 1 \geqslant \frac{1}{\theta}$). Further, if $\theta \in [q_n, q_{n+1}[$ for some $n \geqslant 2$, then by the relations $f_n(x) = x f_{n-1}(x) - 1$, $f_n(\theta) \in C$ and

$$0 \leqslant f_n(\theta) < \frac{1}{\theta},$$

we have $f_n(\theta) = 0$. Hence, $\theta = q_n$ and so $C$ is uniformly discrete. Now, we use the algorithm of [2] to determine the elements of $C \cap ]0, \frac{1}{\theta - 1}[$, where $\theta = q_n$. Clearly, by the inequalities $q_k < \theta$, where $k \in \{1, 2, \ldots, n-1\}$, we have $f_k(\theta) > 0$ and $f_k(\theta) + 2 > 2 > \frac{1}{\theta - 1}$; thus the only element of $C \cap ]0, \frac{1}{\theta - 1}[$ with degree $k$ (as a polynomial in $\theta$) is $f_k(\theta)$ and so the set $C \cap ]-\frac{1}{\theta - 1}, \frac{1}{\theta - 1}[$ contains exactly one element with degree $n$, namely $f_n(\theta) = 0$. Hence,

$$C \cap \left]0, \frac{1}{\theta - 1}\right[ = \left\{1, f_1(\theta), f_2(\theta), \ldots, f_{n-1}(\theta)\right\},$$

and $\gamma(\theta) = \frac{1}{\theta}$, since $1 > f_1(\theta) > f_2(\theta) > \cdots > f_{n-1}(\theta) = \frac{1}{\theta}$.

**Step 2.** *We prove that* $C \cap ]0, \frac{1}{\theta + 1}] \neq \emptyset$ *when* $\theta \notin F_0 := \{q_n, n \geqslant 2\}$, *and solve the equation* $\gamma(\theta) = \frac{1}{\theta + 1}$.

Assume $\theta \notin F_0$. Then, $C \cap \left]0, \frac{1}{\theta}\right[ \neq \emptyset$. By the relations

$$-T_0(C) \subset C$$

and

$$-T_0\left(\left]\frac{1}{\theta+1}, \frac{1}{\theta}\right[\right) \subset \left]0, \frac{1}{\theta+1}\right[,$$

we find that $C \cap \left]0, \frac{1}{\theta+1}\right] \neq \emptyset$, and $\gamma(\theta) = \frac{1}{\theta+1}$ if and only if $C \cap \left]0, \frac{1}{\theta}\right[ = \{\frac{1}{\theta+1}\}$. Now, let $\theta$ be such that $\gamma(\theta) = \frac{1}{\theta+1}$. Then, $\theta > q_2$, because the inequality $\theta - 1 < \frac{1}{\theta}$ yields $\theta - 1 = \frac{1}{\theta+1}$, and the set $C(\sqrt{2})$ is dense in $\mathbb{R}$. Further, if $\theta \in \left]q_n, q_{n+1}\right[$ for some $n \geqslant 2$, then we have $0 < f_n(\theta) < \frac{1}{\theta}$ and so $f_n(\theta) = \frac{1}{\theta+1}$; thus $\theta$ is a root of the polynomial

$$g_{n+1}(x) = (x+1)\bigl(x^n - x^{n-1} - \cdots - x - 1\bigr) - 1 = x^{n+1} - 2\bigl(x^{n-1} + x^{n-2} + \cdots + x + 1\bigr).$$

It is clear that the polynomial $g_{n+1}$ is irreducible over $\mathbb{Q}$, as it is a 2-Einstein polynomial, and can also be written, for $x \neq 1$, as

$$g_{n+1}(x) = f_n(x) + f_{n+1}(x) = \frac{x^n(x-2)(x+1)+2}{x-1},$$

since

$$f_n(x) = \frac{x^n(x-2)+1}{x-1}.$$

It follows by the relations $g_{n+1}(q_{n+1}) = f_n(q_{n+1}) = \frac{1}{q_{n+1}} > 0$ and $g_{n+1}(q_n) = f_{n+1}(q_n) = -1 < 0$ that $g_{n+1}$ has a real root, say $r_{n+1}$, such that

$$q_n < r_{n+1} < q_{n+1};$$

so the sequence $(r_n)_{n \geqslant 3}$ is increasing towards 2 with $r_3 = 1.769\ldots$. Moreover, if we fix $\delta \in \left]1, 2\right[$ and choose $N \in \mathbb{N}$ so that $\delta^N(2-\delta)(\delta-1) > 2$, then we have on the circle $|z| = \delta$ (in the complex plane) that $|z^n(z-2)(z+1)| > 2$ when $n \geqslant N$. It follows by Rouché's theorem that the polynomial $x^n(x-2)(x+1)+2 = (x-1)g_{n+1}(x)$ has $n+1$ roots in the disc $|z| < \delta$, and so the polynomial $g_{n+1}(x)$ has $n$ roots with modulus less than $\delta$. A short computation shows that we can choose $N = 5$ for $\delta = q_2$; thus the polynomial $g_{n+1}$ has exactly $n$ roots with modulus less than $q_2$ when $n \geqslant 5$, and the remaining root, which is $r_{n+1}$, satisfies $r_{n+1} > q_n \geqslant q_2$. Hence, the conjugates of the algebraic integer $r_n$ are of modulus less than $r_n$ for each $n \geqslant 6$. Directly we verify that $r_3$, $r_4$ and $r_5$ are also Perron numbers (it is easy to see that for any $1 < \delta < 2$, there is $N \in \mathbb{N}$ such that the conjugates of the Perron number $r_n$ are in the annulus $\frac{1}{\delta} < |z| < \delta$ for all $n \geqslant N$). Note also that if $u$ and $v$ are two positive roots of the polynomial $g_{n+1}$, where $u \leqslant v$, then

$$u = 2\left(\frac{1}{u} + \frac{1}{u^2} + \cdots + \frac{1}{u^n}\right) \geqslant 2\left(\frac{1}{v} + \frac{1}{v^2} + \cdots + \frac{1}{v^n}\right) = v$$

and so $u = v$; thus $r_{n+1}$ is the only root of $g_{n+1}$ in $]1, 2[$. Similarly as in the case above, by the relations $g_n(x) = f_n(x) + f_{n-1}(x)$, where $n \geqslant 3$, and $r_n > q_n \geqslant q_2$, we easily obtain for $\theta = r_n$ that

$$C \cap \left]0, \frac{1}{\theta - 1}\right[ = \{1, f_1(\theta), f_2(\theta), \dots, f_{n-1}(\theta)\};$$

thus $\gamma(\theta) = f_{n-1}(\theta) = \frac{1}{\theta+1}$, and by Theorem C we have that the set $C$ is discrete (we will see in the proof of the corollaries that $\beta_1(\theta) = 0$; so by the relation $2B_1 \subset C - C$, the set $C$ is not uniformly discrete). Finally, notice that the number $r_n$ has at least a conjugate of modulus $> 1$ (respectively, has no conjugate of modulus 1), because $r_n < 2$ and $r_n$ has norm 2 (respectively, because $r_n$ is not a unit); thus $r_n$ is not a Pisot nor a Salem number.

**Step 3.** *Let $F_0' = \{r_n, n \geqslant 3\}$. We show that $C \cap ]0, \frac{\theta-1}{\theta^2}] \neq \emptyset$ when $\theta \notin F_0 \cup F_0'$, and prove that the equation $\gamma(\theta) = \frac{\theta-1}{\theta^2}$ holds at least for two families of Perron numbers $\theta$.*

Let $\theta \notin F_0 \cup F_0'$. Then, $C \cap ]0, \frac{1}{\theta+1}[ \neq \emptyset$. Similarly as in Step 1, by iterating the real function

$$T_1(x) = \theta^2 x - \theta + 1$$

when it is necessary and when $C \cap ]\frac{\theta-1}{\theta^2}, \frac{1}{\theta+1}[ \neq \emptyset$, we obtain that $C \cap ]0, \frac{\theta-1}{\theta^2}] \neq \emptyset$, since

$$T_1(C) \subset C,$$

$$T_1\left(\left]\sum_{k=1}^{n} \frac{\theta - 1}{\theta^{2k}}, \sum_{k=1}^{n+1} \frac{\theta - 1}{\theta^{2k}}\right]\right) \subset \left]\sum_{k=1}^{n-1} \frac{\theta - 1}{\theta^{2k}}, \sum_{k=1}^{n} \frac{\theta - 1}{\theta^{2k}}\right],$$

where $n \in \mathbb{N}$, and

$$\sum_{k \geqslant 1} \frac{\theta - 1}{\theta^{2k}} = \frac{1}{\theta + 1}.$$

Moreover, we have

$$\gamma(\theta) = \frac{\theta - 1}{\theta^2} \quad \text{if and only if} \quad C \cap \left]0, \frac{1}{\theta + 1}\right[ \subset \left\{\sum_{k=1}^{n} \frac{\theta - 1}{\theta^{2k}},\ n \in \mathbb{N}\right\},$$

as

$$T_1(x) \neq \sum_{k=1}^{n} \frac{\theta - 1}{\theta^{2k}} \quad \text{when } x \neq \sum_{k=1}^{n+1} \frac{\theta - 1}{\theta^{2k}}.$$

Now, let $\theta$ be such that $\gamma(\theta) = \frac{\theta-1}{\theta^2}$. It is clear when $\theta < \sqrt{2}$ that

$$\theta - 1 < \frac{1}{\theta + 1}, \qquad \theta - 1 = \sum_{k=1}^{n} \frac{\theta - 1}{\theta^{2k}}$$

for some $n \geqslant 2$, and $\theta = \sqrt{q_n}$. If $\theta > \sqrt{2}$, then there is $n \geqslant 2$ such that $\theta \in \left]r_n, q_n\right[$, where $r_2 := \sqrt{2}$, or $\theta \in \left]q_n, r_{n+1}\right[$, and by the same arguments as in the above cases we obtain that there is $m \in \mathbb{N}$ such that $\theta$ is respectively a root of one of the polynomials

$$h^+_{m,n}(x) = x^{2m} f_n(x) + \frac{x^{2m} - 1}{x + 1}$$

or

$$h^-_{m,n}(x) = x^{2m} f_n(x) - \frac{x^{2m} - 1}{x + 1}.$$

By Theorems 5.3 and 5.4 of [2], we see that the polynomial $h^-_{1,n}$ has only one root of modulus greater than 1, say $s^-_{1,n}$, and $s^-_{1,n}$ is a Salem number such that $\gamma(s^-_{1,n}) = \frac{s^-_{1,n} - 1}{(s^-_{1,n})^2}$ and $A(s^-_{1,n})$ is discrete. Notice also that

$$q_n < s^-_{1,n} < r_{n+1}.$$

Now, consider the polynomial

$$h^+_{1,n}(x) = x^2 f_n(x) + x - 1.$$

It is clear that $h^+_{1,n}(q_n) = q_n - 1 > 0$. Further, by the identities $x g_n(x) = x f_n(x) + x f_{n-1}(x) = (x + 1) f_n(x) + 1$ we have $h^+_{1,n}(r_n) = r_n^2 f_n(r_n) + r_n - 1 = -\frac{1}{r_n + 1} < 0$; thus the polynomial $h^+_{1,n}$ has a real root, say $s^+_{1,n}$, satisfying

$$r_n < s^+_{1,n} < q_n,$$

and the sequence $(s^+_{1,n})_{n \geqslant 2}$ is increasing towards 2. Writing

$$h^+_{1,n}(x) = x^2 f_n(x) + x - 1 = \frac{x^{n+2}(x - 2) + (2x^2 - 2x + 1)}{x - 1},$$

we obtain identically as for the polynomials $g_n$ that the roots of $h^+_{1,n}$ other than $s^+_{1,n}$ are of modulus less than $r_2$; so the other conjugates of $s^+_{1,n}$ over $\mathbb{Q}$ are of modulus less than $s^+_{1,n}$, and $s^+_{1,n}$ is a Perron number. Similarly as for the case where $\theta = q_n$, a short computation shows when $\theta = s^+_{1,n}$ and $n \geqslant 3$ that

$$C \cap \left]0, \frac{1}{\theta - 1}\right[ = \left\{1, f_1(\theta), \ldots, f_{n-1}(\theta), -f_n(\theta) = \frac{\theta - 1}{\theta^2}, \frac{1}{\theta}\right\};$$

thus $C$ is discrete and $\gamma(\theta) = \frac{\theta - 1}{\theta^2}$, since $\theta > r_3$ and

$$\frac{1}{\theta} > f_{n-1}(\theta) = \frac{f_n(\theta) + 1}{\theta} = \frac{\theta^2 - \theta + 1}{\theta^3} > \frac{\theta - 1}{\theta^2}.$$

Finally, notice when $\theta = s^+_{1,2} = 1.512\ldots$ that $0 < \theta^6 - \theta^5 - \theta^4 - \theta^3 + \theta^2 + \theta + 1 = 0.1\ldots < \frac{\theta-1}{\theta^2} = 0.2\ldots$.

**Step 4.** *Let* $F_1 = \{\theta \in ]1, 2[,\ \gamma(\theta) = \frac{\theta-1}{\theta^2}\}$. *We prove that* $C \cap ]0, \frac{\theta-1}{\theta^2+1}] \neq \emptyset$ *when* $\theta \notin F_0 \cup F'_0 \cup F_1$, *and show that there are infinitely many Perron numbers* $\theta$ *satisfying* $\gamma(\theta) = \frac{\theta-1}{\theta^2+1}$.

Let $\theta \notin F_0 \cup F'_0 \cup F_1$. Then, $C \cap ]0, \frac{\theta-1}{\theta^2}[ \neq \emptyset$. By the relations

$$-T_1(C) \subset C$$

and

$$-T_1\left(\left]\frac{\theta-1}{\theta^2+1}, \frac{\theta-1}{\theta^2}\right[\right) \subset \left]0, \frac{\theta-1}{\theta^2+1}\right[,$$

we obtain $C \cap ]0, \frac{\theta-1}{\theta^2+1}] \neq \emptyset$, and $\gamma(\theta) = \frac{\theta-1}{\theta^2+1}$ if and only if $C \cap ]0, \frac{\theta-1}{\theta^2}[ = \{\frac{\theta-1}{\theta^2+1}\}$. Let $\theta$ be such that $\gamma(\theta) = \frac{\theta-1}{\theta^2+1}$ and $\theta \in ]q_n, s^-_{1,n}[$, where $n \geqslant 3$ (we will see that such a $\theta$ exists). Then, $h^-_{1,n}(\theta) < 0$, as $h^-_{1,n}(1) = f_n(1) < 0$ and $h^-_{1,n}$ has no root in $]1, s^-_{1,n}[$, $0 < f_n(\theta) < \frac{\theta-1}{\theta^2}$ and so $f_n(\theta) = \frac{\theta-1}{\theta^2+1}$; thus $\theta$ is a root of the polynomial

$$l_n(x) = x^{n+1} - x^n - 2\left(x^{n-2} + x^{n-3} + \cdots + x + 1\right).$$

From the identities

$$xl_n(x) = \left(x^2+1\right)f_n(x) - (x-1) = f_n(x) + h^-_{1,n}(x),$$

we have $l_n(q_n) < 0$, $l_n(s^-_{1,n}) = f_n(s^-_{1,n}) > 0$ and so $l_n$ has a real root, say $t_n$, satisfying

$$q_n < t_n < s^-_{1,n};$$

thus the sequence $(t_n)_{n\geqslant 3}$ is increasing towards 2 with $t_3 = 1.873\ldots$. By the same arguments as in the above cases, we obtain that $t_n$ is a Perron number and

$$C(t_n) \cap \left]0, \frac{1}{t_n-1}\right[ = \left\{1, f_1(t_n), \ldots, f_n(t_n), -f_{n+1}(t_n) = \frac{t_n+1}{t_n^2+1}\right\}.$$

Hence, $C(t_n)$ is discrete and $\gamma(t_n) = f_n(t_n) = \frac{t_n-1}{t_n^2+1}$.

**Step 5.** *We use induction to complete the proof.*

Let $F_n$ and $F'_n$ be the sets of the numbers $\theta$ satisfying

$$\gamma(\theta) = \frac{\prod_{0\leqslant i\leqslant n-1}(\theta^{2^i}-1)}{\theta^{2^n}} \quad \text{and} \quad \gamma(\theta) = \frac{\prod_{0\leqslant i\leqslant n-1}(\theta^{2^i}-1)}{\theta^{2^n}+1},$$

where $n \in \mathbb{N}$, respectively. It suffices to show that the following two propositions

$$P_n\colon\quad \text{if } \theta \notin \bigcup_{i=0}^{n-1}\bigl(F_i \cup F_i'\bigr) \cup F_n,\ \text{then } C(\theta) \cap \left]0, \frac{\prod_{0\leqslant i\leqslant n-1}(\theta^{2^i}-1)}{\theta^{2^n}+1}\right] \neq \emptyset$$

and

$$P_n'\colon\quad \text{if } \theta \notin \bigcup_{i=0}^{n}\bigl(F_i \cup F_i'\bigr),\ \text{then } C(\theta) \cap \left]0, \frac{\prod_{0\leqslant i\leqslant n}(\theta^{2^i}-1)}{\theta^{2^{n+1}}}\right] \neq \emptyset$$

are true for all $n$. From Step 4, $P_1$ is true. Further, if $\theta \notin F_0 \cup F_0' \cup F_1 \cup F_1'$ then $C \cap ]0, \frac{\theta-1}{\theta^2+1}[ \neq \emptyset$. By iterating the map

$$T_2(x) = \theta^4 x - (\theta-1)\bigl(\theta^2-1\bigr) = \theta^4 x - \theta^3 + \theta^2 + \theta - 1$$

when $C \cap ]\frac{(\theta-1)(\theta^2-1)}{\theta^4}, \frac{\theta-1}{\theta^2+1}[ \neq \emptyset$, we obtain that $P_1'$ is true, since $T_2(C) \subset C$,

$$T_2\left(\left]\sum_{k=1}^{n}\frac{(\theta-1)(\theta^2-1)}{\theta^{4k}}, \sum_{k=1}^{n+1}\frac{(\theta-1)(\theta^2-1)}{\theta^{4k}}\right]\right)$$

is contained in

$$\left]\sum_{k=1}^{n-1}\frac{(\theta-1)(\theta^2-1)}{\theta^{4k}}, \sum_{k=1}^{n}\frac{(\theta-1)(\theta^2-1)}{\theta^{4k}}\right],$$

where $n \in \mathbb{N}$, and

$$\sum_{k\geqslant 1}\frac{(\theta^2-1)}{\theta^{4k}} = \frac{1}{\theta^2+1}.$$

Identically, by considering the map

$$T_n(x) = \theta^{2^{n+1}}x + \prod_{0\leqslant i\leqslant n}\bigl(\theta^{2^i}-1\bigr),$$

we easily show that the propositions $P_{n+1}$ and $P_{n+1}'$ are true, when $P_n$ and $P_n'$ are so. The relation $T_n(C) \subset C$ follows from the fact that the polynomial $\prod_{0\leqslant i\leqslant n}(x^{2^i}-1)$ has its coefficients in $\{-1, 1\}$ and is of degree $2^{n+1}-1$. □

**Proof of the corollaries.** From the proof of Theorem 2, we have when $\theta \notin F_0 \cup F_0' \cup F_1$ that $C \cap ]0, \frac{\theta-1}{\theta^2+1}] \neq \emptyset$ and so

$$\beta_1(\theta) \leqslant \gamma(\theta) \leqslant \frac{\theta-1}{\theta^2+1} < \min\left\{\frac{1}{5}, \frac{\theta-1}{\theta^2}\right\}.$$

Moreover, if $\theta \in F_0$ (respectively, $\theta \in F_0'$) then $\theta = q_n$ for some $n \geqslant 2$ and $\gamma(q_n) = \frac{1}{q_n}$ (respectively, $\theta = r_n$ for some $n \geqslant 3$ and $\gamma(r_n) = \frac{1}{r_n}$) tends to $\frac{1}{2}$ (respectively, to $\frac{1}{3}$) when $n$ tends to infinity; for $\theta \in F_1$, we have $\gamma(\theta) = \frac{\theta-1}{\theta^2} < \frac{1}{4}$, and in particular when $\theta = s_{1,n}^+$ (or $\theta = s_{1,n}^-$), $\gamma(\theta)$ tends to $\frac{1}{4}$ when $n$ tends to infinity (I am not able to determine whether $\frac{1}{4}$ belongs to the second derived set of $\{\gamma(\theta),\ \theta \in ]1,2[\}$). Recall also that the equality $\beta_1(\theta) = \frac{1}{\theta}$ when $\theta \in F_0$, has been proved in many places and firstly in [5]. Finally, if $\theta \in F_0'$ then by Remark 2 of [3] we have $\beta_1(\theta) = 0$, since $\theta$ is not a root of a polynomial with coefficients in $\{-1, 0, 1\}$. □

**Remark 2.** Let $\theta = 1.7548\ldots$ be the Pisot number root of $x^3 - 2x^2 + x - 1$ ($\theta$ is the square of the smallest Pisot number). Then, $\beta_1(\theta) = \frac{\theta-1}{\theta^2}$ and so Corollary 2 is optimal. From the proof of Theorem 2 we have that the solutions of the equality $\beta_1(\theta) = \frac{\theta-1}{\theta^2}$ are among the numbers $\sqrt{q_n}$, where $n \geqslant 2$, and the roots, say $s_{m,n}^{\pm}$, of $h_{m,n}^+$ and $h_{m,n}^-$ which belong to $]1,2[$. To determine whether $\frac{1}{4}$ is a limit point of $\{\beta_1(\theta), \theta \in ]1,2[\}$ it suffices to consider the numbers $s_{m,n}^{\pm}$, since we have $\beta_1(\sqrt{q_n}) = 0$ by the following proposition: *If* $p \in \mathbb{N}$ *and* $\theta^{\frac{1}{p}} \notin \mathbb{Q}(\theta)$, *then* $\beta_1(\theta^{\frac{1}{p}}) = 0$. Indeed, with the notation of Remark 1, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be the beta-expansion to base $\theta$ of $\alpha - 1$, where $\alpha = \theta^{\frac{1}{p}}$, and let $r_n = \varepsilon_{n+1}\theta^{-1} + \varepsilon_{n+2}\theta^{-2} + \varepsilon_{n+3}\theta^{-3} + \cdots$, where $n \geqslant 0$. Then, $\alpha = 1 + \varepsilon_1\theta^{-1} + \varepsilon_2\theta^{-2} + \cdots + \varepsilon_n\theta^{-n} + r_n\theta^{-n}$, $r_n \in [0,1[$, and the set $\{r_n,\ n \geqslant 0\}$ is not finite because $\alpha \notin \mathbb{Q}(\theta)$ and $Per(\theta) \subset \mathbb{Q}(\theta)$. It follows from the last equality that

$$r_n = \alpha^{np+1} - \alpha^{np} - \varepsilon_1\alpha^{np-p} - \varepsilon_2\alpha^{np-2p} - \cdots - \varepsilon_{n-1}\alpha^p - \varepsilon_n \in B_1(\alpha)$$

and so $B_1(\alpha)$ has a limit point, say $l$. Let $(r_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(r_n)$ such that $n_1 < n_2 < n_3 < \cdots$, $r_{n_i} \neq r_{n_j}$ when $i \neq j$, and $\lim r_{n_k} = l$. Then, $\lim(r_{n_{k+1}} - r_{n_k}) = 0$, and $r_{n_{k+1}} - r_{n_k} \in B_1(\alpha)$, as $p \geqslant 2$ and $p$ does not divide in $\mathbb{Z}$ the numbers $n_k p + 1$.

## 4. Proof of Theorem 3

Note first by Lemma 2.1(b) of [8] (or by Proposition 1(i) of [16]) that Theorem 3 is true when $m \leqslant \lfloor\theta\rfloor - 1$, since in this case the set $B_m$ is discrete. So, assume $m \geqslant \lfloor\theta\rfloor$. By definition we have that $B_m \cap [0, \frac{1}{\theta+1}]$ is finite when $B_m$ is discrete. To prove the converse, we shall first show that $B_m$ is discrete when the set $B_m \cap [0,1]$ is finite. Since $B_m = -B_m$, it suffices to prove that each finite subinterval, say $[0,\varepsilon]$, of $[0,\infty[$ contains at most a finite number of elements of $B_m$. Let $B_m \cap ]0,1] = \{b_1, b_2, \ldots, b_k\}$. We will show that each $b \in B_m \cap [0,\infty[$ can be written as

$$b = \sum_{1\leqslant i\leqslant k} n_i b_i, \tag{3}$$

for some non-negative rational integers $n_1, n_2, \ldots, n_k$. Indeed, if (3) is true, then for each $i \in \{1, 2, \ldots, k\}$ we have

$$n_i \leqslant n_i \frac{b_i}{\beta_m} \leqslant \frac{b}{\beta_m},$$

as $\beta_m = \min\{b_1, b_2, \ldots, b_k\}$, and so $n_i \in \{0, 1, \ldots, \lfloor \frac{\varepsilon}{\beta_m} \rfloor\}$ when $b \in [0, \varepsilon] \cap B_m$; thus there are at most $(1 + \lfloor \frac{\varepsilon}{\beta_m} \rfloor)^k$ elements of $B_m$ in $[0, \varepsilon]$ (this is a quantitative version of what we want to prove). Now, let

$$b \in B_m \cap ]1, \infty[.$$

Since $B_m = B_{m-\lfloor\theta\rfloor} + B_{\lfloor\theta\rfloor} = B_{m-\lfloor\theta\rfloor} + A_{\lfloor\theta\rfloor} - A_{\lfloor\theta\rfloor}$, there exist $z \in B_{m-\lfloor\theta\rfloor} + A_{\lfloor\theta\rfloor}$ and $a \in A_{\lfloor\theta\rfloor}$ such that $b = z - a$; by convention $B_0 + A_{\lfloor\theta\rfloor} = A_{\lfloor\theta\rfloor}$. Considering the beta-expansion of the number $a + 1$, which satisfies $1 \leqslant a + 1 < z$, we deduce that there is $a' \in A_{\lfloor\theta\rfloor}$ such that $a + 1 - a' \in [0, 1[$; thus $a' \in A_{\lfloor\theta\rfloor} \cap ]a, a + 1]$ and so $a' \in A_{\lfloor\theta\rfloor} \cap ]a, z[$. Let $b' = a' - a$ and $b'' = z - a'$. Then,

$$b = b' + b'',$$

$$b' \in B_{\lfloor\theta\rfloor} \cap ]0, 1] \subset B_m \cap ]0, 1]$$

and

$$b'' \in B_m \cap \big]0, b - b'\big[ \subset B_m \cap ]0, b - \beta_m[. \tag{4}$$

It follows when $b'' \leqslant 1$ that $b$ is a sum of two elements of $B_m \cap ]0, 1]$; otherwise we repeat the same process for $b''$ instead of $b$, and by induction we obtain (3). By the relation (4), the process must terminate. To complete the proof of Theorem 3 it is enough to verify that the following two assertions are true: *$B_m$ has a limit point in $[\frac{1}{\theta}, 1] \Rightarrow B_m$ has a limit point in $[0, \frac{1}{\theta}]$*, and *$B_m \cap [\frac{1}{\theta+1}, \frac{1}{\theta}]$ is not finite $\Rightarrow B_m \cap [0, \frac{1}{\theta+1}]$ is not finite*. The first proposition follows easily by iterating the continuous real function $T_0$ defined in the proof of Theorem 2 (Step 1), since $T_0(B_m) \subset B_m$ and $T_0(l)$ is a limit point of $B_m$ when $l$ is so, and the second implication is immediate by considering the injective map $-T_0$. □

**Remark 3.** The constant $\frac{1}{\theta+1}$ in Theorem 3 is not certainly the best one (in fact the initial aim was to prove that $B_m$ is discrete when 0 is not a limit point of $B_m$). For example by considering the functions $T_2(x)$ (defined in the proof of Theorem 2) and $f(x) = \theta^2 x - 1$ in the real variable $x$, we obtain when $\theta < 2$ that $B_m$ is discrete if and only if $B_m \cap [0, \frac{\theta-1}{\theta^2+1}]$ is finite. Finally, notice that Theorem 3 is also true for sets of the form $(A_1 \pm A_1 \pm \cdots \pm A_1) - A_{\lfloor\theta\rfloor}$ (like the one in Theorem 1).

## Acknowledgment

The author would like to thank the referee for careful reading of the paper.

## References

[1] C. Batut, D. Bernardi, H. Cohen, M. Olivier, Pari Calculator, copyright 1989, 1992.

[2] P. Borwein, K.G. Hare, Some computations on the spectra of Pisot and Salem numbers, Math. Comp. 71 (238) (2002) 767–780.

[3] Y. Bugeaud, On a property of Pisot numbers and related questions, Acta Math. Hungar 73 (1996) 33–39.

[4] P. Erdős, I. Joó, M. Joó, On a problem of Tamás Varga, Bull. Soc. Math. France 120 (1992) 507–521.

[5] P. Erdős, I. Joó, V. Komornik, Characterization of the unique expansion $1 = \sum_{i \geqslant 1} q^{n_i}$ and related problems, Bull. Soc. Math. France 118 (1990) 377–390.

[6] P. Erdős, I. Joó, V. Komornik, On the sequence of the numbers of the form $\varepsilon_0 + \varepsilon_1 q + \cdots + \varepsilon_n q^n$, $\varepsilon_i \in \{0, 1\}$, Acta Arith. 83 (1998) 201–210.
[7] P. Erdős, I. Joó, F.J. Schnitzer, On Pisot numbers, Ann. Univ. Sci. Budapest Eotvos Sect. Math. 39 (1996) 95–99.
[8] P. Erdős, V. Komornik, Developments in non integer bases, Acta. Math. Hungar. 79 (1998) 57–83.
[9] C. Frougny, Representations of numbers and finite automata, Math. Systems Theory 25 (1992) 37–60.
[10] C. Frougny, B. Solomyak, Finite beta-expansions, Ergodic Theory Dynam. Systems 12 (1992) 713–723.
[11] W. Parry, On the $\beta$-expansions of real numbers, Acta Math. Acad. Sci. Hungar. 11 (1960) 401–416.
[12] Y. Peres, B. Solomyak, Approximation by polynomials with coefficients ±1, J. Number Theory 84 (2000) 185–198.
[13] A. Rényi, Representations for real numbers and their ergodic properties, Acta Math. Hungar. 8 (1957) 477–493.
[14] K. Schmidt, On periodic expansions of Pisot and Salem numbers, Bull. London Math. Soc. 12 (1980) 269–278.
[15] T. Zaïmi, On an approximation property of Pisot numbers, Acta Math. Hungar. 96 (4) (2002) 309–325.
[16] T. Zaïmi, On an approximation property of Pisot numbers II, J. Théor. Nombres Bordeaux 16 (2004) 239–249.