People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
University of Oum El Bouaghi
Faculty of: Exact Sciences and Sciences of Nature and Life

Thesis

Presented to obtain

3rd Cycle Doctorate

Branch: Mathematics

Specialty: Mathematical Analysis and Applications

Title :

About Inverse Problems In Partial Differential Equation With Integer And Fractional Order

Presented by :

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Publicly defended on : ../../2024

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بسم الله الرحمن الرحيم

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Dedication

To my beloved parents, whose continuous support and encouragement have been my constant source of strength.

To my teachers and mentors, who have inspired me with their wisdom and encouraged me to pursue knowledge and excellence.

And to all those who have walked beside me and believed in me during this academic journey, this work is dedicated to you.

Acknowledgements

First of all, we thank Allah, our Creator, for giving us the strength to accomplish this work.

First and foremost, I would like to express my sincere thanks and deep gratitude to everyone who has contributed, directly or indirectly, to the completion of this modest work.

My sincere thanks and appreciation go to my supervisor, Professor Oussaeif Taki-Eddine, for his assistance and for the trust he placed in me throughout this work. He motivated each step of my work with valuable comments and helped me progress in my research.

I would also like to extend my heartfelt thanks to the jury members who honored me by accepting to evaluate this work.

I express my gratitude to all the teachers who contributed to my education, particularly those from the Department of Mathematics at Oum El Bouaghi University.

Finally, I extend my most sincere thanks to my family; my parents, my sisters, my brothers, my brothers-in-law, my sisters-in-law, and all my friends, especially " Laaraba Roqya", as well as my relatives, who accompanied, helped, supported, and encouraged me throughout the completion of this thesis. I also remember with gratitude my brother's wife and my grandmother.

Abstract

This work aims to investigate various inverse nonlinear problems with both integer and fractional orders for parabolic equations with additional information about the solution to the inverse problem in the form of integral conditions type. For the direct problem, the proof is based on the energy inequality method for the uniqueness of the solution and the density of the image of the operator generated by the considered linear problem for the existence. As well as the application of Banach's fixed-point theorem for study of the inverse problem.

Chapter one provides a review of fundamental concepts and necessary tools for this thesis.

Chapter two addresses to the existence and uniqueness of solution for the inverse problem involving a semi-linear equation with supplementary information of integral overdetermination condition type.

Chapter three is devoted to the existence and uniqueness of the solution of an inverse problem for a super-linear parabolic equation with an integral condition, where the nonlinear term is given by $|u|^p u$. This chapter is more general than the second one.

Chapter four deals with the existence and uniqueness of the solution of an inverse problem for a super linear patial differential equation with a fractional order with integral overdetermination condition.

Finally, the fifth chapter is devoted to the generalization of the previous chapter , where the non-linear term is given by u^p with $p > 1$ where we examine the inverse problem associated with determining the right-hand side of a non linear fractional parabolic equation.

Keywords: Inverse problem, integral condition, fixed-point theorem, energy inequality method, nonlinear parabolic equation, fractional partial differential equation, nonlinear partial differential equations.

Résumé

Ce travail vise à étudier divers problèmes inverses non linéaires pour des équations paraboliques d'ordres entiers et fractionnaires, avec des informations supplémentaires sur la solution sous forme de conditions intégrales. Pour le problème direct, la preuve repose sur la méthode de l'inégalité d'énergie pour l'unicité de la solution et sur la densité de l'image de l'opérateur généré par le problème étudié pour l'existence de la solution, ainsi que sur l'application du théorème du point fixe de Banach pour l'étude du problème inverse.

Le premier chapitre commence par un rappel de quelques concepts fondamentaux et des outils nécessaires pour ce travail.

Le deuxième chapitre traite de l'existence et de l'unicité de la solution d'un problème inverse pour une équation parabolique non linéaire d'ordre entier, avec une information supplémentaire sous forme d'une condition intégrale surdéterminée.

Le troisième chapitre examine l'existence et l'unicité de la solution d'un problème inverse pour une équation parabolique non linéaire avec une condition intégrale, où le terme non linéaire est donné par $|u|^p u$. Ce chapitre est plus général que le deuxième.

Le quatrième chapitre aborde l'existence et l'unicité de la solution d'un problème inverse pour une équation différentielle partielle super-linéaire d'ordre fractionnaire avec une condition supplémentaire de type intégrale.

Enfin, le cinquième chapitre est consacré à la généralisation de l'étude précédente du chapitre quatre, où le terme non linéaire est donné par u^p avec $p > 1$.

Mots-clés: Problème inverse, condition intégrale, théorème du point fixe, méthode de l'inégalité d'énergie, équation parabolique non linéaire, équation différentielle partielle fractionnaire, équation aux dérivées partielles non linéaires.

الملخص

يهدف هذا العمل إلى دراسة العديد من المسائل العكسية للمعادلات التكافئية غير الخطية ذات الرتب الصحيحة والكسرية مع وجود معلومة إضافية من نوع تكامل.

بالنسبة للمسألة المباشرة، يعتمد الإثبات على متراجحة الطاقة لوحدانية الحل وكثافة صورة المؤثر الناتج عن المسألة المدروسة من أجل وجود الحل، بالإضافة إلى تطبيق نظرية النقطة الثابتة لباناخ لدراسة المسألة العكسية.

الفصل الأول نبدأ بالتذكير ببعض المفاهيم الأولية الأساسية والأدوات اللازمة لهذا العمل.

الفصل الثاني يعالج وجود ووحدانية حل مسألة عكسية لمعادلة تكافئية غير خطية ذات رتبة صحيحة مقرونة بمعلومة إضافية من نوع شرط تكامل .

الفصل الثالث يتناول وجود ووحدانية حل مسألة عكسية لمعادلة تكافئية غير خطية مع وجود شرط تكامل إضافي حيث الحد غيرالخطي يعطى بواسطة $|u|^p u$ وهذا الفصل يعتبر كتعميم للنتائج السابقة .

الفصل الرابع يدرس وجود ووحدانية حل مسألة عكسية لمعادلة تفاضلية جزئية غير خطية ذات رتبة كسرية مع وجود شرط إضافي من نوع تكامل.

أخيرا، الفصل الخامس والذي خصص لتعميم الدراسة السابقة من الفصل الرابع، حيث الحد غيرالخطي يعطى بواسطة u^p مع $p > 1$.

الكلمات المفتاحية: المسألة العكسية، شرط التكامل، نظرية النقطة الثابتة، طريقة متراجحة الطاقة، معادلة تكافئية غير خطية، معادلة تفاضلية جزئية كسرية، معادلات تفاضلية جزئية غير خطية.

Contents

Notations

$\forall$	for all
$\exists$	there exists
$\equiv$	equivalent
$\in$	belongs to
$\subset$	subset of
lim	Limit
Σ	summation
$\mathbb{N}$	set of natural numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of positive real numbers
$\mathbb{R}^N$	Euclidean space of N-dimensional vectors
Ω	open bounded subset of $\mathbb{R}^N$
$\partial\Omega$	boundary of Ω
$\overline{\Omega}$	closure of Ω (i.e., Ω plus its boundary)
$d\mathbf{x}$	The derivative with respect to x
$C(\Omega)$	continuous functions from Ω to $\mathbb{R}$
∇u	gradient of u
$C^\infty(\Omega)$	the spaces of infinitely differentiable functions on Ω
$C_0^\infty(\Omega)$	infinitely differentiable functions with compact support on Ω
a.e.	almost everywhere
i.e.	that is.
$\rightarrow$	strong convergence
X	arbitrary Banach space
X'	dual space of the Banach space X
$\langle \cdot, \cdot \rangle_{L^2(\Omega)}$	scalar product on $L^2(\Omega)$
FPP	denotes the fractional parabolic problem
$R(A)$	$\{Au; u \in D(A)\} \subset$ F(The Range of A)
$G(A)$	Graph of the operator A
$D(\mathbb{R})$	The space of $\mathcal{C}^\infty$ on $\mathbb{R}$ with compact support(the space of test functions).
$D'(\mathbb{R})$	The space of distributions (the topological dual of $D(\mathbb{R})$) .
$\mathcal{C}_c$	Space of continuous functions with compact support.
$\square$	The proof is complete

General Introduction

Partial differential equations (PDEs) have been a crucial tool in mathematical modeling since the 18th century, with foundational contributions made by mathematicians such as Euler, d'Alembert, Lagrange, and Laplace. In the past forty years, numerous contemporary physical, mechanical, biological, and technological phenomena and issues have been modeled using partial differential equations (PDEs), either parabolic or hyperbolic, but with non-local conditions. Modeling problems with integral conditions arises in various fields, including plasma physics (particle diffusion in turbulent plasma) [[?]], heat conduction [[?],[?], [?], [?],[?],[?]], thermoelasticity [[?],[?]], some technological processes [[?]], medium oscillations [[?]], groundwater dynamics [[?], [?], [?]], moisture propagation [[?]], chemical engineering [[?]], semiconductor theory [[?]], and mathematical biology [[?]]and demographic models.

Integral conditions are applied to inverse problems in heat conduction theory [[?], [?], [?], [?], [?], [?], [?], [?]]. The inverse problems are new fields that appear in PDEs with unknown functions and additional information.

In science, an inverse problem involves determining unknown causes from observed effects. As a result, this problem is the opposite of the so-called " direct problem", where effects are determined based on known causes. Generally inverse problems are ill-posed due to the absence of one of the three Hadamard criteria(the concept of a well-posed problem is defined as follows: the solution exists, unique and continuously depends on the given data) . Most difficulties in solving ill- posed problems are caused by the instability of the solution.

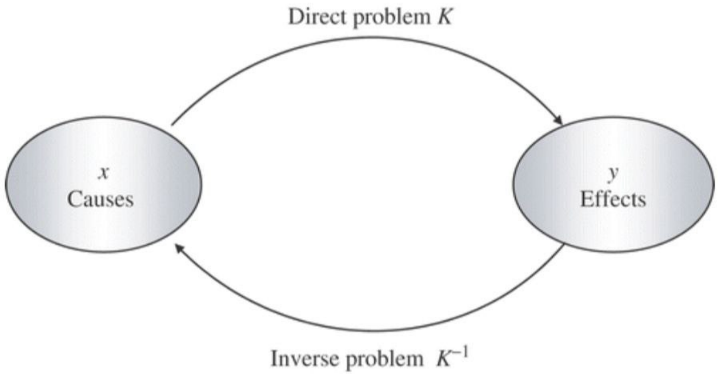

Figure 1. Representation of the direct or forward (given x, find $y = K(x)$) and of inverse problems (given y, find $x = K^{-1}(y)$).

In 1932, Hadamard, a French mathematician who lived from 1865 to 1963, published a well-known article in which he solved the first inverse problem by reconstructing the Cauchy problem's solution from its initial condition. Inverse problems are crucial in fields such as hydrogeology (calculation of hydraulic permeabilities), petroleum engineering (seismic and magnetic prospecting, identification of permeabilities in a reservoir), and medical imaging (X-rays, ultrasound, scanners, etc...)

To define the known function , we need additional information(an a priori condition) in the form of bounds and initial conditions with nonlocal conditions. Thus, Integral boundary conditions are applied when it is not feasible to measure the desired quantity directly at the boundary, but its total or average value is known. Traditional conditions like Dirichlet or Neumann, which are specified at discrete points, may not always be suitable because they depend on contexts where data can be measured at the domain's boundary. In certain situations, it is impractical to define the solution (e.g., pressure, temperature) at specific boundary points because only the average value of the solution is available along the boundary or part of it. The fundamental physical interpretation of integral conditions (such as total energy, average temperature, total mass of impurities, total flux, or moments) has led to increasing interest in these types of problems.

Inverse boundary value problems appear in diverse fields such as mineral exploration, seismology, medicine, biology, and industrial quality control, making them a dynamic and significant area of modern mathematics and physics. The investigation of inverse problems related to parabolic equations governed by nonlocal integral overdetermination conditions began with the examination of equations featuring time-independent coefficients and first and third kind boundary conditions. These studies established theorems that demonstrated the equivalence of the original inverse problem to a second-kind operator equation with a completely continuous operator.

Cannon et al. [[**?**],[**?**]] investigated inverse problems related to perfusion, source control coefficients, and temperature. Kamynin [[**?**]] demonstrated that the inverse problem of finding the forcing term of a parabolic equation, where the leading coefficient that depends on both temporal and spatial variables may be solved under a terminal overdetermination condition. Kamynin [[**?**]] studied the existence of solutions to the initial boundary problem for parabolic equations, M.J. Huntul and Taki-Eddine Oussaeif [[**?**]] investigated the unique solvability of the inverse problem related to determining the source term of a parabolic equation whose leading coefficient depends on time variable under nonlocal integral overdetermination condition of type:

$$\int_{\Omega} \xi(x,t)v(x)dx = \Phi(t), \quad t \in (0,T), \tag{0.0.1}$$

The objective of this thesis is to investigate the inverse problem of determining the right hand- for various problems including integral condition of type (0.0.1), for a new class of problems involving fractional equations with an integral condition of type (0.0.1), and for nonlinear parabolic problems with an integral condition of type (0.0.1).

Using functional analysis, we demonstrate the continuity of dependence, as well as the existence and uniqueness of the solution, based on the formulation of the direct problem. This approach relies on the density of the operator's range, combined with the energy inequality technique, commonly referred to as the method of a priori estimates. The framework of this method can be summarized as follows:
First, the problem is formulated as an operator equation:

$$Lu = F, \qquad u \in D(L); \tag{0.0.2}$$

where the operator L is considered from a Banach space E to a Hilbert space F, both appropriately chosen.

Next, a priori estimates for the operator L are established.
Finally, the density of the range of this operator in the space F is demonstrated.

To ensure the solvability and uniqueness of the inverse problem. We establish certain appropriate conditions and utilize the fixed point theorem to show both the existence and uniqueness of the solution.

Content of the thesis

This thesis includes interesting and original results that have been published in international journals. These can be consulted according to the references cited in the bibliography. The thesis begins with an introduction where a history of the inverse problems studied, the interest, the objective of the theme addressed is presented and it is composed of five chapters presented as follows:

Chapter 1: is devoted to reminders of some fundamental preliminary notions and the tools necessary in this work concerning unbounded linear operators, functional spaces and some technical lemmas.

Chapter 2: We have investigated the inverse problem of a superlinear parabolic Dirichlet equation with an additional integral over-determination condition. In this context, we have used the energy inequality to establish the solvability of the direct problem and applied the fixed point technique to address the inverse problem. Specifically, This chapter focuses on investigating the existence and uniqueness of solutions for the inverse problem related to the superlinear parabolic Dirichlet equation with a second-type integral condition, by reformulating the problem in terms of the fixed point principle.

Chapter 3:It can be regarded as an extension of the previous chapter because the equation being examined is more general than the previous one.

Chapter 4: we have investigated the inverse problem of determining the right-hand side of a nonlinear fractional parabolic equation with an integral overdetermination condition. To ensure solvability, we have employed the same technique used in the previous chapters,

Chapter 5:We have investigated the inverse problem concerning the identification of the right-hand side of a nonlinear fractional parabolic equation. This equation included an integral overdetermination supplementary condition, which we have effectively addressed and analyzed.
This chapter can be considered as an extension of the previous chapter.

Chapter 1

Preliminaries

Introduction

This chapter is dedicated to a review of the essential concepts and fundamental notions of analysis that will be employed throughout this work and will serve as a permanent reference in the subsequent chapters. These important concepts are presented in the form of definitions, theorems, corollaries, and lemmas. For further details, references to the relevant literature will be systematically provided.

Definition 1.0.1. *A map S is considered an extension of T if the domain of T is contained within the domain of S (i.e.,$G(T) \subset G(S)$), and for every u in the domain of T, the values of T and S coincide, meaning $Tu = Su$ for all $u \in D(T)$. In other words, the graph of T is a subset of the graph of S (i.e.,$G(T) \subset G(S)$)).*

Remark 1.0.1. *Not every subspace of $E \times F$ is the graph of an operator.*

Definition 1.0.2. *A map T is said to be closed if its graph $G(T)$ is a closed subset of $E \times F$.*

Definition 1.0.3. *A linear operator T is said to be closable in E if it admits a closed extension. It follows immediately that T is closable in E if and only if the closure $\overline{G(T)}$ of its graph is a graph (since $T \subset \overline{T}$ implies $G(T) \subset G(\overline{T})$, and since the extension $\overline{T}$ is closed, $G(\overline{T})$ is closed. Thus, $G(T) \subset G(\overline{T})$ implies $\overline{G(T)} \subset \overline{G(\overline{T})} = G(\overline{T})$). In other words, T is closable if and only if for any sequence $(u_n)_{n\in\mathbb{N}} \subset D(T)$, where $u_n \to 0$ and $Tu_n \to v$, it follows that $v = 0$. The closed operator $\overline{T}$, whose graph $G(\overline{T}) = \overline{G(T)}$, is called the closure of T. (The fact that $G(\overline{T}) = \overline{G(T)}$ implies that $\overline{G(T)}$ is a graph: $\forall\, (u_n, Tu_n) \in G(T) \Rightarrow$*

$\left(\lim\limits_{n\to+\infty} u_n, \lim\limits_{n\to+\infty} Tu_n\right) \in \overline{G(T)}$ *and* $(0,v) \in \overline{G(T)}$, *which necessitates* $v = 0$ *to ensure that* $\overline{G(T)}$ *is a graph.*

Theorem 1.0.1. ***(Isomorphism Theorem)***
Let E and F be Banach spaces, and let T be a continuous bijective linear operator mapping E onto F. Then, the inverse operator T^{-1} *is also continuous from F to E.*

Theorem 1.0.2. ***(Closed Graph Theorem)***
Let E and F be two Banach spaces. Suppose T is a linear operator from E to F, and its graph $G(T)$ *is closed in the product space* $E \times F$*. Then T is continuous.*

1.1 Relationship Between Orthogonality and Density in Hilbert Spaces

Definition 1.1.1. *[**?**] Let M be a subspace of the Hilbert space F. The orthogonal complement of M, denoted* $M^\perp$*, is defined as*

$$M^\perp = \{f \in F, < f, g >_F = 0, \forall g \in M\}.$$

.

Corollary 1.1.1. *[**?**] Let H be a Hilbert space, and F a subspace of H. Then F is dense in H if and only if* $F^\perp = \{0\}$*. Thus, to demonstrate the density of a subspace F in H, it is sufficient to verify:*

$$[(x \mid y) = 0, \forall x \in F] \Rightarrow y = 0$$

Proposition 1.1.1. *[**?**] Let M be a subspace within the Hilbert space F. The subspace M is dense in F if and only if* $M^\perp = \{0\}$*.*

Proof. First, suppose M is dense in F. Let $f \in M^\perp \subset F$, and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of elements in M that converges to f. We have $\langle f, f_n\rangle_F = 0$ for all $n \in \mathbb{N}$. Taking the limit, we conclude that $\|f\|_F = 0$. Thus, $f = 0$, which gives $M^\perp = \{0\}$.

Conversely, suppose $M^\perp = \{0\}$. Then we have $(M^\perp)^\perp = \{0\}^\perp = F$. Since $M \subset \overline{M}$, implies that $(\overline{M})^\perp \subset M^\perp$, and hence $(M^\perp)^\perp \subset \left((\overline{M})^\perp\right)^\perp$. Since $\bar{M}$ is closed, we have $\left((\overline{M})^\perp\right)^\perp = \overline{M}$, thus $(M^\perp)^\perp \subset \overline{M} \Rightarrow F \subset \overline{M}$. Therefore, $F = \overline{M}$. □

1.2 Functions Traces from Sobolev Spaces

1.2.1 Existence of Trace

Consider the set $L^2(\partial\Omega) = \left\{w \text{ measurable on } \partial\Omega \mid \int_{\partial\Omega} w^2 d\Gamma < +\infty\right\}$. *This is a Hilbert space with the usual inner product.*

Theorem 1.2.1. *[?](**Existence of Trace**) Let Ω be a bounded open set in $\mathbb{R}^N$ with a "sufficiently regular" boundary. Then the trace application*

$$l : C^\infty(\overline{\Omega}) \longrightarrow L^2(\partial\Omega)$$
$$u \longrightarrow lu = u\backslash_{\partial\Omega}$$

Extends by continuity to a continuous linear application, still denoted l, from $H^1(\Omega)$ into $L^2(\partial\Omega)$, and there exists a constant c^* independent of u such that

$$\|lu\|_{L^2(\partial\Omega)} \leq c^*\|u\|_{H^1(\Omega)}.$$

Remark 1.2.1. *[?] The trace application l is not surjective from $H^1(\Omega)$ into $L^2(\partial\Omega)$. However, it is surjective onto $H^{1/2}(\partial\Omega)$, where $H^{1/2}(\partial\Omega)$ is the Sobolev space of fractional index $1/2$.*
We have:

$$H^{1/2}(\partial\Omega) = \left\{w \in L^2(\partial\Omega), \text{ such that } \quad \exists v \in H^1(\Omega), w = l(v)\right\}.$$

1.2.2 Density of $W^{1-\frac{1}{p},p}(\partial\Omega) \in Ł^2(\partial\Omega)$

Proposition 1.2.1. *[?] Let Ω be an open set in $\mathbb{R}^N$ of class C^1. Consequently, $L^2(\partial\Omega)$ contains a dense set $W^{1-\frac{1}{p},p}(\partial\Omega)$.*

Proof. [?] Given that Ω is of class C^1, there exists a continuous linear extension g from $W^{1,p}(\Omega)$ into $W^{1,p}(\mathbb{R}^N)$. For any $U \in W^{1,p}(\Omega)$, we have

$$l(U) = u \in L^2(\partial\Omega).$$

A sequence $\{U_n\}_{n\in\mathbb{N}} \subset D(\mathbb{R}^N)$ exists according to the density of $D(\mathbb{R}^N)$ in $W^{1,p}(\mathbb{R}^N)$, such that:

$$\|U_n - g(U)\|_{W^{1,p}(\mathbb{R}^{\mathbb{N}})} \longrightarrow 0, \quad g(U) \in W^{1,p}(\mathbb{R}^{\mathbb{N}}).$$

Let u_n be the restriction of U_n to Ω (i.e., $\{u_n\}_{n\in\mathbb{N}} \subset D(\Omega)$). Given that $g(U)$ is restricted to Ω by U, we can deduce that:

$$\|u_n - U\|_{W^{1,p}(\Omega)} \longrightarrow 0, \quad U \in W^{1,p}(\Omega).$$

The trace application l's continuity leads to the following conclusion:

$$\|l(u_n - U)\|_{L^2(\partial\Omega)} \leq c^* \|u_n - U\|_{W^{1,p}(\Omega)},$$

where c^* is a positive constant. Then, by the linearity of the trace application l, we have:

$$\|l(u_n) - u\|_{L^2(\partial\Omega)} = \|l(u_n - U)\|_{L^2(\partial\Omega)} \leq c^* \|u_n - U\|_{W^{1,p}(\Omega)}.$$

Which implies:

$$\|l(u_n) - u\|_{L^2(\partial\Omega)} \leq c^* \|u_n - U\|_{W^{1,p}(\Omega)} \longrightarrow 0.$$

Therefore, we find:

$$\|l(u_n) - u\|_{L^2(\partial\Omega)} \longrightarrow 0$$

Since $l(u_n) \in l(W^{1,p}(\Omega)) = W^{1-\frac{1}{p},p}(\partial\Omega)$, we conclude that $W^{1-\frac{1}{p},p}(\partial\Omega)$ is dense in $L^2(\partial\Omega)$. □

1.3 Gronwall Lemma

The Gronwall lemma plays a significant role in the estimation of integro-differential terms and is frequently used to obtain a priori estimates in the norms of the above-mentioned and other spaces.

Lemma 1.3.1. *[?] (**Gronwall**)*
If α and β are non-negative and integrable functions on $(0,T)$, with β being non-decreasing on $(0,T)$, and $\Phi \in L^1(0,T)$, $\Phi > 0$, then if:

$$\alpha(t) \leq \beta(t) + \int_0^t \Phi(s)\alpha(s)ds, \tag{1.3.1}$$

it follows that:

$$\alpha(t) \leq \beta(t) exp(\theta(t)),$$

where

$$\theta(t) = \int_0^t \Phi(s)ds.$$

Proof. We define

$$k(t) = \exp(-\theta(t)) \int_0^t \Phi(s)\alpha(s)ds.$$

Then, for all $t \in [0,T]$, the estimate

$$\frac{\partial}{\partial t}k(t) = \Phi(t)\exp(-\theta(t))\left(\alpha(t) - \int_0^t \Phi(s)\alpha(s)ds\right) \leqslant \Phi(t)\beta(t)\exp(-\theta(t)),$$

results from (**??**) and $\Phi(t) > 0$. with $k(0) = 0$. By definition, integration over $[0,T]$ leads to

$$k(t) \leqslant \int_0^t \Phi(s)\beta(s)\exp(-\theta(s))ds.$$

Again, using (**??**), we have

$$\exp(-\theta(t))(\alpha(t) - \beta(t)) \leqslant \exp(-\theta(t))\int_0^t \Phi(s)\alpha(s)ds = k(t) \leqslant \int_0^t \Phi(s)\beta(s)\exp(-\theta(s))ds.$$

Thus, we find

$$\alpha(t) - \beta(t) \leqslant \int_0^t \Phi(s)\beta(s)\exp(\theta(t) - \theta(s))ds, \tag{1.3.2}$$

if β is non-decreasing on $(0,T)$, from (**??**) and given that $\Phi(t) > 0$, we get

$$\begin{aligned}
\alpha(t) &\leqslant \beta(t) + \int_0^t \Phi(s)\beta(s)\exp(\theta(t) - \theta(s))ds \\
&\leqslant \beta(t)\left[1 + \int_0^t \Phi(s)\exp(\theta(t) - \theta(s))ds\right] \\
&\leqslant \beta(t)\left[1 + \exp(\theta(t))\int_0^t \frac{\partial}{\partial s}[-\exp(-\theta(s))]ds\right]. \\
&\leqslant \beta(t)[1 + \exp(\theta(t))[-\exp(-\theta(t)) + 1]] \\
&\leqslant \beta(t)\exp(\theta(t))
\end{aligned}$$

This completes the lemma's proof. □

1.4 L^p Spaces:Definition and basic properties

Definition 1.4.1. *[**?**]*

Let $p \in \mathbb{R}$ with $1 \leq p < \infty$; the space $L^p(\Omega)$ is efinedas follows:

$$L^p(\Omega) = \left\{f : \Omega \longrightarrow \mathbb{R};\ f \text{ is measurable and } |f|^p \in L^1(\Omega)\right\}.$$

Where

$$\|f\|_{L^p(\Omega)} = \left[\int_\Omega |f(x)|^p dx\right]^{\frac{1}{p}}.$$

Definition 1.4.2. *[?] Let's define*

$L^\infty(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}; f\, mesurable\ and\ \exists \quad a\ constant\ C\ such\ that \quad |f(x)| \leq C\ a.e.\ on \quad \Omega\}$

We denote

$$\|f\|_{L^\infty(\Omega)} = inf\{\ C;\ a.e.\ on\ \Omega\}.$$

Theorem 1.4.1. *[?] (**Holder's Inequality**)*
If $1 \leq p \leq \infty$ and q is the conjugate exponent of p(i.e., $\frac{1}{p} + \frac{1}{q} = 1$), then for $f \in L^p(\Omega)$andg $\in L^q(\Omega)$, Holder's inequality states that: $f.g \in L^1(\Omega)$ and

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}.$$

Remark 1.4.1. *If $p = q = 2$, Holder's Inequality is known as the Cauchy-Schwarz Inequality.*

Definition 1.4.3. *(**the Cauchy's ε-inequality**) is defined as follows:*

$$2|ab| \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \quad a, b \in \mathbb{R}.$$

Definition 1.4.4. *[?] ($L^p(0, T; X)$ **Space**)*

X being a Banach space, we denote by $L^p(0, T; X)$ the space of (classes of) functions $t \rightarrow f(t)$ of $]0, T[\rightarrow X$ which are measurable at values in X and such that

$$\left(\int_0^T \|f(t)\|_X^p \, \mathrm{d}t \right)^{1/p} = \|f\|_{L^p(0,T;X)} < \infty \tag{1.4.1}$$

*if $p = \infty$, we replace the norm (**??**) by*

$$\sup_{t \in]0,T[} \|f(t)\|_X = \|f\|_{L^\infty(0,T;X)}$$

thus normalized the space $L^p(0, T; X)$ is complete . Naturally, we have:

$$L^p\,(0, T; L^p(\Omega)) = L^p(Q)$$

We denote by $C((0, T), L_2(\Omega))$ the space comprising all continuous functions on $(0, T)$ with values in $L_2(\Omega)$. The corresponding norm is given by

$$\|u\|_{C((0,T),L_2(\Omega))} = \max_{0 \leq t \leq T} \|u\|_{L_2(\Omega)} < \infty. \tag{1.4.2}$$

1.5 Sobolev Spaces $W^{1,p}(\Omega)$: definition and fundamental characteristics

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $p \in \mathbb{R}$ such that $1 \leq p \leq \infty$.

Definition 1.5.1. *[?] The Sobolev space $W^{1,p}(\Omega)$ is characterized as follows:*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \setminus \exists g_1, g_2, ..., g_N \quad such \quad that: \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi \quad \forall \varphi \in C_c^\infty(\Omega) \quad \forall i = 1, 2, ..., N \right\}$$

Furthermore, we define:

$$W^{1,2}(\Omega) = H^1(\Omega).$$

For any function $u \in W^{1,p}(\Omega)$, we set:

$$\frac{\partial u}{\partial x_i} = g_i \quad and \quad \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N} \right) = gradu$$

The norm in the space $W^{1,p}(\Omega)$ is given by:

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|_{L^p(\Omega)}$$

Alternatively, this can be expressed using the corresponding norm:

$$\left(\|u\|^p_{L^p(\Omega)} + \sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} \qquad (if \quad 1 \leq p < \infty).$$

The space $H^1(\Omega)$ is defined with the inner product

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2(\Omega)}$$

The associated norm is expressed as:

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \sum_{i=1}^{N} \|\frac{\partial u}{\partial x_i}\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}$$

which is equivalent to the norm in $W^{1,2}(\Omega)$.

Proposition 1.5.1. *[**?**] For $1 \leq p \leq \infty$, the space $W^{1,p}(\Omega)$ is a Banach space. It is separable for $1 \leq p < \infty$ and reflexive for $1 < p < \infty$. Furthermore, $H^1(\Omega)$ is a separable Hilbert space.*

The space $W_0^{1,p}(\Omega)$

Definition 1.5.2. *[**?**] Let $1 \leq p < \infty$. The space $W_0^{1,p}(\Omega)$ is defined the closure of $C_c^1(\Omega)$ in $W^{1,p}(\Omega)$.*
We denote:

$$H_0^1(\Omega) = W_0^{1,2}(\Omega).$$

The space $W_0^{1,p}(\Omega)$, with the norm induced from $W^{1,p}(\Omega)$, is a separable Banach space and is reflexive when $1 < p < \infty$. Additionally, $H_0^1(\Omega)$ is a Hilbert space with the inner product defined in $H^1(\Omega)$.

Corollary 1.5.1. *[**?**](**Poincaré Inequality**).*
Assume that the open set Ω is bounded. Then there exists a constant C (depending on Ω and p) such that

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)} \qquad \forall u \in W_0^{1,p}(\Omega) \qquad (1 \leq p < \infty).$$

In other words, on $W_0^{1,p}(\Omega)$, the quantity $\|\nabla u\|_{L^p(\Omega)}$ is an equivalent norm to the norm $\|u\|_{W^{1,p}(\Omega)}$.

Theorem 1.5.1. *[**?**] Let Ω be a bounded domain in $\mathbb{R}^n$ and let $u \in H_0^1(\Omega)$. Then there exists a positive constant C which depends only on Ω and n such that for all $u \in H_0^1(\Omega)$ we have:*

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}. \tag{1.5.1}$$

1.6 Convergence in $D'(\Omega)$

In what follows, Ω denotes an open subset of $\mathbb{R}^n$, and D' will refer to $D'(\Omega)$.

Definition 1.6.1. *Let $(T_j)_{j\in\mathbb{N}}$ be a sequence in D', and let $T \in D'$. We say that (T_j) converges to T in D' if:*

$$\lim_{j\to+\infty} \langle T_j, \varphi \rangle = \langle T, \varphi \rangle, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Properties

(i) *If $(T_j) \to T$, then for all $\alpha \in \mathbb{N}^n$:*

$$\partial^\alpha T_j \to \partial^\alpha T.$$

(ii) *Convergence in $L^p_{loc}(\Omega)$, $1 \le p \le +\infty$, implies convergence in D'. Indeed, let $(f_j) \subset L^p_{loc}(\Omega)$ such that $f_j \to f$ in L^p_{loc}. Let $\varphi \in C_c^\infty(\Omega)$, and assume supp$(\varphi) \subset K$. Then, by Hölder's inequality:*

$$\left|\langle f_j - f, \varphi\rangle\right| = \left|\int_K (f_j - f)\varphi\, dx\right| \le \int_K |f_j - f||\varphi|\, dx \le \|f_j - f\|_{L_p(K)}\|\varphi\|_{L_q(K)} \to 0,$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

1.7 Fractional Differentiation

Fractional differentiation is a concept that generalizes classical differentiation to a non-integer order. It naturally integrates into the mechanical modeling of materials that retain memory of past transformations. This has led to a significant interest in fractional calculus and analysis over recent decades. While classical differential calculus provides powerful tools for modeling many phenomena in applied sciences, these tools fall short in accounting for the anomalous dynamics present in certain complex systems found in nature or in societal interactions. Experimental results indicate that many processes associated with complex systems exhibit non-local dynamics involving long-term effects. The history of fractional calculus extends from the late 17th century to the present day. Specialists agree that its origin dates back to the end of the year 1695 when L'Hôpital raised a question to Leibniz by inquiring about the meaning of $\frac{d^n y}{dx^n}$ when $n = \frac{1}{2}$.Liouville, who wrote nine papers between 1832 and 1837, is credited with making the first significant effort to give a logical formulation for the fractional derivative. Separately, Riemann put forth what is now known as the "Riemann-Liouville Approach," which proved to be essentially Liouville's methodology. Later, further theories-including those of Weyl, Caputo, and Grunwald-Letnikov-came to light. This hypothesis was regarded as an abstract idea at the time, involving only mathematical calculations and having very few practical applications. Beginning in the 1990s, fractional differential equations started to show up in a variety of domains, including physics, engineering, biology, and mechanics, signaling the shift from pure mathematical formulations to applications.

1.7.1 Special Function

Gamma Function

One of the key functions in fractional calculus is the Gamma function Γ, which holds a crucial role in the development of the theory of fractional calculus.

Definition 1.7.1. *[?] The Eulerian Gamma function (or Eulerian integral of the second kind) is the function denoted by Γ defined by:*

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt$$

where x is any complex number such that $Re(x) > 0$.

Properties of the Gamma Function

Proposition 1.7.1. *For all $x \in \mathbb{R}_*^+$, we have:*

$$\Gamma(x+1) = x \cdot \Gamma(x)$$

.

Definition 1.7.2. *[?]*
The fractional integral of a function $h \in L^1[a,b]$ with order $\sigma \in \mathbb{R}^+$ is given by:

$$I_a^\sigma h(t) = \frac{1}{\Gamma(\sigma)} \int_a^t \frac{h(s)}{(t-s)^{1-\sigma}} ds.$$

Riemann-Liouville Approach

Definition 1.7.3. *[?] Let h be a locally integrable function defined on $[0,T]$ and let $\sigma \in \mathbb{R}^+$. The derivative of order σ of h is defined by:*

1. *Left Riemann-Liouville derivative*

$$ {}_0^R\partial_t^\sigma h(t) := \frac{1}{\Gamma(n-\sigma)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{h(\tau)}{(t-\tau)^{\sigma-n+1}} d\tau \tag{1.7.1}$$

2. *Right Riemann-Liouville derivative*

$$ {}_t^R\partial_T^\sigma h(t) := \frac{(-1)^n}{\Gamma(n-\sigma)} \frac{\partial^n}{\partial t^n} \int_t^T \frac{h(\tau)}{(t-\tau)^{\sigma-n+1}} d\tau \tag{1.7.2}$$

where the integer n is chosen such that: $n-1 < \sigma < n$.

*In general, the non-integer derivative of a constant function in the sense of Riemann-Liouville[**?**] is neither zero nor constant. Instead, we have:*

$$\begin{aligned}
{}^{R}D_t^{\alpha}C &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{C}{(t-\tau)^{\alpha-n+1}}d\tau \\
&= \frac{C}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1}d\tau \\
&= -\frac{C}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left.\frac{(t-\tau)^{n-\alpha}}{n-\alpha}\right|_{\tau=0}^{\tau=t} \\
&= \frac{C}{\Gamma(n-\alpha)(n-\alpha)}\frac{d^n}{dt^n}t^{n-\alpha} \\
&= \frac{C(n-\alpha)(n-\alpha-1)\ldots(n-\alpha-(n-1))}{\Gamma(1-\alpha)(n-\alpha)(n-\alpha-1)\ldots(n-\alpha-(n-1))}t^{-\alpha} \\
&= \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}$$

1.7.2 Fractional Derivatives in the Sense of Caputo

*The theory of fractional derivatives and integrals has developed significantly thanks to the Riemann-Liouville type fractional differentiation [**?**], especially in its applications to pure mathematics (solving integer-order differential equations, defining new functions classes, summarising series, etc.). Nonetheless, the well-known pure mathematical technique needs to be modified in various ways due to contemporary technology. A plethora of studies have emerged, particularly in the fields of solid mechanics and the theory of viscoelasticity, where fractional derivatives are employed to precisely characterize material properties. The formulation of beginning conditions for fractional order differential equations is a necessary consequence of a mathematical modeling based on rheological models. Definitions of fractional derivatives that permit the use of physically comprehensible initial conditions, such as $f(a), f^{'}(a),...$, etc., are necessary for applied issues. Initial value problems with such initial conditions can be solved mathematically, but M. Caputo (in the 1960s) presented a solution in his definition, which he adapted in collaboration with Mainardi within the framework of viscoelasticity theory. This leads to the introduction of a more restrictive fractional derivative compared to the Riemann-Liouville derivative.*

Definition 1.7.4. *Let $\sigma \in \mathbb{R}^+$ and h be a locally integrable function defined on $[0,T]$. The σ -order derivative of h is defined by:*

1. *Left Caputo derivative*

$$ {}_0^C\partial_t^\sigma h(t) := \frac{1}{\Gamma(n-\sigma)} \int_0^t \frac{h^{(n)}(\tau)}{(t-\tau)^{\sigma-n+1}} d\tau. \tag{1.7.3}$$

2. *Right Caputo derivative*

$$ {}_t^C\partial_T^\sigma h(t) := \frac{(-1)^n}{\Gamma(n-\sigma)} \int_t^T \frac{h^{(n)}(\tau)}{(t-\tau)^{\sigma-n+1}} d\tau \tag{1.7.4}$$

with n being a positive integer satisfying the inequality $n-1<\sigma<n$.

1.7.3 Relationship between the Riemann-Liouville and Caputo Derivatives

Let $\alpha \in \mathbb{R}^+$ with $n-1<\alpha<n$, for $n \in \mathbb{N}^*$. Assume χ is a function for which both the Caputo fractional derivative ${}_0^c\partial_t^\alpha \chi(t)$ and the Riemann-Liouville fractional derivative ${}_0^R\partial_t^\alpha \chi(t)$ exist. Then

$$ {}_0^R\partial_t^\alpha \chi(t) = {}_0^c\partial_t^\alpha \chi(t) + \sum_{k=0}^{n-1} \frac{\chi^{(k)}(0)(t)^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{1.7.5}$$

If $n=1$, we have

$$ {}_0^R\partial_t^\alpha \chi(t) = {}_0^c\partial_t^\alpha \chi(t) + \frac{\chi(0)}{\Gamma(1-\alpha)t^\alpha}. \tag{1.7.6}$$

Remark 1.7.1. *If $\chi^{(k)}(0)=0$ for $k=0,1,\ldots,n-1$, then the Riemann-Liouville and Caputo fractional derivatives are the same, i.e.,*

$$ {}_0^R\partial_t^\alpha \chi(t) = {}_0^c\partial_t^\alpha \chi(t)$$

Proposition 1.7.2. *Let $n-1<\alpha<n$, then we have:*

$$\lim_{\alpha\to n} {}^R\partial_t^\alpha \psi(t) = \lim_{\alpha\to n} {}^C\partial_t^\alpha \psi(t) = \psi^{(n)}(t)$$

Proof. Using integration by parts and Proposition (**??**), we obtain:

$$\begin{aligned} {}^R\partial_t^\alpha \psi(t) &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d^n}{dt^n}\right)\int_0^t \frac{\psi(\tau)}{(t-\tau)^{\alpha-n+1}}\,d\tau \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d^n}{dt^n}\right)\left(-\psi(\tau)\frac{(t-\tau)^{n-\alpha}}{n-\alpha}\Big|_{\tau=0}^{\tau=t} + \int_0^t \psi'(\tau)\frac{(t-\tau)^{n-\alpha}}{n-\alpha}\,d\tau\right) \end{aligned}$$

$$= \frac{1}{\Gamma(n-\alpha+1)}\left(\frac{d^n}{dt^n}\right)\left(\psi(0)t^{n-\alpha} + \int_0^t \psi'(\tau)(t-\tau)^{n-\alpha}\,d\tau\right) \tag{1.7.7}$$

Similarly,

$$\begin{aligned}
{}^c\partial_t^\alpha \psi(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{\psi^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\,d\tau \\
&= \frac{1}{\Gamma(n-\alpha)}\left(-\psi^{(n)}(\tau)\frac{(t-\tau)^{n-\alpha}}{n-\alpha}\Big|_{\tau=0}^{\tau=t} + \int_0^t \psi^{(n+1)}(\tau)\frac{(t-\tau)^{n-\alpha}}{n-\alpha}\,d\tau\right)
\end{aligned}$$

$$= \frac{1}{\Gamma(n-\alpha+1)}\left(\psi^{(n)}(0)t^{n-\alpha} + \int_0^t \frac{d\psi^{(n)}(\tau)}{d\tau}(t-\tau)^{n-\alpha}\,d\tau\right) \tag{1.7.8}$$

Taking the limit as $\alpha \to n$ in (**??**) and (**??**), we have:

$$\begin{aligned}
\lim_{\alpha\to n} {}^R\partial_t^\alpha \psi(t) &= \frac{d^n}{dt^n}\left(\psi(0) + \int_0^t \psi'(\tau)\,d\tau\right) \\
&= \psi^{(n)}(t)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\alpha\to n} {}^c\partial_t^\alpha \psi(t) &= \left(\psi^{(n)}(0) + \int_0^t \frac{d\psi^{(n)}(\tau)}{d\tau}\,d\tau\right) \\
&= \psi^{(n)}(t)
\end{aligned}$$

So, we have

$$\lim_{\alpha\to n} {}^R\partial_t^\alpha \psi(t) = \lim_{\alpha\to n} {}^C\partial_t^\alpha \psi(t) = \psi^{(n)}(t)$$

.

□

1.7.4 The spaces ${}^lH^\alpha(Q)$, ${}^rH^\alpha(Q)$ and ${}^cH^\alpha(Q)$

Definition 1.7.5. *[**?**] As the closure of $C_0^\infty(Q)$ with regard to the following norm $\|\psi\|_{{}^lH_0^\alpha(Q)}$ for every real $\alpha > 0$, we define the space ${}^lH_0^\alpha(Q)$:*

$$\|\psi\|_{{}^lH^\alpha(Q)} := \left(\|\psi\|_{L^2(Q)}^2 + |\psi|^2{}_{{}^lH_0^\alpha(Q)}\right)^{\frac{1}{2}} \tag{1.7.9}$$

where

$$|\psi|^2{}_{{}^lH_0^\alpha(Q)} = \left\|{}_0^R\partial_t^\alpha \psi\right\|_{L^2(Q)}$$

Definition 1.7.6. *[**?**] For any real $\alpha > 0$, we define the space ${}^{R}H_0^{\alpha}(Q)$ as the closure of $C_0^{\infty}(Q)$ with respect to the following norm $\|\psi\|_{{}^{R}H_0^{\alpha}(Q)}$:*

$$\|\psi\|_{{}^{R}H_0^{\alpha}(Q)} := \left(\|\psi\|_{L^2(Q)}^2 + |\psi|_{{}^{R}H_0^{\alpha}(Q)}^2\right)^{\frac{1}{2}} \tag{11}$$

where

$$|\psi|_{{}^{R}H_0^{\alpha}(Q)}^2 := \left\|{}_t^R\partial_T^{\alpha}\psi\right\|_{L^2(Q)}^2$$

Lemma 1.7.1. *[**?**, **?**]*

If $\psi \in {}^{l}H^{\alpha}(Q)$ and $v \in C_0^{\infty}(Q)$ for any real $\alpha \in \mathbb{R}_+$, then

$$\left({}^{R}\partial_t^{\alpha}\psi(t), v(t)\right)_{L^2(Q)} = \left(\psi(t), {}_t^R\partial^{\alpha}v(t)\right)_{L^2(Q)}$$

Lemma 1.7.2. *[**?**, **?**]*

For $0 < \alpha < 2, \alpha \neq 1, \psi \in H_0^{\frac{\alpha}{2}}(Q)$, we have :

$${}^{R}\partial_t^{\alpha}\psi(t) = {}^{R}\partial_t^{\frac{\alpha}{2}}\,{}^{R}\partial_t^{\frac{\alpha}{2}}\psi(t)$$

Lemma 1.7.3. *[**?**, **?**] Semi-norms: $|.|_{{}^{l}H^{\alpha}(Q)}, |.|_{{}^{R}H^{\alpha}(Q)}$ and $|.|_{{}^{c}H^{\alpha}(Q)}$ are equivalent if $\alpha \in \mathbb{R}_+$,*
$\alpha \neq n + \frac{1}{2}$. After that, we pose.

$$|\psi|_{{}^{l}H^{\alpha}(Q)} \cong |\psi|_{{}^{R}H^{\alpha}(Q)} \cong |\psi|_{{}^{c}H^{\alpha}(Q)}.$$

Lemma 1.7.4. *[**?**]*

*For every real number $\alpha > 0$, the space ${}^{l}\mathrm{H}_0^{\alpha}(\mathrm{Q})$ equipped with the norm defined in (**??**), is complete.*

Chapter 2

Inverse problem of a semilinear parabolic equation with an integral over determination condition

Introduction

In [?], the authors addressed the existence and uniqueness of the solution to an inverse problem that combines a Dirichlet condition with an integral condition for a linear parabolic equation. This motivated me to investigate a similar non-local problem for a nonlinear parabolic equation, where the nonlinear term y^3 is added to the left-hand side of the studied equation, along with an integral-type condition (??).

2.1 Setting of the problem

In this chapter, our goal is to explore the unique solvability of the inverse problem that involves identifying the pair of functions $\{y(x,t), f(t)\}$ which satisfy the following parabolic equation:

$$y_t(x,t) - a\frac{\partial^2 y(x,t)}{dx^2} + by(x,t) + cy^3(x,t) = f(t)h(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{2.1.1}$$

with the initial condition

$$y(x,0) = \varphi(x), \quad x \in \Omega, \tag{2.1.2}$$

the boundary condition

$$y(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{2.1.3}$$

and the nonlocal over determination condition

$$\int_{\Omega} y(x,t)v(x)dx = E(t), \quad t \in (0,T), \tag{2.1.4}$$

where the functions f, h, E and $y_0(x)$ are known functions and a, b, c are also given constants that verify the following hypothesis:

$$A1 : a > 0, b > 0, c > 0.$$

2.2 Existence and Uniqueness of the Solution to the Direct Problem

2.2.1 A priori estimate

In the rectangle $Q = (0,1) \times (0,T) = \Omega \times (0,T)$, with $T < \infty$, we consider the semilinear parabolic problem

$$(P) \qquad \begin{cases} y_t(x,t) - a\dfrac{\partial^2 y(x,t)}{dx^2} + by(x,t) + cy^3(x,t) = \tilde{f}(x,t), & (x,t) \in \Omega \times (0,T), \\ y(x,0) = \varphi(x), \quad x \in \Omega, \\ y(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \end{cases}$$

$$\mathcal{L}y = y_t - a\frac{\partial^2 y}{dx^2} + by + cy^3 = \tilde{f}(x,t), \tag{2.2.1}$$

with the initial condition

$$ly = y(x,0) = \varphi(x), \qquad x \in \Omega, \tag{2.2.2}$$

and the Dirichlet boundary condition

$$y(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{2.2.3}$$

where the function $\tilde{f}$ is known.

Defined from E to F, the operator L comprises all functions $y(x,t)$ with finite norms in the Banach space E.

$$\|y\|_E^2 = \|y\|^2_{L^\infty(0,T,L^2(\Omega))} + \|\frac{\partial y}{\partial x}\|^2_{L^2(Q)} + \|y\|^2_{L^2(Q)} + \|y\|^4_{L^4(Q)}.$$

Besides, F represents the Hilbert space, which includes all elements $\mathcal{F} = (\tilde{f}, \varphi)$ for which the norm

$$\|\mathcal{F}\|_F^2 = \|\tilde{f}\|_{L^2(Q)}^2 + \|\varphi\|_{L^2(\Omega)}^2.$$

is finite.

Theorem 2.2.1. *Let condition A1 be satisfied. Then for any function $y \in D(L)$, we have the inequality*

$$\|y\|_E \leq C\|Ly\|_F,$$

where C is a positive constant independent of y and $D(L)$ represents the domain of definition of the operator L, which can be defined as follows:

$$D(L) = \{y \setminus y, \frac{\partial y}{\partial x} \in L^2(Q), y \in L^4(Q)\}.$$

Proof. Using $My = y$ and the scalar product in $L^2(Q)$ of (**??**), we obtain

$$\begin{aligned} < \mathcal{L}y, My >_{L^2(Q)} &=< y_t, y >_{L^2(Q)} - a < \frac{\partial^2 y}{dx^2}, y >_{L^2(Q)} + b < y, y >_{L^2(Q)} + c < y^3, y >_{L^2(Q)} \\ &=< \tilde{f}, y >_{L^2(Q)} . \end{aligned} \tag{2.2.4}$$

Integrating (**??**) and applying Cauchy's ε-inequality yield

$$\begin{aligned} &\frac{1}{2}\|y(.,t)\|_{L^2(\Omega)}^2 + a\|\frac{\partial y}{\partial x}\|_{L^2(Q)}^2 + b\|y\|_{L^2(Q)}^2 + c\|y\|_{L^4(Q)}^4 \\ &\qquad \leq \frac{1}{2\varepsilon}\|\tilde{f}\|_{L^2(Q)}^2 + \frac{1}{2}\|\varphi\|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2}\int_0^T \|y\|_{L^2(\Omega)}^2 dt. \end{aligned}$$

Using Gronwall's lemma and the fact that the right-hand side is not related to t, we substitute the left-hand side with its upper bound with respect to t from 0 to T to obtain

$$\|y\|_{L^\infty(0,T,L^2(\Omega))}^2 + \|\frac{\partial y}{\partial x}\|_{L^2(Q)}^2 + \|y\|_{L^2(Q)}^2 + \|y\|_{L^4(Q)}^4 \leq C(\|\tilde{f}\|_{L^2(Q)}^2 + \|\varphi\|_{L^2(\Omega)}^2),$$

where

$$C = \frac{max(\frac{c'}{2}, \frac{c'}{2\varepsilon})}{min(\frac{1}{2}, a, b, c)} \quad \text{and} \quad c' = exp(\frac{\varepsilon T}{2}).$$

Consequently, we have

$$\|y\|_E \leq C\|Ly\|_F. \tag{2.2.5}$$

□

Proposition 2.2.1. *The operator L from E to F has a closure.*

Proof. Let $(y_n)_{n\in\mathbb{N}} \subset D(L)$ be a sequence such that

$$y_n \longrightarrow 0 \qquad in \quad E$$

and

$$Ly_n \longrightarrow (\tilde{f}, \varphi) \qquad in \quad F. \tag{2.2.6}$$

Herein, we should prove that

$$\tilde{f} \equiv 0, \varphi \equiv 0 \quad in \quad F.$$

The convergence of y_n to 0 in E entails that

$$y_n \longrightarrow 0 \quad in \quad D'(Q). \tag{2.2.7}$$

Due to the continuity of the derivation of $D'(Q)$ and the continuous distribution of the function y^3, relation (**??**) implies

$$\mathcal{L}y_n \longrightarrow 0 \qquad in \quad D'(Q). \tag{2.2.8}$$

Also, the convergence of Ly_n to $\tilde{f}$ in $L^2(Q)$ gives

$$\mathcal{L}y_n \longrightarrow \tilde{f} \qquad in \quad D'(Q). \tag{2.2.9}$$

Utilizing the uniqueness of the limit in $D'(Q)$, we can infer from (**??**) and (**??**) that $\tilde{f} \equiv 0$. Consequently, from (**??**), it follows that

$$ly_n \longrightarrow \varphi \qquad in \quad L^2(\Omega).$$

On the other hand, we have

$$\|y_n\|_E \geq \|y_n\|^2_{L^\infty(0,T,L^2(\Omega))},$$

i.e.,

$$\|y_n\|_E \geq \|\varphi\|^2_{L^2(\Omega)}.$$

Immediately, we have

$$y_n \longrightarrow 0 \qquad in \quad E,$$

which implies

$$\|y_n\|^2_E \longrightarrow 0 \qquad in \quad \mathbb{R}.$$

So, we get $\varphi \equiv 0$, and as a result the operator L is closable. □

Definition 2.2.1. *Let $\overline{L}$ represent the closure of L, and $D(\overline{L})$ denote its domain of definition. The solution to the equation*

$$\overline{L}y = F$$

*is referred to as a strong solution to the problem (**??**)-(**??**). Then a priori estimate (**??**) can be extended to the strong solution, i.e., we have the following inequality:*

$$\|y\|_E \leq C\|\overline{L}y\|_F, \forall y \in D(\overline{L}). \tag{2.2.10}$$

Corollary 2.2.1. *The range $R(\overline{L})$ of the operator $\overline{L}$ is closed in F and equal to the closure $\overline{R(L)}$ of $R(L)$.*

Proof. First, we prove the uniqueness of the solution if it exists. Let y_1 and y_2 be two different solutions. If we put $\eta = y_1 - y_2$, then η satisfies

$$(P') \qquad \begin{cases} \eta_t(x,t) - a\dfrac{\partial^2 \eta(x,t)}{\partial x^2} + c(y_1^3 - y_2^3) + b\eta(x,t) = 0, & (x,t) \in Q, \\ \eta(x,0) = 0, \quad x \in \Omega, \\ \eta(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \end{cases} \tag{2.2.11}$$

$$\eta_t(x,t) - a\frac{\partial^2 \eta(x,t)}{\partial x^2} + c(y_1^3 - y_2^3) + b\eta(x,t) = 0, \qquad (x,t) \in Q. \tag{2.2.12}$$

By multiplying (**??**) by η and integrating the result over Ω, we get

$$\int_\Omega \eta_t(x,t).\eta(x,t)dx - a\int_\Omega \frac{\partial^2 \eta}{\partial x^2}.\eta(x,t)dx + c\int_\Omega (y_1^3 - y_2^3)(y_1 - y_2)dx + b\int_\Omega \eta^2(x,t)dx = 0.$$

Consequently, we can get

$$\frac{1}{2}\frac{d}{dt}\|\eta\|^2_{L^2(\Omega)} + a\|\frac{\partial \eta}{\partial x}\|^2_{L^2(\Omega)} + b\|\eta\|^2_{L^2(\Omega)} + c\int_\Omega (y_1^3 - y_2^3)(y_1 - y_2)dx = 0. \tag{2.2.13}$$

As the function λ^3 is a monotone function over Ω, we can conclude that the last term of the left-hand side of (**??**) is positive, so it follows that

$$\frac{d}{dt}\|\eta\|^2_{L^2(\Omega)} \leq 0,$$

which implies that for all $t \in (0,T)$, we have $y_1(t) = y_2(t)$ in E.
Now, we will return to the proof of Corollary **??**. To this end, we let $z \in \overline{R(L)}$. Then there exists a Cauchy sequence $(z_n)_{n\in\mathbb{N}}$ in $R(L)$ such that

$$\lim_{n \longrightarrow +\infty} z_n = z.$$

So, there exists a corresponding sequence $(y_n)_{n\in\mathbb{N}}$ in $D(L)$ such that $Ly_n = z_n$.
Now, let $\varepsilon, n \geq n_0$ and $m, m' \in \mathbb{N}$ such that $m \geq m'$ and $y_m, y_{m'}$ are two solutions, i.e.,

$$Ly_m = \tilde{f} \quad and \quad Ly_{m'} = \tilde{f}.$$

We put $\phi = y_m - y_{m'}$ and we apply to ϕ the same procedure that we used to demonstrate the uniqueness of the solution in the previous step. This yields $\phi = 0$. It means that for all $t \in (0, T)$, we have

$$0 \leq \|y_m(t) - y_{m'}(t)\|_E \leq 0 \tag{2.2.14}$$

$$\Leftrightarrow \forall \varepsilon \geq 0, \quad \exists n_0 \in \mathbb{N} \backslash \forall m, m' \geq n_0 : \|y_m(t) - y_{m'}(t)\|_E \leq \varepsilon.$$

As a result, $(y_n)_n$ is a Cauchy sequence in the Banach space E. So, there is $y \in E$ such that

$$\lim_{n \longrightarrow +\infty} y_n = y.$$

By virtue of the definition of $\overline{L}$ (i.e., $\lim\limits_{n \longrightarrow +\infty} y_n = y$; if $\lim\limits_{n \longrightarrow +\infty} Ly_n = \lim\limits_{n \longrightarrow +\infty} z_n = z$, then $\lim\limits_{n \longrightarrow +\infty} \overline{L}y_n = z$ and since $\overline{L}$ is closed, it follows that $\overline{L}y = z$), the function y verifies

$$y \in D(\overline{L}),\ \overline{L}y = z.$$

Thus $z \in R(\overline{L})$, then

$$\overline{R(L)} \subset R(\overline{L})$$

.
In the same regard, we can also deduce that $R(\overline{L})$ is closed because it is a Banach space.
It remains to prove the reverse inclusion.
For this purpose, we note that $z \in R(\overline{L})$, meaning there exists a sequence $(z_n)_n$ in F, composed of elements from the set $R(\overline{L})$, such that

$$\lim_{n \longrightarrow +\infty} z_n = z.$$

As a result, there exists a corresponding sequence $(v_n)_n \subset D(\overline{L})$ such that

$$\overline{L}v_n = z_n.$$

On the other hand, we have $(v_n)_n$ is a Cauchy sequence in E. So, there is $v \in E$ such that

$$\lim_{n \longrightarrow +\infty} v_n = v, \qquad v \in E.$$

Once again, there exists a corresponding sequence $\{L(v_n)\} \in R(L)$ such that

$$Lv_n = \overline{L}v_n \quad on \quad R(L), \forall n.$$

This implies

$$\lim_{n \longrightarrow +\infty} Lv_n = z.$$

Consequently, $z \in \overline{R(L)}$, and hence we conclude that $R(\overline{L}) \subset \overline{R(L)}$. □

2.2.2 Existence of a solution to the direct problem

To establish the existence of a solution, it is necessary to demonstrate that $R(L)$ is dense in F for every $y \in E$ and for any $\mathcal{F} = (\tilde{f}, \varphi) \in F$.

Theorem 2.2.2. *Suppose that A1 is satisfied. Then for each $\mathcal{F} = (\tilde{f}, \varphi) \in F$, there is a unique strong solution $y = \overline{L}^{-1}\mathcal{F}$ to problem (P).*

Proof. First, we demonstrate that $R(L)$ is dense in F for the particular case where $D(L)$ is reduced to $D_0(L)$. where

$$D_0(L) = \{y, y \in D(L) : ly = 0\}.$$

To this end, we prove the following proposition:

Proposition 2.2.2. *Let the conditions of Theorem ?? be satisfied. If for $w \in L^2(Q)$ and for each $y \in D_0(L)$, we have*

$$\int_Q \mathcal{L}y.wdxdt = 0, \tag{2.2.15}$$

then w vanishes almost everywhere in Q.

Proof. The scalar product of F is defined as follows:

$$(Ly, w)_F = \int_Q \mathcal{L}y.wdxdt + \int_\Omega ly.w_1 dx, \forall y \in D(L). \tag{2.2.16}$$

If we put $y = w$, the equality (**??**) can be written as follows:

$$\int_Q y_t(t,x).y(t,x)dxdt - a\int_Q \frac{\partial^2 y}{\partial x^2}.y(t,x)dxdt + b\int_Q y^2(t,x)dxdt + c\int_Q y^4(t,x)dxdt = 0. \tag{2.2.17}$$

Integrating (**??**) by parts yields

$$a\|\frac{\partial y}{\partial x}\|^2_{L^2(Q)} + b\|y\|^2_{L^2(Q)} + c\|y\|^4_{L^4(Q)} + \frac{1}{2}\|y\|^2_{L^2(\Omega)} = 0.$$

So, we can deduce that $\|y\|^2_{L^2(Q)} = 0$, i.e., $y \equiv 0$ in Q, and hence $w \equiv 0$. □

Now, we return to the proof of Theorem **??**. To this end, we suppose that $W = (w, w_1) \in R^{\perp}(L)$. This implies

$$(Ly, w)_F = \int_Q \mathcal{L}y.wdxdt + \int_\Omega ly.w_1 dx = 0, \forall y \in D(L). \tag{2.2.18}$$

By means of the last proposition and by putting $y \in D_0(L)$, we obtain $w \equiv 0$. Thus, (**??**) becomes

$$\int_\Omega ly.w_1 dx = 0, \ \forall y \in D(L). \tag{2.2.19}$$

The range of the trace operator l is dense in the Hilbert space F, then the equality (**??**) implies that $w_1 = 0$. As a result, we can conclude that $W = 0$, and this completes the proof of Theorem **??**. □

2.3 The inverse problem's existence and uniqueness

In this part, we will assume that the functions appearing in the problem data are measurable and satisfy the following conditions:

$$(H) \qquad \begin{cases} h \in C(0,T,L^2(\Omega)), v \in V = \{v, \dfrac{\partial v}{\partial x} \in L^2(\Omega), v \in L^4(\Omega)\}, \quad E \in W_2^2(0,T) \\ \|h(x,t)\| \leq m; |g^*(t)| \geq r > 0, \quad for \quad r \in \mathbb{R}, \quad (x,t) \in Q, \\ \varphi(x) \in W_2^1(\Omega). \end{cases} \quad .$$

The relation between f and y is defined by the following operator:

$$A : L^2(0,T) \longrightarrow L^2(0,T), \tag{2.3.1}$$

with the expression

$$Af(t) = \frac{1}{g^*}\{a \int_\Omega \frac{\partial y}{\partial x}\frac{\partial v}{\partial x} dx + c \int_\Omega y^3(t,x)v(x)dx\}. \tag{2.3.2}$$

Consequently, the preceding relation between f and y may be expressed in terms of the function f, defined over the interval $L^2(0,T)$, as:

$$f = Af + \mu, \tag{2.3.3}$$

where

$$\mu = \frac{E' + bE}{g^*}. \tag{2.3.4}$$

Theorem 2.3.1. *Assume that the input of data of the inverse problem (??)-(??) verifies condition* (H). *Then the following statements are equivalent:*

- *If the inverse problem (??)-(??) is solved, then so is equation (??).*
- *If equation (??) has a solution and the compatibility condition* $E(0) = \int_\Omega \varphi(x)v(x)dx$ *is true, then the inverse problem (??)-(??) has also a solution.*

Proof.

- Assume that the inverse problem (??)-(??) is solved. We denote its solution by $\{y, f\}$. By multiplying (??) by v and integrating the resulting expression over Ω, we get

$$\frac{d}{dt}\int_\Omega y(t,x)v(x)dx + a\int_\Omega \frac{\partial y}{\partial x}\frac{\partial v}{\partial x}dx + b\int_\Omega y(x,t)v(x)dx + c\int_\Omega y^3(t,x)v(x)dx = f(t)g^*(t). \tag{2.3.5}$$

Using (??) and (??) implies

$$\frac{E' + bE}{g^*} + Af = f.$$

This gives that f solves equation (??).

- According to the assumption, the equation (??) has a solution, say f. By substituting f into equation (??), the resulting relationships (??)-(??) can be then treated as a direct problem with a unique solution. It is yet up to us to show that y verifies the integral over determination (??). By the equation (??), the function y is subject to the following relation:

$$E' + bE + a\int_\Omega \frac{\partial y}{\partial x}\frac{\partial v}{\partial x}dx + c\int_\Omega y^3(t,x)v(x)dx = f(t)g^*(t). \tag{2.3.6}$$

Subtracting equation (??) from (??) yields

$$\frac{d}{dt}\int_\Omega y(t,x)v(x)dx + b\int_\Omega y(x,t)v(x)dx = E' + bE. \tag{2.3.7}$$

Now, integrating the above differential equation and using the compatibility condition $E(0) = \int_\Omega \varphi(x)v(x)dx$ lead us to the conclusion that y satisfies the integral condition (??). As a result, we can conclude that $\{y, f\}$ is the solution of the inverse problem (??)-(??).

□

In the following part, we intend to present certain properties related to the operator A.

Lemma 2.3.1. *If condition (H) holds, then there exists a positive δ for which the operator A is a contracting operator in $L^2(0,T)$.*

Proof. We obtain from (**??**) the following estimate:

$$|Af(t)|^2 \leq \frac{2}{r^2}[a^2\|\frac{\partial y}{\partial x}\|^2_{L^2(\Omega)}\|\frac{\partial v}{\partial x}\|^2_{L^2(\Omega)} + \gamma\|v\|^2_{L^4(\Omega)}\|y\|^4_{L^4(\Omega)}],$$

where $\gamma = \|y\|^2_{L^\infty(0,T,L^4(\Omega))} \geq 0$.
Now, integrating the above equality over $(0,T)$ yields

$$\int_0^T |Af(t)|^2 dt \leq \frac{2}{r^2}max(a^2\|\frac{\partial v}{\partial x}\|^2_{L^2(\Omega)}, \gamma\|v\|^2_{L^4(\Omega)}) \int_0^T (\|\frac{\partial y}{\partial x}\|^2_{L^2(\Omega)} + \|y\|^4_{L^4(\Omega)})dt. \tag{2.3.8}$$

So, we get

$$\|Af\|_{L^2(0,T)} \leq K(\int_0^T (\|\frac{\partial y}{\partial x}\|^2_{L^2(\Omega)} + \|y\|^4_{L^4(\Omega)})dt)^{\frac{1}{2}},$$

where

$$K = \frac{1}{r}\sqrt{2max(a^2\|\frac{\partial v}{\partial x}\|^2_{L^2(\Omega)}, \gamma\|v\|^2_{L^4(\Omega)})}.$$

By multiplying each side of (**??**) by y in $L^2(Q)$ and then integrating the resulting expression by parts, while utilizing Cauchy's ε-inequality and the Poincare inequality, we obtain

$$\frac{1}{2}\|y\|^2_{L^2(\Omega)} + (a - \frac{c''\varepsilon}{2})\|\frac{\partial y}{\partial x}\|^2_{L^2(Q)} + b\|y\|^2_{L^2(Q)} + c\|y\|^4_{L^4(Q)} \leq \frac{m^2}{2\varepsilon}\|f\|^2_{L^2(0,T)} + \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)}, \tag{2.3.9}$$

with $a - \dfrac{c''\varepsilon}{2} > 0$. With the help of passing to the maximum and omitting some terms, we get

$$\int_0^T (\|\frac{\partial y}{\partial x}\|^2_{L^2(\Omega)} + \|y\|^4_{L^4(\Omega)})dt \leq M'\|f\|^2_{L^2(0,T)}, \tag{2.3.10}$$

where

$$M' = \frac{\dfrac{m^2}{2\varepsilon}}{min(a - \dfrac{c''\varepsilon}{2}, c)}.$$

It means that

$$\left(\int_0^T (\|\frac{\partial y}{\partial x}\|^2_{L^2(\Omega)} + \|y\|^4_{L^4(\Omega)})dt\right)^{\frac{1}{2}} \leq M''\|f\|_{L^2(0,T)}, \tag{2.3.11}$$

where $M'' = \sqrt{M'}$. Consequently, we get

$$\|Af\|_{L^2(0,T)} \leq \delta\|f\|_{L^2(0,T)}, \tag{2.3.12}$$

with $\delta = KM''$.
It is obvious from the above assertion that there exists a positive δ such that $\delta < 1$. Thus, inequality (**??**) demonstrates that the operator A is a contracting mapping in $L^2(0,T)$. □

Theorem 2.3.2. *Let the compatibility condition $E(0) = \int_\Omega \varphi(x)v(x)dx$ and the condition (H) hold. Then the following statements are correct:*

- *Given any initial iteration $f_0 \in L^2(0,T)$, the following approximations are valid:*
$$f_{n+1} = \mathcal{A}f_n, \tag{2.3.13}$$
which converge to f in the $L^2(0,T)$-norm.

- *The inverse problem (**??**)-(**??**) has a unique solution $\{y, f\}$.*

Proof.
- We define the following operator $\mathcal{A} : L^2(0,T) \longrightarrow L^2(0,T)$ as
$$\mathcal{A}f = Af + \frac{E' + bE}{g^*}, \tag{2.3.14}$$
where the operator A and the function g^* come from (**??**). Consequently, based on (**??**), the relation (**??**) can be expressed as
$$f = \mathcal{A}f. \tag{2.3.15}$$
Thus, it is sufficient to demonstrate that the operator $\mathcal{A}$ has a fixed point in the space $L^2(0,T)$. Accordingly, we can write
$$\mathcal{A}f_1 - \mathcal{A}f_2 = Af_1 - Af_2.$$
From estimate (**??**), we can deduce that
$$\|\mathcal{A}f_1 - \mathcal{A}f_2\|_{L^2(0,T)} \leq \delta\|f_1 - f_2\|_{L^2(0,T)}. \tag{2.3.16}$$
According to (**??**), we see that $\mathcal{A}$ is a contracting mapping on $L^2(0,T)$. As a result, $\mathcal{A}$ has a unique fixed point f in $L^2(0,T)$ and the successive approximations (**??**) converge to f in $L^2(0,T)$-norm, independent of the initial iteration $f_0 \in L^2(0,T)$.

- This demonstrates that equations (**??**) and (**??**) have a unique solution f in $L^2(0,T)$. The existence of a solution to the main problem is proved by Theorem **??**, but it has to be proven that this solution is unique. Using the demonstration by contradiction, we assume that there are two distinct solutions $\{y_1, f_1\}$ and $\{y_2, f_2\}$ to problem (**??**)-(**??**). To begin, we assert that f_1 is not equal to f_2 almost everywhere on $(0,T)$. If $f_1 = f_2$, then by utilizing the uniqueness theorem for the related direct problem (**??**)-(**??**), we find $y_1 = y_2$ almost everywhere in Q. Given that both pairs have verified (**??**), we infer that the functions f_1 and f_2 are two distinct solutions to equation (**??**), which contradicts the uniqueness of the functions.

□

Corollary 2.3.1. *If the conditions of Theorem* **??** *are satisfied, then the solution f varies continuously with respect to the data μ of the equation (*__??__*).*

Proof. Let μ and ϑ be two sets of data that satisfy the assumptions of Theorem **??** and let f and g be two solutions of the equation (**??**), which correspond to μ and ϑ, respectively. As a result of (**??**), we have

$$f = Af + \mu, \quad g = Ag + \vartheta.$$

By calculating the difference $f - g$ and by using (**??**), we can have:

$$\|f - g\|_{L^2(0,T)} \leq \frac{1}{1-\delta} \|\mu - \vartheta\|_{L^2(0,T)}.$$

Therefore, the proof of this corollary is completed. □

Conclusion *The novel contribution of this manuscript has been successfully made by investigating the solvability of the semilinear parabolic problem with the integral over determination condition for an inverse problem. In addition, we have solved the direct problem by using the "energy inequality" method and accordingly, we have dealt with the inverse problem by using the fixed point technique.*

Chapter 3

A study of a superlinear parabolic Dirichlet problem with unknown coefficient

Introduction

In this chapter, we study the unique solvability of the solution for a superlinear parabolic inverse problem with a nonlinear term $|u|^p u$ of the determination of a pair of functions $\{u(x,t), f(t)\}$. For a direct problem we use the energy inequality method and for the inverse problem we use the fixed point theorem. This chapter can be considered more general than the first chapter.

3.1 Setting of the problem

In the domain $Q = \Omega \times (0,T)$ such that $T > 0$, we consider the following parabolic equation

$$u_t(x,t) - a\Delta u + bu(x,t) + c|u(x,t)|^p u(x,t) = f(t)h(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{3.1.1}$$

where p is a natural number that verify $p > 0$.
with the initial condition

$$u(x,0) = \varphi(x), \quad x \in \Omega, \tag{3.1.2}$$

subject to the boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3.1.3}$$

and the nonlocal over determination condition

$$\int_{\Omega} v(x)u(x,t)dx = E(t), \quad t \in (0,T), \tag{3.1.4}$$

where Ω *is a bounded domain of* $\mathbb{R}^n$ *with a smooth boundary* $\partial\Omega$. *The functions* h, φ, E *and* v *are known and* a, b, c *are constants that verify the following condition:*

$$A1 : a > 0, b > 0, c > 0.$$

3.2 Existence and uniqueness of a direct problem

3.2.1 A priori estimate and uniqueness of the solution

*In this part, we aim to apply the energy inequality method to study the solution of problem (**??**)-(**??**). Our primary objective is to establish the existence and uniqueness of the strong solution to the main direct problem. For a detailed explanation of the method, refer to references [70-81]. The proof is based on the energy inequality and the density of the operator's range, as formulated abstractly in the problem. With this in mind, we proceed to formulate the main problem by considering the following superlinear parabolic Dirichlet problem:*

$$(P) \qquad \begin{cases} u_t(x,t) - a\Delta u + bu(x,t) + c|u(x,t)|^p u(x,t) = \tilde{f}(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,0) = \varphi(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \end{cases}$$

$$\mathcal{L}u = u_t - a\Delta u + bu + c|u|^p u = \tilde{f}(x,t), \tag{3.2.1}$$

with the initial condition

$$l_1 u = u(x,0) = \varphi(x), \qquad x \in \Omega, \tag{3.2.2}$$

and with the Dirichlet boundary condition

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T). \tag{3.2.3}$$

Here, $\tilde{f}$ *is a given function.*
Additionally, the operator L *is defined from* E *to* F, *where* E *is a Banach space, which includes all functions* $u(x,t)$ *having the following finite norms:*

$$\|u\|_E^2 = \|u\|^2_{L^\infty(0,T,L^2(\Omega))} + \|\nabla u\|^2_{L(Q)} + \|u\|^2_{L(Q)} + \|u\|^{p+2}_{L^{p+2}(Q)}.$$

Similarly, F defined above is the Hilbert space, which consists of all elements $\mathcal{F} = (f, \varphi)$ and equipped according to the norm

$$\|\mathcal{F}\|_F^2 = \|\tilde{f}\|_{L^2(Q)}^2 + \|\varphi\|_{L^2(\Omega)}^2.$$

In the following content, we begin by establishing an a priori estimate for problem resolution. For this purpose, we introduce the next theoretical result.

Theorem 3.2.1. *If the assumption A1 is satisfied, then for any function $u \in D(L)$, there exists a positive constant C that is independent of u such that*

$$\|u\|_E \leq C\|\mathcal{F}\|_F,$$

where $D(L)$ is the domain of the definition of the operator L, which is defined by

$$D(L) = \{u/u, u_t, \nabla u, \Delta u \in L^2(Q), u \in L^{p+2}(Q)\}.$$

Proof. To prove this result, we use the scalar product in $L^2(Q)$ of (**??**) and the operator $Mu = u$, where $Q = \Omega \times (0, T)$. This would imply

$$\begin{aligned}\langle \mathcal{L}u, Mu\rangle_{L^2(Q)} &= \langle u_t, u\rangle_{L^2(Q)} - a\langle \Delta u, u\rangle_{L^2(Q)} + \langle bu, u\rangle_{L^2(Q)} + \langle c|u|^p u, u\rangle_{L^2(Q)} \\ &= \langle \tilde{f}, u\rangle_{L^2(Q)}.\end{aligned} \tag{3.2.4}$$

Consequently, we have

$$\langle u_t, u\rangle_{L^2(Q)} = \int_Q u_t(t,x)u(t,x)dtdx = \frac{1}{2}\int_Q \frac{d}{dt}u^2(t,x)dtdx.$$

This immediately yields

$$\langle u_t, u\rangle_{L^2(Q)} = \frac{1}{2}\|u(.,t)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\varphi\|_{L^2(\Omega)}^2 \tag{3.2.5}$$

and

$$-a\langle \Delta u, u\rangle_{L^2(Q)} = -a\int_Q \Delta u(t,x)u(t,x)dtdx.$$

Hence, we obtain

$$-a\langle \Delta u, u\rangle_{L^2(Q)} = a\|\nabla u\|_{L^2(Q)}^2, \tag{3.2.6}$$

$$\langle bu, u\rangle_{L^2(Q)} = b\int_Q u^2(x,t)dxdt = b\|u\|_{L^2(Q)}^2 \tag{3.2.7}$$

and

$$\langle c|u|^p u, u\rangle_{L^2(Q)} = c\int_Q |u(x,t)|^p u^2(x,t)dtdx = c\|u\|_{L^{p+2}(Q)}^{p+2}. \tag{3.2.8}$$

By substituting (**??**)-(**??**) into (**??**), we get

$$\frac{1}{2}\|u(.,t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)} + a\|\nabla u\|^2_{L^2(Q)} + b\|u\|^2_{L^2(Q)} + c\|u\|^{p+2}_{L^{p+2}(Q)} = \int_Q \tilde{f}(t,x)u(t,x)dtdx.$$

If one estimates the last term of the right hand side using $\left(|ab| \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}\right)$, we get

$$\frac{1}{2}\|u(\cdot,t)\|^2_{L^2(\Omega)} + a\|\nabla u\|^2_{L^2(Q)} + b\|u\|^2_{L^2(Q)} + c\|u\|^{p+2}_{L^{p+2}(Q)}$$
$$\leq \frac{1}{2\varepsilon}\|\tilde{f}\|^2_{L^2(Q)} + \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\int_0^T \|u(.,t)\|^2_{L^2(\Omega)}dt.$$

Thus, using Gronwall's lemma yields

$$\frac{1}{2}\|u(.,t)\|^2_{L^2(\Omega)} + a\|\nabla u\|^2_{L^2(Q)} + b\|u\|^2_{L^2(Q)} + c\|u\|^{p+2}_{L^{p+2}(Q)} \leq \frac{c'}{2\varepsilon}\|\tilde{f}\|^2_{L^2(Q)} + \frac{c'}{2}\|\varphi\|^2_{L^2(\Omega)},$$

where

$$c' = exp\left(\frac{\varepsilon T}{2}\right).$$

It should be noted here that the right hand side of the last estimate is independent of t, and so we can replace the left hand side by its upper bound with respect to t from 0 to T. This means

$$\|u\|^2_{L^\infty(0,T,L^2(\Omega))} + \|\nabla u\|^2_{L^2(Q)} + \|u\|^2_{L^2(Q)} + \|u\|^{p+2}_{L^{p+2}(Q)} \leq C(\|\tilde{f}\|^2_{L^2(Q)} + \|\varphi\|^2_{L^2(\Omega)}),$$

where

$$C = \frac{max\left(\frac{c'}{2}, \frac{c'}{2\varepsilon}\right)}{min\left(\frac{1}{2}, a, b, c\right)}.$$

So, we have

$$\|u\|_E \leq C\|Lu\|_F. \tag{3.2.9}$$

□

*Herein, the range of the operator L is denoted by $R(L)$. However, because we do not know anything about $R(L)$ except that $R(L) \subset F$, we have to extend the operator L. Thus, the estimate (**??**) holds for the extension and its range is the whole space F. As a result of this discussion, we state and prove the following proposition.*

Proposition 3.2.1. *The operator $L : E \longrightarrow F$ has a closure.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset D(L)$ be a sequence in which

$$u_n \longrightarrow 0 \qquad in \quad E$$

and

$$Lu_n \longrightarrow (\tilde{f}, \varphi). \tag{3.2.10}$$

Herein, we must demonstrate that

$$\tilde{f} \equiv 0, \varphi \equiv 0 \quad in \quad F.$$

Herein, the convergence of u_n to 0 in E causes

$$u_n \longrightarrow 0 \quad in\ D'(Q). \tag{3.2.11}$$

The relationship (**??**) is regarded very complicated in accordance with the continuity derivation of $D'(Q)$ in $D'(Q)$ and the continuity distribution of the function $|u_n|^p u_n$. This means

$$\mathcal{L}u_n \longrightarrow 0 \qquad in \quad D'(Q). \tag{3.2.12}$$

In addition, the convergence of Lu_n to $\tilde{f}$ in $L^2(Q)$ yields

$$\mathcal{L}u_n \longrightarrow \tilde{f} \qquad in \quad D'(Q). \tag{3.2.13}$$

Hence, we can deduce from (**??**) and (**??**) that $\tilde{f} \equiv 0$. This is because we know the limit in $D'(Q)$ is unique. However, it can be generated from (**??**) that

$$l_1 u_n \longrightarrow \varphi \qquad in \quad L^2(\Omega).$$

On the other hand, we can have

$$\|u_n\|_E = \|u_n(\cdot,t)\|^2_{L^\infty(0,T,L^2(\Omega)} + \|\nabla u_n\|^2_{L^2(Q)} + \|u_n\|^2_{L^2(Q)} + \|u_n\|^{p+2}_{L^{p+2}(Q)}.$$

This consequently implies

$$\|u_n\|_E \geq \|u_n(x,0)\|^2_{L^2(\Omega)},$$

and so

$$\|u_n\|_E \geq \|\varphi\|^2_{L^2(\Omega)}.$$

Now, since we have

$$u_n \longrightarrow 0 \qquad in \quad E,$$

then we can obtain

$$\|u_n\|_E^2 \longrightarrow 0 \qquad in \quad \mathbb{R}.$$

Consequently, we get

$$0 \geq \|\varphi\|_{L^2(\Omega)}^2.$$

Therefore, one might deduce

$$\varphi \equiv 0,$$

which accordingly implies the desired result. □

Definition 3.2.1. *A solution to the operator equation*

$$\overline{L}u = F$$

*is known as a strong solution to problem (**??**)-(**??**).*

In light of the above definition, we may extend a priori estimate to strong solutions, i.e., we define the following estimate:

$$\|u\|_E \leq C\|\overline{L}u\|_F, \forall u \in D(\overline{L}), \tag{3.2.14}$$

where L is the closure of this operator, and D represents domain of definition of L.

Corollary 3.2.1. *The range of the operator $\overline{L}$ is closed within F and equals to the closure of $R(L)$, that is,*

$$R(\overline{L}) = \overline{R(L)}.$$

Proof. First, we intend to demonstrate the uniqueness of the solution if it exists. To do so, we let u_1 and u_2 be two solutions, and $\eta = u_1 - u_2$. Accordingly, η satisfies the following problem:

$$(P') \qquad \begin{cases} \eta_t(x,t) - a\Delta\eta + |u_1|^p u_1 - |u_2|^p u_2 + b\eta(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ \eta(x,0) = 0, \quad x \in \Omega, \\ \eta(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{3.2.15}$$

where the following statement holds:

$$\eta_t(x,t) - a\Delta\eta + |u_1|^p u_1 - |u_2|^p u_2 + b\eta(x,t) = 0, \qquad (x,t) \in \Omega \times (0,T). \tag{3.2.16}$$

Herein, we use the scalar product in $L^2(\Omega)$ of (**??**) and η to obtain

$$\int_\Omega \eta_t(t,x)\eta(t,x)dx - a\int_\Omega \Delta\eta(t,x)\eta(t,x)dx \\ + c\int_\Omega (|u_1|^p u_1 - |u_2|^p u_2)(u_1 - u_2)dx + b\int_\Omega \eta^2(t,x)dx = 0.$$

Consequently, we can have

$$\frac{1}{2}\frac{d}{dt}\|\eta\|^2_{L^2(\Omega)} + a\|\nabla\eta\|^2_{L^2(\Omega)} + c\int_\Omega (|u_1|^p u_1 - |u_2|^p u_2)(u_1-u_2)dx + b\|\eta\|^2_{L^2(\Omega)} = 0 \quad (3.2.17)$$

Since $|\lambda|^p\lambda$ is a monotone function in λ, the last term of the left hand side of (**??**) will be non negative. It follows from (**??**) that

$$\frac{d}{dt}\|\eta\|^2_{L^2(\Omega)} \leq 0,$$

which implies

$$\|\eta\|^2_{L^2(\Omega)} \leq 0,$$

for all $t \in (0,T)$. In other words, we have $\eta(t) = 0$, which demonstrates the uniqueness of the solution, i.e. $u_1(t) = u_2(t)$.

In light of the previous discussion, we are ready now to prove Corollary **??**.

For this purpose, we let $z \in \overline{R(L)}$. This implies the existence of a sequence $(z_n)_{n\in\mathbb{N}}$ in $R(L)$ such that $\lim\limits_{n\longrightarrow+\infty} z_n = z$. Since $(z_n)_{n\in\mathbb{N}}$ is in $R(L)$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $D(L)$ such that $Lu_n = z_n$.

Now, let ε, $n \geq n_0$, and let $m, m' \in \mathbb{N}$, $m \geq m'$ such that u_m and $u_{m'}$ are two solutions, i.e.,

$$Lu_m = \tilde{f} \quad and \quad Lu_{m'} = \tilde{f}.$$

Herein, we assume that $y = u_m - u_{m'}$, then y satisfies the following problem:

$$(P'') \qquad \begin{cases} y_t(x,t) - a\Delta y + |u_m|^p u_m - |u_{m'}|^p u_{m'} + by(x,t) = 0, & (x,t) \in \Omega\times(0,T), \\ y(x,0) = 0, & x \in \Omega, \\ y(x,t) = 0, & (x,t) \in \partial\Omega\times(0,T). \end{cases}$$

Now, by applying the same procedure used to prove the uniqueness issue, we get $y = 0$. This immediately implies

$$0 \leq \|u_m(t) - u_{m'}(t)\|_E \leq 0, \quad (3.2.18)$$

for all $t \in (0,T)$. In other words, we can have

$$\lim_{m,m'\longrightarrow+\infty} \|u_m(t) - u_{m'}(t)\|_E = 0$$

$$\Longleftrightarrow \forall\varepsilon \geq 0, \quad \exists n_0 \in \mathbb{N}\backslash \forall m, m' \geq n_0$$

$$\|u_m(t) - u_{m'}(t)\|_E \leq \varepsilon.$$

Thus, we can conclude that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E, whereby E itself represents a Banach space. Therefore, there exists $u \in E$ such that

$$\lim_{n\longrightarrow+\infty} u_n = u.$$

Now, we use the definition of $\overline{L}$ that says $\lim_{n\longrightarrow+\infty} u_n = u$ in E if

$$\lim_{n\longrightarrow+\infty} Lu_n = \lim_{n\longrightarrow+\infty} z_n = z,$$

then

$$\lim_{n\longrightarrow+\infty} \overline{L}u_n = z$$

as $\overline{L}$ is closed, and so we have $\overline{L}u = z$. This makes the function u satisfies

$$u \in D(\overline{L}),\ \overline{L}u = z.$$

Then, $z \in R(\overline{L})$, and so we have

$$\overline{R(L)} \subset R(\overline{L}).$$

Furthermore, due to $R(\overline{L})$ is a Banach subspace, we conclude that $R(\overline{L})$ is closed.
From this point of view, it is still necessary to show that the opposing party has been included.
To this end, we let z in $R(\overline{L})$.
As a result, we can identify a sequence $(z_n)_n$ in F composed of elements from the set $R(\overline{L})$,which satisfies:

$$\lim_{n\longrightarrow+\infty} z_n = z.$$

As a consequence, there exists a corresponding sequence $(v_n)_n \subset D(\overline{L})$ such that:

$$\overline{L}v_n = z_n.$$

On the other hand, we have $(v_n)_n$ is a Cauchy sequence in F (this result can be established by applying the same procedure used to prove that $(u_n)_n$ is a cauchy sequence in the previous steps). Consequently, there exists $v \in E$

$$\lim_{n\longrightarrow+\infty} v_n = v, \quad v \in E,$$

which implies

$$\lim_{n\longrightarrow+\infty} Lv_n = z.$$

As a consequence, $z \in \overline{R(L)}$, and hence we can conclude

$$R(\overline{L}) \subset \overline{R(L)},$$

which finalizes the proof of the required result. □

3.2.2 The Existence of the solution

In the next part, we will investigate the existence of a solution for the given problem. To achieve this, we must demonstrate that $R(L)$ is dense in F for every $u \in E$ and for any $\mathcal{F} = (\tilde{f}, \varphi) \in F$. To do so, we will introduce and prove the following result.

Theorem 3.2.2. *Assume that the assumption $A1$ is satisfied. Then for every $\mathcal{F} = (\tilde{f}, \varphi) \in F$, there exists a unique strong solution $u = \overline{L}^{-1}\mathcal{F}$ to problem (P).*

Proof. First, we demonstrate that $R(L)$ is dense in F for the particular case where $D(L)$ is reduced to $D_0(L)$. where

$$D_0(L) = \{y, y \in D(L) : ly = 0\}.$$

However, with the aim of achieving this goal, we have to verify the validation of the following claim.

Proposition 3.2.2. *Let us consider that the condition of Theorem **??** is satisfied. If $w \in L^2(Q)$ and for all $u \in D_0(L)$, we have*

$$\int_Q \mathcal{L}u.w dx dt = 0, \tag{3.2.19}$$

then, w vanishes almost everywhere in Q.

Proof. For the purpose of proving this Proposition, we should note that the scalar product of F can be defined as

$$(Lu, w)_F = \int_Q \mathcal{L}u.w dx dt + \int_\Omega l_1 u w_1 dx, \qquad \forall u \in D(L) \tag{3.2.20}$$

Therefore, the equality (**??**) can be reformulated as follows:

$$\begin{aligned} \int_Q u_t(x,t).w(x,t) dx dt - a \int_Q \Delta u.w(x,t) dx dt + b \int_Q u(x,t) w(x,t) dx dt \\ + c \int_Q |u(x,t)|^p u(x,t).w(x,t) dx dt = 0. \end{aligned} \tag{3.2.21}$$

Then, setting $w = u$ yields

$$\int_Q u_t(x,t).u(x,t) dx dt - a \int_Q \Delta u.u(x,t) dx dt + b \int_Q u^2(x,t) dx dt + c \int_Q |u(x,t)|^{p+2} dx dt = 0. \tag{3.2.22}$$

This consequently gives

$$\frac{1}{2}\|u(.,t)\|^2_{L^2(\Omega)} + a\|\nabla u\|^2_{L^2(Q)} + b\|u\|^2_{L^2(Q)} + c\|u\|^{p+2}_{L^{p+2}(Q)} = 0,$$

So, we can conclude

$$\|u\|^2_{L^2(Q)} = 0. \tag{3.2.23}$$

It means that $u \equiv 0$ in Q.
Therefore, $w \equiv 0$ in Q, and hence the Proposition is verified. □

Now, we intend to go back to Theorem **??**. To this end, we should prove the set $R(L)$ is dense in F. For this purpose, we assume the following assumption

$$(Lu, w)_F = \int_Q \mathcal{L}u.wdxdt + \int_\Omega l_1 u w_1 dx = 0, \tag{3.2.24}$$

holds for some $W = (w, w_1) \in R^{\perp}(L)$ and for all $u \in D(L)$. According to the Proposition reported above, if we put $u \in D_0(L)$, we can have $\int_Q \mathcal{L}u.wdxdt = 0$, and hence $w \equiv 0$. As a result, assumption (**??**) becomes

$$\int_\Omega l_1 u w_1 dx = 0, u \in D(L). \tag{3.2.25}$$

As a result, given that the range of the trace operator l_1 is dense throughout the Hilbert space, the equality (**??**) implies that $w_1 = 0$. Thus, we have $W = 0$, which implies $(\overline{R(L)} = F)$. Thus, the proof of this theorem is completed. □

3.3 The invers problem's unique solvability

In this part, the unique solvability of the inverse problem is addressed. To accomplish this goal, we assume that the functions involved in the problem's data are measurable and meet the following conditions:

$$(H) \qquad \begin{cases} h \in C(0,T,L^2(\Omega)), v \in V = \{v, \nabla v \in L^2(\Omega), v \in L^{p+2}(\Omega)\}, & E \in W_2^2(0,T) \\ \|h(x,t)\| \le m; |g^*(t)| \ge r \rangle 0, \quad for \quad r \in \mathbb{R}, \quad (x,t) \in Q, \\ \varphi(x) \in W_2^1(\Omega) \end{cases} .$$

The connection between f and u can be seen as a way to define the following operator:

$$A : L^2(0,T) \longrightarrow L^2(0,T), \tag{3.3.1}$$

which is defined by

$$Af(t) = \frac{1}{g^*}\left\{a\int_\Omega \nabla u \nabla v dx + \int_\Omega |u|^p uv dx\right\}. \tag{3.3.2}$$

Consequently, the previously discussed relationship between f and uit can be expressed in terms of the function f, defined over the interval $L^2(0,T)$, as follows:

$$f = Af + \mu, \tag{3.3.3}$$

where

$$\mu = \frac{E' + bE}{g^*}. \tag{3.3.4}$$

In view of the above lines, we introduce the next theoretical result that aims to demonstrate the unique solvability of the inverse problem.

Theorem 3.3.1. *Assuming that the data function for the inverse problem (**??**)-(**??**) satisfies Condition (H). Then, the following assertions are equivalent:*

1. *If the inverse problem (**??**)-(**??**) is solvable, then equation (**??**) is also solvable.*

2. *If there is a solution to equation (**??**) and the compatibility condition $E(0) = \int_\Omega \varphi(x)v(x)dx$ holds, then the inverse problem (**??**)-(**??**) has a solution.*

Proof. 1. Suppose that the problem (**??**)-(**??**) is solvable with a solution of the form $\{u, f\}$.
Now, by multiplying equation (**??**) by v and then integrating the result over Ω, we get

$$\int_\Omega u_t(x,t)v(x)dx + a\int_\Omega \nabla u \nabla v dx + b\int_\Omega u(x,t)v(x)dx + \int_\Omega |u(x,t)|^p u(x,t)v(x)dx$$
$$= f(t)g^*(t). \tag{3.3.5}$$

Using (**??**) and (**??**) yields

$$\frac{E' + bE}{g^*} + Af = f.$$

This means that f solves equation (**??**), and hence the result holds.

2. By considering the given assumption, we deduce that equation (**??**) has a solution, say f.
   Now, by substituting f into equation (**??**), then the resulting relation (**??**)-(**??**) can be treated as a direct problem having a unique solution. Thus, it remains for us to prove that u satisfies also the integral over determination (**??**). To do so, it should be note that equation (**??**) can yield

$$\frac{d}{dt}\int_\Omega u(x,t)v(x)dx + a\int_\Omega \nabla u\nabla v dx + b\int_\Omega u(x,t)v(x)dx \\ + c\int_\Omega |u(x,t)|^p u(x,t)v(x)dx = f(t)g^*(t). \tag{3.3.6}$$

   On the other hand, as a solution to equation (**??**), the function u satisfies the following relation:

$$E'+bE+a\int_\Omega \nabla u\nabla v dx+b\int_\Omega u(x,t)v(x)dx+c\int_\Omega |u(x,t)|^p u(x,t)v(x)dx = f(t)g^*(t). \tag{3.3.7}$$

   Consequently, subtracting (**??** from (**??**) immediately yields

$$\frac{d}{dt}\int_\Omega u(x,t)v(x)dx + b\int_\Omega u(x,t)v(x)dx = E' + bE. \tag{3.3.8}$$

   By integrating the previous differential equation coupled with taking into account the compatibility condition $E(0) = \int_\Omega \varphi(x)v(x)dx$, we conclude that u satisfies the integral condition (**??**). Therefore, we infer that $\{u, f\}$ is the solution of the inverse problem (**??**)-(**??**), as required.

□

In what follows, we state and prove some properties in connection of the operator A. These properties are formulates as certain theoretical aspects for completeness.

Lemma 3.3.1. *Let as assume that Condition* (H) *holds, then there is a positive* δ *for which the operator* A *is a contraction in* $L^2(0,T)$.

Proof. Based on (**??**), we can get the following estimate:

$$
\begin{aligned}
|Af(t)|^2 &= \left|\frac{1}{g^*}\left\{a\int_\Omega \nabla u\nabla v dx + \int_\Omega |u(x,t)|^p u(x,t)v(x)dx\right\}\right|^2 \\
&\le \frac{2}{r^2}\left[a^2\left(\int_\Omega \nabla u\nabla v dx\right)^2 + \left(\int_\Omega |u(x,t)|^p|u(x,t)v(x)|dx\right)^2\right] \\
&\le \frac{2}{r^2}\left[a^2\|\nabla u\|^2_{L^2(\Omega)}\|\nabla v\|^2_{L^2(\Omega)} + \left(\int_\Omega |u(x,t)|^{p+1}|v|dx\right)^2\right] \\
&\le \frac{2}{r^2}\left[a^2\|\nabla u\|^2_{L^2(\Omega)}\|\nabla v\|^2_{L^2(\Omega)} + \|u\|^{2(p+1)}_{L^{p+2}(\Omega)}\|v\|^2_{L^{p+2}(\Omega)}\right] \\
&\le \frac{2}{r^2}\left[a^2\|\nabla u\|^2_{L^2(\Omega)}\|\nabla v\|^2_{L^2(\Omega)} + \|u\|^{p+2}_{L^{p+2}(\Omega)}\|u\|^{p}_{L^{p+2}(\Omega)}\|v\|^2_{L^{p+2}(\Omega)}\right].
\end{aligned}
$$

Now, we suppose that $\|u\|^p_{L^\infty(0,T,L^{p+2}(\Omega)} = \gamma \ge 0$. Then, we have

$$
|Af(t)|^2 \le \frac{2}{r^2}\left[a^2\|\nabla u\|^2_{L^2(\Omega)}\|\nabla v\|^2_{L^2(\Omega)} + \|u\|^{p+2}_{L^{p+2}(\Omega)}\gamma\|v\|^2_{L^{p+2}(\Omega)}\right].
$$

As a result, integrating the above assertion over $(0,T)$ yields

$$
\int_0^T |Af(t)|^2 dt \quad \le \frac{2}{r^2} max\left(a^2\|\nabla v\|^2_{L^2(\Omega)}, \gamma\|v\|^2_{L^{p+2}(\Omega)}\right)\left(\int_0^T \|\nabla u\|^2_{L^2(\Omega)}dt + \int_0^T \|u(\cdot,t)\|^{p+2}_{L^{p+2}(\Omega)}dt\right). \tag{3.3.9}
$$

Thus, we obtain

$$
\|Af\|^2_{L^2(0,T)} \le K(\int_0^T \|\nabla u\|^2_{L^2(\Omega)}dt + \int_0^T \|u(\cdot,t)\|^{p+2}_{L^{p+2}(\Omega)})dt),
$$

where

$$
K = \frac{2}{r^2} max(a^2\|\nabla v\|^2_{L^2(\Omega)}, \gamma\|v\|^2_{L^{p+2}(\Omega)}).
$$

By taking the equation (**??**) and multiplying both sides by u in $L^2(Q)$, then performing integration by parts on the resulting expression, we can derive the following result

$$
\int_Q u_t(x,t)u(x,t)dxdt - a\int_Q \Delta u u(x,t)dxdt + b\int_Q u^2(x,t)dxdt,
$$

$$
+\int_Q |u(x,t)|^p u^2(x,t)dxdt = \int_Q f(t)h(x,t)udxdt \tag{3.3.10}
$$

where $(x,t) \in \Omega \times (0,T)$. Accordingly, we can obtain

$$\frac{1}{2}\|u\|^2_{L^2(\Omega)}+a\|\nabla u\|^2_{L^2(Q)}+b\|u\|^2_{L^2(Q)}+c\|u\|^{p+2}_{L^{p+2}(Q)} \leq \frac{m^2}{2\varepsilon}\|f\|^2_{L^2(0,T)}+\frac{\varepsilon}{2}\|u\|^2_{L^2(Q)}+\frac{1}{2}\|\varphi\|^2_{L^2(\Omega)} \tag{3.3.11}$$

and so, we can have

$$\frac{1}{2}\|u\|^2_{L^2(\Omega)}+a\|\nabla u\|^2_{L^2(Q)}+(b-\frac{\varepsilon}{2})\|u\|^2_{L^2(Q)}+c\|u\|^{p+2}_{L^{p+2}(Q)} \leq \frac{m^2}{2\varepsilon}\|f\|^2_{L^2(0,T)}+\frac{1}{2}\|\varphi\|^2_{L^2(\Omega)}, \tag{3.3.12}$$

where $0 < \varepsilon < 2b$. Passing to the maximum and omitting some terms yield

$$\int_0^T \left(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^{p+2}_{L^{p+2}(\Omega)}dt\right) \leq M''\|f\|^2_{L^2(0,T)}dt, \tag{3.3.13}$$

where

$$M'' = \frac{\frac{m^2}{2\varepsilon}}{min(a,c)}.$$

Therefore, we obtain

$$\|Af\|_{L^2(0,T)} \leq \delta\|f\|_{L^2(0,T)}, \tag{3.3.14}$$

where $\delta = \sqrt{kM''}$. As a result of the preceding, there exists a positive δ such that

$$\delta < 1, \tag{3.3.15}$$

which demonstrates that the operator A has a contracting mapping on $(L^2(0,T)$ and this completes the proof. □

Theorem 3.3.2. *Assume that Condition (H) and the compatibility condition $E(0) = \int_\Omega \varphi(x)v(x)dx$ are satisfied. Then the following assertions are true:*

1. *The following approximations*

$$f_{n+1} = \mathcal{A}f_n \tag{3.3.16}$$

   converge to f in $L^2(0,T)$-norm with any initial iteration f_0 in $L^2(0,T)$, and for the operator $\mathcal{A}$.

2. *The pair $\{u,f\}$ representing the solution to the inverse problem (**??**)-(**??**) exists uniquely.*

Proof. 1. To prove this result, we use the following nonlinear operator

$$\mathcal{A} : L^2(0,T) \longrightarrow L^2(0,T),$$

which us defined by

$$\mathcal{A}f = Af + \frac{E' + bE}{g^*}, \tag{3.3.17}$$

where the operator A and the function g^* are obtained from (**??**). Based on(**??**), we can express (**??**) as

$$f = \mathcal{A}f. \tag{3.3.18}$$

Now, we have to demonstrate that the operator $\mathcal{A}$ has a fixed point in the space $L^2(0,T)$. To do so, we have to observe that according to the relationship

$$\mathcal{A}f_1 - \mathcal{A}f_2 = Af_1 - Af_2,$$

we can infer, based on the estimate (**??**), the following assertions:

$$\|\mathcal{A}f_1 - \mathcal{A}f_2\|_{L^2(0,T)} = \|Af_1 - Af_2\|_{L^2(0,T)} \leq \delta\|f_1 - f_2\|_{L^2(0,T)}, \tag{3.3.19}$$

in which $\mathcal{A}$ is a contracting mapping on $L^2(0,T)$ based on (**??**) and (**??**). As a result, the operator $\mathcal{A}$ possesses a unique fixed point f in $L^2(0,T)$. Consequently, the successive approximations (**??**) converge to f in $L^2(0,T)$-norm, which is independent of the initial iteration $f_0 \in L^2(0,T)$.

2. Actually, based on the previous discussion, this leads us to conclude that both equations (**??**) and, by extension, (**??**) have a unique solution f in $L^2(0,T)$. Moreover, Theorem **??** guarantees the existence of a solution for the inverse problem outlined in (**??**)-(**??**). So, it remains to prove that this solution is unique. So, by the proof of contrary, we suppose that there are two distinct solutions $\{u_1, f_1\}$ and $\{u_2, f_2\}$ for the main inverse problem. Consequently, we first assert that $f_1 \neq f_2$ almost everywhere on $(0,T)$. If $f_1 = f_2$, then by applying the uniqueness theorem of the corresponding direct problem (**??**-(**??**), we get $u_1 = u_2$ almost everywhere in Q. As both pairs have verified (**??**), we arrive at the conclusion that the functions f_1 and f_2 represents two destinct solutions to equation (**??**), which contradicts the uniqueness of the solution for this equation. This finalizes the proof of the desired result.

□

Corollary 3.3.1. *If the conditions of Theorem* **??** *are satisfies, then the solution* f *depends continuously with respect to the data* μ *of the equation (***??***).*

Proof. Assume that μ and ϑ are two sets of data that satisfy the assumptions of Theorem **??**. Let f and g be two solutions of equation (**??**) that corresponds respectively to μ and ϑ. According to (**??**), we can have

$$f = Af + \mu, \quad g = Ag + \vartheta.$$

Now, let us begin by estimating the difference $f - g$. Then, by using (**??**), we obtain

$$\|f - g\|_{L^2(0,T)} = \|(Af + \mu) - (Ag + \vartheta)\|_{L^2(0,T)} = \|A(f - g) + (\mu - \vartheta)\|_{L^2(0,T)}.$$

Accordingly, we can get

$$\|f - g\|_{L^2(0,T)} \leq \frac{1}{1 - \delta}\|\mu - \vartheta\|_{L^2(0,T)}.$$

As a result, the proof of the corollary is finished. □

Chapter 4

Inverse coefficient super-linear problem for a time fractional para- bolic equation under integral overdetermination condition

Introduction

In this section, we investigate the unique solvability of solution to an inverse problem of a super-linear fractional parabolic(FPP) type where the nonlinear term u^2*, involving the determination of a pair of functions* $\{u(x,t), f(t)\}$ *(we suppose that* $u > 0$*)*

4.1 Setting of the problem

In the rectangular domain $\mathrm{Q} = \Omega \times (0,\ \mathrm{T})$*, where* $\Omega = (0,\ \mathrm{d}), \mathrm{T} < \infty, d > 0$ *and* $0 < \alpha < 1$*. we examine the following fractional parabolic equation:*

$$\mathcal{L}u = {}^{c}\partial_t^{\alpha} u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} + \beta u(x,t) + u^2(x,t) = \tilde{f}(x,t), \tag{4.1.1}$$

(where $\tilde{f}(x,t) = f(t)g(x,t)$*),*
subject to the initial condition:

$$\ell u = u(x,0) = 0, \quad x \in \Omega, \tag{4.1.2}$$

and the Dirichlet boundary condition:

$$u(0,t) = u(d,t) = 0, \quad t \in (0,T). \tag{4.1.3}$$

and the nonlocal integral overdetermination condition:

$$\int_{\Omega} v(x)u(x,t)dx = E(t), \qquad t \in [0,T]. \tag{4.1.4}$$

where $\beta \in \mathbb{R}_*^+$ *and* g *is known function.*

4.2 Solvability of the direct fractional parabolic problem

4.2.1 A priori estimate and the uniqueness of the solution

In this part, we will show the existence and uniqueness of the solution to the problem defined by equations (??)-(??), which can be framed as a solution to the operator equation

$$Lu = \mathcal{F}, \tag{4.2.1}$$

where $\mathrm{L} = (\mathcal{L}, \ell)$, *with domain of definition* $\mathrm{D(L)} = \mathrm{B}$ *which defined by*

$$\mathrm{D(L)} = \left\{ u \backslash \mathrm{u} \in L^2(Q) \cap L^3(Q), \quad {}^c\partial_t^{\alpha} u, \frac{\partial u}{\partial x} \in L^2(Q) \right\}.$$

The operator L *is defined as a mapping from the Banach* B *to* F, *with* B *consisting of all functions* $u(x,t)$ *having a finite norm:*

$$\|u\|_B^2 = \left\| \quad {}^c\partial_t^{\frac{\alpha}{2}} u \right\|_{L^2(Q)}^2 + \left\| \frac{du}{dx} \right\|_{L^2(Q)}^2 + \|u\|_{L^2(Q)}^2 + \|u\|_{L^3(Q)}^3,$$

and F *is the Hilbert space consisting of all elements Fourier* $= (\tilde{f}, 0)$ *for which the norm* $L^2(Q)$ *is finite.*

Theorem 4.2.1. *For each function* $u \in B$, *we have the inequality*

$$\|u\|_B \leq C\|Lu\|_{L^2(Q)}, \tag{4.2.2}$$

where C *is a positive constant independent of* u *.*

Proof. We use the scalar product in $L^2(Q)$ of $\mathcal{L}u$ and the following function:

$$Mu = u(x,t),$$

where

$$\mathrm{Q} = (0,\ \mathrm{d}) \times (0,\ \mathrm{T})$$

$$\begin{aligned}\int_Q \mathcal{L}u \cdot Mu dxdt = & \int_Q {}^{c}\partial_t^{\alpha} u(x,t) \cdot u(x,t) dxdt \\ & - \int_Q \left(\frac{\partial^2 u(x,t)}{\partial x^2} \right) \cdot u(x,t) dxdt \\ & + \beta \int_Q u^2(x,t) dxdt \\ & + \int_Q u^3(x,t) dxdt \\ = & \int_Q \tilde{f}(x,t) u(x,t) dxdt \end{aligned} \tag{4.2.3}$$

As $u(x,0) = 0$, so by applying the Lemma**??**, Lemma**??** and Lemma**??** becomes:

$$\begin{aligned}\int_Q {}^{c}\partial_t^{\alpha} u(x,t) \cdot u(x,t) dxdt = & ({}^{c}\partial_t^{\alpha} u, u)_{L^2(Q)} \\ = & \left({}^{R}\partial_t^{\frac{\alpha}{2}} {}^{R}\partial_t^{\frac{\alpha}{2}} u, u \right)_{L^2(Q)} \quad \text{(According to Lemma ??)} \\ = & \left({}^{R}\partial_t^{\frac{\alpha}{2}} u, {}^{R}_{t}\partial^{\frac{\alpha}{2}} u \right)_{L^2(Q)} \quad \text{(According to Lemma ??)} \\ = & \left\| {}^{c}\partial_t^{\frac{\alpha}{2}} u \right\|^2_{L^2(Q)}. \end{aligned}$$

By applying the relationship ($|ab| \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}$) and using the integral by parts, we find

$$\left\| {}^{c}\partial_t^{\frac{\alpha}{2}} u \right\|^2_{L^2(Q)} + \left\| \frac{du}{dx} \right\|^2_{L^2(Q)} + \left(\beta - \frac{\varepsilon}{2} \right) \|u\|^2_{L^2(Q)} + \|u\|^3_{L^3(Q)} \leq \frac{1}{2\varepsilon} \|\tilde{f}\|^2_{L^2(Q)}.$$

So, for $(\varepsilon < 2\beta)$ we get

$$\min\left(1, \beta - \frac{\varepsilon}{2}\right)\left(\left\| {}^{c}\partial_t^{\frac{\alpha}{2}} u \right\|^2_{L^2(Q)} + \left\| \frac{du}{dx} \right\|^2_{L^2(Q)} + \|u\|^2_{L^2(Q)} + \|u\|^3_{L^3(Q)} \right) \leq \frac{1}{2\varepsilon} \|\widetilde{f}\|^2_{L^2(Q)}.$$

So, we get

$$\left\| \ {}^c\partial_t^{\frac{\alpha}{2}} u \right\|^2_{L^2(Q)} + \left\| \frac{du}{dx} \right\|^2_{L^2(Q)} + \|u\|^2_{L^2(Q)} + \|u\|^3_{L^3(Q)} \le c\|\widetilde{f}\|^2_{L^2(Q)}.$$

With

$$c = \frac{1}{2\varepsilon \min\left(1, \beta - \frac{\varepsilon}{2}\right)}$$

As a result, we obtain that:

$$\|u\|_B \le C\|Lu\|_{L^2(Q)},$$

with

$$C = \sqrt{c}.$$

□

Proposition 4.2.1. *The operator L from B to F has a closure.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset$ D(L) be a sequence satisfying:

$$u_n \to 0 \quad \text{in } B, \tag{4.2.4}$$

and

$$Lu_n \to \mathcal{F} \quad \text{in } F. \tag{4.2.5}$$

We must prove that

$$\tilde{f} \equiv 0.$$

In B, the convergence of u_n to 0 causes :

$$u_n \to 0 \quad \text{in}\,(C_0^\infty(Q))'. \tag{4.2.6}$$

As the continuity of the fractional derivative, along with the of first-order derive (which is a special case of the fractional derivative) within $\left(C_0^\infty(Q)\right)'$, and considering the continuity of the distribution for the function u^2, we find that the relationship in equation (**??**) leads to

$$\mathcal{L}u_n \to 0 \quad \text{in}\,(C_0^\infty(Q))'. \tag{4.2.7}$$

In addition, the convergence of Lv_n to $\tilde{f}$ in $L^2(Q)$ yields:

$$\mathcal{L}v_{n \to} \tilde{f} \quad \text{in } (C_0^\infty(Q))'. \tag{4.2.8}$$

Since we know that the limit in $\left(C_0^\infty(Q)\right)'$ is unique, we can conclude from (**??**) and (**??**) that :

$$\tilde{f} \equiv 0.$$

Consequently, the operator L is closable.

□

Let $\bar{L}$ be the closure of L and let D($\bar{L}$) *represent the domain of definition of $\bar{L}$.*

Definition 4.2.1. *A solution to the operator equation*

$$\bar{L}u = \mathcal{F}.$$

*Is known as a strong solution to problem (**??**)-(**??**). We may extend the a priori estimate to strong solution, i.e; we obtain the estimate*

$$\|u\|_B \leq C\|\bar{L}u\|_F, \quad \forall u \in D(\bar{L}). \tag{4.2.9}$$

Corollary 4.2.1. *The strong solution to the problem (**??**)-(**??**) is unique and depends continuously on $\tilde{f} \in F$.*

Corollary 4.2.2. *The range* R($\bar{L}$) *of the operator $\bar{L}$ is closed within F and equals the closure of* R(L)*, that is, $R(\overline{L}) = \overline{R(L)}$.*

Proof. First, we demonstrate the uniqueness of the solution if it exists Let u_1, u_2 be two solution we put $\eta = u_1 - u_2$, then η satisfies:

$$(P')\begin{cases} {}^c\partial_t^\alpha \eta(x,t) - \frac{\partial^2\eta(x,t)}{\partial x^2} + \beta\eta(x,t) + u_1^2(x,t) - u_2^2(x,t) = 0, & \text{in } Q, \\ \eta(x,0) = 0, \quad x \in \Omega, \\ \eta(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \end{cases}$$

$${}^c\partial_t^\alpha \eta(x,t) - \frac{\partial^2\eta(x,t)}{\partial x^2} + \beta\eta(x,t) + u_1^2(x,t) - u_2^2(x,t) = 0, \quad \text{in Q.} \tag{4.2.10}$$

We use the scalar product in $L^2(\Omega)$ of (**??**) and η we get :

$$\int_\Omega {}^c\partial_t^\alpha \eta(x,t)\cdot\eta(x,t)dx - \int_\Omega \left(\frac{\partial^2\eta(x,t)}{\partial x^2}\right)\cdot\eta(x,t)dx + \beta\int_\Omega \eta^2(x,t)dx$$
$$+ \int_\Omega \left(u_1^2(x,t) - u_2^2(x,t)\right)(u_1(x,t) - u_2(x,t))\,dx = 0.$$

As η(x,0) = 0, so by applying the Lemma **??**, Lemma**??** and Lemma**??** and integrating by parts we get:

$$\| \ {}^c\partial_t^{\overline{\frac{\alpha}{2}}} \eta(.,t)\|^2_{L^2(\Omega)} + \|\frac{d\eta(.,t)}{dx}\|^2_{L^2(\Omega)} + \|\eta(.,t)\|^2_{L^2(\Omega)}$$
$$+ \int_\Omega \left(u_1^2(x,t) - u_2^2(x,t)\right)(u_1(x,t) - u_2(x,t))\,dx = 0 \tag{4.2.11}$$

Since λ^2 is a monotone function in λ (on $\Omega = (0,\ \text{d}\ )$), we can conclude that the last term of the left hand side of (**??**) is positive, so it follows from (**??**) that:

$$\|\eta\|^2_{L^2(\Omega)} = 0.$$

Which implies that for all $t\epsilon(0,T)$ we have:

$$u_1 = u_2.$$

We"ll return to demonstrate the orollary (**??**).
Let $z \in \overline{R(L)}$ so there is a sequence $(\mathrm{z_n})_{\mathrm{n}\in\mathbb{N}}$ in R(L) such as

$$\lim_{n\longrightarrow+\infty} z_n = z$$

So like as $(z_n)_{n\in\mathbb{N}}$ in R(L), there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in D(L) such that

$$Lu_n = z_n.$$

Let $\varepsilon, \mathrm{n} \geq \mathrm{n_0}$ Let $\mathrm{m}, \mathrm{m}' \in \mathbb{N}, m \geq m'$ such as $u_m, u_m,$ be two solutions i,e;

$$Lu_m = \tilde{f} \text{ and } Lu_{m'} = \tilde{f}$$

We put $\mathrm{y} = u_m - u_m,$ then y satisfies

$$(P'')\begin{cases} {}^c\partial_t^\alpha y(x,t) - \left(\frac{\partial^2 y(x,t)}{\partial x^2}\right) + \beta y(x,t) + u_m^2 - u_{m'}^2 = 0, & \text{in } Q, \\ y(x,0) = 0, \quad x \in \Omega, \\ y(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \end{cases}$$

We apply same proceeding which we use to prove the uniqueness of the solution we get $y = 0$ which implies that for all $t \in (0,\ \mathrm{T})$ we have

$$0 \leq \|u_m - u_m,\| \leq 0$$

i;e

$$\forall \varepsilon \geq 0, \exists n_0 \in \mathbb{N} \backslash \forall m, m' \geq n_0; \|u_m - u_m,\| \leq \varepsilon$$

So we can conclude that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E which is a Banach space, then there is $u \in E$ such that

$$\lim_{n\longrightarrow+\infty} u_n = u$$

Using the definition of $\bar{L}(\lim\limits_{n\longrightarrow+\infty} u_n = u$ in E, if $\lim\limits_{n\longrightarrow+\infty} Lu_n = \lim\limits_{n\longrightarrow+\infty} z_n = z$, then $\lim\limits_{n\longrightarrow+\infty} \bar{L}u_n = z$ as like $\bar{L}$ is closed so $\bar{L}u = z$), the function u check

$$u \in D(\bar{L}), \bar{L}u = z$$

then $\mathrm{z} \in R(\bar{L})$ so

$$\overline{R(L)} \subset R(\bar{L})$$

Furthermore, because $R(\bar{L})$ is Banach, we conclude that is closed.
It is still necessary to demonstrate the opposing inclusion.
Let z ∈ R($\bar{L}$), then there is a sequence of $(z_n)_n$ in F composed of elements from the set R($\bar{L}$), such that

$$\lim_{n\longrightarrow+\infty} z_n = z$$

As a consequence, there is a corresponding sequence $(u_n)_{n\in\mathbb{N}} \in D(\bar{L})$ such that

$$\lim_{n\longrightarrow+\infty} \bar{L}u_n = z_n$$

On the other hand we have $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in F (to show this we apply the same method which we use in the previous step). So there exists $u \in E$:

$$\lim_{n\longrightarrow+\infty} u_n = u \quad \text{in } E$$

Once again, there exists a corresponding sequence $L(u_n) \in R(L)$ such that

$$Lu_n = \bar{L}u_n \quad on \quad R(L), \forall n$$

. then

$$\lim_{n\longrightarrow+\infty} Lu_n = z$$

As a consequence z ∈ $\overline{R(L)}$ and we can conclude that

$$\overline{R(L)} = R(\overline{L})$$

□

4.2.2 The existence of a solution

To establish the existence of solutions, we need to show that $R(L)$ is dense in F for every $u \in B$ and for any arbitrary $\mathcal{F} = (\tilde{f}, 0) \in F$.

Theorem 4.2.2. *The problem (**??**)-(**??**) has a solution.*

Proof. The scalar product of F is defined by

$$(Lu, w)_F = \int_Q \mathcal{L}u.wdxdt. \tag{4.2.12}$$

If we put w $\in R^{\perp}(L)$, we get:

$$\int_Q {}^c\partial_t^\alpha u(x,t)w(x,t)dxdt - \int_Q \left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) w(x,t)dxdt + \beta \int_Q u(x,t)w(x,t)dxdt + \int_Q u^2(x,t)\Big)\cdot w(x,t)dxdt = 0.$$

Putting $w = u$ we get:

$$\int_Q {}^c\partial_t^\alpha u(x,t)\cdot u(x,t)dxdt - \int_Q \left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)\cdot u(x,t)dxdt + \beta \int_Q u^2(x,t)dxdt + \int_Q u^3(x,t)dxdt = 0. \tag{4.2.13}$$

Integrating by parts each term of (**??**) and taking account the condition of u, we obtain :

$$\left\| {}^c\partial_t^{\frac{\alpha}{2}} u\right\|^2_{L^2(Q)} + \left\|\frac{du}{dx}\right\|^2_{L^2(Q)} + \beta\|u\|^2_{L^2(Q)} + \|u\|^3_{L^3(Q)} = 0$$

Then

$$\|u\|^2_{L^2(Q)} = 0$$

It means that $u = 0$ in Q which gives $w = O$ in Q , this proves Theorem (**??**) . □

4.3 Existence and uniqueness of the main problem's solution

We assume that the functions appearing in the problem's data are mesurable and satisfy the following conditions:

$$(H)\begin{cases} g \in C\Big((0,T), L^2(\Omega)\Big), v \in W_2^1(\Omega)\cap L^3(Q), E \in W_2^2(0,T), \\ \|g(x,t)\| \le m, \quad |g^*(t)| \ge r > 0, \quad \text{for } r \in \mathbb{R}, \quad (x,t)\in Q. \end{cases}$$

The relation between f and u is given by the following operator

$$A : L^2(0,T) \to L^2(0,T). \tag{4.3.1}$$

With the values

$$(Af(t)) = \frac{1}{g^*}\left\{\int_\Omega \frac{du}{dx}\frac{dv}{dx}dx + \int_\Omega u^2(x,t)\cdot v(x)dx\right\}. \tag{4.3.2}$$

As a result, the previous relationship between f and u can be represented in terms of the function f, defined the interval over $L^2(0,T)$, as :

$$f = Af + W. \tag{4.3.3}$$

Where

$$W = \frac{{}^c\partial_t^\alpha E + \beta E}{g^*}. \tag{4.3.4}$$

And

$$E(0) = 0.$$

Theorem 4.3.1. *Assume that the inverse problem's data functions (**??**)-(**??**) validate the condition (H). Then we have the following statement equivalent:*

*i) If the inverse problem (**??**)-(**??**) is solvable, then equation is also solvable* (**??**)

ii) If equation (**??**) *has a solution and the compatibility condition* E(0) = 0 *hold, the inverse problem (**??**)-(**??**) also has a solution.*

Proof. i) Suppose that the problem (**??**)-(**??**) has been solved. We designate its solution by {u, f}.

Multiplying both side of (**??**) by v and integrating over Ω we get:

$${}^c\partial_t^\alpha \int_\Omega u(x,t) \cdot v(x)dx + \int_\Omega \frac{du}{dx}\frac{dv}{dx}dx + \beta \int_\Omega u(x,t) \cdot v(x)dx + \int_\Omega u^2(x,t) \cdot v(x)dx = f(t)g^*(t) \tag{4.3.5}$$

Using (**??**)and (**??**), we get

$$f = Af + \frac{\beta E + {}^c\partial_t^\alpha E}{g^*}$$

This give us that f solves equation (**??**).

ii) By assuming that equation (**??**) has a solution, say f, and then substituting f into equation (**??**), the resulting relation (**??**)-(**??**) can be viewed as a direct problem with a unique solution.

It remains to show that u satisfies also the condition of integral overdetermination (**??**). By the equation (**??**) the function u is subject to the following relation

$${}^c\partial_t^\alpha E + \int_\Omega \frac{du}{dx}\frac{dv}{dx}dx + \beta E + \int_\Omega u^2(x,t) \cdot v(x)dx = f(t)g^*(t). \tag{4.3.6}$$

Substracting equation (??) from equation (??), we obtain

$$ {}^{c}\partial_t^{\alpha} \int_{\Omega} u(x,t)\cdot v(x)dx + \beta \int_{\Omega} u(x,t)\cdot v(x)dx = {}^{c}\partial_t^{\alpha} E + \beta E. \tag{4.3.7} $$

Integrating the previous equation and taking account the compatibility condition $E(0) = 0$ we conclude that u satisfies the integral condition (??) As a result, we may deduce that $\{u, f\}$ is the solution of the inverse problem (??)-(??).

□

Now, we state some properties of the operator A.

Lemma 4.3.1. *Assuming that condition* (H) *is satisfied, there exists a positive constant δ for which the operator A acts as a contraction in the space $L^2(0,T)$.*

Proof. The following estimate comes from (??)

$$ |Af(t)|^2 \leq \frac{2}{r^2}\left[\left\|\frac{du(.,t)}{dx}\right\|^2_{L^2(\Omega)}\left\|\frac{dv}{dx}\right\|^2_{L^2(\Omega)} + \|u(.,t)\|^3_{L^3(\Omega)}\|u(.,t)\|_{L^3(\Omega)}\|v\|^2_{L^3(\Omega)}\right] $$

We assume that

$$ \|u(.,t)\|_{L^\infty(0,T,L^3(\Omega))} = \Upsilon \geq 0, $$

then we have

$$ |Af(t)|^2 \leq \frac{2}{r^2}\left[\left\|\frac{du(.,t)}{dx}\right\|^2_{L^2(\Omega)}\left\|\frac{dv}{dx}\right\|^2_{L^2(\Omega)} + \Upsilon\|u(.,t)\|^3_{L^3(\Omega)} \quad \|v\|^2_{L^3(\Omega)}\right] $$

Integrating over (0, T) we obtain

$$ \int_0^T |Af(t)|^2 dt \leq \frac{2}{r^2}\max\left(\left\|\frac{dv}{dx}\right\|^2_{L^2(\Omega)}, \Upsilon\|v\|^2_{L^3(\Omega)}\right)\left[\int_0^T \left\|\frac{du(.,t)}{dx}\right\|^2_{L^2(\Omega)} dt + \int_0^T \|u(.,t)\|^3_{L^3(\Omega)}\right) dt\right] \tag{4.3.8} $$

So, we get

$$ \|Af\|_{L^2(0,T)} \leq K\left[\int_0^T \left\|\frac{du(.,t)}{dx}\right\|^2_{L^2(\Omega)} dt + \int_0^T \|u(.,t)\|^3_{L^3(\Omega)}\right) dt\right]^{\frac{1}{2}}. $$

Where

$$ K = \sqrt{\frac{2}{r^2}\max\left(\left\|\frac{dv}{dx}\right\|^2_{L^2(\Omega)}, \Upsilon\|v\|^2_{L^3(\Omega)}\right)} $$

Using the A priori estimate and removing some terms, we gets

$$\left\|\frac{du}{dx}\right\|^2_{L^2(Q)} + \|u\|^3_{L^3(Q)} \leq C\|\tilde{f}\|^2_{L^2(Q)}.$$

So, we get

$$\|Af\|_{L^2(0,T)} \leq \delta\|f\|_{L^2(0,T)}. \tag{4.3.9}$$

With

$$\delta = K\sqrt{C}.$$

From the previous discussion, it is evident that there exists a positive δ, such that

$$\delta < 1.$$

The inequality (**??**) demonstrate that operator A is a contracting mapping on $L^2(0,\ \mathrm{T})$. □

Theorem 4.3.2. *Assume that the assumption* (H) *and the compatibility condition be satisfied, then the inverse problem (**??**)-(**??**) has a unique solution* $\{u, f\}$.

Proof. This implies that the equation (**??**) has a unique solution f in $L^2(0,\ \mathrm{T})$. According to Theorem **??** the existence of a solution to the inverse problem (**??**)-(**??**) is established. This solution's uniqueness has yet to be established. Assume, on the other hand, that there are two destinct solutions to the inverse problem under examination $\{u_1, f_1\}$ and $\{u_2, f_2\}$. Also, if the linear operator A contracts on $L^2(0,\ \mathrm{T})$ from Lemma **??**, this leads to $f_1 = f_2$, which implies that uniqueness theorem for the main direct problem (**??**)-(**??**) gives us $z_1 = z_2$. □

Corollary 4.3.1. *Under the assumptions of Theorem* **??**, *the solution f to equation* (**??**) *is continuously dependent on the data W .*

Proof. Let's examine the data sets ω and v. Both of which satisfy the conditions outlined in Theorem **??**.
Let f and g represent the solutions to equation (**??**) corresponding to the data ω and v, respectively. According to (**??**)

$$f = Af + v$$
$$g = Ag + \omega$$

First, we'll calculate the difference, $f - g$. With the help of (**??**), it's clear that

$$\begin{aligned}\|f-g\|_{L^2(0,T)} &= \|(Af+v)-(Ag+\omega)\|_{L^2(0,T)}\\ &\leq \delta\|f-g\|_{L^2(0,T)} + \|v-\omega\|_{L^2(0,T)}.\end{aligned}$$

As a result, we obtain

$$\|f-g\|_{L^2(0,T)} \leq \frac{1}{1-\delta}\|v-\omega\|_{L^2(0,T)}.$$

□

Chapter 5

Investigation of a superlinear problem for a time fractional para-
bolic equation with integral over-determination condition

Introduction

In this section, we investigate the unique solvability of an inverse problem associated with super-linear fractional parabolic(FPP) equation, involving the determination of a pair of functions $\{u(x,t), f(t)\}$

5.1 Setting of the problem

In the domain $Q = \Omega \times [0,T]$*, we examine the following fractional parabolic equation:*

$$^{C}\partial_t^{\alpha} u(x,t) - \Delta u + \beta u(x,t) + u^p(x,t) = f(t)g(x,t), \quad x \in \Omega, t \in (0,T), \quad (5.1.1)$$

where p *is a natural number that verify* $p > 1$.
with the initial condition

$$u(x,0) = 0, x \in \Omega, \quad (5.1.2)$$

the boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \quad (5.1.3)$$

and the nonlocal condition

$$\int_\Omega v(x)u(x,t)dx = E(t), \qquad t \in [0,T], \tag{5.1.4}$$

where p is a given positive odd number, β is a positive constant g, E are known functions and Ω is a bounded domain in $\mathbb{R}^n$ with a regular boundary $\partial\Omega$.

5.2 Solvability of direct problem

Our objective in this part is to investigate the existence and uniqueness of the solution $u = u(x,t)$ of the FPP within the rectangular domain
$Q = (0,d) \times (0,T)$, where both d and T are finite values.

$$\begin{cases} {}^c\partial_t^\alpha u - (\frac{\partial^2 u(x,t)}{\partial x^2}) + \beta u(x,t) + u^p(x,t) = \tilde{f}(x,t) \ \ in\ Q, \\ u(x,0) = 0, \ \ \forall x \in (0,d), \\ u(0,t) = u(d,t) = 0, \ \ \forall t \in (0.T), \end{cases} \qquad (P)$$

where $0 < \alpha < 1$. This problem enjoys nonlinear FPP of the form

$$\mathcal{L}u =^c \partial_t^\alpha u - \frac{\partial^2 u}{\partial x^2} + \beta u + u^p = \tilde{f},$$

with initial condition

$$lu = u(x,0) = 0, \ \ \forall x \in (0,d),$$

$$u(0,t) = u(d,t) = 0, \ \ \forall t \in (0,T),$$

where $\beta \in \mathbb{R}_*^+$ and $\tilde{f}$ is a given function.
*In this contect, we aim to establish both the existence and uniqueness of the solution to the problem (**??**)-(**??**) which can be framed as the solution to the equation*

$$Lu = \mathcal{F}, \tag{5.2.1}$$

where $L = (\mathcal{L}, l)$ is an operator defined to the Hilbert space F from the Banach space B over the domain $D(L) = B$, which can be determined by

$$D(L) = \left\{ u : u \in L^2(Q) \cap L^{p+1}(Q), \ {}^c\partial_t^{\frac{\alpha}{2}} u, \frac{\partial u}{\partial x} \in L^2(Q) \right\}$$

Here, B represents the Banach space that comprises of all $u(x,t)$ with the following finite norm:

$$\|u\|_B^2 = \|^c\partial_t^{\frac{\alpha}{2}} u\|_{L^2(Q)}^2 + \|\frac{du}{dx}\|_{L^2(Q)}^2 + \|u\|_{L^2(Q)}^2 + \|u\|_{L^{p+1}(Q)}^{p+1}$$

Furthermore, F refers to the Hilbert space that comprises all Fourier elements $(\tilde{f}, 0)$ *satisfying the condition that their norm in* $L^2(Q)$ *is finite.*

Theorem 5.2.1. *For each* $u \in B$*, the inequality*

$$\|u\|_B \leq C\|Lu\|_{L^2(Q)} \tag{5.2.2}$$

holds, where $C \in \mathbb{R}_+$ *so that it is not dependent on the function* u*.*

Proof. To prove this result, we first use the scalar product in $L^2(Q)$, where $Q = (0,d) \times (0,T)$. Also, we apply the following function:

$$Mu = u(x,t),$$

on (**??**) to obtain the following assertion:

$$\begin{aligned}\int_Q \mathcal{L}u.Mudxdt &= \int_Q {}^c\partial_t^\alpha u(x,t).u(x,t)dxdt \\ &- \int_Q (\frac{\partial^2 u(x,t)}{\partial x^2}).u(x,t)dxdt + b\int_Q u^2(x,t)dxdt \\ &+ \int u^{p+1}(x,t)dxdt = \int_Q \tilde{f}(x,t)u(x,t)dxdt.\end{aligned} \tag{5.2.3}$$

Now as $u(x,0) = 0$ and by applying Lemmas **??**, **??** and **??** we get

$$\begin{aligned}\int_Q {}^c\partial_t^\alpha u(x,t).u(x,t)dxdt &= ({}^c\partial_t^\alpha u, u)_{L^2(Q)} \\ &= ({}^R\partial_t^{\frac{\alpha}{2}}\ {}^R\partial_t^{\frac{\alpha}{2}} u, u)_{L^2(Q)} \text{ (By Lemma ??)} \\ &= ({}^R\partial_t^{\frac{\alpha}{2}} u,\ {}^R_t\partial^{\frac{\alpha}{2}} u)_{L^2(Q)} \text{ (By Lemma ??)} \\ &= |u|^2_{{}^cH^\alpha(Q)} \cong |u|^2_{{}^lH^\alpha(Q)} \\ &= \|{}^c\partial_t^{\frac{\alpha}{2}} u\|^2_{L^2(Q)} \text{ (By Lemma ??)}.\end{aligned}$$

By applying on ($|ab| \leq \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}$) coupled with using the integral by parts of the previous equality, we find

$$\begin{aligned}\|{}^c\partial_t^{\frac{\alpha}{2}} u\|^2_{L^2(Q)} + \|\frac{du}{dx}\|^2_{L^2(Q)} + (\beta - \frac{\varepsilon}{2})\|u\|^2_{L^2(Q)} \\ + \|u\|^{p+1}_{L^{p+1}(Q)} \leq \frac{1}{2\varepsilon}\|\tilde{f}\|^2_{L^2(Q)}.\end{aligned}$$

Thus, for ($\varepsilon < 2\beta$) we get

$$\|^{c}\partial_t^{\frac{\alpha}{2}} u\|^2_{L^2(Q)} + \|\frac{du}{dx}\|^2_{L^2(Q)} + \|u\|^2_{L^2(Q)} + \|u\|^{p+1}_{L^{p+1}(Q)} \leq c\|\tilde{f}\|^2_{L^2(Q)}$$

with

$$c = \frac{1}{2\varepsilon min(1, \beta - \frac{\varepsilon}{2})}.$$

As a result, we obtain

$$\|u\|_B \leq C\|Lu\|_{L^2(Q)}$$

with

$$C = \sqrt{c},$$

which completes the proof of this result. □

Proposition 5.2.1. *There is a closure for the operator L that maps from B to F.*

Proof. Consider a sequence $(u_n)_{n\in\mathbb{N}} \subset D(L)$ that converge to zero in the Banach space B, expressed as:

$$u_{n\to}0 \quad \text{in } B \tag{5.2.4}$$

and

$$Lu_{n\to}\mathcal{F} \quad \text{in } F. \tag{5.2.5}$$

Now, we have to show

$$\tilde{f} \equiv 0.$$

To this end, it should be noted that the convergence of u_n to 0 in B causes

$$u_{n\to}0 \quad \text{in } (C_0^\infty(Q))'. \tag{5.2.6}$$

Considering the continuity of the fractional derivative and the first-order derivative of the first-order within $(C_0^\infty(Q))'$, along with the continuous distribution of the function u^p, then (**??**) would involve

$$\mathcal{L}u_{n\to}0 \quad \text{in } (C_0^\infty(Q))'. \tag{5.2.7}$$

In addition, the convergence of Lu_n to f in $L^2(Q)$ yields

$$\mathcal{L}u_{n\to}\tilde{f} \quad \text{in } (C_0^\infty(Q))'. \tag{5.2.8}$$

Since we know that the limit in $(C_0^\infty(Q))'$ is unique, then from (**??**) and (**??**), one might infer

$$\tilde{f} \equiv 0.$$

As a result, the operator L is closable. This could let us to denote $\bar{L}$ and $D(\bar{L})$ as the closure of L and its domain respectively. □

It is necessary here to draw our attention to the fact that the solution to the following operator equation:

$$\bar{L}u = \mathcal{F}$$

*is known as a strong solution of problem (**??**)-(**??**). Hence, one might expand the prior estimate to this solution by proposing the following estimate:*

$$\|u\|_B \leq C\|\bar{L}u\|_F, \quad \forall u \in D(\bar{L}). \tag{5.2.9}$$

Proposition 5.2.2. *The strong solution of problem (**??**)-(**??**) is unique and depends continuously on $\tilde{f} \in F$.*

Proposition 5.2.3. *$R(\bar{L})$ is closed in F and*

$$R(\bar{L}) = \overline{R(L)}.$$

Proof. Our primary objective is to investigate the uniqueness of the solution to the problem (**??**)-(**??**), under the assumption that a solution exists. Let u_1 and u_2 denote two solutions of (**??**)-(**??**). Assume $\eta = u_1 - u_2$, satisfies the following equation:

$$\begin{cases} {}^c\partial_t^\alpha \eta(x,t) - (\frac{\partial^2 \eta(x,t)}{\partial x^2}) + \beta\eta(x,t) + u_1^p - u_2^p = 0 \text{ in } Q, \\ \eta(x,0) = 0, \ \forall x \in (0,d), \\ \eta(x,t) = 0, \ \forall (x,t) \in \partial\Omega \times (0.T). \end{cases} \tag{P'}$$

If we use η as a scalar product in $L^2(\Omega)$ for the equation:

$${}^c\partial_t^\alpha \eta(x,t) - (\frac{\partial^2 \eta(x,t)}{\partial x^2}) + \beta\eta(x,t) + u_1^p - u_2^p = 0 \text{ in } Q, \tag{5.2.10}$$

and integrate over Ω, we obtain:

$$\int_\Omega {}^c\partial_t^\alpha \eta(x,t).\eta(x,t)dx - \int_\Omega (\frac{\partial^2 \eta(x,t)}{\partial x^2}).\eta(x,t)dx \\ + \beta \int_\Omega \eta^2(x,t)dx + \int_\Omega (u_1^p - u_2^p)(u_1 - u_2)dx = 0.$$

Due to $\eta(x,0) = 0$ and by applying Lemmas **??**, **??** and **??** along with integrating by parts, we get:

$$\|{}^c\partial_t^{\frac{\alpha}{2}} \eta\|^2_{L^2(\Omega)} + \|\frac{d\eta}{dx}\|^2_{L^2(\Omega)} + \|\eta\|^2_{L^2(\Omega)} \\ + \int_\Omega (u_1^p - u_2^p)(u_1 - u_2)dx = 0. \tag{5.2.11}$$

Since λ^p is a monotone function in λ (on $\Omega = (0,d)$) and based on some ours analysis, we can conclude that $\int_\Omega (u_1^p - u_2^p)(u_1 - u_2)dx$ of (**??**) is positive. Consequently, it follows from equation (**??**) that

$$\|\eta\|^2_{L^2(\Omega)} = 0,$$

which gives

$$u_1 = u_2, \qquad for \quad all \quad t \in (0,T).$$

Now, we will return to demonstrate the assertions declared in this result. For this purpose, we let $z \in \overline{R(L)}$. Then $\exists (z_n)_{n\in\mathbb{N}}$ a sequence in $R(L)$ such that

$$\lim_n z_n = z.$$

Similarly, $\exists (u_n)_{n\in\mathbb{N}}$ a sequence in $D(L)$ such that

$$Lu_n = z_n.$$

Now, let $\varepsilon, n \geq n_0$ and $m, m' \in \mathbb{N}$, $m \geq m'$ such that u_m and $u_{m'}$ are two solutions, i.e.,

$$Lu_m = \tilde{f} \quad Lu_{m'} = \tilde{f}.$$

Consequently, putting $y = u_m - u_{m'}$ makes y satisfying

$$\begin{cases} {}^c\partial_t^\alpha y(x,t) - (\frac{\partial^2 y(x,t)}{\partial x^2}) + \beta y(x,t) + u_m^p - u_{m'}^p = 0 \text{ in } Q, \\ y(x,0) = 0, \ \forall x \in (0,d), \\ y(x,t) = 0, \ \forall (x,t) \in \partial\Omega \times (0.T). \end{cases} \qquad (P'')$$

By employing the same approach used to examine the solution's uniqueness, we can deduce that $y = 0$, which yields

$$0 \leq \|u_m - u_{m'}\| \leq 0,$$

for all $t \in (0,T)$. In other words, we have $\forall \varepsilon \geq 0$, $\exists n_0 \in \mathbb{N}$ such that $\|u_m - u_{m'}\| \leq \varepsilon$, $\forall m, m' \geq n_0$. Thus, one might conclude that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E, and hence $\exists u \in E$ such that

$$\lim_n u_n = u$$

By using the definition of $\bar{L}$ ($\lim_{n\to+\infty} u_n = u$ in E, if $\lim_{n\to+\infty} Lu_n = \lim_{n\to+\infty} z_n = z$, then $\lim_{n\to+\infty} \bar{L}u_n = z$. Also, as $\bar{L}$ is closed, then $\bar{L}u = z$), we can assert that the function u satisfies

$$\bar{L}u = z \text{ so that } u \in D(\bar{L}).$$

Therefore, $z \in R(\bar{L})$, and hence $\overline{R(L)} \subset R(\bar{L})$.
Furthermore, due to $R(\bar{L})$ is a Banach space, then we conclude that it is closed.
Now, it is still necessary to demonstrate the opposing inclusion.
To accomplish this goal, we take $z \in R(\bar{L})$. Consequently, there exists a sequence $(z_n)_n$ in F composed of elements from $R(\bar{L})$ such that

$$\lim_n z_n = z.$$

As a consequence, $\exists (u_n)_{n\in\mathbb{N}}$, a corresponding sequence in which

$$\lim_n \bar{L}u_n = z_n.$$

Alternatively, we have a Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ in F. So, $\exists u \in E$ such that

$$\lim_n u_n = u \quad in \;\; E.$$

Once again, there exists a corresponding sequence $L(u_n) \in R(L)$ such that

$$Lu_n = \bar{L}u_n \quad on \quad R(L), \forall n$$

. Thus, we have

$$\lim_n Lu_n = z.$$

As a consequence $z \in \overline{R(L)}$, which asserts that $\overline{R(L)} = R(\bar{L})$. □

5.3 Establishing the existence of solutions

*To demonstrate the existence of a solution for the problem(**??**)-(**??**), we need to demonstrate that $R(L)$ is dense in F, $\forall u \in B$ and for any arbitrary $\mathcal{F} = (\tilde{f}, 0) \in F$. To this end, we list the following result.*

Theorem 5.3.1. *Problem (**??**)-(**??**) has a solution.*

In order to show this result, we should first notice that the scalar product of F is defined by

$$(Lu, w)_F = \int_Q \mathcal{L}u.wdxdt. \tag{5.3.1}$$

If one puts $w \in (R(L))^{\perp}$, we get

$$\int_Q {}^c\partial_t^{\alpha} u(x,t)w(x,t)dxdt - \int_Q (\frac{\partial^2 u(x,t)}{\partial x^2}).w(x,t)dxdt$$
$$+\beta \int_Q u(x,t)w(x,t)dxdt + \int_Q u^p(x,t).w(x,t)dxdt = 0.$$

Setting $w = u$ yields

$$\int_Q {}^c\partial_t^\alpha u(x,t).u(x,t)dxdt - \int_Q (\frac{\partial^2 u(x,t)}{\partial x^2}).u(x,t)dxdt + \beta \int_Q u^2(x,t)dxdt + \int_Q u^{p+1}(x,t)dxdt = 0. \tag{5.3.2}$$

By performing integration by parts on each term of equation (??) and considering the given condition of u, we obtain

$$\|{}^c\partial_t^{\frac{\alpha}{2}} u\|^2_{L^2(Q)} + \|\frac{du}{dx}\|^2_{L^2(Q)} + \beta\|u\|^2_{L^2(Q)} + \|u\|^{p+1}_{L^{p+1}(Q)} = 0.$$

which implies $\|u\|^2_{L^2(Q)} = 0$. This means that $u = 0$ in Q, and hence $w = 0$ in Q.

5.4 Existence and uniqueness of the Solution of problem (??)-(??)

In this part, for necessary reasons, we need to define the following function:

$$g^*(t) = \int_\Omega g(x,t).v(x)dx, \quad Q = \Omega \times (0,T). \tag{5.4.1}$$

In addition, we suppose that all functions involved in the problem at hand meet the conditions

$$(H)\begin{cases} g \in C((0,T), L^2(\Omega)), v \in W_2^1(\Omega) \cap L^{p+1}(\Omega), \\ \|g(x,t)\| \leq m, \quad |g^*(t)| \geq r > 0, \end{cases}$$

for $E \in W_2^2(0,T)$, $r \in \mathbb{R}$ and $(x,t) \in Q$. Besides, we also suppose that these functions are generally measurable.
Now, one might notice that the relationship between f and u can be described by

$$A : L^2(0,T) \to L^2(0,T), \tag{5.4.2}$$

where A is a linear operator defined as

$$(Af(t)) = \frac{1}{g^*}\left\{\int_\Omega \frac{du}{dx}\frac{dv}{dx}dx + \int_\Omega u^p(x,t).v(x)dx\right\}. \tag{5.4.3}$$

As a result, the previous relationship between f and u can be represented in terms of the function f, defined the interval over $L^2(0,T)$, as :

$$f(t) = Af(t) + W, \tag{5.4.4}$$

where

$$W = \frac{{}^{c}\partial_t^{\alpha} E + \beta E}{g^*} \tag{5.4.5}$$

in which $E(0) = 0$.

Theorem 5.4.1. *Assume that the inverse problem's data functions (**??**)-(**??**) satisfy condition* (H)*. Then we have the following equivalent statements:*

*(i) If there is a solution of the inverse problem (**??**)-(**??**) then equation (**??**) is solvable.*

*(ii) If there is a solution of (**??**) the compatibility condition* $E(0) = 0$ *is satisfied, then there is also a solution of the inverse problem (**??**)-(**??**).*

Proof. (i) Suppose that problem (**??**)-(**??**)is solved with designating its solution by $\{u, f\}$. Multiplying both sides of (**??**) by v and integrating the result over Ω yield

$$\begin{aligned} {}^{c}\partial_t^{\alpha} \int_{\Omega} u(x,t).v(x)dx + \int_{\Omega} \frac{du}{dx}\frac{dv}{dx}dx \\ + \beta \int_{\Omega} u(x,t)v(x)dx + \int_{\Omega} u^p(x,t).v(x)dx \\ = f(t)g^*(t). \end{aligned} \tag{5.4.6}$$

Now, using (**??**) and (**??**) implies

$$f = Af + \frac{\beta E +^{c} \partial_t^{\alpha} E}{g^*}.$$

This confirms that f solves (**??**).

(ii) By assuming that equation (**??**) has a solution, say f, and then substituting f into equation (**??**), then the resulting relation (**??**)-(**??**) will be viewed as a direct problem with a unique solution. We still need to demonstrate that u meets the condition of integral overdetermination (**??**)as well. To this end, we should note that according to equation (**??**), the function u is contingent on the following assertion:

$$\begin{aligned} {}^{c}\partial_t^{\alpha} E + \int_{\Omega} \frac{du}{dx}\frac{dv}{dx}dx + \beta E + \int_{\Omega} u^p(x,t).v(x)dx \\ = f(t)g^*(t). \end{aligned} \tag{5.4.7}$$

Consequently, subtracting equation (??) from equation (??) yields

$$\begin{aligned} {}^{c}\partial_t^{\alpha} \int_{\Omega} u(x,t).v(x)dx + \beta \int_{\Omega} u(x,t).v(x)dx \\ =^{c} \partial_t^{\alpha} E + \beta E. \end{aligned} \tag{5.4.8}$$

By integrating both sides of (??) and considering $E(0) = 0$, we deduce that u meets the integral condition (??). Therefore, we can immediately infer that $\{u, f\}$ is the solution of the inverse problem (??)-(??), which concludes the proof.

□

In the following content, we intend to discuss some properties of the operator A by proposing the next result. This result would pave the way to establish a further result connected with the existence and uniqueness of solution of (??)-(??).

Lemma 5.4.1. *Assuming condition* (H) *holds, then there exists a positive* δ *such that the operator* A *is a contraction mapping in* $L^2(0,T)$.

Proof. Observe that the following estimate can be inferred from (??) easily:

$$|Af(t)|^2 \leq \frac{2}{r^2} \times \left[\| \frac{du}{dx} \|^2_{L^2(\Omega)} \| \frac{dv}{dx} \|^2_{L^2(\Omega)} + \|u\|^{2p}_{L^{p+1}(\Omega)} \|v\|^2_{L^{p+1}(\Omega)} \right].$$

Now, we assume that

$$\|u\|^p_{L^\infty(0,T,L^{p+1}(\Omega))} = \gamma \geq 0.$$

Then we have

$$|Af(t)|^2 \leq \frac{2}{r^2} \times \left[\| \frac{du}{dx} \|^2_{L^2(\Omega)} \| \frac{dv}{dx} \|^2_{L^2(\Omega)} + \gamma \|u\|^{p+1}_{L^{p+1}(\Omega)} \|v\|^2_{L^{p+1}(\Omega)} \right].$$

By integrating the above inequality over $(0,T)$, we obtain

$$\begin{aligned} \int_0^T |Af(t)|^2 dt \leq \frac{2}{r^2} max \left(\| \frac{dv}{dx} \|^2_{L^2(\Omega)}, \gamma \|v\|^2_{L^{p+1}(\Omega)} \right) \\ \times \left[\int_0^T \| \frac{du}{dx} \|^2_{L^2(\Omega)} dt + \int_0^T \|u\|^{p+1}_{L^{p+1}(\Omega)} dt \right], \end{aligned} \tag{5.4.9}$$

which immediately yields

$$\|Af\|_{L^2(0,T)} \leq K\left[\int_0^T \|\frac{du}{dx}\|^2_{L^2(\Omega)} dt + \int_0^T \|u\|^{p+1}_{L^{p+1}(\Omega)} dt\right]^{\frac{1}{2}}$$

where

$$K = \sqrt{\frac{2}{r^2} max\left(\|\frac{dv}{dx}\|^2_{L^2(\Omega)}, \gamma\|v\|^2_{L^{p+1}(\Omega)}\right)}$$

Then, using the A priori estimate and removing some terms lead us to infer

$$\|\frac{du}{dx}\|^2_{L^2(Q)} + \|u\|^{p+1}_{L^{p+1}(Q)} \leq C\|f\|^2_{L^2(Q)}.$$

This consequently gives

$$\|Af\|_{L^2(0,T)} \leq \delta\|f\|_{L^2(0,T)} \tag{5.4.10}$$

where $\delta = K\sqrt{C}$. Obviously, it can be noticed from the previous discussion that there is a positive δ in which $\delta < 1$. Thus, inequality (**??**) demonstrates that A is a contracting operator on $L^2(0,T)$. □

Theorem 5.4.2. *Assume condition* (H) *and the compatibility condition are satisfied, then there is a unique solution* $\{u, f\}$ *for inverse problem (**??**)-(**??**).*

Proof. It follows that equation (**??**) possesses a unique solution f in $L^2(0,T)$. Also, based on Theorem **??**, the existence of a solution of (**??**)-(**??**) can be established. However, the uniqueness of such a solution has not been yet shown. For this purpose, we assume that there are two different solutions for the inverse problem at hand, denoted as $\{u_1, f_1\}$ and $\{u_2, f_2\}$. In this regard, it should be noted that if the linear operator A contracts on $L(0,T)$ leading to $f_1 = f_2$, then according to Lemma **??**the uniqueness theorem of the solution to the main direct problem (**??**)-(**??**) implies $z_1 = z_2$, and this completes the proof. □

Corollary 5.4.1. *If the same assumptions of Theorem* **??** *hold, then the solution* f *of (**??**) is continuously dependent on the data* W.

Proof. Consider ω and v are two sets of W that satisfy all assumptions of Theorem **??**. Suppose f and g are two solutions of (**??**) for the data ω and v, respectively. Now, based on (**??**), we obtain

$$f = Af + v \quad \text{and} \quad g = Ag + \omega.$$

So, with the help of (??), we can calculate the difference $f-g$ as follows

$$\begin{aligned}\|f-g\|_{L^2(0,T)} &= \|(Af+v)-(Ag+\omega)\|_{L^2(0,T)}\\ &\leq \delta\|f-g\|_{L^2(0,T)}+\|v-\omega\|_{L^2(0,T)}.\end{aligned}$$

As a result, we obtain

$$\|f-g\|_{L^2(0,T)}\leq \frac{1}{1-\delta}\|v-\omega\|_{L^2(0,T)},$$

which finishes the proof. □

Conclusion

This work has studied inverse problems associated with parabolic equations of both integer and fractional orders, incorporating additional integral-type conditions. The existence and uniqueness of solutions were established using Banach's fixed-point theorem. Chapter 2 addressed the existence and uniqueness of solutions for a semi-linear inverse problem, while Chapter 3 explored superlinear equations in a more general context. Chapter 4 focused on parabolic equations of fractional order, and Chapter 5 generalized these results by considering non-linear terms.

We note that many interesting problems remain open to further enrich this study. Here, we cite a few:

1. *We The study of the solutions of nonlinear hyperbolic fractional inverse problems.*

2. *Thus, many interesting perspectives for numerical analysis could make it possible to continue the work undertaken in this thesis, especially in terms of the development of efficient numerical methods, in order to be compatible with non-local conditions of the integral type.*

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