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Comparison between the controls Calculated through HUM and the Low- regret method

Presented by

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شكر وعرفان وتقدير

الحمد لله الذي تتم بنعمته الصالحات، حمدا تقوم به النعم وتزول به النقم ويستجاب به الدعاء ويزيد الله من فضله ما يشاء على إتمام هذا العمل وصلى الله على سيدنا وحبيبنا ورسولنا محمد صلى الله عليه وسلم وبعد فالشكر الأول والأخير

لله عز وجل " لان شكرتم لأزيدنكم " << سورة إبراهيم – الآية 08 –>>*

قال رسول الله صلى الله عليه وسلم "من لم يشكر الناس لم يشكر الله" رواه الترميذي

* اتقدم بجزيل الشكر والامتنان والتقدير لأستاذنا القدير والمميز الأستاذ المؤطر **عيادي عبد الحميد** على إخلاصه وتفانيه في تدريسنا وكذا اثراء موضوع دراستي في جوانبها المختلفة كما أتقدم بجزيل الشكر إلى أخي وزميلي وأستاذي **رزوق عماد** على كل ما قدمه لي من توجيهات ونصائح قيمة.

كذلك جزيل الشكر **للأستاذة عشاب فاطمة والأستاذ حفظ الله عبد الحق** على روح العطاء والمساعدة.

إلى الأستاذ **الطيب تقي الدين أوصيف** كما لا أنسي جميع أساتذة قسم الرياضيات كل باسمه: الأستاذ **سعدي محمد مخلوف الصديق، دهيليس سفيان، مراد أحسن ....**

دمتم خير خلف لخير سلف بارك الله فيكم وسدد خطاكم لما هو أفضل، فخر لي بان تمدرست على يد خيرة الاساتذة سوف تبقون خالدين في ذاكرتي ممتنة لكل ما قدمتموه لنا من علم ومعرفة وعطاء.

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Résumé

Dans ce mémoire, nous allons faire une étude comparative de deux méthodes:

A/ La méthode d'unicité de Hilbert introduite par **J.L.Lions (15) (HUM)** ce qui résoudre le problème de contrôlabilité des systèmes dynamiques lorsque le contrôle est appliquée sur certain données du système.

B/ La deuxième méthode est la méthode de contrôle à moindre regret: ce qui a

L'avantage de résoudre de contrôlabilité des systèmes dynamiques singuliers lorsque le contrôle est appliquée sur certain données du système.

On peut dire que la méthode à moindre –regret généralise dans un sens la méthode de HUM si l'ensemble des contraintes sur le contrôle est d'intérieur vide

Mots clés :contrôle optimale, données incomplètes, contrôle sans regret,contrôle a moindre regret, équation des ondes, méthode HUM

Abstract

In this thesis, we will make a comparative study of two methods :
A/ Hilbert's uniqueness method introduced by J.L.Lions (15) (HUM) which solves the problem of controllability of dynamic systems when control is applied to certain system data.
B/ The second method is the Low- regret control method: which has the advantage of solving the controllability of singular dynamic systems when the control applied to certain system data.
We can say that the Low-regret method generalizes in a sense the HUM method if the set of constraints on the control has empty interior

Keywords :optimum control **,**incomplet data, no-regret control ,low- regret control, wave equation,Hilbert Uniquenesse Method

الملخص

في هذه المذكرة سوف نقوم بدراسة مقارنة لطريقتين:
أ/ طريقة الوحدانية الهيلبرتية HUM التي أنشأها ج.ا. ليونس (15) والتي تحل مشكلة إمكانية التحكم في الأنظمة الديناميكية عند تطبيق التحكم على بيانات نظام معينة.
ب / الطريقة الثانية هي طريقة التحكم الأقل ندم: والتي لها ميزة حل إمكانية التحكم في الأنظمة الديناميكية الفردية عند تطبيق التحكم على بيانات نظام معينة.
يمكننا القول أن الطريقة الأقل ندمًا تُعمم بطريقة ما طريقة HUM إذا كانت مجموعة مجموعة القيود المفروضة على عنصر التحكم تحتوي على مساحات داخلية فارغة

الكلمات المفتاحية: التحكم الأمثل، معطيات غير مكتملة، التحكم دون ندم، التحكم منخفض الندم، معادلة الأمواج طريقة الوحدانية الهيلبرتية.

Table des matières

Notations

$\mathbb{R}$	Set of real numbers.
$\Vert . \Vert_H$	A norm in Banach space H.
$(.,.)_H$	A scalar product in Hilbert space H.
$\vert . \vert_H$	A semi-norm in H.
C^2	The class of functions with continuous first and second derivative.
$\frac{\partial y}{\partial \nu} = \nabla y.\nu$	The conormal derivative.
$\Delta = \sum\limits_{i=1}^{n} \frac{\partial}{\partial x_i}$	The laplacien operator.
$\nabla = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \right)^T$	The gradient operator.
div	Divergence.
$\mathcal{A}^*$	The adjoint operator of $\mathcal{A}$.
$d\Gamma$	Lebesgue measure on boundary Γ.
χ_ω	Characteristic function of the set ω.
$\mathcal{L}(\mathcal{Y}, \mathcal{Z})$	The space of linear bounded operators from $\mathcal{Y}$ to $\mathcal{Z}$.
$\mathcal{D}(Q)$	The space of functions in C^∞ with a compact support in Q.
$\mathcal{D}'(Q)$	The dual space of $\mathcal{D}(Q)$.
$L^2(0, T, H)$	The bounded linear operator space.
a.e.	Almost every where.
PDE	Partial differential equation.
$\rightharpoonup$	Symbol of weak convergence.
iff	If and only if.
$HUM.$	Hilbert Uniquenss Method..

Introduction

Several domains are modeled by dynamic or stationary systems, the sentinel theory is an important tool for the identification of some system data based on control theory
control plays an interesting role in resolving the different systems in the different domains
Bellow we Present the organization of our memory
In the first chapter, we will recall certain mathematical tools for functional analysis, and Sobolev space Also, we present an introduction to the observability and controlability theory.
In the second chapter, we present a description of the HUM method for solving the problem of the controllability system
In the third chapter, we present the standard optimal control theory and we consecrated to study the notion of no-regret control and low-regret of distributed system.
Finally, we conclude by comparison between the HUM method and the low regrets control method.

Chapitre 1

Preliminaries

During this chapter, we will recall the concept and definition of fundamental spaces in functional analysis which contains some essential notion that concerns the spaces L^P
Hilbert and Sobolev, then semigroups and controllability of linear system which is necessary to know to approach the memory suites finally we recall some notions and definition concerning the optimization domain.

1.0.1 Functional analysis reminders

The banach space and the normed space

Definition 1.1 : *Let E a vector space over R. a norm over E is an application from E in R^+*

$x \longrightarrow \|x\|$ verifed :
(i) $\|x\| = 0 \Longleftrightarrow x = 0;$
(ii) $\|\lambda x\| = |\lambda| \, \|x\| \quad \forall x \in E \forall \lambda \in R;$
(iii) $\|x + y\| \leq \|x\| + \|y\| \qquad \forall x, y \in E$ (the triangular quality)
The pair $(x,\|.\|)$ called by anorm vector space
The pair ($x, \|.\|)$ from E in R^+ verifiying the proprties(ii) and (iii), is called semi-norm on E
We call Banach space a complete normed space.

The Hilbert space **Definition** : Let H a vector space over R a scalar product is a bilinear from $H \times H$ in R symetric positive definite
The scaler product satisfies the inequality of Cauchy -Schwarz

$$\left|(u, v) \leq (u, u)^{\frac{1}{2}}(v, v)^{\frac{1}{2}}\right| \forall u, v \in H$$

We also have

$$\|u\| = (u,u)^{\frac{1}{2}}$$

Definition :A Hilbert space is a vector space H endowed with a scalar product (u,v) and which is complete for the associated norm$\|.\|$.

Theorm d 'identification of Hilbert spaces

Theorem 1.1 *: (Riesz-Frechet presentation theorem)*

Given $\varphi \in\ H'$there existe $f \in H$ unique such that

$$\langle \varphi, v \rangle = \int_{\Omega} fv \ , \forall v \in H.$$

Moreover we have

$$\|f\|_{H} = \|\varphi\|_{H'}$$

Lax-milgram theorem

Definition : The bilinear from $a(u,v) : H \times H \to R$ is

(i) continious if

$$\exists c > 0 : |a(u,v)| \leq C \|u\| \|v\| \quad \forall u,v \in H$$

(ii)coerciveif

$$\exists a > 0 : |a(u,v)| \leq \alpha \|v\|^{2} \quad \forall v \in H$$

Theorem 1.2 ***(Lax-milgram)*** *Let $a(u,v)$ a bilinear form,continious and coercive so for all $\varphi \in H$ $\exists! u \in H$ such that*

$$a(u,v) = \langle \varphi, v \rangle \quad \forall v \in H$$

The L^{P} space.

Definition 1.2 *(The Lebesgue spaces) Let $p \in R$ with 1$\leq p < \infty$ the application is defined by :*

$$L^P(\Omega) = \left\{ f : \Omega \to \mathbb{R} \setminus f \text{ is mesurable and } \int_\Omega |f|^p < \infty \right\},$$

We call $\|.\|_p$ the norm of L^Psuch that

$$\|f\|_P = \left[\int |f(x)|^p \, dx \right]^{\frac{1}{p}}$$

Definition : We set $L^\infty(\Omega) = f : \Omega \to \mathbb{R}, f$ is mesurable and there is a constant c such that $|f(x)| \leq c$ a.e .on Ω, we note that

$$\|f\|_\infty = \inf \{c, |f(x)| \leq c \text{ a.e .on } \Omega\}$$

we verify that $\|.\|_\infty$ is a norm

Remark 1.1 *The space L^2 provided with a scalar product*

$$(u, v) = \int_\Omega u(x)v(x)dt, \ \ \forall u, v \in L^2(\Omega),$$

is a Hilbert space.

The $L^P(0, T, V)$ space

Definition 1.3 *Let V a Banach space, we denote by $L^P(0, T, V)$ the space of mesurable function $u :]0, T[\to V$ such that*

$$\|u\|_{L^P(0,T,V)} = (\int_0^T \|u(t)\|_v^p \, dt)^{\frac{1}{p}} \ < \ \infty \quad \text{for } 1 \leq p \ < \ \infty$$

and for $p = \infty$ we have

$$\|u\|_{L^P(0,T,V)} = \sup_{0 \leq t \leq T} essu(t) \, \|u(t)\|_v \ < \ \infty$$

For $1 \leq p \leq \infty$,the space of $L^P(0, T, V)$ is a Banach space.
If V is Hilbert for the scalar product $(., .)_v$, $L^2(0, T, V)$ is a hilbert space for the scalar product

$$(u, v)_{L^2(0,T,V)} = \int_0^T (u(t), v(t))_v dt$$

The reflexive space

Definition 1.4 *Let E be a Banach space and let J be the canonical of E into E". We say that E is reflexive if $J(E) = E$"*

Theorem 1.3 *L^P is reflexive for any $p, \ 1 < p < \infty$.*

Aubin Lions compactness theorem

We give three Banach space B_0, B, B_1

$$B_0 \subset B \subset B_1, B_i \text{ réflexive}, i = 0, 1.$$

$$\text{L'injection } B_0 \to B \text{ is compact}$$

We define

$$W = \left\{ v \backslash v \in L^{P0}(0, T; B_0), v' = \frac{dv}{dt} \in L^{P1}(0, T, B_1) \right\}$$

Where T over and $1 < \ p_i < \infty \quad , i = 0, 1.$
with the norm

$$\|v\|_{L^{P0}(0,T,B_0)} + \|v'\|_{L^{P1}(0,T,B_1)}$$

W is a B■anach space.
Also we have the following result

Theorem 1.4 *Under the previous assumptions l'injection of W into $L^{P0}(0, T, B)$*

The sobolev spaces on H^m order

The $H^1(\Omega)$ space

Definition 1.5 *We note $H^1(\Omega)$ the vector subspace of $L^2(\Omega)$ défined by :*

$$H^1(\Omega) = \{u \in L^2(\Omega), \nabla u \in L^2(\Omega)^n\} = \left\{ u \in L^2(\Omega), \frac{\partial u}{\partial x_i} \in L^2(\Omega), \forall i = 1, ..., n \right\}$$

Remark 1.2 *$H^1(\Omega)$ is a Hilbert space.*

The $H_0^1(\Omega)$ space

Definition 1.6 *The function of $H_0^1(\Omega)$ are the function of $H^1(\Omega)$ which vanich on the boundary $\Gamma = \partial\Omega$.*

$$H_0^1(\Omega) = \{u \in H^1(\Omega), u = 0 \text{ in } \Gamma\}$$

The $H^{-1}(\Omega)$ space

Definition 1.7 *: we note by $H^{-1}(\Omega)$ the topological dual of $H_0^1(\Omega)$,in other word,*

$$H^{\bar{-}1}(\Omega) = (H_0^1(\Omega))' = \ \{T; T \text{ is a continious linear form on } H_0^1(\Omega)\}$$

This space is associated with a dual norm

$$\|T\|_{H^{-1}(\Omega)} = \sup\left\{|\langle T, V\rangle|\,; v \in H_0^1(\Omega) \text{ and } \|v\|_{H^1(\Omega)} \leq 1\right\}$$

The $H^k(\Omega)$ space

Definition 1.8 *We denote by $H^k(\Omega)$ the vector subpace of $L^2(\Omega)$ defined by :*

$$H^k(\Omega) = \{u \in L^2(\Omega), \partial^\alpha u \in L^2(\Omega), \forall\alpha \in \mathbb{N}^n, |\alpha| \leq k\}$$

We provided it with the standar

$$\|u\|_{H^m}^2 = \sum_{|\alpha \leq m|} \|D^\alpha u\|_{L^2(\Omega)}^2$$

The $W^{m.p}$ space
Let be Ω be an open set for $\mathbb{R}^n$ for all $1 \leq p \leq \infty$ and for all $m \in \mathbb{N}$ we can define the space of sobolev

$$W^{m.p}(\Omega) = \{u \in L^p(\Omega), \partial^\alpha u \in L^p(\Omega), \forall\alpha \in \mathbb{N}^n, |\alpha| \leq m\}$$

That one equipped with the norm

$$\|u\|_{W^{m.p}}^p = \sum_{|\alpha \leq m|} \|D^\alpha u\|^\alpha dx \text{if } 1 \leq p < \infty$$

$$\|u\|_{W^{m.\infty}} = \max_{|\alpha \leq m|} \|D^\alpha u\|_\infty$$

Theorem 1.5 ***(Green Formulas)*** *Let $\Omega \subset \mathbb{R}^n$ a bounded and regular domain, and the normal vector unit to the outside $\Gamma = \partial\Omega$.So, for $u \in H^1(\Omega)$ and $v \in H^2(\Omega)$ we have : the first Green's formula*

$$\int_\Omega u\Delta v dx = -\int_\Omega \nabla u \nabla v dx + \int_\Gamma u\frac{\partial v}{\partial\nu} d\Gamma,$$

for $u, v \in H^2(\Omega)$ we have the second formula of Green's :

$$\int_\Omega (u\Delta v - v\Delta u) dx = \int_\Gamma u\frac{\partial v}{\partial\nu} - v\frac{\partial u}{\partial\nu} d\Gamma.$$

Definition 1.9 *Let $U \in \mathbb{R}^n$ a convex set and $f : U \longrightarrow \mathbb{R}$ a strictly convex function in U if*

$$f\left(ty+(1-t)\ x\right) < tf\left(y\right)+(1-t)\,f\left(x\right)\ \forall x, y \in U, \forall t \in [0,1]$$

Definition 1.10 *Let $f : E \to \mathbb{R} \cup \{+\infty\}$ is a proper function, so Legendre transform f^* of f is a function of $E \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$f^*(p) = \sup_{x \in E}\left((p, x) - f(x)\right)$$

Lemma 1.1 ***(Gronwall)*** *Let Ψ, G be continuous in $[0, T]$, with G nondecreasing and $\gamma > 0$. If*

$$\Psi(t) \leq G(t) + \gamma \int_0^t \Psi\left(s\right) ds \ \textit{ for all } t \in [0, T],$$

then

$$\Psi(t) \leq G(t) \exp(\gamma t), \text{ for all } t \in [0, T].$$

Proposition 1.1 ***(Cauchy inequality)***

Let a, b are any real numbers and p, q are real numbers connected by the relationship $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the Cauchy inequality

$$ab \leq \frac{1}{2}\left(a^2 + b^2\right).$$

Proposition 1.2 ***(Poincare inequality)***

For $n = 1$ (1diemension),we have :

$$\|u\|^2_{L^2(I)} \leq \frac{(b-a)^2}{2}\left\|u'\right\|^2_{L^2(I)}, \forall u \in H^1_0(I).$$

For $n \geq 2$, we have :

$$\exists C > 0\, \|u\|^2_{L^2(I)} \leq C\,\|\nabla u\|^2_{L^2(\Omega)^n}\, \forall u \in H^1_0(\Omega)$$

1.1 Controllability of linear system

Overviews of semigroupe theory

We consider cauchy problem :

$$\begin{cases} \frac{dy}{dt} = ay, \quad t \geq 0 \\ \quad y(0) = y_{0,} \end{cases} \tag{1.1}$$

with a is constant and $y_0 \in E$ (E is a Banach space).
The solution of problem (1.1) given by :

$$y(t) = y_0\, exp(at) \text{ such that } \exp(at) = \sum_{k=0}^{\infty} \frac{(at)^k}{k!}$$

Let the following problem :

$$\begin{cases} \frac{dy}{dt} = Ay, \quad t \geq 0 \\ \quad y(0) = y_{0,} \end{cases} \tag{1.2}$$

If A is a bounded linear operato and the space E is a ifinite dimention n then we fined the solution of the problem (1.2)
$y(t) = y_0 \exp(At)$ such that $\exp(At) = \sum_{k=0}^{\infty} \frac{(at)^k}{k!}$we not that $y(t) = y_0 S_1(t), S_1(t) = \exp(At)$, (if $t \in R$), $S_1(t)$ satisfied the proprities of group .
The solution of the problem(1.2) is given by formula :

$$y(t) = y_0 S_2(t), \quad t \in R$$

When A is inbounded linear operator $[E = L^2(R^n)]$
where $S_2(t)$ verifying some propreties of the $exp(At)$

$$\begin{cases} S_2(0) = I, \\ S_2(t+s) = S_{2(t)} S_2(s); \quad \forall t, s \geq 0. \end{cases}$$

Definition 1.11 *A familly $\{S(t)\}_{t\geq 0}$ $\forall x \in y$ of elements $S(t) \in L(y)^2$ for all $t \geq 0$ forms a semigroup of class C^0 in y,if it verifies the following properties :*
1.$S(0) = I$.
2.$S(s+t) = S(s)S(t); \ \forall t, s \geq 0.$
3.$\lim_{t \to 0^+} \|s(t)x - x\|_y = 0, \forall x \in y.$

Proposition 1.3 *Let $\{S(t)\}_{t\geq 0}$ be a semigroup of class C^0in $\mathcal{Y}$ then :*

1.$t \to \|S(t)\|$is bounded in $[0, \alpha]$; $(\alpha > 0)$.
2 the fonction $t \to S(t)x$ is continious in R^+ ,$\forall x \in \mathcal{Y}$
3. $\forall$ W ,$M \in \mathbb{R}$such that $\|S(t)\| \leq Me^{Wt}, t \in R^+$

Definition 1.12 *(type of semigroup) We call type of semigroup $S(t)$ the number W_0defied by :*

$$W_0 = \inf\left\{W \in R; \exists M \geq 1 \text{Such that } \|S(t)\| \leq Me^{Wt}, \forall t \geq 0\right\}$$
$$W_0 = \lim_{t \to 0^+} \frac{\log\|S(t)\|}{t} = \inf \frac{\log\|S(t)\|}{t}$$

The important semigroup

1/ if $\forall t \geq 0\,;\|S(t)\| \leq M \Rightarrow S(t)$ is bounded,

2/ if $\forall t \geq 0\,;\, \|S(t)\| \leq 1 \Rightarrow S(t)$ is contraction,

Definition 1.13 ***(infinitisimal generatot)*** *Let $S(t)$ be a semigroup in E the infinitesimal generator of $S(t)$ is an inbounded operator A defined by*

$$A : D(A) \subset E \to E$$

$$D(A) = \left\{x \in E \text{ such that } \lim_{t \to 0} \frac{S(t) - I}{t} x \text{ exist}\right\}.$$

where

$$Ax = \lim_{t \to 0} \frac{S(t) - I}{t}, \forall x \in D(A)$$

Proposition 1.4 *.$D(A)$ is a subspace of y :*

2. if $x \in D(A)$ so $S(t)x \in D(A)$ then $S(t)Ax = AS(t)x$, $\forall t \geq 0$

3.$\{S(t)\}_{t \geq 0}$ a semigrop of $C^0 \Rightarrow \lim\limits_{h \to 0} \frac{1}{h} \int_t^{t+h} S(\delta)x d\delta = S(t)x, \forall x \in E$ and $t \geq 0$

4.$(\overline{D(A)} = y)$ implique that the set $D(A)$ is dense in y .

5. A is a closed operator and $\overline{D(A)} = E$.

Theorem 1.6 ***(Hille-Yosida Theorem)*** *A linear operator A :$D(A) \subset y \to y$ is the infinitisimal generator of a semigroup $\{S(t)\}_{t \geq 0} \in SG(M, w)$ if and only if :*

1. A is a closed densely defined operator in $\mathcal{Y}$; $(\overline{D(A)} = y)$.

2.$\forall$ $w \geq 0$ and $M \geq 1$ such that $A_w = \{\lambda \in \mathbb{C} : \|A\| < |\lambda|\} \subset p(A)$ and for any $\lambda \in A_w$: $\|(\lambda I + A)^{-n}\| \leq \frac{M}{(\lambda - w)}, \forall n \in \mathbb{N}^*$.

. $(\Rightarrow)$ Its clear.

$(\Leftarrow)$ For each $\lambda > w$: let $A_\lambda = \lambda^2(\lambda I - A)^{-1} - \lambda I \Rightarrow A_\lambda \in L(E)$; we can then construct the semi group :

$$S_t^\lambda = e^{A_\lambda t} = e^{\lambda^{2..t.(\lambda I - A)^{-1} - \lambda t}}$$

$$S_t^\lambda = e^{A_\lambda t} = e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda^2 t)^n}{n}(\lambda I - A)^{-1}$$

We will show that the limite of semi-group S_t^λexixts then $\lambda \to 0$.and that of semi-groupe to find. note that

$$\left\|S_t^\lambda\right\| \leq e^{-\lambda t}\sum_{n=0}\frac{(\lambda^2 t)^n}{n!}\frac{M}{(\lambda - w)^n}$$

$$= M\exp(\frac{\lambda w t}{\lambda - w})$$

It is easy to see that : $A_\lambda A_\mu = A\mu \; A_\lambda$.

Let $x \in D(A)$ we have :

$$S_t^\lambda x - S_t^\mu x = \int_0^t \frac{d}{ds}(S_{t-s}^\mu - S_s^\lambda)xds$$

$$= \int_0^t (S_{t-s}^\mu - S_s^\lambda)(A_\lambda - A_\mu)xds$$

$$\Rightarrow \left\|S_t^\lambda x - S_t^\mu x\right\| \leq M^2\exp(\frac{\mu w t}{\mu - w})\left\|(A_\lambda - A_\mu)x\right\|.\int_0^t \exp(-\frac{(\lambda-\mu)w^2 s}{(\mu - w)(\lambda - w)}ds$$

Selected :$\lambda > \mu$

$$\left\|S_t^\lambda x - S_t^\mu x\right\| \leq M^2\exp(\frac{\mu w t}{\mu - w}).q.\underbrace{\left\|A_\lambda - A_\mu)x\right\|}_{\to 0}$$

(Since $A_\lambda x \to Ax$ because $\lambda, \mu \to \infty$)

So $S_t^\lambda x$ converges strongly towards a limit denoted by :$S_t x$.

It remains that S_t is a semi-group C^0 in the infinitisimal generator $A.t$

*$S_{t+s}x = \lim\limits_{\lambda\to\infty} S_{t+s}^\lambda x = \lim\limits_{\lambda\to\infty} S_t^\lambda S_s^\lambda x = S_t S_s x$;$\forall x \in E; \forall t, s \geq 0$

*$S_0 x = \lim\limits_{\lambda\to\infty} S_0^\lambda x = Ix = x \Rightarrow S_0 = I$

*Strong continuity is a direct consequence of uniform continuity on the compact.

Let $x \in D(A)$

$$S(t)x - x = \lim_{\lambda\to\infty}(e^{A_\lambda t}x - x) = \lim_{\lambda\to\infty}\int_0^t e^{sA_\lambda}A_\lambda ds = \int_0^t S(s)Axds$$

Let B be the generator of $S(t)$ and let $x \in D(A)$

$$\Rightarrow \frac{S(t)x - x}{t} = \lim_{t\to 0}\frac{1}{t}\int_0^t S(s)Axds$$

$$\Rightarrow x \in D(B); Bx = Ax$$

$$\Rightarrow B \supseteq A$$

If $\lambda > w$ we first have $\lambda \in p(A)$ and $\lambda \in p(B)$ according to the necessary condition of **Hille-Yosida** so $A = B$

Thus the theorem is proved ■

1.1.1 Optimization

1.1.2 Global and local maximum or minimum

Let the set of constrainte C such that

$$C = \{x \in R^n \,; g(x) \leq 0 \,, h(x) = 0\} .$$

We suppose that C is non-empty ; an element x of C will be sed realizable.

Definition 1.14 *C is non-empty set of a real Hilbert space H and f is a function C in R We say that $x^* \in C$ satisfies a local munimum of f if*

$$\forall x \in B(x^*, r) \cap C; \;\; f(x^*) \leq f(x),$$

such as

$$B(x^*, r) = \{x \in H \,; \|x - x^*\| \leq r\} .$$

Definition 1.15 *We say that $x^* \in C$ achieves a global minimum of f if*

$$\forall x \in C \;; \;\; f(x^*) \leq f(x),$$

or a global maximum if

$$\forall x \in C \;; \;\; f(x^*) \geq f(x),$$

Proposition 1.5 *If x^*satisfies a (global or local) munimum ;x^*satisfies a (global or local) maximum ; i.e*

$$\max\{f(x), x \in C\} = -\min\{-f(x), x \in C\} .$$

The Controllability-Observability duality

Controllability of localized systems

We consider the following linear system

$$\begin{cases} y'(t) = Ay(t) + Bv(t) & \forall t \in (0,t] \\ y(0) = y_0 \in D(A) & t = 0 \end{cases} \tag{1.3}$$

where $A : R^n \to R^n, B : R^m \to R^m$ are bounded linear with m,n$\in N^*$and $m \leq n, v \in L^2(0,T,U)$,whereas $(U = R^m), y(.) \in \mathcal{Y}, y(0)$ is the initial state.

The systhem has unique continious weak solution on $[0,T]$,If $\mathcal{A}$ auto adjoint with compact resolvent given by "Duhamel " formula :

$$y(t) = S(t-s)Bv(s)ds \quad \forall t \in [0,T],$$

Definition 1.16 *the system* (1.3) *is controlable,if for any two state* y_0 *and* $y_{d.}$*if there exists a control* $u \in L^2(0,t,R^m)$ *that returns the syetem* (1.3) *from* $y_{{}_0}$ *to* y_d *in infinite time* $t \in (0,t]$

i.e $y(T) = y_d$ in other words we have

$$S(t)y_0 + \int_0^T S(t-s)Bv(s)ds = y_d.$$

Definition 1.17 *The dual system*

$$\begin{cases} y'(t) = Ay(t) + Bv(t), \\ y(0) = y_0 \\ z(t) = Cy(t) \end{cases}$$

is called by "the adjoint system".The

Different notion of controllability

1.1.3 Exact controllability

Definition 1.18 *We say that system (1.3) is exactly controllable over the time* $[0.T]$ *if*

$$\forall y_d \in H, \exists v \in L^2(0,T,U) \text{ such that } \quad y_v(T) = y_d.$$

1.1.4 Null controlability

We say that system (1.3) is null controllable over the time $[0.T]$ if :

$$\forall y_d \in H, \exists v \in L^2(0,T,U) \text{ such that } \quad y_v(T) = 0$$

Remark 1.3 *In general the,exact controlability implies null controllability, but the converse is not true.*

1.1.5 approximate controllability

We say that system(1.3) is approximate controllable over the time $[0.T]$ if

$$\forall y_d \in H, \exists v \in L^2(0,T,U)\text{such that } \|y(T) - y_d\|_H \leq \epsilon \text{ ,} \forall\ \epsilon \rangle 0.$$

Chapitre 2

The Hilbert Uniqueness Method

2.0.6 Description of the HUM method

The construction of the Hilbertian spaces adapted to the building of the system according to the criteria of the specific uniqueness of the homogeneous system associated with it ,and the method adopted for that is HUM, the following algorithm describes the basics of applying the HUM method to solving the problem of exact system controllability.

2.0.7 Position of problem

The basic idea is the following :
Assuming that the system is exactly controllable characterize the control that minimizes the associated cost function among the set of admissible controls by an optimality system

2.0.8 Exacte controllability and penalization

2-1 **Orientation**
Let be Ω a bounded domaine of R^n, $n \geq 1$, at the border Γ of class C^2.
we consider the wave equation

$$y" - \Delta y = 0 \quad in \quad Q = \Omega \times [0, T] \tag{2.1}$$

With $T > 0$ fixed
we assume that we can act on the system throuh the intermidiary of the control $v = (x, t)$ on the edge $\Sigma = \Gamma \times [0, T]$, so that

$$y = v \qquad on \quad \Sigma \tag{2.2}$$

let the initial data be

$$y(x,0) = y^0(x); y'(x,0) = y^1(x) \qquad on \ \ \Omega \tag{2.3}$$

let $x^0 \in R^n$, $m(x) = x - x^0$ and $R(x^0) = \underset{x \in \overline{\Omega}}{max} \left| m(x) \right|$

consider the usual partition of the boundary

$$\begin{cases} \Gamma(x^0) \ = x \in \Gamma / m(x).v(x) \succ 0 \\ \Gamma * (x^0) = \Gamma \backslash \Gamma(x^0) \end{cases} \tag{2.4}$$

and

$$\begin{cases} \Sigma(x^0) \ = \Gamma(x^0) \times \ [0,T] \\ \Sigma * (x^0) = \Sigma / \Sigma(x^0) \end{cases} \tag{2.5}$$

let be the exacte controlability of the following equation
if $T > T(x^0) = 2R(x^0)$ for each pair of initial data
$(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$
there is a control

$$v \in \ L^2(\Sigma(x^0))$$

such as the solution $y = y(v)$ in (2.1) (2.2) (2.3) checked $y(T,v) = y^{'}(T,v) = 0$
the facte that the control v is definided $\Sigma(x^0)$ must be interepted as meaning

$$\left\{ \begin{array}{c} y = v \ \ in \ \Sigma(x^0) \\ y = 0 \ in \ \Sigma * (x^0) \end{array} \right| \tag{2.7}$$

For each pair of initial data we have
$\{y^0, y^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$
the set of admissibles controls

$$U_{ad} = \left\{ v \in L^2(\Sigma(x^0) / y(T,v) = y^{'}(T,v) = 0 \ \ in \ \ \Omega \right\} \tag{2.8}$$

contaivse an infinity of element.

we will now show that control given by HUM is that realizes the munimum of the cost fonction

$$J(v) = \frac{1}{2} \int_{\Sigma(x^0)} |v|^2 \, d\Sigma \tag{2.9}$$

on all admissible controls U_{ad} we wll next characterize the control v using the optimality system

2.0.9 Characterization of control

Theorem 2.1 *for each pair of initial data*

$$\left\{y^0; y^1\right\} \in L^2(\Omega) \times H^{-1}(\Omega)$$

Theorem 2.2 *the control $v \in L^2(\Sigma(x^0))$ given by HUM is the one that minimizes the blow function $J(v)$ on all admissible controls $U_{ad.}$*

First step

Theorem 2.3 *we consider the minimization problem*

$$\inf_{v \in U_{ad}} J(v) \tag{2.10}$$

Theorem 2.4 *the (2-10) problem is an optimal control problem with constraint on the state . by a pealization method we define the function*

$$J_\epsilon(v,z) = \frac{1}{2}\int_{\Sigma(x^0)} |v|^2\, d\Sigma + \frac{1}{2\varepsilon}\int_Q (z" - \Delta z)^2 dxdt \tag{2.11}$$

with $\varepsilon > 0, \mathbf{v} \in \mathbf{L}^2(\Sigma(x^0))$ et $\mathbf{z} = \mathbf{z}(\mathbf{x}, \mathbf{t})$ a fonction such as

$$z" - \Delta z \in L^2(Q)$$

$$z(0) = y^0 z'(0) = y^1 \qquad \textbf{in } \Omega$$

$$\begin{cases} z = & v \; in \; \Sigma(x^0) \\ & 0 \;\; in \;\; \Sigma * (x^0) \end{cases} \tag{2.12}$$

$$z(T) = z'(T) = 0 \text{ in } \Omega$$

recess for each $v \in U_{ad}$ the function $y = y(v)$ of (2.1)(2.3) verifies these condition
the term $\frac{1}{2\varepsilon}\int_Q (z" - \Delta z)^2 dxdt$ is a penality term
we consider the optimal control problem

$$\underset{v,z}{inf}\, J_\varepsilon(v,z) \tag{2.13}$$

for each $\varepsilon > 0$ there existe a unique solution $\{u_\varepsilon, z_\varepsilon\}$ of this problem, ie

$$J_\varepsilon(u_\varepsilon, z_\varepsilon\;\;) = \underset{v,z}{inf}\, J_\varepsilon(v,z) \tag{2.14}$$

Second step

note that the sequence $(u_\epsilon)_{\epsilon\rangle 0}$ is bounded in $L^2(\Sigma(x^0))$

Let $v \in U_{ad}$ any and $y = y(v)$ the solution of the problem (2.1) (2.2) (2.3) associated .the couple $\{v, y(v)\}$ is admissible for the minimization problem (2.13) for every $\epsilon > 0$

and so

$$J_{_\epsilon}(u_\epsilon, z_\epsilon) \le J_\epsilon(y, y(v)) \tag{2.15}$$

But as $y(x)$ verifies

$$y^{"} - \triangle y = 0 \quad \text{in } Q \tag{2.16}$$

We see that

$$J\epsilon(v, y(v)) = J(v) \;, \forall \epsilon > 0 \tag{2.17}$$

so

$$J_\epsilon(u_{\epsilon,} z_\epsilon) \le J(v), \forall \epsilon > 0 \tag{2.18}$$

and this for each $v \in U_{ad}$, so we have

$$J_\epsilon(u_{\epsilon,} z_\epsilon) \le \inf_{v \in u_{ad}} J(v), \forall \epsilon > 0 \tag{2.19}$$

Espicially

$$J(u_\varepsilon) \le \inf_{v \in u_{ad}} J(v), \forall \epsilon > 0 \tag{2.20}$$

And, if we put

$$f_\epsilon = \frac{1}{\sqrt{\epsilon}}(z''_\epsilon - \triangle z_\epsilon) \tag{2.21}$$

We have

$$(f_\epsilon) \text{ bounded in } L^2(Q) \tag{2.22}$$

Third step

Quite to extract subsequences we will have

$$u_\epsilon \;{}_{\to 0}\, \widehat{v} \text{ in } L^2(\Sigma(x^0))\; weak \tag{2.23}$$

we moreover

$$\left\| z_{\epsilon}^{"} - \triangle z_{\epsilon} \right\|_{L^2(Q)} \leq C\sqrt{\epsilon},\ \forall \epsilon > 0 \tag{2.4}$$

We fined

$$(z_{\epsilon}) \text{ bounded in,} L^{\infty}(0, T, L^2(\Omega)) \cap W^{1.\infty}(0, T, H^{-1}(\Omega)) \tag{2.5}$$

on particular

$$\|z_{\epsilon}\|_{L^2}(Q) \leq C,\ \forall \epsilon > 0 \tag{2.6}$$

and even it means extracting yet another sub-suite

$$\underset{\epsilon \to 0}{z_{\epsilon}} \to \widehat{y} \text{ in } L^2(Q) \text{ weak} \tag{2.27}$$

according to $(1.12)(2.24)$ we have

$$\widehat{y}^{"} - \triangle \widehat{y} = 0$$

$$\begin{cases} \widehat{y} = \widehat{v} \text{ in } \Sigma(x^0) \\ \widehat{y} = 0 \text{ in } \Sigma * (x^0) \end{cases} \tag{2.28}$$

$$\widehat{y}(T) = \widehat{y}'(T) = 0 \qquad in\ \Omega$$

$$\widehat{y}(0) = y^0; \widehat{y}'(0) = y^1 \qquad in\ \Omega$$

so we have $\widehat{v} \in U_{ad}$

$$J_{\epsilon}(u_{\epsilon}, z_{\epsilon}) \geq J(u_{\epsilon}) \tag{2.29}$$

and after the week loxer semcontinuity of J we have

$$J(\widehat{v}) \leq \liminf J(u_{\epsilon}) \leq \liminf J_{\epsilon}(u_{\epsilon}, z_{\epsilon}) \tag{2.30}$$

of $(2.19)(2.30)$ we concloude

$$J(\widehat{v}) = \inf_{v \in U_{ad}} J(v) \tag{2.31}$$

We have also demonstrated that

$$\lim_{\epsilon \to 0} J(u_\epsilon) = J(\widehat{v}) \tag{3.32}$$

which,combined with(2.23) gives

$$u_{\epsilon \to 0}\, \widehat{v} \text{ in } L^2(\Sigma(x^0)) \quad \text{(strong)} \tag{2.33}$$

Fourth step

consider the sequel

$$P_\epsilon = \frac{1}{\epsilon}(z_\epsilon^{"} - \Delta z_\epsilon)\ , \forall \epsilon > 0 \tag{2.34}$$

.

obviously

$$P_\epsilon = \frac{1}{\epsilon} f_\epsilon, \forall \epsilon > 0 \tag{2.35}$$

wesay that $(f_\epsilon)_{\epsilon \rangle 0}$ i bounded in $L^2(Q)$ but we do not yet have of estimate on $(P_\epsilon)_{\epsilon\, >\, 0}$.
by writing the equation of Euler associated whith the problem of minimization (2.13). we have

$$\int_{\Sigma(x^0)} u_\epsilon v d\,\Sigma - \int_Q p_\epsilon(\zeta" - \Delta\zeta")\, dxdt \ = 0 \tag{2.36}$$

for all solution of

$$\zeta - \Delta\zeta \in L^2(Q)$$

$$\zeta(0) = \zeta^{'}(0) = \zeta(T) = \zeta^{'}(T) = 0$$

$$\begin{cases} \zeta = v & in \quad \Sigma(x^0) \\ \zeta = 0 & in \ \ \Sigma * (x^0) \end{cases}$$

with $v \in L^2(\Sigma x^0))$.
by meanes of the formula f Green we thus obtain

$$\begin{cases} p_\epsilon^{"} - \Delta p_\epsilon = 0 & \text{in } Q. \\ p_\epsilon = 0 & on \ \ \Sigma \\ \frac{\partial p_\epsilon}{\partial v} = u_\epsilon & on \ \ \Sigma(x^0) \end{cases} \tag{2.37}$$

in effect

$$\int_Q p_\varepsilon(\zeta" - \triangle\zeta)\, dxdt = \int_Q (p"_\varepsilon - \triangle\, p_\varepsilon)\zeta dxdt - \int_\Sigma p_\varepsilon \frac{\partial\zeta}{\partial v} d\Sigma + \int_\Sigma \frac{\partial p_\varepsilon}{\partial v}\zeta d\Sigma \tag{2.38}$$

$$= \int_Q (p_\varepsilon^" - \triangle\, p_\varepsilon)\zeta dxdt - \int_\Sigma p_\varepsilon \frac{\partial\zeta}{\partial v} d\Sigma + \int_{\Sigma(x^0)} \frac{\partial p_\varepsilon}{\partial v} v d\Sigma$$

and so, after (2.36)

$$\int_Q (p_\varepsilon^" - \triangle\, p_\varepsilon)\zeta dxdt - \int_\Sigma p_\varepsilon \frac{\partial\zeta}{\partial v} d\Sigma + \int_{\Sigma(x^0)} \frac{\partial p_\varepsilon}{\partial v} v d\Sigma = \int_{\Sigma(x^0)} u_\varepsilon v\, d\Sigma \tag{2.39}$$

which is equivalent

Fiveth step

According to the inverse inequality we obtain

$$\frac{(T - 2R(x^0))}{2} \left\{ |\nabla p_\varepsilon(0)|^2 + |p'_\varepsilon(0)|^2 \right\} \leq \frac{R(x^0)}{2} \int_{\Sigma(x^0)} \left| \frac{\partial p_\varepsilon}{\partial v} \right|^2 d\Sigma = \frac{R(x^0)}{2} \int_{\Sigma(x^0)} |u_\varepsilon|^2\, d\Sigma \tag{2.40}$$

the sequence $(u_\varepsilon)_{\varepsilon > 0}$ being bounded in $L^2(\Sigma(x^0))$ we see that

$$|\nabla p_\varepsilon(0)| + |p'_\varepsilon(0)| \leq 0 \qquad \forall \varepsilon \rangle 0 \tag{2.41}$$

and according to the law of conservation of energy ,we have

$$(p_\varepsilon \ \longrightarrow p \ \text{ in } L^\infty((0, T, H_0^1(\Omega)) \cap W^{1.\infty}(0, T, L^2(\Omega)) \tag{2.42}$$

$$p_\varepsilon \longrightarrow \ p \quad \text{on } L^\infty((0, T, H_0^1(\Omega)) \text{ weak} \tag{2.43}$$

$$p'_\varepsilon \longrightarrow p' \quad \text{on } L^\infty((0, T, L^2(\Omega)) \text{ weak} \tag{2.44}$$

$$\left\{ p_\epsilon(0)\ ,\ p'_\varepsilon(0) \right\} \to \left\{ p(0), p'(0) \right\} \quad \text{on } H_0^1(\Omega) \times L^2(\Omega) \quad \text{weak} \tag{2.45}$$

so the function $p = p(x,t)$ solution of

$$\begin{cases} p^" - \triangle p = 0 & \text{in } Q \\ p = 0 & \text{on } \Sigma \\ \frac{\partial p}{\partial v} = \overrightarrow{v} & \text{on } \Sigma(x^0) \end{cases} \tag{2.46}$$

$$p(0) = p\,;\, p^{'}(0) = p^1 \text{ in } \Omega$$

sixth step

we pose $\Phi = p$; $\Phi^0 = p^0$; $\Phi^1 = p^1$ and $\psi = \widehat{y}$

Acording to (2.28) and (2.46) we have

$$\begin{cases} \Phi" - \triangle\,\Phi = 0 & in \quad Q \\ \Phi = 0 & on \quad \Sigma \\ \Phi(0) = \Phi^0 \;\; ,\Phi^{'}(0) = \Phi^1 & \text{in } \Omega \end{cases} \tag{2.47}$$

$$\psi^{"} - \Delta\psi = 0 \qquad \text{in } Q \tag{2.48}$$

$$\begin{aligned} \psi &= \frac{\partial\psi}{\partial v} && \text{on} && \Sigma(x^0) \\ \psi &= 0 && \text{on} && \Sigma_*(x^0) \end{aligned}$$

on the other hand as

$$\psi(0) = y^0 \qquad ;\psi^{'}(0) = y^1$$

we have

$$\Lambda\left\{\Phi^{0,} \;\; \Phi^1\right\} = \left\{y^1 \;\; , -y^0\right\}. \tag{2.49}$$

d ou Λ is the isomorphism between the spaces $H_0^1(\Omega) \times L^2(\Omega)$ and $H^{-1}(\Omega) \times L^2(\Omega)$ introduced in the application of HUM.

We thus see that the control $\overrightarrow{v}$ which by construction minimizes $J'(v)$ on U_{ad} , is the control given by HUM since

$$\widehat{v} = \frac{\partial p}{\partial v} = \frac{\partial\Phi}{\partial v} \quad .$$

Chapitre 3

Standard optimal control of distributed system

In this chapter, we will study the optimal control of linear PDE's ,(the dimension of space of solution is infinit) We start by the presentationof the
classical theory of the optimal control when we prove the existence,uniquness and characterization of the optimum and we give some examples Then we study the optimal control for a linear system with incomplet data by present the notion of no-regret control introduced by **J.Lions (1992)**, and associated with low-regret control which converges to the no-regret control, then we characterize them and we give example

3.1 Position of problem

Let $\mathcal{Y}$, $\mathcal{U}$ and $\mathcal{Z}$ be infinite dimensional Hilbert spaces of states, controls and observation resp. $\mathcal{U}_{ad}\subset \mathcal{U}$ is a subset of admissible controls supposed non empty, closed and convex.
f is a source function in y . Consider the fwell-posed abstract linear partial differential equation :

$$\mathcal{A}y = f + \mathcal{B}v. \tag{3.1}$$

Where $\mathcal{A} \in \mathcal{L}(\mathcal{Y})$ is a linear partial differential operator stationary or evolutionary (elliptic, parabolic and hyperbolic) makes an isomorphism on $\mathcal{Y}'$ identified to$\mathcal{Y}$, $\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is the control operator.
Our optimal control problem consists in looking for a control function $u \in \mathcal{U}_{ad}$ which minimizes the following cost function

$$J(v) = \|\mathcal{C}y(v) - y_d\|_{\mathcal{Z}}^2 + N \|v\|_{\mathcal{U}}^2 \quad \forall v \in \mathcal{U}_{ad}, \tag{3.2}$$

J : is convex function from $\mathcal{U}_{ad} \subset \mathcal{U}$ to $\mathbb{R} \cup \{+\infty\}$, $\mathcal{C} \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$: the observation operator and N is a symetric definite positive operatore bounded in U
y_d is the fixed observation in $\mathcal{Z}$.
we search u solution of

$$\begin{cases} \text{find } u \in \mathcal{U}_{ad} \text{ such that } J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v), \end{cases} \tag{3.3}$$

Theorem 3.1 ***(Existence and uniqueness of optimal control)***

Let $\mathcal{U}_{ad} \subset \mathcal{U}$ closed and nonempty, J is lower semicontinuous, bounded from below and coercive on $\mathcal{U}_{ad}$. Then there exists a minimizer for J on $\mathcal{U}_{ad}$. Moreover, if J is strictly convex the minimizer is unique.
. 1.**Existence**
Since $v \longrightarrow J(v)$ is continious and J is lower semi-continious convex function on $\mathcal{U}_{ad}$ and even stricly convex ,coercive because $(Nv, v)_u \geq \alpha \|v\|_u^2$, $\alpha > 0$ for every $v \in \mathcal{U}_{ad}$, then there existe u solution to$(3.1) - (3.3)$.Therefore ,we consider minimizing sequence (v_n)

$$J(u) \leq \lim_{n \to +\infty} \inf_{v \in \mathcal{U}_{ad}} J(v_n) = \inf_{v \in \mathcal{U}_{ad}} J(v) = m,$$

then u is aminimizer if J on $\mathcal{U}_{ad}$
2.**Uniqueness**
Suppose that the problem (3.3) admits two distinct solutions u_1, $u_2 \in u_{ad}$. since $\mathcal{U}_{ad}$ convex .We set $u = \frac{u_1+u_2}{2}$, due to strict convexity of J we get

$$J(u) < \frac{1}{2} J(u_1) + \frac{1}{2} J(u_2) = m,$$

This is a condradiction of being that u_1, u_2 are two solutions of (3.3) . Thus (3.3) admits a unique solution. ■

3.2 Optimal systems (Optimal control characterization)

We have by a first order optimality condition :

$$J'(u)(v - u) \geq 0 \qquad \forall v \in \mathcal{U}_{ad},$$

J is Gateaux-differentiable function

$$J'(u)(v-u) = \lim_{t\to 0}\frac{J(u+t(v-u))-J(u)}{t} \text{ for every } v\in\mathcal{U}_{ad},$$

with a calculatation we fined

$$\begin{aligned} J(u+t(v-u)) &= J(u)+t^2\|\mathcal{C}y(v-u)\|_z^2+2t(\mathcal{C}y(u)-y_d,\mathcal{C}y(v-u))_{\mathcal{Z}}+t^2N\|v-u\|_{\mathcal{U}}^2 \\ &\quad +2tN(u,v-u)_{\mathcal{U}}, \end{aligned}$$

which gives

$$\frac{J(u+t(v-u))-J(u)}{t} = t\|\mathcal{C}y(v-u)\|_{\mathcal{Z}}^2+2(\mathcal{C}y(u)-y_d,\mathcal{C}y(v-u))_z+tN\|v-u\|_{\mathcal{U}}^2+2N(u,v-u)_{\mathcal{U}},$$

when $t\longrightarrow 0$ we find

$$J'(u)(v-u) = 2(\mathcal{C}^*(\mathcal{C}y(u)-y_d),y(v-u))_{\mathcal{Y}}+2N(u,v-u)_{\mathcal{U}}\geq 0 \quad \forall v\in\mathcal{U}_{ad}. \tag{3.4}$$

Remark 1.1 A condition of the (3.3) from $J'(u)(v-u)$ is called the variationel inequality $\mathcal{C}^*\in\mathcal{L}(\mathcal{Z},\mathcal{Y})$ is the adjoint of $\mathcal{C}$,$\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$ and introduce the adjoint state $p=p(u)$ given by

$$\mathcal{A}^*p(u) = \mathcal{C}^*(\mathcal{C}y(u)-y_d), \tag{3.5}$$

then

$$\begin{aligned} (\mathcal{C}^*(\mathcal{C}y(u)-y_d),\delta y(v-u))_{\mathcal{Y}} &= (\mathcal{A}^*p(u),\delta y(v-u))_{\mathcal{Y}} = (p(u),\mathcal{A}\delta y(v-u))_y \\ &= (p(u),\mathcal{B}(v-u))_{\mathcal{Y}} = (\mathcal{B}^*p(u),(v-u))_{\mathcal{U}}, \end{aligned}$$

Hence,

$$J'(u)(v-u) = (\mathcal{B}^*p(u)+Nu,v-u)_{\mathcal{U}}\geq 0 \quad \forall v\in\mathcal{U}_{ad}. \tag{3.6}$$

The optimal control problem (3.1) - (3.3) has a unique solution u characterized by the following optimality system

$$\begin{cases} \mathcal{A}y(u) = f+\mathcal{B}u, \\ \mathcal{A}^*p(u) = \mathcal{C}^*(\mathcal{C}y(u)-y_d), \\ (\mathcal{B}^*p+Nu,v-u)_{\mathcal{U}}\geq 0 \qquad \forall v\in\mathcal{U}_{ad}. \end{cases} \tag{3.7}$$

The equation(1) ,(2)from(3.) must be associated to some appropriate boundary and initial condition.
We called the pair $(u, p(u))$ by the optimal pair.

Remark 3.1 *we have no constraintes on control , by space structure of $\mathcal{U}$ (if $\mathcal{U}_{ad} = \mathcal{U}$) we deduce that we also have*

$$J'(u)(v-u) \leq 0 \ \ \forall v \in \mathcal{U}_{ad},$$

and with the previous condition we get

$$J'(u)(v-u) = 0 \ \ \forall v \in \mathcal{U},$$

therefore the optimality system become as following

$$\begin{cases} \mathcal{A}y(u) = f + \mathcal{B}u, \\ \mathcal{A}^* p(u) = \mathcal{C}^*(\mathcal{C}y(u) - y_d), \\ (\mathcal{B}^* p(u) + Nu, v-u)_U = 0 \qquad \forall v \in \mathcal{U}. \end{cases}$$

3.2.1 Examples

1- **Optimal control of an hyperbolic equation :**
Let Ω, Γ,T $\rangle 0$,Q, Σ as in the last subsection let $U = L^2(Q)$ be the space of controls, U_{ad} is the set admissible controls non- empty closed and convex, $\mathcal{B}$ is bounded operator
from to $y = L^2(0\ ,T, H_0^1(\Omega))$.consider the following optimal control problem

$$\inf_{v \in \mathcal{U}_{ad}} J(\boldsymbol{v}) \tag{3.8}$$

with

$$J(v) = \|C_y(v) - Z_d\|_z^2 + (Nv, v)_u \ forevery \ v \in \ \mathcal{U}_{ad}, \tag{3.9}$$

with the following second order hyperbolic PDE

$$\begin{cases} \frac{\delta^2 y}{\partial t^2} + A(x)y = f + \mathcal{B}v & \text{in } Q, \\ y(x,t) = 0 & \text{on } \Sigma \\ y(x,0) = y_0(x), \frac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega, \end{cases} \tag{3.10}$$

where $f \in L^2(Q), y_0 \in H_0^1(\Omega), y_1 \in L^2(\Omega)$,$Zd \in L^2(Q\ )$, N is the positive operator on U and $A(x,t)$ is a seconde order elliptic operator.
First case : C is the canonical injection from $L^2(0,T,H_0^1(\Omega)) \rightarrow \ L^2(L^2(Q)$,then $\ Z = \ L^2(Q)$.

Then , the optimal pair $(u, y(u))$ for **??** ....is characterized by (Lions,1971)

$$\begin{cases} \frac{\partial^2 y(u)}{\partial t} A(x)y(u) = f + Bu, \\ \frac{\partial^2 p}{\partial t^2} + A^*(x)p = y(u) - z_d \qquad \text{in } Q, \\ y(x,t) = 0, p(x,t) = 0, \qquad \text{on } \sum, \\ y(u)(x,0) = y_0(x), \frac{\partial y}{\partial t}(x,0) = y_1(x) \\ p(x,T) = 0, \frac{\partial p}{\partial t}(x,T) = 0 \qquad \text{in } \Omega, \\ \int_0^T \int_\Omega (B^*P + Nu)(v-u)dxdt \geq 0 \quad \text{for every } v \in U_{ad.} \end{cases}$$

Second case : A final observation operator i,e $C\ : L^2(0,T,H_0^1(\Omega)) \rightarrow L^2(\ \Omega)$ with $C_y(x,t) = y(x,T)$, so $Z = L^2(\Omega)$.then the optimal pair $(u, y(u))$ for$(1.8)-(1.10)$
is caracterized by (Lions 1971)

$$\begin{cases} \frac{\partial^2 y(u)}{\partial t} A(x)y(u) = f + Bu, \\ \frac{\partial^2 p}{\partial t^2} + A^*(x)p = 0 \qquad \text{in } Q, \\ y(x,t) = 0, p(x,t) = 0, \qquad \text{on } \sum, \\ y(u)(x,0) = y_0(x), \frac{\partial y}{\partial t}(x,0) = y_1(x) \\ p(x,T) = 0, \frac{\partial p}{\partial t}(x,T) = 0 \qquad \text{in } \Omega, \\ \int_0^T \int_\Omega (B^*P + Nu)(v-u)dxdt \geq 0 \quad \text{for every } v \in U_{ad.} \end{cases}$$

2-Optimal control of a parabolic distributed system :

Let Ω a bounded domain of $\mathbb{R}^n$ with boundary Γ of class C^2, $T > 0$, Consider time space cylinder $Q = \Omega \times [0,T]$, $\Sigma = \Gamma \times [0,T]$ and the Heat equation with Dirichlet boundary condition

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = f + \chi_\omega v & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{3.11}$$

where $f \in L^2(Q)$, $v \in L^2(0,T,L^2(w))$, χ_ω is the characteristic function of ω a bounded open of Ω and $y_0 \in L^2(\Omega)$.

Our optimal control problem consists in looking for a control function $u \in L^2(0,T,L^2(\omega))$ which minimizes the following cost function

$$J(v) = \|y(v) - y_d\|^2_{L^2(Q)} + N\,\|v\|^2_{L^2(0,T,L^2(\omega))}\,,$$

with $y(v)$is the solution of (1.5), $y_d \in L^2(Q)$, $N > 0$. So, we want to characterize the solution of the following optimal control

$$\inf\left\{J(v,y) : (v,y) \in \mathcal{U}_{ad} \times L^2(0,T,H_0^1(\Omega)) \text{ verifies } (1.11)\right\}.$$

In this case
the state space $\mathcal{Y} = L^2(0,T,H_0^1(\Omega))$, the observation space $\mathcal{Z} = L^2(\Omega)$ and the control space $\mathcal{U} = L^2(0,T,L^2(\omega))$.
the observation operator is canonical injection

$$\begin{aligned} \mathcal{C}: \ L^2(0,T,H_0^1(\Omega)) &\longrightarrow L^2(\Omega) \\ y &\longrightarrow y \ , \end{aligned}$$

An optimality condition gives us

$$J'(u)(v-u) = 2(y(u)-y_d, y(v-u))_{L^2(Q)} + 2N(u,v-u)_{L^2(0,T,L^2(\omega))} \geq 0 \quad \forall v \in \mathcal{U}_{ad}.$$

by introducing the adjoint state $p = p(u)$

$$\begin{cases} -\frac{\partial p}{\partial t} - \Delta p = y(u) - y_d & \text{in } Q, \\ p = 0 & \text{on}\Sigma, \\ p(x,T) = 0 & \text{in } \Omega. \end{cases}$$

and using second Green formula we get

$$\begin{aligned} (y(u)-y_d, y(v-u))_{L^2(Q)} &= (-\frac{\partial p}{\partial t} - \Delta p, y(v-u))_{L^2(Q)} \\ &= \int_0^T \int_\Omega -\frac{\partial p}{\partial t} y(v-u) dxdt - \int_0^T \int_\Omega \Delta p y(v-u) dxdt \\ &= \int_0^T \int_\Omega p \left(\frac{\partial y}{\partial t}(v-u) - \Delta y(v-u) \right) dxdt - \int_\Omega [py(v-u)]_0^T \\ &\quad + \int_0^T \int_\Gamma p(\frac{\partial y}{\partial \nu}(v-u) - y(v-u)\frac{\partial p}{\partial \nu}) d\Gamma dt \\ &= \int_0^T \int_\Omega p\chi_\omega (v-u) dxdt - \int_\Omega p(T) y(v-u)\,(T)\, dx \\ &\quad + \int_0^T \int_\Gamma p \frac{\partial y}{\partial \nu}(v-u) d\Gamma dt \\ &= \int_0^T \int_\omega p(v-u) dxdt, \end{aligned}$$

Hence,

$$J'(u)(v-u) = 2\int_0^T \int_\omega (p+Nu)(v-u)dxdt \geq 0 \ \ \forall v \in \mathcal{U}_{ad}.$$

So, the solution of () is characterized by the following optimality system

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = f + \chi_\omega u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega, \\ -\frac{\partial p}{\partial t} - \Delta p = y(u) - y_d & \text{in } Q, \\ p = 0 & \text{on} \Sigma, \\ p(x,T) = 0 & \text{in } \Omega, \\ \int_0^T \int_\omega (p + Nu)(v-u) dx dt \geq 0 & \forall v \in \mathcal{U}_{ad}. \end{cases}$$

3- Optimal control of an elliptic distributed system

Let Ω a bounded domain of $\mathbb{R}^n$ with boundary Γ of class C^2, consider the following Laplace equation with Newman boundary condition

$$\begin{cases} -\Delta y + y = f & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} = v & \text{on } \Gamma, \end{cases} \tag{3.12}$$

$f \in L^2(\Omega)$ is a source function , $v \in L^2(\Gamma)$ is a control function. Associate to () the following cost function

$$J(v) = \|y(v) - y_d\|^2_{L^2(\Omega)} + N \|v\|^2_{L^2(\Gamma)},$$

$y(v)$ is the solution of (1.4), $y_d \in L^2(\Omega)$ is a fixed observation and $N > 0$.

Our optimal control consists a determine $u \in \mathcal{U}_{ad}$ that minimizing $J(v)$. $\inf \{J(v,y) : (v,y) \in \mathcal{U}_{ad} \times H^1(\Omega)$

the state space $\mathcal{Y} = H^1(\Omega)$, the observation space $\mathcal{Z} = L^2(\Omega)$ and the control space $\mathcal{U} = L^2(\Gamma)$.

the observation operator is canonical injection

$$\begin{aligned} \mathcal{C} : \; H^1(\Omega) &\longrightarrow L^2(\Gamma) \\ y &\longrightarrow y \;, \end{aligned}$$

An optimality condition gives us

$$J'(u)(v-u) \geq 0 \forall v \in \mathcal{U}_{ad} \iff 2(y(u) - y_d, y(v-u))_{L^2(\Omega)} + 2N(u, v-u)_{L^2(\Gamma)} \geq 0 \forall v \in \mathcal{U}_{ad}.$$

Now, we introduce an adjoint state $p = p(u)$

$$\begin{cases} -\Delta p + p = y(u) - y_d & \text{in } \Omega, \\ \frac{\partial p}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases}$$

and using the second Green formula (see appendix Theorem (1.5) we get

$$\begin{aligned}
(y(u)-y_d, y(v-u))_{L^2(\Omega)} &= (-\Delta p+p, y(v-u))_{L^2(\Omega)} \\
&= (p, -\Delta y(v-u)+y(v-u))_{L^2(\Omega)} \\
&\quad + \int_\Gamma (p\frac{\partial y}{\partial \nu}(v-u) - y(v-u)\frac{\partial p}{\partial \nu})d\Gamma \\
&= \int_\Gamma p(v-u)d\Gamma.
\end{aligned}$$

Finally, the solution of (1.4) is characterized by the following optimality system :

$$\begin{cases}
-\Delta y(u)+y(u)=f & \text{in}\Omega, \\
\frac{\partial y}{\partial \nu}(u)=u & \text{on } \Gamma, \\
-\Delta p+p=y(u)-y_d & \text{in } \Omega, \\
\frac{\partial p}{\partial \nu}=0 & \text{on } \Gamma, \\
\int_\Gamma (p+Nu)(v-u)d\Gamma \geq 0 & \forall v \in \mathcal{U}_{ad}.
\end{cases}$$

3.3 No-regret control and low-regret control to solve distributed system with missing data

In this chapter, we make an initiation to the theory of the optimal control of problems with incomplt data, where we introduce this leads to difine the notion of no- regret control, low regret control introduced the first time by **J.Lions(1992)**. Mereover, we give existance, uniqueness , and prouve that it converges to the no-regret control,then we characterize them via optimality systems and we give exemple..

3.3.1 Position of problem

We keep the same theorical framework as mentioned in the last paragraphe, the difference here is the presence of missing data. For this reason, we define a new operator $\beta \in \mathcal{L}(F, \mathcal{Y})$ where F is a Hilbert space of uncertainties (missing data), G is a non -empty closed subspce of F,
For $f \in \mathcal{Y}$ the abstract equation related to the control $v \in \mathcal{U}_{ad}$ and the uncertainty $g \in G$ is given by

$$\mathcal{A}y(v,g) = f + \mathcal{B}v + \beta g. \tag{3.13}$$

The equation (3.13) is well posed in $\mathcal{Y}$ and has a unique solution $y(v,g)$, which associate to her the following cost function :

$$J(v,g) = \| \mathcal{C}y(v,g) - y_d \|_{\mathcal{Z}}^2 + N \| v \|_{\mathcal{U}}^2 \qquad \forall v \in \mathcal{U}_{ad}, \forall g \in G, \tag{3.14}$$

as usual, we are concerned with the optimal control of (3.13) and (3.14) is to search u solution of :

$$\inf_{v \in \mathcal{U}_{ad}} J(v,g) \quad \forall g \in G,$$

when$G$ is an infinite dimensional space the problem (3.13) has no sens,this problem is solved by J.Lions they using many notion like no-regtet control (Lions,1992) and pareto control (Lions 1986)

there equivalents is proved in (Nacoulima,Omrane &Velin ,2003).Lions thought to take

$$\inf_{v \in \mathcal{U}_{ad}} \left(\sup_{g \in G} J(v,g) \right), \tag{3.15}$$

but G is an infinite dimensional space we can get $\sup_{g \in G} J(v,g) = +\infty$ and by the way the problem has no sense, to avoid this difficulty **J.Lions** introduce the concept of "*No-regret control*".

Remark 3.2 *2.1 If $G = \{0\}$ then $J(v,g) = J(v,0)$. Therefore, the problem (3.15) becomes a standar optimal control problem*

$$\begin{cases} \text{find } u \in \mathcal{U}_{ad} \text{ such that} \\ \quad J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v). \end{cases}$$

To avoid difficulties arise when we get $\sup_{g \in G} J(v,g) = +\infty$, **J.Lions** thought to take only controls such that $\forall v \in \mathcal{U}_{ad}$

$$\begin{aligned} J(v,g) &\leq J(0,g) \quad ; \qquad \forall g \in G \\ J(v,g) - J(0,g) &\leq 0, \forall g \in G \end{aligned} \tag{3.16}$$

Thus, we can say that $\sup_{g \in G} (J(v,g) - J(0,g))$ exists.

3.4 The no-regret control

Definition 3.1 *[15]We say that $u \in \mathcal{U}_{ad}$ is a no-regret control for (3.13) and (3.14) if u solves*

$$\inf_{v\in\mathcal{U}_{ad}}\left(\sup_{g\in G}\left(J\left(v,g\right)-J(0,g)\right)\right). \tag{3.17}$$

Remark :of course , the next problem is defied only for controls such that

$$\sup_{g\in G}\left(J\left(v,g\right)-J(0,g)\right)<\infty\cdot \tag{3.18}$$

Lemma 3.1 *($Na\,\mathrm{col}\,ima; Ormane$&Velin 2003) For every $u\in\mathcal{U}_{ad}$ and $g\in G$ we have :*

$$J(v,g)-J(0,g)=J(v.0)-J(0,0)+2\left(S(v),g\right)_{G',G} \tag{3.19}$$

where $S(v)=\beta^{*}\xi(v)$ and $\xi(v)$ defined for $v\in\mathcal{U}_{ad}$ by

$$\mathcal{A}^{*}\xi(v)=\mathcal{C}^{*}\mathcal{C}(y(v,0)-y(0,0)).$$

. $\mathcal{A}$ is a linear operator in $\mathcal{Y}$, so :

$$\begin{cases} y(v.g)=y(v,0)+y(0,g)-y(0,0)\\ y(0,g)=y(0,0)+y(0,g)-y(0,0)\end{cases}$$

with $y(v,0)$ and $y(0,g)$ are a solution of (3.13) when $g=0$ and $v=0$ resp.
By the definition of $J(v,g)$ one obtain

$$\begin{aligned} J(v,g) &= \|\mathcal{C}(y(v,0)+y(0,g)-y(0,0))-y_d\|_{\mathcal{Z}}^2+N\,\|v\|_{\mathcal{U}}^2\\ &= J(v,0)+\|\mathcal{C}(y(0,g)-y(0,0)\|_{\mathcal{Z}}^2+2(\mathcal{C}y(v,0)-y_d,\mathcal{C}(y(0,g)-y(0,0)))_{\mathcal{Z}},\end{aligned}$$

and

$$\begin{aligned} J(0,g) &= \|\mathcal{C}(y(0,0)+y(0,g)-y(0,0))-y_d\|_{\mathcal{Z}}^2\\ &= J(0,0)+\|\mathcal{C}(y(0,g)-y(0,0))\|_{\mathcal{Z}}^2+2(\mathcal{C}(y(0,0)-y_d,\mathcal{C}(y(0,g)-y(0,0)))_{\mathcal{Z}},\end{aligned}$$

then

$$J(v,g)-J(0,g)=J(v,0)-J(0,0)+2(\mathcal{C}^{*}\mathcal{C}(y(v,0)-y(0,0)),y(0,g)-y(0,0))_{\mathcal{Y}}. \tag{3.20}$$

Introduce an adjoint state $\xi(v)$ given by $\mathcal{A}^{*}\xi(v)=\mathcal{C}^{*}\mathcal{C}(y(v,0)-y(0,0))$ to write

$$\begin{aligned} J(v,g)-J(0,g) &= J(v,0)-J(0,0)+2(\mathcal{A}^{*}\xi(v),y(0,g)-y(0,0))_{\mathcal{Y}}\\ &= J(v,0)-J(0,0)+2(\xi(v),\mathcal{A}(y(0,g)-y(0,0)))_{\mathcal{Y}}\\ &= J(v,0)-J(0,0)+2(\xi(v),\beta g)_{\mathcal{Y}}=J(v,0)-J(0,0)+2(\beta^{*}\xi(v),g)_{G}\\ &= J(v,0)-J(0,0)+2(S(v),g)_{G',G}\text{ where }S(v)=\beta^{*}\xi(v).\end{aligned}$$

the last equation leads to (3.19)

Remark

1.By(3.19) you can see that condition (3.18) holds iff $v \in k$,where

$$K = \{v \in \mathcal{U}_{ad},\ (S(v), g) = 0\ \forall g \in G\}\,.$$

is a closed subspace of $\mathcal{U}$., Then ,u is a no-regret control iff $u \in k$.

2.The notion of no-regret control could be generalized to no-regret control related to any a fixed control $u_0 \in U_{ad}$, i.e , we want controls v s.t

$$J(v, g) \leq J(u_{0,}g) \quad \forall \ \ g \in G$$

Definition 3.2 *we say that $u \in U_{ad}$ is a no-regret control related to $u \in U_{ad}$ for (3.13) (3.14) if u solve .*

$$\inf \sup(J(v, g) - J(u_{0,}g) \tag{3.21}$$

Unfortunately , the main difficulty with no-regret control arises when we want to characterize the set k,for this reason we shall approximate the no-regret control by a sequence of controls called low regret controls

■

3.5 The low-regret control

One throught to relax (3.17) by making some quadratic perturbation on $J(0, g)$ (Lions 1992), in other words, we search controls v such that

$$J(v, g) \leq J(0, g) + \gamma \left\|g\right\|_G^2,\ \gamma > 0, g \in G$$

Definition 3.3 *(Lions, 1992) We say that $u_\gamma \in U_{ad}$ is a low -regret control for (3.13) (3.14) if u solves*

$$\inf_{v \in \mathcal{U}_{ad}} \sup_{g \in G}(J(v, g) - J(0, g) - \gamma \left\|g\right\|_G^2),\ \gamma > 0 \tag{3.22}$$

Take () into account to get the equivalence between () and

$$\inf_{v \in \mathcal{U}}{}' (J(v, 0) - J(0, 0) + \sup_{g \in G}(2(S(v), g)_G - \gamma \left\|g\right\|_G^2)),$$

e Legendre transform (Aubin, 1984 : p49)for

$$\sup_{g\in G}(2(S(v),g)_G - \gamma\left\|g\right\|_G^2) = \frac{1}{\gamma}\left\|S(v)\right\|_G^2.$$

then

$$\inf_{v\in\mathcal{U}_{ad}} J^\gamma(v) \tag{3.23}$$

where

$$J^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}\left\|S(v)\right\|_G^2. \tag{3.24}$$

Now, we can define the low-regret by

Definition 3.4 *[15]We say that $u_\gamma \in \mathcal{U}_{ad}$ is a low-regret control for (3.13) and (3.14) if u solves*

$$\inf_{v\in\mathcal{U}_{ad}}\sup_{g\in G}(J(v,g) - J(0,g) - \gamma\left\|g\right\|_G^2, \ \gamma > 0).$$

Theorem 3.2 ***:(Low-regret control : existence and uniqueness)***

The problem (3.13) and (3.18) with (3.19) has a unique solution u_γ.

. 1.**Existence**

We have that :

$$J^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma}\left\|S(v)\right\|_G^2 \qquad \forall v \in \mathcal{U}_{ad},$$

$$J^\gamma(v) \geq -J(0,0) = \text{constant},$$

i.e, $\inf\limits_{v\in\mathcal{U}_{ad}} J^\gamma(v)$ exists.

Let be $d_\gamma = \inf\limits_{v\in\mathcal{U}_{ad}} J^\gamma(v)$ .Let a minimizing sequence (v_n^γ) verifying

$$\lim_{n\to\infty} J^\gamma(v_n^\gamma) = \inf_{v\in\mathcal{U}_{ad}} J^\gamma(v) = d_\gamma,$$

we have that :

$$-J(0,0) \leq J^\gamma(v_n^\gamma) = J(v_n^\gamma,0) - J(0,0) + \frac{1}{\gamma}\left\|\beta^*\zeta(v_n^\gamma)\right\|_G^2 \leq d_\gamma + 1,$$

which implies that

$$\left\|Cy(v_n^\gamma,0) - y_d\right\|_{\mathcal{Z}}^2 + N\left\|v_n^\gamma\right\|_{\mathcal{U}}^2 + \frac{1}{\gamma}\left\|\beta^*\zeta(v_n^\gamma)\right\|_G^2 \leq d_\gamma + J(0,0) + 1 = C_\gamma.$$

we deduce that

$$\begin{aligned} \left\|v_n^\gamma\right\|_{\mathcal{U}} &\leq C_\gamma, && \text{(2.8.a)}\\ \left\|Cy(v_n^\gamma,0) - y_d\right\|_{\mathcal{Z}} &\leq C_\gamma, \text{ implies } \left\|Cy(v_n^\gamma,0)\right\|_{\mathcal{Z}} \leq C_\gamma, && \text{(2.8.b)}\\ \left\|\beta^*\zeta(v_n^\gamma)\right\|_G &\leq C_\gamma\sqrt{\gamma}, && \text{(2.8.c)} \end{aligned}$$

C_γ is a constant independent of n.
(v_n^γ) is bounded in compact space $\mathcal{U}_{ad}$ then we can extracting a subsequence still denoting by (v_n^γ) converges weakly to u_γ in $\mathcal{U}_{ad}$, due to isomorphism of $\mathcal{A}$ we deduce that $y\left(v_n^\gamma, 0\right)$ converge weakly to $y\left(u_\gamma, 0\right)$ in $\mathcal{Y}$.
The cost function $J^\gamma\left(v\right)$ is a lower semi continuous

$$J^\gamma\left(u_\gamma\right) \leq \lim_{n\to\infty} \inf_{v\in\mathcal{U}_{ad}} J^\gamma\left(v_n^\gamma\right) = \inf_{v\in\mathcal{U}_{ad}} J^\gamma\left(v\right) = d_\gamma,$$

$$J^\gamma\left(u_\gamma\right) = \inf_{v\in\mathcal{U}_{ad}} J^\gamma\left(v\right).$$

2.Uniqueness

Suppose that the problem (3.18) admits two distinct solutions u_γ^1, u_γ^2. We set $u_\gamma = \frac{u_\gamma^1+u_\gamma^2}{2}$, due to strict convexity of J we get

$$J^\gamma\left(u_\gamma\right) < \frac{1}{2}J^\gamma\left(u_\gamma^1\right) + \frac{1}{2}J^\gamma\left(u_\gamma^2\right) = d_\gamma,$$

we obtain a contradiction with the assumption that u_γ^1, u_γ^2 are two solutions of (3.18). Thus (3.18) admits a unique solution. ■

Theorem 3.3 *The unique low-regret control $u_\gamma$is converge weakly when $\gamma \to 0$ to the unique no-regret control u in $\mathcal{U}_{ad}$.*

. Let u_γ be a low-regret control in $\mathcal{U}_{ad}$ then for all $v \in \mathcal{U}_{ad}$

$$J^\gamma\left(u_\gamma\right) \leq J^\gamma\left(v\right),$$

$$J\left(u_\gamma, 0\right) - J\left(0,0\right) + \frac{1}{\gamma}\left\|\beta^*\zeta\left(u_\gamma\right)\right\|_G^2 \leq J\left(v, 0\right) - J\left(0,0\right) + \frac{1}{\gamma}\left\|\beta^*\zeta\left(v\right)\right\|_G^2 \quad \forall v \in \mathcal{U}_{ad},$$

by implies

$$J\left(u_\gamma, 0\right) + \frac{1}{\gamma}\left\|\beta^*\zeta\left(u_\gamma\right)\right\|_G^2 \leq J\left(v, 0\right) + \frac{1}{\gamma}\left\|\beta^*\zeta\left(v\right)\right\|_G^2 \quad \forall v \in \mathcal{U}_{ad},$$

we choose $v = 0$ to find :

$$J\left(u_\gamma, 0\right) + \frac{1}{\gamma}\left\|\beta^*\zeta\left(u_\gamma\right)\right\|_G^2 = \left\|\mathcal{C}y\left(u_\gamma, 0\right) - y_d\right\|_{\mathcal{Z}}^2 + N\left\|u_\gamma\right\|_{\mathcal{U}}^2 + \frac{1}{\gamma}\left\|\beta^*\zeta\left(u_\gamma\right)\right\|_G^2 \leq J\left(0,0\right) = constant,$$

then

$$\left\|u_\gamma\right\|_{\mathcal{U}} \leq C, \tag{3.25}$$

$$\left\|\mathcal{C}y\left(u_\gamma, 0\right)\right\|_{\mathcal{Z}} \leq C, \tag{3.26}$$

$$\left\|\beta^*\zeta\left(u_\gamma\right)\right\|_G \leq \sqrt{\gamma}C, \tag{3.27}$$

where C is a constant independent of γ.
(u_γ) is bounded in $\mathcal{U}_{ad}$ then we can extract a subsequence still be denoting (u_γ) converges weakly to $u \in \mathcal{U}_{ad}$.
It's clear that for every $v \in \mathcal{U}_{ad}$

$$J(v,g) - J(0,g) - \gamma \|g\|_G^2 \leq J(v,g) - J(0,g) \qquad \forall g \in G,$$

i.e,

$$J(v,g) - J(0,g) - \gamma \|g\|_G^2 \leq \sup_{g\in G} (J(v,g) - J(0,g)) \, \forall g \in G,$$

from another side we have

$$J(u_\gamma,g) - J(0,g) - \gamma \|g\|_G^2 \leq J(v,g) - J(0,g) - \gamma \|g\|_G^2,$$

so

$$J(u_\gamma,g) - J(0,g) - \gamma \|g\|_G^2 \leq \sup_{g\in G} (J(v,g) - J(0,g)) \qquad \forall g \in G,$$

when γ tend to 0 we obtain :

$$J(u,g) - J(0,g) \leq \sup_{g\in G} (J(v,g) - J(0,g)) \qquad \forall g \in G,$$

which means that

$$\sup_{g\in G} (J(u,g) - J(0,g)) = \inf_{v\in\mathcal{U}_{ad}} \left\{ \sup_{g\in G} (J(v,g) - J(0,g)) \right\}.$$

In conclusion, u is a no-regret control. ■

Characterization of the low-regret control

By a first order optimality conditionon we have

$$J^{\gamma\prime}(u_\gamma)(v - u_\gamma) \geq 0 \;\; \forall v \in \mathcal{U}_{ad},$$

where

$$J^{\gamma\prime}(u_\gamma)(v - u_\gamma) = \lim_{h\to 0} \frac{J(u_\gamma + h(v - u_\gamma)) - J(u_\gamma)}{h} \;\; \forall v \in \mathcal{U}_{ad},$$

we have

$$\begin{aligned}
\frac{J(u_\gamma + t(v - u_\gamma)) - J^\gamma(u_\gamma)}{h} \quad = \quad & h \|\mathcal{C}y(v - u_\gamma, 0)\|_{\mathcal{Z}}^2 + hN \|v - u_\gamma\|_{\mathcal{U}}^2 + \frac{h}{\gamma} \|S(v - u_\gamma)\|_G^2 \\
& + 2(\mathcal{C}y(u_\gamma, 0) - y_d, \mathcal{C}y(v - u_\gamma, 0))_{\mathcal{Z}} + 2N(u_\gamma, v - u_\gamma)_{\mathcal{U}} \\
& + \frac{2}{\gamma}(S(u_\gamma), S(v - u_\gamma))_G,
\end{aligned}$$

when $h \to 0$ we find

$$J^{\gamma\prime}(u_\gamma)(v-u_\gamma) = 2(\mathcal{C}y(u_\gamma,0)-y_d, \mathcal{C}y(v-u_\gamma,0))_{\mathcal{Z}} + 2N(u_\gamma, v-u_\gamma)_{\mathcal{U}} + \frac{2}{\gamma}(S(u_\gamma), S\,(v-u_\gamma))_G.$$

By linearityof the operator C in $\mathbb{Z}$ we ge

$$\begin{aligned} J^{\gamma\prime}(u_\gamma)(v-u_\gamma) &= 2(\mathcal{C}y(u_\gamma,0)-y_d, \mathcal{C}y(v,0)-\mathcal{C}y\,(u_\gamma,0))_{\mathcal{Z}} + 2N(u_\gamma, v-u_\gamma)_{\mathcal{U}} + \frac{2}{\gamma}(S(u_\gamma), S\,(v-u_\gamma))_G \\ &= 2(\mathcal{C}^*\left(\mathcal{C}y(u_\gamma,0)-y_d\right), y(v,0)-y\,(u_\gamma,0))_{\mathcal{Y}} + 2N(u_\gamma, v-u_\gamma)_{\mathcal{U}} + \frac{2}{\gamma}(S(u_\gamma), S\,(v-u_\gamma))_G, \end{aligned}$$

$$y(v,0)-y(u_\gamma,0) = y\,(v-u_\gamma,0) - y\,(0,0)\,,$$

then

$$\begin{aligned} J^{\gamma\prime}(u_\gamma)(v-u_\gamma) &= 2(\mathcal{C}^*\left(\mathcal{C}y(u_\gamma,0)-y_d\right), y(v-u_\gamma,0)-y\,(0,0))_{\mathcal{Y}} \\ &\quad + 2N(u_\gamma, v-u_\gamma)_{\mathcal{U}} + \frac{2}{\gamma}(S(u_\gamma), S\,(v-u_\gamma))_G. \end{aligned}$$

The adjoint state $\mathcal{A}^*\xi(u_\gamma) = \mathcal{C}^*\mathcal{C}(y(u_\gamma,0)-y(0,0))$, then

$$(S(u_\gamma), S\,(v-u_\gamma))_G = (\beta^*\xi(u_\gamma), \beta^*\xi(v-u_\gamma))_G = (\beta\beta^*\xi(u_\gamma), \xi(v-u_\gamma))_{\mathcal{Y}}.$$

Introduce the state $\rho_\gamma = \rho(u_\gamma)$ by

$$\mathcal{A}\rho_\gamma = \frac{1}{\gamma}\beta\beta^*\xi(u_\gamma),$$

this leads to the following equality

$$\begin{aligned} (\mathcal{A}\rho_\gamma, \xi(v-u_\gamma))_{\mathcal{Y}} &= (\rho_\gamma, \mathcal{A}^*\xi(v-u_\gamma))_{\mathcal{Y}} = (\rho_\gamma, \mathcal{C}^*\mathcal{C}(y(v-u_\gamma,0)-y(0,0)))_{\mathcal{Y}} \\ &= (\mathcal{C}^*\mathcal{C}\rho_\gamma, y(v-u_\gamma,0)-y(0,0))_{\mathcal{Y}}, \end{aligned}$$

introducing the new adjoint state $p_\gamma = p(u_\gamma)$ by

$$\mathcal{A}^*p_\gamma = \mathcal{C}^*(\mathcal{C}y_\gamma - y_d) + \mathcal{C}^*\mathcal{C}\rho_\gamma,$$

to find

$$\begin{aligned}(\mathcal{A}^*p_\gamma, y\,(v-u_\gamma,0)-y(0,0))_{\mathcal{Y}} &= (p_\gamma, \mathcal{A}(y\,(v-u_\gamma,0)-y(0,0)))_{\mathcal{Y}} \\ &= (p_\gamma, \mathcal{B}\,(v-u_\gamma))_{\mathcal{Y}} \\ &= (\mathcal{B}^*p_\gamma, v-u_\gamma)_{\mathcal{U}},\end{aligned}$$

Hence, the optimality condition is given by

$$J^{\gamma'}(u_\gamma)(v-u_\gamma) = (\mathcal{B}^*p_\gamma + Nu_\gamma, v-u_\gamma)_{\mathcal{U}} \geq 0 \;\; \forall v \in \mathcal{U}_{ad},$$

Finally, the low-regret control is characterized by the following optimality system :

$$\left\{\begin{array}{c} \mathcal{A}y_\gamma = f + Bu_\gamma, \\ \mathcal{A}^*\xi_\gamma = \mathcal{C}^*\mathcal{C}(y_\gamma - y(0,0)), \\ \mathcal{A}\rho_\gamma = \frac{1}{\gamma}\beta\beta^*\xi_\gamma, \\ \mathcal{A}^*p_\gamma = \mathcal{C}^*(\mathcal{C}y_\gamma - y_d) + \mathcal{C}^*\mathcal{C}\rho_\gamma, \\ (\mathcal{B}^*p_\gamma + Nu_\gamma, v-u_\gamma)_{\mathcal{U}} \geq 0 \;\; \forall v \in \mathcal{U}_{ad}. \end{array}\right. \tag{3.28}$$

where $y\,(u_\gamma, 0) = y_\gamma$, $\xi(u_\gamma) = \xi_\gamma$.

3.5.1 Characterization of the no-regret control

To get the optimality system of no-regret control we pass to limit when $\gamma \to 0$ in the system (3.10)

$$\left\{\begin{array}{c} \mathcal{A}y = f + Bu, \\ \mathcal{A}^*\zeta = \mathcal{C}^*\mathcal{C}y\,(u,0) - y_d, \\ \mathcal{A}\rho = \beta\lambda, \lambda \in G, \\ \mathcal{A}^*p = \mathcal{C}^*\left(\mathcal{C}y\,(u,0) - y_d\right) + \mathcal{C}^*\mathcal{C}\rho, \\ (\mathcal{B}^*p + Nu, v-u)_{\mathcal{U}} \geq 0 \;\; \forall v \in \mathcal{U}_{ad}. \end{array}\right. \tag{3.29}$$

where $y\,(u,0) = y$, $\xi(u) = \xi$.

3.6 Example

1-optimal control of an elliptic distributed system with missing Newmann boundary condition :

Let Ω be an open bounded set of $\mathbb{R}^n$ with smooth boundary Γ. Consider the following elliptic equation :

$$\begin{cases} -\Delta y + y = f + v & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} = g & \text{on } \Gamma, \end{cases} \tag{3.30}$$

where $v \in L^2(\Omega)$, $g \in G = L^2(\Gamma)$, $f \in L^2(\Omega)$ and $y(v, g) \in H^{\frac{3}{2}}(\Omega) \subset L^2(\Omega)$ is the unique solution of this system depend on v and g. Associate to (3.30) the following cost function :

$$J(v, g) = |y(v, g) - y_d|^2_{L^2(\Gamma)} + N \|v\|^2_{L^2(\Omega)} \, \forall g \in G, \tag{3.31}$$

where $y_d \in L^2(\Gamma)$, $N > 0$ and $|.|_{L^2(\Gamma)}$ denote the semi norm in $L^2(\Gamma)$.
Here, we have that : $\mathcal{Y} = L^2(\Omega)$ is the state space, $\mathcal{U} = L^2(\Omega)$ is the control space, $\mathcal{Z} = L^2(\Gamma)$ is the observation space, $G = L^2(\Gamma)$ is the uncertainties space, the observation operator $\mathcal{C}$:

$$\begin{aligned} \mathcal{C} : \ L^2(\Omega) &\longrightarrow L^2(\Gamma) \\ y &\longrightarrow y|_{\Gamma} \end{aligned}$$

β : is the uncertainties operator

$$\begin{aligned} \beta : \ L^2(\Gamma) &\longrightarrow L^2(\Omega) \\ g &\longrightarrow y(0, g) \end{aligned}$$

where $y(0, g)$ is solution of (3.30) when $v = 0$.

Definition 3.5 *We say that u is a no-regret control for (3.30) and (3.31) iff u is solution of :*

$$\inf_{v \in L^2(\Omega)} \left(\sup_{g \in L^2(\Gamma)} J(v, g) - J(0, g) \right).$$

We need the characterization of a no-regret control. Therefore, for all $v \in L^2(\Omega)$ and $g \in L^2(\Gamma)$, we have :

$$J(v, g) = J(v, 0) + |y(0, g) - y(0, 0)|^2_{L^2(\Gamma)} + 2(y(v, 0) - y_d, y(0, g) - y(0, 0))_{L^2(\Gamma)},$$

and

$$J(0, g) = J(0, 0) + |y(0, g) - y(0, 0)|^2_{L^2(\Gamma)} + 2(y(0, 0) - y_d, y(0, g) - y(0, 0))_{L^2(\Gamma)},$$

so :

$$J(v, g) - J(0, g) = J(v, 0) - J(0, 0) + 2(y(v, 0) - y(0, 0), y(0, g) - y(0, 0))_{L^2(\Gamma)}.$$

Let's introduce the adjoint state $\xi = \xi(u)$

$$\begin{cases} -\Delta\zeta + \zeta = 0 & \text{in } \Omega, \\ \frac{\partial \zeta}{\partial \nu} = y(u,0) - y(0,0) & \text{on } \Gamma, \end{cases}$$

using the second Green formula(see appendix Theorem 4.3), we obtained :

$$\begin{aligned} (-\Delta\zeta + \zeta, y(0,g) - y(0,0))_{L^2(\Omega)} &= \int_\Omega (-\Delta\zeta + \zeta)(y(0,g) - y(0,0))\, dx \\ &= \int_\Gamma \zeta g d\Gamma. \end{aligned}$$

So :

$$J(v,g) - J(0,g) = J(v,0) - J(0,0) + 2(\zeta, g)_{L^2(\Gamma)}.$$

Let's define the low-regret control

Definition 3.6 *We say that u_γ is a low-regret control for* (3.30) *and* (3.31) *iff u_γ is solution of :*

$$\inf_{v \in L^2(\Omega)} \left(\sup_{g \in L^2(\Gamma)} J(v,g) - J(0,g) - \gamma \|g\|^2_{L^2(\Gamma)} \right).$$

We have

$$\begin{aligned} \sup_{g \in L^2(\Gamma)} \left(J(v,g) - J(0,g) - \gamma \|g\|^2_{L^2(\Gamma)} \right) &= J(v,0) - J(0,0) + \sup_{v \in L^2(\Gamma)} \left(2(\zeta, g)_{L^2(\Gamma)} - \gamma \|g\|^2_{L^2(\Gamma)} \right) \\ &= J(v,0) - J(0,0) + \frac{1}{\gamma} \|\zeta(v)\|^2_{L^2(\Gamma)}. \end{aligned}$$

Hence, we define the new following cost function related to the problem of low regret given by

$$J^\gamma(v) = J(v,0) - J(0,0) + \frac{1}{\gamma} \|\zeta(v)\|^2_{L^2(\Gamma)}. \tag{3.32}$$

Then our problem optimal becomes

$$\begin{cases} \text{find } u_\gamma \in \mathcal{U} \text{ such that} \\ J^\gamma(u_\gamma) = \inf_{v \in \mathcal{U}} J^\gamma(v). \end{cases} \tag{3.33}$$

Theorem 3.4 ***(The existence and uniqueness of a low-regret control)***

The problem (3.30) and (3.32) with(3.33) has a unique solution u_γ.

. The cost function $J^\gamma(v)$ is coercive and strictly convex in $L^2(\Omega)$ which implies the existence and uniqueness of u_γ. ■

3.6.1 Characterization of the low-regret control :

We have by a first order optimality conditionon

$$J^{\gamma'}(u_\gamma)\left(v-u_{_\gamma}\right)=0 \qquad \forall v\in L^2(\Omega)\,,$$

i.e,

$$\begin{aligned} J^{\gamma'}(u_\gamma)\left(v-u_{_\gamma}\right) &= 2(y\,(u_\gamma,0)-y_d,y\,(v-u_\gamma,0)-y\,(0,0))_{L^2(\Gamma)}+2N(u_{_\gamma},v-u_\gamma)_{L^2(\Omega)} \\ &\quad +\frac{2}{\gamma}(\zeta\left(u_{_\gamma}\right),\zeta\left(v-u_{_\gamma}\right))_{L^2(\Gamma)}=0 \quad \forall v\in L^2(\Omega)\,, \end{aligned}$$

Introduce the state $\rho_\gamma=\rho(u_\gamma)$ by

$$\begin{cases} -\Delta\rho_\gamma+\rho_\gamma=0 & \text{in } \Omega, \\ \frac{\partial\rho_\gamma}{\partial\nu}=\frac{1}{\gamma}\zeta\,(u_\gamma) & \text{on } \Gamma, \end{cases}$$

multiply the first equation of the last one by $\zeta\,(v-u_\gamma)$ and apply second Green formula (see appendix Theorem 4.3) to get :

$$\begin{aligned} \left(-\Delta\rho_\gamma+\rho_\gamma,\zeta\,(v-u_\gamma)\right)_{L^2(\Omega)} &= \int_\Omega \left(-\Delta\rho_\gamma+\rho_\gamma\right)\zeta\,(v-u_\gamma)\,dx \\ &= \int_\Gamma \left(\rho_\gamma\,(y\,(v,0)-y\,(0,0))-\zeta(v-u_\gamma)\frac{\partial\rho_\gamma}{\partial\nu}\right)d\Gamma, \end{aligned}$$

Hence,

$$\begin{aligned} J^{\gamma'}(u_\gamma)\left(v-u_{_\gamma}\right) &= (\rho_\gamma+y\,(u_\gamma,0)-y_d,y\,(v-u_\gamma,0)-y\,(0,0))_{L^2(\Gamma)}+N(u_{_\gamma},v-u_\gamma)_{L^2(\Omega)} \\ &= 0\;\forall v\in\mathcal{U}. \end{aligned}$$

We introduce another adjoint state $p_\gamma=p(u_\gamma)$ given by:

$$\begin{cases} -\Delta p_\gamma+p_\gamma=0 & \text{in } \Omega, \\ \frac{\partial p_\gamma}{\partial\nu}=\rho_\gamma+y\,(u_\gamma,0)-y_d & \text{on } \Gamma. \end{cases}$$

Again, we have

$$
\begin{aligned}
\Theta &= \left(-\Delta p_\gamma + p_\gamma, y\left(v-u_\gamma,0\right) - y\left(0,0\right)\right)_{L^2(\Omega)} \\
&= \int\limits_\Omega \left(-\Delta p_\gamma + p_\gamma\right)\left(y\left(v-u_\gamma,0\right) - y\left(0,0\right)\right) dx \\
&= \int\limits_\Omega p_\gamma(\left(-\Delta + I\right)\left(y\left(v-u_\gamma,0\right) - y\left(0,0\right)\right))dx \\
&\quad + \int\limits_\Gamma (p_\gamma \frac{\partial\left(y\left(v-u_\gamma,0\right) - y\left(0,0\right)\right)}{\partial\nu} \\
&\quad - \left(y\left(v-u_\gamma,0\right) - y\left(0,0\right)\right)\frac{\partial p_\gamma}{\partial\nu})d\Gamma,
\end{aligned}
$$

Then

$$
J^{\gamma'}\left(u_\gamma\right)\left(v-u_{_\gamma}\right) = (p_{_\gamma} + Nu_{_\gamma}, v-u_\gamma)_{L^2(\Omega)} = 0,\ \forall v \in L^2\left(\Omega\right).
$$

we can say

$$
\begin{aligned}
p_{_\gamma} + Nu_{_\gamma} &= 0 \text{ in } L^2\left(\Omega\right), \\
p_{_\gamma} + Nu_{_\gamma} &= 0 \text{ a.e in } \Omega.
\end{aligned}
$$

Then the low-regret control is characterized by the following optimality system :

$$
\begin{cases}
-\Delta y\left(u_\gamma,0\right) + y\left(u_\gamma,0\right) = f + u_\gamma & \text{in } \Omega, \\
\frac{\partial y(u_\gamma,0)}{\partial\nu} = 0 & \text{on } \Gamma, \\
-\Delta\zeta\left(u_\gamma\right) + \zeta\left(u_\gamma\right) = 0 & \text{in } \Omega, \\
\frac{\partial\zeta_\gamma}{\partial\nu} = y\left(u_\gamma,0\right) - y\left(0,0\right) & \text{on } \Gamma, \\
-\Delta\rho_\gamma\left(u_\gamma\right) + \rho_\gamma\left(u_\gamma\right) = 0 & \text{in } \Omega, \\
\frac{\partial\rho_\gamma}{\partial\nu} = \frac{1}{\gamma}\zeta\left(u_\gamma\right) & \text{on } \Gamma, \\
-\Delta p_\gamma\left(u_\gamma\right) + p_\gamma\left(u_\gamma\right) = 0 & \text{in } \Omega, \\
\frac{\partial p_\gamma}{\partial\nu} = \rho_\gamma + y\left(u_\gamma,0\right) - y_d & \text{on } \Gamma, \\
p_{_\gamma} + Nu_{_\gamma} = 0 & \text{a.e in } \Omega.
\end{cases}
$$

To get a no-regret control characterization we pass to limit when $\gamma \to 0$ in the last system we obtain :

$$\begin{cases} -\Delta y\left(u,0\right)+y\left(u,0\right)=f+u & \text{in } \Omega, \\ \frac{\partial y(u,0)}{\partial \nu}=0 & \text{on } \Gamma, \\ -\Delta \zeta\left(u\right)+\zeta\left(u\right)=0 & \text{in } \Omega, \\ \frac{\partial \zeta}{\partial \nu}=y\left(u,0\right)-y\left(0,0\right) & \text{on } \Gamma, \\ -\Delta \rho\left(u\right)+\rho\left(u\right)=0 & \text{in } \Omega, \\ \frac{\partial \rho}{\partial \nu}=\lambda & \text{on } \Gamma, \\ -\Delta p\left(u\right)+p\left(u\right)=0 & \text{in } \Omega, \\ \frac{\partial p}{\partial \nu}=\rho+y\left(u,0\right)-y_d & \text{on } \Gamma, \\ p+Nu=0 & \text{a.e in } \Omega. \end{cases}$$

with the following limits

$$\begin{aligned} \lim_{\gamma\to 0} u_\gamma &= u,\ \lim_{\gamma\to 0} y\left(u_\gamma,0\right)=y\left(u,0\right),\ \lim_{\gamma\to 0}\xi_\gamma=\xi, \\ \lim_{\gamma\to 0}\rho\left(u_\gamma\right) &= \rho\left(u\right),\ \lim_{\gamma\to 0}\frac{1}{\gamma}\zeta\left(u_\gamma\right)=\lambda\in G,\ \lim_{\gamma\to 0}p\left(u_\gamma\right)=p\left(u\right). \end{aligned}$$

3.6.2 The comparison between the controls calculated through HUM and the low regrets method

After the comprehensive and in- depth study of the two method, we can draw the following compariso

The no-regrete control

Avantages	The inconviniences
If U_{ad} d'interior empty you can use this methode	not constractive

The no-regret control is used for the singulars systems

The HUH methode

Avantages	The inconviniences
1/The methode is constructive if $U_{ad}=H$	1/ If U_{ad} d'interior empty you cannot use this methode
	2/ If U_{ad} d'interior non- empty (slater) theon-methode is not constructive

In another way

The advantages of the HUM method :

If $U_{ad}=H$ the corole is identifiable with the conjoint state of systems for systems satisfying the MISOHATA hypotheses

The inconviniences :

if U_{ad} empty this method does not work

if U_{ad} is not empty (slatar) the method gives a duality between the control and the conjoint state of systems

The advantages of the low- regret method :

The advantage of the least regret method ensures control existence even in the empty U_{ad} case and it gives characterization equations for singular systems

The inconviniences :

Is not constrictive

Conclusion

Generaly,we conclude that the HUM méthode is used for the regularts systemes,and the no-regret methode is used for the singulars sysrems

When $U_{ad} = H$ or d'intérior non-empty(slater) we can use the HUM méthode

Bibliographie

[1] Apolaya, R. F. (1994). Exact consrollability for temporlally wave equation. Portugaliae Mathematica, 51(4), 475-488.

[2] Ayadi, A., & Hafdallah, A. (2016). Averaged optimal control for some evolution problems with missing parameter. Technical Report. DOI :10.13140/RG.2.1.3002.8563.

[3] Brokate,M.(2015). Modern methods of nonlinear optimization. Zentrum Mathematik, TU Munchen.

[4] Jacob, B., & Omrane, A. (2010). Optimal control for age-structured population dynamics of incomplete data. Journal of mathematical analysis and applications, 370(1), 42-48.

[5] Hafdallah, A., & Ayadi, A. (2019). Optimal control of electromagnetic wave displacement with an unknown velocity of propagation. International Journal of Control, 92(11), 2693-2700.

[6] Hafdallah, A., & Ayadi, A. (2018). Optimal control of some hyperbolic equations with missing data (Doctoral dissertation, university of Oum-El-Bouaghi).

[7] Hafdhallah,A. (2019).Introduction à la théorie de controle des systèmes.(gruadate course, university larbi tebessi of tebessa).

[8] Hafdallah, A. (2020). On the optimal control of linear systems depending upon a parameter and with missing data. Nonlinear Studies, 27(2), 457-469.

[9] LE DIPLÔME, D. M. Braik Abdelkader (Doctoral dissertation, Université de Mostaganem).

[10] Lions, J. L. (1971). Optimal control of systems governed by partial differential equations problèmes aux limites.

[11] Lions, J. L. (1992). Contrôle à moindres regrets des systèmes distribués. Comptes rendus de l'Académie des sciences. Série 1, Mathématique, 315(12), 1253-1257.

[12] Lohéac, J., & Zuazua, E. (2017). Averaged controllability of parameter dependent conservative semigroups. Journal of Differential Equations, 262(3), 1540-1574.

[13] Medeiros, L. A., MIRANDA, M. M., & Louredo, A. T. (2013). Introduction exact control theory : Method Hum. EDUEPB, Campina Grande.

[14] Mophou, G., Foko Tiomela, R. G., & Seibou, A. (2018). Optimal control of averaged state of a parabolic equation with missing boundary condition. International Journal of Control, 1-12.

[15] Nakoulima, O., Omrane, A., & Velin, J. (2003). On the pareto control and no-regret control for distributed systems with incomplete data. SIAM journal on control and optimization, 42(4), 1167-1184.

[16] Raymond, J. P. (2013). Optimal control of partial differential equations. Université Paul Sabatier, Internet.

[17] Salsa, S. (2016). Partial differential equations in action : from modelling to theory (Vol. 99). Springer.

[18] Savage, L. J. (1972). The foundations of statistics. Courier Corporation.

[19] Touchette, H. (2005). Legendre-Fenchel transforms in a nutshell. URL http ://www. maths. qmul. ac. uk/∼ht/archive/lfth2. pdf.

[20] Zuazua, E. (2014). Averaged control. Automatica, 50(12),pp 3077-3087.

[21] H.BREZIS,Functional analysis ,sobolev spaces and partial differential equation ,Springer ,2011

[22] Achab Fatma (2020) Optimal control of various linear PDE with incomplete data.

[23] Achouak Bekkai et Lina menasria (2016) .Méthode HUM pour la controlabilité exacte des EDP hyperboliques de type ondes

[24] Kallab Debbih Sandra Mallal (2021) ,Approximate solution for the Heat equation with delay time and missing data.