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Soumia Harkat

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Asymptotic behaviour of solutions to some nonlinear problems

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Thèse *pour l'obtention du diplôme de Doctorat en Mathématiques*
Option: *Mathématiques Appliquées*

Par:

SOUMIA HARKAT

Intitulée:

Comportement asymptotique des solutions de certains problèmes non linéaire

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Abstract:

Applying an asymptotic method, the existence of the minimal solution to some variational elliptic inequalities defined on bounded or unbounded domains is established. As well, the large time behaviour of the solution to some evolution problems on time-dependent domains becoming unbounded in many directions when t tends to infinity is dealt with. The convergence and its rate are also investigated with respect to the growth rate of the domain when $t \to \infty$. The steady state solution and its existence for nonlinear parabolic problems is already investigated when we deal with the variational elliptic inequalities. Since the convergence cannot be expected on the whole domain correctors are built to describe the asymptotic behaviour, of the solution of Heat equation, in the distant regions.

Keywords:

Variational inequalities, parabolic problems, elliptic problems, heat equations, monotone operator, noncylindrical domains, unbounded domains, minimal solution, asymptotic behavior, stability, rate of convergence, weak maximum, correctors.

العنوان: **السلوك المقارب لحلول بعض المسائل الغير خطية**

ملخص:

بتطبيق طريقة مقاربة قمنا بدراسة وجود الحل الأدنى لبعض المتراجحات التناقصية المتغايرة والمعرفة على ميادين محدودة أو غير محدودة. وقمنا أيضا بمعالجة السلوك المقارب لحلول بعض المسائل التطورية المعرفة على ميادين متعلقة بالزمن والتي تصبح غير محدودة حينما يتقدم بها الزمن.

بالإضافة إلى ذلك تم تناول التقارب و معدله مع الأخذ بعين الاعتبار قيمة تزايد ميدان الدراسة لما $t\to\infty$ وللإشارة فان وجود الحلول المستقرة الخاصة بالمسائل التكافئية غير الخطية تم دراستها أثناء معالجة المتراجحات التناقصية المتغايرة. و بما أن التقارب لا يمكن توقع حدوثه على كامل الميدان, تم إنشاء بعض المصححات التي تصف لنا السلوك المقارب لحلول معادلة الحرارة في جوار حدود ميدان التعريف.

الكلمات المفتاحية: المتراجحات المتغايرة, المسائل التكافئية, المسائل التناقصية, معادلة الحرارة, المؤثرات الرتيبة, ميادين متعلقة بالزمن, ميادين غير محدودة, الحل الأدنى, السلوك المقارب، الاستقرار, معدل التقارب, الحل الضعيف, المصححات.

Titre: **Comportement asymptotique des solutions de certains problèmes non linéaire**

Résumé:

En appliquant une méthode asymptotique, l'existence de la solution minimale de certaines inéquations variationnelles elliptiques, définies sur des domaines bornés ou non bornés est établie. En outre, le comportement asymptotique des solutions de problèmes d'évolution est traité, lorsque le temps et certaines directions de la variable spatiale tendent vers l'infini simultanément. La convergence et son taux sont également étudiés en tenant compte du taux de croissance du domaine lorsque $t \to \infty$. L'existence de la solution stationnaire pour les problèmes paraboliques non linéaires est déjà étudiée lorsque nous traitons les inéquations variationnelles elliptiques. Comme la convergence ne peut pas être prévue sur le domaine entier, des correcteurs sont construits pour décrire le comportement asymptotique de la solution de l'équation de la chaleur dans les régions éloignées.

Mots Clés:

Inéquations variationnelles, problèmes paraboliques, problèmes elliptiques, équations de chaleur, opérateur monotone, domaines non-cylindriques, domaines non bornés, solution minimale, comportement asymptotique, stabilité, taux de convergence, maximum faible, correcteurs.

Contents

Introduction

This thesis is mainly devoted to study some topics in the theory of nonlinear variational inequalities and moving boundary problems, where the solution of variational inequalities is the possible steady state solution of the moving boundary problems. The existence of the minimal solution to some nonlinear variational elliptic inequalities defined on bounded or unbounded domains will be dealt with. Also, the asymptotic behaviour of the solution to some linear and nonlinear parabolic problems defined on noncylindrical domains becoming unbounded in many directions when the time t tends to infinity will be investigated. Furthermore, the geometry of the noncylindrical domain can be single out to analyze the rate of convergence. Since the convergence cannot be expected on the whole domain the corrector results, for the Heat equation in the case of cylindrical symmetries, are given.

There are many different methods that have been used to prove the existence and comparison results for nonlinear elliptic boundary value problems, among them the regularization method. This method consists of approximating the posed problem by a family of problems that are constructed according to a certain regularization criteria. In this context, the main part of this work is concerned with the existence of the minimal solution to some variational elliptic inequalities when the strict monotonicity that guarantees the uniqueness of the solution is violated. We will investigate variational inequalities defined on bounded domains where the existence of solutions is already ensured and we will deal with the minimal solution. Variational inequalities defined on unbounded domains, for which both the existence of solutions and the minimal solution have to be dealt with, will be also investigated.

Since the involved operators are assumed to be only monotone or even noncoercive, the comparison between the solutions of such inequalities is not always possible and the weak maximum principle for example might fail. However, by using the regularization technique presented in [7, 15–19, 21–36, 42, 43], we can now perturb the concerned inequalities to construct an approximating family of elliptic nonlinear problems defined on large cylindrical domains with new (strictly) monotone operators. When the size of the cylinders becomes unbounded we get our variational inequalities which also means that the minimal solution will be obtained as a limit of the solutions to the perturbed problems. Noting that, this technique is used in [34–36] to investigate either the existence of solutions of some variational inequalities with linear operator in unbounded domains, or the asymptotic behaviour of variational inequalities with nonhomogeneous Dirichlet boundary condition and pointwise constraints, as the domain becomes unbounded. In the case of unbounded domains we apply the same argument to deduce the existence of nonnegative solutions and their minimal solution.
To simplify the presentation and explain more clearly the main idea, let us illustrate the procedure of this approach in the following example. Consider the variational problem

$$\begin{cases} u \in \mathcal{K}, \\ \displaystyle\int_\alpha^\beta a\left(u'\right)(v-u)'\,dx \geq \int_\alpha^\beta f\,(v-u)\,dx \quad \forall v \in \mathcal{K}, \end{cases} \tag{P_∞}$$

where $\mathcal{K}$ is a closed convex subset of $H_0^1(\alpha,\beta)$, a is a Caratheodory function satisfying suitable monotonicity, coercivity and growth conditions and f is a nonnegative function in $L^2(\alpha,\beta)$. It is well know (see [14,48]) that (P_∞) admits at least one solution, whereas the uniqueness cannot be expected in general (see Example 1.1).
The approach used for studying the existence of minimal nonnegative solution to (P_∞) consists to introduce $(P_\ell)_{\ell>0}$, a family of problems with strictly monotone operators defined on cylinders $\Omega_\ell = (-\ell,\ell)\times(\alpha,\beta)$ becoming unbounded when $\ell\to\infty$, as follows

$$\begin{cases} u_\ell \in \mathcal{K}_\ell, \\ \displaystyle\int_{-\ell}^{\ell}\int_{\alpha}^{\beta} \left(\partial_y u_\ell \partial_y (v-u_\ell) + a(\partial_x u_\ell)\,\partial_x(v-u_\ell)\right) dxdy \geq \int_{-\ell}^{\ell}\int_{\alpha}^{\beta} f\,(v-u_\ell)\,dxdy \quad \forall v\in\mathcal{K}_\ell, \end{cases} \tag{P_ℓ}$$

where

$$\mathcal{K}_\ell = \left\{ v\in H_0^1\left((-\ell,\ell)\times(\alpha,\beta)\right) \mid v(y,.)\in\mathcal{K} \text{ a.e. in } (-\ell,\ell)\right\}$$

is a closed convex subset of $H_0^1\left((-\ell,\ell)\times(\alpha,\beta)\right)$. The basic idea of this approach is to apply the weak maximum principle in order to find the minimal solution of (P_∞). Since the operator $A_\ell = -\partial_y^2 - \partial_x a$ is strictly monotone we can show by comparing the problem (P_ℓ) with (P_∞) that

$$u_\ell(y,x) \leq u(x) \text{ for a.e.}(x,y)\in\Omega_\ell \text{ and for any } u \text{ solution to } (P_\infty).$$

Formally, if we pass to the limit when $\ell\to\infty$, the function limit u_∞ is a solution to (P_∞). Moreover, it follows from above that this limit is also the minimal solution.

The existence of solutions or even extremal solutions of coercive and noncoercive variational inequalities have been investigated by many authors. In [2,10], the sub-supersolution method has been used to prove the existence of solutions and extremal solutions, confined between their sub and supersolutions, for a class of noncoercive variational inequalities involving monotone operators. Various methods, as topological fixed point approaches, bifurcation techniques, recession arguments and variational approaches are adapted to deal with the solvability of noncoercive variational inequalities (see [3,4,55,60,64] and the references therein). In [11,12] the existence of maximal and minimal solutions for some quasi-linear elliptic equations with pseudomonotone operators are proved, by using different methods, under fairly general conditions. Also, comparison results for maximal and minimal solutions are proved in the same papers. More information and details about this type of variational inequalities can be found in [10] and the references therein. Our approach is totally different from the above arguments, it presents a new approximation technique for proving the existence of minimal solution of some variational inequalities. Here we also take into account bounded and unbounded domains and sometimes the hypotheses are related to the construction.

Over the past few years, parabolic equations in noncylindrical domains have been the subject of intense research. Overall, these problems are interesting, not only from the point of view of the general theory of PDE's but also due to various applications in biology, physics and engineering (cf. [37–39, 49, 62] and the references therein). Many important results about the existence, uniqueness, regularity, asymptotic behaviour..., for these problems are investigated in many papers (see [**?**, 5, 8, 13, 37–41, 44, 47, 49, 51, 54, 56–58, 61, 62, 65] and the references therein). The main novelty of the remaining part of this work is its focus on the study of the asymptotic behaviour of some nonhomogeneous parabolic initial-boundary value problems defined on noncylindrical domains becoming unbounded in many directions when $t\to\infty$. That is to say our noncylindrical domains $Q\subset\mathbb{R}^{n+1}$ are defined by

$$Q = \bigcup\nolimits_{0<t<\infty} \{t\}\times\Omega_t, \tag{1}$$

where $\Omega_t \subset \mathbb{R}^n$ is a nondecreasing sequence of bounded domains in the inclusion sense that becomes unbounded in some sense when $t \to \infty$. In fact, this problem is not classical in the sense that some spatial variables are also becoming unbounded when the time t becomes very large. Also, the stability study is considered for nonhomogeneous parabolic problems for which the convergence of their solutions towards a steady state solution is certainly held only far away from the boundary layer. However, one can not expect that this convergence holds on the whole domain Ω_t since the solution and its limit do not belong necessarily to the same space. That is to say if u, the solution of the considered problem, has a limit in $W_0^{1,p}(\Omega_t)$ (i.e. $W_0^{1,p}(\mathbb{R}^m \times \omega)$ where u is extended by 0 outside Ω_t) then this limit belongs to $W_0^{1,p}(\mathbb{R}^m \times \omega)$. Although (as we will see below), the candidate limit is a solution of an elliptic problem defined on $\mathbb{R}^m \times \omega$ where the homogeneous Dirichlet boundary conditions are not necessarily preserved. This is not the case for almost of the existing works (see [13,40,41,51,62]) where the asymptotic behaviour is established for bounded sections Ω_t that converge in some sense to a bounded domain. However, in [47, 58, 65] the size of sections Ω_t can also become unbounded when $t \to \infty$ but for homogenous or "approximately homogenous" (i.e. the applied forces decay in some sense as $t \to \infty$) parabolic equation which is, in general, less challenging since the limit is zero. The convergences in these papers are held on the whole Ω_t (i.e. the solution and its limit are in the same space). In order to recover the convergence on the whole domain Ω_t, we need to construct correctors that will act near the boundary layer. When $t \to \infty$, the correctors with the solution of the limit problem will give a good approximation of u in $W^{1,p}(\Omega_t)$ with a convenient rate of convergence. In this respect, it is worthwhile to note that the steady state problem in [13, 40, 41, 47, 51, 58, 62, 65] is unique, i.e. there is only one problem that describes the unique limit. While, here we have to consider the steady state problem for the limit behaviour far away from the boundary layer and we need to define another different problem to describe the limit on the neighborhood of the boundary layer.

Recently the asymptotic behaviour of the following nonhomogeneous heat equation in noncylindrical domains is investigated in [44]

$$\begin{cases} u' - \Delta u = f & \text{in } Q_t, \\ u = 0 & \text{on } \Gamma_t, \\ u(0,.) = u_0 & \text{on } \Omega_0, \end{cases}$$

where

$$Q_t = \bigcup_{0<s\leq t} \{s\} \times \{(-t_0 - s, t_0 + s) \times (0,1)\} \text{ with } t_0 > 0,$$

$f \in L^2(0,1)$ and $u_0 \in L^2(\Omega_0)$. It is clear that the state variable domain is becoming unbounded when the time goes to infinity. Furthermore, an iteration technique is used in [44] to show that

$$u \to u_\infty \text{ in } H^1(O) \text{ as } t \to \infty, \tag{2}$$

where O is any bounded domain of $\mathbb{R} \times (0,1)$ and u_∞ is the unique solution to

$$\begin{cases} -\partial_{x_2}^2 u_\infty = f & \text{in } (0,1), \\ u_\infty = 0 & \text{on } \{0,1\}. \end{cases} \tag{3}$$

Since the data assumed in [44] are only depending on the bounded space variable, the limit problem (3) is well defined and the convergence in (2) cannot be obtained in $H_0^1((-t_0 - t, t_0 + t) \times (0,1))$ whenever $f \neq 0$ (i.e. $u_\infty \notin H_0^1(\mathbb{R} \times (0,1))$. Nevertheless, our goal is to study the stability of linear parabolic problems defined on a large class of noncylindrical domains where the data are depending on all space variables. Of course here the existence results of the limit problem is not classical as we

will explain in the sequel. Also we are interested here in the study of some corrector results for the heat equation in the cylindrical symmetries case.
Now, since the argument used for the linear problem is strictly linear (it can not be extended to the nonlinear case) the rest of this work is devoted to the study of the asymptotic behavior, for large values of the time, of solutions to quasilinear parabolic equations with monotone operator.

The main body of this thesis consists of three chapters. In the first chapter, we investigate the existence of the minimal solution to some variational elliptic inequalities. We start with the coercive monotone case. Some comparison results between different solutions are established as tools to pass to the limit in the perturbed problems and show the existence of the minimal solution defined on bounded domains. The section 1.2 prepares the way to deal with more general problems involving noncoercive operators in the section 1.3. The section 1.4 is devoted to adapt the tools and the argument used in the bounded case to show the existence of nonnegative solutions and their minimal solutions for a more complicated variational inequalities involving unbounded domains. Both cases, coercive and noncoercive operators, are handled. Comparison results between minimal solutions are also proved under only a monotonicity assumption in all sections.
In the second chapter we investigate the asymptotic behaviour of the solution of linear parabolic problems defined on large class of noncylindrical domains. Since here the data are assumed to be dependent on all space variables, the candidate limit problem is an elliptic problem defined on an unbounded domain. Then, as the existence results of such problems is not classical, section 2.2 is devoted to study the existence of the candidate limit solution by using the idea introduced in [17, 23]. In section 2.3, we investigate the asymptotic behaviour of the considered parabolic problem, when $t \to \infty$, taking into account the fact that the size of the domain may also go to infinity in some directions. Of course the convergence and its rate are depending on the geometry of the noncylindrical domains. Here also, some model problems for which we can describe the behaviour of u by simpler functions are considered. However the immediate extension of the results obtained in [44] are also given in the last subsection. As already mentioned above, the convergence can not be expected on the whole domain Ω_t, the section 2.4 is dedicated to the establishment of corrector results in order to get a complete asymptotic description for the heat equation, in the case of cylindrical symmetries. The rate of convergence is deeply treated taking into account the behaviour of the boundary layer and the convergence time interval.
The last chapter is devoted to the study of large time behaviour of solutions to quasilinear parabolic equations with monotone operator. In section 3.1 we will establish some useful estimate results that come from the weak maximum principle method. Then, we will use the Minty-Browder technique to show under some assumptions on the data that the limit problem is elliptic and defined on a lower dimensional domain. The above argument turn our attention to an interesting result concerning monotone operator. So, Theorem 3.2 ensures that the set of solutions of quasilinear elliptic problem with monotone operator is bounded and attains its infimum. Finally, in the last section, we will assume a new hypotheses and we close this chapter by determining the convergence rate estimate.

In order to make the thesis as self-contained as possible, let us fix some preliminary notations. We start by introducing the general ones.

- $:=$ is the equal by definition.
- $\rightharpoonup$ indicates the weak convergence.
- $\to$ designates the strong convergence.
- $\liminf$ denotes the infimum limit.

- lim sup is the supremum limit.
- $\sim$ means the classical equivalence between the functions.
- $|\alpha|$ and $[\alpha]$ are respectively the absolute value and the integer part of $\alpha \in \mathbb{R}$.
- ∇ stands for the gradient operator.
- Δ is the Laplace operator.
- div is the divergence operator.
- $x \cdot y$ denotes the Euclidean scalar product of $x, y \in \mathbb{R}^n$.
- C denotes a generic positive constant, not necessarily the same at each occurrence.
- p is a real number strictly bigger than 1 and q is its conjugate, i.e. $\frac{1}{p} + \frac{1}{q} = 1$.
- *int* stands for the interior of the domain.
- $\bar{\Omega}$ and $\partial\Omega$ denote respectively the closure and the boundary of domain Ω.
- $\Omega' \subset\subset \Omega$ means that the domain Ω' is strictly included in Ω, i.e. $\overline{\Omega'} \subset \Omega$.
- $mes\,(A)$ means the measure of the set A.
- A^c stands for the complement of a set A in $\mathbb{R}$.
- χ_A denotes the characteristic function of a set A.
- $dist\,(A, B)$ is the distance between two sets A and B.

For real functions f and g, we set

- $f^{\pm} := \max(0, \pm f)$.
- supp$f$ is the support of f.
- $f(x) = O(g(x))$ when $x \to x_0$ $(x_0 \in \bar{\mathbb{R}})$, if there exists a constant $C > 0$ such that $|f(x)| \leq C\,|g(x)|$ for all x in a neighborhood of x_0.
- $f \approx g$, when $x \to x_0$ $(x_0 \in \bar{\mathbb{R}})$ means that $f(x) = O(g(x))$ and $g(x) = O(f(x))$.
- $\mathcal{T}_{h,x_i} f(x) := f(x_1, \ldots, x_i + h, \ldots, x_n)$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $h \in \mathbb{R}$.

Now let us conclude this section by the notation of some function spaces and their tools. Let Ω be an open set in $\mathbb{R}^n$ and X be a real Banach space. Then we denote by

- $|.|_X$ the norm of X,
- X' the dual of X,
- $< ., . >_X$ the duality product between X and X',

- $C^{\infty}(\Omega)$ the space of infinitely differentiable functions in Ω,
- $\mathcal{D}(\Omega)$ the space of $C^{\infty}(\Omega)$ functions with compact support in Ω,
- $L^p(\Omega; X)$ the Banach space of equivalence classes of X-valued Bochner p-integrable functions $f : \Omega \to X$ normed by $|f|_{L^p(\Omega;X)} := \left(\int_\Omega |f|_X^p \, dx\right)^{\frac{1}{p}}$,
- $|f|_{L^\infty(\Omega;X)}$ the Banach space of X-valued measurable, essentially bounded functions with the norm $|f|_{L^\infty(\Omega;X)} := \inf\{C > 0 \mid |f|_X \le C \text{ a.e. on } \Omega\}$,
- $L^p_{loc}(\Omega; X)$ the space of local p-integrable functions on Ω,
- $L^p(\Omega) := L^p(\Omega; \mathbb{R})\,;\ |.|_{L^p(\Omega)} := |.|_{p,\Omega}$,
- $W^{1,p}(\Omega; X)$ the X-valued Sobolev space of order one, i.e. the space of Bochner p-integrable functions with distributional derivatives of first order in $L^p(\Omega; X)$,
- $|f|_{W^{1,p}(\Omega;X)} := \left(\int_\Omega |f|_X^p \, dx + \int_\Omega |\nabla f|_X^p \, dx\right)^{\frac{1}{p}}$ the norm of $W^{1,p}(\Omega; X)$,
- $W^{1,p,q}(\Omega; X, X') := \{u \in L^p(\Omega; X), \nabla u \in (L^q(\Omega; X'))^n\}$,
- $W^{1,p}(\Omega) := W^{1,p}(\Omega; \mathbb{R})\,;\ |.|_{W^{1,p}(\Omega)} := |.|_{1,p,\Omega}$,
- $W^{1,p}_{loc}(\Omega)$ the local Sobolev space,
- $W^{1,p}_0(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$,
- $W^{-1,q}(\Omega) := \left(W^{1,p}_0(\Omega)\right)'$; $|.|_{W^{-1,q}(\Omega)} := |.|_{-1,p,\Omega}$; $< .,. >_{W^{1,p}_0(\Omega)} := < .,. >_\Omega$,
- $H^1(\Omega) := W^{1,2}(\Omega)\,;\ H^1_0(\Omega) := W^{1,2}_0(\Omega)\,;\ H^{-1}(\Omega) := W^{-1,2}(\Omega)$.

Chapter 1

Minimal solution of some variational inequalities

In this chapter, we deal with the existence of the minimal solution to some variational elliptic inequalities defined on bounded or unbounded domains. The minimal solution is obtained as limit of solutions to some classical variational inequalities defined on domains becoming unbounded when some parameter tends to infinity. The considered quasilinear operators are only monotone (not strictly) and noncoercive. The weak maximum principle is used, under only a monotonicity assumption, to prove some comparison results between minimal solutions.

1.1 Variational inequalities in bounded domains

Let us start by describing and defining the setting of the problem. Let Ω be a bounded open set of $\mathbb{R}^n$. We denote by $\mathcal{K}$ a closed convex subset of $W_0^{1,p}(\Omega)$ containing 0 and satisfying

$$\max(u,v),\ \min(u,v) \in \mathcal{K} \quad \forall u,v \in \mathcal{K}. \tag{1.1}$$

Note that we can also write

$$u \vee v := \max(u,v) = u + (v-u)^+ \text{ and } u \wedge v := \min(u,v) = v - (v-u)^+.$$

This type of lattice convex sets usually occur in applications. For example:
- *equations;* $\mathcal{K} = W_0^{1,p}(\Omega)$,
- *obstacle problems;* $\mathcal{K} = \left\{ u \in W_0^{1,p}(\Omega) : u(x) \geq \psi(x),\ \text{for a.e. } x \in \Omega_0 \right\}$, Ω_0 is a subset of Ω and ψ is a given function on Ω_0,
- *elasto-plastic torsion problem;* $\mathcal{K} = \left\{ u \in W_0^{1,p}(\Omega) : |\nabla u(x)| \leq c,\ \text{for a.e. } x \in \Omega_0 \right\}$, $c \geq 0$. (See [1, 10, 14, 63]).

Now, let $a(x,\xi) = (a_i(x,\xi))_{1\leq i\leq n}$ and $a_0(x,\xi)$ be a family of Carathéodory functions defined on $\Omega\times\mathbb{R}^{n+1}$ and satisfying suitable coerciveness, monotonicity and growth conditions, i.e. for all

$\xi = (\xi_i)_i, \xi' = (\xi'_i)_i \in \mathbb{R}^{n+1}$ and for a.e. x in Ω, there exist nonnegative constants α, β such that

$$\sum_{0\leq i\leq n} a_i(x,\xi)\xi_i \geq \alpha \sum_{1\leq i\leq n} |\xi_i|^p, \tag{1.2}$$

$$\sum_{0\leq i\leq n} \left(a_i(x,\xi) - a_i(x,\xi')\right)\left(\xi_i - \xi'_i\right) \geq 0, \tag{1.3}$$

$$(x,\xi) \longmapsto a_i(x,\xi) \text{ is measurable on } \Omega \times \mathbb{R}^{n+1},\ i = 0,\dots,n, \tag{1.4}$$

$$\xi \longmapsto a_i(x,\xi) \text{ is continuous on } \mathbb{R}^{n+1},\ i = 0,\dots,n, \tag{1.5}$$

$$|a_i(x,\xi_0,\xi_1,\dots,\xi_n)| \leq \vartheta(x) + \beta \sum_{0\leq i\leq n} |\xi_i|^{p-1} \text{ with } \vartheta \in L^q(\Omega). \tag{1.6}$$

Then for f in $L^q(\Omega)$, we consider u solution of the following nonlinear variational inequality

$$\begin{cases} u \in \mathcal{K}, \\ \langle Au, v-u\rangle_\Omega \geq \displaystyle\int_\Omega f\,(v-u)\,dx \quad \forall v \in \mathcal{K}, \end{cases} \tag{1.7}$$

where A is a nonlinear operator defined from $W_0^{1,p}(\Omega)$ into its dual by

$$Au\,(x) := -\operatorname{div} a(x,u,\nabla u) + a_0(x,u,\nabla u). \tag{1.8}$$

Here, for simplicity we set $|.|_{1,p,\Omega} := |.|_{1,p}$, $|.|_{p,\Omega} := |.|_p$ and note that the above duality is equivalent to

$$\langle Au, v\rangle_\Omega = \int_\Omega a(x,u,\nabla u)\cdot\nabla v dx + \int_\Omega a_0(x,u,\nabla u) v dx \quad \forall v \in W_0^{1,p}(\Omega).$$

So, the existence of a solution for this problem is a classical result due to [14,48], while the uniqueness cannot be guaranteed in general, as shown in the following example. Moreover the minimal solution, which is established in Theorem 1.1, for the considered example problem is given.

Example 1.1. *For $n = 1$ and $p = 2$. Consider in (1.7) $a_0 = 0$ and let $a : \mathbb{R} \to \mathbb{R}$ be a single-valued function which graph is depicted in the following figure.*

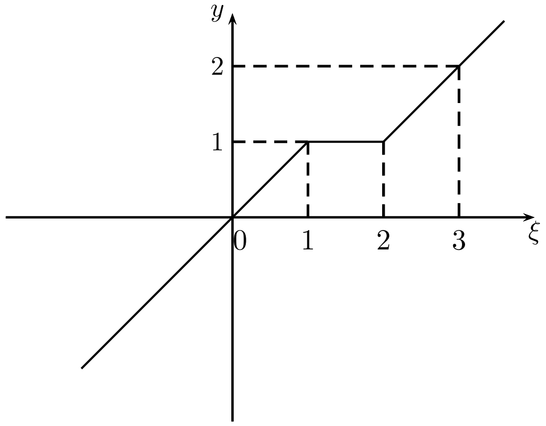

Figure 1.1: The function a.

Clearly

$$|a(\xi)| \leq |\xi|$$

and

$$a(\xi)\cdot\xi = \begin{cases} \xi^2 \text{ if } \xi \leq 1, \\ \xi \geq \frac{\xi^2}{2} \text{ if } 1 \leq \xi \leq 2, \\ (\xi-1)\xi \geq \frac{\xi^2}{2} \text{ if } 2 \leq \xi, \end{cases}$$

so that a *satisfies the above assumptions. Let us choose* $\mathcal{K} = H_0^1(-1,1)$ *and* $f = 2\chi_{(-\frac{1}{2},1)}$ *in (1.7) and consider the following Dirichlet boundary value problem in* $\Omega = (-1,1)$

$$\begin{cases} -(a(u'))' = 2\chi_{(-\frac{1}{2},1)} \quad \text{in } \Omega, \\ u(\pm 1) = 0. \end{cases} \tag{1.9}$$

Then, let us find all solutions to (1.9) in the weak sense, i.e. such that

$$\begin{cases} u \in H_0^1(\Omega), \\ \int_\Omega a(u')\, v' dx = \int_{-\frac{1}{2}}^1 2v dx \quad \forall v \in H_0^1(\Omega). \end{cases}$$

If u *be a solution to (1.9), one has*

$$-(a(u'))' = 0 \quad \text{in } \left(-1,-\frac{1}{2}\right), \quad -(a(u'))' = 2 \quad \text{in } \left(-\frac{1}{2},1\right).$$

This implies

$$a(u') = c_1 \quad \text{in } \left(-1,-\frac{1}{2}\right), \quad a(u') = -2x + c_0 \quad \text{in } \left(-\frac{1}{2},1\right),$$

where c_0 *and* c_1 *are constants. In order to guarantee (1.9) in the distributional sense,* $a(u')$ *has to be continuous i.e.* $1 + c_0 = c_1$*. We claim that* $c_0 = 0$*. Indeed, let us suppose* $c_0 \neq 0$ *and consider the three cases below.*

i. *Suppose first that* $c_0 < 0$*. It follows that* $a(u') < 1$ *a.e. on* $(-1,1)$*. Thus we have*

$$a(u') = u' = 1 + c_0 \quad \text{in } \left(-1,-\frac{1}{2}\right), \quad a(u') = u' = -2x + c_0 \quad \text{in } \left(-\frac{1}{2},1\right).$$

Therefore, due to the zero boundary condition at the point -1*,*

$$u(x) = (1+c_0)(x+1) \quad \text{in } \left(-1,-\frac{1}{2}\right),$$

and for some constant c_2

$$u(x) = -x^2 + c_0 x + c_2 \quad \text{in } \left(-\frac{1}{2},1\right).$$

Due to the zero boundary condition at 1 *we have* $c_2 = 1 - c_0$ *and*

$$u(x) = -x^2 + c_0 x + 1 - c_0 \quad \text{in } \left(-\frac{1}{2},1\right).$$

The continuity of u *at* $-\frac{1}{2}$ *implies that* $c_0 = \frac{1}{8}$ *which contradicts the fact that* $c_0 < 0$*.*

ii. *Suppose now that* $0 < c_0 < 3$. *Then we have*

$$\frac{-1}{2} < \frac{c_0 - 1}{2} < 1,\ a\left(u'\right) > 1 \text{ for } x \in \left(-1, \frac{c_0 - 1}{2}\right),\ a\left(u'\right) < 1 \text{ for } x \in \left(\frac{c_0 - 1}{2}, 1\right).$$

This implies

$$\begin{aligned} a\left(u'\right) &= u' - 1 = 1 + c_0 \quad \text{in } \left(-1, -\frac{1}{2}\right), \\ a\left(u'\right) &= u' - 1 = -2x + c_0 \quad \text{in } \left(-\frac{1}{2}, \frac{c_0 - 1}{2}\right), \\ a\left(u'\right) &= u' = -2x + c_0 \quad \text{in } \left(\frac{c_0 - 1}{2}, 1\right) \end{aligned}$$

and thus

$$u(x) = \begin{cases} (2 + c_0)\, x + c_2 \quad \text{in } \left(-1, -\frac{1}{2}\right), \\ -x^2 + (1 + c_0)\, x + c_3 \quad \text{in } \left(-\frac{1}{2}, \frac{c_0 - 1}{2}\right), \\ -x^2 + c_0 x + c_4 \quad \text{in } \left(\frac{c_0 - 1}{2}, 1\right). \end{cases}$$

Since $u(\pm 1) = 0$ *and* u *is continuous at the point* $-\frac{1}{2}$, *it follows that* $c_2 = 2 + c_0$, $c_3 = c_0 + \frac{7}{4}$ *and* $c_4 = 1 - c_0$. *We use now the continuity of* u *at the point* $\frac{c_0 - 1}{2}$ *to obtain* $c_0 = -\frac{1}{10}$, *which is impossible.*

iii. *Assume finally that* $c_0 \geq 3$. *Then one has* $a(u') > 1$ *a.e. on* $(-1, 1)$ *and hence we have since* $a(u') = u' - 1$

$$u' = (2 + c_0)\, \chi_{\left(-1, -\frac{1}{2}\right)} + (-2x + c_0 + 1)\, \chi_{\left(-\frac{1}{2}, 1\right)}.$$

This implies that

$$u(x) = (2 + c_0)(x + 1)\, \chi_{\left(-1, -\frac{1}{2}\right)} + \left(-x^2 + (c_0 + 1)\, x - c_0\right) \chi_{\left(-\frac{1}{2}, 1\right)}.$$

Since u *is continuous, it follows that* $c_0 = -\frac{7}{8}$ *which is also impossible here.*

Thus we have necessarily $c_0 = 0$ *and*

$$a\left(u'\right) = 1 \text{ in } \left(-1, -\frac{1}{2}\right),\ a\left(u'\right) = -2x \text{ in } \left(-\frac{1}{2}, 1\right).$$

Since $a(u') < 1$ *a.e. on* $\left(-\frac{1}{2}, 1\right)$ *and* $u(1) = 0$ *we derive*

$$u(x) = 1 - x^2 \quad \text{on } \left(-\frac{1}{2}, 1\right),\ u' \in [1, 2] \quad \text{on } \left(-1, -\frac{1}{2}\right).$$

Thus, the family of solutions to (1.9) is given by

$$u(x) = \left(\int_{-1}^{x} w(s)\, ds\right) \chi_{\left(-1, -\frac{1}{2}\right)} + \left(1 - x^2\right) \chi_{\left(-\frac{1}{2}, 1\right)},$$

where $w \in L^2\left(-1, -\frac{1}{2}\right)$ *is such that*

$$w \in [1, 2] \ \text{a.e.on} \left(-1, -\frac{1}{2}\right),\ \int_{-1}^{-\frac{1}{2}} w(s)\, ds = \frac{3}{4}.$$

It is here clear that the weak maximum principle does not hold. Then in particular since $u' \in [1,2]$, $u\left(-\frac{1}{2}\right) = \frac{3}{4}$ *and* $u(-1) = 0$, *we have for* $x \in \left(-1, -\frac{1}{2}\right)$

$$x+1 \le u(x) = \int_{-1}^{x} u'(s)\,ds \le 2(x+1), \quad \frac{7}{4} + 2x \le u(x) = \frac{3}{4} - \int_{x}^{-\frac{1}{2}} u'(s)\,ds \le \frac{5}{4} + x.$$

This gives for minimal solution (Fig. 2)

$$u_m(x) = (x+1)\,\chi_{\left(-1,-\frac{3}{4}\right)} + \left(\frac{7}{4} + 2x\right)\chi_{\left(-\frac{3}{4},-\frac{1}{2}\right)} + \left(1-x^2\right)\chi_{\left(-\frac{1}{2},1\right)}$$

and as maximal solution

$$u_M(x) = 2(x+1)\,\chi_{\left(-1,-\frac{3}{4}\right)} + \left(\frac{5}{4} + x\right)\chi_{\left(-\frac{3}{4},-\frac{1}{2}\right)} + \left(1-x^2\right)\chi_{\left(-\frac{1}{2},1\right)}.$$

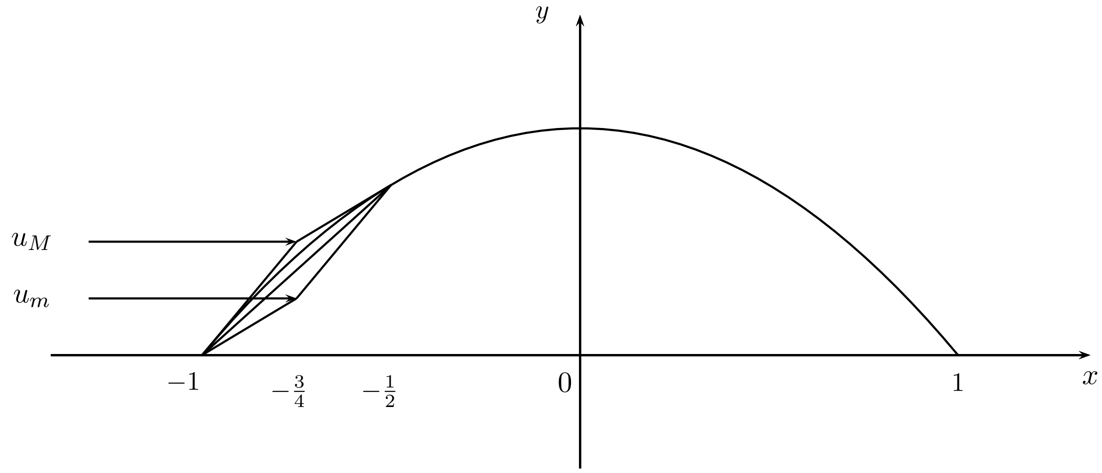

Figure 1.2: Solutions of (1.9).

In fact, thanks to the following lemma u_m *is also the minimal solution of the variational inequality*

$$\left|\begin{array}{l} u \in \mathcal{K} := \left\{u \in H_0^1(\Omega) : u \ge 0 \ \text{ for a.e. } x \in \Omega\right\}, \\ \displaystyle\int_\Omega a\left(u'\right)(v-u)'\,dx \ge \int_{-\frac{1}{2}}^{1} 2(v-u)\,dx \quad \forall v \in \mathcal{K}. \end{array}\right.$$

Lemma 1.1. *Let* Ω *be an open bounded interval,* $f \in L^q(\Omega)$ *and* a *is a monotone continuous function on* $\mathbb{R}$ *such that for some* $\alpha_0, \beta_0 > 0$,

$$a(\xi)\,\xi \ge \alpha_0 |\xi|^2, \quad |a(\xi)| \le \beta_0(1+|\xi|) \quad \forall \xi \in \mathbb{R}.$$

Assume that the following Dirichlet problem

$$\left\{\begin{array}{l} u \in H_0^1(\Omega), \\ \displaystyle\int_\Omega a\left(u'\right)v'\,dx = \int_\Omega f v\,dx \quad \forall v \in H_0^1(\Omega), \end{array}\right. \tag{1.10}$$

has a positive minimal solution $\underline{u}$*, then* $\underline{u}$ *is also a minimal solution of the variational inequality*

$$\begin{cases} u \in \mathcal{K}, \\ \int_\Omega a\left(u'\right)(v-u)'\,dx \geq \int_\Omega f\,(v-u)\,dx \quad \forall v \in \mathcal{K}, \end{cases} \tag{1.11}$$

where $\mathcal{K} := \left\{u \in H_0^1(\Omega) : u \geq 0 \ \ \textit{for a.e. } x \in \Omega\right\}.$

Proof. Since $\underline{u}$ also solution to (1.11), let us show that $\underline{u} \leq u$ for any u solution to (1.11). Let u be a solution to (1.11). Taking $v = 2u,\ v = 0$ in (1.11) one sees that

$$\int_\Omega a\left(u'\right)u'dx = \int_\Omega fudx$$

and thus u satisfies

$$\int_\Omega a\left(u'\right)v'dx \geq \int_\Omega fvdx = \int_\Omega a\left(\underline{u}'\right)v'dx \quad \forall v \in \mathcal{K}.$$

Taking $v = (\underline{u} - u)^+$ one derives

$$\int_{\{\underline{u}>u\}} \left(a\left(\underline{u}'\right) - a\left(u'\right)\right)(u-\underline{u})'\,dx \leq 0,$$

that is to say- due to the monotonicity of a

$$\left(a\left(\underline{u}'\right) - a\left(u'\right)\right)\left(u' - \underline{u}'\right) = 0 \text{ on } \{\underline{u} > u\}.$$

If $u' \neq \underline{u}'$ one has $a\left(\underline{u}'\right) = a\left(u'\right)$ and also of course when $u' = \underline{u}'$. Thus it holds $a\left(\underline{u}'\right) = a\left(u'\right)$ on $\{\underline{u} > u\}$. It follows that $a\left((u \wedge \underline{u})'\right) = a\left(\underline{u}'\right)$ in Ω and thus

$$\begin{cases} -\left(a\left((u \wedge \underline{u})'\right)\right)' = -\left(a\left(\underline{u}'\right)\right)' = f, \\ u \wedge \underline{u} \in H_0^1(\Omega). \end{cases}$$

Since $\underline{u}$ is the minimal solution to (1.11) one has

$$u \geq u \wedge \underline{u} \geq \underline{u}.$$

This achieves the proof of Lemma. □

Let $\ell > 0$ be a real number. We denote by Ω_ℓ the cylinder defined as

$$\Omega_\ell = (-\ell, \ell) \times \Omega.$$

The points in $\mathbb{R}^{n+1}$ are denoted by (y, x) with $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and the gradient operator defined over $\mathbb{R}^{n+1}$ is also denoted by

$$\nabla' = (\partial_y, \nabla) \text{ with } \nabla = \left(\partial_{x_1}, \partial_{x_2}, ..., \partial_{x_n}\right).$$

We set

$$\mathcal{K}_\ell = \left\{v \in W_0^{1,p}(\Omega_\ell) \mid v(y,.) \in \mathcal{K} \text{ a.e. in } (-\ell, \ell)\right\}.$$

This is a closed convex subset of $W_0^{1,p}(\Omega_\ell)$. Then for $f \in L^q(\Omega)$, let u_ℓ be the solution of the following variational inequality

$$\begin{cases} u_\ell \in \mathcal{K}_\ell, \\ \int_{\Omega_\ell} |\partial_y u_\ell|^{p-2} \partial_y u_\ell \partial_y (v - u_\ell)\, dxdy + \int_{-\ell}^{\ell} \langle Au_\ell, v - u_\ell \rangle_\Omega dy \geq \int_{\Omega_\ell} f(x)(v - u_\ell)\, dxdy \quad \forall v \in \mathcal{K}_\ell. \end{cases} \tag{1.12}$$

It is clear that all the foregoing hypotheses assumed on the monotone operator A can be adapted to the operator $-\partial_y \left(|\partial_y v|^{p-2} \partial_y v \right) + Av$ in addition to the strict monotonicity. Therefore, there exists a unique solution u_ℓ of (1.12).

Formally, if we pass to the limit when $\ell \to \infty$, the function limit $\tilde{u}$ is a solution to (1.7). In Theorem 1.1, the main result of this section, we will prove that when f is nonnegative this limit is also the minimal solution to Problem (1.7).

The following lemmas will be used in the proof of our main result.

Lemma 1.2. *Suppose that $f \in L^q(\Omega)$ is nonnegative and the assumptions (1.1)-(1.6) are satisfied. Then*

- (i) *$(u_\ell)_{\ell>0}$ is a nondecreasing sequence of nonnegative functions bounded above by any solution of Problem (1.7),*
- (ii) *for all $\ell_0 > 0$, there exists a constant $C(\ell_0)$ independent of ℓ such that*

$$|u_\ell|_{1,p,\Omega_{\ell_0}} \leq C(\ell_0).$$

Proof. (i) Taking $v = u_\ell^+ \in \mathcal{K}_\ell$, the nonnegative part of u_ℓ, in (1.12) we get

$$-\int_{\Omega_\ell} \left|\partial_y u_\ell^-\right|^p dxdy - \int_{-\ell}^{\ell} \langle A(-u_\ell^-), (-u_\ell^-)\rangle_\Omega dy \geq \int_{\Omega_\ell} f(x) u_\ell^- dxdy \geq 0,$$

where $u_\ell^- = u_\ell^+ - u_\ell$. Using the coerciveness condition (1.2), we easily obtain

$$\int_{\Omega_\ell} \left|\partial_y u_\ell^-\right|^p dxdy + \int_{-\ell}^{\ell} \alpha \left|u_\ell^-\right|_{1,p}^p dy \leq 0.$$

Hence, we derive that u_ℓ is nonnegative. The same result for u follows by testing (1.7) with u^+ and following the same argument as above. Of course this is one of comparison principle that still takes place when the strict monotonicity is missing, since it only uses the coerciveness condition (1.2). Now, we prove that the sequence $(u_\ell)_{\ell>0}$ is nondecreasing in ℓ and bounded. Let $\ell < \ell'$. Extending u_ℓ by 0 on $\Omega_{\ell'}$ and since $u_{\ell'}$ is nonnegative we take $v = u_\ell - (u_\ell - u_{\ell'})^+ \in \mathcal{K}_\ell$ in (1.12) and $v = u_{\ell'} + (u_\ell - u_{\ell'})^+ \in \mathcal{K}_{\ell'}$ in (1.12) written for $u_{\ell'}$ and add the two inequalities, it comes

$$\int_{\Omega_\ell} \left(|\partial_y u_\ell|^{p-2} \partial_y u_\ell - |\partial_y u_{\ell'}|^{p-2} \partial_y u_{\ell'} \right) \partial_y (u_\ell - u_{\ell'})^+ dxdy + \int_{-\ell}^{\ell} \langle Au_\ell - Au_{\ell'}, (u_\ell - u_{\ell'})^+ \rangle_\Omega dy \leq 0.$$

Thanks to the monotonicity condition (1.3) we deduce

$$\int_{\Omega_\ell} \left(|\partial_y u_\ell|^{p-2} \partial_y u_\ell - |\partial_y u_{\ell'}|^{p-2} \partial_y u_{\ell'} \right) \partial_y (u_\ell - u_{\ell'})^+ dxdy \leq 0.$$

This implies -see [18]-

$$\int_{\Omega_\ell} \left(|\partial_y u_\ell| + |\partial_y u_{\ell'}|\right)^{p-2} \left|\partial_y \left(u_\ell - u_{\ell'}\right)^+\right|^2 dxdy \leq 0$$

and this leads to

$$\partial_y \left(u_\ell - u_{\ell'}\right)^+ = 0 \text{ in } \Omega_\ell.$$

Applying the Poincaré inequality we get

$$\left(u_\ell - u_{\ell'}\right)^+ = 0 \text{ in } \Omega_\ell,$$

which shows that u_ℓ is a nondecreasing sequence in ℓ. On the other hand, taking $v = u + \left(u_\ell\left(y, .\right) - u\right)^+ \in \mathcal{K}$ as a test function in (1.7) we derive, for almost all y,

$$\langle Au, \left(u_\ell\left(y, .\right) - u\right)^+\rangle_\Omega \geq \int_\Omega f\left(u_\ell\left(y, .\right) - u\right)^+ dx.$$

Integrating in y we obtain

$$\int_{-\ell}^{\ell} \langle Au, \left(u_\ell - u\right)^+\rangle_\Omega dy \geq \int_{\Omega_\ell} f\left(u_\ell - u\right)^+ dxdy.$$

One has also $u_\ell - \left(u_\ell - u\right)^+ \in \mathcal{K}_\ell$ and from (1.12) we derive

$$-\int_{\Omega_\ell} |\partial_y u_\ell|^{p-2} \partial_y u_\ell \partial_y \left(u_\ell - u\right)^+ dxdy - \int_{-\ell}^{\ell} \langle Au_\ell, \left(u_\ell - u\right)^+\rangle_\Omega dy \geq -\int_{\Omega_\ell} f\left(u_\ell - u\right)^+ dxdy.$$

Adding the last two inequalities and using the fact that u is independent of y and (1.3) we arrive to

$$\int_{\Omega_\ell} \left(|\partial_y u_\ell|^{p-2} \partial_y u_\ell - |\partial_y u|^{p-2} \partial_y u\right) \partial_y \left(u_\ell - u\right)^+ dxdy \leq 0.$$

One concludes as above that $\left(u_\ell - u\right)^+ = 0$, which means that $u_\ell \leq u$ a.e. on Ω_ℓ and for all u solution to (1.7).
(ii) Let now $\ell_0 > 0$ and $\rho \in \mathcal{D}\left(-2\ell_0, 2\ell_0\right)$ such that

$$0 \leq \rho \leq 1 \text{ and } \rho = 1 \text{ on } \left(-\ell_0, \ell_0\right). \tag{1.13}$$

Then $u_\ell - \rho^p\left(u_\ell - u\right) \in \mathcal{K}_\ell$ and from (1.12), we derive

$$\begin{aligned}\int_{\Omega_\ell} |\partial_y u_\ell|^{p-2} \partial_y u_\ell \partial_y \left\{-\rho^p\left(u_\ell - u\right)\right\} dxdy + \int_{-\ell}^{\ell} \langle Au_\ell, -\rho^p\left(u_\ell - u\right)\rangle_\Omega dy &\geq \int_{\Omega_\ell} f\left\{-\rho^p\left(u_\ell - u\right)\right\} dxdy \\ &\geq 0.\end{aligned}$$

Using the growth condition (1.6) and the fact that u is independent of y it follows that

$$\begin{aligned}&\int_{\Omega_\ell} \rho^p |\partial_y u_\ell|^p dxdy + \int_{-\ell}^{\ell} \rho^p \langle Au_\ell, u_\ell\rangle_\Omega dy \\ &\qquad \leq -p\int_{\Omega_\ell} |\partial_y u_\ell|^{p-2} \partial_y u_\ell \rho^{p-1} \partial_y \rho\left(u_\ell - u\right) dxdy + \int_{-\ell}^{\ell} \rho^p \langle Au_\ell, u\rangle_\Omega dy \\ &\qquad \leq C\int_{\Omega_\ell} |\partial_y u_\ell|^{p-1} \rho^{p-1} u dxdy + C\int_{-\ell}^{\ell} \rho^p \left(|u_\ell|_{1,p}^{p-1} + 1\right) |u|_{1,p}\, dy,\end{aligned}$$

for some constant C independent of ℓ (recall $u \geq u_\ell$). Applying the Young inequality we easily get from (1.2) that

$$\int_{\Omega_\ell} \rho^p \left|\partial_y u_\ell\right|^p dxdy + \alpha \int_{-\ell}^{\ell} \rho^p \left|u_\ell\right|_{1,p}^p dy \leq \varepsilon \left(\int_{\Omega_\ell} \rho^p \left|\partial_y u_\ell\right|^p dxdy + \int_{-\ell}^{\ell} \rho^p \left|u_\ell\right|_{1,p}^p dy\right) + C_\varepsilon \left(\int_{\Omega_{2\ell_0}} |u|^p \, dxdy + \int_{-\ell}^{\ell} \rho^p \left(|u|_{1,p}^p + 1\right) dy\right).$$

Since $\rho = 1$ on $(-\ell_0, \ell_0)$, choosing ε small enough we get

$$\int_{\Omega_{\ell_0}} |u_\ell|^p + \left|\nabla' u_\ell\right|^p dxdy \leq C(\ell_0).$$

This achieves the proof of Lemma 1.2. □

Now, using the above lemma we can prove the following:

Lemma 1.3. *Under the assumptions of Lemma 1.2, the solution u_ℓ of (1.12) converges to $\tilde{u}$, as ℓ goes to ∞, a solution of (1.7).*

Proof. We start by applying Lemma 1.2, it follows that u_ℓ is converging towards some function $\tilde{u}$. Next, we show that $\tilde{u}$ is independent of y. We use the idea introduced, for example, in [32]. Let $h > 0$. The functions $\mathcal{T}_{\pm h} u_\ell(y,x)$ and $\left(\mathcal{T}_{\pm h} u_\ell(y,x) - u_{\ell+h}(y,x)\right)^+$ are supported in the closure of $\Omega_\ell^{\pm h} := (-\ell \mp h, \ell \mp h) \times \Omega$. For simplicity, we write $\mathcal{T}_{\pm h} u_\ell(y,x)$ instead of $\mathcal{T}_{\pm h,y} u_\ell(y,x)$. Then from (1.12), we have

$$\int_{\Omega_\ell^{\pm h}} \left|\partial_y \mathcal{T}_{\pm h} u_\ell\right|^{p-2} \partial_y \mathcal{T}_{\pm h} u_\ell \partial_y \left(v - \mathcal{T}_{\pm h} u_\ell\right) dxdy + \int_{-\ell \mp h}^{\ell \mp h} \langle A \mathcal{T}_{\pm h} u_\ell, v - \mathcal{T}_{\pm h} u_\ell \rangle dy \geq \int_{\Omega_\ell^{\pm h}} f(x) \left(v - \mathcal{T}_{\pm h} u_\ell\right) dxdy \quad \forall v \in \mathcal{K}_{\ell,\pm h}, \tag{1.14}$$

where

$$\mathcal{K}_{\ell,\pm h} := \{\mathcal{T}_{\pm h} v \mid v \in \mathcal{K}_\ell\} = \left\{v \in W_0^{1,p}\left(\Omega_\ell^{\pm h}\right) \mid v(y,.) \in \mathcal{K} \text{ a.e. in } (-\ell \mp h, \ell \mp h)\right\}.$$

Choosing $v = \mathcal{T}_{\pm h} u_\ell - \left(\mathcal{T}_{\pm h} u_\ell - u_{\ell+h}\right)^+ \in \mathcal{K}_{\ell,\pm h}$ in (1.14) and $v = u_{\ell+h} + \left(\mathcal{T}_{\pm h} u_\ell - u_{\ell+h}\right)^+ \in \mathcal{K}_{\ell+h}$ in (1.12) written for $u_{\ell+h}$ and then adding the two inequalities, we obtain

$$\int_{\Omega_\ell^{\pm h}} \left(\left|\partial_y \mathcal{T}_{\pm h} u_\ell\right|^{p-2} \partial_y \mathcal{T}_{\pm h} u_\ell - \left|\partial_y u_{\ell+h}\right|^{p-2} \partial_y u_{\ell+h}\right) \partial_y \left(\mathcal{T}_{\pm h} u_\ell - u_{\ell+h}\right)^+ dxdy + \int_{-\ell \mp h}^{\ell \mp h} \langle A \mathcal{T}_{\pm h} u_\ell - A u_{\ell+h}, \left(\mathcal{T}_{\pm h} u_\ell - u_{\ell+h}\right)^+ \rangle dy \leq 0.$$

Using the monotonicity condition (1.3) we deduce

$$\int_{\Omega_\ell^{\pm h}} \left(\left|\partial_y \mathcal{T}_{\pm h} u_\ell\right|^{p-2} \partial_y \mathcal{T}_{\pm h} u_\ell - \left|\partial_y u_{\ell+h}\right|^{p-2} \partial_y u_{\ell+h}\right) \partial_y \left(\mathcal{T}_{\pm h} u_\ell - u_{\ell+h}\right)^+ dxdy \leq 0.$$

By the same arguments as in Lemma 1.2, one can show that

$$u_\ell\left(y\pm h,x\right)\leq u_{\ell+h}\left(y,x\right).$$

Passing to the limit as $\ell\to\infty$, we get

$$\tilde{u}\left(y+h,x\right)\leq\tilde{u}\left(y,x\right),$$

where h can be now chosen arbitrary. This of course implies that

$$\tilde{u}\left(y,x\right)=\tilde{u}\left(x\right).$$

Finally, we use the Minty-Browder technique to show that the limit $\tilde{u}$ is a solution to (1.7). Let $\ell_0\in\mathbb{R}$, for ℓ large enough, it follows from Lemma 1.2 that $\left|\partial_y u_\ell\right|^{p-2}\partial_y u_\ell$ and $\left\{a_i\left(x,u_\ell,\nabla u_\ell\right)\right\}_{i=0,\cdots,n}$ are bounded in the Banach space $L^q\left(\Omega_{\ell_0}\right)$. Therefore

$$\begin{aligned}&u_\ell\to\tilde{u},\quad\nabla u_\ell\rightharpoonup\nabla\tilde{u}\ \text{ in } L^p\left(\Omega_{\ell_0}\right),\\&\left|\partial_y u_\ell\right|^{p-2}\partial_y u_\ell\rightharpoonup d,\quad a_i\left(x,u_\ell,\nabla u_\ell\right)\rightharpoonup d_i\ \text{in } L^q\left(\Omega_{\ell_0}\right).\end{aligned}\tag{1.15}$$

The two first convergences hold for the whole sequence since $\left(u_\ell\right)_{\ell>0}$ is nondecreasing, which guarantees the uniqueness of the limit and the last two convergences hold up to subsequence. Once the limit is uniquely identified, the previous convergences will also take place for the whole sequence. Now let ϕ be a nonnegative function in $\mathcal{D}\left(-\ell_0,\ell_0\right)$, then under the above convergences we claim, up to subsequence

$$\lim_{\ell\to\infty}\int_{-\ell_0}^{\ell_0}\phi\langle Au_\ell,u_\ell\rangle_\Omega dy=\int_{\Omega_{\ell_0}}\phi\sum_{0\leqslant i\leqslant n}d_i\partial_{x_i}\tilde{u}dxdy,\qquad(\partial_{x_0}\tilde{u}=\tilde{u}),\tag{1.16}$$

$$\lim_{\ell\to\infty}\int_{\Omega_{\ell_0}}\phi\left|\partial_y u_\ell\right|^p dxdy=0.\tag{1.17}$$

The last limit means that $d=0$. Indeed, using the monotonicity condition (1.3) we get

$$\langle Au_\ell,u_\ell\rangle_\Omega\geq\langle Au_\ell,\tilde{u}\rangle_\Omega+\langle A\tilde{u},u_\ell-\tilde{u}\rangle_\Omega.$$

Thus one easily derives

$$\liminf_{\ell\to\infty}\int_{-\ell_0}^{\ell_0}\phi\langle Au_\ell,u_\ell\rangle_\Omega dy\geq\liminf_{\ell\to\infty}\int_{-\ell_0}^{\ell_0}\phi\langle Au_\ell,\tilde{u}\rangle_\Omega dy=\int_{\Omega_{\ell_0}}\phi\sum_{0\leqslant i\leqslant n}d_i\partial_{x_i}\tilde{u}dxdy.\tag{1.18}$$

On the other hand, since $u_\ell-\frac{\phi}{|\phi|_\infty}\left(u_\ell-\tilde{u}\right)\in\mathcal{K}_\ell$ (note that $\tilde{u}\in\mathcal{K}$) one has from (1.12)

$$\int_{\Omega_{\ell_0}}\left|\partial_y u_\ell\right|^{p-2}\partial_y u_\ell\partial_y\left\{\phi\left(u_\ell-\tilde{u}\right)\right\}dxdy+\int_{-\ell_0}^{\ell_0}\phi\langle Au_\ell,u_\ell-\tilde{u}\rangle_\Omega dy\leq\int_{\Omega_{\ell_0}}\phi f\left(u_\ell-\tilde{u}\right)dxdy\leq 0$$

and thus

$$\int_{\Omega_{\ell_0}}\phi\left|\partial_y u_\ell\right|^p dxdy+\int_{-\ell_0}^{\ell_0}\phi\langle Au_\ell,u_\ell\rangle_\Omega dy\leq\int_{\Omega_{\ell_0}}\left|\partial_y u_\ell\right|^{p-2}\partial_y u_\ell\partial_y\phi\left(\tilde{u}-u_\ell\right)dxdy+\int_{-\ell_0}^{\ell_0}\phi\langle Au_\ell,\tilde{u}\rangle_\Omega dy.$$

Passing to the lim sup as $\ell \to \infty$, we get

$$\limsup_{\ell\to\infty}\left[\int_{\Omega_{\ell_0}}\phi\left|\partial_y u_\ell\right|^p dxdy+\int_{-\ell_0}^{\ell_0}\phi\langle Au_\ell,u_\ell\rangle_\Omega dy\right]\le\int_{\Omega_{\ell_0}}\phi\sum_{0\leqslant i\leqslant n}d_i\partial_{x_i}\tilde{u}dxdy.$$

Combining this with (1.18) we end up with (2.16) and (1.17).
Now noting that for $\psi\in\mathcal{K}$ and nonnegative function $\phi\in\mathcal{D}\left(-\frac{\ell_0}{2},\frac{\ell_0}{2}\right)$ $(\phi\not\equiv 0)$, $u_\ell+\frac{\phi}{|\phi|_\infty}(\psi-u_\ell)\in\mathcal{K}_\ell$ and thus we can take it as a test function in (1.12) to get

$$\int_{\Omega_{\frac{\ell_0}{2}}}\left|\partial_y u_\ell\right|^{p-2}\partial_y u_\ell\partial_y\left\{\phi\left(\psi-u_\ell\right)\right\}dxdy+\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}\phi\langle Au_\ell,\psi-u_\ell\rangle_\Omega dy\ge\int_{\Omega_{\frac{\ell_0}{2}}}f\phi\left(\psi-u_\ell\right)dxdy.$$

By the monotonicity condition (1.3), this leads to

$$\int_{\Omega_{\frac{\ell_0}{2}}}\left|\partial_y u_\ell\right|^{p-2}\partial_y u_\ell\partial_y\left\{\phi\left(\psi-u_\ell\right)\right\}dxdy+\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}\phi\langle A\psi,\psi-u_\ell\rangle_\Omega dy\ge\int_{\Omega_{\frac{\ell_0}{2}}}f\phi\left(\psi-u_\ell\right)dxdy.$$

From (1.17) it is clear $\partial_y u_\ell\to 0$ in $L^p\left(\Omega_{\frac{\ell_0}{2}}\right)$, thus passing to the limit in the above inequality as $\ell\to\infty$ yields

$$\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}\phi\langle A\psi,\psi-\tilde{u}\rangle_\Omega dy\ge\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}\phi\int_\Omega f\left(\psi-\tilde{u}\right)dxdy.$$

This implies

$$\langle A\psi,\psi-\tilde{u}\rangle_\Omega\ge\int_\Omega f\left(\psi-\tilde{u}\right)dx\quad\forall\psi\in\mathcal{K}.$$

Choosing $\psi=\tilde{u}+t\left(v-\tilde{u}\right)$, where $0<t<1$ and $v\in\mathcal{K}$ we deduce

$$\langle A\left(\tilde{u}+t\left(v-\tilde{u}\right)\right),v-\tilde{u}\rangle_\Omega\ge\int_\Omega f\left(v-\tilde{u}\right)dx\quad\forall v\in\mathcal{K}.$$

Passing to the limit as $t\to 0$, taking into account the Carathéodory condition (1.4)-(1.5), we get

$$\langle A\tilde{u},v-\tilde{u}\rangle_\Omega\ge\int_\Omega f\left(v-\tilde{u}\right)dx\quad\forall v\in\mathcal{K}.$$

Then the Lemma 3.5 is proved. □

Remark 1.1. *Through this Lemma we are also practically answering the following question. What about the limit of the solution to (1.12) when its limit problem has more than one solution?*

We are now ready to prove the main result of this section.

Theorem 1.1. *Suppose that $f\in L^q(\Omega)$ is nonnegative and the assumptions (1.1)-(1.6) are satisfied. Then, there exists a minimal solution of (1.7), i.e.*

$$\tilde{u}\left(x\right)=\min\left\{u\left(x\right),\ u\ \textit{solution to}(1.7)\right\},\quad\tilde{u}\in\mathcal{K}$$

is solution to (1.7). Moreover, if u_1 and u_2 are the minimal solutions of (1.7) obtained by replacing f with f_1 and f_2 respectively, then if $f_1\le f_2$ we have $u_1\le u_2$.

Proof. Let u be an arbitrary solution of the problem (1.7) and u_ℓ be the solution to (1.12). Then from Lemma 1.2 we have

$$u_\ell (y,x) \leq u(x) \text{ for a.e. } (y,x) \in \Omega_\ell. \tag{1.19}$$

Passing to the limit as $\ell \to \infty$, we derive from Lemma 3.5 that $u_\ell (y,.)$ converges towards some $\tilde{u} \in \mathcal{K}$ solution to (1.7). Combining this with (1.19) yields

$$\tilde{u} \leq u \text{ a.e. in } \Omega.$$

This means that $\tilde{u}$ is the minimal solution of the problem (1.7). Furthermore, let $u_{\ell,1}$ and $u_{\ell,2}$ be the solutions of (1.12) obtained if we replace f by f_1 and f_2 respectively. Taking $v = u_{\ell,1} - (u_{\ell,1} - u_{\ell,2})^+$ and $v = u_{\ell,2} + (u_{\ell,1} - u_{\ell,2})^+$ in (1.12) for f_1 and f_2 respectively, we get

$$\int_{\Omega_\ell} \left(|\partial_y u_{\ell,1}|^{p-2} \partial_y u_{\ell,1} - |\partial_y u_{\ell,2}|^{p-2} \partial_y u_{\ell,2} \right) \partial_y (u_{\ell,1} - u_{\ell,2})^+ \, dxdy + \int_{-\ell}^{\ell} \langle Au_{\ell,1} - Au_{\ell,2}, (u_{\ell,1} - u_{\ell,2})^+ \rangle_\Omega dy$$
$$\leq \int_{\Omega_\ell} (f_1 - f_2)(u_{\ell,1} - u_{\ell,2})^+ \, dxdy \leq 0.$$

Using the monotonicity condition (1.3) we obtain

$$\int_{\Omega_\ell} \left(|\partial_y u_{\ell,1}|^{p-2} \partial_y u_{\ell,1} - |\partial_y u_{\ell,2}|^{p-2} \partial_y u_{\ell,2} \right) \partial_y (u_{\ell,1} - u_{\ell,2})^+ \, dxdy \leq 0.$$

This implies

$$u_{\ell,1} \leq u_{\ell,2} \text{ in } \Omega_\ell.$$

Passing to the limit as $\ell \to \infty$, using the above argument we get

$$u_1 \leq u_2 \text{ in } \Omega.$$

This completes the proof of Theorem 1.1. □

Remark 1.2. *The results of Theorem 1.1 remain true for a nonnegative distribution f in $W^{-1,q}(\Omega)$.*

1.2 Noncoercive variational inequalities

Employing the results of the previous section, we aim to extend the study to more general variational inequalities. We keep here the notation and assumptions of section 1.1 and consider the following extension of the nonlinear problem (1.7)

$$\begin{cases} u \in \mathcal{K}, \\ \langle Au, v-u \rangle_\Omega \geq \displaystyle\int_\Omega F(x,u)(v-u)\, dx \quad \forall v \in \mathcal{K}, \end{cases} \tag{1.20}$$

where $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a nonnegative function satisfying

$$\begin{cases} F(x,\cdot) : \mathbb{R} \to \mathbb{R} \text{ is continuous and nondecreasing for a.e. } x \in \Omega, \\ F(\cdot, r) : \Omega \to \mathbb{R} \text{ is measurable } \forall r \in \mathbb{R}, \end{cases} \tag{1.21}$$

$$F(x,u) \in L^q(\Omega), \quad \forall u \in L^{p^*}(\Omega), \; \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \tag{1.22}$$

The existence of a solution $u \in \mathcal{K}$ to the problem (1.20) requires more assumptions and it is investigated in different situations (see [10]). Here we will give a general condition related to our technique of construction but the existence of the minimal solution remains our main goal.

Let us first define the sequence of functions u_n as follows

$$\begin{cases} u_0 = 0, \\ u_n \in \mathcal{K}, \\ \langle Au_n, v - u_n \rangle_\Omega \geq \int_\Omega F(x, u_{n-1})(v - u_n)\, dx \quad \forall v \in \mathcal{K}, \end{cases} \tag{1.23}$$

where u_n is the minimal solution of the variational inequality in the last line of (1.23). Its existence is guaranteed by Theorem 1.1 since $F(x, u_{n-1}) \in L^q(\Omega)$. It is clear that the sequence of functions $(u_n)_{n\in\mathbb{N}}$ is nonnegative and in particular we have

$$F(x, u_0) \leq F(x, u_1) \quad a.e.\ x \in \Omega,$$

since F is nondecreasing in the second variable. This implies that

$$u_1 \leq u_2 \quad a.e.\ x \in \Omega,$$

thanks to the comparison principle of the minimal solutions shown in Theorem 1.1. By induction we can easily show that $(u_n)_{n\in\mathbb{N}}$ is nondecreasing sequence, i.e.

$$u_{n-1} \leq u_n \quad a.e.\ x \in \Omega, \quad \forall n \geq 1.$$

Let us then denote by u_∞ the pointwise nonnegative limit of $(u_n)_{n\in\mathbb{N}}$ which is not necessarily in $L^p(\Omega)$ and may equal ∞. We also denote

$$F_\infty := \lim_{n\to\infty} F(., u_n),$$

which may also be infinite on some subset. Assume that

$$F_\infty \in L^q(\Omega). \tag{1.24}$$

Note that the above assumption is satisfied, for example, if $\sup_{r\geq 0} F(., r) \in L^q(\Omega)$. Then the following lemma gives a characteristic property about the existence of the solution to (1.20) related to the above scheme.

Lemma 1.4. *Let F be a nonnegative function satisfying the hypotheses (1.21), (1.22) and suppose that the assumptions (1.2)-(1.6) are fulfilled. If (1.24) is satisfied then u_∞, the limit of u_n, belongs to $\mathcal{K}$ and is a solution to (1.20).*

Proof. Let $\tilde{u}_\infty \in \mathcal{K}$ be the minimal solution of

$$\begin{cases} \tilde{u}_\infty \in \mathcal{K}, \\ \langle A\tilde{u}_\infty, v - \tilde{u}_\infty \rangle_\Omega \geq \int_\Omega F_\infty (v - \tilde{u}_\infty)\, dx \quad \forall v \in \mathcal{K}. \end{cases} \tag{1.25}$$

The existence of $\tilde{u}_\infty$ is insured by Theorem 1.1. Then since

$$F_\infty \geq F(., u_{n-1}) \quad \text{a.e. on } \Omega, \quad \forall n \in \mathbb{N} - \{0\}$$

and thanks to the comparison principle of the minimal solutions of Theorem 1.1, we deduce that

$$u_n \leq \tilde{u}_\infty \;\; a.e. \text{ on } \Omega, \quad \forall n \in \mathbb{N}. \tag{1.26}$$

We can now affirm that u_∞ is finite almost everywhere and

$$u_\infty \leq \tilde{u}_\infty,\, F_\infty = F(\cdot, u_\infty) \quad \text{a.e. on } \Omega,$$

which, in particular, proves that $u_\infty \in L^p(\Omega)$. All this is not enough to pass to the limit in (1.23). We need the boundedness of the gradient. Then taking $v = \tilde{u}_\infty$ in (1.23), using the coerciveness, the growth condition (1.2) and (1.6), (1.26) with the monotonicity of F we obtain

$$\begin{aligned}\alpha |u_n|_{1,p}^p \leq \langle Au_n, u_n\rangle_\Omega &\leq \int_\Omega F(x, u_{n-1})(u_n - \tilde{u}_\infty)\, dx + \langle Au_n, \tilde{u}_\infty\rangle_\Omega \\ &\leq C\left(|u_n|_{1,p}^{p-1} + 1\right)|\tilde{u}_\infty|_{1,p},\end{aligned}$$

for some constant C independent of n. Applying the Young inequality we end up with

$$|u_n|_{1,p} \leq C,$$

for some constant C independent of n. Combining now all the above results, it follows, as $n \to \infty$ that

$$u_n \to u_\infty \text{ in } \Omega, \quad u_n \to u_\infty \text{ in } L^p(\Omega), \quad u_n \rightharpoonup u_\infty \text{ in } W^{1,p}(\Omega). \tag{1.27}$$

Since $\mathcal{K}$ is a closed convex subset of $W_0^{1,p}(\Omega)$, it is weakly closed and we have in particular $u_\infty \in \mathcal{K}$. Now in order to pass to the limit in (1.23), we use the Minty-Browder technique and exploit the above convergences. From the monotonicity condition (1.3) and (1.23), it follows for $w \in \mathcal{K}$ that

$$\int_\Omega F(x, u_{n-1})(u_n - w)\, dx + \langle Aw, w - u_n\rangle_\Omega \geq \langle Aw - Au_n, w - u_n\rangle_\Omega \geq 0.$$

Passing to the limit as $n \to \infty$, we get from (1.27) that

$$\langle Aw, w - u_\infty\rangle_\Omega \geq \int_\Omega F(x, u_\infty)(w - u_\infty)\, dx.$$

The continuity and the monotonicity of F in the second variable, (1.26) and (1.27) are used to ensure the strong convergence of $F(\cdot, u_n)$ in $L^q(\Omega)$. Taking $w = u_\infty + t(v - u_\infty)$ with $0 < t < 1$ and $v \in \mathcal{K}$, then passing to the limit as $t \to 0$ yields

$$\langle Au_\infty, v - u_\infty\rangle_\Omega \geq \int_\Omega F(x, u_\infty)(v - u_\infty)\, dx \quad \forall v \in \mathcal{K}.$$

This completes the proof of Lemma 1.4. □

We will see in the following theorem that (1.24) is more than just a simple condition and u_∞ is more than just a simple solution of (1.20).

Theorem 1.2. *Under the assumptions (1.1)-(1.6), (1.21), (1.22), we have the equivalence between the following assertions* *i) (1.20) has at least one solution,*

ii) (1.20) has a minimal solution,

iii) the hypothesis (1.24) holds.

Moreover if the hypothesis (1.24) holds then u_∞, the limit of u_n, belongs to $\mathcal{K}$ and is the minimal solution to (1.20), i.e.

$$u_\infty (x) = \min \{u (x) ,\ u \textit{ solution to } (1.20)\} \quad a.e. \quad on\ \Omega. \tag{1.28}$$

Proof. From the above lemma we can see that

$$iii) \Rightarrow i) \Leftarrow ii).$$

In the following we will show

$$iii) \Leftarrow i) \Rightarrow ii),$$

which achieves the equivalence of the above assertions and (1.28) simultaneously. Suppose that (1.20) has a solution $\bar{u} \in \mathcal{K}$. Let $u \in \mathcal{K}$ be the minimal solution of

$$\langle Au, v - u\rangle_\Omega \geq \int_\Omega F (x, \bar{u}) (v - u) \, dx \quad \forall v \in \mathcal{K}.$$

Here $\bar{u}$ is considered as a data while u is the unknown solution. The existence of u is insured by Theorem 1.1. Since $\bar{u} \geq 0$ a.e. on Ω (which also implies that $F (., \bar{u}) \geq F (., 0)$ a.e. on Ω) and thanks to the comparison principle of the minimal solutions of Theorem 1.1, we can show by induction, that

$$u_n \leq u \leq \bar{u} \ \text{ a.e. on } \Omega, \quad \forall n \in \mathbb{N}. \tag{1.29}$$

Then it follows, by the monotonicity of F in the second variable, that

$$F (., u_n) \leq F (., \bar{u}) \quad \text{a.e. on } \Omega, \quad \forall n \in \mathbb{N},$$

which shows (1.24). Then thanks to the preceding lemma and (1.29) we derive that u_∞, the limit of u_n, belongs to $\mathcal{K}$ and is a solution to (1.20) and

$$u_\infty \leq \bar{u} \ \ a.e. \ \text{ on } \Omega, \quad \forall n \in \mathbb{N}.$$

Since $\bar{u}$ is an arbitrary solution of (1.20) we deduce that u_∞ is the minimal solution of (1.20) and the proof is achieved. □

Remark 1.3. *It is possible to assume that $F (x, 0) \geq 0$ and $F (x, \cdot)$ is nondecreasing only on $\mathbb{R}^+$ for a.e. $x \in \Omega$, or to assume that F is defined on $\Omega \times \mathbb{R}^+$, to keep all the above results.*

Let us now give some immediate variant of the above result. Assume that $\bar{F} : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is a Carathéodory function satisfying the following Lipschitz condition:

$$\left|\bar{F} (x, r) - \bar{F} (x, s)\right| \leq k |r - s| , \quad a.e. \ \ x \in \Omega, \quad \forall r, s \in \mathbb{R}^+, \tag{1.30}$$

for some constant $k > 0$, and in addition

$$\bar{F} (x, 0) \in L^q (\Omega) \ \text{ is a nonnegative function.} \tag{1.31}$$

Then, as a consequence of Theorem 3.6 (recall Remark 2.1) we have

Corollary 1.1. *Assume that the assumptions (1.1)-(1.6), (1.30) and (1.31) are satisfied and for* $p \geq \frac{2n+2}{n+3}$ *there exists a solution to*

$$\begin{cases} u \in \mathcal{K}, \\ \langle Au, v-u \rangle_\Omega \geq \int_\Omega \bar{F}(x,u)(v-u)\,dx \quad \forall v \in \mathcal{K}. \end{cases} \tag{1.32}$$

Then (1.32) has a minimal solution, i.e.

$$\tilde{u}(x) = \min\{u(x),\ u \text{ solution to (1.32)}\}, \quad \tilde{u} \in \mathcal{K} \tag{1.33}$$

is a solution to (1.32).

Proof. We first check that $F(x,r) = \bar{F}(x,r) + kr$ satisfies almost all the hypotheses assumed on F above. Since we have from (1.30)

$$0 \leq F(x,r) - F(x,s) \leq 2k(r-s) \quad a.e.\ \ x \in \Omega,\ \ r > s \geq 0, \tag{1.34}$$

it is obvious to see that $F(x,.)$ is continuous and nondecreasing on $\mathbb{R}^+$ for a.e. $x \in \Omega$. We also deduce from the above inequality and (1.31) that

$$F(x,r) \geq F(x,0) \geq 0 \quad a.e.\ \ x \in \Omega,\ \ \forall r \geq 0.$$

Now if $u \in L^{p^*}(\Omega)$ we have (1.21) i.e.

$$F(x,u) = F(x,u) - F(x,0) + F(x,0) \in L^q(\Omega).$$

This will follow from (1.31) and (1.34), since $u \in L^q(\Omega)$. Indeed. By the Sobolev embedding theorem when $p \geq \frac{2n}{n+1} \Leftrightarrow \frac{1}{p} - \frac{1}{n} \leq 1 - \frac{1}{p} \Leftrightarrow p^* \geq q$, this is the case.
Consider the following variational problem

$$\begin{cases} u \in \mathcal{K}, \\ \langle Au, v-u \rangle_\Omega + \int_\Omega ku(v-u)\,dx \geq \int_\Omega F(x,u)(v-u)\,dx \quad \forall v \in \mathcal{K}. \end{cases} \tag{1.35}$$

In fact this is nothing else then our problem (1.32). On the other hand, we can easily check that the operator $\hat{A} = A + kId$ (Id is the identity mapping) satisfies the assumptions (1.2)-(1.5). The hypothesis (1.6) is assumed to ensure the boundedness of A. $\hat{A}$ is also bounded thanks to the assumption $p \geq \frac{2n}{n+1}$ i.e. for any $u, v \in W_0^{1,p}(\Omega)$,

$$\begin{aligned} \langle Au, v \rangle_\Omega + k \int_\Omega uv\,dx &\leq C\left(|u|_{1,p} + 1\right)|v|_{1,p} + k\,|u|_{q,\Omega}\,|v|_{p,\Omega} \\ &\leq C\left(|u|_{1,p} + 1\right)|v|_{1,p}, \end{aligned}$$

since $W^{1,p}(\Omega) \subset L^q(\Omega)$. □

Remark 1.4. *It is assumed that* $p \geq \frac{2n+2}{n+3}$ *to have* $L^{p^*}(\Omega_\ell) \subset L^q(\Omega_\ell)$ *which ensures that the arguments of the previous section can be adapted.*

1.3 Variational inequalities in unbounded domains

This section is devoted to study the existence of nonnegative solutions and their minimal solutions for some quasilinear variational inequalities in unbounded domains. We first investigate variational inequalities with coercive operators.

1.3.1 Variational inequalities with coercive operator

Let ω be a bounded open subset of $\mathbb{R}^{n-1}$ $(n \geq 2)$ and $\mathcal{K}_\omega$ be a lattice closed convex subset of $W_0^{1,p}(\omega)$ containing 0 i.e.

$$0,\ \max(u,v), \min(u,v) \in \mathcal{K}_\omega \quad \forall u,v \in \mathcal{K}_\omega. \tag{1.36}$$

For $x \in \mathbb{R}\times\omega$ we denote by x_1 the first coordinate of x and by X_2 the $n-1$ last ones, i.e.

$$x = (x_1, X_2) \text{ with } X_2 = (x_2, ..., x_n).$$

Also, we denote by $\mathcal{K}$ the closed convex subset of $W_{loc}^{1,p}(\mathbb{R}\times\overline{\omega})$ defined by

$$\mathcal{K} := W_{loc}^{1,p}(\mathbb{R};\mathcal{K}_\omega) := \left\{ v \in W_{loc}^{1,p}(\mathbb{R}\times\overline{\omega}) \mid v = 0 \text{ on } \mathbb{R}\times\partial\omega \text{ and } v(x_1,.) \in \mathcal{K}_\omega \text{ for a.e. } x_1 \in \mathbb{R} \right\}.$$

Then for a nonnegative f in $L_{loc}^q(\mathbb{R}, L^q(\omega))$, we consider the following nonlinear variational inequality defined on the infinite cylinder $\Omega = \mathbb{R}\times\omega$

$$\begin{cases} u \in \mathcal{K}, \\ \displaystyle\int_{\mathbb{R}\times\omega} a(x,u,\nabla u)\cdot\nabla(\varphi(v-u))\,dx + \int_{\mathbb{R}\times\omega} a_0(x,u,\nabla u)\varphi(v-u)\,dx \geq \int_{\mathbb{R}\times\omega} f\varphi(v-u)\,dx, \\ \qquad\qquad\qquad\qquad \forall v \in \mathcal{K},\ \forall\varphi \in \mathcal{D}(\mathbb{R}),\ \varphi \geq 0. \end{cases} \tag{1.37}$$

Note that if we have more smoothness on $a_1(x,u,\nabla u)$, i.e. if $\partial_{x_1}a_1(x,u,\nabla u) \in L_{loc}^q(\mathbb{R}, L^q(\omega))$ the above variational inequality can be written as

$$\int_\omega \sum_{2\leqslant i\leqslant n} a_i(x,u,\nabla u)\partial_{x_i}(v-u)(x_1,.)\,dX_2 + \int_\omega \{a_0(x,u,\nabla u) - \partial_{x_1}a_1(x,u,\nabla u)\}(v-u)(x_1,.)\,dX_2$$
$$\geq \int_\omega f(v-u)(x_1,.)\,dX_2 \quad \forall v \in \mathcal{K},\ \text{a.e. } x_1 \text{ in } \mathbb{R}.$$

Since the domain is unbounded and f is not necessarily in the dual of $W_0^{1,p}(\mathbb{R}\times\omega)$, the existence of nonnegative solutions to problem (1.37) is not an ordinary issue. Once this is ensured, we can then look for the minimal nonnegative solution. Here, we will use the same approach as in Section 1.1 to prove these existence results. To this end, in addition to the hypotheses (1.2)-(1.6) assume that

$$\begin{aligned} a_i(x_1, X_2, \xi_0, 0, \xi_2, \cdots, \xi_n) &= a_i(X_2, \xi_0, 0, \xi_2, \cdots, \xi_n) \\ &:= a_i(X_2, \xi_0, \xi_2, \cdots, \xi_n) \quad \forall\xi_j \in \mathbb{R},\ j = 0,2,...,n,\ i = 0,...,n. \end{aligned} \tag{1.38}$$

That is to say if $\xi_1 = 0$ then the coefficients a_i for $i = 0,...,n$ are independent of x_1. We also assume that there exists $h \in L^q(\omega)$ such that

$$f(x_1, X_2) \leq h(X_2) \text{ for a.e. } (x_1, X_2) \in \mathbb{R}\times\omega. \tag{1.39}$$

For $\ell > 0$, we set

$$\Omega_\ell = (-\ell, \ell)^2 \times \omega$$

and for simplicity we also set $\langle .,. \rangle_{\ell,\omega}$ instead of $\int_{-\ell}^{\ell} \langle .,. \rangle_\omega \, dx_1$. We denote by (y, x_1, X_2) the points in Ω_ℓ and by $\mathcal{K}_\ell$ the closed convex subset of $W_0^{1,p}(\Omega_\ell)$ defined by

$$\mathcal{K}_\ell := \left\{ v \in W_0^{1,p}(\Omega_\ell) \mid v(y, x_1, .) \in \mathcal{K}_\omega, \text{ a.e. in } (-\ell, \ell)^2 \right\}.$$

Then consider u_ω and u_ℓ respectively solutions of the following variational inequalities

$$\begin{cases} u_\omega \in \mathcal{K}_\omega, \\ \langle A_\omega u_\omega, v - u_\omega \rangle_\omega \geq \int_\omega h \, (v - u_\omega) \, dX_2 \quad \forall v \in \mathcal{K}_\omega, \end{cases} \tag{1.40}$$

and

$$\begin{cases} u_\ell \in \mathcal{K}_\ell, \\ \int_{\Omega_\ell} |\partial_y u_\ell|^{p-2} \, \partial_y u_\ell \partial_y \, (v - u_\ell) \, dxdy + \int_{-\ell}^{\ell} \langle A u_\ell, v - u_\ell \rangle_{\ell,\omega} \, dy \geq \int_{\Omega_\ell} f(x_1, X_2) \, (v - u_\ell) \, dxdy, \\ \qquad \forall v \in \mathcal{K}_\ell, \end{cases} \tag{1.41}$$

where $A_\omega u = -\sum_{2 \leqslant i \leqslant n} \partial_{x_i} a_i(X_2, u, \nabla_{X_2} u) + a_0(X_2, u, \nabla_{X_2} u)$ and A is the nonlinear operator given by (2.2). Under the above assumptions, the problems (1.40) and (1.41) have respectively a solution $u_\omega \in \mathcal{K}_\omega$ and a unique solution $u_\ell \in \mathcal{K}_\ell$. Then, we have

Theorem 1.3. *Suppose that the assumptions (1.2)-(1.6), (1.36), (1.38) and (1.39) are satisfied. Then* $(u_\ell)_{\ell > 0}$ *is a nonnegative nondecreasing sequence in* ℓ *converging towards some* $\tilde{u}$*, as* ℓ *goes to* ∞*, a nonnegative solution of (1.37).*

Proof. The proof will be broken into several steps.
Step 1. u_ℓ is nondecreasing sequence in ℓ bounded by any solution u_ω to (1.40).
Since f is nonnegative, arguing as in Lemma 1.2, it follows easily that u_ℓ and u_ω are nonnegative and $(u_\ell)_\ell$ is nondecreasing sequence. Now, taking $v = u_\omega + (u_\ell(y, x_1, .) - u_\omega)^+ \in \mathcal{K}_\omega$ as a test function in (1.40) then integrating on $(-\ell, \ell)^2$ and taking into account the assumption (1.38), we derive

$$\int_{-\ell}^{\ell} \left\langle A u_\omega, (u_\ell - u_\omega)^+ \right\rangle_{\ell,\omega} dy \geq \int_{\Omega_\ell} h \, (u_\ell - u_\omega)^+ \, dxdy.$$

At the same time, choosing $v = u_\ell - (u_\ell - u_\omega)^+ \in \mathcal{K}_\ell$ in (1.41). Then, adding the resulting inequality with the above one yields

$$\int_{\Omega_\ell} |\partial_y u_\ell|^{p-2} \, \partial_y u_\ell \partial_y \, (u_\ell - u_\omega)^+ \, dxdy + \int_{-\ell}^{\ell} \left\langle A u_\ell - A u_\omega, (u_\ell - u_\omega)^+ \right\rangle_{\ell,\omega} dy \\ \leq \int_{\Omega_\ell} (f - h) \, (u_\ell - u_\omega)^+ \, dxdy.$$

Using the monotonicity condition (1.3) and the assumption (1.39), we get

$$\int_{\Omega_\ell} |\partial_y u_\ell|^{p-2}\, \partial_y u_\ell \partial_y \left(u_\ell - u_\omega\right)^+ dxdy \leq 0.$$

Since u_ω is independent of y, this implies that

$$\int_{\Omega_\ell} \left(|\partial_y u_\ell|^{p-2}\, \partial_y u_\ell - |\partial_y u_\omega|^{p-2}\, \partial_y u_\omega \right) \partial_y \left(u_\ell - u_\omega\right)^+ dxdy \leq 0.$$

One derives easily $(u_\ell - u_\omega)^+ = 0$ and thus

$$u_\ell \leq u_\omega. \tag{1.42}$$

Step 2. $\tilde{u}$, the pointwise limit of u_ℓ, *is independent of* y. It is now established that

$$0 \leq u_\ell \leq u_{\ell'} \leq u_\omega \quad \forall \ell \leq \ell', \tag{1.43}$$

then it follows that u_ℓ possesses a pointwise nonnegative limit that we will denote by $\tilde{u}$ i.e.

$$u_\ell \to \tilde{u} \text{ in } \mathbb{R}^2 \times \omega. \tag{1.44}$$

Following the same arguments as in the proof of Lemma 3.5 we get

$$u_\ell\left(y+h, x_1, X_2\right) \leq u_{\ell+h}\left(y, x_1, X_2\right) \quad \forall h \in \mathbb{R}, \text{ a.e. in } \Omega_\ell^h := (-\ell-h, \ell-h) \times (-\ell, \ell) \times \omega.$$

Passing to the limit as $\ell \to \infty$, we easily obtain

$$\tilde{u}\left(y, x_1, X_2\right) = \tilde{u}\left(x_1, X_2\right).$$

Step 3. *For all* $\ell_0 > 0$, *there exists a constant* C_{ℓ_0} *independent of* ℓ *such that*

$$|u_\ell|_{1,p,\Omega_{\ell_0}} \leq C_{\ell_0}. \tag{1.45}$$

Let $\varrho \in \mathcal{D}\left((-2\ell_0, 2\ell_0)^2\right)$ such that

$$0 \leq \rho \leq 1 \text{ and } \rho = 1 \text{ on } \left(-\ell_0, \ell_0\right)^2.$$

Taking $v = u_\ell - \varrho^p\left(u_\ell - u_\omega\right) \in \mathcal{K}_\ell$ in (1.41) then following the same arguments as in the proof of the second part of Lemma 1.2 we end up with (1.45).

Step 4. $\tilde{u}$ *is a solution to* (1.37). The proof of this part is based on the Minty-Browder technique and follows the proof of the last step of Lemma 3.5 with some modifications. First, for $\ell_0 > 0$ we have from (1.44) and (1.45)

$$u_\ell \to \tilde{u} \;\text{ in } L^p\left(\Omega_{\ell_0}\right) \text{ and } u_\ell \rightharpoonup \tilde{u} \;\text{ in } W^{1,p}\left(\Omega_{\ell_0}\right). \tag{1.46}$$

Note that since

$$\tilde{\mathcal{K}}_{\ell_0} := \left\{ v \in W^{1,p}\left(\Omega_{\ell_0}\right) \mid v(y, x_1, .) \in \mathcal{K}_\omega, \text{ a.e. in } \left(-\ell_0, \ell_0\right)^2 \right\}$$

is closed and convex, it is also weakly closed and by consequence $\tilde{u} \in \tilde{\mathcal{K}}_{\ell_0}$ i.e. $\tilde{u}(x_1, .) \in \mathcal{K}_\omega$ a.e. in $(-\ell_0, \ell_0)$. Then by using the above convergence results, we can prove as in (2.16) and (1.17) that

$$\lim_{\ell\to\infty}\int_{\Omega_{\ell_0}} \phi\left\{a(x,u_\ell,\nabla u_\ell)\cdot\nabla u_\ell + a_0(x,u_\ell,\nabla u_\ell)u_\ell\right\}dxdy = \int_{\Omega_{\ell_0}}\phi\sum_{0\leqslant i\leqslant n} d_i\partial_{x_i}\tilde{u}dxdy, \tag{1.47}$$

$$\lim_{\ell\to\infty}\int_{\Omega_{\ell_0}} \phi\left|\partial_y u_\ell\right|^p dxdy \to 0, \qquad \forall\phi\in\mathcal{D}\left((-\ell_0,\ell_0)^2\right),\ \phi\geq 0, \tag{1.48}$$

where d_i is the weak limit of $a_i(x,u_\ell,\nabla u_\ell)$ in $L^q(\Omega_{\ell_0})$.

Now, let $\phi \not\equiv 0$ be a nonnegative function in $\mathcal{D}\left(\left(-\frac{\ell_0}{2},\frac{\ell_0}{2}\right)^2\right)$ and $w \in \mathcal{K}$. Taking $v = u_\ell + \frac{\phi}{|\phi|_\infty}(w-u_\ell) \in \mathcal{K}_\ell$ as a test function in (1.41) we get

$$\int_{\Omega_{\frac{\ell_0}{2}}} \left|\partial_y u_\ell\right|^{p-1}\partial_y u_\ell\partial_y\left(\phi\left(w-u_\ell\right)\right)dxdy + \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}\left\langle Au_\ell,\phi\left(w-u_\ell\right)\right\rangle_{\frac{\ell_0}{2},\omega}dy \geq \int_{\Omega_{\frac{\ell_0}{2}}}\phi f\left(w-u_\ell\right)dxdy.$$

Passing to the limit as $\ell\to\infty$, taking into account (1.46), (1.47) and (1.48) we obtain

$$\int_{\Omega_{\frac{\ell_0}{2}}}\sum_{0\leqslant i\leqslant n} d_i\partial_{x_i}\left(\phi\left(w-\tilde{u}\right)\right)dxdy \geq \int_{\Omega_{\frac{\ell_0}{2}}}\phi f\left(w-\tilde{u}\right)dxdy. \tag{1.49}$$

On the other hand, we claim by using the Minty-Browder technique that

$$\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}\left\langle A\tilde{u},\phi\left(w-\tilde{u}\right)\right\rangle_{\frac{\ell_0}{2},\omega}dy \geq \int_{\Omega_{\frac{\ell_0}{2}}}\sum_{0\leqslant i\leqslant n} d_i\partial_{x_i}\left(\phi\left(w-\tilde{u}\right)\right)dxdy. \tag{1.50}$$

Indeed. Let $t > 0$ and ψ be a nonnegative function in $\mathcal{D}\left(\left(-\frac{\ell_0}{2},\frac{\ell_0}{2}\right)^2\right)$, then it follows from the monotonicity condition (1.3) that

$$\begin{aligned}
&\int_{\Omega_{\frac{\ell_0}{2}}}\psi\left\{a\left(x,\tilde{u}+t\phi\left(w-\tilde{u}\right),\nabla\left(\tilde{u}+t\phi\left(w-\tilde{u}\right)\right)\right)-a(x,u_\ell,\nabla u_\ell)\right\}\cdot\nabla\left(\tilde{u}-u_\ell\right)dxdy\\
&+\int_{\Omega_{\frac{\ell_0}{2}}}\psi\left\{a_0\left(x,\tilde{u}+t\phi\left(w-\tilde{u}\right),\nabla\left(\tilde{u}+t\phi\left(w-\tilde{u}\right)\right)\right)-a_0(x,u_\ell,\nabla u_\ell)\right\}\left(\tilde{u}-u_\ell\right)dxdy\\
&+t\int_{\Omega_{\frac{\ell_0}{2}}}\psi\left\{a\left(x,\tilde{u}+t\phi\left(w-\tilde{u}\right),\nabla\left(\tilde{u}+t\phi\left(w-\tilde{u}\right)\right)\right)-a(x,u_\ell,\nabla u_\ell)\right\}\cdot\nabla\left(\phi\left(w-\tilde{u}\right)\right)dxdy\\
&+t\int_{\Omega_{\frac{\ell_0}{2}}}\psi\left\{a_0\left(x,\tilde{u}+t\phi\left(w-\tilde{u}\right),\nabla\left(\tilde{u}+t\phi\left(w-\tilde{u}\right)\right)\right)-a_0(x,u_\ell,\nabla u_\ell)\right\}\phi\left(w-\tilde{u}\right)dxdy\geq 0.
\end{aligned}$$

Passing to the limit as $\ell\to\infty$, taking into account (1.46) and (1.47) we get

$$\int_{\Omega_{\frac{\ell_0}{2}}}\psi\sum_{0\leqslant i\leqslant n}\left(a_i\left(x,\tilde{u}+t\phi\left(w-\tilde{u}\right),\nabla\left(\tilde{u}+t\phi\left(w-\tilde{u}\right)\right)\right)-d_i\right)\partial_{x_i}\left(\phi\left(w-\tilde{u}\right)\right)dxdy\geq 0.$$

Letting $t \to 0$, we easily obtain

$$\int_{\Omega_{\frac{\ell_0}{2}}} \psi \sum_{0\leqslant i\leqslant n} \left(a_i\left(x,\tilde{u},\nabla\tilde{u}\right)-d_i\right)\partial_{x_i}\left(\phi\left(w-\tilde{u}\right)\right)dxdy \geq 0, \quad \forall\psi\in\mathcal{D}\left(\left(-\frac{\ell_0}{2},\frac{\ell_0}{2}\right)^2\right), \ \psi\geq 0.$$

This is nothing else (1.50).
Now, combining (1.49) and (1.50) we end up with

$$\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \left\langle A\tilde{u},\phi\left(w-\tilde{u}\right)\right\rangle_{\frac{\ell_0}{2},\omega} dy \geq \int_{\Omega_{\frac{\ell_0}{2}}} \phi f\left(w-\tilde{u}\right)dxdy, \ \forall w\in\mathcal{K}, \ \phi\in\mathcal{D}\left(\left(-\frac{\ell_0}{2},\frac{\ell_0}{2}\right)^2\right), \ \phi\geq 0. \quad (1.51)$$

Taking $\phi\left(y,x_1\right)=\tilde{\varphi}\left(y\right)\varphi\left(x_1\right)$ in (1.51) where $\tilde{\varphi}\not\equiv 0$ and φ are nonnegative functions in $\mathcal{D}\left(-\frac{\ell_0}{2},\frac{\ell_0}{2}\right)$, we derive

$$\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \tilde{\varphi}\left(y\right)\left\langle A\tilde{u},\varphi\left(w-\tilde{u}\right)\right\rangle_{\frac{\ell_0}{2},\omega} dy \geq \int_{\Omega_{\frac{\ell_0}{2}}} \tilde{\varphi}\left(y\right)\varphi f\left(w-\tilde{u}\right)dxdy.$$

Since $\tilde{u}$ is independent of y this implies that

$$\left\langle A\tilde{u},\varphi\left(w-\tilde{u}\right)\right\rangle_{\frac{\ell_0}{2},\omega} \geq \int_{\left(-\frac{\ell_0}{2},\frac{\ell_0}{2}\right)\times\omega} \varphi f\left(w-\tilde{u}\right)dx, \ \forall w\in\mathcal{K}, \ \forall\varphi\in\mathcal{D}\left(-\frac{\ell_0}{2},\frac{\ell_0}{2}\right), \ \varphi\geq 0$$

and since ℓ_0 is arbitrary this means that

$$\int_{\mathbb{R}\times\omega} a(x,u,\nabla u)\cdot\nabla\left(\varphi\left(w-u\right)\right)dx + \int_{\mathbb{R}\times\omega} a_0(x,u,\nabla u)\varphi\left(w-u\right)dx \geq \int_{\mathbb{R}\times\omega} f\varphi\left(w-u\right)dx,$$

$$\forall w\in\mathcal{K}, \ \forall\varphi\in\mathcal{D}\left(\mathbb{R}\right), \ \varphi\geq 0,$$

which completes the proof of Theorem 1.3. □

As we saw above, $\tilde{u}$ is more than a usual solution and we can state the following result

Theorem 1.4. *Suppose that the assumptions (1.2)-(1.6), (1.36), (1.38) and (1.39) are satisfied. Then, there exists a minimal nonnegative solution to (1.37), i.e.* $\tilde{u}=\min\left\{u \text{ solution to (1.37)}, \ u\geq 0\right\}\in\mathcal{K}$ *is a solution to (1.37). Moreover, let* u_1 *and* u_2 *be the minimum of nonnegative solutions to (1.37) obtained by replacing* f *with* f_1 *and* f_2 *respectively. Then if* $f_1\leq f_2$ *we have* $u_1\leq u_2$.

Proof. Let u be an arbitrary nonnegative solution of (1.37) and u_ℓ be the solution of the problem (1.41). Since $u\geq 0$ choosing $v=u+\left(u_\ell-u\right)^+\in\mathcal{K}$ as a test function in (1.37), taking into account the fact that $\operatorname{supp}(u_\ell-u)^+\subset\Omega_\ell$, then integrating on $(-\ell,\ell)$ we get

$$\int_{-\ell}^{\ell} \left\langle Au,\left(u_\ell-u\right)^+\right\rangle_{\ell,\omega} dy \geq \int_{\Omega_\ell} f\left(u_\ell-u\right)^+ dxdy, \quad (1.52)$$

we took φ in (1.37) such that $\varphi=1$ on $\left(-\ell,\ell\right)$. At the same time, testing (1.41) by $v=u_\ell-\left(u_\ell-u\right)^+\in\mathcal{K}_\ell$ then summing the produced inequality with (1.52) to obtain

$$\int_{\Omega_\ell} \left|\partial_y u_\ell\right|^{p-2}\partial_y u_\ell\partial_y\left(u_\ell-u\right)^+ dxdy + \int_{-\ell}^{\ell} \left\langle Au_\ell-Au,\left(u_\ell-u\right)^+\right\rangle_{\ell,\omega} dy \leq 0. \quad (1.53)$$

Using the monotonicity condition (1.3) we get

$$\int_{\Omega_\ell} \left(|\partial_y u_\ell|^{p-2} \partial_y u_\ell - |\partial_y u|^{p-2} \partial_y u \right) \partial_y (u_\ell - u)^+ \, dxdy \le 0,$$

since u is independent of y. By consequence, it follows that

$$u_\ell \le u \ \text{ a.e. in } \Omega_\ell. \tag{1.54}$$

Passing to the limit as $\ell \to \infty$, we derive from Theorem 1.3 that

$$\tilde{u} \le u \text{ a.e. in } \mathbb{R} \times \omega.$$

This means that $\tilde{u}$ is the minimal nonnegative solution of (1.37). For the proof of the maximum principle, we denote by u_ℓ^1 and u_ℓ^2 the converging sequences defined above as solutions to (1.41) for f_1 and f_2 respectively. Since $f_1 \le f_2$ we have $u_\ell^1 \le u_\ell^2$. Passing to the limit, we deduce that $u_1 \le u_2$ and this completes the proof of Theorem 1.4. □

Remark 1.5. *The maximum principle showed in the above theorem can be extended as follows. Let u_1 be the nonnegative minimal solution to (1.37) for some $f = f_1$ satisfying (1.39), u_2 be an arbitrary nonnegative solution to (1.37) for some $f = f_2$ that does not necessary satisfy (1.39). If we assume that $f_1 \le f_2$, then $u_1 \le u_2$. Indeed, taking $v = u_\ell^1 - \left(u_\ell^1 - u_2\right)^+ \in \mathcal{K}_\ell$ (resp. $v = u_2 + \left(u_\ell^1 - u_2\right)^+ \in \mathcal{K}$) as a test function in (1.41) for $f = f_1$ (resp. in (1.37) for $f = f_2$), choosing in (1.37) $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi = 1$ on $(-\ell, \ell)$ and integrating on $(-\ell, \ell)$. Then summing the produced inequalities we derive, as supp$\left(u_\ell^1 - u_2\right)^+ \subset \Omega_\ell$,*

$$\int_{\Omega_\ell} \left|\partial_y u_\ell^1\right|^{p-2} \partial_y u_\ell^1 \partial_y \left(u_\ell^1 - u_2\right)^+ dxdy + \int_{-\ell}^{\ell} \left\langle A u_\ell^1 - A u_2, \left(u_\ell^1 - u_2\right)^+ \right\rangle_{\ell,\omega} dy$$
$$\le \int_{\Omega_\ell} \left(f_1(x) - f_2(x)\right) \left(u_\ell^1 - u_2\right)^+ dxdy.$$

This implies, as u_2 is independent of y,

$$\int_{\Omega_\ell} \left(\left|\partial_y u_\ell^1\right|^{p-2} \partial_y u_\ell^1 - |\partial_y u_2|^{p-2} \partial_y u_2 \right) \partial_y \left(u_\ell^1 - u_2\right)^+ dxdy \le 0,$$

we also used the monotonicity condition (1.3). Thus

$$u_1 = \lim_{\ell \to \infty} u_\ell^1 \le u_2.$$

Consider now the following nonlinear elliptic problem defined on the infinite cylinder $\mathbb{R}\times\omega$

$$\begin{cases} u \in W_{loc}^{1,p}(\mathbb{R}\times\overline{\omega}), \ u = 0 \text{ on } \mathbb{R}\times\partial\omega, \\ Au = f \text{ in } \mathbb{R}\times\omega. \end{cases} \tag{1.55}$$

The solutions is understood in the following sense

$$\int_{\mathbb{R}\times\omega} a(x, u, \nabla u) \cdot \nabla \phi dx + \int_{\mathbb{R}\times\omega} a_0(x, u, \nabla u) \phi dx = \int_{\mathbb{R}\times\omega} f \phi dx \ \ \forall \phi \in \mathcal{D}(\mathbb{R} \times \omega). \tag{1.56}$$

We take $\mathcal{K}_\omega = W_0^{1,p}(\omega)$, then any solution of problem (1.37) for $\mathcal{K}_\omega = W_0^{1,p}(\omega)$ is a solution of (1.55) and vice versa. Thus the existence of nonnegative solutions of problem (1.55) is proved in Theorem 1.3. Indeed, to see this let $u \in \mathcal{K}$ be a solution of (1.37). By choosing in (1.37) $v = u \pm \phi$ with $\phi \in \mathcal{D}(\mathbb{R}\times\omega)$ and $\varphi = 1$ on the support of ϕ, we end up with the above weak formulation of problem (1.55). The converse is an immediate consequence of (1.56). Thus, we have the following result as an immediate consequence of Theorem 1.4.

Corollary 1.2. *Under the assumptions of Theorem 1.4, there exists a minimal nonnegative solution to (1.55), i.e.* $\tilde{u} = \min\{u \text{ solution to (1.55)}, u \geq 0\}$ *is a solution to (1.55). Moreover, if* u_1 *and* u_2 *are the minimal nonnegative solutions to (1.55) obtained by replacing* f *with* f_1 *and* f_2 *respectively. Then if* $f_1 \leq f_2$*, we have* $u_1 \leq u_2$.

1.3.2 Noncoercive variational inequalities

We keep the notation and the assumptions of the subsection 1.3.1. Then we consider the following nonlinear variational inequality defined on the infinite cylinder $\mathbb{R}\times\omega$

$$\begin{cases} u \in \mathcal{K}, \\ \displaystyle\int_{\mathbb{R}\times\omega} a(x,u,\nabla u)\cdot\nabla\left(\varphi\left(v-u\right)\right)dx + \int_{\mathbb{R}\times\omega} a_0(x,u,\nabla u)\varphi\left(v-u\right)dx \\ \qquad \displaystyle\geq \int_{\mathbb{R}\times\omega} F\left(x,u\right)\varphi\left(v-u\right)dx \quad \forall v\in\mathcal{K},\ \forall\varphi\in\mathcal{D}\left(\mathbb{R}\right),\ \varphi\geq 0, \end{cases} \tag{1.57}$$

where the function F is defined as in Section 1.2, replacing Ω by $\mathbb{R}\times\omega$. In addition, we assume that

$$h\left(X_2,r\right) := \sup_{x_1\in\mathbb{R}} F\left(x_1,X_2,r\right),$$

satisfies (1.22), i.e.

$$h\left(X_2,u\right)\in L^q\left(\omega\right),\quad \forall u\in L^{p^*}\left(\omega\right). \tag{1.58}$$

Of course, here also, we do not have any proof neither on the existence of nonnegative solutions of problem (1.57) nor on their minimal solution. We claim that we can define sequences $(\hat{u}_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ such that $\hat{u}_0 = u_0 = 0$ and for $n \geq 1$, $\hat{u}_n \in \mathcal{K}_\omega$, $u_n \in \mathcal{K}$ are respectively the minimal nonnegative solutions to

$$\langle A_\omega\hat{u}_n, v-\hat{u}_n\rangle_\omega \geq \int_\omega h(X_2,\hat{u}_{n-1})\left(v-\hat{u}_n\right)dX_2 \quad \forall v\in\mathcal{K}_\omega \tag{1.59}$$

and

$$\int_{\mathbb{R}\times\omega} a(x,u_n,\nabla u_n)\cdot\nabla\left(\varphi\left(v-u_n\right)\right)dx + \int_{\mathbb{R}\times\omega} a_0(x,u_n,\nabla u_n)\varphi\left(v-u_n\right)dx$$
$$\geq \int_{\mathbb{R}\times\omega} F\left(x,u_{n-1}\right)\varphi\left(v-u_n\right)dx \quad \forall v\in\mathcal{K},\ \forall\varphi\in\mathcal{D}\left(\mathbb{R}\right),\ \varphi\geq 0. \tag{1.60}$$

The existence of the minimal solution $\hat{u}_n \in \mathcal{K}_\omega$ (resp. $u_n \in \mathcal{K}$), to (1.59) (resp. to (1.60)) will be ensured by Theorem 1.1 (resp. Theorem 1.4). More precisely we have:

Lemma 1.5. *Let F and h be nonnegative functions satisfy the above hypotheses. Then under the assumptions (1.2)-(1.6) and (1.36), the sequences $(\hat{u}_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ are well defined and nondecreasing sequences satisfying*

$$u_n \leq \hat{u}_n, \quad F(x, u_n) \leq h(X_2, \hat{u}_n) \quad \forall n \in \mathbb{N}, \text{ a.e. on } \mathbb{R}\times\omega. \tag{1.61}$$

Proof. Of course u_0, $\hat{u}_0$ satisfy clearly (1.61). Suppose that u_{n-1}, $\hat{u}_{n-1}$ are defined and satisfy (1.61). One has, by (1.58), $h(X_2, \hat{u}_{n-1}) \in L^q(\omega)$ and $\hat{u}_n$ exists by Theorem 1.1. Moreover by (1.61) written for $n-1$ one has

$$F(x, u_{n-1}) \leq h(X_2, \hat{u}_{n-1}) \in L^q(\omega)$$

and by (1.39) the existence of u_n follows. Arguing as in Theorem 1.3 (step 1) one deduces easily that $u_n \leq \hat{u}_n$ and (1.61) follows. The monotonicity follows by induction from our previous comparison principles. □

Now setting $h_\infty = \lim_{n\to\infty} h(\cdot, \hat{u}_n)$, $F_\infty = \lim_{n\to\infty} F(\cdot, u_n)$ and we assume that

$$h_\infty \in L^q(\omega). \tag{1.62}$$

Then we have

Theorem 1.5. *Let F be a nonnegative function satisfy the above hypotheses. Then, under the assumptions (1.2)-(1.6), (1.36) and (1.62), $(u_n)_{n\in\mathbb{N}}$ is a nondecreasing and converging sequence towards $\tilde{u} \in \mathcal{K}$, as n goes to ∞, a minimal nonnegative solution to (1.57).*

Proof. Since F is nondecreasing we have $F(\cdot, u_{n-1}) \leq F_\infty$, $\forall n \geq 1$, a.e. in $\mathbb{R}\times\omega$ and it follows easily, by using the maximum principle showed in Theorem 1.4, that

$$u_n \leq \tilde{u}_\infty, \tag{1.63}$$

where $\tilde{u}_\infty \in \mathcal{K}$ is the minimal solution to (1.37) with $f = F_\infty$. Note that $F_\infty \leq h_\infty$ a.e. in $\mathbb{R}\times\omega$ and h_∞ is independent of x_1 which insures (1.39). Using Lemma 2.2 we establish that u_n possesses a pointwise limit that we will denote by $\tilde{u} \in L^p_{loc}(\mathbb{R}\times\overline{\omega})$, i.e. $u_n \to \tilde{u}$ on $\mathbb{R}\times\omega$. Then, in order to get more estimates, we test (1.60) by $v = u_n - \varphi^p(u_n - \tilde{u}_\infty)$ and choose φ such that $\varphi = 1$ on $(-\ell_0, \ell_0)$, we get, using the growth conditions (1.2) and (1.6)

$$\begin{aligned}
&\int_{\mathbb{R}\times\omega} \varphi^p a(x, u_n, \nabla u_n)\cdot\nabla u_n dx + \int_{\mathbb{R}\times\omega} \varphi^p a_0(x, u_n, \nabla u_n) u_n dx \\
&\leq \int_{\mathbb{R}\times\omega} F(x, u_{n-1})\varphi^p(u_n - \tilde{u}_\infty)\,dx \int_{\mathbb{R}\times\omega} a(x, u_n, \nabla u_n)\nabla(\varphi^p \tilde{u}_\infty)\,dx + \int_{\mathbb{R}\times\omega} a_0(x, u_n, \nabla u_n)\varphi^p \tilde{u}_\infty dx \\
&\qquad - p\int_{\mathbb{R}\times\omega} \partial_{x_1}\varphi\varphi^{p-1} a_1(x, u_n, \nabla u_n) u_n dx \\
&\leq C\int_{\mathbb{R}\times\omega} \varphi^p\left(|u_n|^p + |\tilde{u}_\infty|^p + |\nabla\tilde{u}_\infty|^p + 1\right)dx + C\int_{\mathbb{R}\times\omega} |\partial_{x_1}\varphi|^p\left(|u_n|^p + |\tilde{u}_\infty|^p\right)dx \\
&\qquad + \frac{\alpha}{2}\int_{\mathbb{R}\times\omega} \varphi^p|\nabla u_n|^p dx,
\end{aligned}$$

for some positive constant C independent of n. Finally, thanks to the coerciveness assumption and (1.63), we end up with

$$|u_n|_{1,p,(-\ell_0,\ell_0)\times\omega} \leq C(\ell_0). \tag{1.65}$$

Since ℓ_0 is arbitrary we deduce as, $n \to \infty$, that

$$u_n \to \tilde{u} \text{ in } L^p_{loc}\left(\mathbb{R}{\times}\overline{\omega}\right),\ u_n \rightharpoonup \tilde{u} \text{ in } W^{1,p}_{loc}\left(\mathbb{R}{\times}\overline{\omega}\right),\ \tilde{u} \in \mathcal{K}. \tag{1.66}$$

Now, by following the same arguments as in the proof of Theorem 1.3 replacing u_ℓ by u_n, it is easy to show that $\tilde{u}$ is a nonnegative solution to (1.57). We also claim that $\tilde{u}$ is the minimal solution. Indeed, let u be a nonnegative solution to (1.57). Here we can not use the maximum principle mentioned in Theorem 1.4, since for $f = F(\cdot, u)$ the hypothesis (1.39) is not surely insured. By consequence, we are not sure that for $f = F(\cdot, u)$ the problem (1.37) has a minimal solution. Here Remark 1.5 will do the trick. Applying iteratively the maximum principle, showed in Remark 1.5, it follows that $u_n \leq u$. Then letting $n \to \infty$, we end up with our minimal solution. This completes the proof of Theorem 2.1. □

Remark 1.6. *Let $F_1,\ F_2 : \mathbb{R} \times \omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions and monotone nondecreasing in the last variable. Assume that F_1 is nonnegative and satisfy the assumptions (1.58), (1.62) and F_2 satisfies the following coerciveness and growth conditions: for all $r \in \mathbb{R}$ and for a.e. x in $\mathbb{R} \times \omega$, there exists a nonnegative constant β' such that*

$$\begin{aligned} &F_2\left(x, r\right) r \geq 0, \\ &\left|F_2\left(x, r\right)\right| \leq \vartheta\left(x\right) + \beta' \left|r\right|^{p-1} \ \ \textit{with } \vartheta \in L^q_{loc}(\mathbb{R}{\times}\overline{\omega}). \end{aligned}$$

Then Theorem 2.1 remains true for $F = F_1 - F_2$, since the problem (1.57) is equivalent to

$$\begin{cases} u \in \mathcal{K}, \\ \displaystyle\int_{\mathbb{R}\times\omega} a(x, u, \nabla u) \cdot \nabla\left(\varphi\left(v - u\right)\right) dx + \int_{\mathbb{R}\times\omega} \left(a_0(x, u, \nabla u) + F_2\left(x, u\right)\right) \varphi\left(v - u\right) dx \\ \qquad\qquad \displaystyle\geq \int_{\mathbb{R}\times\omega} F_1\left(x, u\right) \varphi\left(v - u\right) dx \quad \forall v \in \mathcal{K},\ \forall \varphi \in \mathcal{D}\left(\mathbb{R}\right),\ \varphi \geq 0. \end{cases}$$

Chapter 2

Long time behaviour of parabolic equations in time-dependent growing domains

In this chapter, we study the large time behaviour of the solution to parabolic problems defined on noncylindrical domains becoming unbounded in many directions when t tends to infinity. That is to say, the state variable domains are also becoming unbounded when $t \to \infty$. Since the steady state problem is elliptic and defined on unbounded domains, we need to define in which sense the solution is understood and to deal with its existence. The convergence and its rate are also investigated with respect to the growth rate of the domain when $t \to \infty$. As the convergence cannot be expected on the whole domain, correctors are built to describe the asymptotic behaviour in the distant regions.

2.1 Setting the problem

Let $\hat{Q}$ be an unbounded open subset of $\mathbb{R}_+ \times \mathbb{R}^m$ and ω be an open bounded subset of $\mathbb{R}^{n-m}$, $1 \leq m < n$. Consider Q the domain of $\mathbb{R}^{n+1}$ given by

$$Q = \hat{Q} \times \omega,$$

whose boundary is denoted by Γ. A generic point (s, x) in Q is denoted by (s, X_1, X_2) with

$$s > 0, \quad X_1 = (x_1, \ldots, x_m), \quad X_2 = (x_{m+1}, \ldots, x_n).$$

For $t > 0$, we set

$$\hat{Q}_t = \left\{(s, X_1) \in \hat{Q} \mid s \leq t\right\}, \quad Q_t = \{(s, x) \in Q \mid s \leq t\},$$
$$\hat{\Omega}_t = \left\{X_1 \in \mathbb{R}^m \mid (t, X_1) \in \hat{Q}\right\}, \quad \Omega_t = Q \cap \{s = t\},$$
$$\hat{\Omega}_0 = int\left\{X_1 \in \mathbb{R}^m \mid (0, X_1) \in \overline{\hat{Q}}\right\}, \quad \Omega_0 = int\left(\overline{Q} \cap \{s = 0\}\right),$$
$$\Gamma_t = (\Gamma \setminus \Omega_0) \cap \overline{Q}_t.$$

Note that when it is needed we can also set $\Omega_t = \hat{\Omega}_t \times \omega$, i.e. we omit the component t. We assume that the boundary Γ is smooth enough and the sequence of nonempty sets $\hat{\Omega}_t$ is non-decreasing and becoming unbounded when $t \to \infty$ in the following sense

$$\emptyset \not\equiv \hat{\Omega}_0 \subset\subset \hat{\Omega}_{t_1} \subset\subset \hat{\Omega}_{t_2} \quad \text{for } 0 < t_1 < t_2 \text{ and } \lim_{t\to\infty} dist\left(\hat{\Omega}_0, \partial\hat{\Omega}_t\right) = \infty. \tag{2.1}$$

We are interested here in the study of the asymptotic behaviour of the following nonhomogeneous parabolic initial-boundary value problem

$$\begin{cases} u' - \operatorname{div}(A\nabla u) = f & \text{in } Q_t, \\ u = 0 & \text{on } \Gamma_t, \\ u(0,\cdot) = u_0 & \text{in } \Omega_0, \end{cases} \tag{2.2}$$

where $f \in L^2_{loc}\left(\mathbb{R}^m; L^2(\omega)\right)$, $u_0 \in L^2(\Omega_0)$ and $A = (a_{ij})_{i,j=1,\ldots,n}$ is a $n \times n - matrix$ in $L^\infty(\mathbb{R}^m \times \omega)$ satisfying

$$\beta_1 |\zeta|^2 \leq A\zeta \cdot \zeta \leq \beta_2 |\zeta|^2 \quad \forall \zeta \in \mathbb{R}^n,\ a.e.\ x \in \mathbb{R}^m \times \omega, \tag{2.3}$$

for some positive constants β_1 and β_2. The solution of such problem is unique (see [54]) and it is understood in the following weak sense: find a solution $u \in \mathcal{F}(Q_t)$ such that

$$< u', v >_{V_t} + \int_0^t \int_{\Omega_s} A\nabla u \cdot \nabla v dxds = \int_0^t \int_{\Omega_s} fvdxds \quad \forall v \in \mathcal{F}(Q_t), \tag{2.4}$$

where the corresponding evolution space is defined as

$$\mathcal{F}(Q_t) = \left\{u \in V_t;\ u' \in V_t'\right\} \text{ with } V_t = \left\{v \in L^2(Q_t) \mid v(s,.) \in H^1_0(\Omega_s),\ \forall s \leq t\right\},$$

V_t' is the dual of V_t, i.e. $v \in V_t'$ iff $v(s,.) \in H^{-1}(\Omega_s),\ \forall s \leq t$ and $|v(s,.)|_{-1,2,\Omega_s} \in L^2(0,t)$. In what follows, we write $< .,. >$ instead of $< .,. >_{V_t}$.

2.2 Limit problem

Since here there are no cylindrical symmetries assumptions as in [44] the natural limit candidate may also depends on X_1. When we consider t and X_1 go to ∞ we aim to check that u behaves as u_∞ solution to the following elliptic problem defined on the strip $\mathbb{R}^m \times \omega$ by

$$\begin{cases} -\operatorname{div}(A\nabla u_\infty) = f & \text{in } \mathbb{R}^m \times \omega, \\ u_\infty = 0 & \text{on } \mathbb{R}^m \times \partial\omega. \end{cases} \tag{2.5}$$

The domain, on which Problem (2.5) is defined, is not bounded and there is no summability of the data, i.e. the source term f does not necessary belong to $L^2(\mathbb{R}^m \times \omega)$. This prevents to apply, for example, a classical tool as the Lax-Milgram theorem that ensures the existence and the uniqueness of the solution. In fact, we have to state clearly in which sense the solution is intended according to the data of the main problem (2.2). For example, problems as (2.5) may have more than one solution (see [17]). Let us first take care of the problem (2.5). Consider the following family of elliptic problems defined on bounded domains Ω_t. So, for $t > 0$ we consider $u^t \in H^1_0(\Omega_t)$ the unique solution to

$$\int_{\Omega_t} A\nabla u^t \cdot \nabla v dx = \int_{\Omega_t} fvdx \quad \forall v \in H^1_0(\Omega_t). \tag{2.6}$$

This family consists in parametrized problems defined on cylindrical domains becoming unbounded. This theory is extensively investigated by Chipot and his collaborators, in particular when some cylindrical symmetries are assumed (see [7, 15, 18, 22, 27, 35, 42, 43, 46] and the references therein). In [17, 23], the same issue is handled for the Stokes and some quasilinear model problems and the same argument is workable here with a slight difference. In the following we show that, under suitable assumptions, u^t converges to a unique solution to (2.5) in some sense that will be defined below and we give a brief sketch about the argument used in [17, 23].

Theorem 2.1. *Let B be an arbitrary bounded set in $\mathbb{R}^m$ and define for t large enough*

$$d_t = dist\left(B, \partial\hat{\Omega}_t\right).$$

We assume that (2.3) and (2.1) are satisfied and that

$$f \in L^2_{loc}\left(\mathbb{R}^m; L^2(\omega)\right) \text{ with } \sum_{k=0}^{\infty} e^{-\gamma d_k} |f|_{2,\Omega_{k+2}} < \infty, \tag{2.7}$$

for some $\gamma > 0$ small enough. Then u^t converges, in $H^1(B \times \omega)$, towards u_∞ solution to

$$\begin{cases} u_\infty \in V, \\ -\operatorname{div}(A\nabla u_\infty) = f \quad \text{in } \mathbb{R}^m \times \omega, \end{cases} \tag{2.8}$$

where

$$V = \left\{u \in L^2_{loc}\left(\mathbb{R}^m; L^2(\omega)\right) \mid \nabla u \in L^2_{loc}\left(\mathbb{R}^m; L^2(\omega)\right),\ u_\infty = 0 \quad \text{on } \mathbb{R}^m \times \partial\omega\right\}.$$

Remark 2.1. *The hypothesis (2.7) is independent of the choice of B. Indeed, let B_1, B_2 be two bounded sets in $\mathbb{R}^m$ and set, for t large enough,*

$$d^1_t = dist\left(B_1, \partial\hat{\Omega}_t\right), \quad d^2_t = dist\left(B_2, \partial\hat{\Omega}_t\right).$$

We can easily show that

$$d^2_t - \overline{dist}(B_2, B_1) \le d^1_t \le d^2_t + \overline{dist}(B_1, B_2),$$

where

$$\overline{dist}(B_1, B_2) = \sup_{X_1 \in B_2} dist(B_1, X_1).$$

This implies that the two series involved in the hypothesis (2.7) and written for d^1_t and d^2_t are equivalent.

Remark 2.2. *The hypothesis (2.7) is ensured if we assume that*

$$\int_1^{\infty} e^{-\gamma d_{t-1}} |f|_{2,\Omega_{t+2}}\, dt < \infty.$$

Proof. We denote by

$$\bar{B}^t_k = \left\{X_1 \in \hat{\Omega}_t \mid dist(X_1, B) \le k\right\},\ k \in \mathbb{N}.$$

Let k be a positive integer with $k \le [d_t] - 1$ ([.] denotes the integer part) and $\rho_k : \mathbb{R}^m \to \mathbb{R}$ be a family of smooth cut-off functions such that

$$\rho_k = 1 \text{ on } \bar{B}^t_k, \quad \rho_k = 0 \text{ on } \mathbb{R}^m \backslash \bar{B}^t_{k+1},\ 0 \le \rho_k \le 1 \text{ and } |\nabla_{X_1}\rho_k| \le C,$$

where C is a constant independent of $t,\ k$ and X_1. We can show that $(u^t)_{t>0}$ is a Cauchy sequence in $H^1(B\times\omega)$ by taking $v=\rho_k^2\left(u^{t+\alpha}-u^t\right)$ as a test function in (2.6) written for t and $t+\alpha$ with $\alpha>0$ and then arguing as in [17,23] using an iteration technique (for $\alpha\leq 1$ and then for α arbitrary). That is to say, for some positive constants C and $\tilde{\gamma}$ independent of t and α, we get for $\alpha\leq 1$

$$\left|\nabla\left(u^{t+\alpha}-u^t\right)\right|_{2,B\times\omega}\leq Ce^{-\tilde{\gamma}d_t}\left|f\right|_{2,\Omega_{t+\alpha}}$$

and for α arbitrary, we have

$$\left|\nabla\left(u^{t+\alpha}-u^t\right)\right|_{2,B\times\omega}\leq C\sum_{k=0}^{[\alpha]}e^{-\tilde{\gamma}d_{t+k}}\left|f\right|_{2,\Omega_{t+k+1}}\leq C\sum_{k=[t]}^{\infty}e^{-\tilde{\gamma}d_k}\left|f\right|_{2,\Omega_{k+2}}.$$

Choosing $\gamma<\tilde{\gamma}$, it follows that $\left(u^t\right)_{t>0}$ is a Cauchy sequence and converges to some function u_∞ in $H^1(B\times\omega)$. In fact, u_∞ is independent of B since it is arbitrary in $\mathbb{R}^m$. Taking, for t large enough, $v\in H_0^1(B\times\omega)$ in (2.6) and passing to the limit we infer that u_∞ is a solution of (2.8). $\square$

Remark 2.3. *If $f\in L^2_{loc}\left(\mathbb{R}^m;L^2(\omega)\right)$ such that*

$$\left|f\right|_{2,\Omega_{k+2}}=O\left(d_k^{\ell_1}\right)\ \textit{(or more general }\left|f\right|_{2,\Omega_{k+2}}=O\left(e^{\ell_2 d_k}\right)\textit{)},\tag{2.9}$$

for some $\ell_1>0$ and $0<\ell_2<\tilde{\gamma}$, then $\left(u^t\right)_{t>0}$ converges towards the unique solution to

$$\begin{cases} u_\infty\in V,\\ -\operatorname{div}\left(A\nabla u_\infty\right)=f\quad \textit{in } \mathbb{R}^m\times\omega,\ \left|\nabla u_\infty\right|_{2,\Omega_t}=O\left(d_t^{\ell_1}\right)\ \textit{(or } O\left(e^{\ell_2 d_t}\right)\textit{)}.\end{cases}\tag{2.10}$$

Moreover, it holds

$$\left|\nabla\left(u^t-u_\infty\right)\right|_{2,B\times\omega}\leq Ce^{-\theta d_t},$$

for some positive constants C and θ. (See [17] for the uniqueness).

Remark 2.4. *In fact, it would have been easier if we had assumed that Ω_t was the ball centred in the origin with radius t and $d_t=t-c$ where c is some positive constant. This, of course, weakens the hypothesis (2.7). Nevertheless, the above study investigates the asymptotic behaviour of the sequence $(u^t)_{t>0}$ defined on arbitrary domains Ω_t and the constraint in (2.10) is more general.*

Remark 2.5. *It is also possible to assume that f belongs to the following space*

$$\tilde{V}=\left\{v\in\mathcal{D}'\left(\mathbb{R}^m\times\omega\right)\mid v\in H^{-1}\left(\Omega_t\right)\ \forall t>0\right\}.$$

In this case the existence of $u_\infty\in V$ solution to (2.10) is strongly related to hypotheses of type (2.9), replacing the $L^2(\Omega_{k+2})$ norm by the norm of $H^{-1}(\Omega_{k+2})$. It is clear from the above theorem that these type of assumptions are sufficient conditions for existence (replacing the L^2-data by the H^{-1} ones). Now if u_∞ is a solution to (2.10), we have for $v\in H_0^1(\Omega_t)$

$$\begin{aligned}\langle f,v\rangle_{\Omega_t}&=\int_{\Omega_t}A\nabla u_\infty\cdot\nabla v\,dx\\&\leq\beta_2\left|\nabla u_\infty\right|_{2,\Omega_t}\left|\nabla v\right|_{2,\Omega_t}\\&\leq Cd_t^{\ell_1}\left|\nabla v\right|_{2,\Omega_t}\quad\textit{(or } Ce^{\ell_2 d_t}\left|\nabla v\right|_{2,\Omega_t}\textit{)}.\end{aligned}$$

This implies that $\left|f\right|_{-1,2,\Omega_t}=O\left(d_t^{\ell_1}\right)$ (or $O\left(e^{\ell_2 d_t}\right)$).

In the rest of the paper we need the weak maximum principle for Problem (2.8) which is an immediate consequence of the above limit.

Lemma 2.1. *If the function* $f \in L^2_{loc}\left(\mathbb{R}^m; L^2(\omega)\right)$ *is nonnegative and satisfies (2.9), the solution* u_∞ *of (2.10) is also nonnegative. Moreover, if we choose two functions* $f_1, f_2 \in L^2_{loc}\left(\mathbb{R}^m; L^2(\omega)\right)$ *such that*

$$f_1 \leq f_2 \quad \text{a.e. on } \mathbb{R}^m \times \omega$$

and we also assume that the difference $u^1_\infty - u^2_\infty$ *satisfies the constraint in (2.10) where* $u^1_\infty, u^2_\infty \in V$ *are solutions to (2.8) replacing* f *by* f_1, f_2 *respectively, then*

$$u^1_\infty \leq u^2_\infty \quad \text{a.e. on } \mathbb{R}^m \times \omega. \tag{2.11}$$

Proof. The first point is immediate since u_∞ is a limit of the positive sequence $(u^t)_{t>0}$ solution to (2.6) defined on a bounded domain where we can apply the weak maximum principle (see for instance [16]). The proof of (2.11) is based on the previous point and the fact that $u^2_\infty - u^1_\infty$ is a solution to (2.10) for $f = f_2 - f_1$. □

Remark 2.6. *The hypothesis (2.9), for* ℓ_2 *arbitrary, does not ensure neither the uniqueness nor the weak maximum principle for the problem (2.8). Indeed, the solution* $u_\infty = e^{2\pi x_1}\sin(2\pi x_2) \in V$ *of the following problem, introduced in [17],*

$$\begin{cases} -\Delta u_\infty = 0 & \text{in } \mathbb{R} \times (0,1), \\ u_\infty = 0 & \text{on } \mathbb{R} \times \{0,1\}, \end{cases}$$

is not unique since θu_∞ *is also a solution for all* $\theta \in \mathbb{R}$ *and it changes the sign on* $\mathbb{R} \times (0,1)$*. Note that the above solution satisfies the constraint in (2.10) but not for* $\ell_2 < \tilde{\gamma} = 1$*. In order to distinguish it from the other solutions, the limit of* $\left(u^t\right)_{t>0}$ *will be called the* limit solution *of (2.5).*

2.3 Interior asymptotic behaviour

This section is devoted to the study of the asymptotic behaviour of the solution u to (2.2) when $t \to \infty$. This allows to take into account the limit of the solution when X_1 also goes to ∞ simultaneously with $t \to \infty$, since the domains Ω_s become unbounded.

2.3.1 Limit behaviour

Here we will focus the study on the convergence far away from the boundary layer $\partial\hat{\Omega}_t \times \omega$. As it will be clarified later, we need local estimates to avoid the unimportant regions located near the boundary layer, since the required convergences are not affected by these far away regions. In fact, the measure of these regions may become very large when $t \to \infty$ and hence we lose the convenient estimates. For example if $\hat{\Omega}_t = \left(-t, e^{rt}\right)$ we have to avoid estimates on intervals as $\left(ct, e^{rt}\right)$ where $c, r > 0$ and r is large enough. In general to do this, we apply the weak maximum principle to the equation obtained by comparing (2.2) and (2.8). Here we may not run the above idea since we cannot ensure that $u_\infty - u_0$ keep the same sign in Ω_0 even if u_∞, u_0 do. However, a closely useful approach may serve the purpose as what we will see in the next lemma.

Lemma 2.2. *Under the assumptions (2.3), (2.1) and (2.7). If we suppose that u_0 and f are nonnegative (resp. nonpositive), then u, the weak solution of (2.4), is nonnegative (resp. nonpositive). However, if we suppose that f and u_0 do not necessary keep the same sign and set*

$$M^- = \{u \geq u_{\infty,-}\}^c, \qquad M^+ = \{u \leq u_{\infty,+}\}^c, \qquad M_t^{\pm} = M^{\pm} \cap \Omega_t.$$

Then we have for every $t > 0$

$$|u(t,.)|_{2,M_t^{\pm}} \leq |u_{\infty,\pm}|_{2,M_t^{\pm}} + \left|(u_0 - u_{\infty,\pm})^{\pm}\right|_{2,\Omega_0}, \tag{2.12}$$

$$\int_0^t |\nabla u|^2_{2,M_s^{\pm}} ds \leq 2\int_0^t |\nabla u_{\infty,\pm}|^2_{2,M_s^{\pm}} ds + \frac{1}{\beta_1}\left|(u_0 - u_{\infty,\pm})^{\pm}\right|^2_{2,\Omega_0}, \tag{2.13}$$

where $u_{\infty,+}$ (resp. $u_{\infty,-}$) is the limit solution of (2.5), obtained by replacing f by f^+ (resp. by $-f^-$). In addition, for an arbitrary domain $\mathcal{O} \subset \mathbb{R}^m \times \omega$, there exists a positive constant C' independent of t such that

$$|u(t,.)|_{2,\mathcal{O}\cap\Omega_t} \leq C'\left(1 + |u_\infty|_{2,\mathcal{O}}\right).$$

Proof. Suppose firstly that f and u_0 are nonnegative (resp. nonpositive) almost everywhere. Then the positivity (resp. the negativity) of u is obtained by testing (2.4) with u^- (resp. u^+) and showing that it vanishes. Now, let $u_{\infty,+}$ and $u_{\infty,-}$ be the solutions of (2.5) replacing f by f^+ and $-f^-$ respectively, i.e.

$$\begin{cases} -\operatorname{div}(A\nabla u_{\infty,\pm}) = \pm f^{\pm} & \text{in } \mathbb{R}^m \times \omega, \\ u_{\infty,\pm} = 0 & \text{on } \mathbb{R}^m \times \partial\omega. \end{cases} \tag{2.14}$$

Multiplying (2.14) by $v \in \mathcal{F}(Q_t)$ and integrating on Q_t, we get

$$\int_0^t \int_{\Omega_s} A\nabla u_{\infty,\pm} \cdot \nabla v dx ds = \int_0^t \int_{\Omega_s} \left(\pm f^{\pm}\right) v dx ds.$$

Comparing this with (2.4) we deduce

$$< (u - u_{\infty,\pm})', v > + \int_0^t \int_{\Omega_s} A\nabla(u - u_{\infty,\pm}) \cdot \nabla v dx ds = \int_0^t \int_{\Omega_s} \left(f - \left(\pm f^{\pm}\right)\right) v dx ds \quad \forall v \in \mathcal{F}(Q_t). \tag{2.15}$$

Since $f^{\pm} \geq 0$ it follows from Lemma 2.1 that $u_{\infty,+} \geq 0$ and $u_{\infty,-} \leq 0$. Hence we can test (2.15) with $(u - u_{\infty,\pm})^{\pm} \in \mathcal{F}(Q_t)$ to get

$$|(u - u_{\infty,\pm})(t,.)|^2_{2,M_t^{\pm}} + 2\beta_1 \int_0^t |\nabla(u - u_{\infty,\pm})(s)|^2_{2,M_s^{\pm}} ds \leq \left|(u_0 - u_{\infty,\pm})^{\pm}\right|^2_{2,\Omega_0}.$$

This leads to (2.12) and (2.13). Now for an arbitrary domain $\mathcal{O} \subset \mathbb{R}^m \times \omega$, we have

$$\begin{aligned} |u(t,.)|_{2,\mathcal{O}\cap\Omega_t} &= \left|\left(\chi_{M^-\cap\mathcal{O}} + \chi_{\{u_{\infty,-}\leq u\leq u_{\infty,+}\}\cap\mathcal{O}} + \chi_{M^+\cap\mathcal{O}}\right) u(t,.)\right|_{2,\Omega_t} \\ &\leq |u(t,.)|_{2,M_t^-\cap\mathcal{O}} + |u(t,.)|_{2,\{u_{\infty,-}\leq u\leq u_{\infty,+}\}\cap\mathcal{O}\cap\Omega_t} + |u(t,.)|_{2,M_t^+\cap\mathcal{O}} \\ &\leq |u(t,.) - u_{\infty,-}|_{2,M_t^-\cap\mathcal{O}} + |u(t,.) - u_{\infty,+}|_{2,M_t^+\cap\mathcal{O}} + |u_{\infty,-}|_{2,M_t^-\cap\mathcal{O}} \\ &+ |u_{\infty,+}|_{2,M_t^+\cap\mathcal{O}} + |u_{\infty,+} - u_{\infty,-}|_{2,\{u_{\infty,-}\leq u\leq u_{\infty,+}\}\cap\mathcal{O}\cap\Omega_t} \\ &\leq \left|(u_0 - u_{\infty,-})^-\right|_{2,\Omega_0} + \left|(u_0 - u_{\infty,+})^+\right|_{2,\Omega_0} + 2|u_\infty|_{2,\mathcal{O}} \\ &\leq C'\left(1 + |u_\infty|_{2,\mathcal{O}}\right), \end{aligned}$$

for some positive constant C' independent of t and $\mathcal{O}$. This concludes the proof. □

Now, in order to study the asymptotic behaviour of the solution u to (2.2) we define a family of cut-off functions, consistent with the stability of evolution problems (we take into account the time as a coordinate not as a parameter as in Section 2.2), that we will use to apply some iteration technique. Let us start by defining the following subsets that play an essential role to run the iterative argument. Let $\hat{\Omega}$ be a bounded domain in $\mathbb{R}^m$ that may depend on t and such that its size is not necessary bounded when $t \to \infty$. Of course, as we are interested in the local limit behaviour when $t \to \infty$ we can always assume, when the time is large, that

$$\vartheta = \vartheta_t := dist\left(\bar{\Gamma}_t, \{t\} \times \hat{\Omega}\right) \to \infty \quad \text{as} \quad t \to \infty,$$

where $\bar{\Gamma}_t = \partial \hat{Q}_t \diagdown \hat{\Omega}_t$. Note that we have $\vartheta \leq t$, since either $dist\left(\hat{\Omega}_0, \{t\} \times \hat{\Omega}\right) = t$, or a segment between each point in $\hat{\Omega}$ and its projection on the axis $t = 0$ crosses the boundary $\bar{\Gamma}_t$. If it is needed we can also assume that $\vartheta < dist\left(\bar{\Gamma}_t, \{t\} \times \hat{\Omega}\right)$ but it has to go to infinity when $t \to \infty$. Then, we set

$$S = S_0^t := \left\{(s, X_1) \in \hat{Q}_t \mid dist\left((s, X_1), \{t\} \times \hat{\Omega}\right) \leq \vartheta\right\}$$

and for $1 \leq \mu < [\vartheta]$ we define

$$S_\mu = S_\mu^t := \left\{(s, X_1) \in S_{\mu-1} \mid dist\left((s, X_1), \partial S_{\mu-1} \diagdown \hat{\Omega}_t\right) \geq 1\right\}.$$

The required cut-off function ϱ_μ is nonnegative, smooth and defined on $(0, t) \times \mathbb{R}^m$ as

$$0 \leq \varrho_\mu \leq 1, \;\; \varrho_\mu = 1 \text{ on } \mathcal{S}_{\mu+1}, \;\; \varrho_\mu = 0 \text{ outside } \mathcal{S}_\mu, \;\; \left|\partial_t \rho_\mu\right|, \left|\nabla_{X_1} \rho_\mu\right| \leq c_0,$$

for some positive constant c_0 independent of μ and t (see, e.g., Figure 2.1).

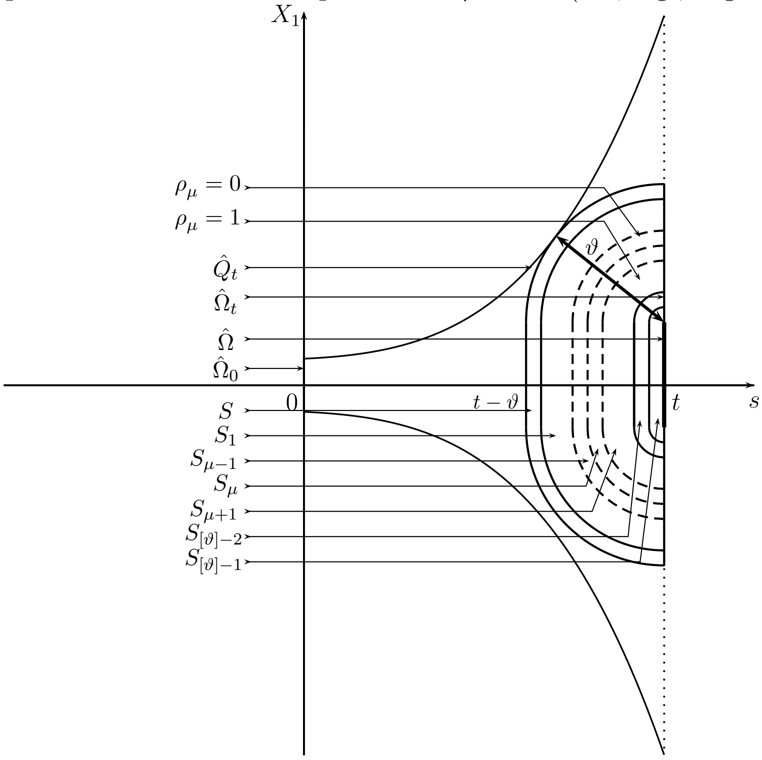

Figure 2.1: An example of the domain Q and the corresponding S.

Now we are able to state the following main result.

Theorem 2.2. *Under the assumptions (2.3), (2.1) and (2.7), for t large enough, there exist positive constants C and λ independent of t such that*

$$|(u-u_\infty)(t)|^2_{2,\hat{\Omega}\times\omega}, |\nabla(u-u_\infty)|^2_{2,(t-1,t)\times\hat{\Omega}\times\omega} \leq C\vartheta e^{-\lambda\vartheta}\left(1+|u_\infty|^2_{2,\mathcal{N}_\vartheta\times\omega}\right),$$

where $\mathcal{N}_\vartheta$ is the $\vartheta-$neighborhood of $\hat{\Omega}$ in $\mathbb{R}^m$, u is the weak solution of (2.2) and u_∞ is the limit solution of (2.5).

Proof. It is clear from the definition of the cut-off functions that $\rho^2_\mu(u-u_\infty) \in \mathcal{F}(Q_t)$ can be considered as a test function in (2.15) since $\rho_\mu = 0$ on Γ_t. Thus, it follows

$$\begin{aligned}&< \left(\rho_\mu(u-u_\infty)\right)', \rho_\mu(u-u_\infty) > + \int_0^t\int_{\Omega_s}\rho^2_\mu A\nabla(u-u_\infty)\cdot\nabla(u-u_\infty)\,dxds\\ &= \int_0^t\int_{\Omega_s}\partial_t\rho_\mu\rho_\mu(u-u_\infty)^2\,dxds - 2\int_0^t\int_{\Omega_s}\rho_\mu(u-u_\infty)A\nabla(u-u_\infty)\cdot\nabla\rho_\mu dxds. \qquad (2.16)\end{aligned}$$

Using the integration by parts formula, (2.3), the Cauchy-Schwarz and Young's inequalities we obtain

$$\begin{aligned}&\frac{1}{2}\left|\rho_\mu(u-u_\infty)(t)\right|^2_{2,\Omega_t} + \left(\beta_1-\frac{\beta_2}{\varepsilon}\right)\int_0^t\int_{\Omega_s}\rho^2_\mu\left|\nabla(u-u_\infty)\right|^2 dxds\\ &\qquad\leq \int_0^t\int_{\Omega_s}\left(\left|\partial_t\rho_\mu\right|\rho_\mu + \beta_2\varepsilon\left|\nabla_{X_1}\rho_\mu\right|^2\right)(u-u_\infty)^2\,dxds,\end{aligned}$$

for some positive constant $\varepsilon > \frac{\beta_2}{\beta_1}$. Note that we took into account the values of the cut-off functions on the initial boundary of Q_t, i.e. $\rho_\mu = 0$ on Ω_0. Since ρ_μ is constant inside $S_{\mu+1}$ and is vanishing outside S_μ we derive

$$|(u-u_\infty)(t)|^2_{2,\mathcal{N}_{\vartheta-\mu}} + |\nabla(u-u_\infty)|^2_{2,S_{\mu+1}\times\omega} \leq c_\varepsilon\int_{(S_\mu\backslash S_{\mu+1})\times\omega}(u-u_\infty)^2\,dxds, \qquad (2.17)$$

with $c_\varepsilon = \frac{c_0(1+c_0\beta_2\varepsilon)}{\min\left(\frac{1}{2},\left(\beta_1-\frac{\beta_2}{\varepsilon}\right)\right)}$. On the other hand, applying the Poincaré inequality we deduce in particular that

$$|\nabla(u-u_\infty)|^2_{2,S_{\mu+1}\times\omega} \leq c_\varepsilon c_\omega|\nabla(u-u_\infty)|^2_{2,S_\mu\times\omega} - c_\varepsilon c_\omega|\nabla(u-u_\infty)|^2_{2,S_{\mu+1}\times\omega}, \qquad (2.18)$$

which implies

$$|\nabla(u-u_\infty)|^2_{2,S_{\mu+1}\times\omega} \leq K_\varepsilon|\nabla(u-u_\infty)|^2_{2,S_\mu\times\omega},$$

where $K_\varepsilon = \frac{c_\omega c_\varepsilon}{1+c_\omega c_\varepsilon} < 1$. Iterating the above inequality for $\mu = 1, ..., [\vartheta]-2$, we obtain

$$\begin{aligned}|\nabla(u-u_\infty)|^2_{2,S_{[\vartheta]-1}\times\omega} &\leq K_\varepsilon|\nabla(u-u_\infty)|^2_{2,S_{[\vartheta]-2}\times\omega}\\ &\leq ...\\ &\leq (K_\varepsilon)^{[\vartheta]-2}|\nabla(u-u_\infty)|^2_{2,S_1\times\omega}\\ &\leq (K_\varepsilon)^{\vartheta-3}|\nabla(u-u_\infty)|^2_{2,S_1\times\omega}.\end{aligned}$$

Setting $\lambda_\varepsilon = -\ln K_\varepsilon > 0$ and using again (2.17), the above estimate can be rewritten as follows

$$|(u-u_\infty)(t)|^2_{2,\hat{\Omega}}, |\nabla(u-u_\infty)|^2_{2,S_{[\vartheta]-1}\times\omega} \leq \frac{c_\varepsilon}{(K_\varepsilon)^3}e^{-\lambda_\varepsilon\vartheta}|u-u_\infty|^2_{2,(S\backslash S_1)\times\omega}.$$

The last estimate explains what we meant above by local estimates ignoring the distant regions of Q_t, i.e. we need just to consider $\nu-$neighborhoods of the domain $\hat{\Omega}$ $(\nu < \vartheta)$ on which we dealt with the convergence and its rate. Now we apply Lemma 3.1

$$\begin{aligned}
&|(u-u_\infty)(t)|^2_{2,\hat{\Omega}}, |\nabla(u-u_\infty)|^2_{2,S_{[\vartheta]-1}\times\omega} \\
&\le \frac{2c_\varepsilon}{(K_\varepsilon)^3} e^{-\lambda_\varepsilon\vartheta}\left(|u|^2_{2,(S\setminus S_1)\times\omega} + |u_\infty|^2_{2,(S\setminus S_1)\times\omega}\right) \\
&\le \frac{2c_\varepsilon}{(K_\varepsilon)^3} e^{-\lambda_\varepsilon\vartheta}\left(\int_{t-\vartheta}^t |u|^2_{2,(\hat{\Omega}_s\cap\{s\}\times\mathcal{N}_\vartheta)\times\omega}\, ds + \int_{\mathcal{N}_\vartheta\times\omega}\int_{t-\vartheta}^t |u_\infty|^2\, dsdx\right) \\
&\le \frac{2\left(1+2\left(C'\right)^2\right)c_\varepsilon}{(K_\varepsilon)^3} e^{-\lambda_\varepsilon\vartheta}\left(\vartheta + \int_{\mathcal{N}_\vartheta\times\omega}\int_{t-\vartheta}^t |u_\infty|^2\, dsdx\right).
\end{aligned}$$

This implies that

$$|(u-u_\infty)(t)|^2_{2,\hat{\Omega}}, |\nabla(u-u_\infty)|^2_{2,S_{[\vartheta]-1}\times\omega} \le C_\varepsilon\vartheta e^{-\lambda_\varepsilon\vartheta}\left(1+|u_\infty|^2_{2,\mathcal{N}_\vartheta\times\omega}\right),$$

where $C_\varepsilon = \frac{2\left(1+2(C')^2\right)c_\varepsilon}{(K_\varepsilon)^3}$. Then Theorem 2.2 is proved for each choice of $\varepsilon > \frac{\beta_2}{\beta_1}$. □

Remark 2.7. *i. Since λ_ε depends on ε continuously, the above theorem remains true for all $\lambda < \hat{\lambda} := \sup_{\varepsilon > \frac{\beta_2}{\beta_1}} \lambda_\varepsilon$. Even if it is possible to deduce that the above sup is a maximum, we cannot confirm that this value is the optimal λ since this depends on the used techniques. For example if we write (2.17) without the first norm we can only take $c_\varepsilon = \frac{c_0(1+c_0\beta_2\varepsilon)}{\beta_1 - \frac{\beta_2}{\varepsilon}}$.*

ii. It is clear that if $|u_\infty|_{2,\mathcal{N}_\vartheta\times\omega}$ grows polynomially or even exponentially with $|u_\infty|_{2,\mathcal{N}_\vartheta\times\omega} = O\left(e^{\delta\vartheta}\right)$ for $\delta < \sup_{\varepsilon > \frac{\beta_2}{\beta_1}} \lambda_\varepsilon$, Theorem 2.2 shows that u tends to u_∞ exponentially as $\vartheta \to \infty$.

iii. It is possible to show firstly that $u(t)$ is a Cauchy sequence in $L^2\left((t-1,t); H^1_{loc}(\mathbb{R}^m \times \bar{\omega})\right)$ and by consequence its limit exists, but we cannot use it to get a sharp estimate as above. In fact we will be limited to $L^2(Q_t)$ estimate of f where the measure of non-suitable regions will be considered and we cannot benefit from the maximum principle results.

iv. We can apply the same techniques of Theorem 2.2 to get the same estimate on larger time intervals, i.e. we may estimate the norm $|\nabla(u-u_\infty)|_{2,(t-\eta,t)\times\hat{\Omega}\times\omega}$ for some constant $\eta > 0$.

2.3.2 Some Applications

We have seen that the solutions u to (2.2) and u_∞ to (2.5) behave similarly when $t \to \infty$. However, it has also been shown that the limit problem (2.5), which is also the steady-state problem of (2.2), is depending on the structure of the data (see [15,17,27,31]). We will consider here two particular cases for which we can describe the behaviour of the solution u at the limit by a simpler form of Problem (2.5). We will investigate the case of periodic structure and the case of cylindrical symmetries. This allows also to have more clear vision about the solution of Problem (2.5).

Periodic structures

In this part we will investigate the case of periodic data and we will show that the periodicity of the data forces u to be periodic at the limit when $t, X_1 \to \infty$. We assume that $(a_{ij})_{i,j=1,\dots,n}$ and f are periodic in X_1 with period P, i.e.

$$a_{ij}(X_1 + Pe_k, X_2) = a_{ij}(X_1, X_2), \quad f(X_1 + Pe_k, X_2) = f(X_1, X_2), \qquad (2.19)$$
$$\text{a.e. } x = (X_1, X_2) \in \mathbb{R}^m \times \omega, \ k = 1, \dots, m.$$

((e_k) denotes the canonical basis of $\mathbb{R}^m$). We denote by $\mathcal{C}$ the period cell

$$\mathcal{C} = (0, P)^m.$$

Note that for a periodic function $f \in L^2_{loc}\left(\mathbb{R}^m; L^2(\omega)\right)$ we have in particular $f \in L^2(\mathcal{C} \times \omega)$.

Let us consider u_∞ solution to

$$\begin{cases} u_\infty \in H^1_{0,per}(\mathcal{C} \times \omega, \partial\omega), \\ \displaystyle\int_{\mathcal{C}\times\omega} A\nabla u_\infty \cdot \nabla v dx = \int_{\mathcal{C}\times\omega} f v dx \quad \forall v \in H^1_{0,per}(\mathcal{C} \times \omega, \partial\omega), \end{cases} \qquad (2.20)$$

where $H^1_{0,per}(\mathcal{C} \times \omega, \partial\omega)$ is defined by

$$H^1_{0,per}(\mathcal{C} \times \omega, \partial\omega) = \{v \in H^1(\mathcal{C} \times \omega) \mid v = 0 \text{ on } \mathcal{C} \times \partial\omega, \ v(x + Pe_k) = v(x),$$
$$\forall x \in \partial\mathcal{C} \times \omega \cap \{x_k = 0\}, \ k = 1, \dots, m\}.$$

It is clear that, under the assumptions (2.3) and (2.19), the problem (2.20) has a unique solution $u_\infty \in H^1_{0,per}(\mathcal{C} \times \omega, \partial\omega)$. We suppose from now on that u_∞ is extended by periodicity P on the whole strip $\mathbb{R}^m \times \omega$. This extended function is nothing else than the limit solution of (2.5) as it is shown in the following.

Lemma 2.3. *The extended function u_∞ is the unique limit solution to (2.5) satisfying the constraint*

$$|\nabla u_\infty|_{2,\mathcal{B}_t} = O\left(t^{m/2}\right), \qquad (2.21)$$

where $\mathcal{B}_t = (-tP, tP)^m \times \omega$. In particular, we have

$$\int_{\mathbb{R}^m\times\omega} A\nabla u_\infty \cdot \nabla v dx = \int_{\mathbb{R}^m\times\omega} f v dx \quad \forall v \in \mathcal{D}(\mathbb{R}^m \times \omega). \qquad (2.22)$$

Proof. The integral identity (2.22) is shown in [31, Lemma 2.4] and (2.21) is coming from the periodicity of u_∞. □

Using now Theorem 2.2 and taking into account Lemma 2.3 we get

$$|\nabla (u - u_\infty)|^2_{2,S_{[\vartheta]-1}\times\omega} \le C\vartheta e^{-\lambda\vartheta}\left(1 + |u_\infty|^2_{2,\mathcal{N}_\vartheta\times\omega}\right), \qquad (2.23)$$

where C, λ are positive constants independent of t. Assume that the minimal covering $\left(\overline{\mathcal{C}_k}\right)_{k\in I}$ of $\hat{\Omega}(t) \subset \hat{\Omega}_t$, consisting of the translation of $\mathcal{C}$, satisfies

$$\hat{\Omega} \subset \bigcup_{k\in I} \overline{\mathcal{C}_k}, \ \mathcal{C}_k \cap \mathcal{C}_j = \emptyset \text{ for } k \neq j, \ card(I) = O\left(e^{\gamma\vartheta}\right) \text{ as } t \to \infty, \qquad (2.24)$$

for some small enough $\gamma > 0$. Then there exists a covering $\left(\overline{\mathcal{C}_k}\right)_{k\in J}$ of $\mathcal{N}_\vartheta$ such that

$$card(J) = O\left(\vartheta^m e^{\gamma\vartheta}\right) \text{ as } t \to \infty.$$

It follows from the above inequalities that

$$\begin{aligned}
|\nabla(u - u_\infty)|^2_{2,S_{[\vartheta]-1}\times\omega} &\le C\vartheta e^{-\lambda\vartheta}\left(1 + \int_{\cup_{k\in J}\mathcal{C}_k\times\omega} u_\infty^2 dx\right) \\
&\le C\vartheta e^{-\lambda\vartheta}\left(1 + \sum_{k\in J}\int_{\mathcal{C}_k\times\omega} u_\infty^2 dx\right) \\
&\le C\vartheta e^{-\lambda\vartheta} card(J)\int_{\mathcal{C}\times\omega} u_\infty^2 dx \\
&\le C\vartheta^{m+1} e^{-(\lambda-\gamma)\vartheta}.
\end{aligned}$$

We choose γ small enough such that $\hat{\lambda} > \gamma$. Then we can state the following result.

Corollary 2.1. *For t large enough and under the hypotheses (2.3), (2.1), (2.19) and (2.24), there exist positive constants C and β independent of t such that*

$$|(u - u_\infty)(t)|_{2,\hat{\Omega}\times\omega}, |\nabla(u - u_\infty)|_{2,(t-\eta,t)\times\hat{\Omega}\times\omega} \le Ce^{-\beta\vartheta} \quad \forall t \ge 0,$$

where η is a positive constant, u and u_∞ are respectively the solutions of (2.2) and (2.20).

Let us now consider some examples. Set $\hat{\Omega}_t = (-P(t+t_0), P(t+t_0))^m$ with some constant $t_0 > 0$. Then we have $card(I) \le (2[t+t_0]+2)^m$. Thus we can choose ϑ such that $t = O\left(e^{\gamma\vartheta}\right)$ for some small enough $\gamma > 0$, and the result of Corollary 2.1 follows. For a more general example, the largest $\hat{\Omega} = (-PL(t), PL(t))^m$ ($L(t)$ is a function of t) that ensures the results of Corollary 2.1 is controlled by the condition $L(t) = O\left(e^{\gamma\vartheta}\right)$, for some small enough $\gamma > 0$.

Cylindrical symmetries

We now consider the case where the data are independent of X_1 to establish the immediate extension of the results obtained in [44]. Setting

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix};\ A_{11} \text{ is a } m\times m\text{-matrix},\ A_{22} \text{ is a } (n-m)\times(n-m)\text{-matrix},$$

and we suppose in addition to the assumption (2.3) that $A_{12} = A_{12}(X_2)$, $A_{22} = A_{22}(X_2)$ and $f \in L^2(\omega)$ are only depending on X_2. By the Lax–Milgram theorem there exists a unique u_∞ solution to

$$u_\infty \in H_0^1(\omega),\ \int_\omega A_{22}\nabla_{X_2}u_\infty\cdot\nabla_{X_2}v dX_2 = \int_\omega fv dX_2 \ \ \forall v \in H_0^1(\omega). \tag{2.25}$$

Since u_∞ and A_{12} are independent of t and X_1, it follows easily from (2.25) that

$$\int_{\mathbb{R}^m\times\omega} A\nabla u_\infty\cdot\nabla v dx = \int_{\mathbb{R}^m\times\omega} fv dx \ \ \forall v \in \mathcal{D}(\mathbb{R}^m\times\omega),$$

where $u_\infty(X_1, X_2) = u_\infty(X_2)$ is extended as a function on $\mathbb{R}^m \times \omega$ satisfying

$$|\nabla u_\infty|_{2,\mathcal{B}_t} = (2t)^{m/2} |\nabla_{X_2} u_\infty|_{2,\omega} = O\left(t^{m/2}\right),$$

for $\mathcal{B}_t = (-t,t)^m \times \omega$. Thus the extended function u_∞ is the unique limit solution to (2.5). Using now Theorem 2.2, we end up with

$$|\nabla (u - u_\infty)|^2_{2,S_{[\vartheta]-1}\times\omega} \leq C\vartheta e^{-\lambda\vartheta}\left(1 + |u_\infty|^2_{2,\mathcal{N}_\vartheta\times\omega}\right),$$

where C, λ are positive constants independent of t. As u_∞ is independent of t and X_1, we have

$$|\nabla (u - u_\infty)|^2_{2,S_{[\vartheta]-1}\times\omega} \leq C\vartheta e^{-\lambda\vartheta} mes\left(\mathcal{N}_\vartheta\right).$$

Finally, we can state the following Corollary.

Corollary 2.2. *For t large enough and under the above hypotheses, there exist positive constants C and λ independent of t such that*

$$|(u - u_\infty)(t)|^2_{2,\hat{\Omega}\times\omega}, \ |\nabla (u - u_\infty)|^2_{2,(t-\eta,t)\times\hat{\Omega}\times\omega} \leq Ce^{-\lambda\vartheta}\vartheta mes\left(\mathcal{N}_\vartheta\right),$$

where η is a positive constant, u and u_∞ are the solutions of (2.2) and (2.25) respectively.

Remark 2.8. *If $\hat{\Omega}$ is a star-shaped domain, the above estimates can be written as*

$$|(u - u_\infty)(t)|^2_{2,\hat{\Omega}\times\omega}, \ |\nabla (u - u_\infty)|^2_{2,(t-\eta,t)\times\hat{\Omega}\times\omega} \leq C\vartheta^{m+1} e^{-\lambda\vartheta} mes\left(\hat{\Omega}\right).$$

2.4 Corrector results

In order to get a complete description, we here deal with the asymptotic behaviour of the solution of the heat equation near the parts of the boundary becoming very far away when $t \to \infty$, for which no information is given about the behaviour of the solution in the above study. This, of course, leads to define correctors depending on time. We have succeeded in establishing that these correctors do not necessary keep the same nature as the solution u, i.e. even if they depend on the time t, they are not necessarily solutions to parabolic problems.

2.4.1 Construction of correctors

Let us then adapt the framework in order to construct the correctors. Let $\alpha \in C^1[0,\infty)$ be a positive and strictly nondecreasing function such that

$$\alpha(0) > 0, \ \lim_{t\to\infty} \alpha(t) = \infty, \ \lim_{t\to\infty} \alpha'(t) = \delta, \tag{2.26}$$

for some nonnegative constant δ. We set

$$\hat{\Omega}_t = (-\alpha(t), \alpha(t)).$$

Also, we assume that $f \in L^2(\omega)$ which means that

$$|f|_{2,\Omega_s} = O\left((\alpha(s))^{\frac{1}{2}}\right). \tag{2.27}$$

As we saw above, for $u_0 \in L^2(\Omega_0)$, there exist a unique solution u to

$$\begin{cases} \langle u', v\rangle + \int_0^t \int_{\Omega_s} \nabla u \cdot \nabla v dx ds = \int_0^t \int_{\Omega_s} f v dx ds \quad \forall v \in \mathcal{F}(Q_t), \\ u(0,.) = u_0, \quad u \in \mathcal{F}(Q_t), \end{cases} \tag{2.28}$$

and a unique limit solution u_∞ to

$$\begin{cases} u_\infty \in H_0^1(\omega), \\ -\Delta_{X_2} u_\infty = f \quad \text{in } \omega. \end{cases} \tag{2.29}$$

It is also shown that

$$u(t) \to u_\infty \text{ in } H^1(O) \text{ as } t \to \infty,$$

where O is a bounded domain of $\mathbb{R} \times \omega$. However, as it is explained in the introduction, one cannot expect in general that

$$|u(t) - u_\infty|_{1,2,\Omega_t} \to 0 \text{ as } t \to \infty. \tag{2.30}$$

To recover the convergence on the whole Ω_t, let us construct a simple function $w(t)$ as a corrector of $u - u_\infty$ which describes the behaviour of u near the end sections $\{-\alpha(t), \alpha(t)\} \times \omega$, i.e. it satisfies

$$(u - u_\infty - w)(t) \to 0 \text{ in } L^2(\Omega_t) \text{ and } |(u - u_\infty - w)(s)|_{1,2,\Omega_s} \to 0 \text{ in } L^2(t', t) \text{ as } t \to \infty,$$

where t' is depending on t and tends to infinity when $t \to \infty$.

We denote by $\mathcal{I}_s^\pm$ the half cylinders

$$\mathcal{I}_s^+ = (s, \infty) \times \omega, \ \mathcal{I}_s^- = (-\infty, -s) \times \omega.$$

Besides, we consider $w_\pm$ the solution of the following linear elliptic problems

$$\begin{cases} -\Delta w_\pm \pm \delta \partial_{x_1} w_\pm = 0 \text{ in } \mathcal{I}_0^\pm, \\ w_\pm = -u_\infty \text{ on } \{0\} \times \omega, \ w_\pm = 0 \text{ on } \partial \mathcal{I}_0^\pm \backslash \{0\} \times \omega, \end{cases} \tag{2.31}$$

where the limit δ is given in (2.26). The solution of the above problem exists and is unique. In fact, it can be written as $w_\pm = \tilde{u}_\pm - \varrho(x_1) u_\infty \in H^1(\mathcal{I}_0^\pm)$ where ϱ is a compact supported smooth function with $\varrho(0) = 1$ and $\tilde{u}_\pm \in H_0^1(\mathcal{I}_0^\pm)$ is the unique solution of the following homogeneous Dirichlet boundary value problem

$$\int_{\mathcal{I}_0^\pm} \nabla \tilde{u}_\pm \cdot \nabla v \pm \delta \partial_{x_1} \tilde{u}_\pm v dx = \int_{\mathcal{I}_0^\pm} \nabla(\varrho(x_1) u_\infty) \cdot \nabla v \pm \delta \partial_{x_1}(\varrho(x_1) u_\infty) v dx \quad \forall v \in \mathcal{F}(\mathcal{I}_0^\pm).$$

Then, we introduce as a corrector the function $w \in H^1(Q_t)$ defined by

$$w(t, x_1, X_2) = \rho(x_1) \bar{w}_+(t, x_1, X_2) + \rho(-x_1) \bar{w}_-(t, x_1, X_2),$$

where

$$\bar{w}_+(t, x_1, X_2) = w_+(\alpha(t) - x_1, X_2), \ \bar{w}_-(t, x_1, X_2) = w_-(-\alpha(t) - x_1, X_2)$$

and ρ is the function whose graph is depicted in the following figure.

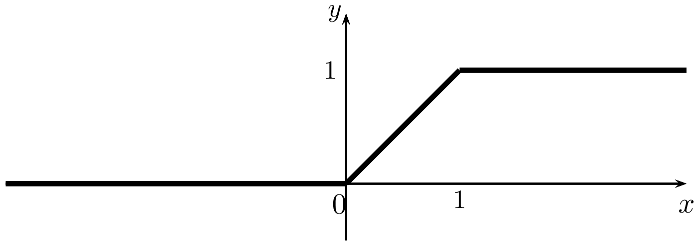

Figure 2.1: The function ρ.

Moreover let us set

$$\Omega_{k,t}^{+} = \{t\} \times (0, k\alpha(t)) \times \omega,\ \Omega_{k,t}^{-} = \{t\} \times (-k\alpha(t), 0) \times \omega, \quad k \in \mathbb{N}\backslash\{0\}.$$

For simplicity of notation, we set

$$\Omega_t^{\pm} = \Omega_{1,t}^{\pm}$$

and we keep the same notation for $k > 1$. Besides we set

$$\mathcal{U}_t^{\pm} = \left\{v \in H^1\left(\Omega_t^{\pm}\right) \mid v = 0 \text{ on } \partial\Omega_t^{\pm}\backslash\{0\} \times \omega\right\}$$

and for $v \in \mathcal{U}_t^{+}$ (resp. $v \in \mathcal{U}_t^{-}$), we also denote by $\hat{v}$ and $\tilde{v}$ the functions defined as

$$\hat{v}(x_1, X_2) = \begin{cases} v(x_1, X_2) & \text{if } x_1 \geq 0, \\ v(-x_1, X_2) & \text{if } x_1 < 0, \end{cases} \qquad \tilde{v}(x_1, X_2) = \begin{cases} v(x_1, X_2) & \text{if } x_1 \leq 0, \\ v(-x_1, X_2) & \text{if } x_1 > 0. \end{cases}$$

Then, we have the following useful lemmas that are proved in [19] for $\delta = 0$. Here we need to repeat the proof because of the new term which requires more details.

Lemma 2.4. *For all $t > 0$, we have the following*

(*i*)

$$\int_{\Omega_s^+} \nabla\bar{w}_+ \cdot \nabla v dx - \delta \int_{\Omega_s^+} \partial_{x_1}\bar{w}_+ v dx = -\int_{\Omega_t^-} \nabla\bar{w}_+ \cdot \nabla\hat{v} dx + \delta \int_{\Omega_t^-} \partial_{x_1}\bar{w}_+ \hat{v} dx \quad \forall v \in \mathcal{U}_t^{+},$$

(*ii*)

$$\int_{\Omega_t^-} \nabla\bar{w}_- \cdot \nabla v dx + \delta \int_{\Omega_t^-} \partial_{x_1}\bar{w}_- v dx = -\int_{\Omega_t^+} \nabla\bar{w}_- \cdot \nabla\tilde{v} dx - \delta \int_{\Omega_t^+} \partial_{x_1}\bar{w}_- \tilde{v} dx \quad \forall v \in \mathcal{U}_t^{-}.$$

Proof. (*i*) Let us consider $v \in \mathcal{U}_s^{+}$. It is clear that $\hat{v}(\alpha(s) - x_1, X_2) \in H_0^1\left(\Omega_{2,s}^{+}\right)$. Thus, it follows from (2.31) that

$$\begin{aligned} &\int_{\Omega_s^+} \nabla w_+ \cdot \nabla\hat{v}(\alpha(s) - x_1, X_2)\, dx + \delta \int_{\Omega_s^+} \partial_{x_1} w_+ \hat{v}(\alpha(s) - x_1, X_2)\, dx \\ &= -\int_{\Omega_{2,s}^+\backslash\Omega_s^+} \nabla w_+ \cdot \nabla\hat{v}(\alpha(s) - x_1, X_2)\, dx - \delta \int_{\Omega_{2,s}^+\backslash\Omega_s^+} \partial_{x_1} w_+ \hat{v}(\alpha(s) - x_1, X_2)\, dx. \end{aligned} \tag{2.32}$$

Making the change of variable $x_1' = \alpha(s) - x_1$, we get

$$\int_{\Omega_s^+} \nabla w_+ \cdot \nabla \hat{v}\big(\alpha(s) - x_1, X_2\big)\, dx + \delta \int_{\Omega_s^+} \partial_{x_1} w_+ \hat{v}\big(\alpha(s) - x_1, X_2\big)\, dx$$
$$= \int_{\Omega_s^+} \nabla \bar{w}_+ \cdot \nabla v \, dx - \delta \int_{\Omega_s^+} \partial_{x_1} \bar{w}_+ v \, dx$$

and

$$\int_{\Omega_{2,s}^+ \backslash \Omega_s^+} \nabla w_+ \cdot \nabla \hat{v}\big(\alpha(s) - x_1, X_2\big)\, dx + \delta \int_{\Omega_{2,s}^+ \backslash \Omega_s^+} \partial_{x_1} w_+ \hat{v}\big(\alpha(s) - x_1, X_2\big)\, dx$$
$$= \int_{\Omega_s^-} \nabla \bar{w}_+ \cdot \nabla \hat{v} \, dx - \delta \int_{\Omega_s^-} \partial_{x_1} \bar{w}_+ \hat{v} \, dx.$$

The proof of (i) follows then from (2.32).

(ii) We will use the same technique used for point (i). Then for $v \in \mathcal{U}_s^-$, we have $\tilde{v}(-\alpha(s) - x_1, X_2) \in H_0^1\left(\Omega_{2,s}^-\right)$ and from (2.31) we obtain

$$\int_{\Omega_s^-} \nabla w_- \cdot \nabla \tilde{v}\big(-\alpha(s) - x_1, X_2\big)\, dx - \delta \int_{\Omega_s^-} \partial_{x_1} w_- \tilde{v}\big(-\alpha(s) - x_1, X_2\big)\, dx$$
$$= -\int_{\Omega_{2,s}^- \backslash \Omega_s^-} \nabla w_- \cdot \nabla \tilde{v}\big(-\alpha(s) - x_1, X_2\big)\, dx + \delta \int_{\Omega_{2,s}^- \backslash \Omega_s^-} \partial_{x_1} w_- \tilde{v}\big(-\alpha(s) - x_1, X_2\big)\, dx.$$

Making this time the change of variable $x_1' = -(\alpha(s) + x_1)$, we end up with the second identity in the above lemma which also finishes the proof. □

Lemma 2.5. *There exist positive constants C_1 and $\tilde{\lambda}$ independent of t such that*

$$\int_{\mathcal{I}_{\alpha(t)-1}^{\pm}} |\nabla w_\pm|^2 \, dx \le C_1 e^{-\tilde{\lambda}\alpha(t)} \int_{\mathcal{I}_0^{\pm}} |\nabla w_\pm|^2 \, dx.$$

Proof. Let ρ be the function depicted in the figure (2.1). For t large enough we may assume that $\alpha(t) \ge 2$ and for $0 \le i \le \alpha(t) - 2$ we denote by

$$\rho_{\pm,i}(x_1) = \rho(\pm(x_1 \mp i)) \quad \forall x_1 \in \mathbb{R}.$$

Since $\rho_{\pm,i} w_\pm \in H_0^1(\mathcal{I}_0^\pm)$ it follows from (2.31) that

$$\int_{\mathcal{I}_0^\pm} \nabla w_\pm \cdot \nabla\left(\rho_{\pm,i} w_\pm\right) dx \pm \delta \int_{\mathcal{I}_0^\pm} \rho_{\pm,i} \partial_{x_1} w_\pm w_\pm dx = 0,$$

which implies

$$\int_{\mathcal{I}_0^\pm} \rho_{\pm,i} |\nabla w_\pm|^2 \, dx = -\int_{\mathcal{I}_0^\pm} \partial_{x_1} \rho_{\pm,i} \partial_{x_1} w_\pm w_\pm dx \pm \frac{\delta}{2} \int_{\mathcal{I}_0^\pm} \partial_{x_1} \rho_{\pm,i} (w_\pm)^2 \, dx.$$

In fact, this can be proved using the Green formula and the fact that $\rho_{\pm,i} = 0$ or $w_\pm = 0$ on the boundary of $\mathcal{I}_0^\pm$. Now from the definition of ρ we derive

$$\int_{\mathcal{I}_{i+1}^\pm} |\nabla w_\pm|^2 \, dx \le \int_{\mathcal{I}_i^\pm \setminus \mathcal{I}_{i+1}^\pm} |\partial_{x_1} w_\pm| \, |w_\pm| \, dx + \frac{\delta}{2} \int_{\mathcal{I}_i^\pm \setminus \mathcal{I}_{i+1}^\pm} (w_\pm)^2 \, dx$$
$$\le \frac{\varepsilon}{2} \int_{\mathcal{I}_i^\pm \setminus \mathcal{I}_{i+1}^\pm} |\partial_{x_1} w_\pm|^2 \, dx + \frac{1}{2} \left(\delta + \frac{1}{\varepsilon}\right) \int_{\mathcal{I}_i^\pm \setminus \mathcal{I}_{i+1}^\pm} (w_\pm)^2 \, dx,$$

for some constant $\varepsilon > 0$. Applying the Poincaré inequality in ω to the last integral, we get for some positive constant c_ω

$$\int_{\mathcal{I}_{i+1}^\pm} |\nabla w_\pm|^2 \, dx \le \frac{\varepsilon}{2} \int_{\mathcal{I}_i^\pm \setminus \mathcal{I}_{i+1}^\pm} |\partial_{x_1} w_\pm|^2 \, dx + \frac{c_\omega}{2} \left(\delta + \frac{1}{\varepsilon}\right) \int_{\mathcal{I}_i^\pm \setminus \mathcal{I}_{i+1}^\pm} |\nabla w_\pm|^2 \, dx$$
$$\le \frac{1}{2} \left[\varepsilon + c_\omega \left(\delta + \frac{1}{\varepsilon}\right)\right] \int_{\mathcal{I}_i^\pm} |\nabla w_\pm|^2 \, dx - \frac{1}{2} \left[\varepsilon + c_\omega \left(\delta + \frac{1}{\varepsilon}\right)\right] \int_{\mathcal{I}_{i+1}^\pm} |\nabla w_\pm|^2 \, dx.$$

Consequently,

$$\int_{\mathcal{I}_{i+1}^\pm} |\nabla w_\pm|^2 \, dx \le r_\varepsilon \int_{\mathcal{I}_i^\pm} |\nabla w_\pm|^2 \, dx,$$

where $r_\varepsilon = \frac{\varepsilon^2 + c_\omega \delta \varepsilon + c_\omega}{\varepsilon^2 + (c_\omega \delta + 2)\varepsilon + c_\omega}$. Iterating the above inequality for $k = 0, \ldots, [\alpha(t)] - 2$ we end up with

$$\int_{\mathcal{I}_{\alpha(t)-1}^\pm} |\nabla w_\pm|^2 \, dx \le r_\varepsilon^{[\alpha(t)]-1} \int_{\mathcal{I}_0^\pm} |\nabla w_\pm|^2 \, dx$$
$$\le C_\varepsilon e^{-\tilde{\lambda}_\varepsilon \alpha(t)},$$

where C_ε and $\tilde{\lambda}_\varepsilon$ are positive constants given by

$$C_\varepsilon = \frac{1}{(r_\varepsilon)^2} \int_{\mathcal{I}_0^\pm} |\nabla w_\pm|^2 \, dx \text{ and } \tilde{\lambda}_\varepsilon = -\ln r_\varepsilon = \ln \left(1 + \frac{2}{\varepsilon + c_\omega \left(\delta + \frac{1}{\varepsilon}\right)}\right). \tag{2.33}$$

Taking $\tilde{\lambda} = \max_{\varepsilon > 0} \tilde{\lambda}_\varepsilon$, this achieves the proof of Lemma 2.5. □

Let now p and q be positive real valued functions such that

$$q(t) \to \infty, \; t - (p(t) + q(t)) \to \infty \quad \text{as } t \to \infty. \tag{2.34}$$

Note that $p(t)$ may also tend to ∞ when $t \to \infty$. Let us first state the following theorem establishing the main estimate error in its general form.

Theorem 2.3. *Suppose that (2.26) and (2.34) are fulfilled, that $f \in L^2(\omega)$ and that $u_0 \in L^2(\Omega_0)$. Then there exist positive constants C_2, C_3 and σ independent of t such that*

$$\int_{\Omega_t} |(u - u_\infty - w)(t,.)|^2 \, dx, \; \int_{t-p(t)}^t \int_{\Omega_s} |\nabla (u - u_\infty - w)|^2 \, dx ds$$
$$\le C_2 (1 + \alpha(t - ([p(t)] + [q(t)] - 2))) e^{-\sigma q(t)}$$
$$+ C_3 \sum_{k=1}^{[q(t)]-2} ([p(t)] + k + 1) e^{-\sigma k} \left[\max_{s \in (t - ([p(t)]+k+1), t)} |\alpha' - \delta|^2 + e^{-\tilde{\lambda}\alpha(t - ([p(t)]+k+1))}\right], \tag{2.35}$$

where $\tilde{\lambda}$ is the constant whose existence is proved in Lemma 2.5.

Proof. For $t, k > 0$ with $k + 1 < t$, we denote by $\tilde{\rho}_k$ the time dependent continuous function defined on $\mathbb{R}^+$ as

$$\tilde{\rho}_k = 0 \text{ on } (0, t-k-1)\,,\ \tilde{\rho}_k \text{ is linear on } (t-k-1, t-k)\,,\ \tilde{\rho}_k = 1 \text{ on } (t-k, \infty)\,.$$

It is clear that

$$\tilde{\rho}_k^2 (u - u_\infty - w) \in \mathcal{F}(Q_t) \text{ and that } 0 \leq (\tilde{\rho}_k)' \leq 1.$$

Thus, from (2.28), (2.29) and since u_∞ is independent of t and x_1, we drive

$$< (u - u_\infty)', \tilde{\rho}_k^2 (u - u_\infty - w) > + \int_0^t \int_{\Omega_s} \nabla (u - u_\infty) \cdot \nabla \left(\tilde{\rho}_k^2 (u - u_\infty - w) \right) dxds = 0.$$

This can also be written as follows

$$\begin{aligned}
&< (\tilde{\rho}_k (u - u_\infty - w))', \tilde{\rho}_k (u - u_\infty - w) > + \int_0^t \int_{\Omega_s} |\tilde{\rho}_k \nabla (u - u_\infty - w)|^2 dxds \\
&= \int_0^t \int_{\Omega_s} \partial_t \tilde{\rho}_k \tilde{\rho}_k (u - u_\infty - w)^2 dxds - \int_0^t \int_{\Omega_s} \tilde{\rho}_k^2 \partial_t w (u - u_\infty - w) dxds \\
&- \int_0^t \int_{\Omega_s} \tilde{\rho}_k^2 \nabla w \cdot \nabla (u - u_\infty - w) dxds \\
&= \int_0^t \int_{\Omega_s} \partial_t \tilde{\rho}_k \tilde{\rho}_k (u - u_\infty - w)^2 dxds + \int_0^t \int_{\Omega_s^+} \alpha' \tilde{\rho}_k^2 \rho(x_1) \partial_{x_1} \bar{w}_+ (u - u_\infty - \rho(x_1) \bar{w}_+) dxds \\
&- \int_0^t \int_{\Omega_s^-} \alpha' \tilde{\rho}_k^2 \rho(-x_1) \partial_{x_1} \bar{w}_- (u - u_\infty - \rho(-x_1) \bar{w}_-) dxds - \int_0^t \int_{\Omega_s} \tilde{\rho}_k^2 \nabla w \cdot \nabla (u - u_\infty - w) dxds.
\end{aligned}$$

Applying the integration by parts formula and taking into account the definitions of ρ and $\tilde{\rho}_k$, we deduce

$$\begin{aligned}
&\frac{1}{2} |(u - u_\infty - w)(t, .)|_{2,\Omega_t}^2 + \int_{t-k-1}^t \int_{\Omega_s} |\tilde{\rho}_k \nabla (u - u_\infty - w)|^2 dxds \\
&= \int_{t-k-1}^{t-k} \int_{\Omega_s} \tilde{\rho}_k (u - u_\infty - w)^2 dxds + \int_{t-k-1}^t \int_{\Omega_s^+} \left(\alpha' - \delta\right) \tilde{\rho}_k^2 \rho(x_1) \partial_{x_1} \bar{w}_+ (u - u_\infty - \rho(x_1) \bar{w}_+) dxds \\
&- \int_{t-k-1}^t \int_{\Omega_s^-} \left(\alpha' - \delta\right) \tilde{\rho}_k^2 \rho(-x_1) \partial_{x_1} \bar{w}_- (u - u_\infty - \rho(-x_1) \bar{w}_-) dxds \\
&+ \int_{t-k-1}^t \tilde{\rho}_k^2 \left(\int_{\Omega_s^+} \rho(x_1) \left(\delta \partial_{x_1} \bar{w}_+ (u - u_\infty - \rho(x_1) \bar{w}_+) - \nabla \bar{w}_+ \cdot \nabla (u - u_\infty - \rho(x_1) \bar{w}_+) \right) dx \right) ds \\
&- \int_{t-k-1}^t \tilde{\rho}_k^2 \int_{\Omega_s^-} \rho(-x_1) \left(\delta \partial_{x_1} \bar{w}_- (u - u_\infty - \rho(-x_1) \bar{w}_-) + \nabla \bar{w}_- \cdot \nabla (u - u_\infty - \rho(-x_1) \bar{w}_-) \right) dxds \\
&- \int_{t-k-1}^t \tilde{\rho}_k^2 \int_{(0,1)\times\omega} \bar{w}_+ \partial_{x_1} (u - u_\infty - \rho(x_1) \bar{w}_+) dxds \\
&+ \int_{t-k-1}^t \tilde{\rho}_k^2 \int_{(-1,0)\times\omega} \bar{w}_- \partial_{x_1} (u - u_\infty - \rho(-x_1) \bar{w}_-) dxds,
\end{aligned} \tag{2.36}$$

where the positive constant δ is the limit of α', as $t \to \infty$. Then we apply Lemma 2.4 to deal with the terms of the fourth and the fifth lines in the above identity. We derive

$$\begin{aligned}
&\int_{t-k-1}^{t} \tilde{\rho}_k^2 \int_{\Omega_s^+} \rho(x_1)\big(-\nabla \bar{w}_+ \cdot \nabla\big(u - u_\infty - \rho(x_1)\bar{w}_+\big) + \delta\partial_{x_1}\bar{w}_+\big(u - u_\infty - \rho(x_1)\bar{w}_+\big)\big)\, dx\, ds \\
&= \int_{t-k-1}^{t} \tilde{\rho}_k^2 \bigg(\int_{\Omega_s^-} \nabla \bar{w}_+ \cdot \nabla\big(u - \widehat{u_\infty - \rho}(x_1)\bar{w}_+\big)\, dx \\
&\quad - \delta \int_{\Omega_s^-} \partial_{x_1}\bar{w}_+\big(u - \widehat{u_\infty - \rho}(x_1)\bar{w}_+\big)\, dx\bigg)\, ds \\
&\quad + \int_{t-k-1}^{t} \tilde{\rho}_k^2 \int_{(0,1)\times\omega} \big(\rho(x_1) - 1\big)\big(-\nabla \bar{w}_+ \cdot \nabla\big(u - u_\infty - \rho(x_1)\bar{w}_+\big) \\
&\quad + \delta\partial_{x_1}\bar{w}_+\big(u - u_\infty - \rho(x_1)\bar{w}_+\big)\big)\, dx\, ds
\end{aligned}$$

and

$$\begin{aligned}
&- \int_{t-k-1}^{t} \tilde{\rho}_k^2 \int_{\Omega_s^-} \rho(-x_1)\big(\nabla \bar{w}_- \cdot \nabla\big(u - u_\infty - \rho(-x_1)\bar{w}_-\big) \\
&\quad + \delta\partial_{x_1}\bar{w}_-\big(u - u_\infty - \rho(-x_1)\bar{w}_-\big)\big)\, dx\, ds \\
&= \int_{t-k-1}^{t} \tilde{\rho}_k^2 \bigg(\int_{\Omega_s^+} \nabla \bar{w}_- \cdot \nabla\big(u - \widetilde{u_\infty - \rho}(-x_1)\bar{w}_-\big)\, dx \\
&\quad + \delta \int_{\Omega_s^+} \partial_{x_1}\bar{w}_-\big(u - \widetilde{u_\infty - \rho}(-x_1)\bar{w}_-\big)\, dx\bigg)\, ds \\
&\quad + \int_{t-k-1}^{t} \tilde{\rho}_k^2 \int_{(-1,0)\times\omega} \big(1 - \rho(-x_1)\big)\nabla \bar{w}_- \cdot \nabla\big(u - u_\infty - \rho(-x_1)\bar{w}_-\big)\, dx\, ds \\
&\quad + \int_{t-k-1}^{t} \tilde{\rho}_k^2 \int_{(-1,0)\times\omega} \delta\big(1 - \rho(-x_1)\big)\partial_{x_1}\bar{w}_-\big(u - u_\infty - \rho(-x_1)\bar{w}_-\big)\, dx\, ds.
\end{aligned}$$

Going back to (2.36), using the above identities and the inequalities of Cauchy-Shwartz, Poincaré and Young, it follows from the definition of ρ that

$$\begin{aligned}
&\frac{1}{2}\left|(u - u_\infty - w)(t,.)\right|_{2,\Omega_t}^2 + \int_{t-k-1}^{t} \int_{\Omega_s} \left|\tilde{\rho}_k \nabla (u - u_\infty - w)\right|^2 dxds \\
&\leq \int_{t-k-1}^{t-k} \int_{\Omega_s} \tilde{\rho}_k (u - u_\infty - w)^2\, dxds \\
&+ \frac{\varepsilon}{2} \max\nolimits_{s\in(t-k-1,t)} \left|\alpha'(s) - \delta\right|^2 \int_{t-k-1}^{t} \int_{\Omega_s^+} |\nabla \bar{w}_+|^2\, dx + \int_{\Omega_s^-} |\nabla \bar{w}_-|^2\, dxds \\
&+ \frac{\varepsilon(1+\delta)}{2} \int_{t-k-1}^{t} \Bigg(\int_{\Omega_s^-} |\nabla \bar{w}_+|^2\, dx + \int_{\Omega_s^+} |\nabla \bar{w}_-|^2\, dx + \int_{(0,1)\times\omega} \left(|\nabla \bar{w}_+|^2 + |\bar{w}_+|^2\right) dx \\
&+ \int_{(-1,0)\times\omega} \left(|\nabla \bar{w}_-|^2 + |\bar{w}_-|^2\right) dx\Bigg)\, ds + \frac{3 + c'_\omega(1+2\delta)}{2\varepsilon} \int_{t-k-1}^{t} \int_{\Omega_s} \left|\tilde{\rho}_k \nabla (u - u_\infty - w)\right|^2 dxds,
\end{aligned} \tag{2.37}$$

where $\varepsilon > \dfrac{3 + c'_\omega(1+2\delta)}{2}$ and c'_ω is the Poincaré constant. Note that

$$\int_{\Omega_s^-} |\nabla \widehat{v}|^2\, dx = \int_{\Omega_s^+} |\nabla v|^2\, dx \text{ and } \int_{\Omega_s^+} |\nabla \widetilde{v}|^2\, dx = \int_{\Omega_s^-} |\nabla v|^2\, dx \qquad \forall v \in H_0^1(\Omega_s).$$

Making the change of variables $x_1' = \alpha(s) - x_1$ and $x_1' = -\alpha(s) - x_1$, we obtain

$$\int_{t-k-1}^{t} \int_{\Omega_s^+} |\nabla \bar{w}_+|^2 \, dx + \int_{\Omega_s^-} |\nabla \bar{w}_-|^2 \, dxds \leq (1+k) \left(\int_{\mathcal{I}_0^+} |\nabla w_+|^2 \, dx + \int_{\mathcal{I}_0^-} |\nabla w_-|^2 \, dx \right). \tag{2.38}$$

On the other hand, using the Poincaré inequality and Lemma 2.5, it follows that

$$\begin{aligned}
&\int_{t-k-1}^{t} \int_{\Omega_s^-} |\nabla \bar{w}_+|^2 \, dx + \int_{\Omega_s^+} |\nabla \bar{w}_-|^2 \, dx + \int_{(0,1)\times\omega} \left(|\nabla \bar{w}_+|^2 + |\bar{w}_+|^2 \right) dxds \\
&+ \int_{t-k-1}^{t} \int_{(-1,0)\times\omega} \left(|\nabla \bar{w}_-|^2 + |\bar{w}_-|^2 \right) dxds \\
&\leq \left(2 + c_\omega'' \right) \int_{t-k-1}^{t} \int_{\mathcal{I}_{\alpha(s)-1}^+} |\nabla w_+|^2 \, dx + \int_{\mathcal{I}_{\alpha(s)-1}^-} |\nabla w_-|^2 \, dxds \\
&\leq C_1 \left(2 + c_\omega'' \right) \left(\int_{\mathcal{I}_0^+} |\nabla w_+|^2 \, dx + \int_{\mathcal{I}_0^-} |\nabla w_-|^2 \, dx \right) \times \left(\int_{t-k-1}^{t} e^{-\tilde{\lambda}\alpha(s)} ds \right) \\
&\leq C_1 \left(2 + c_\omega'' \right) \left(\int_{\mathcal{I}_0^+} |\nabla w_+|^2 \, dx + \int_{\mathcal{I}_0^-} |\nabla w_-|^2 \, dx \right) (1+k)\, e^{-\tilde{\lambda}\alpha(t-k-1)}.
\end{aligned} \tag{2.39}$$

Combining (2.37), (2.38) and (2.39) we get

$$\begin{aligned}
&\int_{t-k}^{t} \int_{\Omega_s} |\nabla (u - u_\infty - w)|^2 \, dxds \\
&\leq C_\varepsilon \left(\int_{t-k-1}^{t-k} \int_{\Omega_s} |(u - u_\infty - w)|^2 \, dxds + (1+k) \left(\max_{s\in(t-k-1,t)} \left|\alpha' - \delta\right|^2 + e^{-\tilde{\lambda}\alpha(t-k-1)} \right) \right) \\
&\leq C_\varepsilon \left(c_\omega' \int_{t-k-1}^{t-k} \int_{\Omega_s} |\nabla (u - u_\infty - w)|^2 \, dxds + (1+k) \left(\max_{s\in(t-k-1,t)} \left|\alpha' - \delta\right|^2 + e^{-\tilde{\lambda}\alpha(t-k-1)} \right) \right),
\end{aligned} \tag{2.40}$$

where

$$\begin{aligned}
C_\varepsilon = \max &\left(\frac{2\varepsilon}{2\varepsilon - 3 - c_\omega'(1+2\delta)}, \frac{\varepsilon^2}{2\varepsilon - 3 - c_\omega'(1+2\delta)} \left(\int_{\mathcal{I}_0^+} |\nabla w_+|^2 \, dx + \int_{\mathcal{I}_0^-} |\nabla w_-|^2 \, dx \right), \right. \\
&\left. \frac{\varepsilon^2 C_1 (1+\delta) \left(2 + c_\omega''\right)}{2\varepsilon - 3 - c_\omega'(1+2\delta)} \left(\int_{\mathcal{I}_0^+} |\nabla w_+|^2 \, dx + \int_{\mathcal{I}_0^-} |\nabla w_-|^2 \, dx \right) \right).
\end{aligned}$$

This implies, in particular, that

$$\begin{aligned}
&\int_{t-k}^{t} \int_{\Omega_s} |\nabla (u - u_\infty - w)|^2 \, dxds \\
&\leq r_\varepsilon' \int_{t-k-1}^{t} \int_{\Omega_s} |\nabla (u - u_\infty - w)|^2 \, dxds + r_\varepsilon' (1+k) \left[\max_{s\in(t-k-1,t)} \left|\alpha' - \delta\right|^2 + e^{-\tilde{\lambda}\alpha(t-k-1)} \right],
\end{aligned}$$

with $r_\varepsilon' = \dfrac{C_\varepsilon \max(1, c_\omega')}{1 + C_\varepsilon \max(1, c_\omega')}$. Iterating k from $[p(t)] + 1$ to $[p(t)] + [q(t)] - 1$ in the above formula, we can use (2.40) to obtain

$$
\begin{aligned}
&\int_{t-([p(t)]+1)}^{t}\int_{\Omega_s}\left|\nabla\left(u-u_\infty-w\right)\right|^2 dxds \le r'_\varepsilon \int_{t-([p(t)]+2)}^{t}\int_{\Omega_s}\left|\nabla\left(u-u_\infty-w\right)\right|^2 dxds\\
&+r'_\varepsilon\left([p(t)]+2\right)\left[\max_{s\in(t-([p(t)]+2),t)}\left|\alpha'-\delta\right|^2+e^{-\tilde{\lambda}\alpha(t-([p(t)]+2))}\right]\\
&\le \left(r'_\varepsilon\right)^2\int_{t-([p(t)]+3)}^{t}\int_{\Omega_s}\left|\nabla\left(u-u_\infty-w\right)\right|^2 dxds\\
&+\left(r'_\varepsilon\right)^2\left([p(t)]+3\right)\left[\max_{s\in(t-([p(t)]+3),t)}\left|\alpha'-\delta\right|^2+e^{-\tilde{\lambda}\alpha(t-([p(t)]+3))}\right]\\
&+r'_\varepsilon\left([p(t)]+2\right)\left[\max_{s\in(t-([p(t)]+2),t)}\left|\alpha'-\delta\right|^2+e^{-\tilde{\lambda}\alpha(t-([p(t)]+2))}\right]\\
&\le\dots\\
&\le\left(r'_\varepsilon\right)^{[q(t)]-3}\int_{t-([p(t)]+[q(t)]-2)}^{t}\int_{\Omega_s}\left|\nabla\left(u-u_\infty-w\right)\right|^2 dxds\\
&+\sum_{k=1}^{[q(t)]-3}\left(r'_\varepsilon\right)^k\left([p(t)]+k+1\right)\left[\max_{s\in(t-([p(t)]+k+1),t)}\left|\alpha'-\delta\right|^2+e^{-\tilde{\lambda}\alpha(t-([p(t)]+k+1))}\right]\\
&\le C_\varepsilon\left(r'_\varepsilon\right)^{[q(t)]-3}\int_{t-([p(t)]+[q(t)]-1)}^{t-([p(t)]+[q(t)]-2)}\int_{\Omega_s}\left|(u-u_\infty-w)\right|^2 dxds\\
&+\left(C_\varepsilon+1\right)\sum_{k=1}^{[q(t)]-2}\left(r'_\varepsilon\right)^k\left([p(t)]+k+1\right)\left[\max_{s\in(t-([p(t)]+k+1),t)}\left|\alpha'-\delta\right|^2+e^{-\tilde{\lambda}\alpha(t-([p(t)]+k+1))}\right].
\end{aligned}
$$

Applying Lemma 3.1 to the last integral in the above formula and using (2.38), we get

$$
\begin{aligned}
&\int_{t-([p(t)]+[q(t)]-1)}^{t-([p(t)]+[q(t)]-2)}\int_{\Omega_s}\left|(u-u_\infty-w)\right|^2 dxds\\
&\le C_0\int_{t-([p(t)]+[q(t)]-1)}^{t-([p(t)]+[q(t)]-2)}\left(\int_{\Omega_s}|u|^2 dx+\int_{\Omega_s}|u_\infty|^2 dx+\int_{\Omega_s^+}|\bar{w}_+|^2 dx+\int_{\Omega_s^-}|\bar{w}_-|^2 dx\right)ds\\
&\le C_0\left(1+2\left(C'\right)^2\right)\left[\int_{t-([p(t)]+[q(t)]-1)}^{t-([p(t)]+[q(t)]-2)}\int_{\Omega_s}|u_\infty|^2 dxds+1+\int_{I_0^+}|w_+|^2 dx+\int_{I_0^-}|w_-|^2 dx\right]\\
&\le C_0\left(1+2\left(C'\right)^2\right)\left[2\alpha\left(t-([p(t)]+[q(t)]-2)\right)\int_\omega|u_\infty|^2 dX_2+1+\int_{I_0^+}|w_+|^2 dx+\int_{I_0^-}|w_-|^2 dx\right],
\end{aligned}
$$

where C_0 is a positive constant independent of t. Using the change of variable $x_1' = \alpha(s) - x_1$ and $x_1' = -(\alpha(s) + x_1)$, we end up with

$$
\begin{aligned}
&\int_{t-p(t)}^{t}\int_{\Omega_s}\left|\nabla\left(u-u_\infty-w\right)\right|^2 dxds\le C'_\varepsilon\left(1+\alpha\left(t-([p(t)]+[q(t)]-2)\right)\right)e^{-\sigma_\varepsilon q(t)}\\
&+\left(C_\varepsilon+1\right)\sum_{k=1}^{[q(t)]-2}\left([p(t)]+k+1\right)e^{-\sigma_\varepsilon k}\left[\max_{s\in(t-([p(t)]+k+1),t)}\left|\alpha'-\delta\right|^2+e^{-\tilde{\lambda}\alpha(t-([p(t)]+k+1))}\right],
\end{aligned}
$$

where

$$\sigma_\varepsilon = -\ln r'_\varepsilon \text{ and } C'_\varepsilon = \frac{C_0\left(1+2\left(C'\right)^2\right)C_\varepsilon}{\left(r'_\varepsilon\right)^4}\max\left(2\int_\omega |u_\infty|^2 dX_2, 1+\int_{I_0^+}|w_+|^2 dx + \int_{I_0^-}|w_-|^2 dx\right).$$

Note that we can always improve σ_ε when ε varies along the interval $\left(\frac{3+c'_\omega(1+2\delta)}{2}, \infty\right)$. This ends the proof. ☐

2.4.2 Sharper analysis

A deeper analysis shows more clearly what the above theorem hides as a convergence rate. The simplest convergence result can be state as follows.

Corollary 2.3. *In addition to the assumptions of Theorem 2.3, suppose that*

$$\alpha\left(t - ([p(t)] + [q(t)] - 2)\right)e^{-\sigma q(t)} \to 0 \quad \text{as } t \to \infty, \tag{2.41}$$

$$p(t)\max\nolimits_{s\in(t-(p(t)+q(t)-1),t)}\left|\alpha' - \delta\right|^2,\ p(t)e^{-\tilde{\lambda}\alpha(t-(p(t)+q(t)-1))} \to 0 \quad \text{as } t \to \infty. \tag{2.42}$$

Then it results that

$$\int_{\Omega_t}|(u-u_\infty-w)(t,.)|^2 dx,\ \int_{t-p(t)}^t |u-u_\infty-w|^2_{1,2,\Omega_s} ds \to 0 \quad \text{as } t\to\infty. \tag{2.43}$$

Proof. Since $\alpha'(t) \to \delta$ and $\alpha(t) \to \infty$ when $t \to \infty$, it follows from (2.42) that

$$(p(t)+1)\max\nolimits_{s\in(t-(p(t)+q(t)-1),t)}\left|\alpha'-\delta\right|^2,\ (p(t)+1)\,e^{-\tilde{\lambda}\alpha(t-(p(t)+q(t)-1))} \to 0, \quad \text{as } t\to\infty. \tag{2.44}$$

In fact (2.44) is more workable than (2.41) when we deal with the rate of convergence. Now going back to (2.35) we derive

$$\begin{aligned}
&\sum_{k=1}^{[q(t)]-2}([p(t)]+k+1)\,e^{-\sigma k}\left(\max\nolimits_{s\in(t-([p(t)]+k+1),t)}\left|\alpha'-\delta\right|^2 + e^{-\tilde{\lambda}\alpha(t-([p(t)]+k+1))}\right)\\
&\le \left(\max\nolimits_{s\in(t-(p(t)+q(t)-1),t)}\left|\alpha'-\delta\right|^2 + e^{-\tilde{\lambda}\alpha(t-(p(t)+q(t)-1))}\right)\sum_{k=1}^{[q(t)]-2}(p(t)+k+2)\,e^{-\sigma k}\\
&\le C\,(p\,(t)+1)\left(\max\nolimits_{s\in(t-(p(t)+q(t)-1),t)}\left|\alpha'-\delta\right|^2 + e^{-\tilde{\lambda}\alpha(t-(p(t)+q(t)-1))}\right) \to 0 \quad \text{as } t\to\infty,
\end{aligned}$$

where C is a positive constant independent of t. Thus, the corollary is an immediate consequence of Theorem 2.3, (2.41) and (2.44). ☐

Remark 2.9. *For any choice of α, we can ensure some convergence results, of course with respect to a convenient time interval length. This is the case for $p(t) = cst$ and $q(t) = \frac{t}{2}$. This last choice allows to treat the rate of convergence of the term $\int_{\Omega_t}|(u-u_\infty-w)(t,.)|^2 dx$ since it is independent of the choice of p, i.e. we have for some constant $\kappa > 0$*

$$|(u-u_\infty-w)(t,.)|^2_{2,\Omega_t} = O\left(\min\left(e^{-\kappa\alpha\left(\frac{t}{2}\right)}, \max\nolimits_{s\in\left(\frac{t}{2},t\right)}\left|\alpha'-\delta\right|^2\right)\right). \tag{2.45}$$

Of course, this also means that the above norm always tends to 0.

Since the rate of convergence in Theorem 2.3 depends on the rate of convergence of α' towards δ, the following lemma gives more information about the last one.

Lemma 2.6. *If $\delta = 0$, we never have $\alpha'(t) = O\left(\frac{1}{t^\theta}\right)$, for $\theta > 1$ and t large.*

Proof. Suppose by contradiction that $\alpha'(t) = O\left(\frac{1}{t^\theta}\right)$ with $\theta > 1$, then there exists a positive constant C such that, for t much bigger than 0,

$$\alpha'(t) \leq \frac{C}{t^\theta}.$$

Integrating over some interval $[s,t]$, we derive

$$\alpha(t) \leq \alpha(s) + \frac{C}{1-\theta}\left(\frac{1}{t^{\theta-1}} - \frac{1}{s^{\theta-1}}\right).$$

Passing to the limit when $t \to \infty$, we get $\lim_{t\to\infty} \alpha(t) < \infty$ which is a contradiction and hence the lemma holds. ☐

Remark 2.10. *If $\delta \neq 0$, it is possible to find α such that $|\alpha' - \delta| = O\left(e^{-\theta t}\right)$ with $\theta > 0$ (for instance $\alpha(t) = \delta t - e^{-\theta t}$).*

Thanks to the above remark, we will look for the possible exponential rate of convergence that can be ensured by Theorem 2.3. The largest convergence time interval will be also dealt with in the following Corollary.

Corollary 2.4. *If $\delta \neq 0$, suppose that the assumptions of Theorem 2.3 are satisfied and that*

$$\frac{t-p(t)}{\ln p(t)} \to \infty, \quad p(t)\max_{s\in\left(\frac{t-p(t)}{2},t\right)} \left|\alpha' - \delta\right|^2 \to 0 \quad \text{as } t \to \infty. \tag{2.46}$$

Then we have

$$\int_{t-p(t)}^{t} |u - u_\infty - w|^2_{1,2,\Omega_s}\, ds \to 0 \quad \text{as } t \to \infty. \tag{2.47}$$

Moreover if α' converges to δ exponentially, i.e.

$$\left|\alpha'(t) - \delta\right| \leq Ce^{-\beta t} \quad \text{for some } C, \beta > 0, \tag{2.48}$$

we have

$$|(u - u_\infty - w)(t,.)|_{2,\Omega_t}, \int_{\frac{t}{2}}^{t} |u - u_\infty - w|^2_{1,2,\Omega_s}\, ds \leq Ce^{-\tilde{\beta}t} \quad \text{for some } C, \tilde{\beta} > 0. \tag{2.49}$$

Proof. According to Theorem 2.3, it is enough to check that the limits (2.41), (2.44) hold and to ensure that there exists q satisfying (2.34). Since $\delta \neq 0$, it follows that $\alpha(t) \approx t$ when $t \to \infty$. Therefore, we have

$$\alpha(t - ([p(t)] + [q(t)] - 2))\, e^{-\sigma q(t)} = O\left((t - p(t) - q(t))\, e^{-\sigma q(t)}\right), \tag{2.50}$$

$$(p(t)+1)\, e^{-\tilde{\lambda}\alpha(t-(p(t)+q(t)-1))} = O\left((p(t)+1)\, e^{-\theta(t-(p(t)+q(t)))}\right), \tag{2.51}$$

for some positive constant θ, whenever $t - p(t) - q(t) \to \infty$ when $t \to \infty$. Taking $q(t) = t - p(t) - \mu \ln p(t)$, for μ large enough, we derive

$$\alpha\left(t - ([p(t)] + [q(t)] - 2)\right) e^{-\sigma q(t)} = O\left(\ln p(t)\, e^{-\sigma(t-p(t)-\mu \ln p(t))}\right),$$
$$(p(t)+1)\, e^{-\tilde{\lambda}\alpha(t-(p(t)+q(t)-1))} = O\left(e^{-\theta' \ln p(t)}\right),$$

where θ' is a positive constant independent of t. Using (2.46) we end up with (2.41), (2.44) and thus we have (2.47). Now, for the exponential rate of convergence (2.49) we consider (2.48) and take $p(t) = \frac{t}{2}$, $q(t) = \frac{t}{4}$ in (3.39) and (2.51). Then the corollary 2.4 follows from (2.35), (2.41) and (2.44). □

Remark 2.11. *i. In fact it was supposed that $p(t) \to \infty$ when $t \to \infty$ in Corollary 2.4 to consider the largest interval of convergence in (2.47).*

ii. Note that if $p(t) = t - a \ln t$, for a small enough, Theorem 2.3 does not give any information about the convergence (2.47) since the last term in (2.44) tends to ∞.

iii. Instead of the assumption $\frac{t-p(t)}{\ln p(t)} \to \infty$, in Corollary 2.4, it is enough to assume that $t - p(t) + \mu \ln p(t) \to \infty$ for a constant μ large enough.

Now we will investigate the case $\delta = 0$. Of course here the rate of convergence is at most polynomial with power less or equal than 1. However since we cannot go beyond $\frac{1}{t}$ in the rate of convergence of $\alpha'(t)$ (see Lemma 2.6), we assume that

$$\alpha'(t) \approx \frac{\ln^{\delta_1}(t)}{t^{l_1}} \text{ with } 0 \le l_1 \le 1 \text{ and } \delta_1 \in \mathbb{R}. \tag{2.52}$$

For the convergence time interval, the convenient choice for the present study can be given as

$$p(t) \approx t^{l_2} \ln^{\delta_2}(t) \text{ with } l_2 \le 1 \text{ and } \delta_2 \in \mathbb{R}. \tag{2.53}$$

Of course, to ensure that $p(t) < t$ and $\alpha'(t) \to 0$ when $t \to \infty$ we have to assume that $\delta_2 \in (-\infty, 0)$ if $l_2 = 1$ and $\delta_1 \in (-\infty, 0)$ if $l_1 = 0$. However, we can easily check that

$$t - p(t) \approx t \text{ when } t \to \infty.$$

Then we have the following result.

Corollary 2.5. *In addition to the assumptions of Theorem 2.3, we assume that (2.52) and (2.53) are satisfied. Then we have (2.47) with the rates of convergence summarized in the tables below, according to the values of the parameters.*
− *If* $0 \le l_1 < 1$

Parameters		*Rate of convergence*
l_2	δ_2	
$(-\infty, 0)$	/	$O\left(\frac{\ln^{2\delta_1}(t)}{t^{2l_1}}\right)$
0	$(-\infty, 0]$	
$l_2 = 2l_1,\ l_1 \ne 0$	$(-\infty, -2\delta_1)$	$O\left(\ln^{(\delta_2+2\delta_1)}(t)\right)$
$l_2 = l_1 = 0$	$(0, -2\delta_1)$	
$(0, 2l_1)$	/	$O\left(\frac{\ln^{(\delta_2+2\delta_1)}(t)}{t^{2l_1-l_2}}\right)$.
$l_2 = 0,\ l_1 \ne 0$	$(0, \infty)$	

− *If* $l_1 = 1$ *and* $\delta_1 \geq -1$

Cases	*Parameters*		*Rate of conv.*
A_1		$\delta_1 = -1$	$O\left(\frac{1}{\ln^{\tilde{\lambda}r}(t)}\right)$
A_2	$l_2 < 0,$	$\delta_1 \in (-1,0)$	$O\left(\frac{1}{t^{\tilde{\lambda}r}\ln^{\delta_1}(t)}\right)$
A_3	*or* $l_2 = 0$ *and* $\delta_2 \leq 0$	$\delta_1 = 0$	$O\left(\frac{1}{t^{\min(2,\tilde{\lambda}r)}}\right)$
A_4		$\delta_1 > 0$	$O\left(\frac{\ln^{2\delta_1}(t)}{t^2}\right)$
A_5	$l_2 = \tilde{\lambda}r$	$\delta_2 < 0,\ \delta_1 = 0$	$O\left(\ln^{\delta_2}(t)\right)$
A_6	$l_2 \in (0,1] \cap \left(0, \tilde{\lambda}r\right),$ *or* $l_2 = 0$ *and* $\delta_2 > 0$	$\delta_1 = 0$	$O\left(\frac{\ln^{\delta_2}(t)}{t^{\min(2,\tilde{\lambda}r)-l_2}}\right)$
A_7	$l_2 = 0$	$0 < \delta_2 < \tilde{\lambda}r,\ \delta_1 = -1$	$O\left(\ln^{\left(\delta_2 - \tilde{\lambda}r\right)}(t)\right)$
A_8	$l_2 \in (0,1],$ *or* $l_2 = 0$ *and* $\delta_2 > 0$	$\delta_1 > 0$	$O\left(\frac{\ln^{(\delta_2+2\delta_1)}(t)}{t^{2-l_2}}\right)$
A_9	$l_2 = 0$	$\delta_2 > 0,\ \delta_1 \in (-1,0)$	$O\left(\frac{\ln^{\delta_2}(t)}{t^{\tilde{\lambda}r}\ln^{\delta_1}(t)}\right).$

The constant r *is coming from the assumptions (2.52) and (2.53).*
Keeping the same assumptions, except (2.53) (which is not needed), then if $l_1 = 1$ *and* $\delta_1 \geq -1$ *the rate of convergence of* $|(u - u_\infty - w)(t,.)|^2_{2,\Omega_t} \to 0$ *is as in the above cases* $A_1,\ A_2,\ A_3,\ A_4$ *and if* $0 \leq l_1 < 1$ *we have*

$$|(u - u_\infty - w)(t,.)|_{2,\Omega_t} = O\left(\frac{\ln^{\delta_1}(t)}{t^{l_1}}\right).$$

Proof. Firstly, assume that $0 \leq l_1 < 1$. Then if $\delta_1 < 0$ (resp. $\delta_1 \geq 0$), there exist positive constants c_1, c_2 such that, for t large enough,

$$\frac{c_1}{t^{l_1} t^{\frac{1}{2}(1-l_1)}} \leq \alpha'(t) \leq \frac{c_2}{t^{l_1}} \quad \left(\text{resp. } \frac{c_1}{t^{l_1}} \leq \alpha'(t) \leq \frac{c_2}{t^{\frac{l_1}{2}}}\right).$$

Integrating over the interval $(0,t)$ we derive

$$c_1 t^{\frac{1}{2}(1-l_1)} \leq \alpha(t) \leq c_2 t^{1-l_1} \quad \left(\text{resp. } c_1 t^{1-l_1} \leq \alpha(t) \leq c_2 t^{1-\frac{l_1}{2}}\right).$$

Using this and taking $q(t) = \frac{t-p(t)}{2}$ we get, in both cases, for some $\tilde{\beta}_1, \tilde{\beta}_2 > 0$

$$\begin{cases} \alpha\left(t - ([p(t)] + [q(t)] - 2)\right) e^{-\sigma q(t)} = O\left(e^{-\tilde{\beta}_1 t}\right), \\ (p(t)+1)\, e^{-\tilde{\lambda}\alpha(t-(p(t)+q(t)-1))} = O\left((p(t)+1)\, e^{-\tilde{\lambda} c_1 t^{\frac{1}{2}(1-l_1)}}\right) = O\left(e^{-\tilde{\beta}_2 t^{\frac{1}{2}(1-l_1)}}\right). \end{cases} \tag{2.54}$$

Note that if $l_2 < 0$, or $l_2 = 0$ and $\delta_2 \leq 0$, $p(t)$ is bounded. Then, from (2.52) we deduce

$$(p(t)+1) \max_{s \in (t-(p(t)+q(t)+1),t)} \left|\alpha'\right|^2 = O\left(\frac{\ln^{2\delta_1}(t)}{t^{2l_1}}\right). \tag{2.55}$$

Next, if $l_2 > 0$, or $l_2 = 0$ and $\delta_2 > 0$, we get from (2.52) and (2.53)

$$(p(t)+1)\max_{s\in(t-(p(t)+q(t)+1),t)}\left|\alpha'\right|^2 = O\left(\frac{\ln^{(2\delta_1+\delta_2)}(t)}{t^{2l_1-l_2}}\right). \tag{2.56}$$

To deal with the global rate of convergence we will only take into account the estimates (2.55), (2.56) and we distinguish six cases. If $l_2 < 0$, or $l_2 = 0$ and $\delta_2 \le 0$ the rate of convergence is as in (2.55). Then, if $l_2 = 2l_1$, $l_1 \neq 0$ and $\delta_2 < -2\delta_1$ we obtain (2.47) with $O\left(\ln^{(\delta_2+2\delta_1)}(t)\right)$ as a rate of convergence. The same rate of convergence still takes place for the forth case $l_2 = l_1 = 0$ and $0 < \delta_2 < -2\delta_1$. For the fifth and the last cases $0 < l_2 < 2l_1$, or $l_2 = 0$, $l_1 \neq 0$ and $\delta_2 > 0$, we also have (2.47) and the rate of convergence is $O\left(\frac{\ln^{(\delta_2+2\delta_1)}(t)}{t^{2l_1-l_2}}\right)$.

Now for $l_1 = 1$, to guarantee that $\alpha(t) \to \infty$ when $t \to \infty$ we need to assume that $\delta_1 \ge -1$. That is to say

$$\begin{aligned} \alpha(t) &\approx \ln^{\delta_1+1}(t) \quad \text{for } \delta_1 > -1, \\ \alpha(t) &\approx \ln(\ln t) \quad \text{for } \delta_1 = -1. \end{aligned} \tag{2.57}$$

This implies, by choosing $q(t) = \frac{t-p(t)}{2}$, that the first estimate in (2.54), (2.55) and (2.56) still hold (with $l_1 = 1$). Here the second term in (2.54) requires a deeper analysis since we lose the exponential rate of convergence. We will take into account the above two first cases with the last two ones and we omit the third and the fourth ones.

- For the first two cases $l_2 < 0$, or $l_2 = 0$ and $\delta_2 \le 0$ (i.e. $p(t)$ is bounded) there exists a constant $r > 0$ (coming from (2.57)) such that

$$(p(t)+1)\,e^{-\tilde{\lambda}\alpha(t-(p(t)+q(t)-1))} = \begin{cases} O\left(\frac{1}{t^{\tilde{\lambda}r}\ln^{\delta_1}(t)}\right) & \text{for } \delta_1 > -1, \\ O\left(\frac{1}{\ln^{\tilde{\lambda}r}(t)}\right) & \text{for } \delta_1 = -1. \end{cases}$$

Combining the above estimates with (2.55), we end up with the following rates of convergence

$$O\left(\frac{1}{\ln^{\tilde{\lambda}r}(t)}\right) \text{ for } \delta_1 = -1, \quad O\left(\frac{1}{t^{\tilde{\lambda}r}\ln^{\delta_1}(t)}\right) \text{ for } -1 < \delta_1 < 0,$$
$$O\left(\frac{1}{t^{\min(2,\tilde{\lambda}r)}}\right) \text{ for } \delta_1 = 0, \quad O\left(\frac{\ln^{2\delta_1}(t)}{t^2}\right) \text{ for } \delta_1 > 0,$$

which shows $A_1, \ldots, A_4$.

- Now we pass to the fifth and the last cases $0 < l_2 \le 1$, or $l_2 = 0$ and $\delta_2 > 0$ (i.e. $p(t)$ is unbounded). Using (2.53) and (2.57), it is obvious that

$$(p(t)+1)\,e^{-\tilde{\lambda}\alpha(t-(p(t)+q(t)-1))} = \begin{cases} O\left(\frac{\ln^{\delta_2}(t)}{t^{-l_2+\tilde{\lambda}r}\ln^{\delta_1}(t)}\right) & \text{for } \delta_1 > -1, \\ O\left(\frac{\ln^{(\delta_2-\tilde{\lambda}r)}(t)}{t^{-l_2}}\right) & \text{for } \delta_1 = -1. \end{cases}$$

It is clear that, when $\delta_1 < 0$ and $l_2 \neq 0$ we can not deduce any information about the convergence.

However, by using (2.56) and the above estimate, the rate of convergence in (2.47) is given by

$$O\left(\ln^{\delta_2}(t)\right) \text{ for } l_2 = \tilde{\lambda} r,\ \delta_2 < 0 \text{ and } \delta_1 = 0,$$

$$O\left(\frac{\ln^{\delta_2}(t)}{t^{\min\left(2,\tilde{\lambda} r\right)-l_2}}\right) \text{ for } \left(0 < l_2 < \tilde{\lambda} r \text{ or } (l_2 = 0,\ \delta_2 > 0)\right) \text{ and } \delta_1 = 0,$$

$$O\left(\ln^{\left(\delta_2-\tilde{\lambda} r\right)}(t)\right) \text{ for } l_2 = 0,\ 0 < \delta_2 < \tilde{\lambda} r \text{ and } \delta_1 = -1,\ O\left(\frac{\ln^{(\delta_2+2\delta_1)}(t)}{t^{2-l_2}}\right) \text{ for } \delta_1 > 0,$$

$$O\left(\frac{\ln^{\delta_2}(t)}{t^{\tilde{\lambda} r}\ln^{\delta_1}(t)}\right) \text{ for } l_2 = 0,\ \delta_2 > 0 \text{ and } -1 < \delta_1 < 0,$$

which shows the remaining cases $A_5 - A_9$.
Next, in order to specify the rate of convergence (2.45), we only need to take care of the case when p is bounded, i.e. when $l_2 < 0$, or $l_2 = 0$ and $\delta_2 \leq 0$. This means that we only need to compare between the rate of convergence of $\frac{\ln^{2\delta_1}(t)}{t^{2l_1}}$ and $e^{-\tilde{\lambda} c_1 t^{\frac{1}{2}(1-l_1)}}$ when $0 \leq l_1 < 1$. When $l_1 = 1$ we compare between $\frac{\ln^{2\delta_1}(t)}{t^2}$ and

$$\begin{cases} O\left(\frac{1}{t^{\tilde{\lambda} r}\ln^{\delta_1}(t)}\right) & \text{for } \delta_1 > -1, \\ O\left(\frac{1}{\ln^{\tilde{\lambda} r}(t)}\right) & \text{for } \delta_1 = -1. \end{cases}$$

This completes the proof of the corollary 2.5. □

Remark 2.12. *Note that the omission of the case $l_2 = 1$ and $\delta_2 = 0$ from (2.53) is not justified except if the equivalence $t - p(t) \approx t$ is violated. We can guarantee this equivalence for example for $p(t) = \frac{t}{2}$ and by consequence we have the corresponding rates of convergence.*

Remark 2.13. *If the parabolic equation is homogeneous (i.e. $f = 0$), then $u_\infty = w = 0$. This means that there is no boundary layer and in the above convergences we only need to replace $u - u_\infty - w$ by u. In fact, this is true since u and its limit are in the same space $H_0^1(\mathbb{R} \times \omega)$.*

Chapter 3

Asymptotic behaviour of quasilinear parabolic equations in moving boundary domains

This chapter is devoted to study the stability of some quasilinear parabolic equations in a noncylindrical domain Q that expands as t increases. Here the Minty-Browder technique, based on some special estimates, will be used to prove the convergence theorems. Also, some sufficient conditions are formulated in order to improve the rate of convergence.

3.1 Setting the problem

We keep the notation of the chapter 2. We denote by Q the noncylindrical domain of $\mathbb{R}^+ \times \mathbb{R}^n$ defined by

$$Q = \left\{ (x,t) \mid x \in \hat{\Omega}_t \times \omega_t,\ t > 0 \right\},$$

where $\hat{\Omega}_t$ and ω_t are nondecreasing sequences of open sets of $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$ respectively, $1 \leq m < n$, such that $\hat{\Omega}_t$ (resp. ω_t) is becoming unbounded (resp. bounded) when $t \to \infty$, i.e.

$$\emptyset \not\equiv \hat{\Omega}_0 \times \omega_0 \subset\subset \hat{\Omega}_{t_1} \times \omega_{t_1} \subset\subset \hat{\Omega}_{t_2} \times \omega_{t_2} \qquad \forall 0 < t_1 < t_2,$$
$$\lim_{t\to\infty} dist\left(\hat{\Omega}_0, \partial\hat{\Omega}_t\right) = \infty \text{ and } \lim_{t\to\infty} dist\left(\omega_0, \partial\omega_t\right) < \infty.$$

We set

$$\forall t \geq 0,\ Q_t = \{(s,x) \in Q \mid s \leq t\},\ \Omega_t = \hat{\Omega}_t \times \omega_t,\ \Gamma_t = (\partial Q \diagdown \Omega_0) \cap \overline{Q}_t,$$
$$\omega = \bigcup_{t>0} \omega_t$$

and for $x \in \Omega_t$ we denote by X_1 the m first coordinates of x and by X_2 the $n-m$ last ones, i.e.

$$x = (X_1, X_2) \quad \text{with } X_1 = (x_1, \ldots, x_m),\ X_2 = (x_{m+1}, \ldots, x_n).$$

Also we use the notation

$$\nabla_{X_1} u = (\partial_{x_1} u, \ldots, \partial_{x_m} u),\ \nabla_{X_2} u = \left(\partial_{x_{m+1}} u, \ldots, \partial_{x_n} u\right).$$

Now for $u_0 \in L^2(\Omega_0)$ and $f \in L^q(\omega)$, we consider the following quasilinear parabolic boundary value problem

$$\begin{cases} u' + \mathcal{A}u = u' - \sum_{1\leq i\leq n} \partial_{x_i} a_i(x,u,\nabla u) = f(X_2) & \text{in } Q_t, \\ u = 0 & \text{on } \Gamma_t, \\ u(0,.) = u_0 & \text{on } \Omega_0, \end{cases} \tag{3.1}$$

where $a(x,\xi) = (a_i(x,\xi))_{1\leq i\leq n}$ is a family of Carathéodory functions defined on $\mathbb{R}^n \times \mathbb{R}^{n+1}$ and satisfy suitable coerciveness, monotonicity and growth conditions, i.e. for all $\xi = (\xi_i)_i$, $\xi' = (\xi'_i)_i \in \mathbb{R}^{n+1}$, $k = 1,\dots,n$ and for a.e. x in $\mathbb{R}^n$, there exist positive constants α_1 and α_2 such that

$$\sum_{1\leq i\leq n} a_i(x,\xi)\xi_i \geq \alpha_1 \sum_{1\leq i\leq n} |\xi_i|^p, \tag{3.2}$$

$$\sum_{1\leq i\leq n} \left(a_i(x,\xi) - a_i(x,\xi')\right)\left(\xi_i - \xi'_i\right) \geq 0, \tag{3.3}$$

$$(x,\xi) \longmapsto a_k(x,\xi) \text{ is measurable on } \mathbb{R}^n \times \mathbb{R}^{n+1}, \tag{3.4}$$

$$\xi \longmapsto a_k(x,\xi) \text{ is continuous on } \mathbb{R}^{n+1} \text{ for a.e. } x \text{ in } \mathbb{R}^n, \tag{3.5}$$

$$|a_k(x,\xi_0,\xi_1,\dots,\xi_n)| \leq \alpha_2 \left(g(X_2) + \sum_{0\leq i\leq n} |\xi_i|^{p-1} \right) \text{ with } g \in L^q_{loc}(\mathbb{R}^{n-m}). \tag{3.6}$$

In addition to the above assumptions, we assume that for $1 \leq i \leq n$, the coefficient a_i satisfies

$$a_i(x,\xi_0,0,\dots,0,\xi_{m+1},\dots,\xi_n) = a_i(X_2,\xi_0,0,\dots,0,\xi_{m+1},\dots,\xi_n) := a_i(X_2,\xi_0,\xi_{m+1},\dots,\xi_n). \tag{3.7}$$

Then we set

$$\mathcal{A}_\omega u_\infty := -\sum\nolimits_{m+1\leq i\leq n} \partial_{x_i} a_i(X_2, u_\infty, \nabla_{X_2} u_\infty). \tag{3.8}$$

Formally, if we pass to the limit in (3.1), when $t \to \infty$, the candidate limit is a solution to an elliptic problem defined on a lower dimensional domain ω as follows

$$\begin{cases} \mathcal{A}_\omega u_\infty = f(X_2) & \text{in } \omega, \\ u_\infty = 0 & \text{on } \partial\omega. \end{cases} \tag{3.9}$$

Of course, the existence of u a weak solution to (3.1) is unique and the limit problem (3.9) has at least one solution, which is also unique if $\mathcal{A}_\omega$ is strictly monotone (for the existence results see [9, 50, 52, 54, 57, 59]). The solution of each problem is understood in the following weak sense: find a solution $u \in \mathcal{F}(Q_t)$ (resp. $u_\infty \in W_0^{1,p}(\omega)$) such that

$$\left| \begin{array}{l} \langle u', v\rangle + \int_0^t \langle \mathcal{A}u, v\rangle_{\Omega_s} ds = \int_0^t \int_{\Omega_s} f(X_2) v dx ds \ \forall v \in \mathcal{F}(Q_t), \\ u(0,.) = \ u_0, \end{array} \right. \tag{3.10}$$

and

$$\langle \mathcal{A}_\omega u, v\rangle_\omega = \int_\omega \sum_{m+1\leq i\leq n} a_i(X_2, u_\infty, \nabla_{X_2} u_\infty)\partial_{x_i} v \, dX_2 = \int_\omega f(X_2) v \, dX_2 \quad \forall v \in W_0^{1,p}(\omega), \tag{3.11}$$

where

$$\mathcal{F}(Q_t) = \left\{ u \in V_t;\ u' \in V'_t \right\} \text{ with } V_t = \left\{ v \in L^2(Q_t) \mid v(s,.) \in W_0^{1,p}(\Omega_s),\ \forall s \leq t \right\},$$

V'_t is its dual.

3.2 Estimate results

We start with some useful estimates of the solution u of (3.1) that will play an essential role in the proof of our main results. Let $u_0^1, u_0^2 \in L^2(\Omega_0)$ and $f_1, f_2 \in L^q(\mathbb{R}^m \times \omega)$. Then we have

Proposition 3.1 (The Weak Maximum Principal). *Under the assumptions (3.2)-(3.6), let u_1 (resp. u_2) be the weak solution of (3.1), when we replace (u_0, f) by (u_0^1, f_1) (resp. (u_0^2, f_2)). If we suppose that $u_0^1 \leq u_0^2$ a.e. on Ω_0 and $f_1 \leq f_2$ a.e. on $\mathbb{R}^m \times \omega$, then for every $t > 0$ it holds that*

$$u_1 \leq u_2 \quad a.e.\ on\ \Omega_t. \tag{3.12}$$

We obtain the same result for the problem (3.9), if we substitute the monotonicity of $\mathcal{A}_\omega$ by its strict monotonicity and of course $f_1, f_2 \in L^q(\omega)$.

Proof. By the comparison of the two identities obtained from (3.10), replacing (u_0, f) by (u_0^1, f_1) and (u_0^2, f_2), we deduce

$$< (u_1 - u_2)', v > + \int_0^t \langle \mathcal{A}u_1 - \mathcal{A}u_2, v \rangle_{\Omega_s} ds = \int_0^t \int_{\Omega_s} (f_1 - f_2)\, v dx ds, \tag{3.13}$$

for all $v \in \mathcal{F}(Q_t)$. Since $v = (u_1 - u_2)^+ \in \mathcal{F}(Q_t)$, we can take it as a test function in (3.13). Then, from the inequality $f_1 \leq f_2$ we have

$$< (u_1 - u_2)', (u_1 - u_2)^+ > + \int_0^t \int_{\Omega_s} \sum_{1 \leq i \leq n} (a_i(x, u_1, \nabla u_1) - a_i(x, u_2, \nabla u_2)) \partial_{x_i} (u_1 - u_2)^+ dx ds \leq 0.$$

The monotonicity condition (3.3) allows to write

$$< (u_1 - u_2)', (u_1 - u_2)^+ > \leq 0.$$

Using the integration by parts formula (see [53]) and the assumption $u_0^1 \leq u_0^2$ we obtain

$$\left|(u_1 - u_2)^+ (t, .)\right|_{2,\Omega_t}^2 \leq \left|\left(u_0^1 - u_0^2\right)^+\right|_{2,\Omega_0}^2 = 0,$$

this shows that $(u_1 - u_2)^+ (t, .) = 0$ and by consequence (3.12). The proof of the second part of the proposition is similar. □

Now we use this proposition to prove the following Lemmas. First, we suppose that

$$u_0 \in L^\infty(\Omega_0) \text{ with } M = |u_0|_{\infty,\Omega_0} \tag{3.14}$$

and we set

$$\frac{1}{p_0} - \frac{1}{n - m} = \frac{1}{2}.$$

Then, we have

Lemma 3.1. *Under the assumptions (3.2)-(3.7) and (3.14), we have the following assertions.*

(i) *Assume that f and u_0 are nonnegative (or nonpositive for an analogous statement), then u_∞ and u are also nonnegative (or nonpositive). Moreover, for $p \geq p_0$, it holds that*

$$u \leq u_\infty + M \;\; (or\ u_\infty - M \leq u).$$

(ii) *Let $u_{\infty,+}$ (resp. $u_{\infty,-}$) be the weak solution of (3.9), obtained if we replace f by f^+ (resp. $-f^-$), then for $p \geq p_0$ we have*

$$u_{\infty,-} - M \leq u \leq u_{\infty,+} + M.$$

Proof. (i) Taking $v = u_\infty^- \in W_0^{1,p}(\omega)$ in (3.11) we get

$$\left\langle \mathcal{A}_\omega u_\infty, u_\infty^- \right\rangle_\omega = \int_\omega f(X_2) u_\infty^- dX_2 \geq 0.$$

Using the coerciveness condition (3.2), we obtain

$$\alpha_1 \int_\omega \left|\nabla_{X_2} u_\infty^-\right|^p dX_2 \leq 0,$$

then from the Poincaré inequality we have

$$\int_\omega \left|u_\infty^-\right|^p dX_2 \leq 0.$$

Hence, we derive that u_∞ is nonnegative. For the positivity of u, we test (3.10) by u^-, we derive by using the integration by parts formula and following the same argument as above that $u \geq 0$.
Now taking $v \in \mathcal{F}(Q_t)$ as a test function in (3.11), integrating in X_1 and t directions then comparing with (3.10), we get

$$\left\langle u', v \right\rangle + \int_0^t \int_{\Omega_s} \sum_{1\leq i\leq n} \left[a_i(x,u,\nabla u) - a_i(x,u_\infty,\nabla u_\infty)\right] \partial_{x_i} v dxds$$
$$= -\int_0^t \int_{\Omega_s} \sum_{1\leq i\leq m} a_i(x,u_\infty,\nabla u_\infty) \partial_{x_i} v dxds.$$

As $\nabla_{X_1} u_\infty = 0$ it follows from (3.7) that

$$\left\langle u', v \right\rangle + \int_0^t \int_{\Omega_s} \sum_{1\leq i\leq n} \left[a_i(x,u,\nabla u) - a_i(x,u_\infty,\nabla u_\infty)\right] \partial_{x_i} v dxds = -\int_{\Gamma_t} \sum_{1\leq i\leq m} a_i(x,u_\infty,\nabla u_\infty) v\nu_i dx$$
$$= 0. \tag{3.15}$$

In addition, since $p \geq p_0$ and $u_\infty \geq 0$, we also have $(u - u_\infty - M)^+ \in \mathcal{F}(Q_t)$. Therefore we can take it as a test function in (3.15) to get

$$\left\langle (u - u_\infty - M)', (u - u_\infty - M)^+ \right\rangle$$
$$+ \int_0^t \int_{\Omega_s} \sum_{1\leq i\leq n} \left[a_i(x,u,\nabla u) - a_i(x,u_\infty,\nabla u_\infty)\right] \partial_{x_i} (u - u_\infty - M)^+ dxds = 0.$$

This implies

$$\left\langle \left((u-u_\infty-M)^+\right)', (u-u_\infty-M)^+\right\rangle + \int_{u-u_\infty>M}\sum_{1\le i\le n}\left(a_i(x,u,\nabla u)-a_i(x,u_\infty,\nabla u_\infty)\right)\partial_{x_i}(u-u_\infty)\,dxds=0.$$

Using the integration by parts formula and the monotonicity condition (3.3), we deduce

$$\left|(u-u_\infty-M)^+(t,.)\right|^2_{2,\Omega_t}\le\left|(u_0-u_\infty-M)^+\right|^2_{2,\Omega_0}=0,$$

whence

$$(u-u_\infty-M)^+=0.$$

This shows that $u\le u_\infty+M$. For the second case where $u_0, f\le 0$, we use the same approach choosing the positive or negative parts of the above test functions.

(ii) Let u_+ (resp. u_-) be the weak solution of (3.1), obtained when we replace (u_0,f) by $\left(u_0^+, f^+\right)$ (resp. $\left(-u_0^-, -f^-\right)$). Noting that

$$-u_0^-\le u_0\le u_0^+ \text{ and } -f^-\le f\le f^+,$$

then from Proposition 3.1, it holds that

$$u_-\le u\le u_+$$

and thanks to (i), we get

$$u_{\infty,-}-M\le u_-,\ u_+\le u_{\infty,+}+M.$$

This completes the proof of the lemma. □

Remark 3.1. *Set $F=-\sum_{i=m+1}^n\partial_{x_i}a_i(X_2,0,0)\in W^{-1,q}(\omega)$. Using Proposition 3.1 we deduce that u_∞ is nonnegative whenever $f\ge F$ in the distributional sense. In addition if u_0 is nonnegative, u is also nonnegative. Further, if F is a function in $L^q(\omega)$, we can replace f^+ (resp. f^-) by $F+(f-F)^+$ (resp. $(F-f)^+-F$) in the proofs whenever f^+ (resp. f^-) is used and by consequence we can cover Lemma 3.1.*

Remark 3.2. (i) *For a more general class of operators, as*

$$\begin{aligned}\check{\mathcal{A}}u&=\mathcal{A}u+a_0(x,u,\nabla u)=-\sum_{1\le i\le n}\partial_{x_i}a_i(x,u,\nabla u)+a_0(x,u,\nabla u),\\ \check{\mathcal{A}}_\omega u_\infty&=\mathcal{A}_\omega u_\infty+a_0(X_2,u_\infty,\nabla_{X_2}u_\infty)=-\sum_{m+1\le i\le n}\partial_{x_i}a_i(X_2,u_\infty,\nabla_{X_2}u_\infty)+a_0(X_2,u_\infty,\nabla_{X_2}u_\infty),\end{aligned}\tag{3.16}$$

the foregoing hypotheses assumed on the operator $\mathcal{A}$ can be adapted to the above operators (see [66]), although the argument used to show the above estimate is not workable for $u_0\in L^\infty(\Omega_0)$. In order to run the same argument and get an equivalent result we use the sub-supersolution to control the initial condition. Assume that

$$\underline{u}_\infty\le u_0\le\bar{u}_\infty,\tag{3.17}$$

where $\bar{u}_\infty\in W^{1,p}(\omega)$ is any weak positive supersolution and $\underline{u}_\infty\in W^{1,p}(\omega)$ is any weak negative subsolution of problem (3.9), then the solution of Problem (3.1) stays between the same sub-supersolution.

In fact, the sets of positive supersolutions and negative subsolutions are not empty, since there exists at least one solution of Problem (3.9), replacing f by f^+ for the first set and for the second one we just replace f by $-f^-$ to find a subsolution. Next, let u_+ (resp. u_-) be the weak solution of (3.1), obtained if we replace u_0 by $\bar{u}_\infty$ (resp. $\underline{u}_\infty$) taking into account the operators defined in (3.16). Then, comparing the equations satisfied by u_+ and $\bar{u}_\infty$(resp. u_- and $\underline{u}_\infty$) yields

$$\left\langle (u_+ - \bar{u}_\infty)', v \right\rangle + \int_0^t \int_{\Omega_s} \sum_{0\leq i\leq n} \left[a_i(x, u_+, \nabla u_+) - a_i(X_2, \bar{u}_\infty, \nabla_{X_2}\bar{u}_\infty)\right] \partial_{x_i} v dx ds \leq 0, \tag{3.18}$$

respectively,

$$\left\langle (\underline{u}_\infty - u_-)', v \right\rangle + \int_0^t \int_{\Omega_s} \sum_{0\leq i\leq n} \left[a_i(X_2, \underline{u}_\infty, \nabla_{X_2}\underline{u}_\infty) - a_i(x, u_-, \nabla u_-)\right] \partial_{x_i} v dx ds \leq 0, \tag{3.19}$$

for all $v \in \mathcal{F}(Q_t)$ and $v \geq 0$. Since $p \geq p_0$, $\bar{u}_\infty \geq 0$ and $\underline{u}_\infty \leq 0$, it follows that $(u_+ - \bar{u}_\infty)^+$, $(\underline{u}_\infty - u_-)^+ \in \mathcal{F}(Q_t)$, we can take them as test functions in (3.18) and (3.19) respectively. Using the integration by parts and the monotonicity condition (3.3) we deduce

$$\left|(u_+ - \bar{u}_\infty)^+ (t,.)\right|^2_{2,\Omega_t}, \left|(\underline{u}_\infty - u_-)^+ (t,.)\right|^2_{2,\Omega_t} \leq 0,$$

which implies that

$$\underline{u}_\infty \leq u_-,\ u_+ \leq \bar{u}_\infty.$$

Consequently, we derive easily from Proposition 3.1 that

$$\underline{u}_\infty \leq u \leq \bar{u}_\infty. \tag{3.20}$$

(*ii*) *The forthcoming results of this section can be easily extended to the operators (3.16) but for simplicity reasons, we keep the initial choice $\mathcal{A}$, while (3.20) will be used again.*

For $1 < p < p_0$, the solutions of problem (3.9) do not necessary belong to $L^2(\omega)$, and hence the results of Lemma 3.1 and Remark 3.2 cannot be proved by the same argument. To secure these results, we assume that the operator $\mathcal{A}$ is strictly monotone, i.e.

$$\sum_{1\leq i\leq n} \left(a_i(x,\xi) - a_i(x,\xi')\right)\left(\xi_i - \xi_i'\right) > 0. \tag{3.21}$$

Then, we have

Lemma 3.2. *For $1 < p < p_0$. Suppose that Assumptions (3.2), (3.4)-(3.7) and (3.21) hold. Then we have the following assertions.*

(*i*) *Under the assumption (3.14), let $u_{\infty,-}, u_{\infty,+} \in W_0^{1,p}(\omega)$ be respectively solutions of problem (3.9) replacing f by $-f^-$ and f^+. Then we have*

$$u_{\infty,-} - M \leq u \leq u_{\infty,+} + M. \tag{3.22}$$

(*ii*) *For $\underline{u}_\infty \leq u_0 \leq \bar{u}_\infty$, we have*

$$\underline{u}_\infty \leq u \leq \bar{u}_\infty,$$

where $\bar{u}_\infty \in W^{1,p}(\omega)$ (resp. $\underline{u}_\infty \in W^{1,p}(\omega)$) is any weak supersolution (resp. subsolution) of problem (3.9) replacing f by f^+ (resp. $-f^-$).

Proof. (i) Using Lemma 3.1, we obtain

$$u_{\infty,-}, u_- \leq 0, \; u_{\infty,+}, u_+ \geq 0 \text{ and } u_- \leq u \leq u_+.$$

Now, we follow the same argument as in the proof of Lemma 3.1, we end up with (3.22). In fact, since $u_{\infty,-} \in W_0^{1,p}(\omega)$, there exists a negative sequence $(\varphi_k)_k \subset \mathcal{D}(\omega)$ such that

$$\varphi_k \to u_{\infty,-}, \text{ as } k \to \infty, \text{ in } W^{1,p}(\omega).$$

Taking $v = (u_- - \varphi_k + M)^- \in \mathcal{F}(Q_t)$ in the following identity

$$\langle u'_-, v\rangle + \int_0^t \int_{\Omega_s} \sum_{1\leq i\leq n} [a_i(x, u_-, \nabla u_-) - a_i(x, u_{\infty,-}, \nabla u_{\infty,-})]\, \partial_{x_i} v dxds = 0,$$

we get, from the integration by parts formula

$$\int_0^t \int_{\Omega_s} \sum_{1\leq i\leq n} (a_i(x, u_-, \nabla u_-) - a_i(x, u_{\infty,-}, \nabla u_{\infty,-}))\, \partial_{x_i} (u_- - \varphi_k + M)^- \, dxds \geq 0.$$

Passing to the limit as $k \to \infty$, we eventually obtain

$$\int_{u_- - u_{\infty,-} < -M} \sum_{1\leq i\leq n} (a_i(x, u_-, \nabla u_-) - a_i(x, u_{\infty,-}, \nabla u_{\infty,-}))\, \partial_{x_i} (u_- - u_{\infty,-})\, dxds \leq 0.$$

Since the operator $\mathcal{A}$ is strictly monotone, it follows that

$$u_- \geq u_{\infty,-} - M.$$

For the second inequality, we use the same above approach we derive that

$$u_+ \leq u_{\infty,+} + M.$$

(ii) The proof of this point is in principle the same as the proof of Remark 3.2, apart from a few exceptions. So, let $\tilde{u}_+$ (resp. $\tilde{u}_-$) be the weak solution of (3.1) obtained if we choose $0 \leq u_0 \leq \bar{u}_\infty$ (resp. $0 \geq u_0 \geq \underline{u}_\infty$) and replace f by f^+ (resp.$-f^-$). Since $\bar{u}_\infty \geq 0$ (resp. $\underline{u}_\infty \leq 0$), there exists a positive (resp. negative) sequence $(\varphi_k)_k$ (resp. $(\psi_k)_k$) such that

$$(\varphi_k)_k, (\psi_k)_k \subset C^\infty(\omega) \text{ and } \phi_k \to \bar{u}_\infty \text{ (resp. } \psi_k \to \underline{u}_\infty) \text{ in } W^{1,p}(\omega), \text{ as } k \to \infty.$$

Taking $v = (\tilde{u}_+ - \varphi_k)^+, (\psi_k - \tilde{u}_-)^+ \in \mathcal{F}(Q_t)$, respectively as test function in the following inequalities

$$\langle (\tilde{u}_+)', v\rangle + \int_0^t \int_{\Omega_s} \sum_{0\leq i\leq n} [a_i(x, \tilde{u}_+, \nabla \tilde{u}_+) - a_i(X_2, \bar{u}_\infty, \nabla_{X_2} \bar{u}_\infty)]\, \partial_{x_i} v dxds \leq 0,$$

$$\langle (-\tilde{u}_-)', v\rangle + \int_0^t \int_{\Omega_s} \sum_{0\leq i\leq n} [a_i(X_2, \underline{u}_\infty, \nabla_{X_2} \underline{u}_\infty) - a_i(x, \tilde{u}_-, \nabla \tilde{u}_-)]\, \partial_{x_i} v dxds \leq 0 \;\; \forall v \in \mathcal{F}(Q_t), \; v \geq 0.$$

Then, using the integration by parts formula, we get from the fact $\tilde{u}_+ \geq 0$ and $\tilde{u}_- \leq 0$ that

$$\int_0^t \int_{\Omega_s} \sum_{0\leq i\leq n} \left[a_i(x,\tilde{u}_+,\nabla\tilde{u}_+) - a_i(X_2,\bar{u}_\infty,\nabla_{X_2}\bar{u}_\infty)\right] \partial_{x_i} \left(\tilde{u}_+ - \varphi_k\right)^+ dxds \leq \frac{1}{2}\int_{\Omega_0} u_0 \left(u_0 - \varphi_k\right)^+ dx$$

and

$$\int_0^t \int_{\Omega_s} \sum_{0\leq i\leq n} \left[a_i(X_2,\underline{u}_\infty,\nabla_{X_2}\underline{u}_\infty) - a_i(x,\tilde{u}_-,\nabla\tilde{u}_-)\right] \partial_{x_i} \left(\psi_k - \tilde{u}_-\right)^+ dxds \leq -\frac{1}{2}\int_{\Omega_0} u_0 \left(\psi_k - u_0\right)^+ dx.$$

Passing to the limit as $k \to \infty$, taking into account $\underline{u}_\infty \leq u_0$ (resp. $u_0 \leq \bar{u}_\infty$) we derive

$$\int_0^t \int_{\Omega_s} \sum_{0\leq i\leq n} \left[a_i(x,\tilde{u}_+,\nabla\tilde{u}_+) - a_i(X_2,\bar{u}_\infty,\nabla_{X_2}\bar{u}_\infty)\right] \partial_{x_i} \left(\tilde{u}_+ - \bar{u}_\infty\right)^+ dxds \leq 0,$$

respectively

$$\int_0^t \int_{\Omega_s} \sum_{0\leq i\leq n} \left[a_i(X_2,\underline{u}_\infty,\nabla_{X_2}\underline{u}_\infty) - a_i(x,\tilde{u}_-,\nabla\tilde{u}_-)\right] \partial_{x_i} \left(\underline{u}_\infty - \tilde{u}_-\right)^+ dxds \leq 0.$$

Finally, since the operator $\mathcal{A}$ is strictly monotone we have

$$\underline{u}_\infty \leq \tilde{u}_- \text{ and } \tilde{u}_+ \leq \bar{u}_\infty.$$

This achieves the proof. □

Since the argument used in this work requires to pass by some monotonicity properties of the solution, the following lemma plays an essential role in this way. Let us write for simplicity, $\mathcal{T}_{\lambda,t}u := \mathcal{T}_\lambda u$. Then we have

Lemma 3.3. *Suppose that the hypotheses (3.2)-(3.6) hold and there exists a strictly decreasing sequence of positive numbers t_i with positive limit $\tilde{t}$. If*

$$u\left(t_i, x\right) \geq u\left(\tilde{t}, x\right) \quad \forall i \in \mathbb{N} \quad \textit{for a.e. } x \in \Omega_{\tilde{t}},$$

then for non-negative data f and u_0, the solution $u\left(., x\right)$ is non-decreasing beyond $\tilde{t}$ for almost all x, i.e.

$$u\left(t', x\right) \leq u\left(t, x\right) \;\; \forall t, t' > 0, \;\; \tilde{t} \leq t' \leq t \; \textit{and for a.e. } x \in \Omega_{t'}.$$

Proof. Let t, t' and λ be positive numbers such that $t > t' > \lambda$. Since $W_0^{1,p}\left(\Omega_s\right) \subset W_0^{1,p}\left(\Omega_{s+(t-t')}\right)$, $\forall s \leq t$, it follows from (3.1) that

$$< \left(\mathcal{T}_{(t-t')}u - u\right)', v >_{\Omega_s} + \int_{\Omega_s} \sum_{1\leq i\leq n} \left(a_i(x,\mathcal{T}_{(t-t')}u,\nabla\mathcal{T}_{(t-t')}u) - a_i(x,u,\nabla u)\right) \partial_{x_i} v dx = 0,$$

for all $v \in W_0^{1,p}(\Omega_s)$. As f and u_0 are non-negative, it follows from Lemma 3.1 that $u \geq 0$. Thus we can take $v = \left(\mathcal{T}_{(t'-t)}u - u\right)^-(s) \in W_0^{1,p}(\Omega_s)$ as a test function in the above identity to get

$$< \left(\mathcal{T}_{(t-t')}u - u\right)', \left(\mathcal{T}_{(t-t')}u - u\right)^- >_{\Omega_s} + \int_{\Omega_s} \sum_{1\leq i\leq n} \left[a_i(x, \mathcal{T}_{(t-t')}u, \nabla \mathcal{T}_{(t-t')}u) - a_i(x, u, \nabla u)\right] \partial_{x_i} \left(\mathcal{T}_{(t-t')}u - u\right)^- dx = 0.$$

Integrating over $(t'-\lambda, t')$, using the integration by parts formula and the monotonicity condition (3.3), we derive

$$\int_{\Omega_{t'}} \left|\left(u(t,.) - u(t',.)\right)^-\right|^2 dx \leq \int_{\Omega_{t'-\lambda}} \left|\left(u(t-\lambda,.) - u(t'-\lambda,.)\right)^-\right|^2 dx. \tag{3.23}$$

In fact this results that $u(.,x)$ is increasing for a.e. $x \in \Omega_{t'}$. Indeed, suppose by contradiction that $u(.,x)$ is not increasing in t. Then there exist $t > t' > \tilde{t}$ such that

$$\int_{\Omega_{t'}} \left|\left(u(t,.) - u(t',.)\right)^-\right|^2 dx > 0.$$

Since the function $r \to \left|\left(u(r,.) - u(t',.)\right)^-\right|_{2,\Omega_{t'}}$ is continuous, we can choose $\gamma > 0$ with $t - \gamma \geq t'$ such that

$$\left|\left(u(r,.) - u(t',.)\right)^-\right|_{2,\Omega_{t'}} > 0 \ \forall r \in (t-\gamma, t+\gamma).$$

By choosing t_i such that $t_i - \tilde{t} < \gamma$, there exists $k \in \mathbb{N}$ such that $t' + k\left(t_i - \tilde{t}\right) \in (t-\gamma, t+\gamma)$. Thus

$$\left|\left(u\left(t' + k\left(t_i - \tilde{t}\right),.\right) - u(t',.)\right)^-\right|_{2,\Omega_{t'}} > 0.$$

Combining this with (3.23) it follows that

$$\begin{aligned}
0 &< \left|\left(u\left(t' + k\left(t_i - \tilde{t}\right),.\right) - u(t',.)\right)^-\right|_{2,\Omega_{t'}} \\
&\leq \left|\left(u\left(\tilde{t} + k\left(t_i - \tilde{t}\right),.\right) - u(\tilde{t},.)\right)^-\right|_{2,\Omega_{\tilde{t}}} \\
&\leq \left|\left(u\left(\tilde{t} + k\left(t_i - \tilde{t}\right),.\right) - u\left(\tilde{t} + (k-1)\left(t_i - \tilde{t}\right),.\right)\right)^-\right|_{2,\Omega_{\tilde{t}}} \\
&+ \ldots + \left|\left(u(t_i,.) - u(\tilde{t},.)\right)^-\right|_{2,\Omega_{\tilde{t}}} \\
&\leq k\left|\left(u(t_i,.) - u(\tilde{t},.)\right)^-\right|_{2,\Omega_{\tilde{t}}} = 0.
\end{aligned}$$

This is a contradiction and Lemma 3.3 is proved. □

Let $\tilde{u}_0$ be a positive subsolution of an elliptic problem defined on Ω_0 as follows

$$\tilde{u}_0 \in W_0^{1,p}(\Omega_0), \int_{\Omega_0} \sum_{1\leq i\leq n} a_i(x, \tilde{u}_0, \nabla \tilde{u}_0)\partial_{x_i} v dx \leq \int_{\Omega_0} f v dx \quad \forall v \in W_0^{1,p}(\Omega_0), \ v \geq 0. \tag{3.24}$$

Then we have the following result as an immediate consequence of this Lemma. In fact, it services as a model example of Lemma 3.3.

Corollary 3.1. *For $p \geq p_0$. Assume that $u_0 = \tilde{u}_0$ and the hypotheses of Lemma 3.3 hold. Then $u(.,x)$ is increasing for almost all x.*

Proof. It is enough to prove that $u(t,x) \geq \tilde{u}_0,\ \forall t > 0$ and for a.e. $x \in \Omega_0$. Then the corollary obviously follows from Lemma 3.3. Let $v \in W_0^{1,p}(\Omega_0)$. It is clear that the 0−extension $\tilde{v}$ of v is an element of $\mathcal{F}(Q_t)$, thus we can take it as a test function in (3.10) to get

$$\left\langle u', \tilde{v} \right\rangle + \int_0^t \int_{\Omega_s} \sum_{1\leq i\leq n} a_i(x,u,\nabla u)\partial_{x_i}\tilde{v}dxds = \int_0^t \int_{\Omega_s} f(X_2)\tilde{v}dxds.$$

Comparing this with (3.24), we derive

$$\int_0^t < (\tilde{u}_0 - u)', v >_{\Omega_0} ds + \int_0^t \int_{\Omega_0} \sum_{1\leq i\leq n} \left(a_i(x,\tilde{u}_0,\nabla\tilde{u}_0) - a_i(x,u,\nabla u)\right)\partial_{x_i}vdxds \leq 0. \tag{3.25}$$

Since $\tilde{u}_0 \in W_0^{1,p}(\Omega_0)$, testing (3.25) with $(\tilde{u}_0 - u)^+ \in W_0^{1,p}(\Omega_0)$, using the integration by parts and the monotonicity condition (3.3) we deduce

$$\left|(\tilde{u}_0 - u(t,.))^+\right|_{2,\Omega_0}^2 \leq 0,$$

of course, $f \geq 0$ and $\tilde{u}_0 \geq 0$ which follows from Lemma 3.1 are used. Hence, we derive that

$$\tilde{u}_0(x) \leq u(t,x) \quad \text{a.e. } x \in \Omega_0.$$

This completes the proof. □

Let us conclude this subsection by dealing with the energy estimates. As a boundary layer occurs on a neighborhood of the boundary $\partial\hat{\Omega}_t \times \omega$, we will first take into account the energy of the problem on subdomains located far from this boundary. Of course since the main tool (Lemma 3.1) is ready, let us develop this more precisely. For a fixed $t > 0$, let λ be a positive constant with $\lambda < t - 1$, O be an open bounded domain of $\hat{\Omega}_{t-\lambda}$ such that its 1-neighborhood O_1 is also in $\hat{\Omega}_{t-(\lambda+1)}$ and let ϱ be a smooth cut-off function defined on $(0,t)\times\mathbb{R}^m$ such that

$$\rho = 1 \text{ on } (t-\lambda,t)\times O, \quad \rho = 0 \text{ on } (0,t)\times\mathbb{R}^m \backslash (t-\lambda-1,t)\times O_1,$$
$$0 \leq \rho \leq 1,\ |\partial_t\rho|, |\nabla_{X_1}\rho| \leq c,$$

where c is a positive constant. Taking $v = u\varrho^p \in \mathcal{F}(Q_t)$ in (3.10), we obtain

$$< \left(\varrho^{\frac{p}{2}}u\right)', \varrho^{\frac{p}{2}}u > + \int_0^t \int_{\Omega_s} \sum_{1\leq i\leq n} \varrho^p a_i(x,u,\nabla u)\partial_{x_i}udxds = \int_0^t \int_{\Omega_s} \varrho^p fudxds$$
$$+ \frac{p}{2}\int_0^t \int_{\Omega_s} \partial_t\varrho\varrho^{p-1}u^2dxds - p\int_0^t \int_{\Omega_s} \varrho^{p-1} \sum_{1\leq i\leq m} a_i(x,u,\nabla u)\partial_{x_i}\varrho udxds.$$

Applying the integration by parts formula, taking into account the fact that $\varrho(0) = 0$ on Ω_0 and using the coerciveness and the growth conditions (3.2) and (3.6), we get

$$\frac{1}{2}\left|\left(\varrho^{\frac{p}{2}}u\right)(t,.)\right|_{2,\Omega_t}^2 + \alpha_1\int_0^t \int_{\Omega_s} \sum_{1\leq i\leq n} \rho^p \left|\partial_{x_i}u\right|^p dxds \ \leq \int_0^t \int_{\Omega_s} \varrho^p fudxds + \frac{p}{2}\int_0^t \int_{\Omega_s} \partial_t\varrho\varrho^{p-1}u^2dxds$$
$$+ p\alpha_2\int_0^t \int_{\Omega_s} \varrho^{p-1} \sum_{1\leq i\leq m} \left(g(X_2) + |u|^{p-1} + \sum_{1\leq j\leq n} \left|\partial_{x_j}u\right|^{p-1}\right)|u|\left|\partial_{x_i}\varrho\right| dxds.$$

Using the Young inequality and according to the definition of ϱ we get

$$|u(t,.)|^2_{2,O\times\omega_t} + \int_{t-\lambda}^{t}\int_{O\times\omega_t} |\nabla u|^p \, dxds \ \le C \int_{t-\lambda-1}^{t}\int_{O_1\times\omega} \left(|f|^q + |g|^q + u^2 + |u|^p\right) dxds, \tag{3.26}$$

for some positive constant C independent of t. By virtue of Lemma 3.1, we have

$$|u| \le \max\left(|u_{\infty,+} + M|, |u_{\infty,-} - M|\right).$$

Combining this with (3.26) yields

$$|u(t,.)|^2_{2,O\times\omega_t} + |u|^p_{L^p(t-\lambda,t;W^{1,p}(O\times\omega_t))} \le C(\lambda+1)\, mes\left(O_1\right). \tag{3.27}$$

This implies that

$$|u(t,.)|^2_{2,O\times\omega} + |u|^p_{L^p(t-\lambda,t;W^{1,p}(O\times\omega))} \le C(\lambda+1)\, mes\left(O_1\right). \tag{3.28}$$

Using now (3.6) we have for any $v \in W_0^{1,p}(O\times\omega)$

$$\begin{aligned}
\left|\langle \mathcal{A}u, v\rangle_{O\times\omega}\right| &= \left|\int_{O\times\omega} \sum_{1\le i\le n} (a_i(x,u,\nabla u)\partial_{x_i} v dx\right| \\
&\le \alpha_2 \int_{O\times\omega}\left(\sum_{0\le j\le n} \left|\partial_{x_j} u\right|^{p-1} + g(X_2)\right) \sum_{1\le i\le n} |\partial_{x_i} v|\, dx.
\end{aligned}$$

Applying the Hölder inequality ($q(p-1)=p$) we obtain

$$\left|\langle \mathcal{A}u, v\rangle_{O\times\omega}\right| \le C\left(\int_{O\times\omega}\sum_{0\le i\le n} |\partial_{x_i} u|^p \, dx + \int_{O\times\omega} |g(X_2)|^q \, dx\right)^{\frac{1}{q}} \left(\int_{O\times\omega}\sum_{1\le i\le n} |\partial_i v|^p \, dx\right)^{\frac{1}{p}}.$$

Consequently

$$|\mathcal{A}u|^q_{-1,p,O\times\omega} \le C^q \left(\int_{O\times\omega}\sum_{0\le i\le n} |\partial_{x_i} u|^p \, dx + \int_{O\times\omega} |g(X_2)|^q \, dx\right).$$

Integrating over $(t-\lambda, t)$ and using (3.28) we derive

$$|\mathcal{A}u|^q_{L^q(t-\lambda,t;W^{-1,q}(O\times\omega))} \le C(\lambda+1)\, mes\left(O_1\right). \tag{3.29}$$

On the other hand it follows easily from (3.6) and (3.28) that

$$\int_{t-\lambda}^{t}\int_{O\times\omega} |(a_i(x,u,\nabla u)|^q \, dxds \le C(\lambda+1)\, mes\left(O_1\right). \tag{3.30}$$

Now due to (3.29) we can easily see that

$$|u'|^q_{L^q(t-\lambda,t;W^{-1,q}(O\times\omega))} \le C(\lambda+1)\, mes\left(O_1\right). \tag{3.31}$$

Finally, we can state the following lemma.

Lemma 3.4. *Under the assumptions of Lemma 3.1. Let λ be a positive constant and O be any bounded domain of $\mathbb{R}^m$, then for t sufficiently large, there exists a constant C depending only on ω such that*

$$
\begin{aligned}
&|u|^p_{L^p(t-\lambda,t;W^{1,p}(O\times\omega))} \leq C(\lambda+1)\, mes\,(O_1)\,,\\
&|a_i(x,u,\nabla u)|^q_{L^q(t-\lambda,t;L^q(O\times\omega))} \leq C(\lambda+1)\, mes\,(O_1)\,,\\
&|\mathcal{A}u|^q_{L^q(t-\lambda,t;W^{-1,q}(O\times\omega))} \leq C(\lambda+1)\, mes\,(O_1)\,,\\
&\left|u'\right|^q_{L^q(t-\lambda,t;W^{-1,q}(O\times\omega))} \leq C(\lambda+1)\, mes\,(O_1)\,,
\end{aligned}
$$

where $O_1 \subset \hat{\Omega}_{t-\lambda}$ is the 1-neighbourhood of O.

Remark 3.3. *For $1 < p < p_0$, the results of Corollary 3.1 (resp. Lemma 3.4) remain the same if the operator $\mathcal{A}$ is strictly monotone and the hypotheses of Lemma 3.3 are satisfied (resp. the solution u of (3.1) is increasing positive (or decreasing negative) function with respect t and also the assumptions of Lemma 3.2 are fulfilled).*

3.3 Asymptotic behaviour

In the following we will deal with the limit behaviour of u as $t \to \infty$, taking into account that X_1 may also go to ∞. First we take $u_0 = 0$, or $u_0 = \tilde{u}_0$ (solution to (3.24)) and we prove that the equation in the steady state problem is elliptic defined on $\mathbb{R}^m \times \omega$, then by showing that the limit is independent of X_1 we end up with (3.9) as a steady state problem and identify the limit. The last step is devoted to extend the above results to problems with more general initial condition satisfying some new assumptions.

As it is mentioned above, we start with proving that the limit of u exists and satisfies an elliptic equation defined on $\mathbb{R}^m \times \omega$. Of course, in this step we investigate the case $u_0 = 0$.

Lemma 3.5. *Under the assumptions of Lemma 3.1 (or Lemma 3.2), suppose that f is positive and $u_0 = 0$. Then, the solution u of (3.1) converges to $\tilde{u}_\infty$, as t goes to ∞, solution to*

$$
\begin{cases}
-\sum\limits_{1\leq i\leq n} \partial_{x_i} a_i(x,\tilde{u}_\infty,\nabla\tilde{u}_\infty) = f \text{ in } \mathbb{R}^m \times \omega,\\
\tilde{u}_\infty = 0 \text{ on } \mathbb{R}^m \times \partial\omega.
\end{cases}
\tag{3.32}
$$

The smoothness and the boundary conditions of the above solution can be expressed as

$$
\tilde{u}_\infty \in W^{1,p}_0(O\times\omega,\partial\omega)\,, \tag{3.33}
$$

for any bounded domain O in $\mathbb{R}^m$ and

$$
W^{1,p}_0(O\times\omega,\partial\omega) = \left\{ v \in W^{1,p}(O\times\omega)\,;\ v = 0 \text{ on } O\times\partial\omega \right\}.
$$

Proof. First, we apply Lemmas 3.1 and 3.3 (or 3.2 and 3.3), it follows that $u(.,x)$ is positive increasing function for almost all $x \in \mathbb{R}^m \times \omega$ and bounded above by u_∞ a solution of (3.9), consequently it converges towards some function $\tilde{u}_\infty$. Next, we use the Minty-Browder technique to show that the limit $\tilde{u}_\infty$ is a solution to (3.32). Let O be a bounded domain of $\mathbb{R}^m$, for t large enough, it follows from Lemma 3.4 that $\mathcal{T}_{t-1}u$, $\mathcal{A}(\mathcal{T}_{t-1}u)$ and $a_i(x,\mathcal{T}_{t-1}u,\nabla\mathcal{T}_{t-1}u)$ are bounded in the Banach spaces

$W^{1,p,q}\left(0,1;W_0^{1,p}(O\times\omega,\partial\omega),W^{-1,q}(O\times\omega)\right)$, $L^q\left(0,1;W^{-1,q}(O\times\omega)\right)$ and $L^q((0,1)\times O\times\omega)$ respectively. Without loss of generality we considered the time interval $(0,1)$, but according to the estimates of Lemma 3.4, we can take any interval (α,β), where α,β are positive constants. Therefore

$$\begin{aligned}
&\mathcal{T}_{t-1}u \to \tilde{u}_\infty \ \text{ in } L^p((0,1)\times O\times\omega),\\
&\mathcal{T}_{t-1}u \rightharpoonup \tilde{u}_\infty \ \text{ in } L^p\left(0,1;W_0^{1,p}(O\times\omega,\partial\omega)\right),\\
&(\mathcal{T}_{t-1}u)' \rightharpoonup 0 \ \text{ in } L^q\left(0,1;W^{-1,q}(O\times\omega)\right),\\
&\mathcal{A}(\mathcal{T}_{t-1}u) \rightharpoonup \chi \ \text{ in } L^q\left(0,1;W^{-1,q}(O\times\omega)\right),\\
&a_i(x,\mathcal{T}_{t-1}u,\nabla\mathcal{T}_{t-1}u) \rightharpoonup h_i \ \text{ in } L^q((0,1)\times O\times\omega).
\end{aligned} \tag{3.34}$$

The strong convergence holds for the whole sequence since u is increasing, while the weak convergence hold up to a subsequence. Once the limit are uniquely identified, the previous convergences will take place for the whole sequence. Now consider $v\in D(O\times\omega)$, taking its $0-$extension $\tilde{v}$ as a test function in (3.1), integrating over $(t-1,t)$ and making the change of variable $s=s-(t-1)$, we obtain

$$\left\langle(\mathcal{T}_{t-1}u)',v\right\rangle_{\mathcal{D}}+\int_0^1\int_{O\times\omega}\sum_{1\le i\le n}a_i(x,\mathcal{T}_{t-1}u,\nabla(\mathcal{T}_{t-1}u))\partial_{x_i}v\,dxds=\int_0^1\int_{O\times\omega}f(X_2)v\,dxds. \tag{3.35}$$

Then we can pass to the limit $t\to\infty$ in (3.35) to get from (3.34) and the fact that $\tilde{u}_\infty$ and v are independent of t

$$\chi=f \quad \text{in } \mathcal{D}'(\mathbb{R}^m\times\omega). \tag{3.36}$$

At the same time, we have

$$\chi=-\sum_{1\le i\le n}\partial_{x_i}h_i \ \text{ in } \mathcal{D}'((0,1)\times\mathbb{R}^m\times\omega). \tag{3.37}$$

In fact, for $\phi\in\mathcal{D}((0,1)\times O\times\omega)$ we have

$$\begin{aligned}
\langle\chi,\phi\rangle_{\mathcal{D}} &= -\lim_{t\to\infty}\left\langle\sum_{1\le i\le n}\partial_{x_i}a_i(x,\mathcal{T}_{t-1}u,\nabla(\mathcal{T}_{t-1}u)),\phi\right\rangle_{\mathcal{D}}\\
&=\lim_{t\to\infty}\int_0^1\int_{O\times\omega}\sum_{1\le i\le n}a_i(x,\mathcal{T}_{t-1}u,\nabla(\mathcal{T}_{t-1}u))\partial_{x_i}\phi\,dxds\\
&=\int_0^1\int_{O\times\omega}\sum_{1\le i\le n}h_i\partial_{x_i}\phi\,dxds\\
&=\left\langle-\sum_{1\le i\le n}\partial_{x_i}h_i,\phi\right\rangle_{\mathcal{D}}.
\end{aligned}$$

Thus, the lemma is achieved, if it is shown that

$$\mathcal{A}\tilde{u}_\infty=-\sum_{1\le i\le n}\partial_{x_i}h_i. \tag{3.38}$$

Let φ be a positive function in $\mathcal{D}(O\times\omega)$ such that $\varphi=1$ on $\tilde{O}\times\tilde{\omega}$ where $\tilde{O}\times\tilde{\omega}$ is any subdomain in $O\times\omega$ with $\tilde{O}\times\tilde{\omega}\subset\subset O\times\omega$. Since $\tilde{u}_\infty\in W^{1,p}(O\times\omega)$, there exists a sequence $(\phi_k)_k\subset C^\infty(O\times\omega)$ such that

$$\phi_k\to\tilde{u}_\infty,\ \text{in } W^{1,p}(O\times\omega)\,,\ \text{as } k\to\infty.$$

Choosing $v=\varphi^p(\mathcal{T}_{t-1}u-\phi_k)$ as a test function in (3.35) we derive

$$\begin{aligned}&\int_0^1\left\langle(\mathcal{T}_{t-1}u)',\varphi^p\mathcal{T}_{t-1}u\right\rangle_{O\times\omega}ds+\int_0^1\int_{O\times\omega}\sum_{1\le i\le n}a_i(x,\mathcal{T}_{t-1}u,\nabla(\mathcal{T}_{t-1}u))\partial_{x_i}(\varphi^p\mathcal{T}_{t-1}u)\,dxds\\&=\int_0^1\left\langle(\mathcal{T}_{t-1}u)',\varphi^p\phi_k\right\rangle_{O\times\omega}ds+\int_0^1\int_{O\times\omega}\sum_{1\le i\le n}a_i(x,\mathcal{T}_{t-1}u,\nabla(\mathcal{T}_{t-1}u))\partial_{x_i}(\varphi^p\phi_k)\,dxds\\&+\int_0^1\int_{O\times\omega}\varphi^p f(X_2)(\mathcal{T}_{t-1}u-\phi_k)\,dxds.\end{aligned}$$

Passing to the lim sup as $t,k\to\infty$, using (3.34) and taking into consideration that u is increasing positive function with respect t, we get

$$\begin{aligned}&\limsup_{t\to\infty}\int_0^1\int_{O\times\omega}\sum_{1\le i\le n}a_i(x,\mathcal{T}_{t-1}u,\nabla(\mathcal{T}_{t-1}u))\partial_{x_i}(\varphi^p\mathcal{T}_{t-1}u)\,dxds\\&\qquad\le\int_0^1\int_{O\times\omega}\sum_{1\le i\le n}h_i\partial_{x_i}(\varphi^p\tilde{u}_\infty)\,dxds.\end{aligned}$$

Thanks to the strong convergence of $\mathcal{T}_{t-1}u$ in $L^p((0,1)\times O\times\omega)$, this also implies that

$$\limsup_{t\to\infty}\int_0^1\int_{O\times\omega}\varphi^p\sum_{1\le i\le n}a_i(x,\mathcal{T}_{t-1}u,\nabla(\mathcal{T}_{t-1}u))\partial_{x_i}(\mathcal{T}_{t-1}u)\,dxds\le\int_0^1\int_{O\times\omega}\varphi^p\sum_{1\le i\le n}h_i\partial_{x_i}\tilde{u}_\infty dxds.\tag{3.39}$$

On the other hand, it follows from the monotonicity condition (3.3) and (3.34) that

$$\begin{aligned}&\liminf_{t\to\infty}\int_0^1\int_{O\times\omega}\varphi^p\sum_{1\le i\le n}a_i(x,\mathcal{T}_{t-1}u,\nabla(\mathcal{T}_{t-1}u))\partial_{x_i}(\mathcal{T}_{t-1}u)\,dxds\\&\ge\liminf_{t\to\infty}\int_0^1\int_{O\times\omega}\varphi^p\sum_{1\le i\le n}h_i\partial_{x_i}\tilde{u}_\infty dxds\\&=\liminf_{t\to\infty}\int_0^1\int_{O\times\omega}\varphi^p\sum_{1\le i\le n}a_i(x,\mathcal{T}_{t-1}u,\nabla(\mathcal{T}_{t-1}u))\partial_{x_i}\tilde{u}_\infty dxds\\&+\liminf_{t\to\infty}\int_0^1\int_{O\times\omega}\varphi^p\sum_{1\le i\le n}a_i(x,\tilde{u}_\infty,\nabla\tilde{u}_\infty)\partial_{x_i}(\mathcal{T}_{t-1}u-\tilde{u}_\infty)\,dxds.\end{aligned}$$

Combining this with (3.39), we obtain

$$\lim_{t\to\infty}\int_0^1\int_{O\times\omega}\varphi^p\sum_{1\le i\le n}a_i(x,\mathcal{T}_{t-1}u,\nabla(\mathcal{T}_{t-1}u))\partial_{x_i}(\mathcal{T}_{t-1}u)\,dxds=\int_0^1\int_{O\times\omega}\varphi^p\sum_{1\le i\le n}h_i\partial_{x_i}\tilde{u}_\infty dxds.\tag{3.40}$$

Finally, we pass to the limit in the following inequality, which is coming from the monotonicity condition (3.3)

$$\int_0^1 \int_{O\times\omega} \varphi^p \sum_{1\leq i\leq n} \left(a_i(x,\mathcal{T}_{t-1}u,\nabla\left(\mathcal{T}_{t-1}u\right)) - a_i(x,v,\nabla v)\right)\partial_{x_i}\left(\mathcal{T}_{t-1}u - v\right)dxds \geq 0,$$

for all $v\in W^{1,p}\left(O\times\omega\right)$, we derive

$$\int_0^1 \int_{O\times\omega} \varphi^p \sum_{1\leq i\leq n} \left(h_i - a_i(x,v,\nabla v)\right)\partial_{x_i}\left(\tilde{u}_\infty - v\right)dxds \geq 0. \tag{3.41}$$

Taking $v=\tilde{u}_\infty-\tau w$, where $\tau>0$ and $w\in\mathcal{D}\left(\tilde{O}\times\tilde{\omega}\right)$, we get

$$\int_0^1 \int_{\tilde{O}\times\tilde{\omega}} \varphi^p \sum_{1\leq i\leq n} \left(h_i - a_i(x,\left(\tilde{u}_\infty-\tau w\right),\nabla\left(\tilde{u}_\infty-\tau w\right))\right)\partial_{x_i} w dxds \geq 0.$$

Passing to the limit as $\tau\to 0$ taking into account that $\varphi=1$ on $\tilde{O}\times\tilde{\omega}$ it follows that

$$\int_0^1 \int_{\tilde{O}\times\tilde{\omega}} \sum_{1\leq i\leq n} \left[h_i - a_i(x,\tilde{u}_\infty,\nabla\tilde{u}_\infty)\right]\partial_{x_i} w dxds \geq 0, \quad \forall w\in\mathcal{D}\left(\tilde{O}\times\tilde{\omega}\right).$$

This implies that

$$\mathcal{A}\tilde{u}_\infty = -\sum_{1\leq i\leq n}\partial_{x_i}h_i.$$

Combining this with (3.36) and (3.37) we derive the equation in (3.32). The Lemma is proved. □

Remark 3.4. *Assume that*

$$\mathcal{T}_{t-1}u\to\tilde{u}_\infty \text{ in } L^2\left((0,1)\times O\times\omega\right), \tag{3.42}$$

for any bounded domain O of $\mathbb{R}^m$. Then the Lemma 3.5 remain valid even if we lost the monotonicity of u (i.e. for a non-vanishing initial condition). Indeed. It is enough to proof that the identity (3.40) holds without using the fact that u is monotone. So, let φ be a positive function in $\mathcal{D}\left((0,1)\times O\times\omega\right)$ such that $\varphi=1$ on $\left(\frac{1}{2},1\right)\times\tilde{O}\times\tilde{\omega}$ where $\tilde{O}\times\tilde{\omega}$ is any subdomain in $O\times\omega$ with $\tilde{O}\times\tilde{\omega}\subset\subset O\times\omega$. Since $\tilde{u}_\infty\in L^2\left((0,1)\times O\times\omega\right)$, we take $v=\varphi^p\left(\mathcal{T}_{t-1}u-\tilde{u}_\infty\right)$ as a test function in (3.35) we derive

$$\begin{aligned}
&\int_0^1 \int_{O\times\omega} \sum_{1\leq i\leq n} a_i(x,\mathcal{T}_{t-1}u,\nabla\left(\mathcal{T}_{t-1}u\right))\partial_{x_i}\left(\varphi^p\mathcal{T}_{t-1}u\right)dxds \\
&= -\int_0^1 \left\langle\left(\varphi^{\frac{p}{2}}\left(\mathcal{T}_{t-1}u-\tilde{u}_\infty\right)\right)', \varphi^{\frac{p}{2}}\left(\mathcal{T}_{t-1}u-\tilde{u}_\infty\right)\right\rangle_{O\times\omega} ds \\
&+ \frac{p}{2}\int_0^1\int_{O\times\omega}\partial_t\varphi\varphi^{p-1}\left(\mathcal{T}_{t-1}u-\tilde{u}_\infty\right)^2 dxds + \int_0^1\int_{O\times\omega} f(X_2)\varphi^p\left(\mathcal{T}_{t-1}u-\tilde{u}_\infty\right)dxds \\
&+ \int_0^1 \int_{O\times\omega} \sum_{1\leq i\leq n} a_i(x,\mathcal{T}_{t-1}u,\nabla\left(\mathcal{T}_{t-1}u\right))\partial_{x_i}\left(\varphi^p\tilde{u}_\infty\right)dxds.
\end{aligned}$$

Applying the integration by parts formula in the first integral of the right hand side, taking into account that $\rho(0) = 0$, then passing to the limit when $t \to \infty$, using (3.34) and (3.42), we get

$$\lim_{t\to\infty} \int_0^1 \int_{O\times\omega} \varphi^p \sum_{1\le i\le n} a_i(x, \mathcal{T}_{t-1}u, \nabla(\mathcal{T}_{t-1}u))\partial_{x_i}(\mathcal{T}_{t-1}u)\, dxds$$
$$= \int_0^1 \int_{O\times\omega} \varphi^p \sum_{1\le i\le n} h_i \partial_{x_i}\tilde{u}_\infty dxds. \tag{3.43}$$

Hence, the rest of the proof of Lemma 3.5 follows the standard argument taking $v = \tilde{u}_\infty - \tau\phi(t)\, w$ in (3.41), where $\tau > 0$, $\phi \in \mathcal{D}\left(\frac{1}{2}, 1\right)$ and $w \in \mathcal{D}\left(\tilde{O} \times \tilde{\omega}\right)$.

Now we pass to the next step to show that the limit $\tilde{u}_\infty$ is independent of X_1 and it is a solution to (3.9). Let $h \in \mathbb{R}^m$, we denote by Ω_s^h the translated of Ω_s by the vector h, i.e.

$$\Omega_s^h = \{(X_1 - h, X_2) \mid x = (X_1, X_2) \in \Omega_s\}.$$

By choosing τ large enough such that $\Omega_s^h \subset \Omega_{s+\tau}$ and $v \in W_0^{1,p}\left(\Omega_s^h\right)$, we can rewrite (3.1) as

$$< (\mathcal{T}_\tau u)', v >_{\Omega_s^h} + \int_{\Omega_s^h} \sum_{1\le i\le n} a_i(x, \mathcal{T}_\tau u, \nabla \mathcal{T}_\tau u)\partial_{x_i} v dx = \int_{\Omega_s^h} f v dx.$$

Again rewriting (3.1) by making the change of variable $X_1 = X_1 - h$, then comparing the resulting identities, we deduce

$$< (u(s, X_1 + h, X_2) - \mathcal{T}_\tau u(s,x))', v >_{\Omega_s^h}$$
$$+ \int_{\Omega_s^h} \sum_{1\le i\le n} [a_i(X_1 + h, X_2, u(s, X_1 + h, X_2), \nabla u(s, X_1 + h, X_2)$$
$$-a_i(x, \mathcal{T}_\tau u(s,x), \nabla \mathcal{T}_\tau u(s,x)\, \partial_{x_i} v]\, dx = 0 \;\forall v \in W_0^{1,p}\left(\Omega_s^h\right). \tag{3.44}$$

We need here to assume that the operator $\mathcal{A}$ is monotone in the following sense

$$\sum_{1\le i\le n} \left(a_i(X_1, X_2, \xi) - a_i(X_1', X_2, \xi')\right)\left(\xi_i - \xi_i'\right) \ge 0 \;\forall(\xi, \xi') \in \left(\mathbb{R}^{n+1}\right)^2, \; a.e. \left(X_1, X_1', X_2\right) \in \mathbb{R}^{2m} \times \omega. \tag{3.45}$$

Lemma 3.1 ensures that $u \ge 0$ which allows to take $v = (u(s, X_1 + h, X_2) - \mathcal{T}_\tau u(s,x))^+ \in W_0^{1,p}\left(\Omega_s^h\right)$ as a test function in (3.44). Thanks to the monotonicity (3.45) we obtain

$$< (u(s, X_1 + h, X_2) - \mathcal{T}_\tau u(s,x))', (u(s, X_1 + h, X_2) - \mathcal{T}_\tau u(s,x))^+ >_{\Omega_s^h} \le 0.$$

Integrating over $(0, t)$, applying the integration by parts formula and taking into consideration that $u_0 = 0$ and $u \ge 0$, we derive

$$\int_{\Omega_t^h} \left|(u(t, X_1 + h, X_2) - u(t + \tau, x))^+\right|^2 dx \le 0.$$

This of course means that

$$u(t, X_1 + h, X_2) \le u(t + \tau, x), \quad \text{for } a.e.\; x \in \Omega_t^h.$$

Passing to the limit as $t \to \infty$, we eventually obtain

$$\tilde{u}_\infty (X_1 + h, X_2) \le \tilde{u}_\infty (X_1, X_2), \quad \forall h \in \mathbb{R}^m.$$

This lets us to say that $\tilde{u}_\infty$ is independent of X_1. Hence (3.33) means exactly $\tilde{u}_\infty \in W_0^{1,p}(\omega)$ which gives a sense to the boundary conditions in (3.9). To summarize, we can state

Lemma 3.6. *Under the assumptions of Lemma 3.5, suppose that (3.45) is fulfilled. Then*

$$u(t,.) \to \tilde{u}_\infty, \text{ as } t \to \infty, \quad \text{a.e. in } \mathbb{R}^m \times \omega,$$

where $\tilde{u}_\infty \in W_0^{1,p}(\omega)$ is a solution to (3.9).

Before to conclude this subsection by dealing with more general initial conditions, let us mention some remarks and useful consequences.

Remark 3.5. *All the above results still take place if we consider instead of $u_0 = 0$, an initial condition equals to any subsolution of the elliptic problem (3.24) defined on Ω_0 ($u_0 = \tilde{u}_0$, solution to (3.24)).*

Remark 3.6. *Some independence property is included in the monotonicity type condition (3.45) that can be expressed as*

$$\sum_{1\le i\le n} a_i(X_1, X_2, \xi)\xi_i, \text{ is independent of } X_1, \ \forall \xi \in \mathbb{R}^{n+1}. \tag{3.46}$$

Indeed, it is enough to take $\xi' = (1 \pm \varepsilon)\xi$ in (3.45), for positive constant ε, then we derive

$$\pm \sum_{1\le i\le n} \left(a_i(X_1, X_2, \xi) - a_i(X_1', X_2, (1\pm\varepsilon)\xi)\right)\xi_i \ge 0, \quad \forall \xi \in \mathbb{R}^{n+1}, \ \left(X_1, X_1', X_2\right) \in \mathbb{R}^m \times \mathbb{R}^m \times \omega.$$

Passing to the limit $\varepsilon \to 0$, we derive (3.46).

Since when an operator as $\mathcal{A}_\omega$ is not strictly monotone, the corresponding problem may have more than one solution. The above argument turns our attention to an interesting result concerning the monotone operators.

Theorem 3.2. *Let Ω be a bounded open set in $\mathbb{R}^n$, $n \ge 1$, $\mathcal{B}$ be a quasilinear monotone and coercive operator defined as A_ω and satisfies the Carathéodory and the growth conditions and $u \in W_0^{1,p}(\Omega)$, $p \ge p_0$, be a solution to*

$$\begin{cases} \mathcal{B}u = f & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.47}$$

where $f \in L^q(\Omega)$ is a positive function. Then the above problem has a minimal solution, i.e.

$$\tilde{u} = \min\{u; \ u \text{ is solution to (3.47)}\}$$

is also a solution to (3.47).

Proof. Let u be an arbitrary solution of (3.47) and w be the solution of the following parabolic problem defined on the noncylindrical domain as

$$\begin{cases} w' - \partial_y\left(|\partial_y w|^{p-2}\partial_y w\right) + \mathcal{B}w = f & \text{in } Q_t, \\ w = 0 & \text{on } \Gamma_t, \\ w(0,.) = 0 & \text{on } \Omega_0, \end{cases} \tag{3.48}$$

where Q_t here has an accurate form that is to say

$$Q_t = \left\{(s, y, x) \in \mathbb{R}^2 \times \Omega \mid -t_0 - s \leq y \leq t_0 + s,\ 0 \leq s \leq t\right\} \text{ with } t_0 > 0.$$

The coordinate y will play the role of X_1. It is clear that the problem (3.48) satisfies all hypotheses (3.2)-(3.7) and (3.45) under which the above results are fulfilled. Then, it follows from (3.17) and (3.20) that

$$w(t, .) \leq u \qquad a.e. \text{ on } \Omega. \tag{3.49}$$

Passing to the limit, we derive from Lemma 3.6 that $w(t, .)$ converges towards some $\tilde{u} \in W_0^{1,p}(\Omega)$ a solution of (3.47). Combining this with (3.49) we get

$$\tilde{u} \leq u \;\; a.e. \text{ on } \Omega.$$

This completes the Proof of Theorem 3.2. □

In the following $\tilde{u}_\infty$ denotes the minimal solution of Problem (3.9). Finally, we conclude this section by the following general convergence results.

Theorem 3.3. *Suppose that Assumptions (3.2), (3.4)-(3.7) and (3.45) hold. Then for $p \geq p_0$, $0 \leq u_0 \leq \tilde{u}_\infty$ and nonnegative f, the solution u of (3.1) converges towards $\tilde{u}_\infty$, as t goes to ∞, the minimal solution to (3.9), i.e.*

$$\begin{gathered} u(t, .) \to \tilde{u}_\infty \;\; a.e. \text{ in } \mathbb{R}^m \times \omega, \\ \mathcal{T}_{t-1}u \to \tilde{u}_\infty \;\; \text{in } L^p\left((0,1) \times O \times \omega\right), \\ \mathcal{T}_{t-1}u \rightharpoonup \tilde{u}_\infty \;\; \text{in } L^p\left(0,1; W^{1,p}(O \times \omega)\right), \end{gathered}$$

where O is any bounded set in $\mathbb{R}^m$. (The above convergence hold for the whole sequence). Moreover if $\mathcal{A}$ satisfies the S^+ property, i.e. for an operator $\mathcal{A} : B \to B'$ (B is a real reflexive Banach space) we have

$$u_i \rightharpoonup u \text{ in } B,\ \limsup_{i \to \infty} \langle \mathcal{A}u_i - \mathcal{A}u, u_i - u \rangle \leq 0 \text{ implies } u_i \to u, \text{ in } B,$$

then,

$$\mathcal{T}_{t-1}u \to \tilde{u}_\infty \quad \text{in } L^p\left(0,1; W^{1,p}(O \times \omega)\right).$$

Proof. Let $\tilde{u}$ be the weak solution of (3.1) obtained when we choose $u_0 = 0$. Since $0 \leq u_0 \leq \tilde{u}_\infty$, it follows thanks to Proposition 3.1 that

$$\tilde{u} \leq u \leq \tilde{u}_\infty.$$

Passing to the limit taking into account Lemma 3.6, we derive

$$\begin{gathered} u(t, .) \to \tilde{u}_\infty \;\; a.e. \text{ in } \mathbb{R}^m \times \omega, \\ \mathcal{T}_{t-1}u \to \tilde{u}_\infty \;\; \text{in } L^2\left((0,1) \times O \times \omega\right), \end{gathered} \tag{3.50}$$

for any bounded set $O \subset \mathbb{R}^m$. On the other hand using the estimates of Lemma 3.4, we get

$$\mathcal{T}_{t-1}u \rightharpoonup \tilde{u}_\infty \;\; \text{in } L^p\left(0,1; W^{1,p}(O \times \omega)\right). \tag{3.51}$$

This of course holds for the whole sequence since the limit is unique (the minimal solution). Even here, we lost the monotonicity of u, the convergence (3.42) still takes place thanks to (3.50) and also Lemma 3.5 remain valid. Consequently, it follows from (3.43) that

$$\lim_{t\to\infty}\int_0^1\int_{O\times\omega}\varphi^p\sum_{1\le i\le n}\left(a_i(x,\mathcal{T}_{t-1}u,\nabla\left(\mathcal{T}_{t-1}u\right))-a_i(x,\tilde{u}_\infty,\nabla\tilde{u}_\infty)\right)\partial_{x_i}\left(\mathcal{T}_{t-1}u-\tilde{u}_\infty\right)dxds=0,$$

where $\tilde{O}\times\tilde{w}$ is any subdomain in $O\times\omega$. This implies that

$$\lim_{t\to\infty}\int_{\frac{1}{2}}^1\int_{\tilde{O}\times\tilde{\omega}}\sum_{1\le i\le n}\left(a_i(x,\mathcal{T}_{t-1}u,\nabla\left(\mathcal{T}_{t-1}u\right))-a_i(x,\tilde{u}_\infty,\nabla\tilde{u}_\infty)\right)\partial_{x_i}\left(\mathcal{T}_{t-1}u-\tilde{u}_\infty\right)dxds=0.$$

Now if we take into account the property S^+, we derive

$$\mathcal{T}_{t-1}u\to\tilde{u}_\infty,\ \text{ in } L^p\left(\frac{1}{2},1;W^{1,p}\left(\tilde{O}\times\tilde{\omega}\right)\right).$$

As it is mentioned above we can take any interval (α,β) thanks to the translation. Thus the theorem 3.3 is proved. □

Remark 3.7. *As a model example of the operator $\mathcal{A}$ satisfies the properties of the above theorem we consider the $p-$laplace. In fact we have just to check the S^+ property. If $p\ge 2$ the $p-$laplace operator is uniformly monotone and by consequence the S^+ property is fulfilled (see [66]). Now, if $1<p<2$, let u_i be a sequence in $L^p\left(0,1;W^{1,p}\left(\Omega\right)\right)$, Ω is any bounded set of $\mathbb{R}^n$, such that*

$$u_i\rightharpoonup u \text{ in } L^p\left(0,1;W^{1,p}\left(\Omega\right)\right) \text{ and } \limsup_{i\to\infty}\left\langle\Delta_p u_i-\Delta_p u,u_i-u\right\rangle\le 0. \tag{3.52}$$

The last limit leads to

$$\lim_{i\to\infty}\left\langle\Delta_p u_i-\Delta_p u,u_i-u\right\rangle=0. \tag{3.53}$$

Since the $p-$Laplace is monotone. We denote by χ the weak limit of $\Delta_p u$ in $L^q\left(0,1;W^{-1,q}\left(\Omega\right)\right)$ that exists by (3.52). From (3.53) we derive that

$$\lim_{i\to\infty}\left\langle\Delta_p u_i,u_i\right\rangle=\left\langle\chi,u\right\rangle_{\mathcal{D}'}.$$

Hence, the Minty-Browder technique allows to show that

$$|u_i|_{1,p,\Omega}\to|u|_{1,p,\Omega}\quad \text{in } L^p\left(0,1;W^{1,p}\left(\Omega\right)\right).$$

Now since the L^p space is uniformly convex, the weak limit (3.52) is strong.

In what follows we will establish the convergence rate estimate for the solution u of (3.1). To this end, we assume that the operator $\mathcal{A}$ is uniformly monotone, i.e.

$$\exists\delta\ge 0:\sum_{1\le i\le n}\left(a_i(x,\xi)-a_i(x,\xi')\right)\left(\xi_i-\xi_i'\right)\ge\delta\sum_{1\le i\le n}\left|\xi_i-\xi_i'\right|^p,\ \text{ a.e. } (x,\xi,\xi')\in\mathbb{R}^n\times\left(\mathbb{R}^{n+1}\right)^2. \tag{3.54}$$

Also, we substitute the assumption (3.6), by

$$|a_i(x,\xi_0,\xi_1,\ldots,\xi_n)|\le C\sum_{1\le i\le m}|\xi_i|^{p-1}, \tag{3.55}$$

for $i = 1, ..., m$ and keeping it for $i = m + 1, ..., n$. Next, we introduce the following subsets that play an essential role in the proof of theorem below. For $t \geq r > 0$, we denote by $\mathcal{B}_r$ the $r-$neighborhood of $\hat{\Omega}_0$ and we set

$$d_t := dist\left(\partial\hat{\Omega}_0, \partial\hat{\Omega}_t\right), \ \vartheta_t := \min(t, d_t).$$

Also, we define

$$S_r^t = \left\{(s, X_1) \mid t - r < s \leq t, \ X_1 \in \mathcal{B}_{(d_s+r-\vartheta_t)} \cap \mathcal{B}_r\right\}.$$

Then we are in a position to state the following

Theorem 3.4. *Under Assumptions (3.2), (3.4)-(3.5), (3.7), (3.14) and (3.54)-(3.55): For* $2 < p < 2 + \frac{2}{m}$*, there exists a constant* $C > 0$ *independent of* t*, such that*

$$\int_{\mathcal{S}_{\frac{\vartheta_t}{2}} \times \omega_t} \sum_{1 \leq i \leq n} |\partial_{x_i}(u - u_\infty)|^p \, dxds \leq \frac{C}{(\vartheta_t)^\lambda}. \tag{3.56}$$

where u *and* u_∞ *are the solutions of (3.1) and (3.9) respectively and* $\lambda = \frac{2}{p-2} - m$*.*

Proof. Let r be a positive constant with $r \geq \frac{p}{p-2}$ and ϱ be a nonnegative smooth function of $(0, t) \times \mathbb{R}^m$ such that

$$0 \leq \varrho \leq 1, \ \varrho = 1 \text{ on } \mathcal{S}^t_{\frac{\vartheta_t}{2}}, \ \varrho = 0 \text{ outside } \mathcal{S}^t_{\vartheta_t}, \ |\partial_t \rho|, \ |\nabla_{X_1}\rho| \leq \frac{c}{\vartheta_t},$$

where c is some positive constant independent of t. It is clear that

$$\varrho^r (u - u_\infty)(s, .) \in W_0^{1,p}(\Omega_s), \text{ for } a.e. \ s \in (0, t).$$

Hence, from (3.15) we derive

$$\left\langle \left(\varrho^{\frac{r}{2}} (u - u_\infty)\right)', \varrho^{\frac{r}{2}}(u - u_\infty)\right\rangle + \int_0^t \int_{\Omega_s} \varrho^r \sum_{1 \leq i \leq n} (a_i(x, u, \nabla u) - a_i(x, u_\infty, \nabla u_\infty)) \, \partial_{x_i}(u - u_\infty) dxds$$
$$= \frac{r}{2} \int_0^t \int_{\Omega_s} \partial_t \varrho \varrho^{r-1} (u - u_\infty)^2 \, dxds - r \int_0^t \int_{\Omega_s} \varrho^{r-1}(u - u_\infty) \sum_{1 \leq i \leq m} a_i(x, u, \nabla u) \partial_{x_i} \varrho dxds.$$

Applying the integration by parts formula taking into account the fact that $\varrho(0) = 0$, we deduce

$$\frac{1}{2} \left| \varrho^{\frac{r}{2}} (u - u_\infty)(t, .)\right|^2_{2,\Omega_t} + \int_0^t \int_{\Omega_s} \varrho^r \sum_{1 \leq i \leq n} (a_i(x, u, \nabla u) - a_i(x, u_\infty, \nabla u_\infty)) \, \partial_{x_i}(u - u_\infty) dxds$$
$$= \frac{r}{2} \int_0^t \int_{\Omega_s} \partial_t \varrho \varrho^{r-1} (u - u_\infty)^2 \, dxds - r \int_0^t \int_{\Omega_s} \varrho^{r-1}(u - u_\infty) \sum_{1 \leq i \leq m} a_i(x, u, \nabla u) \partial_{x_i} \varrho dxds.$$

Since $r \geq \frac{p}{p-2}$, it follows from the uniform monotonicity and the growth conditions (3.54) and (3.55) that

$$\frac{1}{2} \left| \varrho^{\frac{r}{2}} (u - u_\infty)(t, .)\right|^2_{2,\Omega_t} + \delta \int_{\mathcal{S}_{\vartheta_t} \times \omega} \varrho^r \sum_{1 \leq i \leq n} |\partial_{x_i}(u - u_\infty)|^p \, dxds$$
$$\leq \frac{C}{\vartheta_t} \left(\int_{\mathcal{S}_{\vartheta_t} \times \omega} \varrho^{\frac{2r}{p}} (u - u_\infty)^2 \, dxds + \int_{\mathcal{S}_{\vartheta_t} \times \omega} \sum_{1 \leq i \leq m} |\partial_{x_i} u|^{p-2} |\partial_{x_i}(u - u_\infty)| \, \varrho^{\frac{2r}{p}} |u - u_\infty| \, dxds \right).$$

Noticing that $\frac{1}{p}+\frac{1}{p}+\frac{p-2}{p}=1$, using the Hölder and Poincaré inequalities we derive

$$\int_{\mathcal{S}_{\vartheta_t}\times\omega_t}\varrho^r\sum_{1\le i\le n}|\partial_{x_i}(u-u_\infty)|^p\,dxds\le\frac{C}{\vartheta_t}\left((\vartheta_t)^{\frac{(m+1)(p-2)}{p}}+\left(\int_{\mathcal{S}_{\vartheta_t}\times\omega}|\nabla u|^p\,dxds\right)^{\frac{p-2}{p}}\right)$$
$$\times\left(\int_{\mathcal{S}_{\vartheta_t}\times\omega}\varrho^r\sum_{1\le i\le n}|\partial_{x_i}(u-u_\infty)|^p\,dxds\right)^{\frac{2}{p}}.$$

Recalling (3.27), we deduce

$$\int_{\mathcal{S}_{\vartheta_t}\times\omega_t}\varrho^r\sum_{1\le i\le n}|\partial_{x_i}(u-u_\infty)|^p\,dxds\ \le\frac{C}{(\vartheta_t)^k}\left(\int_{\mathcal{S}_{\vartheta_t}\times\omega}\varrho^r\sum_{1\le i\le n}|\partial_{x_i}(u-u_\infty)|^p\,dxds\right)^{\frac{2}{p}},$$

where $k=\frac{2(m+1)}{p}-m$. Therefore, using the definition of ϱ we arrive at (3.56). This completes the proof of Theorem 3.4. □

Remark 3.8. *Note that the Theorem 3.4 still holds for the case where* $\mathcal{A}=\operatorname{div}(A(|\nabla u|)\nabla u)$, $A:[0,\infty)\to[0,\infty)$ *is a continuous function satisfying the following conditions:*

$$0<A(\xi)\le\lambda_1\xi^{p-2},\quad\forall\xi\in(0,\infty),$$
$$(A(|h_1|)h_1-A(|h_2|)h_2)(h_1-h_2)\ge\lambda_2|h_1-h_2|^p\quad\forall h_1,h_2\in\mathbb{R}^n,$$

with some constants $\lambda_1,\lambda_2>0$.

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