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Control and stability of certain evolution problems (PDEs)

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Abstract

The theory of nonlinear KDV, whose discoverers are usually considered to be Korteweg and de Vries, is still a young science, although research in this direction was carried out even in the 19th century, mainly in connection with the problems of gas and hydrodynamics. For example, the work of T. Oh [40], who observed local well-posedness for problem of coupled KdV-type systems in the periodic/non-periodic cases. Dates back to 1895, the Korteweg-de Vries equation representing the basis of the mathematical description of dynamics of solutions was obtained by [37]. This type of equations describes the propagation of waves on water with small dispersion and small nonlinearity. It serves as a model equation for any physical system with an approximate dispersion. Equations of the KdV or Burgers type play an extremely important role in the theory of nonlinear waves in the study of weakly nonlinear long-wave processes in media with dispersion and (or) dissipation. Below in our thesis, the study of the well-posedness for some non linear (Kdv) type problems in the Gevrey spaces and Bourgain spaces is considered. We first proposed and treated the solution of (Kdv) type equation in Bourgain spaces. Then a coupled periodic (Kdv) system is considered. The last one is for coupled system of(mKdv) type equations on the line. These systems of the KdV equations can be considered with specific physical examples related to plasma physics, gas and hydrodynamics, and radio physics.

Keywords

Local and global solutions approximate conservation law, Banach fixed point Theorem, Well-posedness, Gevrey spaces, Bourgain spaces, Time regularity.

Résumé

La théorie du KDV non linéaire, dont les découvreurs sont généralement considérés comme Korteweg et de Vries, est encore une science jeune, bien que des recherches dans ce sens aient été menées meme au XIXe siecle, principalement en relation avec les problemes de gaz et d'hydrodynamique. Par exemple, le travail de T. Oh [40], qui a observé l'existence local pour le probleme des systemes couplés de type KdV dans les cas périodiques/non périodiques. Datant de 1895, l'équation de Korteweg-de Vries représentant la base de la description mathématique de la dynamique des solutions a été obtenue par [37]. Ce type d'équations décrit la propagation des ondes sur l'eau avec une faible dispersion et une faible non linéarité. Il sert d'équation modele pour tout systeme physique avec une dispersion approximative. Les équations de type KdV ou Burgers jouent un role extremement important dans la théorie des ondes non linéaires dans l'étude des processus a ondes longues faiblement non linéaires dans les milieux a dispersion et (ou) dissipation. Ci-dessous dans notre these, l'étude du bien-posé pour certains problemes de type non linéaire (Kdv) dans les espaces de Gevrey et les espaces de Bourgain est considérée. Nous avons d'abord proposé et traité la solution de l'équation de type (Kdv) dans les espaces de Bourgain. Ensuite, un systeme périodique couplé (Kdv) est considéré. La derniere concerne les équations de type systeme couplé (mKdv) sur la droite. Ces systemes d'équations KdV peuvent etre considérés avec des exemples physiques spécifiques liés a la physique des plasmas, aux gaz et a l'hydrodynamique, et a la radiophysique.

Mots clés

Solutions locales et globales, loi de conservation approximative, théoreme de point fixe de Banach, Bien posé, espaces de Gevrey, espaces de Bourgain, régularité temporelle.

Publications

- A. Elmansouri, Kh. Zennir, A. Boukarou and O. Zehrour, Analytic Gevrey well-posedness and regularity for class of coupledperiodicKdVsystems of Majda-Biello type, Applied Sciences, Vol. 24, 2022, pp. 117-130.

http://www.mathem.pub.ro/apps/v24/A24.htm.
The article is detailed in Chapter 02.

Conference

- Elmansouri AOUATEF Presented an ORAL COMMUNICATION entitled: Analytic Gevrey Well-Posedness And Regularity For Class Of Coupled Periodic Kdv Systems Of Majda-Biello Type

International Conference on Pure and Applied Mathematics IC-PAM'21, May 26-27, 2021, Ouargla, Algeria

- Elmansouri AOUATEF has participated with an oral presentation entitled: Novel lower Bounds on the Radius of Spatial Analyticity for the KdV type equations

in the NationalConferenceonAp- plied Mathematics and Didactics NCAMD2021,Constantine-Algeria.26Juin,2021.

- Elmansouri AOUATEF " has presented a talk entitled "A couple periodic KDV system in Gevery-Bourgain spaces" During The First Conference on Mathematics and Applications of Mathematics held on June 30 and July 01, 2021, Jijel, Algeria.

Acknowledgement and dedication

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Contents

Symbols and Acronyms

1. Ω: is an oben subset of $\mathbb{R}^n$.
2. $\mathbb{C}(\Omega :, \mathbb{R}^n)$: the space of continuous fonction from Ω to $\mathbb{R}^n$.
3. $C(\Omega)$ the space of continuous functions from Ω to $\mathbb{R}$.
4. $C_b\left(\overline{\Omega}\right)$ the space of all continuous and bounded functions on $\overline{\Omega}$.
5. $D_i u\left(x\right) = u_i\left(x\right) = \dfrac{\partial u\left(x\right)}{\partial x_i}$ the partial derivative of u with respect to $x_i\left(1 \leq i \leq n\right)$.
6. $C_0^\infty\left(\Omega\right)$ or $\mathfrak{D}\left(\Omega\right)$, is the space of the infinitely differentiable functions.
7. $C^k\left(\Omega\right)$ is the space of functions u which are k times derivable and whose derivation of order k is continuous on Ω.
8. $\mathcal{G}_{\sigma,\delta,s}(\mathbb{T}_\gamma) = \mathcal{G}_{\sigma,\delta,s}$ is the analytic Gevrey spaces with $\gamma \geq 1$.
9. $X^{\beta}_{\sigma,\delta,s,b}(\mathbb{T}_\gamma \times \mathbb{R}) = X^{\beta}_{\sigma,\delta,s,b}$the analytic Gevrey-Bourgain spaces.
10. $G^\sigma(\mathbb{T})$ class of Gevrey function of order σ.
11. $L^1\left(\Omega\right)$ the space of integrable functions on Ω with values in $\mathbb{R}$.
12. $L^p\left(\Omega\right)$ is the set$\left\{f : \Omega \to \mathbb{R},\ f \text{ measurable and } \int\limits_{\Omega} \left|f\left(x\right)\right|^p dx < +\infty\right\}$.
13. $L^\infty\left(\Omega\right)$ is theset $\{f : \Omega \to \mathbb{R},\ f \text{ measurable, } \exists c > 0, \text{ so that } \left|f\left(x\right)\right| \leq c \quad \text{a.e. on } \Omega\}$.
14. $X^T_{\delta,s,b}$, denotes the restriction of $X_{\delta,s,b}$ onto finite time interval $[-T, T]$, $T > 0$.
15. $W^{1,p}\left(\Omega\right) = \left\{u \in L^p\left(\Omega\right);\ \text{such that } \partial_i u \in L^p\left(\Omega\right), 1 \leq i \leq n\right\}$.
16. $W^{m,p}\left(\Omega\right) = \left\{u \in L^p\left(\Omega\right),\ \text{such that } \partial^\alpha u \in L^p\left(\Omega\right),\ \forall \alpha, |\alpha| \leq m\right\}$.
17. H^s is the Sobolev space.

Introduction

The subject of studying the well-posedness and regularity of some partial differential equations by proving the existence of unique solutions in analytic Gevrey spaces and Bourgain type spaces, is an important problem in applications of science and engineering. Nonlinear wave phenomena have been studied by many scientists such as: Poisson, Stokes, Airy, Rayleigh, Boussinesq, and Riemann. However, as a modern science, great progress was known in the late 1960s and early 1970s, which became the years of rapid development due to the development of computer technology, which made it possible to approach the direct numerical solution of the partial differential equations that described the propagation of waves in various media.

Burgers types spaces are an essential part in nonlinear wave theory for studying weakly nonlinear long-wave processes in mediums with dispersion and (or) dissipation. The coupled system of the KdV equations is discussed using specific physical examples from plasma physics, gas, hydrodynamics, and radio physics. Generally speaking, this equation is an evolution type equation, it was developed in several studies, one of which is the Korteweg-de Vries-Kuramoto Sivashinsky equation, which arises as a model for long waves in a viscous fluid falling down an inclined plane, see [7, 8, 33, 38, 15, 39]. It is considered as a particular case of the Benney-Lin equation.

See also [41, 39] when $= 0$.

Another example related to the Korteweg-de Vries-Burgers equation considered in [31], the authors demonstrated that for given data in H^s, $s > 1$, the problem is locally well-posed. For other problems close results were found in the generalized Ostrovsky-Stepanyams-Tsimring equation, please check [41, 44, 46].

In [44] for 1D Dirac-Klein-Gordon equations, an effective method for analyzing lower bounds on the radius of analyticity, including these problems, was developed, it was applied in [43] to the modified Kawahara equation and in [34] to the non-periodic KdV equation. (For more details, please see [13, 14]).

Korteweg and de Vries proposed long surface waves of water in a narrow, shallow channel. And in [5], J.Biello, and A. Majda treat a system that is a simplified approximate model of the behavior of particular atmospheric Rossby waves, the system in chapter 2 is inspired by them.

The present work extends previous studies and introduces a new class of problems with conditions other than local boundaries. It also expands on the results obtained so far. This work presents new findings on analytic Gevrey spaces, where there have been limited results for this type of system until now.

It is well known that it is not new to study the KdV equations in the classical Sobolev spaces H^s, there are many discussed results according to different value of the exponent s.(See [11], [18], [31], [32], [35], [36])

It is very important to study KdV equation on these spaces, because Gevrey functions on the circle belong to every Sobolev spaces, especially when it comes from a system of coupled equations with nonlinearity.

To motivate our work, we review some related results. In [23], G. Hannah, H. Himonas, and A. Petronilho, proposed:

$$\begin{cases} u_t + u_{xxx} + u^k u_x = 0,\ x \in \mathbb{T},\ t \in \mathbb{R} \\ u(x,0) = \phi(x), \end{cases}$$

with $k = 1, 2, 3$ and they proved existence and regularity results in Gevrey spaces, if $\phi(x)$ belongs to $G^\sigma(\mathbb{T})$. However, in [21], a periodic Cauchy problem for a KdV equation with dispersion of order $m = 2j + 1,\ j > 0$ is proposed

$$\begin{cases} \partial_t u + \partial_x^{2j+1} u + u \partial_x u = 0,\ x \in \mathbb{T},\ t \in \mathbb{R} \\ u(x,0) = \phi(x). \end{cases}$$

An extension to previous works, [31], [27], [36], the aurthors showed that the local in time well-posedness holds when $\phi(x) \in G^\sigma$. Moreover, they showed that the solution is not necessarily G^σ in t and belongs to $G^{m\sigma}(\mathbb{R})$ near zero for any x in the circle.
In [28], for $k = 1, 2, 3, \ldots$ an initial-value problem for the generalized Burgers equation is considered

$$\begin{cases} u_t + u_{xx} + u^k u_x = 0, \ x \in \mathbb{T}, \ t \in \mathbb{R} \\ u(x, 0) = u_0(x), \end{cases}$$

and its well-posedness in $G^{\sigma,\delta,s}$ and the regularity properties of the solution in G^σ in x and in $G^{2\sigma}$ in t is studied in the case when $u_0(x)$ belongs to a class of analytic Gevrey spaces.
Very recently, the same research team issued a series of articles of related results with regard to existence and uniqueness and different regularity's properties in analytic Gevrey spaces (See [7], [8]).

We mention in particular the article *[A.Boukarou, K. Guerbati, Kh.Zennir, S.Alodhaibi, ;S. Alkhalaf, Well-posedness and time regularity for a system of modified Korteweg-de Vries-type equations in analytic Gevrey spaces, Mathematics, 2020, 8, 809]*.

Our thesis is structured as follows:

Chapter 1: In this chapter we introduced necessary initial concepts, and reminders of basic function spaces, such as Sobolev,Gevrey and Bourgain spaces.
Also, we recorded some preliminary estimates that are essential to prove the well-posedness result and devlop our results .

Chapter 2: Our purpose in this chapter is to study the well-posedness and regularity for the coupled Kotewege-de Varie (KdV) system

$$\begin{cases} u_t + u_{xxx} + ww_x = 0 \\ w_t + \beta w_{xxx} + (uw)_x = 0, \quad x \in \mathbb{T}_\gamma \, t \in \mathbb{R}, \ 0 < \beta < 1 \\ (u, w) \mid_{t=0} = (u_0, w_0). \end{cases}$$

where $\mathbb{T}_\gamma = [0, 2\pi\gamma)$ for some $\gamma \geq 1$.
In the first part of this chapter, we prove some essential preliminary estimates.
Next, we demonstrate that the previous Cauchy problem has a unique solution in an analytic Gevrey space.

$$(u, w) \in C\left([-T, T], \mathcal{G}^{\sigma,\delta,s}\right) \times C\left([-T, T], \mathcal{G}^{\sigma,\delta,s}\right),$$

In the last part, Gevrey regularity $G^{3\sigma}$ of the solution in time variable t is provided and the failure of G^d regularity in t is shown.

Chapter 3: In this chapter we study the will-posedness of the nonlinear Cauchy problem with the KdV type equations ,

$$\begin{cases} \partial_t w + \partial_x^3 w + \eta(t)\mathcal{L}w + \partial_x (w)^k = 0, \ (x, t) \in \mathbb{R}^2, \quad k = 2, 4 \\ w(x, 0) = w_0(x), \end{cases}$$

where $\eta(t) \equiv \eta \ sgn(t)$ and η is a positive constant, $w(x, t)$ are real-valued.
We start by define the function spaces, linear estimates and multi-linear estimates. Then, we prove using contraction mapping principle that the solution of this problem is in $X^T_{\delta,s,b}$.
Finally, we use the existence of approximate conservation law to prove that

$$w \in C\left([-T', T'], G^{\delta(T'),s}\right) \text{ with } \delta(T') = \min\left\{\delta_0, C_1 T'^{(1/a_k - \sigma_0)}\right\},$$

.

Chapter 4: In the last chapter of this thesis, we consider the following coupled system of mKdV-type equations on the line,

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(uv^2) = 0, \\ \partial_t v + \beta\partial_x^3 v + \partial_x(u^2 v) = 0, \quad (x,t) \in \mathbb{R}^2, \quad 0 < \beta < 1 \\ u(x,0) = u_0(x), \\ v(x,0) = v_0(x), \end{cases}$$

Also in this chapter we prove using the linear estimates, trilinear estimates and contraction mapping principle, that the previous problem has a unique solution

$$(u,v) \in C\left([0,T], G^{\delta,s}\right) \times C\left([0,T], G^{\delta,s}\right)$$

we prove also the existence of approximate conservation law to prove that the solution

$$(u,v) \in C\left([0,T'], G^{\delta(T'),s}\right) \times C\left([0,T'], G^{\delta(T'),s}\right),$$

with

$$\delta(T') = \min\left\{\delta_0, C_1 T'^{-(2+\sigma_0)}\right\},$$

Chapter 1

Preliminary

1- Continuous function spaces

2- L^p Spaces

3- Sobolev Spaces

4- Analytic Gevrey spaces and Bourgain spaces

1.1 Continuous function spaces

Let $x = (x_1, x_2, ..., x_n)$ denotes the generic point of an open set Ω of $\mathbb{R}^n$. Let u be a function defined from Ω to $\mathbb{R}^n$.

Definition 1. *Let: $C(\Omega)$ denote the space of continuous functions from Ω to $\mathbb{R}$, $C(\Omega, \mathbb{R}^m)$ the space of continuous functions from Ω to $\mathbb{R}^m$ and $C_b\left(\overline{\Omega}\right)$ the space of all continuous and bounded functions on $\overline{\Omega}$, the space $C_b\left(\overline{\Omega}\right)$ is equipped with the norm $\|.\|_\infty$;*

$$\|u\|_\infty = \sup_{x\in\overline{\Omega}} |u(x)|$$

For $k \geq 1$ integer, $C^k(\Omega)$ is the space of functions u which are k times derivable and whose derivation of order k is continuous on Ω. $C_c^k(\Omega)$ is the set of functions of $C^k(\Omega)$ whose support is compact and contained in Ω.
We also define $C^k\left(\overline{\Omega}\right)$ as the set of restrictions to $\overline{\Omega}$ of elements from $C^k(\mathbb{R}^n)$ or as being the set of functions of $C^k(\Omega)$, such that for all $0 \leq j \leq k$, and for all $x_0 \in \partial\Omega$, the limit $\lim\limits_{x\to x_0} D_j u(x)$ exists and depends only on x_0.
$C_0^\infty(\Omega)$ or $\mathfrak{D}(\Omega)$, is the space of the infinitely differentiable functions, with compact supports called test function space.
The Hölder space $C^{k,\alpha}(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$ and $k \geq 0$ an integer, $0 < \alpha \leq 1$, consists of those real or complex-valued k-times continuously differentiable functions f on Ω verifying

$$|f^\beta(x) - f^\beta(y)| \leq C\|x - y\|^\alpha$$

where $C > 0$, $|\beta| \leq k$.

1.2 L^p Spaces

Definition 2. *Let: Ω be an open set of $\mathbb{R}^n$, equipped with the Lebesgue measure dx. We denote by $L^1(\Omega)$ the space of integrable functions on Ω with values in $\mathbb{R}$, it is provided with the norm:*

$$\|u\|_{L^1} = \int_\Omega |u(x)|\, dx.$$

Let $p \in \mathbb{R}$ with $1 \leq p < +\infty$, we define the space $L^p(\Omega)$ by

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R},\ f \ \textit{measurable and} \int_\Omega |f(x)|^p \, dx < +\infty \right\}$$

equipped with norm

$$\|u\|_{L^p} = \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

We also define the space $L^\infty(\Omega)$

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R},\ f \ \textit{measurable}, \ \exists c > 0, \ \textit{so that} \ |f(x)| \leq c \quad \textit{a.e.} \ x \ \textit{on} \ \Omega \},$$

it will be equipped with the essential-sup norm

$$\|u\|_{L^\infty} = \underset{x \in \Omega}{ess\sup} |u(x)| = \inf \{ c \ ; \ |u(x)| \leq c \quad \textit{for almost every} \ x \ \textit{on} \ \Omega \}.$$

We say that a function $f : \Omega \to \mathbb{R}$ belongs to $L^p_{loc}(\Omega)$ if $\mathbf{1}_K f \in L^p(\Omega)$ for any compact $K \subset \Omega$.

Theorem 1. *(Holder's inequality)*

If f and g are measurable functions, then if $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\int |f| \ |g| \, dx \leq \left(\int |f|^p dx \right)^{\frac{1}{p}} \left(\int |g|^q dx \right)^{\frac{1}{q}}.$$

Theorem 2. *(Dominated convergence theorem)*

Let $\{f_n\}_{n \geq 1}$ be a series of functions of $L^1(\Omega)$ converging almost everywhere to a measurable function f. It is assumed that there exists $g \in L^1(\Omega)$ such that for all $n \geq 1$, we get

$$|f_n| \leq g \quad \textit{a.e on} \ \Omega.$$

Then $f \in L^1(\Omega)$ and

$$\lim_{n \to +\infty} \|f_n - f\|_{L^1} = 0, \ \textit{and} \quad \int_\Omega f(x) \, dx = \lim_{n \to +\infty} \int_\Omega f_n(x) \, dx.$$

Remark 1. *if $\{f_n\}_{n \geq 1}$ is a Cauchy sequence in $L^p(\Omega)$,then there exists $f \in L^p(\Omega)$ and a subsequence which converges to $f \in L^p(\Omega)$*

and $\lim_{n \to +\infty} \|f_n - f\|_{L^p} = 0$

$L^p(\Omega)$ is complete, so it is a banach space.

1.3 Sobolev spaces

1.3.1 Weak derivative

Definition 3. *Let* Ω *be an open set of* $\mathbb{R}$*, and* $1 \leq i \leq n$*. A function* $u \in L^1_{loc}(\Omega)$ *has an* i^{th} *weak derivative in* $L^1_{loc}(\Omega)$ *if there exists* $f_i \in L^1_{loc}(\Omega)$ *such that for all* $\varphi \in C_0^\infty(\Omega)$ *we have*

$$\int_\Omega u(x)\,\partial_i\varphi(x)\,dx = -\int_\Omega f_i(x)\,\varphi(x)\,dx.$$

This leads to say that the i^{th} *derivative within the meaning of distributions of* u *belongs to* $L^1_{loc}(\Omega)$*, we write*

$$\partial_i u = \frac{\partial u}{\partial x_i} = f_i$$

1.3.2 $W^{1,p}(\Omega)$ spaces

Let Ω be a bounded or unbounded open set of $\mathbb{R}^n$, and $p \in \mathbb{R}$, $1 \leq p \leq +\infty$.

Definition 4. *The space* $W^{1,p}(\Omega)$ *is defined by*

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega);\ \text{such that } \partial_i u \in L^p(\Omega), 1 \leq i \leq n\}$$

where $\partial_i u$ *is the* i^{th} *weak derivative of* $u \in L^1_{loc}(\Omega)$.

For $1 \leq p < +\infty$ *we define the space* $W_0^{1,p}(\Omega)$ *as being the closure of* $\mathcal{D}(\Omega)$ *in* $W^{1,p}(\Omega)$*, and we write*

$$W_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{1,p}}.$$

1.3.3 $W^{m,p}(\Omega)$ Spaces

Let Ω be an open set of $\mathbb{R}^n, m \geq 2$ integer number and p real number such that $1 \leq p \leq +\infty$.

We define the space $W^{m,p}(\Omega)$ as following

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega),\ \text{such that } \partial^\alpha u \in L^p(\Omega),\ \forall\alpha, |\alpha| \leq m\}$$

where $\alpha \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + ... + \alpha_n$ the length of α and $\partial^\alpha u = \partial_1^{\alpha_1}...\partial_n^{\alpha_n}$ is the weak derivative of a function $u \in L^1_{loc}(\Omega)$ in the sense of definition (3).

The space $W^{m,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{m,p}} = \|u\|_{L^p} + \sum_{0<|\alpha|\leq m} \|\partial^\alpha u\|_{L^p}.$$

For $p = 2$, the space $W^{m,2}(\Omega)$ is noted $H^m(\Omega)$.

1.4 Fourier Transform Formula

There are many different forms of the Fourier Transform for integrable functions on R that you may find in various sources.

Definition 5. *The fourier transforme of function $f(x)$ at frequency ξ, is denoted by $F(f) = \widehat{f}(\xi)$ such that:*

$$\widehat{f}(\xi) = \int_{-\infty}^{+\infty} f(x) \exp(-i2\pi\xi x) dx.$$

Or

$$\widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \exp(-iwx) dx.$$

The Inverse Fourier Transform of function $\widehat{f}(\xi)$ is denoted by $F^{-1}(\widehat{f}) = f(x)$ such that:

$$f(x) = \int_{-\infty}^{+\infty} \widehat{f}(\xi) \exp(i2\pi\xi x) d\xi.$$

Or

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \widehat{f}(w) \exp(iwx) dw.$$

Proposition 1. *The fourier transforme of the k^{th} derivative of function $f(x)$ is:*

$$\widehat{f^{(k)}}(\xi) = (i\xi)^k \widehat{f}(\xi)$$

The Fourier transform of the product of two functions $f(x).g(x)$ is:

$$\widehat{f(x).g(x)} = (\widehat{f} * \widehat{g})(\xi),$$

and

$$\widehat{(f * g)(x)} = (\widehat{f}(\xi).\widehat{g}(\xi))$$

The Fourier transform makes it possible to define certain operators directly through their Fourier transform as:

$$\widehat{S(t).f(\xi)} = \exp(it\xi^3)\widehat{f}(\xi)$$

1.5 The Gevrey and Bourgain spaces

In this part, we present the Gevrey classes and the spaces of functions that we will use in this thesis.

Let Ω be an open set and $\sigma \geqslant 1$ be a fixed real number.

Definition 6. *We call Gevrey space of order σ denoted $G^\sigma(\Omega)$, the set of functions $f(x) \in C^\infty(\Omega)$ with the following property:*

- for every compact subset $K \subset \Omega$, $\exists\ C > 0$, such that for all α and $x \in K$

$$|\partial^\alpha f(x)| \leqslant C^{\|\alpha\|+1}(\alpha!)^\sigma.$$

It is sometimes useful to use the equivalent estimate:

$$|\partial^\alpha f(x)| \leqslant TC^{\|\alpha\|}(\alpha!)^\sigma.$$

Such that T and C are two positive constants independent of α.

Proposition 2. *Any Gevrey function of order 1 is an analytic function in Ω.*

So $G^1(\Omega) = A(\Omega)$ is the space of all analytic functions in Ω.

Proposition 3.

$$\forall s \leqslant t, \quad G^s(\Omega) \subset G^t(\Omega)$$

$$A(\Omega) \subset \bigcap_{\sigma>1} G^\sigma(\Omega),$$

and

$$\bigcup_{\sigma \geqslant 1} G^\sigma(\Omega) \subset C^\infty(\Omega).$$

The space $X_{s,b}$ is well-suited for capturing the dispersive smoothing effect of the operator $\partial_t - ih(D)$ away from the characteristic hypersurface $\tau = h(\xi)$.

By analogy with the relation ship $G^{\delta,s} = e^{-\delta\|D\|}(H^s)$, we define the Gevrey-modified Bourgain space $X_{\delta,s,b}$, for $\delta > 0$ by

$$X_{\delta,s,b} = e^{-\delta\|D\|}(X_{s,b}),$$

with norm

$$\|u\|_{X_{\delta,s,b}} = \left\| e^{\delta\|\xi\|} \langle \xi \rangle^s \langle \tau - h(\xi) \rangle^b \widehat{u}(\xi,\tau) \right\|_{L^2_{\xi,\tau}}.$$

Note that $X_{\delta,s,b}$ is well-defined, since $e^{-\delta\|D\|} = \mathcal{F}^{-1} e^{-\delta\|\cdot\|}\mathcal{F}$ who maps $X_{s,b}$ in to itself, for $\delta \geqslant 0$.

The restriction of $X_{s,b}$ to a time-slab $(0,T) \times \mathbb{R}^d$ is denoted $X^T_{s,b}$. This is a Banach space when equipped with the norm

$$\|u\|_{X^T_{s,b}} = \inf \left\{ \|v\|_{X_{s,b}} : v \in X_{s,b} \text{ and } u = v \text{ on } (0,T) \times \mathbb{R}^d \right\}.$$

The restriction $X^T_{\delta,s,b}$ is similarly defined, and then we clearly have

$$X^T_{\delta,s,b} = e^{-\delta\|D\|}(X^T_{s,b}),$$

hence the well-known properties of $X_{s,b}$ and its restrictions carry over to $X_{\delta,s,b}$ simply by the substitution $u \longrightarrow e^{\delta\|D\|}u$. the analytic Gevrey spaces with $\gamma \geq 1$ are given by $\mathcal{G}_{\sigma,\delta,s}(\mathbb{T}_\gamma) = \mathcal{G}_{\sigma,\delta,s}$. For $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, let us define

$$\mathcal{G}_{\sigma,\delta,s}(\mathbb{T}_\gamma) = \left\{ f \in L^2(\mathbb{T}_\gamma); \|f\|^2_{\mathcal{G}^{\sigma,\delta,s}(\mathbb{T}_\gamma)} = \sum_{k \in \mathbb{Z}} e^{2\delta|k|^{1/\sigma}} \langle k \rangle^{2s} |\widehat{f}(k)|^2 d\xi < \infty \right\},$$

where $\langle \cdot \rangle = (1 + | \cdot |)$.
At a time, the analytic Gevrey-Bourgain spaces $X^{\beta}_{\sigma,\delta,s,b}(\mathbb{T}_\gamma \times \mathbb{R}) = X^{\beta}_{\sigma,\delta,s,b}$ and $X_{\sigma,\delta,s,b}(\mathbb{T}_\gamma \times \mathbb{R}) = X_{\sigma,\delta,s,b}$ are defined by

$$\|u\|_{X_{\sigma,\delta,s,b}(\mathbb{T}_\gamma\times\mathbb{R})} = \left(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{2\delta|k|^{1/\sigma}} \langle k\rangle^{2s} \left\langle \tau - k^3 \right\rangle^{2b} \mid \widehat{u}(k,\tau) \mid^2 d\tau \right)^{\frac{1}{2}},$$

$$\|w\|_{X^{\beta}_{\sigma,\delta,s,b}(\mathbb{T}_\gamma\times\mathbb{R})} = \left(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{2\delta|k|^{1/\sigma}} \langle k\rangle^{2s} \left\langle \tau - \beta k^3 \right\rangle^{2b} \mid \widehat{w}(k,\tau) \mid^2 d\tau \right)^{\frac{1}{2}}.$$

to introduce slightly smaller spaces $Y_{\sigma,\delta,s}(\mathbb{T}_\gamma \times \mathbb{R}) = Y_{\sigma,\delta,s}$ and $Y^{\beta}_{\sigma,\delta,s}(\mathbb{T}_\gamma \times \mathbb{R}) = Y^{\beta}_{\sigma,\delta,s}$ defined via the norms

$$\|u\|_{Y_{\sigma,\delta,s}} = \|u\|_{X_{\sigma,\delta,s,1/2}} + \|e^{\delta|k|^{1/\sigma}} \langle k\rangle^{s} \widehat{u}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})}$$

and

$$\|w\|_{Y^{\beta}_{\sigma,\delta,s}} = \|w\|_{X^{\beta}_{\sigma,\delta,s,1/2}} + \|e^{\delta|k|^{1/\sigma}} \langle k\rangle^{s} \widehat{w}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})}$$

For any interval $I \subset \mathbb{R}$, we define the localized spaces $Y^{I}_{\sigma,\delta,s} = Y_{\sigma,\delta,s}(\mathbb{T}_\gamma \times I)$ and $Y^{\beta,I}_{\sigma,\delta,s} = Y^{\beta}_{\sigma,\delta,s}(\mathbb{T}_\gamma \times I)$ with the norms

$$\|u\|_{Y^{I}_{\sigma,\delta,s}} = inf\left\{ \|U\|_{Y_{\sigma,\delta,s}}; U|_{(\mathbb{T}_\gamma\times I)} = u \right\}$$

and

$$\|w\|_{Y^{\beta,I}_{\sigma,\delta,s}} = inf\left\{ \|W\|_{Y^{\sigma,\beta}_{\delta,s}}; W|_{(\mathbb{T}_\gamma\times I)} = w \right\}$$

For $s \in \mathbb{R}$, $\sigma \geq 1$ and $\delta > 0$, we have, for all $T > 0$

$$Y_{\sigma,\delta,s}(\mathbb{T}_\gamma \times \mathbb{R}) \hookrightarrow C\left(\mathbb{R}, \mathcal{G}_{\sigma,\delta,s}(\mathbb{T}_\gamma)\right),$$

and

$$Y^{\beta}_{\sigma,\delta,s}(\mathbb{T}_\gamma \times \mathbb{R}) \hookrightarrow C\left(\mathbb{R}, \mathcal{G}_{\sigma,\delta,s}(\mathbb{T}_\gamma)\right).$$

Define the spaces $Z_{\sigma,\delta,s}(\mathbb{T}_\gamma \times \mathbb{R}) = Z_{\sigma,\delta,s}$ and $Z^{\beta}_{\sigma,\delta,s}(\mathbb{T}_\gamma \times \mathbb{R}) = Z^{\beta}_{\sigma,\delta,s}$ via the norms

$$\|u\|_{Z_{\sigma,\delta,s}} = \|u\|_{X_{\sigma,\delta,s,-1/2}} + \|e^{\delta|k|^{1/\sigma}} \langle k\rangle^{s} \left\langle \tau - k^3 \right\rangle^{-1} \widehat{u}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})},$$

and

$$\|w\|_{Z^{\beta}_{\sigma,\delta,s}} = \|w\|_{X^{\beta}_{\sigma,\delta,s,-1/2}} + \|e^{\delta|k|^{1/\sigma}} \langle k\rangle^{s} \left\langle \tau-\beta k^{3}\right\rangle^{-1} \widehat{w}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})}.$$

The completion of the Schwartz class $S(\mathbb{R}^2)$ is given by $X_{\delta,s,b}(\mathbb{R}^2) = X_{\delta,s,b}$, with respect to the

$$\begin{aligned}\|v\|_{X_{\delta,s,b}(\mathbb{R}^2)} &= \|Av\|_{X_{s,b}} \equiv \|W(-t)Av\|_{H^{s,b}} = \| \langle\rho\rangle^{b} \langle\zeta\rangle^{s} \widehat{W(-t)A}v(\zeta,\rho)\|_{L_{\zeta,\rho}} \\ \\ &= \|e^{\delta|\zeta|} \langle\rho-\zeta^3\rangle^{b} \langle\zeta\rangle^{s} \widehat{v}(\zeta,\rho)\|_{L_{\zeta,\rho}} \\ &= \left(\int_{\mathbb{R}^2} e^{2\delta|\zeta|} \left\langle\rho-\zeta^3\right\rangle^{2b} \langle\zeta\rangle^{2s} \mid \widehat{v}(\zeta,\rho)\mid^2 d\zeta d\rho\right)^{\frac{1}{2}},\end{aligned}$$

The spaces $X^T_{\delta,s,b}$, denotes the restriction of $X_{\delta,s,b}$ onto finite time interval $[-T,T]$, $T>0$ and equipped with the norm

$$\|v\|_{X^T_{\delta,s,b}} = \inf\left\{\| V \|_{X_{\delta,s,b}(\mathbb{R}^2)} : V\in X_{\delta,s,b}, v(t)=V(t) for -T\le t\le T\right\}.$$

We define the needed spaces beginning by the spaces of analytic Gevrey functions that contain our initial data. For $s\in\mathbb{R}$ and $\delta>0$ let

$$G^{\delta,s}(\mathbb{R}) = \left\{f\in L^2(\mathbb{R}); \|f\|^2_{G^{\delta,s}(\mathbb{R})} = \int e^{2\delta|\zeta|} \langle\zeta\rangle^{2s} |\widehat{f}(\zeta)|^2 d\zeta < \infty\right\},$$

where $\langle\cdot\rangle = (1+|\cdot|)$. For all $0<\delta'<\delta$ and $s,s'\in\mathbb{R}$, we have

$$G^{\delta,s}(\mathbb{R}) \subset G^{\delta',s'}(\mathbb{R}) \ i.e \ \|f\|_{G^{\delta',s'}(\mathbb{R})} \le c_{s,s',\delta,\delta'} \|f\|_{G^{\delta,s}(\mathbb{R})},$$

is the embedding property of the Gevrey spaces.

We, then define the analytic Bourgain spaces related to the modified Korteweg-de Vries type equations. The completion of the Schwartz class $S(\mathbb{R}^2)$ is given by $X^{\beta}_{\delta,s,b}(\mathbb{R}^2)$, for $s,b\in\mathbb{R}$, $\delta>0$ and $\sigma\ge 1$, subjected to the norm

$$\|w\|_{X^{\beta}_{\delta,s,b}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} e^{2\delta|\zeta|} \langle\zeta\rangle^{2s} \left\langle\eta-\beta\zeta^3\right\rangle^{2b} \mid \widehat{w}(\zeta,\eta)\mid^2 d\zeta d\eta\right)^{\frac{1}{2}},$$

Sometimes we use the definition $X^1_{\delta,s,b} = X_{\delta,s,b}$, where

$$\|w\|_{X_{\delta,s,b}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} e^{2\delta|\zeta|} \langle\zeta\rangle^{2s} \left\langle\eta-\zeta^3\right\rangle^{2b} \mid \widehat{w}(\zeta,\eta)\mid^2 d\zeta d\eta\right)^{\frac{1}{2}},$$

For any interval I, we define the localized spaces $X^{\beta,I}_{\sigma,\delta,s,b} = X^{\beta}_{\sigma,\delta,s,b}(\mathbb{R}\times I)$ with norm

$$\|w\|_{X^{\beta}_{\delta,s,b}(\mathbb{R}\times I)} = inf\left\{\|W\|_{X^{\beta}_{\delta,s,b}}; W|_{\mathbb{R}\times I} = w\right\}$$

1.6 Well Posed Problems

The mathematical term 'well-posed problem' stems from a definition given by Hadamard, meaning that an important physical problem must have three attributes:

• (Existence) It has a solution,

• (Uniqueness) It has only one solution,

• (Stability) The solution's behavior changes continuously with the initial data .

In our problems: for example the third problem is locally well posed if:

• Existence: for each $(u_0, v_0) \in \mathcal{G}^{\delta,s} \times \mathcal{G}^{\delta,s}$ there exists $T > 0$ and function $(u, v) \in C\left([0, T], \mathcal{G}^{\delta,s}\right) \times C\left([0, T], \mathcal{G}^{\delta,s}\right)$, satisfying PDEs .

• Uniqueness : The problem has at most one solution in $C\left([0, T], \mathcal{G}^{\delta,s}\right) \times C\left([0, T], \mathcal{G}^{\delta,s}\right)$

• Stability (Continuous dependence of the initial data):The map $(\Lambda[u, v], \Gamma[u, v])$ is continuous from: $\mathcal{G}^{\delta,s} \times \mathcal{G}^{\delta,s}$ to $C\left([0, T], \mathcal{G}^{\delta,s}\right) \times C\left([0, T], \mathcal{G}^{\delta,s}\right)$.

• If T can be taken large, the problem is globally well-posed in: $C\left([0, T], G^{\delta(T),s}\right) \times C\left([0, T], G^{\delta(T),s}\right)$, for any $T > 0$.

1.7 Contraction Mapping Theorem

The next result from The Banach fixed-point theorem (also known as the contraction mapping theorem) which is a useful tool in the metric space theory, it ensures the existence and uniqueness of fixed points and provides a technique for finding them.

Definition 7. *A map* $T : (X, d) \to (X, d)$ *on a metric space*(X, d) *is a contraction on* (X, d), *if and only if for some positif constant* $k < 1$,

$$for\ all\ x, y \in X, d(Tx, Ty) \leq kd(x, y),$$

.

Theorem 3. *(The Banach fixed-point theore)*
Let (X, d) *be a complete metric space, and* $T : (X, d) \to (X, d)$ *be a contraction on* (X, d)*, then* T *has a unique fixed-point* $x \in X$*:*

$$T(x) = x$$

Remark 2. *The solution to the differential equation can be expressed as a fixed point of a suitable integral operator.*
It can be used to prove existence and uniqueness of solutions to integral equations. meaning that Banach's fixed point theorem show that this integral operator has a unique fixed point.

Chapter 2

Analytic Gevrey Well-Posedness and regularity of Coupled Periodic KdV Systems

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2.1 Statement of the problem and overview

Our purpose in this chapter is to study the well-posedness and regularity for the coupled Kotewege-de Varie (KdV) system.

$$\begin{cases} u_t + u_{xxx} + ww_x = 0 \\ w_t + \beta w_{xxx} + (uw)_x = 0, \quad x \in \mathbb{T}_\gamma \, t \in \mathbb{R}, \ 0 < \beta < 1 \\ (u, w) \mid_{t=0} = (u_0, w_0). \end{cases} \tag{2.1.1}$$

where $\mathbb{T}_\gamma = [0, 2\pi\gamma)$ for some $\gamma \geq 1$.

It is well known that it is not new to study the KdV equations in the classical Sobolev spaces H^s, there are many discussed results according to different value of the exponent s. see([11], [18], [17], [31], [32], [35], [36]). And Since Gevrey functions on the circle belong to every Sobolev spaces, it is very important to study KdV equation on these spaces, especially when it comes from a system of coupled equations with nonlinearity.

In the first part of this chapter, we demonstrate that the unique solution of (2.1.1) is well-posed in analytic an appropriate Gevrey spaces. Next, as a last section $G^{3\sigma}$ regularity in t is given and the failure of G^d regularity in t is shown.

2.2 Preliminary estimates and Function spaces

We often use without mention, in the whol manuscript, nation $A+$ and we mean $A + \varepsilon$, for arbitrarily small $\epsilon \ll 1$. It is now necessary to recall a definition of the needed spaces, where the analytic Gevrey spaces with $\gamma \geq 1$ are given by $\mathcal{G}_{\sigma,\delta,s}(\mathbb{T}_\gamma) = \mathcal{G}_{\sigma,\delta,s}$. For $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, let us define:

$$\mathcal{G}_{\sigma,\delta,s}(\mathbb{T}_\gamma) = \left\{ f \in L^2(\mathbb{T}_\gamma); \|f\|^2_{\mathcal{G}^{\sigma,\delta,s}(\mathbb{T}_\gamma)} = \sum_{k\in\mathbb{Z}} e^{2\delta|k|^{1/\sigma}} \langle k \rangle^{2s} |\widehat{f}(k)|^2 d\xi < \infty \right\},$$

where $\langle \cdot \rangle = (1 + |\cdot|)$.

At a time, the analytic Gevrey-Bourgain spaces $X^\beta_{\sigma,\delta,s,b}(\mathbb{T}_\gamma \times \mathbb{R}) = X^\beta_{\sigma,\delta,s,b}$ and $X_{\sigma,\delta,s,b}(\mathbb{T}_\gamma \times \mathbb{R}) = X_{\sigma,\delta,s,b}$ are defined by

$$\|u\|_{X_{\sigma,\delta,s,b}(\mathbb{T}_\gamma\times\mathbb{R})} = \left(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{2\delta|k|^{1/\sigma}} \langle k \rangle^{2s} \left\langle \tau - k^3 \right\rangle^{2b} \mid \widehat{u}(k,\tau) \mid^2 d\tau \right)^{\frac{1}{2}},$$

$$\|w\|_{X^{\beta}_{\sigma,\delta,s,b}(\mathbb{T}_\gamma\times\mathbb{R})} = \left(\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} e^{2\delta|k|^{1/\sigma}} \langle k\rangle^{2s} \left\langle \tau-\beta k^3\right\rangle^{2b} \mid \widehat{w}(k,\tau)\mid^2 d\tau\right)^{\frac{1}{2}}.$$

The proof of local well-posedness is based on the iteration in the spaces $X_{\sigma,\delta,s,1/2}\times X^{\beta}_{\sigma,\delta,s,1/2}$. However, these spaces barely fails to be in $C\left([-T,T],\mathcal{G}^{\sigma,\delta,s}\right)\times C\left([-T,T],\mathcal{G}^{\sigma,\delta,s}\right)$. This led us to consider introducing slightly smaller spaces $Y_{\sigma,\delta,s}(\mathbb{T}_\gamma\times\mathbb{R})=Y_{\sigma,\delta,s}$ and $Y^{\beta}_{\sigma,\delta,s}(\mathbb{T}_\gamma\times\mathbb{R})=Y^{\beta}_{\sigma,\delta,s}$ defined via the norms:

$$\|u\|_{Y_{\sigma,\delta,s}} = \|u\|_{X_{\sigma,\delta,s,1/2}} + \|e^{\delta|k|^{1/\sigma}} \langle k\rangle^s \widehat{u}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})}$$

and

$$\|w\|_{Y^{\beta}_{\sigma,\delta,s}} = \|w\|_{X^{\beta}_{\sigma,\delta,s,1/2}} + \|e^{\delta|k|^{1/\sigma}} \langle k\rangle^s \widehat{w}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})}$$

For any interval $I\subset\mathbb{R}$, we define the localized spaces $Y^{I}_{\sigma,\delta,s}=Y_{\sigma,\delta,s}(\mathbb{T}_\gamma\times I)$ and $Y^{\beta,I}_{\sigma,\delta,s}=Y^{\beta}_{\sigma,\delta,s}(\mathbb{T}_\gamma\times I)$ with the norms

$$\|u\|_{Y^{I}_{\sigma,\delta,s}} = inf\left\{\|U\|_{Y_{\sigma,\delta,s}}; U|_{(\mathbb{T}_\gamma\times I)}=u\right\}$$

and

$$\|w\|_{Y^{\beta,I}_{\sigma,\delta,s}} = inf\left\{\|W\|_{Y^{\sigma,\beta}_{\delta,s}}; W|_{(\mathbb{T}_\gamma\times I)}=w\right\}$$

For $s\in\mathbb{R}$, $\sigma\geq 1$ and $\delta>0$, we have, for all $T>0$

$$Y_{\sigma,\delta,s}(\mathbb{T}_\gamma\times\mathbb{R}) \hookrightarrow C\left(\mathbb{R},\mathcal{G}_{\sigma,\delta,s}(\mathbb{T}_\gamma)\right),$$

and

$$Y^{\beta}_{\sigma,\delta,s}(\mathbb{T}_\gamma\times\mathbb{R}) \hookrightarrow C\left(\mathbb{R},\mathcal{G}_{\sigma,\delta,s}(\mathbb{T}_\gamma)\right).$$

Define the spaces $Z_{\sigma,\delta,s}(\mathbb{T}_\gamma\times\mathbb{R})=Z_{\sigma,\delta,s}$ and $Z^{\beta}_{\sigma,\delta,s}(\mathbb{T}_\gamma\times\mathbb{R})=Z^{\beta}_{\sigma,\delta,s}$ via the norms

$$\|u\|_{Z_{\sigma,\delta,s}} = \|u\|_{X_{\sigma,\delta,s,-1/2}} + \|e^{\delta|k|^{1/\sigma}} \langle k\rangle^s \left\langle \tau-k^3\right\rangle^{-1} \widehat{u}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})},$$

and

$$\|w\|_{Z^{\beta}_{\sigma,\delta,s}} = \|w\|_{X^{\beta}_{\sigma,\delta,s,-1/2}} + \|e^{\delta|k|^{1/\sigma}} \langle k\rangle^s \left\langle \tau-\beta k^3\right\rangle^{-1} \widehat{w}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})}.$$

2.2.1 Linear estimates

Owing to the Fourier transform with respect to x of (2.1.1), we obtain a differential equation and then solving it in t. We consider for two operators Θ, Ξ, the following integral system which is equivalent to (2.1.1)

$$\begin{cases} \Theta[u,w](t) = S(t)u_0 - \displaystyle\int_0^t S(t-\nu)G_1(\nu)d\nu \\ \\ \Xi[u,w](t) = S_\beta(t)w_0 - \displaystyle\int_0^t S_\beta(t-\nu)G_2(\nu)d\nu, \end{cases} \tag{2.2.1}$$

where $S(t) = e^{-t\partial_x^3}$ and $S_\beta(t) = e^{-t\beta\partial_x^3}$. The nonlinear terms are defined by $G_1(\nu) = \partial_x(\frac{w^2}{2})(\nu)$ and $G_2(\nu) = \partial_x(uw)(\nu)$.

Lemma 1. *Let $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$. For some constant $C > 0$ and any time interval I which contains $t = 0$ and has length $\mid I \mid \leq 1$, we have*

$$\|S(t)u_0\|_{Y^I_{\sigma,\delta,s}} \leq C\ \|u_0\|_{\mathcal{G}_{\sigma,\delta,s}},$$

and

$$\|S_\beta(t)w_0\|_{Y^{\beta,I}_{\sigma,\delta,s}} \leq C\ \|w_0\|_{\mathcal{G}_{\sigma,\delta,s}},$$

for all $w_0, u_0 \in \mathcal{G}_{\sigma,\delta,s}$.

Lemma 2. *Let $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, then for some constant $C > 0$ and any time interval I which contains $t = 0$ and has length $\mid I \mid \leq 1$, we have*

$$\left\| \int_0^t S(t-\nu)G_1(\nu)\mathrm{d}\nu \right\|_{Y^I_{\sigma,\delta,s}} \leq C\|G_1(\nu)\|_{Z^I_{\sigma,\delta,s}},$$

and

$$\left\| \int_0^t S_\beta(t-\nu)G_2(\nu)\mathrm{d}\nu \right\|_{Y^{\beta,I}_{\sigma,\delta,s}} \leq C\|G_2(\nu)\|_{Z^{\beta,I}_{\sigma,\delta,s}}.$$

Proof. For the proof of Lemma 1 and Lemma 2 see (Lemma 7.1 and 7.2 in [17]) □

2.2.2 Bilinear estimates

In the following Lemma, we state the desired bilinear estimate.

Lemma 3. *Let* $s \geq \min\{1, \frac{1}{2} + \frac{1}{2}\varrho_1 +\}$, $\delta > 0$ *and* $\sigma \geq 1$, *then*

$$\| \partial_x(w_1 w_2) \|_{Z_{\sigma,\delta,s}} \leq C_0(\gamma) \| w_1 \|_{Y^{\beta}_{\sigma,\delta,s}} \| w_2 \|_{Y^{\beta}_{\sigma,\delta,s}},$$

where

$$C_0(\gamma) = \begin{cases} \gamma^{\frac{1}{2}+\frac{1}{2}\varrho_1+}, \; for \; 0 \leq \varrho_1 < 1, \\ \\ \gamma^{0+}, \; for \; \varrho_1 \geq 1. \end{cases}$$

Proof. Define the operator A by

$$\widehat{Aw}^x(\xi, t) = e^{\delta|\xi|^{1/\sigma}} \widehat{w}^x(\xi, t). \tag{2.2.2}$$

we have

$$\begin{aligned} e^{\delta|k|^{1/\sigma}} \widehat{w_1 w_2} &= (2\pi)^{-2} e^{\delta|k|^{1/\sigma}} \widehat{w_1} * \widehat{w_2} \\ &\leq (2\pi)^{-2} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} e^{\delta|k-k_1|^{1/\sigma}} \widehat{w_1}(k-k_1, \tau-\tau_1) e^{\delta|k_1|^{1/\sigma}} \widehat{w_2}(k_1, \tau_1) d\tau_1 \\ &= \widehat{Aw_1 Aw_2} \end{aligned}$$

Since $\delta \mid k \mid^{1/\sigma} \leq \delta \mid k - k_1 \mid^{1/\sigma} + \delta \mid k_1 \mid^{1/\sigma}, \quad \forall \sigma \geq 1.$

Then

$$\begin{aligned} \|\partial_x(w_1 w_2)\|_{Z_{\sigma,\delta,s}} &= \|\partial_x(w_1 w_2)\|_{X_{\sigma,\delta,s,-1/2}} \\ &+ \|e^{\delta|k|^{1/\sigma}} \langle k \rangle^s \langle \tau - k^3 \rangle^{-1} \widehat{\partial_x(w_1 w_2)}(k, \tau)\|_{L^2_k(\mathbb{T}/\gamma) L^1_\tau(\mathbb{R})} \\ &\leq \|\partial_x(A w_1 A w_2)\|_{X_{s,-1/2}} \\ &+ \| \langle k \rangle^s \langle \tau - k^3 \rangle^{-1} \partial_x(\widehat{Aw_1 A}w_2)(k, \tau)\|_{L^2_k(\mathbb{T}/\gamma) L^1_\tau(\mathbb{R})} \\ &= \| \partial_x(A w_1 A w_2) \|_{Z_s}. \end{aligned}$$

Now, by using Proposition 3.7 of [40], there exists $C_0(\gamma)$ where

$$C_0(\gamma) = \begin{cases} \gamma^{\frac{1}{2}+\frac{1}{2}\varrho_1+}, \ for\ 0 \leq \varrho_1 < 1, \\ \\ \gamma^{0+}, \ for\ \varrho_1 \geq 1. \end{cases}$$

such that

$$\begin{aligned} \| \partial_x(Aw_1Aw_2) \|_{Zs} &\leq C_0(\gamma) \| Aw_1 \|_{Y_s^\beta} \| Aw_2 \|_{Y_s^\beta} \\ \\ &= C_0(\gamma) \| w_1 \|_{Y^\beta_{\sigma,\delta,s}} \| w_2 \|_{Y^\beta_{\sigma,\delta,s}} . \end{aligned}$$

□

Lemma 4. *Let* $s \geq \min\{1, \frac{1}{2} + \frac{1}{2}\max\{\varrho_2, \varrho_3\}+\}$, $\delta > 0$ *and* $\sigma \geq 1$, *we have*

$$\| \partial_x(uw) \|_{Z^\beta_{\sigma,\delta,s}} \leq C_1(\gamma) \| u \|_{Y_{\sigma,\delta,s}} \| w \|_{Y^\beta_{\sigma,\delta,s}}, \tag{2.2.3}$$

where

$$C_1(\gamma) = \begin{cases} \gamma^{\frac{1}{2}+\frac{1}{2}\max\{\varrho_2,\varrho_3\}+}, \ for\ 0 \leq \max\{\varrho_2, \varrho_3\} < 1, \\ \\ \gamma^{0+}, \ for\ \max\{\varrho_2, \varrho_3\} \geq 1. \end{cases}$$

Proof. We have

$$\begin{aligned} \|\partial_x(uw)\|_{Z^\beta_{\sigma,\delta,s}} &= \|\partial_x(uw)\|_{X^\beta_{\sigma,\delta,s,-1/2}} \\ \\ &+\|e^{\delta|k|^{1/\sigma}} \langle k\rangle^s \langle \tau - \beta k^3\rangle^{-1} \widehat{\partial_x(uw)}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})} \\ \\ &\leq \|\partial_x(AuAw)\|_{X^\beta_{s,-1/2}} \\ \\ &+\| \langle k\rangle^s \langle \tau - \beta k^3\rangle^{-1} \partial_x\widehat{(AuAw)}(k,\tau)\|_{L^2_k(\mathbb{T}/\gamma)L^1_\tau(\mathbb{R})} \\ \\ &=\| \partial_x(AuAw) \|_{Z^\beta_s} . \end{aligned}$$

Now, by using Proposition 3.8 of [40], there exists $C_1(\gamma)$ where

$$C_1(\gamma) = \begin{cases} \gamma^{\frac{1}{2}+\frac{1}{2}\max\{\varrho_2,\varrho_3\}+}, \ for \ 0 \leq \max\{\varrho_2, \varrho_3\} < 1, \\ \\ \gamma^{0+}, \ for \ \max\{\varrho_2, \varrho_3\} \geq 1. \end{cases}$$

such that

$$\begin{aligned} \| \partial_x(AuAw) \|_{Z_s^\beta} &\leq C_1(\gamma) \| Au \|_{Y_s} \| Aw \|_{Y_s^\beta} \\ \\ &= C_1(\gamma) \| u \|_{Y_{\sigma,\delta,s}} \| w \|_{Y^\beta_{\sigma,\delta,s}} . \end{aligned}$$

□

The following corollaries will be essential in our proof of the main theorem. For more detail see proof of lemma 6 and corollary 1 in [28].

Corollary 1. *Under the hypotheses of Lemma 3, for any time interval I, we have*

$$\| \partial_x(w_1 w_2) \|_{Z^I_{\sigma,\delta,s}} \leq C_0(\gamma) \mid I \mid^{0+} \| w_1 \|_{Y^{\beta,I}_{\sigma,\delta,s}} \| w_2 \|_{Y^{\beta,I}_{\sigma,\delta,s}},$$

Corollary 2. *Under the hypotheses of Lemma 4, for any time interval I, we have*

$$\| \partial_x(uw) \|_{Z^{\beta,I}_{\sigma,\delta,s}} \leq C_1(\gamma) \mid I \mid^{0+} \| u \|_{Y^I_{\sigma,\delta,s}} \| w \|_{Y^{\beta,I}_{\sigma,\delta,s}},$$

2.3 Local well-posedness and Proof

We are now ready to estimate all terms in (2.2.1) by using the trilinear estimates in the above Lemmas. We define the spaces

$$\mathcal{Y}_{\sigma,\delta,s} = Y^I_{\sigma,\delta,s} \times Y^{\beta,I}_{\sigma,\delta,s}, \ \ \mathcal{Z}_{\sigma,\delta,s} = Z^I_{\sigma,\delta,s} \times Z^{\beta,I}_{\sigma,\delta,s} \quad \text{and} \quad \mathcal{B}^{\sigma,\delta,s} = \mathcal{G}^{\sigma,\delta,s} \times \mathcal{G}^{\sigma,\delta,s},$$

with norms

$$\|(u,w)\|_{\mathcal{Y}_{\sigma,\delta,s}} = max\{\|u\|_{Y^I_{\sigma,\delta,s}}, \|w\|_{Y^{\beta,I}_{\sigma,\delta,s}}\},$$

and similar for $\mathcal{Z}_{\sigma,\delta,s}$ and $\mathcal{B}^{\sigma,\delta,s}$.

Theorem 4. *Let* $s \geq \min\{1, s_0+\}$, $s_0 = \frac{1}{2} + \frac{1}{2}\max\{\varrho_1, \varrho_2, \varrho_3\}$ $0 < \beta < 1, \gamma \geq 1, \sigma \geq 1, \delta > 0$ *and* $(u_0, w_0) \in \mathcal{B}^{\sigma,\delta,s}$. *There exists* $T > 0$, *which depends on* (u_0, w_0), *such that the Cauchy problem (2.1.1) has a unique solution*

$$(u, w) \in C\left([-T, T], \mathcal{G}^{\sigma,\delta,s}\right) \times C\left([-T, T], \mathcal{G}^{\sigma,\delta,s}\right). \tag{2.3.1}$$

We will show that $\Theta \times \Xi$ is a contraction on the ball $\mathcal{Y}(0, R)$ to $\mathcal{Y}(0, R)$.

Lemma 5. *Let* $s \geq \min\{1, s_0+\}$, $\sigma \geq 1$ *and* $\delta > 0$. *Then, for all* $(u_0, w_0) \in \mathcal{B}^{\sigma,\delta,s}$, *such that the map* $\Theta \times \Xi : \mathcal{Y}(0, R) \to \mathcal{Y}(0, R)$ *is a contraction, where* $\mathcal{Y}(0, R)$ *is given by*

$$\mathcal{Y}(0, R) = \{(u, w) \in \mathcal{Y}_{\sigma,\delta,s}; \|(u, w)\|_{\mathcal{Y}_{\sigma,\delta,s}} \leq R\},$$

where $R = 2C\|(u_0, w_0)\|_{\mathcal{B}^{\sigma,\delta,s}}$.

Proof. Combining Lemma (1), (2) and Corollary 1,2 we obtain

$$\begin{aligned} \| \Theta[u, w] \|_{Y^I_{\sigma,\delta,s}} &\leq C \| u_0 \|_{\mathcal{G}^{\sigma,\delta,s}} + CC_0(\gamma) \| w \|^2_{Y^{\beta,I}_{\sigma,\delta,s}} \\ &\leq C \| (u_0, w_0) \|_{\mathcal{B}^{\sigma,\delta,s}} + CC(\gamma)T^\epsilon \| (u, w) \|^2_{\mathcal{Y}_{\sigma,\delta,s}}, \end{aligned} \tag{2.3.2}$$

and

$$\begin{aligned} \| \Xi[u, w] \|_{Y^{\beta,I}_{\sigma,\delta,s}} &\leq C \| w_0 \|_{\mathcal{G}^{\sigma,\delta,s}} + CC_1(\gamma) \| u \|_{Y^I_{\sigma,\delta,s}} \| w \|_{Y^{\beta,I}_{\sigma,\delta,s}} \\ &\leq C \| (u_0, w_0) \|_{\mathcal{B}^{\sigma,\delta,s}} + CC(\gamma)T^\epsilon \| (u, w) \|^2_{\mathcal{Y}_{\sigma,\delta,s}}. \end{aligned} \tag{2.3.3}$$

Where $C(\gamma) = \max\{C_0(\gamma), C_1(\gamma)\}$. Therefore, from (2.3.2) and (2.3.3), we obtain

$$\| (\Theta[u, w], \Xi[u, w]) \|_{\mathcal{Y}_{\sigma,\delta,s}} \leq C \| (u_0, w_0) \|_{\mathcal{B}^{\sigma,\delta,s}} + CC(\gamma)T^\epsilon \| (u, w) \|^2_{\mathcal{Y}_{\sigma,\delta,s}}.$$

For all $(u, w) \in \mathcal{Y}(0, R)$, we have

$$\| (\Theta[u, w], \Xi[u, w]) \|_{\mathcal{Y}_{\sigma,\delta,s}} \leq R,$$

where $T^\epsilon \leq \frac{1}{4CC(\gamma)R}$.
Thus, $\Theta \times \Xi : \Big(\mathcal{Y}(0, R), \mathcal{Y}(0, R)\Big)$, is a contraction, since

$$\| (\Theta[u, w] - \Theta[u^*, w^*], \Xi[u, w] - \Xi[u^*, w^*]) \|_{\mathcal{Y}_{\sigma,\delta,s}} \leq \frac{1}{2} \| (u - u^*, w - w^*) \|_{\mathcal{Y}_{\sigma,\delta,s}}.$$

The rest of the proof (Of Theorem 4) follows a standard argument, see [9]. □

2.4 Gevrey's regularity

Replacing t with $-t$ we can write our system as follows

$$\begin{cases} \partial_t u = \partial_x^3 u + w\partial_x w, \\ \partial_t w = \beta\partial_x^3 w + \partial_x(uw), \qquad 0 < \beta < 1 \\ u(x,0) = u_0(x), \\ w(x,0) = w_0(x), \end{cases} \tag{2.4.1}$$

2.4.1 Gevrey-3σ regularity in time

Theorem 5. *Let $s \geq \min\{1, s_0+\}$, for $0 < \beta < 1$, $\delta > 0$, $\sigma \geq 1$ and $(u,w) \in C\left([-T,T]; \mathcal{G}^{\sigma,\delta,s}\right) \times C\left([-T,T]; \mathcal{G}^{\sigma,\delta,s}\right)$ be the solution of (2.4.1). Then $(u,w) \in \mathcal{G}^{3\sigma}([-T,T]) \times \mathcal{G}^{3\sigma}([-T,T])$ in the time variable t.*

In order to prove Theorem 5 it is enough to prove the following results.

Proposition 4. *Let $n, k \in \mathbb{Z}_+^*$, we have*

$$|\partial_t^n \partial_x^k u(x,t)| \leq C^{n+k+1}((3n+k)!)^\sigma M^n, \quad \forall x \in \mathbb{T}_\gamma, \tag{2.4.2}$$

and

$$|\partial_t^n \partial_x^k w(x,t)| \leq C^{n+k+1}((3n+k)!)^\sigma M^n, \quad \forall x \in \mathbb{T}_\gamma, \tag{2.4.3}$$

where $M = C^2 + \frac{C}{2^\sigma}$.

Proof. We will use the proof by induction on n. For $n = 0$, inequality (2.4.2) follows from the following result.

$$|\partial_x^k u(x,t)| \leq C^{k+1}(k!)^\sigma, \quad \forall x \in \mathbb{T}, \forall k \in \mathbb{Z}_+^*, \tag{2.4.4}$$

and

$$|\partial_x^k w(x,t)| \leq C^{k+1}(k!)^\sigma, \quad \forall x \in \mathbb{T}, \forall k \in \mathbb{Z}_+^*. \tag{2.4.5}$$

For the proof of these inequalities, one can see Proposition 5 in [21]. We now suppose that (2.4.2) holds for all derivatives in t of order $\leq n$ and $k \in \mathbb{Z}_+^*$ and we then prove that (2.4.2) holds for $n+1$ and $k \in \mathbb{Z}_+^*$. We have from (2.4.1) that

$$\begin{aligned}
\partial_t^{n+1}\partial_x^k u &= \partial_t^n\partial_x^{k+3}u + \partial_t^n\partial_x^k(w\partial_x w) \\
&= \partial_t^n\partial_x^{k+3}u + \partial_t^n\left(\sum_{q=0}^{k}\binom{k}{q}\partial_x^{k-q}w\partial_x^{q+1}w\right) \\
&= \partial_t^n\partial_x^{k+3}u + \left(\sum_{q=0}^{k}\binom{k}{q}(\partial_t^n\partial_x^{k-q}w)(\partial_x^{q+1}w)\right) \\
&+ \left(\sum_{q=0}^{k}\binom{k}{q}(\partial_x^{k-q}w)(\partial_t^n\partial_x^{q+1}w)\right) \\
&+ \left(\sum_{m=1}^{n-1}\sum_{q=0}^{k}\binom{n}{m}\binom{k}{q}(\partial_t^{n-m}\partial_x^{k-q}w)(\partial_t^m\partial_x^{q+1}w)\right),
\end{aligned} \tag{2.4.6}$$

and

$$\begin{aligned}
\partial_t^{n+1}\partial_x^k w &= \beta\partial_t^n\partial_x^{k+3}w + \partial_t^n\partial_x^{k+1}(uw) \\
&= \beta\partial_t^n\partial_x^{k+3}u + \partial_t^n\left(\sum_{q=0}^{k}\binom{k}{q}\partial_x^{k-q}u\partial_x^{q+1}w\right) \\
&= \beta\partial_t^n\partial_x^{k+3}w + \left(\sum_{q=0}^{k}\binom{k}{q}(\partial_t^n\partial_x^{k-q}u)(\partial_x^{q+1}w)\right) \\
&+ \left(\sum_{q=0}^{k}\binom{k}{q}(\partial_x^{k-q}u)(\partial_t^n\partial_x^{q+1}w)\right) \\
&+ \left(\sum_{m=1}^{n-1}\sum_{q=0}^{k}\binom{n}{m}\binom{k}{q}(\partial_t^{n-m}\partial_x^{k-q}u)(\partial_t^m\partial_x^{q+1}w)\right).
\end{aligned} \tag{2.4.7}$$

We will proof (2.4.6) and (2.4.7) will be similar. By using the induction assumption, we obtain

$$\begin{aligned} | \partial_t^n \partial_x^{k+3} u | &\leq C^{n+k+3+1}((3n+k+3)!)^\sigma M^n \\ &= C^{(n+1)+k+1}((3(n+1)+k)!)^\sigma M^n C^2. \end{aligned} \tag{2.4.8}$$

For the second term in (2.4.6), by using the induction assumption, we obtain

$$\begin{aligned} \left| \sum_{q=0}^{k} \binom{k}{q} (\partial_t^n \partial_x^{k-q} w)(\partial_x^{q+1} w) \right| &\leq \sum_{q=0}^{k} \frac{k!}{q!(k-q)!} C^{n+k-q+1}((3n+k-q)!)^\sigma M^n C^{q+1+1}((q+1)!)^\sigma \\ &\leq C^{n+k+3} M^n \sum_{q=0}^{k} \frac{(k!)^\sigma}{(q!(k-q)!)^\sigma}((3n+k-q)!)^\sigma ((q+1)!)^\sigma \\ &\leq \frac{1}{3} C^{(n+1)+k+1}((3(n+1)+k)!)^\sigma M^n \frac{C}{2^\sigma}. \end{aligned} \tag{2.4.9}$$

For the third term of (2.4.6), we have

$$\begin{aligned} \left| \sum_{q=0}^{k} \binom{k}{q} (\partial_x^{k-q} w)(\partial_t^n \partial_x^{q+1} w) \right| &\leq \sum_{q=0}^{k} \frac{k!}{q!(k-q)!} C^{k-q+1}((k-q)!)^\sigma M^n C^{n+q+1+1}((3n+q+1)!)^\sigma \\ &\leq C^{n+k+3} M^n \sum_{q=0}^{k} \frac{(k!)^\sigma}{(q!(k-q)!)^\sigma}((k-q)!)^\sigma ((3n+q+1)!)^\sigma \\ &\leq \frac{1}{3} C^{(n+1)+k+1}((3(n+1)+k)!)^\sigma M^n \frac{C}{2^\sigma}. \end{aligned} \tag{2.4.10}$$

For the fourth term in (2.4.6), we have

$$\left|\sum_{m=1}^{n-1}\sum_{q=0}^{k}\binom{n}{m}\binom{k}{q}(\partial_t^{n-m}\partial_x^{k-q}u)(\partial_t^{m}\partial_x^{q+1}w)\right|$$

$$\leq \sum_{m=1}^{n-1}\sum_{q=0}^{k}\binom{n+k}{m+q}C^{n-m+k-q+1}((3(n-m)+k-q)!)^{\sigma}M^{n-m}C^{m+q+1+1}((3m+q+1)!)^{\sigma}M^{m}$$

$$\leq C^{n+k+3}M^{n}\sum_{m=1}^{n-1}\sum_{q=0}^{k}\frac{(n+k)!}{(m+q)!(n+k-m-q)!}((3(n-m)+k-q)!)^{\sigma}((3m+q+1)!)^{\sigma} \tag{2.4.11}$$

$$\leq C^{n+k+3}M^{n}\sum_{m=1}^{n-1}\sum_{q=0}^{k}\frac{((n+k)!)^{\sigma}}{((m+q)!(n+k-m-q)!)^{\sigma}}((3(n-m)+k-q)!)^{\sigma}((3m+q+1)!)^{\sigma}$$

$$\leq \frac{1}{3}C^{(n+1)+k+1}((3(n+1)+k)!)^{\sigma}M^{n}\frac{C}{2^{\sigma}}.$$

To obtain the proof in detail see proof of Lemma 4.2 in [24]. Finally by using (2.4.6), (2.4.8), (2.4.9),(2.4.10) and (2.4.11) we obtain

$$|\partial_t^n\partial_x^k u(x,t)| \leq C^{n+k+1}((3n+k)!)^{\sigma}M^{n}, \quad \forall x\in\mathbb{T}_{\gamma},$$

taking $k=0$ we obtain

$$|\partial_t^n u(x,t)| \leq C^{n+1}((3n)!)^{\sigma}M^{n} \leq L^{n+1}(n!)^{3\sigma}, \quad for\ L>0, \forall x\in\mathbb{T}_{\gamma},$$

i.e. $u\in\mathcal{G}^{3\sigma}([-T,T])$ in time variable. We have also

$$|\partial_t^n\partial_x^k w(x,t)| \leq C^{n+k+1}((3n+k)!)^{\sigma}M^{n}, \quad \forall x\in\mathbb{T}_{\gamma},$$

taking $k=0$ we obtain

$$|\partial_t^n w(x,t)| \leq C^{n+1}((3n)!)^{\sigma}M^{n} \leq L^{n+1}(n!)^{3\sigma}, \quad for\ L>0, \forall x\in\mathbb{T}_{\gamma},$$

i.e. $w\in\mathcal{G}^{3\sigma}([-T,T])$ in time variable t. □

2.4.2 Failure of Gevrey-d regularity in time

The next Lemma will de useful in order to estimate the higher-order derivatives of solution with respect to t.

Lemma 6. *[24] If (u, w) is a solution to (2.4.1) then for every $n \in \{1, 2, ...\}$ we have*

$$\partial_t^n u = \partial_x^{3n} u + \sum_{m=1}^{n} \sum_{|\lambda|+2m=3n} C_\lambda^m (\partial_x^{\lambda_1} u) \cdots (\partial_x^{\lambda_m} w), \tag{2.4.12}$$

and

$$\partial_t^n w = \partial_x^{3n} w + \sum_{m=1}^{n} \sum_{|\lambda|+2m=3n} C_\lambda^m (\partial_x^{\lambda_1} u) \cdots (\partial_x^{\lambda_m} w). \tag{2.4.13}$$

Definition 8. *Let $\{\omega_k\}$ be a sequence of positive numbers. We denote by $\mathcal{C}(\omega_k)$ the class of all functions $h(x)$, infinitely differentiable on $[-1, 1]$, for each of which there is $C > 0$ such that:*

$$|h^{(k)}(x)| \leq C^{k+1} \omega_k, \quad x \in [-1, 1] \text{ and } k = 0, 1, 2, ...$$

Lemma 7. *([23]) For any $\sigma > 1$ and any sequence of complex numbers $\{\varphi_k\}$, satisfying*

$$|\varphi_k| \leq C_1^{k+1} k^{k\sigma}, C_1 > 0,$$

there exists a function $h(x) \in \mathcal{C}(k^{k\sigma})$ for which $h^{(k)}(0) = \varphi_k$.

This result will be used for the sequence of real numbers

$$|h^{(k)}(x)| \leq C^{k+1} k^{k\sigma} \leq C^{k+1} (k!)^\sigma e^{k\sigma}, \quad k = 0, 1, 2, ...$$

where $h(x) \in \mathcal{C}(k^{k\sigma})$ such that $h^{(k)}(0) = \varphi_k = (k!)^\sigma$.

We choose $u_0, w_0 \in \mathcal{G}_c^\sigma(-2, 2)$ such that

$$\begin{cases} \theta(x) = 1 \ for \ \mid x \mid \leq 1 \\ and \\ \theta(x) = 0 \ for \ \mid x \mid > 2, \end{cases}$$

by modifying $h(x)$ to become having a compact support in $(-1, 1)$.

If u_0 and w_0 are extensions of θh, then we have $u_0, w_0 \in \mathcal{G}^\sigma([-T, T])$.Consequently, we obtain the relations:

$$u_0^{(k)}(0) = h^{(k)}(0) = (k!)^\sigma \text{ and } w_0^{(k)}(0) = h^{(k)}(0) = (k!)^\sigma.$$

Theorem 6. *Let $s \geq \min\{1, s_0+\}$ for $0 < \beta < 1$, $\delta > 0$, $\sigma \geq 1$. The real-valued solution to (2.4.1) with real-valued initial data $(u_0, w_0) \in \mathcal{G}^{\sigma,\delta,s} \times \mathcal{G}^{\sigma,\delta,s}$ may not be in $\mathcal{G}^d([-T,T]) \times \mathcal{G}^d([-T,T])$, with $1 \leq d < 3\sigma$, in the time variable t.*

Proof. By using (2.4.12) and (2.4.13) we get

$$\begin{aligned}
\partial_t^n u(0,0) &= \partial_x^{3n} u(0,0) + \sum_{m=1}^{n} \sum_{|\lambda|+2m=3n} C_\lambda^m (\partial_x^{\lambda_1} u(0,0)) \cdots (\partial_x^{\lambda_m} v(0,0)) \\
&= u_0^{3n}(0) + \sum_{m=1}^{n} \sum_{|\lambda|+2m=3n} C_\lambda^m (\partial_x^{\lambda_1} u_0(0)) \cdots (\partial_x^{\lambda_m} w_0(0)) \\
&\geq u_0^{3n}(0) = ((3n)!)^\sigma \geq (n!)^{3\sigma},
\end{aligned}$$

and

$$\begin{aligned}
\partial_t^n w(0,0) &= \partial_x^{3n} w(0,0) + \sum_{m=1}^{n} \sum_{|\lambda|+2m=3n} C_\lambda^m (\partial_x^{\lambda_1} u(0,0)) \cdots (\partial_x^{\lambda_m} w(0,0)) \\
&= v_0^{3n}(0) + \sum_{m=1}^{n} \sum_{|\lambda|+2m=3n} C_\lambda^m (\partial_x^{\lambda_1} u_0(0)) \cdots (\partial_x^{\lambda_m} w_0(0)) \\
&\geq w_0^{3n}(0) = ((3n)!)^\sigma \geq (n!)^{3\sigma},
\end{aligned}$$

we have proved that $(u(0,\cdot), w(0,\cdot)) \notin \mathcal{G}^d([-T,T]) \times \mathcal{G}^d([-T,T])$
for $1 \leq d < 3\sigma$ and for t near 0. ∎

Chapter 3

Novel lower Bounds on the Radius of Spatial Analyticity for the KdV type equations

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3.1 Statement of the problem and main results

In this chapter we study the existence and uniqueness of the solution of KdV type equations in Bourgain type spaces, we consider the nonlinear Cauchy problem

$$\begin{cases} \partial_t w + \partial_x^3 w + \eta(t)\mathcal{L}w + \partial_x (w)^k = 0, \ (x,t) \in \mathbb{R}^2, \quad k = 2,4 \\ \\ w(x,0) = w_0(x), \end{cases} \tag{3.1.1}$$

where $\eta(t) \equiv \eta \ sgn(t)$ and η is a positive constant, $w(x,t)$ are real-valued.
The linear operator $\mathcal{L}$ is defined via the Fourier transform by

$$\widehat{\mathcal{L}f}(\zeta) = \phi(\zeta)\widehat{f}(\zeta).$$

With ϕ we denote the phase function as follows

$$\phi(\nu) = \sum_{j=0}^{n} \sum_{i=0}^{2m} c_{i,j} \nu^i |\nu|^j, \ c_{i,j} \in \mathbb{R}, \ c_{2m,n} = -1,$$

where, there is a constant c such that $\phi(\zeta) < c$.

This problem is a continuation of the previous works. The first novelty here is to study the question of well-posedness of(3.1.1) for initial data $w_0(x)$ that is analytic on the line and can be extended as holomorphic functions in a strip around the $x-$axis. A class of suitable analytic functions for our analysis is the analytic Gevrey spaces $G^{\delta,s}(\mathbb{R})$ introduced by Foias and Temam [19], which may be defined as:

$$G^{\delta,s}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}); \|w_0\|_{G^{\delta,s}}(\mathbb{R}) < \infty \right\},$$

where:

$$\|f\|^2_{G^{\delta,s}(\mathbb{R})} = \int_{\mathbb{R}} e^{2\delta|\xi|} \langle \zeta \rangle^{2s} |\widehat{w_0}(\zeta)|^2 d\zeta,$$

for $s \in \mathbb{R}$, $\delta \geq 0$ and $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$.
If $\delta = 0$ the space $G^{\delta,s}$ coincides with the standard Sobolev space H^s.
For all $0 < \delta' < \delta$ and $s, s' \in \mathbb{R}$, we have:

$$G^{\delta,s}(\mathbb{R}) \subset G^{\delta',s'}(\mathbb{R}) \ i.e \ \|f\|_{G^{\delta',s'}(\mathbb{R})} \leq c_{s,s',\delta,\delta'} \|f\|_{G^{\delta,s}(\mathbb{R})}, \tag{3.1.2}$$

is the embedding property of the Gevrey spaces.

Proposition 5. *(Paley-Wiener Theorem, [33]) Let $\delta > 0$, $s \in \mathbb{R}$. Then $f \in G^{\delta,s}$ if and only if it is the restriction to the real line of a function F which is holomorphic in the strip $\{x + iy : \ x, y \in \mathbb{R}, |y| < \delta\}$ and satisfies:*

$$\sup_{|y|<\delta} \|F(x+iy)\|_{H^s_x} < \infty.$$

In the view of the Paley-Wiener theorem, it is natural to take initial data in $G^{\delta,s}$ and obtain a better understanding of the behavior of solution as we try to extend it globally in time. It means that given $w_0 \in G^{\delta,s}$ for some initial radius $\delta > 0$ we want to estimate the behavior of the radius of analyticity $\delta(T)$ over time.
The first result according to the local well-posedness is given in the next theorem.

Theorem 7. *Let $\delta > 0$ and $s > a_k, k = 2, 4$. Then for any $w_0 \in G^{\delta,s}$, there exists $T = T(\| w_0 \|_{G^{\delta,s}}) > 0$ and a unique solution w of (3.1.1) on $(-T, T)$ such that*

$$w \in C\left([-T, T], G^{\delta,s}\right).$$

Moreover the solution depends w_0, where

$$T = \frac{c_0}{(1+ \| w_0 \|^{k-1}_{G^{\delta,s}})^{\beta}},$$

here $a_2 = -3/4$, $a_4 = -1/6$, for some constants $c_0 > 0$ and $\beta > 1$ depending only on s. Furthermore, the solution w satisfies

$$\|w\|_{X^T_{\delta,s,b}} \leq 2C\|w_0\|_{G^{\delta,s+p(b-1/2)}} \quad 1/2 < b < 1,$$

with constant $C > 0$ depending only on s and b.

The second result for problem (3.1.1) is given in the next theorem.

Theorem 8. *Let $s > a_k, k = 2, 4$ and $\delta_0 > 0$. Assume that $w_0 \in G^{\delta_0,s}$, then the solution in Theorem 7 can be extended to be global in time and for any $T' > 0$, we have:*

$$w \in C\left([-T', T'], G^{\delta(T'),s}\right) \ \text{with } \delta(T') = \min\left\{\delta_0, C_1 T'^{-1/\kappa}\right\},$$

where $\sigma_0 > 0$ can be taken arbitrarily small and $C_1 > 0$ is a constant depending on w_0, δ_0 and s . (Here "T'" have nothing to do with the time derivative.)

This chapter is organized as follows. In sections 2, we define the function spaces, linear estimates and bilinear estimates. In section 3 we prove Theorem 7, using the bilinear estimate and the linear estimate, together with contraction mapping principle. In section 4, we prove the existence of a fundamental approximate conservation law. In the final section 5, Theorem 8 will be proven using the approximate conservation law.

3.2 Preliminary estimates and Function spaces.

3.2.1 Function spaces.

Now we introduce the analytic Gevrey-Bourgain spaces associated to the KdV equation.
First, we consider the linear Kdv equation

$$\begin{cases} v_t + v_{xxx} = 0, \quad x,t \in \mathbb{R} \\ \\ v(x,0) = v_0. \end{cases} \tag{3.2.1}$$

The solution of (3.2.1) is given by $v(x,t) = \left[W(t)v_0\right](x)$, where

$$\widehat{W(t)v_0}(\zeta) = e^{it\xi^3} v_0(\zeta).$$

The completion of the Schwartz class $S(\mathbb{R}^2)$ is given by $X_{\delta,s,b}(\mathbb{R}^2) = X_{\delta,s,b}$, with respect to the

$$\begin{aligned} \|v\|_{X_{\delta,s,b}(\mathbb{R}^2)} &= \|Av\|_{X_{s,b}} \equiv \|W(-t)Av\|_{H^{s,b}} = \| \langle \rho \rangle^b \langle \zeta \rangle^s \widehat{W(-t)Av}(\zeta,\rho)\|_{L_{\zeta,\rho}} \\ \\ &= \|e^{\delta|\zeta|} \left\langle \rho - \zeta^3 \right\rangle^b \langle \zeta \rangle^s \widehat{v}(\zeta,\rho)\|_{L_{\zeta,\rho}} \\ &= \left(\int_{\mathbb{R}^2} e^{2\delta|\zeta|} \left\langle \rho - \zeta^3 \right\rangle^{2b} \langle \zeta \rangle^{2s} \mid \widehat{v}(\zeta,\rho) \mid^2 d\zeta d\rho \right)^{\frac{1}{2}}, \end{aligned}$$

where the operator A, defined by

$$\widehat{Av}^x(\zeta,t) = e^{\delta|\zeta|}\widehat{v}^x(\zeta,t). \tag{3.2.2}$$

For $\delta = 0$, the spaces $X_{0,s,b}$ coincides with the Bourgain spaces $X_{s,b}$.

The spaces $X^T_{\delta,s,b}$, denotes the restriction of $X_{\delta,s,b}$ onto finite time interval $[-T,T]$, $T>0$ and equipped with the norm:

$$\|v\|_{X^T_{\delta,s,b}} = \inf\left\{\| V \|_{X_{\delta,s,b}(\mathbb{R}^2)}: V \in X_{\delta,s,b}, v(t)=V(t) for -T\leq t\leq T\right\}.$$

Now we consider the IVP(3.2.3) associated to the linear parts of (3.1.1)

$$\begin{cases} v_t + v_{xxx} + \eta(t)Lv = 0, \quad x,t\in\mathbb{R} \\ \\ v(0)=v_0. \end{cases} \tag{3.2.3}$$

The solution to (3.2.3) is given by $v(x,t)=[S(t)v_0](x)$, where

$$\widehat{S(t)v_0}(\zeta) = e^{it\zeta^3+\eta|t|\phi(\zeta)}v_0(\zeta),$$

the semigroup $S(t)$ can be written as $S(t)=W(t)U(t)$ where $\widehat{U(t)u_0}(\zeta)=e^{\eta|t|\phi(\zeta)}u_0(\zeta)$ and $W(t)$ is the unitary group associated to the KdV equation.

3.2.2 Linear estimates

To prove our main results, we need some multi-linear estimate in the analytic Gevrey-Bourgain spaces. Note that the spaces $X_{\delta,s,b}$ are continuously embedded in $C\left(\mathbb{R}, G^{\delta,s}(\mathbb{R})\right)$,
provided $b>1/2$.
The proofs of the next Lemmas, for $\delta=0$, are developed in [15] and [35], for $\delta>0$ using operator A in (3.2.2).

Lemma 8. *Let $b>\frac{1}{2}$, $s\in\mathbb{R}$ and $\delta\geq 0$. Then $X_{\delta,s,b}\subset C\left(\mathbb{R}, G^{\delta,s}(\mathbb{R})\right)$ and*

$$\sup_{t\in\mathbb{R}}\|w(t)\|_{G^{\delta,s}} \leq C\|w\|_{X_{\delta,s,b}}, \tag{3.2.4}$$

where C depends only on b.

Lemma 9. *Let $s\in\mathbb{R}$, $\delta\geq 0$ and $-\frac{1}{2}<b\leq b'<\frac{1}{2}$. Then for any $T>0$ we have*

$$\|w\|_{X^T_{\delta,s,b}} \leq CT^{b'-b}\|w\|_{X^T_{\delta,s,b'}}, \tag{3.2.5}$$

where C depends only on b and b'.

Lemma 10. *Let $s \in \mathbb{R}$, $\delta \geq 0$, $-\frac{1}{2} < b < \frac{1}{2}$ and $T > 0$. Then, for any time interval $I \subset [-T, T]$, we have*

$$\|\chi_I(t)w\|_{X_{\delta,s,b}} \leq C\|w\|_{X^T_{\delta,s,b}}, \tag{3.2.6}$$

where $\chi_I(t)$ is the characteristic function of I, and C depends only on b.

Next, consider the linear Cauchy problem (3.2.7), for given $G(x,t)$ and $w_0(x)$,

$$\begin{cases} \partial_t w + \partial_x^3 w + \eta(t)\mathcal{L}w = G, \qquad k = 2,4 \\ \\ w(x,0) = w_0(x). \end{cases} \tag{3.2.7}$$

By Duhamel's principle the solution can be then written as

$$w(t) = S(t)w_0 - \int_0^t S(t-\mu)\left(\partial_x w^k(\mu)\right) d\mu. \tag{3.2.8}$$

Lemma 11. *Let $s \in \mathbb{R}$, $b > 1/2$, $-1/2 < b' \leq 0$, $\delta \geq 0$, $p = 2m + n$ and $0 < T \leq 1$, then there is a constant $C > 0$, such that*

$$\|S(t)w_0\|_{X_{\delta,s,b}} \leq C\|w_0\|_{G^{\delta,s+p(b-1/2)}}. \tag{3.2.9}$$

If $1/2 < b \leq b'/3 + 2/3$ then

$$\left\| \int_0^t S(t-\mu)F(\mu)\mathrm{d}\mu \right\|_{X_{\delta,s,b}} \leq C\|F\|_{X_{\delta,s,b'}}. \tag{3.2.10}$$

3.2.3 Multi-linear estimate

Corollary 3. *Let $k = 2, 4$, $\delta \geq 0$ and $s > a_k$. There exist $\gamma \in (1/2, 1)$ and $r(s) > 0$ such that if b and b' are two numbers satisfying $1/2 < b \leq b' + 1 < \gamma$ and $b' + 1/2 \leq r(s)$ then for $w \in X_{\delta,s,b}$, the following estimate holds*

$$\| \partial_x(w^k) \|_{X_{\delta,s,b'}} \leq C \| w \|^k_{X_{\delta,s,b}}, \tag{3.2.11}$$

where $a_2 = -3/4$ and $a_4 = -1/6$.

Proof. For $k=4$, we observe, by considering the operator A in (3.2.2), that

$$\begin{aligned} e^{\delta|\zeta|}\widehat{w_1 w_2 w_3 w_4} &= (2\pi)^{-2} e^{\delta|\zeta|}\widehat{w_1} * \widehat{w_2} * \widehat{w_3} * \widehat{w_4} \\ &\le (2\pi)^{-2}\int_{\mathbb{R}^6} e^{\delta|\zeta-\zeta_1|}\widehat{w_1}(\zeta-\zeta_1,\rho-\rho_1)e^{\delta|\zeta_1-\zeta_2|}\widehat{w_2}(\zeta_1-\zeta_2,\rho_1-\rho_2) \\ &\quad e^{\delta|\zeta_2-\zeta_3|}\widehat{w_3}(\zeta_2-\zeta_3,\rho_2-\rho_3)e^{\delta|\zeta_3|}\widehat{w_4}(\zeta_3,\rho_3)d\zeta_1 d\zeta_2 d\zeta_3 d\rho_1 d\rho_2 \rho_3 \\ &= (\widehat{Aw_1 Aw_2 Aw_3 Aw_4}), \end{aligned}$$

since

$$\delta \mid \zeta \mid \le \delta \mid \zeta-\zeta_1 \mid +\delta \mid \zeta_1-\zeta_2 \mid +\delta \mid \zeta_2-\zeta_3 \mid +\delta \mid \zeta_3 \mid .$$

Thus, we have

$$\| \partial_x(w_1 w_2 w_3 w_4) \|_{X_{\delta,s,b'}} \le \| \partial_x(Aw_1 Aw_2 Aw_3 Aw_4) \|_{X_{s,b'}} .$$

For $k=2$, we have

$$\| \partial_x(w_1 w_2) \|_{X_{\delta,s,b'}} \le \| \partial_x(Aw_1 Aw_2) \|_{X_{s,b'}} .$$

Thanks to the Proposition 2.10 in [15], for $k=4$ we get

$$\begin{aligned} \| \partial_x(Aw_1 Aw_2 Aw_3 Aw_4) \|_{X_{s,b'}} &\le C \| Aw_1 \|_{X_{s,b}} \| Aw_2 \|_{X_{s,b}} \| Aw_3 \|_{X_{s,b}} \| Aw_4 \|_{X_{s,b}} \\ &= C \| w_1 \|_{X_{\delta,s,b}} \| w_2 \|_{X_{\delta,s,b}} \| w_3 \|_{X_{\delta,s,b}} \| w_4 \|_{X_{\delta,s,b}}, \end{aligned}$$

for $k=2$ and by Proposition 2.10 in [15], we get

$$\| \partial_x(Aw_1 Aw_2) \|_{X_{s,b'}} \le C \| Aw_1 \|_{X_{s,b}} \| Aw_2 \|_{X_{s,b}} = C \| w_1 \|_{X_{\delta,s,b}} \| w_2 \|_{X_{\delta,s,b}} .$$

□

3.3 Proof of Theorem 7

Now, we are ready to estimate all terms in (3.2.8) by using multi-linear estimate in the above lemmas. Let $\delta > 0$, $s - p(b - 1/2) > a_k, k = 2, 4$, and $w_0 \in G^{\delta,s}$, with $b = 1/2 + \epsilon$, $b' = -1/2 + 4\epsilon$ and $0 < \epsilon \ll 1$ satisfying

$$0 < \epsilon < \min\left\{\frac{s - a_k}{p}, \frac{1}{4}(\gamma - \frac{1}{2}), \frac{r(s)}{4}\right\},$$

where $p = 2m + n'\gamma$ and $r(s)$ are as in Corollary 3.

3.3.1 Existence of solution

To construct the local solution w of (3.1.1), we proceed by an iteration argument in the space $X^T_{\delta,s,b}$.

Let $\{w^{(n)}\}_{n=0}^{\infty}$ be the sequence defined by

$$\begin{cases} \partial_t w^{(0)} + \partial_x^3 w^{(0)} + \eta(t)\mathcal{L}w^{(0)} = 0, \\ \\ w^{(0)}(0) = w_0, \end{cases}$$

and for $n \in \{1, 2, \cdots\}$, we have

$$\begin{cases} \partial_t w^{(n)} + \partial_x^3 w^{(n)} + \eta(t)\mathcal{L}w^{(n)} = -(w^{(n-1)})^k, \\ \\ w^{(n)}(0) = w_0. \end{cases}$$

Based on Lemma 11, we have

$$w^{(0)}(x,t) = S(t)w_0(x),$$

and

$$w^{(n)}(x,t) = S(t)w_0(x) - \int_0^t S(t-\mu)\partial_x \left(w^{(n-1)}(x,\mu)\right)^k d\mu.$$

Then, from Lemma 9 and Lemma 11, we have

$$\begin{aligned} \|w^{(0)}\|_{X^T_{\delta,s-p(b-1/2),b}} &\leq C\|w_0\|_{G^{\delta,s}} \\ \|w^{(n)}\|_{X^T_{\delta,s-p(b-1/2),b}} &\leq C\|w_0\|_{G^{\delta,s}} + CT^{b'-b}\|\partial_x \left(w^{(n-1)}\right)^k \|_{X^T_{\delta,s-p(b-1/2),b'}} \\ &\leq C\|w_0\|_{G^{\delta,s}} + CT^{b'-b}\|w^{(n-1)})\|^k_{X^T_{\delta,s-p(b-1/2),b}}, \end{aligned} \tag{3.3.1}$$

with $1/2 < b \leq b' + 1 < \gamma$ and $b' + 1/2 \leq r(s)$. By induction, it follows that

$$\|w^{(n)}\|_{X^T_{\delta,s-p(b-1/2),b}} \leq 2C\|w_0\|_{G^{\delta,s}}, \tag{3.3.2}$$

for all n, if $T \in (0, 1]$ is chosen so small that

$$T \leq \frac{1}{(2^{2k+1}C^k\|w_0\|^{k-1}_{G^{\delta,s}})^{\frac{1}{b'-b}}}.$$

Using Corollary 3 together with (3.3.1) and (3.3.2), we get

$$\begin{aligned} \|w^{(n)} - w^{(n-1)}\|_{X^T_{\delta,s-p(b-1/2),b}} &\leq CT^{b'-b}\left\|\partial_x \left[\left(w^{(n-1)}\right)^k - \left(w^{(n-2)}\right)^k\right] \right\|_{X^T_{\delta,s-p(b-1/2),b'}} \\ &\leq CT^{b'-b}\|f\|_{X^T_{\delta,s-p(b-1/2),b}}\|w^{(n-1)} - w^{(n-2)})\|_{X^T_{\delta,s,b}} \\ &\leq \frac{1}{2}\|w^{(n-1)} - w^{(n-2)})\|_{X^T_{\delta,s-p(b-1/2),b}}, \end{aligned}$$

where

$$f = \left(w^{(n-1)}\right)^3 + \left(w^{(n-2)}\right)^3 + \left(w^{(n-1)}\right)\left(w^{(n-2)}\right)^2 + \left(w^{(n-2)}\right)\left(w^{(n-1)}\right)^2,$$

for $k = 4$ and

$$f = \left(w^{(n-1)}\right) + \left(w^{(n-2)}\right),$$

for $k = 2$. It follows that the sequence converges to a solution w verifying (3.3.2).

3.3.2 Continuous dependence on the initial data

Suppose that w and v are two solutions to (3.1.1) with w_0, v_0, respectively. Then, with T and for any T' such that $0 < T' < T$, we get

$$\| w - v \|_{X^{T'}_{\delta,s-p(b-1/2),b}} \leq C \| w_0 - v_0 \|_{G^{\delta,s}} + \frac{1}{2} \| w - v \|_{X^{T'}_{\delta,s-p(b-1/2),b}},$$

provided $\| w_0 - v_0 \|_{G^{\delta,s}}$ is sufficiently small. This ends the proof of continuous dependence.

3.3.3 The uniqueness

Uniqueness of solution in $C([-T,T],G^{\delta,s})$ can be proved by the following standard argument. Suppose that $w,v \in C([-T,T],G^{\delta,s})$ are two solutions of (3.1.1) with $w(\cdot,0) = v(\cdot,0)$ in $G^{\delta,s}$. Setting $u = w - v$, we see that u solves the Cauchy problem

$$\partial_t u + \partial_x^3 u + \eta(t)\mathcal{L}u + \partial_x(w^k - v^k) = 0, \quad u(0) = 0.$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}\|u(t,\cdot)\|_{L^2}^2 = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u^2(t,x)dx = \int_{\mathbb{R}} u(t,x)\partial_t u(t,x)dx = -\int_{\mathbb{R}} u(t,x)\partial_x(w^k - v^k)dx,$$

since we have

$$\int_{\mathbb{R}} u(t,x)\partial_x^3 u(t,x)dx = \eta(t)\int_{\mathbb{R}} u(t,x)\mathcal{L}u(t,x)dx = 0. \tag{3.3.3}$$

Thanks to equation (3.3.3), we have

$$\frac{d}{dt}\|u(t,\cdot)\|_{L^2}^2 = -2\int_{\mathbb{R}} u(t,x)\partial_x\left[f(t,x)u(t,x)\right]dx,$$

where $f = w^3 + v^3 + wv^2 + vw^2$ for $k = 4$ and $f = w + v$ for $k = 2$.
Integrating by parts the last integral we obtain

$$\frac{d}{dt}\|u(t,\cdot)\|_{L^2}^2 = -\int_{\mathbb{R}} \partial_x f(t,x)u^2(t,x)dx,$$

from which we deduce the inequality

$$\left|\frac{d}{dt}\|u(t,\cdot)\|_{L^2}^2\right| \leq \|\partial_x f\|_{L^\infty}\|u(t)\|_{L^2}^2. \tag{3.3.4}$$

Since $w,v \in C([-T,T],G^{\delta,s})$ we have that w and v are continuous in t on the compact set $[-T,T]$ and are $G^{\delta,s}$ in x. Thus, we can conclude that

$$\|\partial_x f\|_{L^\infty} \leq c < \infty. \tag{3.3.5}$$

Therefore, from (3.3.4) and (3.3.5) we obtain the differential inequality

$$\left|\frac{d}{dt}\|u(t,\cdot)\|_{L^2}^2\right| \leq c\|u(t)\|_{L^2}^2, \quad |t| \leq T.$$

Solving it gives

$$\|u(t)\|_{L^2}^2 \le e^c \|u(0)\|_{L^2}^2, \quad |t| \le T. \tag{3.3.6}$$

Since $\|u(0)\|_{L^2}^2 = 0$, from (3.3.6) we obtain that $u(t) = 0$, $-T \le t \le T$ or $w = v$.

3.4 Approximate conservation law

Our goal in this section is to show an approximate conservation law for a solution of (3.1.1) based on the conservation the $L^2(\mathbb{R})$ norm of solution of the equation in Theorem 9.

Theorem 9. *Let $\kappa \in [0, -a_k)$ and T be as in Theorem 7, there exist $b \in (1/2, 1)$ and $C > 0$, such that for any $\delta > 0$ and any solution $w \in X^T_{\delta,0,b}$ to the Cauchy problem (3.1.1) on the time interval $[0, T]$, we have the estimate*

$$\sup_{t\in[0,T]} \|w(t)\|^2_{G^{\delta,0}} \le \|w(0)\|^2_{G^{\delta,0}} + C\delta^\kappa \|w\|^{k+1}_{X^T_{\delta,0,b}}. \tag{3.4.1}$$

Moreover, we have

$$\sup_{t\in[0,T]} \|w(t)\|^2_{G^{\delta,0}} \le \|w(0)\|^2_{G^{\delta,0}} + C\delta^\kappa \|w(0)\|^{k+1}_{G^{\delta,p(b-1/2)}}. \tag{3.4.2}$$

We need the following estimate.

Lemma 12. *Given $\kappa \in [0, -a_k)$, there exist $b \in (1/2, 1)$ and $C > 0$, such that for all $T > 0$ and $u \in X_{\delta,0,b}$, we have*

$$\|G\|_{X_{0,b-1}} \le C\delta^\kappa \|w\|^k_{X_{\delta,0,b}},$$

where $G = \partial_x \left[(Aw)^k - A(w)^k\right]$ and the operator A given by (3.2.2).

Proof. We distinguish two cases:

1. For $k = 2$. Let $G = \partial_x \left[(Aw)^2 - A(w)^2\right]$. Then,

$$\|G\|_{X_{0,b-1}} \quad = \quad \|\partial_x \left[(Aw)^2 - A(w)^2\right]\|_{X_{0,b-1}}$$

$$= \| \frac{\zeta}{\langle \rho-\zeta^3 \rangle^{1-b}} \int_{\mathbb{R}^2} \left(e^{\delta|\zeta_1|} e^{\delta|\zeta-\zeta_1|} - e^{\delta|\zeta|} \right) \widehat{w}(\zeta_1, \rho_1) \widehat{w}(\zeta-\zeta_1, \rho-\rho_1) d\zeta_1 d\rho_1 \|_{L^2_{\zeta,\rho}}. \quad (3.4.3)$$

Using (3.4.3) and the following estimate (see [45])

$$e^{\delta|\alpha|} e^{\delta|\beta|} - e^{\delta|\alpha+\beta|} \leq [2\delta \min(|\alpha|, |\beta|)]^\theta e^{\delta|\alpha|} e^{\delta|\beta|}, \quad \theta \in [0,1],$$

and

$$\min(|\zeta_1|, |\zeta-\zeta_1|) \leq 2 \frac{\langle \zeta_1 \rangle \langle \zeta - \zeta_1 \rangle}{\langle \zeta \rangle}.$$

For $\kappa \in [0, 3/4) \subset [0,1]$, one can see that

$$\|G\|_{X_{0,b-1}} \leq$$

$$\leq \| \frac{\zeta}{\langle \rho - \zeta^3 \rangle^{1-b}} \int_{\mathbb{R}^2} [2\delta \min(|\zeta_1|, |\zeta-\zeta_1|)]^\theta e^{\delta|\zeta_1|} e^{\delta|\zeta-\zeta_1|} \widehat{w}(\zeta_1, \rho_1) \widehat{w}(\zeta-\zeta_1, \rho-\rho_1) d\zeta_1 d\rho_1 \|_{L^2_{\zeta,\rho}}$$

$$\leq (2\delta)^\kappa \| \frac{\zeta \langle \zeta \rangle^{-\kappa}}{\langle \rho - v^3 \rangle^{1-b}} \int_{\mathbb{R}^2} e^{\delta|\zeta_1|} \langle \zeta_1 \rangle^\kappa \widehat{w}(\zeta_1, \rho_1) e^{\delta|\zeta-\zeta_1|} \langle \zeta - \zeta_1 \rangle^\kappa \widehat{w}(\zeta-\zeta_1, \rho-\rho_1) d\zeta_1 d\rho_1 \|_{L^2_{\zeta,\rho}}.$$

Now by taking $s = -\kappa \in (-3/4, 0]$ we obtain

$$\|G\|_{X_{0,b-1}} \leq C\delta^\kappa \|w\|^2_{X_{\delta,0,b}}.$$

2. For $k = 4$. Let $G = \partial_x [(Aw)^4 - A(w)^4]$, using Lemma 8 in [45], for $\kappa \in [0, 1/6) \subset [0,1]$, we obtain

$$\|G\|_{X_{0,b-1}} \leq C\delta^\kappa \|w\|^4_{X_{\delta,0,b}}.$$

□

Proof. (Of Theorem 9) Let $V(t,x) = Aw(t,x)$ which is real-valued since the multiplier A is even and u is real-valued. Applying A to (3.1.1) we obtain

$$\partial_t V + \partial_x^3 V + \eta(t) \mathcal{L} V + k V^{k-1} \partial_x V = G,$$

where

$$G = \partial_x \left[(Aw)^k - A(w)^k \right], \quad k = 2, 4,$$

then

$$\int_{\mathbb{R}} V\partial_t V dx + \int_{\mathbb{R}} V\partial_x^3 V dx + \eta(t)\int_{\mathbb{R}} V\mathcal{L}V dx + k\int_{\mathbb{R}} V^k\partial_x V dx = \int_{\mathbb{R}} VG dx.$$

Noting that $\partial_x^j V(x,t) \longrightarrow 0$ as $|x| \longrightarrow \infty$ (see [46]), we can use integration by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} V^2 dx = \int_{\mathbb{R}} VG dx. \tag{3.4.4}$$

Integrating (3.4.4) with respect to $t \in [0,T]$, we obtain

$$\int_{\mathbb{R}} V^2(T,x)dx = \int_{\mathbb{R}} V^2(0,x)dx + 2\int_{\mathbb{R}^2} \chi_{[0,T]}(t)VG dx dt.$$

Thus,

$$\|w(T)\|^2_{G^{\delta,0}} = \|w(0)\|^2_{G^{\delta,0}} + 2\int_{\mathbb{R}^2} \chi_{[0,T]}(t)VG dx dt.$$

By using Holder's inequality, Lemma 10, Lemma 12 and the fact that $1-b<b$, since $b>1/2$, we obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}^2} \chi_{[0,T]}(t)VG dx dt\right| &\leq \|\chi_{[0,T]}(t)V\|_{X_{0,1-b}}\|\chi_{[0,T]}(t)G\|_{X_{0,b-1}} \\
&\leq \|V\|_{X^T_{0,1-b}}\|G\|_{X^T_{0,b-1}} \\
&\leq CT^\kappa\|w\|^{k+1}_{X^T_{\delta,0,b}}.
\end{aligned}$$

□

3.5 Proof of Theorem 8

Let $\delta_0 > 0$, $s > a_k$, $\kappa \in (0, -a_k)$ be fixed and $w_0 \in G^{\delta_0,s}$. Since the invariance property of the KdV-type equation under the reflection $(t,x) \to (-t,-x)$, we can restrict to $t > 0$. Then, we have to prove that the solution w of (3.1.1) satisfies:

$$w \in C\left([0,T'], G^{\delta(T'),s}\right),$$

where

$$\delta(T') = \min\left\{\delta_0, C_1 T'^{-1/\kappa}\right\}, \quad for\ all\ T' > 0,$$

and $C_1 > 0$ is a constant depending on w_0, δ_0, s and κ. By Theorem 7, there is a maximal time $T^* = T^*(w_0, \delta_0, s) \in (0, \infty]$ such that:

$$w \in C\left([0,T^*], G^{\delta_0,s}\right).$$

If $T^* = \infty$, it is done.

If $T^* < \infty$, as we assume henceforth, it remains to prove

$$w \in C\left([0,T'], G^{C_1 T'^{-1/\kappa},s}\right), \quad for\ all\ T' \geq T^*. \tag{3.5.1}$$

3.5.1 The case s=0

Fix $T' \geq T^*$. We will show that, for $\delta > 0$ sufficiently small

$$\sup_{t\in[0,T']} \|w(0)\|^2_{G^{\delta,0}} \leq 2\|w(0)\|^2_{G^{\delta_0,0}}.$$

In this case, by Theorems 7 and Theorem 9 with

$$T = \frac{c_0}{(1 + 2 \parallel w(0) \parallel^{k-1}_{G^{\delta_0,0}})^{\beta}},$$

the smallness conditions on δ will be

$$\delta < \delta_0 \quad and \quad \frac{2T'}{T} C\delta^{\kappa} 2^{\frac{k+1}{2}} \parallel w(0) \parallel^{k-1}_{G^{\delta_0,0}} \leq 1, C > 0. \tag{3.5.2}$$

Here C is the constant in Theorems 9. By induction, we check that

$$\sup_{t\in[0,nT]} \|w(t)\|^2_{G^{\delta,0}} \leq \|w(0)\|^2_{G^{\delta,0}} + nC\delta^{\kappa} 2^{\frac{k+1}{2}} \|w(0)\|^{k+1}_{G^{\delta_0,0}}, \tag{3.5.3}$$

and

$$\sup_{t\in[0,nT]} \|w(t)\|^2_{G^{\delta,0}} \leq 2\|w(0)\|^2_{G^{\delta_0,0}}, \tag{3.5.4}$$

for $n \in \{1,\ldots,m+1\}$, where $m \in \mathbb{N}$ is chosen, so that $T' \in [mT,(m+1)T)$, this m exists. By Theorem 7 and the definition of T^*, we have

$$T < \frac{c_0}{(1+ \parallel w(0) \parallel^{k-1}_{G^{\delta_0,0}})^\beta} < T^*, \quad \textit{hence} \quad T < T'.$$

In the first step, we cover the interval $[0,T]$ and by Theorem 9, we have

$$\begin{aligned}
\sup_{t\in[0,T]} \|w(t)\|^2_{G^{\delta,0}} &\leq \|w(0)\|^2_{G^{\delta,0}} + C\delta^\kappa \|w(0)\|^{k+1}_{G^{\delta,p(b-1/2)}} \\
&\leq \|w(0)\|^2_{G^{\delta,0}} + C\delta^\kappa \|w(0)\|^{k+1}_{G^{\delta_1,0}} \\
&\leq \|w(0)\|^2_{G^{\delta,0}} + C\delta^\kappa \|w(0)\|^{k+1}_{G^{\delta_0,0}},
\end{aligned}$$

since $\delta < \delta_1 \leq \delta_0$, we used

$$\|w(0)\|_{G^{\delta,0}} \leq \|w(0)\|_{G^{\delta_0,0}},$$

and

$$\|w(0)\|_{G^{\delta,p(b(1/2-\epsilon)}} \leq C\|w(0)\|_{G^{\delta_1,0}} \leq C\|w(0)\|_{G^{\delta_0,0}}.$$

This verifies (3.5.3) for $n = 1$ and now,(3.5.4) follows using again $\|w(0)\|_{G^{\delta,0}} \leq \|w(0)\|_{G^{\delta_0,0}}$ as well as $C\delta^\kappa\|w(0)\|^{k-1}_{G^{\delta_0,0}} \leq 1$.

Suppose now that (3.5.3) and (3.5.4) hold for some $n \in \{1,\ldots,m\}$ and we prove that it holds for $n+1$.

We estimate

$$\begin{aligned}
\sup_{t\in[nT,(n+1)T]} \|w(t)\|^2_{G^{\delta,0}} &\leq \|w(nT)\|^2_{G^{\delta,0}} + C\delta^\kappa \|w(nT)\|^{k+1}_{G^{\delta,p(b-1/2)}} \\
&\leq \|w(nT)\|^2_{G^{\delta,0}} + C\delta^\kappa 2^{\frac{k+1}{2}} \|w(0)\|^{k+1}_{G^{\delta_1,0}} \\
&\leq \|w(nT)\|^2_{G^{\delta,0}} + C\delta^\kappa 2^{\frac{k+1}{2}} \|w(0)\|^{k+1}_{G^{\delta_0,0}} \\
&\leq \|w(0)\|^2_{G^{\delta,0}} + nC\delta^\kappa 2^{\frac{k+1}{2}} \|w(0)\|^{k+1}_{G^{\delta_0,0}} + C\delta^\kappa 2^{\frac{k+1}{2}} \|w(0)\|^{k+1}_{G^{\delta_0,0}},
\end{aligned}$$

verifying (3.5.3) with n replaced by $n+1$. To get (3.5.4) with n replaced by $n+1$, it is then enough to have

$$(n+1)C\delta^\kappa 2^{\frac{k+1}{2}} \|w(0)\|^{k-1}_{G^{\delta_0,0}} \leq 1,$$

but this holds by (3.5.2), since $n+1 \leq m+1 \leq \frac{T'}{T} + 1 < \frac{2T'}{T}$.

Finally, the condition (3.5.2) is satisfied for $\delta \in (0, \delta_0)$ such that

$$\frac{2T'}{T} C\delta^\kappa 2^{\frac{k+1}{2}} \parallel w(0) \parallel^{k-1}_{G^{\delta_0,0}} = 1.$$

Thus, $\delta = C_1 T'^{-\frac{1}{\kappa}}$, where

$$C_1 = \left(\frac{c_0}{C 2^{\frac{k+3}{2}} \parallel w(0) \parallel^{k-1}_{G^{\delta_0,0}} (1 + 2 \parallel w(0) \parallel^{k-1}_{G^{\delta_0,0}})^\beta} \right)^{\frac{1}{\kappa}}.$$

3.5.2 The General Case

For all s, by(3.1.2), we have $w_0 \in G^{\delta_0,s} \subset G^{\delta_0/2,0}$.

For case $s = 0$, it is proved, there is a $T_1 > 0$, such hat

$$w \in C\left([0, T_1), G^{\delta_0/2,0}\right),$$

and

$$w \in C\left([0, T'], G^{2\sigma T'^{-1/\kappa},0}\right), \quad for\ T' \geq T_1,$$

where $\sigma > 0$ depends on w_0, δ_0 and κ. Applying again the embedding (3.1.2), we now conclude that

$$w \in C\left([0, T_1), G^{\delta_0/4,s}\right),$$

and

$$w \in C\left([0, T'], G^{\sigma T'^{-1/\kappa},s}\right), \quad for\ T' \geq T_1,$$

which imply (3.5.1). The proof of Theorem 8 is now completed.

Chapter 4

On the radius of spatial analyticity for coupled system of mKdV-type equations on the line

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4.1 Statement of the problem and main results

In this chapter we extend the previous results and propose a coupled system of mKdV-type equations on the line.

To begin with, we consider the initial value problem,

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(uv^2) = 0, \\ \partial_t v + \beta\partial_x^3 v + \partial_x(u^2 v) = 0, \quad (x,t) \in \mathbb{R}^2, \quad 0 < \beta < 1 \\ u(x,0) = u_0(x), \\ v(x,0) = v_0(x), \end{cases} \tag{4.1.1}$$

For the mKdV equation, many problems have been studied and many results have been found. It is proved that the mKdV equation is locally [36] and globally [18] well-posed in $H^s(\mathbb{T})$ for $s \geq 1/2$. Global well-posedness in $L^2(\mathbb{T})$ was shown in [32].

For $0 < \beta < 1$, the author in [12] proved that the IVP (4.1.1) is locally well posed for given data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $s > -\frac{1}{2}$. Tadahiro in [40] used the Fourier transform restriction norm method and proved that the IVP (4.1.2)

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(v^2) = 0, \quad u(x,0) = u_0(x), \\ \partial_t v + \beta\partial_x^3 v + \partial_x(uv) = 0, \quad v(x,0) = v_0(x), \quad 0 < \beta < 1, \end{cases} \tag{4.1.2}$$

is locally well posed for data with regularity $s \geq 0$.

For $\beta = 1$, the system (4.1.2) reduces to a special case of a broad class of nonlinear evolution equations considered by Ablowiz et al.[1] in the inverse scattering context. In this case, the well-posedness issues along with existence and stability of solitary waves for this system are widely studied in the literature, using the technique developed by Kenig in [35],[36], Well-posedness for the non-periodic gKdV equation in spaces of analytic functions has been proved by Grujic and Kalisch [22]. Using the analytic spaces $G^{\theta,s}$ introduced by Foias and Temam [19] and which are defined by the norm

$$\|f\|^2_{G^{\theta,s}} = \int e^{2\theta(1+|\zeta|)}(1+|\zeta|)^{2s}|\widehat{f}(\zeta)|^2 d\zeta < \infty$$

A class of suitable analytic functions for our analysis is the analytic Gevrey class $G^{\delta,s}(\mathbb{R})$ introduced by Foias and Temam [19], which may be defined as

$$G^{\delta,s}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}); \|f\|_{G^{\delta,s}}(\mathbb{R}) < \infty \right\},$$

where

$$\|f\|^2_{G^{\delta,s}(\mathbb{R})} = \int_{\mathbb{R}} e^{2\delta|\zeta|} \langle\zeta\rangle^{2s} |\widehat{f}(\zeta)|^2 d\zeta,$$

Proposition 6. *(Paley-Wiener Theorem) [33] Let $\delta > 0$, $s \in \mathbb{R}$. Then $f \in G^{\delta,s}$ if and only if it is the restriction to the real line of a function F which is holomorphic in the strip $\{x + iy : \ x, y \in \mathbb{R}, |y| < \delta\}$ and satisfies*

$$\sup_{|y|<\delta} \|F(x+iy)\|_{H^s_x} < \infty.$$

Remark 3. *In the view of the Paley-Wiener Theorem, it is natural to take initial data in $G^{\delta,s}$ and obtain a better understanding of the behavior of solution as we try to extend it globally in time. It means that given $w_0 \in G^{\delta,s}$ for some initial radius $\delta > 0$ we want to estimate the behavior of the radius of analyticity $\delta(T)$ over time.*

The first main result about the well-posedness of the initial value problem associated with a system consisting mKDV-type equations (4.1.1) in analytic spaces reads as follows.

Theorem 10. *Let $\delta > 0$ and $s > -\frac{1}{2}$. Then for any $(u_0, v_0) \in G^{\delta,s} \times G^{\delta,s}$, there exists $T = T(\|(u_0, v_0)\|_{G^{\delta,s}\times G^{\delta,s}})$ and a unique solution (u, v) of (4.1.1) on $[0, T]$ such that*

$$(u, v) \in C\left([0,T], G^{\delta,s}\right) \times C\left([0,T], G^{\delta,s}\right)$$

Moreover the solution depends (u_0, v_0), where

$$T = \frac{1}{(16C^3 + 16C^3 \parallel (u_0, v_0) \parallel^2_{G^{\delta,s}\times G^{\delta,s}})^{1/\epsilon}},$$

Furthermore, the solution w satisfies

$$\|(u,v)\|_{X_{\delta,s,b}\times X^{\beta}_{\delta,s,b}} \leq 2C\|(u_0,v_0)\|_{G^{\delta,s}\times G^{\delta,s}}, \quad b = \frac{1}{2} + \epsilon,$$

with constant $C > 0$ depending only on s and b.

An effective method for studying lower bounds on the radius of analyticity, including this type of problem, is was introduced in [44] for 1D Dirac-Klein-Gordon equations. It was applied in [33] to the modified Kawahara equation and in [22] to the non-periodic KdV equation. (For more details, please see [22, 46]).
The second result for problem (4.1.1) is given in the next Theorem.

Theorem 11. *Let $s > -\frac{1}{2}, 0 < \beta < 1$ and $\delta_0 > 0$. Assume that $(u_0, v_0) \in G^{\delta,s} \times G^{\delta,s}$, then the solution in Theorem 10 can be extended to be global in time and for any $T' > 0$, we have:*

$$(u, v) \in C\left([0, T'], G^{\delta(T'),s}\right) \times C\left([0, T'], G^{\delta(T'),s}\right),$$

with

$$\delta(T') = \min\left\{\delta_0, C_1 T'^{-1/\kappa}\right\},$$

where $\sigma_0 > 0$ can be taken arbitrarily small and $C_1 > 0$ is a constant depending on w_0, δ_0, and s

.

This chapter is organized as follows. In Sections 2, we define the function spaces, linear estimates and bilinear estimates. In Section 3 we prove Theorem 10, using the trilinear estimate and the linear estimate, together with contraction mapping principle. In Section 4, we prove the existence of a fundamental approximate conservation law. In the last section, Theorem 11 will be proved using the approximate conservation law.

4.2 Preliminary estimates and Function spaces.

4.2.1 Function spaces.

We define the needed spaces beginning by the spaces of analytic Gevrey functions that contain our initial data. For $s \in \mathbb{R}$ and $\delta > 0$ let

$$G^{\delta,s}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}); \|f\|^2_{G^{\delta,s}(\mathbb{R})} = \int e^{2\delta|\zeta|} \langle\zeta\rangle^{2s} |\widehat{f}(\zeta)|^2 d\zeta < \infty \right\}, \tag{4.2.1}$$

where $\langle\cdot\rangle = (1 + |\cdot|)$. For all $0 < \delta' < \delta$ and $s, s' \in \mathbb{R}$, we have

$$G^{\delta,s}(\mathbb{R}) \subset G^{\delta',s'}(\mathbb{R}) \ i.e \ \|f\|_{G^{\delta',s'}(\mathbb{R})} \le c_{s,s',\delta,\delta'} \|f\|_{G^{\delta,s}(\mathbb{R})}, \tag{4.2.2}$$

is the embedding property of the Gevrey spaces.

We, then define the analytic Bourgain spaces related to the modified Korteweg-de Vries type equations. The completion of the Schwartz class $S(\mathbb{R}^2)$ is given by $X^{\beta}_{\delta,s,b}(\mathbb{R}^2)$, for $s, b \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, subjected to the norm

$$\|w\|_{X^{\beta}_{\delta,s,b}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} e^{2\delta|\zeta|} \langle\zeta\rangle^{2s} \left\langle \eta - \beta\zeta^3 \right\rangle^{2b} | \widehat{w}(\zeta, \eta) |^2 \, d\zeta d\eta \right)^{\frac{1}{2}}, \tag{4.2.3}$$

Sometimes we use the definition $X^1_{\delta,s,b} = X_{\delta,s,b}$, where

$$\|w\|_{X_{\delta,s,b}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} e^{2\delta|\zeta|} \langle\zeta\rangle^{2s} \left\langle \eta - \zeta^3 \right\rangle^{2b} \mid \widehat{w}(\zeta,\eta) \mid^2 d\zeta d\eta\right)^{\frac{1}{2}}, \tag{4.2.4}$$

For any interval I, we define the localized spaces $X^{\beta,I}_{\sigma,\delta,s,b} = X^{\beta}_{\sigma,\delta,s,b}(\mathbb{R}\times I)$ with norm

$$\|w\|_{X^{\beta}_{\delta,s,b}(\mathbb{R}\times I)} = inf\left\{\|W\|_{X^{\beta}_{\delta,s,b}}; W|_{\mathbb{R}\times I} = w\right\} \tag{4.2.5}$$

4.2.2 Linear Estimates

To present the proof of the theorems, we start with trilinear estimate (4.2.3) and (4.2.4) defined in the analytic Bourgain spaces. It is well known that the spaces $X^{\beta}_{\delta,s,b}$ is continuously embedded in $C\left([0,T], G^{\delta,s}\right)$, provided $b > 1/2$. Let us begin with the embedded result in the next Lemma.

Lemma 13. *Let $b > \frac{1}{2}$, $s \in \mathbb{R}$ and $\delta > 0$, Then, for all $T > 0$ we have*

$$X_{\delta,s,b} \hookrightarrow C\left([0,T], G^{\delta,s}\right)$$

.

Proof. First, we observe that the operator A defined by

$$\widehat{Aw}^{x}(\zeta,t) = e^{\delta|\zeta|}\widehat{w}^{x}(\zeta,t), \tag{4.2.6}$$

satisfies

$$\|w\|_{X_{\delta,s,b}} = \|Aw\|_{X_{s,b}} \quad and \quad \|w\|_{G^{\delta,s}} = \|Aw\|_{H^s},$$

where $X_{s,b}$ is introduced in [33]. We observe that Aw belongs to $C(\mathbb{R}, H^s)$ and for some $C > 0$, we have

$$\| Aw \|_{C(\mathbb{R},H^s)} \leq C \| Aw \|_{X_{s,b}} .$$

Thus, it follows that $w \in \mathrm{C}\big([0,T], G^{\sigma,\delta,s}\big)$ and

$$\| w \|_{C\left([0,T],G^{\delta,s}\right)} \leq C \| w \|_{X^{\beta}_{\delta,s,b}} .$$

□

Taking the Fourier transform with respect to x of the Cauchy problems (4.1.1), after an ordinary calculation. We localize it t by using a cut-off function, satisfying $\psi \in C_0^\infty$, with $\psi = 1$ in $[-1,1]$, supp$\psi \subset [-2,2]$ and $\psi_T(t) = \psi(\frac{t}{T})$, we consider the operator Λ, Γ given by

$$\begin{cases} \Lambda[u,v](t) = \psi(t)S(t)u_0 - \psi_T(t)\displaystyle\int_0^t S(t-\nu)\partial_x F_1(\nu)d\nu, \\ \\ \Gamma[u,v](t) = \psi(t)S_\beta(t)v_0 - \psi_T(t)\displaystyle\int_0^t S_\beta(t-\nu)\partial_x F_2(\nu)d\nu, \end{cases} \tag{4.2.7}$$

where $S(t) = e^{-t\partial_x^3}$ and $S_\beta(t) = e^{-t\beta\partial_x^3}$ are the unitary groups associated with the linear problems. The non linear terms define by $F_1 = (uv^2)$ and $F_2 = (u^2v)$.

Lemma 14. *Let $s, b \in \mathbb{R}$ and $\delta > 0$. For some constant $C > 0$, we have*

$$\|\psi(t)S(t)u_0\|_{X_{\delta,s,b}} \leq C\ \|u_0\|_{G^{\delta,s}}, \tag{4.2.8}$$

$$\|\psi(t)S_\beta(t)v_0\|_{X^\beta_{\delta,s,b}} \leq C\ \|v_0\|_{G^{\delta,s}}, \tag{4.2.9}$$

for all $u_0, v_0 \in G^{\delta,s}$

Proof. By definition, we have

$$\begin{aligned} \psi(t)S_\beta(t)u_0 &= C\psi(t)\int_{\mathbb{R}} e^{i(x\zeta + t\beta\zeta^3)}\widehat{u_0}(\zeta)d\zeta \\ &= C\int_{\mathbb{R}^2} e^{i(x\zeta+t\eta)}\widehat{\psi}(\eta - \beta\zeta^3))\widehat{u_0}(\zeta)d\zeta d\eta, \end{aligned}$$

it follows that

$$\begin{aligned} &\| \psi(t)S(t)u_0 \|^2_{X^\beta_{\delta,s,b}} \\ &= C\int_{\mathbb{R}^2} e^{2\delta|\zeta|}\,(1+\mid \zeta \mid)^{2s}\left(1+\mid \eta - \beta\zeta^3 \mid\right)^{2b} \mid \widehat{\psi}(\eta-\beta\zeta^3) \mid^2\mid \widehat{u_0}(\zeta) \mid^2 d\zeta d\eta \\ &= C\int_{\mathbb{R}} e^{2\delta|\zeta|}\,(1+\mid \zeta \mid)^{2s} \mid \widehat{u_0}(\zeta) \mid^2 \left(\int_{\mathbb{R}} \mid \widehat{\psi}(\eta-\beta\zeta^3) \mid^2 \left(1+\mid \eta-\beta\zeta^3 \mid\right)^{2b} d\eta\right) d\zeta. \end{aligned}$$

We use the fact that $b > 1/2$ to get

$$\int_{\mathbb{R}} \mid \widehat{\psi}(\eta-\beta\zeta^3) \mid^2 \left(1+ \mid \eta-\beta\zeta^3 \mid\right)^{2b} d\eta$$

$$\leq C\int_{\mathbb{R}} \mid \widehat{\psi}(\eta-\beta\zeta^3) \mid^2 d\eta + C\int_{\mathbb{R}} \mid \widehat{\psi}(\eta-\beta\zeta^3) \mid^2 \left(1+ \mid \eta-\beta\zeta^3 \mid\right)^{2b} d\eta \leq C.$$

□

Lemma 15. *Let $s \in \mathbb{R}$, $-\frac{1}{2} < b' \leq 0 \leq b < b'+1$, $0 \leq T \leq 1$ and $\delta > 0$, then for some constant $C > 0$, we have*

$$\left\|\psi_T(t)\int_0^t S(t-\nu)\partial_x F_1(x,\nu)\mathrm{d}\nu\right\|_{X_{\delta,s,b}} \leq CT^{1-b+b'}\|\partial_x F_1\|_{X_{\delta,s,b'}}. \tag{4.2.10}$$

and

$$\left\|\psi_T(t)\int_0^t S_\beta(t-\nu)\partial_x F_2(x,\nu)\mathrm{d}\nu\right\|_{X^\beta_{\delta,s,b}} \leq CT^{1-b+b'}\|\partial_x F_2\|_{X^\beta_{\delta,s,b'}}. \tag{4.2.11}$$

Proof. Define $W = \psi_T(t)\int_0^t S_\beta(t-\nu)\partial_x F_1(x,\nu)\mathrm{d}\nu$. Let us consider the operator A given by (4.2.6), then we have

$$\begin{aligned}\widehat{AW}^x(\zeta,t) &= \psi_T(t)\int_0^t \left(e^{-i(t-\nu)\beta\zeta^3}\right) e^{\delta|\zeta|}\widehat{\partial_x F_1}^x(\zeta,\nu)\mathrm{d}\nu,\\ &= \psi_T(t)\int_0^t [S_\beta(t-\widehat{\nu)(\partial}_x AF_1)]^x(\zeta,\nu)\mathrm{d}\nu.\end{aligned}$$

Thus,

$$\| W \|_{X^\beta_{\delta,s,b}} = \| AW \|_{X^\beta_{s,b}} = \left\|\psi_T(t)\int_0^t S_\beta(t-\nu)\partial_x AF_1(x,\nu)d\nu\right\|_{X^\beta_{s,b}}.$$

Using Lemma 2.1 in [33], we have

$$\left\|\psi_T(t)\int_0^t S_\beta(t-\nu)\partial_x AF_1(x,\nu)d\nu\right\|_{X^\beta_{s,b}} \leq CT^{1-b+b'}\|\partial_x AF_1\|_{X^\beta_{s,b'}} = CT^{1-b+b'}\|\partial_x F_1\|_{X^\beta_{\delta,s,b'}},$$

□

Lemma 16. *Let $\Theta \in S(\mathbb{R})$ be a Schwartz function in time, $s \in \mathbb{R}$ and $\delta \geq 0$. If $-\frac{1}{2} < b \leq b' < \frac{1}{2}$ then for any $T > 0$ we have,*

$$\|\Theta_T(t)w\|_{X_{\delta,s,b}} \leq CT^{b'-b}\|w\|_{X_{\delta,s,b'}},$$

and

$$\|\Theta_T(t)w\|_{X^{\beta}_{\delta,s,b}} \leq CT^{b'-b}\|w\|_{X^{\beta}_{\delta,s,b'}},$$

where C depends only on b and b'.

Proof. The proof of the Lemma 16 for $\delta = 0$ can be found in Lemma 3.2 of [35], for $\delta > 0$ as one merely has to replace w by Aw, where the operator define in (4.2.6) □

Lemma 17. *([45]) Let $s \in \mathbb{R}$, $\delta \geq 0$, $-\frac{1}{2} < b < \frac{1}{2}$ and $T > 0$. Then, for any time interval $I \subset [0, T]$, we have*

$$\|\chi_I(t)w\|_{X_{\delta,s,b}} \leq C\|w\|_{X^{T}_{\delta,s,b}},$$

and

$$\|\chi_I(t)w\|_{X^{\beta}_{\delta,s,b}} \leq C\|w\|_{X^{\beta,T}_{\delta,s,b}},$$

where $\chi_I(t)$ is the characteristic function of I, and C depends only on b.

4.2.3 Trilinear estimates

The following Lemma states the desired trilinear estimate.

Lemma 18. *Let $s > -\frac{1}{2}$, $\delta > 0$, $b > \frac{1}{2}$ and b' be as in Lemma 15. Then*

$$\| \partial_x(uv^2) \|_{X_{\delta,s,b'}} \leq C \| u \|_{X_{\delta,s,b}} \| v \|^2_{X^{\beta}_{\delta,s,b}} .$$

and

$$\| \partial_x(u^2v) \|_{X^{\beta}_{\delta,s,b'}} \leq C \| u \|^2_{X_{\delta,s,b}} \| v \|_{X^{\beta}_{\delta,s,b}} .$$

Proof. We observe, by considering the operator A in (4.2.6), that

$$\begin{aligned} e^{\delta|\zeta|}\widehat{uuv} &= (2\pi)^{-2}e^{\delta|\zeta|}\widehat{u} * \widehat{u} * \widehat{v} \\ &\leq (2\pi)^{-2}\int_{\mathbb{R}^4} e^{\delta|\zeta-\zeta_1|}\widehat{u}(\zeta-\zeta_1, \eta-\eta_1)e^{\delta|\zeta_1-\zeta_2|}\widehat{u}(\zeta_1-\zeta_2, \eta_1-\eta_2) \\ &\quad e^{\delta|\zeta_2|}\widehat{v}(\zeta_2, \eta_2)d\zeta_1 d\zeta_2 d\eta_1 d\eta_2 \\ &= \widehat{AuAuAv}, \end{aligned}$$

since $\delta \mid \zeta \mid\leq \delta \mid \zeta-\zeta_1 \mid +\delta \mid \zeta_1-\zeta_2 \mid +\delta \mid \zeta_2 \mid$. Then

$$\begin{aligned}\| \partial_x(u^2v) \|_{X^{\beta}_{\delta,s,b'}} &= \| e^{\delta|\zeta|} \langle\zeta\rangle^s \left\langle \eta-\beta\zeta^3\right\rangle^b \widehat{\partial_x(uuv)}(\zeta,\eta) \|_{L^2_{\zeta,\eta}} \\ &\leq \| \partial_x(AuAuAv) \|_{X^{\beta}_{s,b'}} .\end{aligned}$$

Now, by using Proposition 2.3 of [33], there exists $C>0$ such that

$$\begin{aligned}\| \partial_x(AuAuAv) \|_{X^{\beta}_{s,b'}} &\leq C \parallel Au \|^2_{X_{s,b}} \| Av \|_{X^{\beta}_{s,b}} \\ &= C \parallel u \|^2_{X_{\delta,s,b}} \| v \|_{X^{\beta}_{\delta,s,b}} .\end{aligned}$$

□

4.3 Proof of Theorem 10

4.3.1 Existence of solution

We are now ready to estimate all the terms in (4.2.7) by using the trilinear estimates in the above Lemmas. We define spaces

$$B_{\delta,s,b} = X_{\delta,s,b} \times X^{\beta}_{\delta,s,b} \quad \text{and} \quad N^{\delta,s} = G^{\delta,s} \times G^{\delta,s}$$

with norms $\|(u,v)\|_{B_{\delta,s,b}} = max\{\|u\|_{X_{\delta,s,b}}, \|v\|_{X^{\beta}_{\delta,s,b}}\}$ and similar for $N^{\delta,s}$.

Lemma 19. *Let $s>-\frac{1}{2}$ and $\sigma\geq 1$, $b>\frac{1}{2}$. Then, for all $(u_0,v_0)\in N^{\delta,s}$ and $0<T<1$, with some constant $C>0$, we have*

$$\| (\Lambda[u,v],\Gamma[u,v]) \|_{B_{\delta,s,b}} \leq C \left(\| (u_0,v_0) \|_{N^{\delta,s}} + T^{\epsilon} \| (u,v) \|^3_{B_{\delta,s,b}} \right), \tag{4.3.1}$$

and

$$\begin{aligned}&\|(\Lambda[u,v]-\Lambda[u^*,v^*],\Gamma[u,v]-\Gamma[u^*,v^*])\|_{B_{\delta,s,b}} \\ &\leq CT^{\epsilon}\|(u-u^*,v-v^*)\|_{B_{\delta,s,b}} \Big(\|(u,v)\|^2_{B_{\delta,s,b}} \\ &+\|(u,v)\|_{B_{\delta,s,b}}\|(u^*,v^*)\|_{B_{\delta,s,b}} + \|(v^*,v^*)\|^2_{B_{\delta,s,b}} \Big).\end{aligned} \tag{4.3.2}$$

for all $(u,v),(u^*,v^*)\in B_{\delta,s,b}$.

Proof. To prove estimate (4.3.1), we follow

$$\begin{aligned}\| \Lambda[u,v] \|_{X_{\delta,s,b}} &\leq C \| u_0 \|_{G^{\delta,s}} + CT^{\epsilon} \| u \|_{X_{\delta,s,b}} \| v \|^2_{X^{\beta}_{\delta,s,b}} \\ &\leq C \| (u_0,v_0) \|_{N^{\delta,s}} + CT^{\epsilon} \| (u,v) \|^3_{B_{\delta,s,b}},\end{aligned} \tag{4.3.3}$$

and

$$\begin{aligned}\| \Gamma[u,v] \|_{X^{\beta}_{\delta,s,b}} &\leq C \| v_0 \|_{G^{\delta,s}} + CT^{\epsilon} \| u \|^2_{X_{\delta,s,b}} \| v \|_{X^{\beta}_{\delta,s,b}} \\ &\leq C \| (u_0,v_0) \|_{N^{\delta,s}} + CT^{\epsilon} \| (u,v) \|^3_{B_{\delta,s,b}} .\end{aligned} \tag{4.3.4}$$

Therefore, from (4.3.3) and (4.3.4), we obtain

$$\| (\Lambda[u,v], \Gamma[u,v]) \|_{B_{\delta,s,b}} \leq C \left(\| (u_0,v_0) \|_{N^{\delta,s}} + T^{\epsilon} \| (u,v) \|^3_{B_{\delta,s,b}} \right), \tag{4.3.5}$$

For the estimate (4.3.2), we observe that

$$\Lambda[u,v] - \Lambda[u^*,v^*] = \psi_T(t) \int_0^t S(t-\nu)\partial_x \left(uv^2 - u^* v^{*2} \right)(x,\nu)d\nu,$$

and

$$\Gamma[u,v] - \Gamma[u^*,v^*] = \psi_T(t) \int_0^t S_{\beta}(t-\nu)\partial_x \left(u^2 v - u^{*2} v^* \right)(x,\nu)d\nu,$$

where

$$\omega = \partial_x (u^2 v - u^{*2} v^*) = \partial_x \left[v(u+u^*)(u-u^*) + u^{*2}(v-v^*) \right]$$

and

$$\omega' = \partial_x (uv^2 - u^* v^{*2}) = \partial_x \left[u(v+v^*)(v-v^*) + v^{*2}(u-u^*) \right]$$

□

We will show that $\Lambda \times \Gamma$ is a contraction on the ball $\mathbb{B}(0,R)$ to $\mathbb{B}(0,R)$.

Lemma 20. *Let $s \geq -\frac{1}{4}$, $\delta > 0$ and $b > \frac{1}{2}$. Then, for all $(u_0,v_0) \in N^{\delta,s}$, such that the map $\Lambda \times \Gamma : \mathbb{B}(0,R) \to \mathbb{B}(0,R)$ is a contraction, where $\mathbb{B}(0,R)$ is given by*

$$\mathbb{B}(0,R) = \{(u,v) \in B_{\delta,s,b}; \|(u,v)\|_{B_{\delta,s,b}} \leq R\},$$

with $R = 2C\|(u_0,v_0)\|_{N^{\delta,s}}$

Proof. From Lemma 19, for all $(u,v) \in \mathbb{B}(0,R)$, we have

$$\| (\Lambda[u,v], \Gamma[u,v]) \|_{B_{\delta,s,b}} \leq C \| (u_0,v_0) \|_{N^{\delta,s}} + CT^{\epsilon} \| (u,v) \|^3_{B_{\delta,s,b}} \leq \frac{R}{2} + CT^{\epsilon} R^3.$$

We choose T sufficiently small such that $T^{\epsilon} \leq \frac{1}{4CR^2}$, hence,

$$\| (\Lambda[u,v], \Gamma[u,v]) \|_{B_{\delta,s,b}} \leq R, \quad \forall (u,v) \in \mathbb{B}(0,R).$$

Thus, $\Lambda \times \Gamma$ maps $\mathbb{B}(0,R)$ into $\mathbb{B}(0,R)$, which is a contraction, since

$$\| (\Lambda[u,v] - \Lambda[u^*,v^*], \Gamma[u,v] - \Gamma[u^*,v^*] \|_{B_{\delta,s,b}}$$

$$\leq CT^{\epsilon} \| (u-u^*, v-v^*) \|_{B_{\delta,s,b}} \Big(\| (u,v) \|^2_{B_{\delta,s,b}}$$

$$+ \|(u,v)\|_{B_{\delta,s,b}} \|(u^*,v^*)\|_{B_{\delta,s,b}} + \|(v^*,v^*)\|^2_{B_{\delta,s,b}} \Big),$$

$$\leq 3CT^{\epsilon} R^2 \| (u-u^*, v-v^*) \|_{B_{\delta,s,b}}$$

$$\leq \frac{3}{4} \| (u-u^*, v-v^*) \|_{B_{\delta,s,b}},$$

for all $(u,v) \in \mathbb{B}(0,R)$. Hence, $(\Lambda, \Gamma) : \mathbb{B}(0,R) \to \mathbb{B}(0,R)$ is a contraction. □

4.3.2 The uniqueness

From the fixed point argument used above, we have uniqueness of the solution of $(\Lambda[u,v], \Gamma[u,v]) = (u,v)$ in the set $\mathbb{B}(0,R)$. For the proof of uniqueness in the whole space $B_{\delta,s,b} = X_{\delta,s,b} \times X^{\beta}_{\delta,s,b}$, one can see [12].

4.3.3 Continuous dependence of the initial data

To prove continuous dependence of the initial data we will prove the following.

Lemma 21. *Let $s > -\frac{1}{2}$, and $\delta > 0$, $b > \frac{1}{2}$. Then, for all $(u_0,v_0), (u_0^*, v_0^*) \in N^{\delta,s}$, if (u,v) and (u^*,v^*) are two solutions to (4.1.1) corresponding to initial data (u_0,v_0) and (u_0^*,v_0^*), We have*

$$\|(u-u^*, v-v^*)\|_{C([0,T],G^{\delta,s})^2} \leq 4C_0 C \|(u_0 - u_0^*, v_0 - v_0^*)\|_{N^{\delta,s}}.$$

Proof. If (u,v) and (u^*,v^*) are two solutions to (4.1.1), corresponding to initial data (u_0,v_0) and (u_0^*,v_0^*), we have from Lemma 13

$$\|u-u^*\|_{C([0,T],G^{\delta,s})} \leq C_0\|u-u^*\|_{X_{\delta,s,b}}.$$

and

$$\|v-v^*\|_{C([0,T],G^{\delta,s})} \leq C_0\|v-v^*\|_{X^{\beta}_{\delta,s,b}}.$$

By taking $(u,v),(u^*,v^*) \in \mathbb{B}(0,R)$ and $T^{\epsilon} \leq \frac{1}{4CR}$,

$$\| \; u-u^* \; \|_{X_{\sigma,\delta,s,b}} \quad \leq C \; \| \; (u_0-u_0^*, v_0-v_0^*) \; \|_{N^{\delta,s}} + \frac{3}{4} \; \| \; (u-u^*, v-v^*) \; \|_{B_{\delta,s,b}} \; .$$

and

$$\| \; v-v^* \; \|_{X^{\beta}_{\delta,s,b}} \quad \leq C \; \| \; (u_0-u_0^*, v_0-v_0^*) \; \|_{N^{\delta,s}} + \frac{3}{4} \; \| \; (u-u^*, v-v^*) \; \|_{B_{\delta,s,b}} \; .$$

Thus

$$\| \; (u-u^*, v-v^*) \; \|_{B_{\delta,s,b}} \quad \leq 4C \; \| \; (u_0-u_0^*, v_0-v_0^*) \; \|_{N^{\delta,s}},$$

then

$$\|(u-u^*, v-v^*)\|_{C([0,T],G^{\delta,s})^2} \leq 4C_0C\|(u_0-u_0^*, v_0-v_0^*)\|_{N^{\delta,s}}.$$

□

This completes the prove of Theorem 10.

4.4 Approximate Conservation Law

We start by recalling that

$$\|(u,v)\|_{L^2} = \int_{\mathbb{R}} (u^2+v^2)dx$$

is conserved for a solution (u,v) of (4.1.1). Our goal in this section is to show an approximate conservation law for a solution to (4.1.1) based on the conservation the $L^2(\mathbb{R})$ norm of solution in Theorem 12.

Theorem 12. *Let $\kappa \in [0, \frac{1}{2})$ and $0 < T_1 < T_0 < 1$, T_0 be as in Theorem 10 with $s = 0$, there exist $b = \frac{1}{2} + \epsilon$ and $C > 0$, such that for any $\delta > 0$ and any solution $(u, v) \in B^{T_0}_{\delta,0,b}$ to the Cauchy problem (4.1.1) on the time interval $[0, T_1]$, we have the estimate*

$$\sup_{t\in[0,T_1]} \|(u(t), v(t))\|^2_{N^{\delta,0}} \leq \|(u(0), v(0))\|^2_{N^{\delta,0}} + C\delta^\kappa \|(u, v)\|^4_{B_{\delta,0,b}}. \tag{4.4.1}$$

Moreover, we have

$$\sup_{t\in[0,T_1]} \|(u(t), v(t))\|^2_{N^{\delta,0}} \leq \|(u(0), v(0))\|^2_{N^{\delta,0}} + C\delta^\kappa \|(u(0), v(0)\|^4_{N^{\delta,0}}. \tag{4.4.2}$$

We need the following estimate.

Lemma 22. *Given $\kappa \in [0, \frac{1}{2})$, there exist $b = \frac{1}{2} + \epsilon$, $C > 0$ and $(u, v) \in B_{\delta,0,b}$, we have*

$$\|(G_1, G_2)\|_{B_{0,b-1}} \leq C\delta^\kappa \|(u, v)\|^3_{B_{\delta,0,b}},$$

where $G_1 = \partial_x [(AuAvAv) - A(uv^2)]$, $G_2 = \partial_x [(AuAuAv) - A(u^2v)]$ and the operator A given by (4.2.6).

Proof. Let $L_1 = (AuAvAv) - A(uv^2)$. Then

$$\|G_1\|_{X_{0,b-1}} = \left\| \frac{\zeta}{\langle \eta - \zeta^3 \rangle^{1-b}} \widehat{L_1}(\zeta, \eta) \right\|_{L^2_{\zeta,\eta}} = \left(\int_{\mathbb{R}^2} \frac{|\ \zeta\ |^2}{\langle \eta - \zeta^3 \rangle^{2(1-b)}} \ |\ \widehat{L_1}(\zeta, \eta)\ |^2 \ d\zeta d\eta \right)$$

We shall calculate the Fourier transform of L_1

$$\left|\widehat{L_1}(\zeta, \eta)\right| = \left|(\widehat{AuAvAv}) - \widehat{A(uv^2)}\right| = C\left|(e^{\delta|\zeta|}\widehat{u} * e^{\delta|\zeta|}\widehat{v} * e^{\delta|\zeta|}\widehat{v}*)(\zeta, \eta) - e^{\delta|\zeta|}(\widehat{u} * \widehat{v} * \widehat{v}*)(\zeta, \eta)\right|$$

$$= C\Bigg| \int_{\mathbb{R}^4} \Big(e^{\delta|\zeta_1|}\widehat{u}(\zeta_1, \eta_1) e^{\delta|\zeta_2|}\widehat{v}(\zeta_2, \eta_2) e^{\delta|\zeta-\zeta_1-\zeta_2|}\widehat{v}(\zeta - \zeta_1 - \zeta_2, \eta - \eta_1 - \eta_2)$$

$$-e^{\delta|\zeta|}\widehat{u}(\zeta_1, \eta_1)\widehat{v}(\zeta_2, \eta_2)\widehat{v}(\zeta - \zeta_1 - \zeta_2, \eta - \eta_1 - \eta_2)\Big) d\zeta_1 d\zeta_2 d\eta_1 d\eta_2 \Bigg|$$

$$\leq C \int_{\mathbb{R}^4} \left(e^{\delta|\zeta_1|} e^{\delta|\zeta_2|} e^{\delta|\zeta-\zeta_1-\zeta_2|} - e^{\delta|\zeta|}\right) |\widehat{u}(\zeta_1, \eta_1)\widehat{v}(\zeta_2, \eta_2)\widehat{v}(\zeta - \zeta_1 - \zeta_2, \eta - \eta_1 - \eta_2)| d\zeta_1 d\zeta_2 d\eta_1 d\eta_2$$

Now using Corollary 7.3 in [22], let $\theta \in [0, 1]$

$$e^{\delta|\zeta_1|} e^{\delta|\zeta_2|} e^{\delta|\zeta-\zeta_1-\zeta_2|} - e^{\delta|\zeta|} \leq \left[4\delta \frac{\langle \zeta - \zeta_1 - \zeta_2 \rangle \langle \zeta_1 \rangle \langle \zeta_2 \rangle}{\langle \zeta \rangle}\right]^\theta e^{\delta|\zeta_1|} e^{\delta|\zeta_2|} e^{\delta|\zeta-\zeta_1-\zeta_2|},$$

For $\kappa \in [0, \frac{1}{2}) \subset [0,1]$, one can see that

$$\|G_1\|^2_{X_{0,b-1}} = \left\| \tfrac{\zeta}{\langle \eta - \zeta^3 \rangle^{1-b}} \widehat{L_1}(\zeta,\eta) \right\|^2_{L^2_{\zeta,\eta}}$$

$$\leq (C4\delta)^{2\kappa} \int_{\mathbb{R}^2} \frac{|\zeta|^2}{\langle \eta - \zeta^3 \rangle^{2(1-b)}} \left[\int_{\mathbb{R}^4} \left(\frac{\langle \zeta - \zeta_1 - \zeta_2 \rangle \langle \zeta_1 \rangle \langle \zeta_2 \rangle}{\langle \zeta \rangle} \right)^{\kappa} e^{\delta|\zeta_1|} e^{\delta|\zeta_2|} e^{\delta|\zeta - \zeta_1 - \zeta_2|} \right.$$

$$\left. |\widehat{u}(\zeta_1,\eta_1)\widehat{v}(\zeta_2,\eta_2)\widehat{v}(\zeta-\zeta_1-\zeta_2,\eta-\eta_1-\eta_2)| d\zeta_1 d\zeta_2 d\eta_1 d\eta_2 \right]^2 d\zeta d\eta$$

$$= C(4\delta)^{2\kappa} \left\| \tfrac{\zeta \langle \zeta \rangle^{-\kappa}}{\langle \eta - \zeta^3 \rangle^{1-b}} \int_{\mathbb{R}^4} e^{\delta|\zeta_1|} \langle \zeta_1 \rangle^{\kappa} \widehat{u}(\zeta_1,\eta_1) e^{\delta|\zeta_2|} \langle \zeta_2 \rangle^{\kappa} \widehat{v}(\zeta_2,\eta_2) \right.$$

$$\left. \cdot e^{\delta|\zeta-\zeta_1-\zeta_2|} \langle \zeta - \zeta_1 - \zeta_2 \rangle^{\kappa} \widehat{v}(\zeta-\zeta_1-\zeta_2,\eta-\eta_1-\eta_2) d\zeta_1 d\zeta_2 d\eta_1 d\eta_2 \right\|^2_{L^2_{\zeta,\eta}}$$

Now by taking $s = -\kappa \in (-\frac{1}{2}, 0]$ we obtain

$$\|G_1\|_{X_{0,b-1}} \leq C(4\delta)^{\kappa} \left\| \tfrac{\zeta \langle \zeta \rangle^{s}}{\langle \eta - \zeta^3 \rangle^{1-b}} \int_{\mathbb{R}^4} \frac{e^{\delta|\zeta_1|}\widehat{u}(\zeta_1,\eta_1)}{\langle \zeta_1 \rangle^{s}} \frac{e^{\delta|\zeta_2|}\widehat{v}(\zeta_2,\eta_2)}{\langle \zeta_2 \rangle^{s}} \right.$$

$$\left. \cdot \frac{e^{\delta|\zeta-\zeta_1-\zeta_2|}\widehat{v}(\zeta-\zeta_1-\zeta_2,\eta-\eta_1-\eta_2)}{\langle \zeta - \zeta_1 - \zeta_2 \rangle^{s}} d\zeta_1 d\zeta_2 d\eta_1 d\eta_2 \right\|_{L^2_{\zeta,\eta}}$$

$$\leq C(4\delta)^{\kappa} \left\| \tfrac{\zeta \langle \zeta \rangle^{s}}{\langle \eta - \zeta^3 \rangle^{1-b}} \int_{\mathbb{R}^4} \frac{e^{\delta|\zeta_1|} \langle \eta_1 - \zeta_1^3 \rangle^{b} \widehat{u}(\zeta_1,\eta_1)}{\langle \zeta_1 \rangle^{s} \langle \eta_1 - \zeta_1^3 \rangle^{b}} \frac{e^{\delta|\zeta_2|} \langle \eta_2 - \beta\zeta_2^3 \rangle^{b} \widehat{v}(\zeta_2,\eta_2)}{\langle \zeta_2 \rangle^{s} \langle \eta_2 - \beta\zeta_2^3 \rangle^{b}} \right.$$

$$\left. \cdot \frac{e^{\delta|\zeta-\zeta_1-\zeta_2|} \langle \eta - \eta_1 - \eta_2 - \beta(\zeta-\zeta_1-\zeta_2)^3 \rangle^{b} \widehat{v}(\zeta-\zeta_1-\zeta_2,\eta-\eta_1-\eta_2)}{\langle \zeta - \zeta_1 - \zeta_2 \rangle^{s} \langle \eta - \eta_1 - \eta_2 - \beta(\zeta-\zeta_1-\zeta_2)^3 \rangle^{b}} d\zeta_1 d\zeta_2 d\eta_1 d\eta_2 \right\|_{L^2_{\zeta,\eta}}$$

then

$$\begin{aligned} \|G_1\|_{X_{0,b-1}} &\leq C\delta^{\kappa} \|Au\|_{X_{0,b}} \|Av\|^2_{X^{\beta}_{0,b}} = C\delta^{\kappa} \|u\|_{X_{\delta,0,b}} \|v\|^2_{X^{\beta}_{\delta,0,b}} \\ &\leq C\delta^{\kappa} \|(u,v)\|^3_{B_{\delta,0,b}} \end{aligned} \tag{4.4.3}$$

Now let $L_2 = (AuAuAv) - A(u^2v)$. Then

$$\begin{aligned}\|G_2\|_{X_{0,b-1}} &\le C\delta^\kappa \|Au\|^2_{X_{0,b}}\|Av\|_{X^\beta_{0,b}} = C\delta^\kappa\|u\|^2_{X_{\delta,0,b}}\|v\|_{X^\beta_{\delta,0,b}} \\ &\le C\delta^\kappa\|(u,v)\|^3_{B_{\delta,0,b}}\end{aligned} \tag{4.4.4}$$

bay (4.4.3) and (4.4.4) we get

$$\|(G_1,G_2)\|_{B_{0,b-1}} \le C\delta^\kappa\|(u,v)\|^3_{B_{\delta,0,b}},$$

□

Proof. (Of Theorem 12) Let $U(t,x) = Au(t,x), V(t,x) = Av(t,x)$ which are real-valued since the multiplier A is even and u, v are real-valued. Applying A to (4.1.1) we obtain

$$\partial_t U + \partial_x^3 U + \partial_x(UV^2) = G_1, \tag{4.4.5}$$

$$\partial_t U + \partial_x^3 U + \partial_x(U^2V) = G_2, \tag{4.4.6}$$

where $G_1 = \partial_x\left[(AuAvAv) - A(uv^2)\right]$, $G_2 = \partial_x\left[(AuAuAv) - A(u^2v)\right]$ We multiply both sides of (4.4.5) by U, (4.4.6) by V and integrate with respect to space variable, we get

$$\int_{\mathbb{R}} U\partial_t U dx + \int_{\mathbb{R}} U\partial_x^3 U dx + \int_{\mathbb{R}} U\partial_x(UV^2)dx = \int_{\mathbb{R}} UG_1 dx.$$

$$\int_{\mathbb{R}} V\partial_t V dx + \int_{\mathbb{R}} V\partial_x^3 V dx + \int_{\mathbb{R}} V\partial_x(U^2V)dx = \int_{\mathbb{R}} VG_2 dx.$$

then

$$\int_{\mathbb{R}}(U\partial_t U + V\partial_t V)dx + \int_{\mathbb{R}}(U\partial_x^3 U + V\partial_x^3 V)dx + \int_{\mathbb{R}}\left[U\partial_x(UV^2) + V\partial_x(U^2V)\right]dx = \int_{\mathbb{R}}(UG_1 + VG_2)dx.$$

$$\int_{\mathbb{R}}(U\partial_t U + V\partial_t V)dx + \int_{\mathbb{R}}\partial_x(\partial_x U\partial_x U + \partial_x V\partial_x V)dx + \int_{\mathbb{R}}\partial_x(U^2V^2)dx = \int_{\mathbb{R}}(UG_1 + VG_2)dx.$$

Noting that $\partial_x^j U(x,t) \longrightarrow 0$ as $|x| \longrightarrow \infty$ (see [45]), we use integration by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(U^2 + V^2)dx = \int_{\mathbb{R}}(UG_1 + VG_2)dx.$$

Integrating the last equality with respect to $t \in [0, T_1]$, we obtain

$$\int_{\mathbb{R}}(U^2(T_1,x) + V^2(T_1,x))dx = \int_{\mathbb{R}}(U^2(0,x) + V^2(0,x))dx + 2\int_{\mathbb{R}^2}\chi_{[0,T_1]}(t)(UG_1 + VG_2)dxdt.$$

Thus,

$$\|u(T_1)\|^2_{G^{\delta,0}} + \|v(T_1)\|^2_{G^{\delta,0}} = \|u(0)\|^2_{G^{\delta,0}} + \|v(0)\|^2_{G^{\delta,0}} + 2\left|\int_{\mathbb{R}^2} \chi_{[0,T_1]}(t)(UG_1 + VG_2)dxdt\right|.$$

By using Holder's inequality, Lemma 16, Lemma 17 and the fact that $\frac{1}{2} < 1-b < \frac{1}{2}$, $\frac{1}{2} < b-1 < \frac{1}{2}$, since $b > \frac{1}{2} + \epsilon$, we obtain

$$\begin{aligned}
\left|\int_{\mathbb{R}^2} \chi_{[0,T_1]}(t)(UG_1 + VG_2)dxdt\right| &\leq \|\chi_{[0,T_1]}(t)U\|_{X_{0,1-b}}\|\chi_{[0,T_1]}(t)G_1\|_{X_{0,b-1}} \\
&+ \|\chi_{[0,T_1]}(t)V\|_{X^{\beta}_{0,1-b}}\|\chi_{[0,T_1]}(t)G_2\|_{X^{\beta}_{0,b-1}} \\
&\leq C\|U\|_{X^{T_1}_{0,1-b}}\|G_1\|_{X^{T_1}_{0,b-1}} + C\|V\|_{X^{\beta,T_1}_{0,1-b}}\|G_2\|_{X^{\beta,T_1}_{0,b-1}} \\
&\leq C\|\Theta_{T_1}U\|_{X_{0,1-b}}\|\Theta_{T_1}G_1\|_{X_{0,b-1}} + C\|\Theta_{T_1}V\|_{X^{\beta}_{0,1-b}}\|\Theta_{T_1}G_2\|_{X^{\beta}_{0,b-1}} \\
&\leq C\|U\|_{X_{0,1-b}}\|G_1\|_{X_{0,b-1}} + C\|V\|_{X^{\beta}_{0,1-b}}\|G_2\|_{X^{\beta}_{0,b-1}}.
\end{aligned}$$

where $\Theta_{T_1} = 1$ for $t \in [0, T_1]$, we can conclude from Lemma22

$$\begin{aligned}
\left|\int_{\mathbb{R}^2} \chi_{[0,T_1]}(t)(UG_1 + VG_2)dxdt\right| &\leq C\|U\|_{X_{0,1-b}}\|G_1\|_{X_{0,b-1}} + C\|V\|_{X^{\beta}_{0,1-b}}\|G_2\|_{X^{\beta}_{0,b-1}} \\
&\leq C\delta^{\kappa}\|u\|^2_{X_{\delta,0,b}}\|v\|^2_{X^{\beta}_{\delta,0,b}} + C\delta^{\kappa}\|u\|^2_{X_{\delta,0,b}}\|v\|^2_{X^{\beta}_{\delta,0,b}} \\
&= 2C\delta^{\kappa}\|u\|^2_{X_{\delta,0,b}}\|v\|^2_{X^{\beta}_{\delta,0,b}} \\
&\leq 2C\delta^{\kappa}\|(u,v)\|^4_{B_{\delta,0,b}}.
\end{aligned}$$

Therefore,

$$\|u(T_1)\|^2_{G^{\delta,0}} + \|v(T_1)\|^2_{G^{\delta,0}} \leq \|u(0)\|^2_{G^{\delta,0}} + \|v(0)\|^2_{G^{\delta,0}} + 2C\delta^{\kappa}\|(u,v)\|^4_{B_{\delta,0,b}}.$$

$$2\|(u(T_1), v(T_1))\|^2_{N^{\delta,0}} \leq 2\|(u(0), v(0))\|^2_{N^{\delta,0}} + 2C\delta^\kappa\|(u, v)\|^4_{B_{\delta,0,b}}.$$

$$\sup_{t\in[0,T_1]} \|(u(t), v(t))\|^2_{N^{\delta,0}} \leq \|(u(0), v(0))\|^2_{N^{\delta,0}} + C\delta^\kappa\|(u, v)\|^4_{B_{\delta,0,b}}.$$

Finally, by using the condition in theorem (10), we conclude that:

$$\sup_{t\in[0,T_1]} \|(u(t), v(t))\|^2_{N^{\delta,0}} \leq \|(u(0), v(0))\|^2_{N^{\delta,0}} + C\delta^\kappa\|(u(0), v(0))\|^4_{N^{\delta,0}}.$$

□

4.5 Proof of Theorem 11

Let $\delta_0 > 0$, $s > -\frac{1}{2}$, $\kappa \in (0, \frac{1}{2})$ be fixed, and $(u_0, v_0) \in N^{\delta_0, s}$. Then, we have to prove that the solution (u, v) of (4.1.1) satisfies

$$(u, v) \in C\left([0, T'], G^{\delta(T'), s}\right) \times C\left([0, T'], G^{\delta(T'), s}\right),$$

where

$$\delta(T') = \min\left\{\delta_0, C_1 T'^{-1/\kappa}\right\}, \quad for\ all\ T' > 0,$$

and $C_1 > 0$ is a constant depending on u_0, v_0, δ_0, s and κ. By Theorem 10, there is a maximal time $T^* = T^*(u_0, v_0, \delta_0, s) \in (0, \infty]$, such that

$$(u, v) \in C\left([0, T^*], G^{\delta_0, s}\right) \times C\left([0, T^*], G^{\delta_0, s}\right).$$

If $T^* = \infty$, we are done. If $T^* < \infty$, as we assume henceforth, it remains to prove

$$(u, v) \in C\left([0, T'], G^{C_1 T'^{-1/\kappa}, s}\right) \times C\left([0, T'], G^{C_1 T'^{-1/\kappa}, s}\right), \quad for\ all\ T' \geq T^*. \tag{4.5.1}$$

4.5.1 The case s=0

Fix $T' \geq T^*$.we will show that, for $\delta > 0$ sufficiently small

$$\sup_{t \in [0, T']} \|(u(0), v(0))\|^2_{N^{\delta,0}} \leq 2\|(u(0), v(0))\|^2_{N^{\delta_0,0}}.$$

In this case, by Theorems 10 and Theorem 12 with

$$T_0 = \frac{1}{(16C^3 + 32C^3 \parallel (u(0), v(0)) \parallel^2_{N^{\delta_0,0}})^{1/\epsilon}}.$$

The smallness conditions on δ will be

$$\delta < \delta_0 \ \ and \ \ \frac{2T'}{T_0} C\delta^\kappa 2^2 \parallel (u(0), v(0)) \parallel^2_{N^{\delta_0,0}} \leq 1, \ C > 0. \tag{4.5.2}$$

Here C is the constant in Theorems 12. By induction, we check that

$$\sup_{t \in [0, nT_0]} \|(u(t), v(t))\|^2_{N^{\delta,0}} \leq \|(u(0), v(0))\|^2_{N^{\delta,0}} + nC\delta^\kappa 2^2 \|(u(0), v(0))\|^4_{N^{\delta_0,0}}. \tag{4.5.3}$$

$$\sup_{t\in[0,nT_0]} \|(u(t),v(t))\|^2_{N^{\delta,0}} \leq 2\|(u(0),v(0))\|^2_{N^{\delta_0,0}}, \tag{4.5.4}$$

for $n \in \{1,\dots,m+1\}$, where $m \in \mathbb{N}$ is chosen, so that $T' \in [mT_0,(m+1)T_0)$. This m does exist, since by Theorem 10 and the definition of T^*, we have

$$T_0 < \frac{1}{(16C^3 + 16C^3 \parallel (u(0),v(0)) \parallel^2_{N^{\delta_0,0}})^{1/\epsilon}} < T^*, \quad hence \;\; T_0 < T'.$$

In the first step, we cover the interval $[0,T_0]$, and by Theorem 12, we have

$$\begin{aligned}\sup_{t\in[0,T_0]} \|(u(t),v(t))\|^2_{N^{\delta,0}} &\leq \|(u(0),v(0))\|^2_{N^{\delta,0}} + C\delta^\kappa \|(u(0),v(0))\|^4_{N^{\delta,0}} \\ &\leq \|(u(0),v(0))\|^2_{N^{\delta,0}} + C\delta^\kappa \|(u(0),v(0))\|^4_{N^{\delta_0,0}},\end{aligned}$$

since $\delta \leq \delta_0$, we used

$$\|(u(0),v(0))\|_{N^{\delta,0}} \leq \|(u(0),v(0))\|_{N^{\delta_0,0}}.$$

This verifies (4.5.3) for $n=1$ and now, (4.5.4) follows using again $\|(u(0),v(0))\|_{N^{\delta,0}} \leq \|(u(0),v(0))\|_{N^{\delta_0,0}}$ as well as

$$C\delta^\kappa \|(u(0),v(0))\|^2_{N^{\delta_0,0}} \leq 1.$$

Suppose now that (4.5.3) and (4.5.4) hold for some $n \in \{1,\dots,m\}$ and we prove that it holds for $n+1$.

We estimate

$$\begin{aligned}\sup_{t\in[nT_0,(n+1)T_0]} \|(u(t),v(t))\|^2_{N^{\delta,0}} &\leq \|w(nT_0)\|^2_{N^{\delta,0}} + C\delta^\kappa \|(u(nT_0),v(nT_0))\|^4_{N^{\delta,0}} \\ &\leq \|(u(nT_0),v(nT_0))\|^2_{N^{\delta,0}} + C\delta^\kappa 2^2 \|(u(0),v(0))\|^4_{N^{\delta_0,0}} \\ &\leq \|(u(0),v(0))\|^2_{N^{\delta,0}} + nC\delta^\kappa 2^2 \|(u(0),v(0))\|^4_{N^{\delta_0,0}} \\ &+ C\delta^\kappa 2^2 \|(u(0),v(0))\|^4_{N^{\delta_0,0}},\end{aligned}$$

verifying (4.5.3) with n replaced by $n+1$. To get (4.5.4) with n replaced by $n+1$, it is then enough to have

$$(n+1)C\delta^\kappa 2^2 \|(u(0),v(0))\|^2_{N^{\delta_0,0}} \leq 1,$$

but this holds by (4.5.2), since $n+1 \leq m+1 \leq \frac{T'}{T_0}+1 < \frac{2T'}{T_0}$. Finally, the condition (4.5.2) is satisfied for $\delta \in (0, \delta_0)$ such that

$$\frac{2T'}{T_0} C\delta^\kappa 2^2 \parallel (u(0), v(0)) \parallel^2_{N^{\delta_0,0}} = 1.$$

Thus, $\delta = C_1 T'^{-\frac{1}{\kappa}}$, where

$$C_1 = \left(\frac{1}{C2^3 \parallel (u(0), v(0)) \parallel^2_{N^{\delta_0,0}} (16C^3 + 32C^3 \parallel (u(0), v(0)) \parallel^2_{N^{\delta_0,0}})^{1/\epsilon}} \right)^{\frac{1}{\kappa}}.$$

4.5.2 The General Case

For all s, by (4.2.2), we have $u_0, v_0 \in G^{\delta_0,s} \subset G^{\delta_0/2,0}$. For case $s = 0$, it is proved, there is a $T_2 > 0$, such that:

$$(u,v) \in C\left([0,T_2), G^{\delta_0/2,0}\right) \times C\left([0,T_2), G^{\delta_0/2,0}\right),$$

and

$$(u,v) \in C\left([0,T'], G^{2\sigma T'^{-1/\kappa},0}\right) \times C\left([0,T'], G^{2\sigma T'^{-1/\kappa},0}\right), \quad for\ T' \geq T_2,$$

where $\sigma > 0$ depends on u_0, v_0, δ_0 and κ. Applying again the embedding (4.2.2), we now conclude that

$$(u,v) \in C\left([0,T_2), G^{\delta_0/4,s}\right) \times C\left([0,T_2), G^{\delta_0/4,s}\right),$$

and

$$(u,v) \in C\left([0,T'], G^{\sigma T'^{-1/\kappa},s}\right) \times C\left([0,T'], G^{\sigma T'^{-1/\kappa},s}\right), \quad for\ T' \geq T_2$$

which imply (4.5.1). The proof of Theorem 11 is now completed.

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