

# Numerical Simulation and Identification of Fractional Systems using Digital Adjustable Fractional Order Integrator

A. Djouambi, A. Charef, A. Voda

**Abstract**— A new method for numerical simulation and parameter identification of fractional order models is presented in this paper. A Digital Adjustable Fractional Integrator is proposed to improve the numerical simulation of fractional order systems. The main feature of this tool consists in: the important reduction of simulation run time, operate as a one-step forward predictor and, consequently, it can be used for real-time numerical simulation and identification of different fractional order systems using the same structure. The consistence, convergence and stability of the proposed method are proved. Finally, some numerical results are presented to demonstrate the effectiveness of the proposed approximation.

## I. INTRODUCTION

Fractional order systems, also known as fractional filters, have been introduced long ago in various fields of science such as viscoelasticity and biomedical engineering [1], [2], [3], chemistry [4]; [5]; [6] and, particularly, in the fields of signal processing [7]; [8]; [9] and automatic control [10]; [11]; [12]; [13]; [14]. Hence, their analog and digital simulation and implementation are important research topics.

A major difficulty with fractional models is its time-domain simulation. Often, the analytical solution of a model's output is not simple to compute. During the last 20 years, many methods are developed in order to simulate non-integer order fractional systems. Two types of methods can be considered. The first one, also called direct methods, are based on a numerical approximation of the non-integer order operator. The second ones, called indirect methods, are based on the simulation of the continuous fractional model with the help of a specific operator or representation [15]; [16]; [17]; [18]; [19], [20].

In direct methods, the fractional derivation is replaced by a numerical approximation in order to obtain a recurrent equation directly used for simulation [21]. Different types of approximations can be used in this context; the most commonly used method is that directly related to the Grunwald-Letnikov (GL) definition [11]. This method is very simple to use. However, the simulation requires, for each step, the computation of sums of increasing dimension with time. This makes real-time simulation hard to achieve and amplify greatly any noise present in the data. This is a

real constraint when this approximation is used for model identification and parameter estimation.

In [20] a design of an analog variable fractional order differentiator and integrator, in a given frequency band, was presented. The main feature of this analog variable fractional order integrator or differentiator is that its frequency characteristics can be changed without redesigning a new one. This gives a useful tool to approximate and simulate different fractional systems using fixed fractional order filter structure.

In this paper, a new technique for numerical simulation of fractional systems using Digital Adjustable Fractional Integrator (DAFI), obtained by discretisation of an optimal analog variable fractional integrator, will be proposed. This technique leads to one-step forward recurrent equation directly used for numerical simulation and parameters identification of different fractional order systems with a very high speed compared to the classical methods. Moreover, the memory is considerably reduced and the accuracy is increased. Practical use of embedded processors for fractional digital filtration of signals in real time is then possible.

This paper is divided in 5 parts. The first one presents the analog fractional order approximation. The second part is devoted to the presentation of the proposed DAFI. Parts three presents the main methods used in order to simulate fractional systems. In the fourth part, a new technique for parameters identification of fractional systems using DAFI is presented. Simulation examples are illustrated in the fifth part to illustrate the effectiveness of the proposed method.

## II. ANALOG RATIONAL FUNCTION APPROXIMATION OF FRACTIONAL ORDER INTEGRATOR

The analog fractional order integrator is represented by the following irrational transfer function:

$$G(s) = s^{-\alpha}, \quad \alpha \in \mathcal{R}^+ \quad (1)$$

In a given frequency band of interest  $[\omega_L, \omega_H]$ , equation (1) can be approximated by a rational transfer functions as [22]:

$$G(s) = s^{-\alpha} \cong \sum_{i=0}^N \frac{h_i(\alpha)}{1 + s/p_i(\alpha)}, \quad 0 < \alpha < 1 \quad (2)$$

For a given frequencies  $\omega_c$  and  $\omega_{max}$  such that  $\omega_c \ll \omega_L$  and  $\omega_{max} \gg \omega_H$  and a given approximation error  $\varepsilon$  dB, the poles  $p_i$  and the residues  $h_i$ , for  $i=0,1,\dots,N$  of the above approximation are given by :

\*This paper has been partly supported by the ANR project CONFOC-AO  
 A.Djouambi is with the Department of Science and Technology, University Larbi Ben M'Hidi of Oum El-Bouaghi, Oum El-Bouaghi 04000, Algeria, (e-mail : djouambi\_abdelbaki@yahoo.fr).

A.Charef is with the Signal Processing Laboratory, University Mentouri of Constantine, Route de Ain El Bey, Constantine 25000, Algeria.

A.Voda is with the University Joseph Fourier Grenoble 1/CNRS, Grenoble, Image Parole Signal Automatique (GIPSA-lab), UMR 5216, B.P. 46, F-38402 St Martin d'Hères, France.

$$p_i(\alpha) = p_0(ab)^i \quad (3)$$

$$h_i(\alpha) = K \frac{\prod_{j=0}^{N_i-1} \left(1 - \frac{(ab)^{(i-j)}}{a}\right)}{\prod_{j=0, j \neq i}^N \left(1 - (ab)^{(i-j)}\right)} \quad (4)$$

where  $a, b, p_0, K$ , and  $N$  are given by:

$$a = 10^{\left\lfloor \frac{\varepsilon}{10(1-\alpha)} \right\rfloor}, \quad b = 10^{\left\lfloor \frac{\varepsilon}{10\alpha} \right\rfloor}, \quad ab = 10^{\left\lfloor \frac{\varepsilon}{10\alpha(1-\alpha)} \right\rfloor},$$

$$p_0 = \omega_c 10^{\left\lfloor \frac{\varepsilon}{20\alpha} \right\rfloor}, \quad K = 1/\omega_c^\alpha \quad (5)$$

$$N = \text{Integer} \left[ \frac{\log\left(\frac{\omega_{max}}{p_0}\right)}{\log(ab)} \right] + 1 \quad (6)$$

### III. DIGITAL ADJUSTABLE FRACTIONAL ORDER INTEGRATOR

From equations (3) and (4) we can easily see that the poles and the residues of the rational function approximation of the fractional order operators depend on the fractional order  $\alpha$ . Charef *et al.* in [20] proposed a design of an Analog Variable Fractional order Integrator (AVFI) that can be implemented by the following structure:

$$G(s) = s^{-\alpha} \cong \sum_{i=0}^N \frac{h_i(\alpha)}{(1+s/p_i(\alpha_0))} \quad (7)$$

where  $0 < \alpha < 1$  and  $\alpha_0$  is a fixed real number such that  $0 < \alpha_0 < 1$ .

In this section, a generalization of (7) is proposed. Using some mathematic manipulations, it can be easily demonstrated that the optimal values of  $\alpha_0$  giving the best approximation of  $G(s)$  for  $\alpha \in \mathbb{R}$  is  $\alpha_0 = 0.5$ , therefore, the residues  $h_i$  can be given by:

$$h_i(\alpha) = [w_c 10^{\varepsilon(0.2\alpha-0.1)}]^{-\alpha} \frac{\prod_{j=0}^{N-1} (1-10^{0.4\varepsilon(i-j-\alpha)})}{\prod_{j=0, j \neq i}^{N-1} (1-10^{0.4\varepsilon(i-j)})} \quad (8)$$

The mathematical development of this optimization is outside the scope of the present paper, and it will be the topics of a further publications.

The main feature of this adjustable analog fractional order filter is that their frequency characteristics depend only on the parameter  $\alpha$  with fixed real negative poles  $-p_i(\alpha_0)$ . Consequently, their implementation is simple, stable and valid for the whole range of the fractional order  $\alpha$ .

For a given values of  $w_c$  and  $\varepsilon$ , equation (7) can be written as:

$$G(s) = \frac{Y_\alpha(s)}{Y(s)} = \sum_{i=0}^N \frac{h_i(\alpha)}{(1+\tau_i s)} \quad (9)$$

where  $h_i(\alpha)$  is given by (8) for  $\alpha \in \mathbb{R}$  and

$$\tau_i = \frac{1}{p_i(0.5)} \quad (10)$$

From equation (9) we can write:

$$Y_\alpha(s) = \sum_{i=0}^N h_i(\alpha) Y_i(s) \quad (11)$$

$$\text{where } Y_i(s) = \frac{Y(s)}{(1+\tau_i s)} \quad (12)$$

The Z transform of equation (12) with ZOH block gives:

$$y_i(k) = \frac{1-e^{-T/\tau_i}}{z-e^{-T/\tau_i}} y(k) \quad (13)$$

where  $T$  is the sampling time.

Equation (13) leads to a one-step forward regression given by:

$$y_i(k) = e^{-\frac{T}{\tau_i}} y_i(k-1) + \left(1 - e^{-\frac{T}{\tau_i}}\right) y(k-1) \quad (14)$$

Therefore, the numerical fractional order integration  $y_\alpha$  of  $y_i$  can be evaluated as:

$$y_\alpha(k) = \sum_{i=0}^N h_i(\alpha) y_i(k) \quad (15)$$

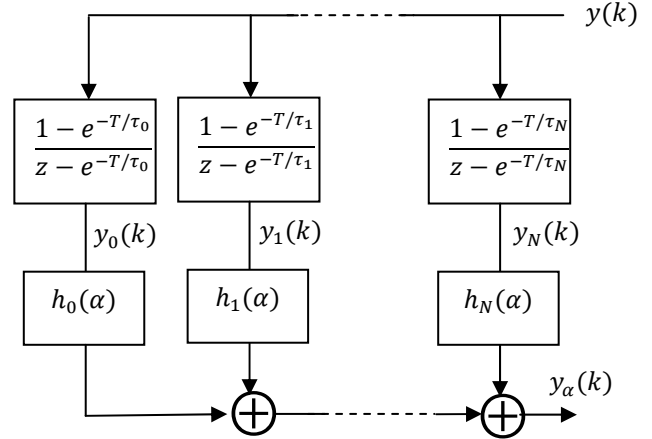


Figure 1. Digital adjustable fractional order integrator (DAFI)

Figure 1 presents the block diagram of the proposed digital filter implementing the adjustable digital fractional order integrator of equation (15).

### IV. NUMERICAL SIMULATION OF FRACTIONAL SYSTEMS USING DAFI

Consider the generalized continuous fractional transfer function given by

$$G(s) = \frac{y(s)}{u(s)} = \frac{\sum_{m=0}^{m_b} b_m s^{-\beta m}}{1 + \sum_{n=1}^{n_a} a_n s^{-\alpha n}} \quad (16)$$

where  $\alpha_n, \beta_m$ , ( $n = 1, \dots, n_a$ ,  $m = 0, \dots, m_b$ ) are real positive numbers and  $a_n, b_m$  are real parameters.

Using the DAFI proposed in section 3, a one-step forward linear regression for numerical simulation of fractional system (16) can be obtained as:

$$y(k) = -\sum_{n=1}^{n_a} a_n y_{\alpha_n}(k) + \sum_{m=0}^{m_b} b_m u_{\beta_m}(k) \quad (17)$$

where  $y_{\alpha_n}$ ,  $n=1, 2, \dots, n_a$ , is given by equation (15) and

$$u_{\beta_m}(k) = \sum_{i=0}^N h_i(\beta_m) u_i(k), \quad m=1, 2, \dots, m_b, \quad (18)$$

$$u_i(k) = e^{-\frac{T}{\tau_i}} u_i(k-1) + \left(1 - e^{-\frac{T}{\tau_i}}\right) u(k-1) \quad (19)$$

Figure 2 presents the simulation scheme of fractional order models represented by the given structure (16).

#### V. FRACTIONAL SYSTEM IDENTIFICATION USING DAFI

Consider time domain identification and parameter estimation of model (16) using  $K$  data pairs  $\{u(k), y(k)\}$ , where  $y(k)$  represents noise disturbed values of true output. Prior knowledge is generally used to fix the fractional integrations orders  $\alpha_1, \dots, \alpha_{n_a}, \beta_0, \dots, \beta_{m_b}$ .

The proposed estimation method consists of computing fractional integrations of input/output signals from sampled

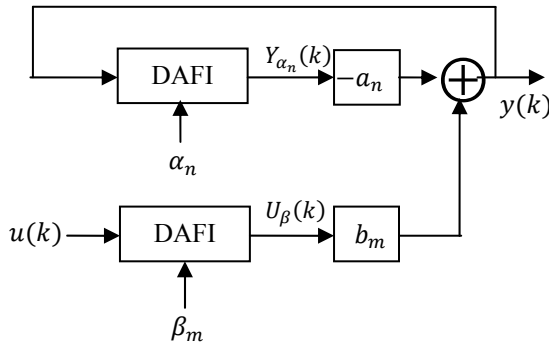


Figure 2. Simulation scheme using DAFI filter.

data using the DAFI presented in section 3. Consequently, a one-step forward linear regression can be obtained using equation (17). Then, the output can be written in a linear regression form as:

$$y(k) = \varphi^t(k)\theta \quad (20)$$

where parameters and regression vectors are respectively given by:

$$\theta = [a_1 \dots a_{n_a}, b_0 \dots b_{m_b}]^t \quad (21)$$

$$\varphi(k) = [-y_{\alpha_1}(k) \dots -y_{\alpha_{n_a}}(k), u_{\beta_0}(k) \dots u_{\beta_{m_b}}(k)]^t \quad (22)$$

Estimated parameters  $\hat{\theta}_k$  can be obtained using the prediction error method minimizing the least squares criterion:

$$\hat{\theta}_k = \arg \min_{\theta} \left\{ \frac{1}{k} \sum_{i=1}^k [y(i) - \hat{y}(i, \theta_k)]^2 \right\} \quad (23)$$

where

$$\hat{y}(i, \hat{\theta}_k) = \varphi^t(i) \hat{\theta}_k \quad (24)$$

Figure 3 summarizes the principle of the proposed technique.

The solution of (23) is given by the classical least squares:

$$\hat{\theta} = (\varphi^t \varphi)^{-1} \varphi^t Y \quad (25)$$

where

$$\varphi = [\varphi(1) \varphi(2) \dots \varphi(k)] \quad (26)$$

$$Y = [y(1) y(2) \dots y(k)]^t \quad (27)$$

A recursive version of (25) denoting by recursive least squares estimation can be given as [23]:

$$\begin{aligned} \hat{\theta}_{k+1} &= \hat{\theta}_k + P_k \varphi(k) \epsilon(k) \\ P_{k+1} &= P_k - \frac{P_k \varphi(k) \varphi^t(k) P_k}{1 + \varphi^t(k) P_k \varphi(k)} \end{aligned} \quad (28)$$

$$\epsilon(k) = \frac{y(k) - \varphi^t(k) \hat{\theta}_k}{1 + \varphi^t(k) P_k \varphi(k)}$$

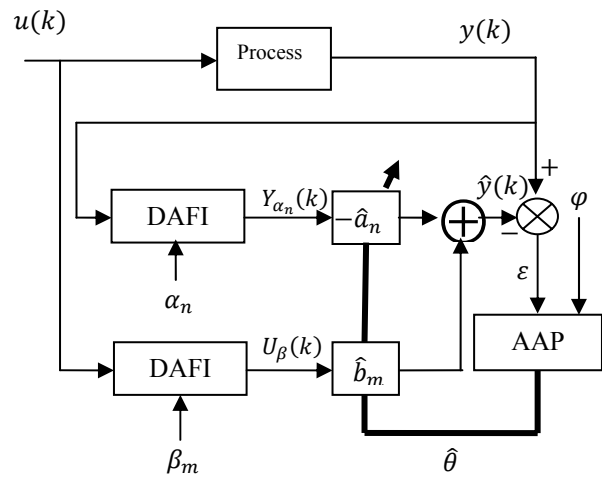


Figure 3. Identification scheme using DAFI filter.

#### VI. ILLUSTRATIVE EXAMPLES

##### A. Example 1

Consider the fractional integrator given by:

$$G_1(s) = \frac{1}{s^{0.25}}$$

This integrator was chosen for comparing the various approximation methods, since the analytical expression of the step response is known as [21]:

$$y_1(t) = \frac{1}{\Gamma(1.25)} t^{0.25}$$

where  $\Gamma$  is the gamma function.

The step response of  $G_1(s)$  can be evaluated using the analytical method, the GL approximation and the proposed DAFI presented in figure 1 within the frequency range  $[10^{-6}, 10^6]$  rad/s and a sampling time  $T = 0.1$ .

Figure 4 shows a comparison between the frequency response of  $G_1$  and its DAFI approximation. It can be seen that the curves are superposed in the frequency band of interest. Figure 5 shows that the step response using the proposed filter is in good agreement with the analytical solution where the GL method fits only its initial values. Also, the simulation with this approach is more than 1000 times faster than the GL method.

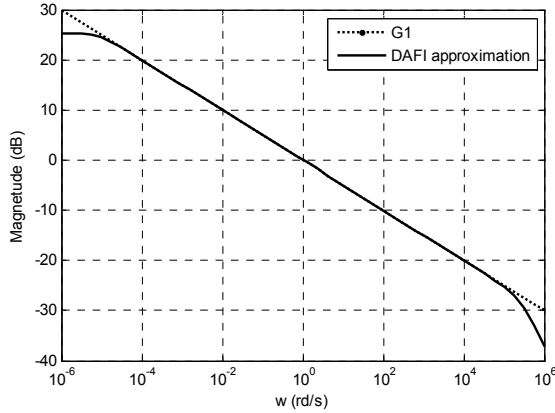


Figure 4. Bode plots of  $G_1$  and its approximation using DAFI.

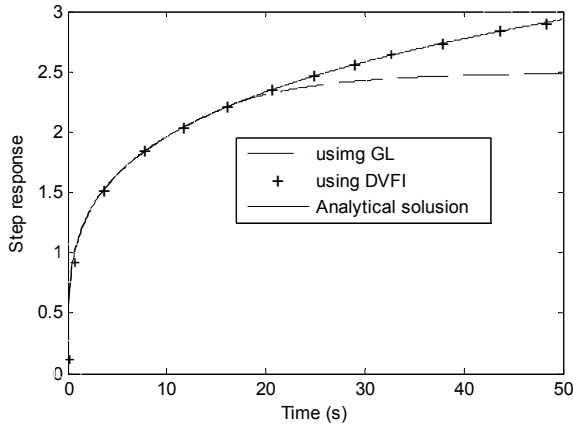


Figure 5. Step responses of  $G_1$  using the GL method, analytical method and the proposed DAFI.

### B. Example 2

Consider the following fractional continuous linear system:

$$G_2(s) = \frac{4}{s^{1.5} + 0.5s + 4}$$

Figure 6 compares the step responses of  $G_2$  using the proposed DAFI and the GL method. It can be seen that the GL method presents a steady state error due to the accumulated approximation error over time where the proposed method reduce it to zero in conformity with the analytical solution.

### C. Example 3

Let us consider the more general fractional continuous linear system:

$$G_3(s) = \frac{s^{0.8} + 4}{s^{1.5} + 0.5s^{0.7} + 4}$$

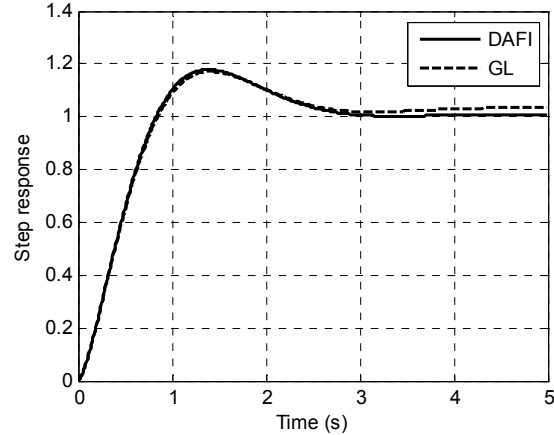


Figure 6. Numerical simulation of  $G_2$  using GL approximation (Dashed) and DAFI (Continued).

Figure 7 shows the step responses with the proposed filter and the GL method. It can be seen that the GL method gives an important steady state error where the ADFI response converges to the true theoretical final value.

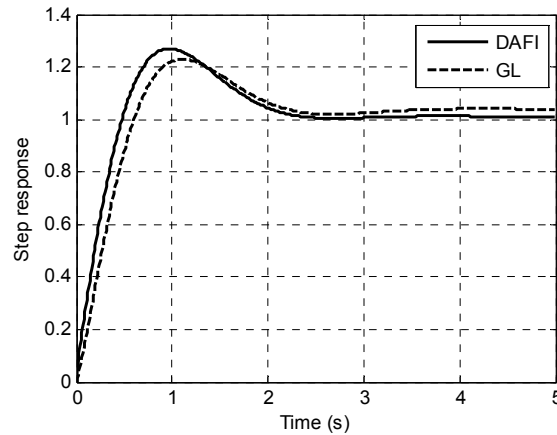


Figure 7. Numerical simulation of  $G_3$  using GL approximation (Dashed) and DAFI (Continued).

### D. Example 4

Let us now consider parameter identification of the following fractional order model:

$$G_4(s) = \frac{b_0}{s^{2\alpha} + a_1s^\alpha + a_2}$$

where  $\alpha = 0.7$ , and  $b_0$ ,  $a_1$  and  $a_2$  are real numbers.

We simulate the fractional order model  $G_4$  with a giving values  $b_0 = 1$ ,  $a_1 = 0.5$  and  $a_2 = 2$  using a pseudo random binary sequence input to generate 500 data pairs  $\{u(k), y(k)\}$ . For this purpose, a frequency interval equal to

12 decades is used, with  $\omega_b = 10^{-6}$  rad/s,  $\omega_h = 10^6$  rad/s and a sampling time  $T = 0.01$ s.

Output are disturbed by a white noise with signal to noise ratio:  $S/N = 20$ dB.

We perform the parameter estimation of  $G_4$  using recursive least squares method with both the proposed DAFI filter and the GL approximation. Results are shown in figure 8 and table I. It can be seen clearly that the proposed method allows us to restore accurately the parameters of the studied system.

In order to study the convergence and the precision of the proposed algorithm, Monte Carlo simulation is performed using 50 realizations of input and output data pairs with the same excitation  $\{u(k)\}$ , but with different white noises with the same ratio  $S/N$ . Obtained results are presented in Table II. We verify that the estimations with DAFI give the best results.

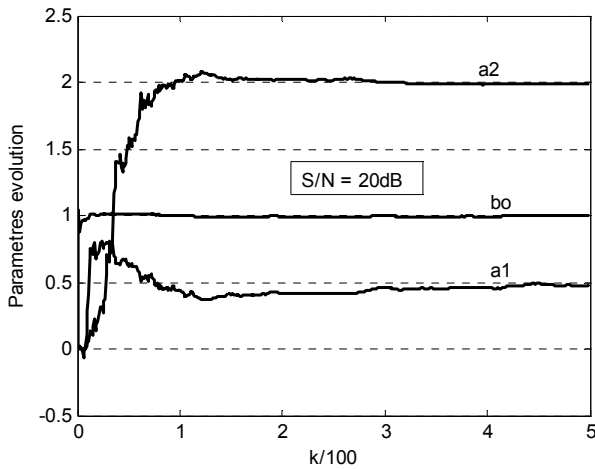


Figure 8. Evolution of the estimated parameters using recursive least square method using DAFI filter.

TABLE I EXACT AND ESTIMATED PARAMETERS OF  $G_4$

	$a_1$	$a_1$	$b_0$
Exact values	0.5	2	1
Estimation with DAFI	0.4781	1.9878	0.9975
Estimation with GL	0.8820	1.5864	0.9769

TABLE II MONTE CARLO SIMULATION (MEAN OF 50 REALIZATIONS)

	$a_1$	$a_1$	$b_0$
Exact values	0.5	2	1
Estimation with DAFI	0.4993	1.9902	1.0040
Estimation with GL	0.7781	1.6878	0.9975

## VII. CONCLUSION

In this paper, a very simple and straightforward design technique for an optimal digital adjustable fractional order integrator s-m, for  $\alpha \in \mathcal{R}^+$ , has been developed based on

discretization of their stable adjustable analogue rational function approximation in a given frequency band of interest.

The main feature of this filter is that their frequency characteristics can be adjusted using only fractional order  $\alpha$  with fixed structure. Consequently, their implementation is simple, stable and useful for numerical simulation of any fractional order system. This technique improves the accuracy, the speed and the stability of the simulation algorithm.

The good results obtained from the four design examples show the efficiency and the effectiveness of the proposed method. This type of implementation will play an important role in the identification of fractional systems, signal processing and control, especially for the online identification of non-stationary systems and tuning of fractional  $PI^\alpha D^\beta$  controllers.

Practical use of embedded processors implementing this filter for real time data acquisition and signal filtering is an important perspective of this work.

## ACKNOWLEDGMENT

This paper has been partly supported by the ANR project CONFOC-AO

## REFERENCES

- [1] Magin, R.L., "Fractional Calculus in Bioengineering," Begell House, Connecticut, 2006.
- [2] Craiem, D. and R.L. Armentano, "A fractional derivative model to describe arterial viscoelasticity," *Biorheology*, 44, 2007, pp 251-263,
- [3] Koh, C.G. and Kelly, J.M., "Application of Fractional Derivatives to Seismic Analysis of Based-Isolated Models", *Earthquake Engineering and Structural Dynamics*, vol. 19, 1990, pp 229-241,.
- [4] M. Ichise, Y. Nagayanagi and T. Kojima, "An Analog Simulation of Non-Integer Order Transfer Functions for Analysis of Electrode Processes," *J. Electro anal. Chem.*, 33, 1971, pp 253-256.
- [5] H. H. Sun and B. Onaral, "A Unified Approach to Represent Metal Electrode Polarization," *IEEE Trans. Biomed. Eng.*, 30, 1983, pp 399-406.
- [6] P. J. Torvik and R. L. Bagley, "On the appearance of the fractional derivative in the behavior of real materials," *ASME Journal of Applied Mech*, 51, 1984, pp 294-298.
- [7] Y. Ferdi, J. P. Herbeuval, A. Charef and B. Boucheham, "wave detection using fractional digital differentiation," *ITBM-RBM Innovation et Technologie en Biologie et Médecine*, vol. 24, N° 5, 2003, pp 273-280.
- [8] B. Mathieu, P. Melchior, A. Oustaloup, C. Ceyral, "Fractional differentiation for edge detection," *Signal Processing*, 83, 2003, pp. 2421-2432.
- [9] A. Goutas, Y. Ferdi, J-P. Herbeuval, M. Boudraa and B. Boucheham, "Digital fractional order differentiation-based algorithm for P and T-waves detection and delineation," *ITBM-RBM*, 26, 2005, pp.127-132.
- [10] A. Oustaloup, *La dérivation non entière: théorie, synthèse et applications*, Hermès, Paris, 1995.
- [11] I. Podlubny, *Fractional Order Systems and  $PI^\alpha D^\beta$  Controllers*, *IEEE Transactions on Automatic Control*, Vol. 44, N°1, 1999, pp 208-214.
- [12] I. Petras, B. M. Vinagre, "Practical Application of Digital Fractional-Order Controller to Temperature Control," *Acta Montanistica Slovaca*, 7 (2), 2002, pp 131-137.
- [13] S. Ladaci and A. Charef, "On Fractional Adaptive Control," *Nonlinear Dynamics*, vol. 43, N° 4, 2006, pp 365-378

- [14] C. A. Monje, B. M. Vinagre, V. Feliu, Y.Q. Chen, "Tuning and Autotuning of Fractional Order Controllers for Industry Applications," *Control Engineering Practice* Vol.16, 2008, pp 798-812.
- [15] A. Oustaloup. et al. , *Représentation et identification par modèle non entier*, LAVOISIER, 2005.
- [16] M. A. Al-Alaoui, M.A., " Novel IIR differentiator from the Simpson Integration rule," *IEEE Transactions on Circuits and Systems I. Fundamental Theory and Applications*, 41 (2), 1994, pp 186-187.
- [17] B. M. Vinagre, I. Podlubny, A. Hernandez, V. Feliu, "Some approximations of fractional order operators used in control theory and applications," *Fractional Calculus & Applied Analysis*, 3(3), 2000, pp. 231–248.
- [18] I. Petras, I. Podlubny, P. O'Leary, L. Dorcak, "Analogue fractional-order controllers: Realization, tuning and implementation," *Proceedings of the ICC'2001*, 2001, pp. 9 - 14, Krynica, Poland.
- [19] Y. Q. Chen, and Kevin L. Moore, "Discretization Schemes for Fractional-Order Differentiators and Integrators. *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, 49(3): 2002, pp 363-367.
- [20] A. Charef and D. Idiou, "Design of analog variable fractional order differentiator and integrator, *Nonlinear Dyn*, DOI 10.1007/s11071-012-0370-x, Springer Science+Business Media B.V. 2012.
- [21] M. Aoun, R. Malti, F. Levron and A. Oustaloup, "Numerical simulations of fractional systems" Proceedings of DETC.03 Chicago, Illinois, USA, September 2-6, 2003.
- [22] A. Charef, "Analogue realization of fractional order integrator, differentiator and fractional  $PI^{\alpha}D^{\beta}$  controllers. *IEE Proc., Control Theory Appl.* 153(6), 2006, pp 714-720.
- [23] I. Landau, R. Lozano, M. M'Saad, *Adaptive Control*. Cambridge, London, England: Springer; 1998.