Linear Quadratic-Based Computed-Torque Control of an Actuated Pendulum with Friction

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***Abstract*—Abstract—In this paper, the trajectory tracking control of an actuated pendulum with friction using LQ-based computed-torque controller is studied. The control scheme consists of feedback linearization loop to cancel nonlinear dynamics and LQ as outer-loop controller. The dynamic model is developed using Euler-Lagrange formulation and the stability behavior is analyzed for both open and closed-loop system. Simulation result show that LQ-based computed-torque controller is effective in terms of computing optimal feedback gains that minimize the position and velocity tracking errors.**

***Index Terms*—optimal control; linear quadratic regulator; computed-torque; stability; pendulum.**

I. Introduction

Robot manipulators are considered one of the most essential elements for industrial automation and manufacturing nowadays, where robots are required to perform various tasks such as welding, painting, packing, assembly applications and so on. Generally, those tasks require following a desired trajectory by robots with high precision and repeatability. However, tracking problems can be challenging, since robot manipulators consists of hard and coupled nonlinearities, and subjected to nonlinear dynamics [2] [6]. As a result it is difficult to design a robust controller that is capable of minimizing trajectory tracking errors and maintaining the energy cost as low as possible. [1] [3]

A common approach to deal with robot trajectory tracking problem is computed-torque control, where the exact dynamics can be used to cancel nonlinearities by transforming the nonlinear system to equivalent linear one. Then, linear control techniques can be implemented as outer-loop controllers such as Proportional-Integral-Derivative (PID) and Iinear Quadratic Regulators (LQR) [2] [7].

PID controller is a commonly used control feedback scheme for industrial applications, due to its simplicity and efficiency of dealing with vast real world control problems. PID controller involves three different parameters, namely, the proportional, integral and derivative gains, which must be tunned to meet the required performance. However, for some high demanding application, it's difficult to select the optimal PID gains that satisfy critical performance.In order to overcome this kind of problems, optimal controllers such as Linear-Quadratic Regulators can be implemented to achieve the best possible performance with respect to the given criteria. [4] [5] [8]

This paper is organized as follows. In section two, the dynamic equations of motion are developed for an actuated pendulum with friction system using Euler-Lagrange formulation, along with stability behavior analysis. In section Three, trajectory tracking problem is formulated and LQ Computed-Torque controller is developed for the given system. the simulation results of the proposed controller are presented in section four,. Finally, conclusions are given in the last section.

II. Mathematical Modeling

A. Dynamics

Consider an actuated pendulum with friction as shown in Fig. 1. The system consists of point-mass m, attached a rigid link of length L (assumed to be of negligible mass), which pivots about the origin $(0,0)$ by an angle θ, under the effect of gravitational force mg, input torque τ, and viscous friction of a coefficient $b > 0$.

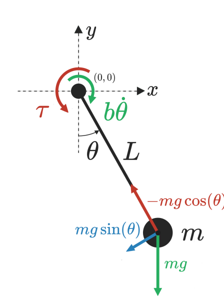

Fig. 1. Actuated pendulum with friction.

Position vector of the point-mass can be obtained as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} L\sin(\theta) \\ -L\cos(\theta) \end{bmatrix} \tag{1}$$

which can be differentiated with respect to time, and having velocity vector as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} L\cos(\theta) \\ L\sin(\theta) \end{bmatrix} \dot{\theta} \tag{2}$$

The Lagrangian $\mathcal{L}$ is defined to be the difference between the kinetic and potential energy $\mathcal{K}$ and $\mathcal{P}$, respectively, where

$$\mathcal{K} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}mL^2\dot{\theta}^2 \tag{3}$$

$$\mathcal{P} = mg(L - y) = mgL\left[1 - \cos(\theta)\right] \tag{4}$$

Using the Lagrange-Eular formulation, the equation of motion can be written as

$$\begin{aligned} \tau &= \frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\theta}}\right) - \frac{d\mathcal{L}}{d\theta} + \tau_f \\ &= mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin(\theta) \end{aligned} \tag{5}$$

or in state-space form as

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) = x \end{aligned} \tag{6}$$

where

$$\begin{aligned} &x = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^T, u = \tau, \\ &f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{L}\sin(\theta) - \frac{b}{mL^2}\dot{\theta} \end{bmatrix}, \\ &g(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{mL^2} \end{bmatrix}. \end{aligned} \tag{7}$$

B. Stability analysis

To conclude the stability behavior of the system, we first find a set of states $\bar{x}$ (equilibrium points), that satisfy $\dot{x} = 0$ in the absence of an input (i.e. $u = 0$). Then, we linearize the system at $\bar{x}$, by deriving the Jacobian matrix from the dynamics and substituting x with $\bar{x}$. The equilibrium points of the system are easily seen to be

$$\bar{x} = (k\pi, 0), k \in \mathbb{Z} \tag{8}$$

and the Jacobian matrix can be obtained as

$$J = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos(\theta) & -\frac{b}{mL^2} \end{bmatrix} \tag{9}$$

Thus, the linearized model of the system at equilibrium points becomes

$$A = \begin{bmatrix} 0 & 1 \\ (-1)^{k+1}\frac{g}{L} & -\frac{b}{mL^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -D & T \end{bmatrix} \tag{10}$$

where T and D represent the trace and determinant of the matrix A, respectively, which yields eigenvalues of the form

$$\lambda_{1,2} = \frac{T}{2} \pm \sqrt{\frac{T^2}{4} - D} \tag{11}$$

The system's stability behavior can be summarized as

- If $D < 0$, the eigenvalues are real and of opposite sign, and the phase portrait is a saddle (which is always unstable)
- If $0 < D < \frac{T^2}{4}$, the eigenvalues are real, distinct, and of the same sign, and the phase portrait is a node, stable if $T < 0$, unstable if $T > 0$
- If $0 < \frac{T^2}{4} < D$, the eigenvalues are neither real nor purely imaginary, and the phase portrait is a spiral, stable if $T < 0$, unstable if $T > 0$

The phase portrait of the system is shown in Fig. 2, assuming that $m = 0.1(kg)$, $L = 0.15(m)$, $g = 9.81(\frac{m}{s^2})$, and $b = 5 \times 10^{-3}(\frac{N.m.s}{rad})$.

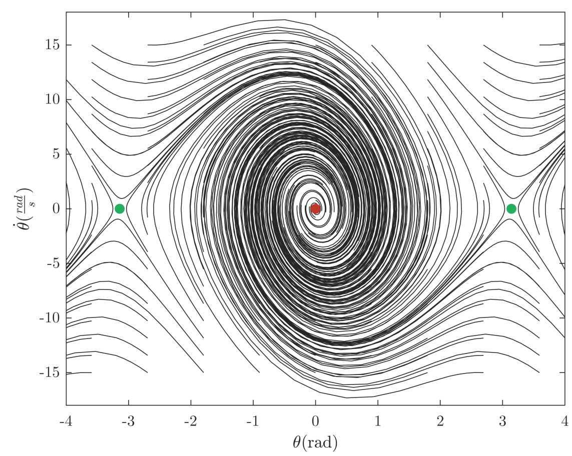

Fig. 2. Phase portrait of an actuated pendulum with friction.

III. Tracking Control Problem

The tracking control problem can be stated as follow: let θ_d, $\dot{\theta}_d$ and $\ddot{\theta}_d$ be a desired trajectory that the pendulum is required to follow. Compute an input u, so that the tracking errors $e = \theta_d - \theta$, $\dot{e} = \dot{\theta}_d - \dot{\theta}$ and $\ddot{e} = \ddot{\theta}_d - \ddot{\theta}$ converge to zero, with all involved signals remain bounded.

A. Computed-Torque Control

A common approach to deal with tracking control problems of robotic systems is computed-torque control (CTC), or inverse dynamics control as referred to in the robotics literature. This approach is based on feedback linearization, which involves using the exact inverse dynamics model to transform the nonlinear system to equivalent linear system, so that linear control techniques can be implemented. Computed-torque control input is straightforward driven from (5) as

$$\begin{aligned} u &= \alpha(x) + \beta(x)v \\ &= mgL\sin(\theta) + b\dot{\theta} + mL^2 v \end{aligned} \tag{12}$$

where v is outer-loop control input. Substituting (12) in (6), yields equivalent open-loop linear system as

$$\begin{aligned} \dot{x} &= Ax + Bv \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \end{aligned} \tag{13}$$

ISBN: 978-9931-9728-1-5

We can easily verify that it has eigenvalues $\lambda_{1,2} = 0$, due to the pair of integral action (Brunovsky canonical form). Therefore, it can be stabilized by state feedback control law $v = -Kx$, on condition that the system is controllable from input v. The controlability matrix can be expressed as

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{14}$$

Having a full rank of $n = 2$, makes the system controllable from input v.

B. LQ Computed-Torque Control

Linear–quadratic regulator (LQR) is an optimal control technique, which provides high performance state-feedback gains in terms of stabilizing the closed-loop system at the origin or an operating point with low cost effort. Given the system in (13), the quadratic performance index for LQR is expressed as

$$J = \frac{1}{2}\int_0^\infty \left(Q_1\theta^2 + Q_2\dot{\theta}^2 + Rv^2\right) dt \tag{15}$$

where $Q_1, Q_2 \geq 0$ and $R > 0$ are parameters to be selected for meeting the required performance. Due to the form of A and B matrices, the solution of the Riccati equation is straightforward, which yields the optimal stabilizing feedback gains as

$$K_1 = \sqrt{\frac{Q_1}{R}}, K_2 = \sqrt{2K_1 + \frac{Q_2}{R}} \tag{16}$$

LQ can be implemented as outer-loop control in computer-torque scheme for the trajectory tracking problem. By selecting a control input

$$v = \ddot{\theta}_d + K_1 e + K_2 \dot{e} \tag{17}$$

and substituting in (13), the closed-loop error dynamics system can be written as

$$\begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K_1 & -K_2 \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix}. \tag{18}$$

with eigenvalues of the form

$$\lambda_{1,2} = -\frac{K_2}{2} \pm \sqrt{\frac{K_2^2}{4} - K_1} \tag{19}$$

Therefore, as long as K_1 and K_2 are strictly positive (implying that $Q_1 > 0$), the close-loop error dynamics system is asymptotically stable.

IV. Simulation and Results

This section demonstrates and discuss the performance of LQ-based computer-torque controller for the trajectory tracking problem. Physical parameters of the studied system are assumed to be $m = 0.1(kg)$, $L = 0.15(m)$, $g = 9.81(\frac{m}{s^2})$, and $b = 5 \times 10^{-3}(\frac{N.m.s}{rad})$. To test the effectiveness of the proposed controller, we have chosen two sets of parameter as $Q_1 = Q_2 = 20, R = 1$ and $Q_1 = Q_2 = 5, R = 10$, which yield equivalent feedback gains as $K = [4.72, 5.38]$ and $K = [0.71, 1.38]$, respectively. Finally, for the purpose of this work, the desired trajectory is chosen as

$$\theta_d = \frac{1}{10}\cos\left(\frac{\pi}{2}t\right), \dot{\theta}_d = -\frac{\pi}{20}\sin\left(\frac{\pi}{2}t\right), \ddot{\theta}_d = -\frac{\pi^2}{4}\theta_d.$$

The control scheme of LQ-based computed-torque is shown in Fig. 3. Using dedicated simulation software, yields the results shown in Fig. 4 and 5, which represent position and velocity trajectory tracking performance for different controller parameters, respectively.

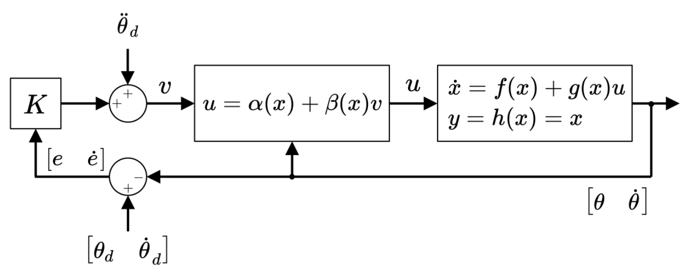

Fig. 3. LQ-based computed-torque control scheme.

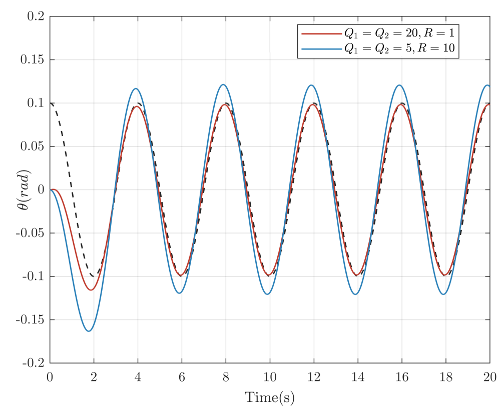

Fig. 4. Position tracking.

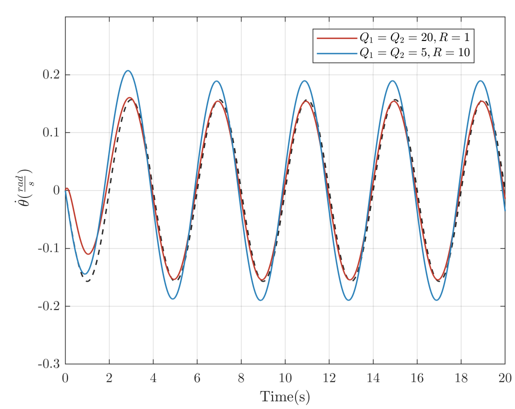

Fig. 5. Velocity tracking.

For $Q_1 = Q_2 = 20$ and $R = 1$, the closed-loop eigenvalues are real, distinct, and negative, which result in a relatively small steady-state tracking error with no overshooting for both the position and velocity. However, in case of $Q_1 = Q_2 = 5$ and $R = 10$, the closed-loop eigenvalues are complex with negative real parts, which result a small overshooting in both the position and velocity tracking performance.

V. Conclusions

In this paper, LQ-based computed-torque controller was presented for trajectory problem of an actuated pendulum with friction system. Simulation results show that the proposed controller is efficient in terms of minimizing tracking errors with considerations of different parameters. As future work for this control technique, LQ parameters will be tuned using deep reinforcement learning for computing optimal feedback gains that minimize both tracking errors and the applied torque.

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