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Common fixed point theorems of several functions in metric spaces and applications to control theory

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Dedication

This study is wholeheartedly dedicated to my beloved parents, who have been my source of inspiration and gave me strength when I thought of giving up, who continually provide theirmoral, spiritual, emotional, support.
For their encouragement and their continual prayers.
To my wife who supported and encourage me to finalize this work.
To my dear daughters HANINE and *NOUHA H. E.*
To my dear sons MOHAMED OUAIL and AHMED OUASSIM.
To my family.
To all my friends.

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Abstract

The purpose of our thesis concentrates on the study of the fixed point theory of mappings, which is one of the most important branches of mathematics, with numerous applications to the solution of various kinds of equations (differential, integral, partial differential), control theory, dynamic programming, game theory, etc.

We were interested in this work to find some applications involving new generalizations theorems of fixed point in a metric space.
Our work includes two principal results:

1. We have established unique fixed point theorems for a self-mapping in complete metric spaces and that the fixed point problem is well-posed. Examples are provided to illustrate the validity of our results and we give some remarks about the papers [3], [4] and [31]. Afterwards, we apply our theorem 3.1 to study the possibility of optimally controlling the solution of an ordinary differential equation via dynamic programming.

2. We have established coincidence and common fixed points theorem for two self-mappings in complete metric spaces. Our theorem generalizes theorem 1 of [31]. Suitable examples are provided to illustrate the validity of our results. We have applied our theorem 4.1 to establish the existence of common solutions of a system of two functional equations arising in dynamic programming.

Keywords: Fixed point, common fixed point, coincidence point, metric space, contractive condition, weakly compatible, control theory, dynamic programming.

ملخص

الهدف من أطروحتنا هو دراسة نظرية النقطة الثابتة للتطبيقات، والتي تعد أحد أهم فروع الرياضيات، مع تطبيقات عديدة لحل أنواع مختلفة من المعادلات (التفاضلية، التكاملية، التفاضلية الجزئية)، نظرية التحكم، البرمجة الديناميكية، نظرية الألعاب، إلخ.

كنا مهتمين في هذا العمل بالعثور على بعض التطبيقات التي تتضمن تعميمات لنظريات جديدة للنقطة الثابتة في فضاء متري.

يتضمن عملنا هذا نتيجتين رئيسيتين:

1. لقد أثبتنا نظريات النقطة الثابتة الوحيدة لتطبيق ذاتي في فضاءات مترية تامة وأن مشكلة النقطة الثابتة مطروحة بشكل جيد ودعمنا هذا بأمثلة لتوضيح صحة نتائجنا وقدمنا أيضا بعض الملاحظات حول الأوراق [3]، [4] و[31]. بعد ذلك، طبقنا النظرية 3 .1 الخاصة بنا لدراسة إمكانية التحكم الأمثل في حل معادلة تفاضلية عادية عبر البرمجة الديناميكية.

2. أثبتنا أيضا نظرية لنقاط التطابق و النقاط الثابتة المشتركة لتطبيقين ذاتيين في فضاءات مترية تامة، حيث تعمل نظريتنا على تعميم النظرية1 من [31]. ولقد قدمنا أمثلة مناسبة لتوضيح صحة نتائجنا، وكذلك طبقنا نظريتنا 4 .1 لإثبات وجود حلول مشتركة لنظام من معادلتين دالتين ناشئتين في البرمجة الديناميكية.

الكلمات المفتاحية: النقطة الثابتة، النقطة الثابتة المشتركة، نقطة التطابق، الفضاء المتري، شرط التقلص، التطبيقات المتوافقة بضعف، نظرية التحكم، البرمجة الديناميكية.

Résumé

Le but de notre thèse se concentre sur l'étude de la théorie du point fixe des applications, qui est l'une des branches les plus importantes des mathématiques, avec de nombreuses applications à la résolution de divers types d'équations (différentielle, intégrale, différentielle partielle), théorie du contrôle, programmation dynamique, théorie des jeux, etc.

Nous nous sommes intéressés dans ce travail de trouver des applications impliquant de nouveaux théorèmes de point fixe de généralisations dans un espace métrique.

Notre travail comprend deux résultats principals:

1. Nous avons prouvé des théorèmes de point fixe uniques pour une auto-applications dans des espaces métriques complets et que le problème du point fixe est bien posé. Ensuite, des exemples sont fournis pour illustrer la validité de notre résultats et nous avons donné quelques remarques sur les articles [3], [4] and [31]. Ensuite, nous avons appliqué notre théorème 3.1 pour étudier la possibilité de contrôler de manière optimale la solution d'une équation différentielle ordinaire via la programmation dynamique.

2. Nous avons prouvé un théorème des points fixes communs et coincidence pour deux auto-applications dans des espaces métriques complets. Notre théorème généralise le théorème 1 de [31]. Des exemples appropriés sont fournis pour illustrer la validité de nos résultats. Nous avons appliqué notre théorème 4.1 pour établir l'existence de solutions communes d'un système de deux équations fonctionnelles surgissant en programmation dynamique.

Mots-clé: Point fixe, point fixe commun, point de coincidence, espace métrique, contractive condition, faiblement compatible, theorie de contrôle, programmation dynamique.

Contents

General introduction

What does fixed point mean?

We say that $u \in X$, X is a nonempty set, is a fixed point of $T : X \to X$ if $T(u) = u$, in other words, the action of the map T on the point u leaves it invariant.

The theorem which guarantees the existence of this stationary point is the fixed point theorem. The interest of this theorem goes far beyond the purely geometric framework, where we find its applications in various fields of mathematics, mathematical physics, chemical reactions, economic theory, etc.

The theorem is now 108 years old, but it is not that old. its founder is the mathematician L. Brouwer (1881-1966).

However, the most essential result in the fixed point theorem reported in literature started from the famous Banach's fixed point theorem called Banach contraction principle, see [7]. The author of this very constructive result is S. Banach (1922). This theorem was a concretization of previous work in particular those of E. Picard who had long before used the method of successive approximations to solve many problems expressed in terms of differential, integral, or partial derivative equations.

Furthermore, this theorem provided a technique for solving a variety of applied problems in mathematical sciences and engineering. Subsequently, the superb result of Banach was extended and generalized by several authors using various contractive conditions in different spaces.

On the other hand, Connell [19] gave an example of a metric space (X, d) such that (X, d) is not complete and every contraction on X has a fixed point. Thus, Banach's fixed point theorem cannot characterize the metric completeness of (X, d). A mapping T on a metric space (X, d) is called Kannan if there exists $\alpha \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \alpha(d(x, Tx) + d(y, Ty)) \text{ for all } x, y \in X.$$

In the year 1968, Kannan [29] proved that if (X, d) is complete, then T has a unique fixed point in X. Kannan [30] provided examples which show that Kannan's fixed point theorem is independent of the Banach contraction principle and Kannan mapping does not have to be continuous. Kannan's theorem is also very interesting because Subrahmanyam [61] demonstrated that Kannan's theorem characterizes the metric completeness, that is a metric space (X, d) is complete if and only if every Kannan mapping on X has a fixed point. Several authors generalized Kannan's fixed point theorem, see [3], [17], [20], [22], [23], [24], [25], [34], [43], [47] and [50].

Suzuki [62] categorized the theorems which ensure the existence of a fixed point of a mapping T into the following four types.

(T_1) Leader-type [35]: T has a unique fixed point and $\{T^n x\}$ converges to the fixed point for all $x \in X$. Such a mapping is called a Picard operator PO, see Picard [48] and Rus [56, 55].

(T_2) Unnamed type: T has a unique fixed point and $\{T^n x\}$ does not necessarily converge to the fixed point.

(T_3) Subrahmanyam-type [61]: T may have more than one fixed point and $\{T^n x\}$ converges to a fixed point for all $x \in X$. Such a mapping is called a weakly Picard operator WPO, see Rus [54, 57].

It is evident that any PO is a WPO, but the converse is not true in general, see Berinde [10].

(T_4) Caristi-type [12, 13]: T may have more than one fixed point and $\{T^n x\}$ does not necessarily converge to a fixed point.

Inspired by the above, we were able to produce some results in the field of fixed point theory, especially in coincidence and common fixed points theorem in the metric space. We were also able to make some remarks. As part of this thesis, we have published research papers in two different journals.

This work is made up of four chapters, the first chapter, as in any thesis, contains the essential elements that we will need for the following chapters. In this chapter we will recall some metric fixed point theorems of one mapping. While, when handling more than one function, it requires some commutativity and compatibility between these functions. Also, we will recall definitions of Picard-Jungck operators types's, the control theory and dynamic programming, etc.

In the second chapter, we will present two new types of fixed point theorems of single-valued mappings which were showed by Khojasteh et al. [31] and Rhoades [52], which belongs to (T_3).

In the third chapter, we will find our work published in [6]. In this publication, inspired by theorem 2.1, it is our purpose in this chapter to prove unique fixed point theorems for a self-mapping in complete metric spaces and that the fixed point problem is well-posed. Examples are furnished to illustrate the validity of our results and we give some remarks about the papers, [3], [4] and [31]. Also, we applied the theorem 3.2 to study the possibility of controlling the solution of an ordinary differential equation by dynamic programming.

The fourth chapter is dedicated to present our work in [5], where we demonstrate coincidence and common fixed points theorem for two self-mappings in complete metric spaces. Our theorem generalizes theorem 1 of [31]. Examples are furnished to illustrate the validity of our results and we have applied theorem 4.1 to establish the existence of common solutions of a system of two functional equations arising in dynamic programming.

Chapter 1

Preliminaries

The aim of this chapter is to recall the concept of contractive mappings types's, some metric fixed point theorems, commutative and compatible mappings and definitions of partially ordered set, fixed point, coincidence point, common fixed point, weakly compatible of mappings, Picard-Jungck operators types's. Also, to present the concepts of viscosity solutions, uniqueness, control theory and dynamic programming.

The chapter is organized as follows:

1. Contractivity and compatibility
2. Picard-Jungck operators
3. Hamilton-Jacobi equation

1.1 Contractivity and compatibility of mappings

1.1.1 Contractive mappings types's

Definition 1.1 *Let (X, d) be a metric space. A mapping $A : X \to X$ is said to be Lipschitz if there exists a real number $k \geqslant 0$ such that for all $x, y \in X$ we have*

$$d(Ax, Ay) \leq kd(x, y),$$

a) The smallest k for which the above inequality holds is the Lipschitz constant of A, *b) If $k < 1$, we say that A is a contraction,*

c) If $k = 1$, we say that A is nonexpansive,

d) Finally, A is said to be contractive if for all $x, y \in X$ and $x \neq y$ we obtain

$$d(Ax, Ay) < d(x, y).$$

Remark 1.1 *Contraction implies contractive implies non-expansive implies Lipschitz and all these functions are uniformly continuous.*

1.1.2 Some metric fixed point theorems

The purpose of this paragraph is to recall some metric fixed point theorems as theorem Brouwer, Schauder, many variations of the Banach contraction principle and the results of these extensions have been established by several authors, among them Boyd-Wong, Meir-Keeler, and others. Also, we present the theorem of Kannan of fixed point for non-continuous mappings.

Theorem 1.1 *(L. Brouwer 1910) Let $K \subset X$ be a nonempty, compact and convex part of $\mathbb{R}^n$ and $f : K \to K$ a continuous function. Then there exists $x \in X$ such that $f(x) = x$.*

Theorem 1.2 *(S. Banach 1922) [7] Let A be a contraction on a non-empty complete metric space (X, d). Then, A has unique fixed point $x \in X$.*

Proof. Define the iterate sequence $x_{n+1} = A(x_n)$. By induction on n, we have

$$d(x_{n+1}, x_n) \leq k^n d(x_1, x_0).$$

If $n \in \mathbb{N},\ m \geqslant 1$, we get

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, x_{n+m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq (k^{n+m} + \cdots + k^n) d(x_1, x_0) \\ &\leq \frac{k^n}{1-k} d(x_1, x_0). \end{aligned}$$

Therefore

$$|d(x_{n+m}, x_n)| \leqslant \frac{k^n}{1-k}|d(x_1, x_0)| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, x_n is a Cauchy sequence and admits a limit $z \in X$ because X is complete. Since A is continuous, we have $S(z) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} x_{n+1} = z$. The uniqueness. Suppose that $w \in X$ is another fixed point of A. So

$$d(z, w) = d(S(z), S(w)) \leq kd(z, w),$$

thus,

$$d(z, w) \leq 0,$$

which is contradiction, then A has a unique fixed point. ■

The Schauder fixed-point theorem is an extension of the Brouwer fixed-point theorem to topological vector spaces, which may be of infinite dimension.

Theorem 1.3 *(J. Schauder 1930) Let K be a nonempty, compact convex closed subset of a Hausdorff topological vector space V and T is a continuous mapping of K into itself such that $T(K)$ is contained in a compact subset of K, then T has a fixed point.*

D. Boyd and J. Wong in 1969 have attempted to generalize Banach contraction principle by replacing the Lipschitz constant by some real valued functions.

Theorem 1.4 *(Boyd-Wong 1969) Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying the following $\varphi-$contraction*

$$\forall x, y \in X, d(T(x), T(y) \leq \varphi(d(x, y)).$$

Then T has a unique fixed point $u \in X$ and

$$\lim_{n\to\infty} T^n(x) = u \textit{ for all } x \in X.$$

The next theorem extends Boyd-Wong's result, it generalizes the Banach contraction principle.

Theorem 1.5 *(Meir-Keeler 1969) [42] Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying the following condition: $\forall \epsilon > 0$, $\exists \delta > 0$ such that:*

$$\epsilon \leq d(x; y) < \epsilon + \delta \Longrightarrow d(T(x), T(y)) < \epsilon.$$

Then T admits a unique fixed point u in X. Moreover, for all $x \in X$.

Theorem 1.6 *(J. caristi 1976) [12] Let (X, d) be a complete metric space and a mapping $T : X \to X$ satisfying the following condition: there is a function: $\phi : X \to \mathbb{R}_+$ semi-continuous below such that:*

$$d(x, Tx) \leq \phi(x) - \phi(Tx) \text{ for all } x \in X.$$

Then T admits a fixed point.

Now we present the theorem of Kannan. It was the first result in literature which guarantees the existence and uniqueness of fixed points for non-continuous mappings.

Theorem 1.7 *(R. kannan 1968) [29] Let (X, d) is complete metric space and a mapping $T : X \to X$. Suppose there exists $\alpha \in [0, 1/2)$ such that:*

$$d(Tx, Ty) \leq \alpha(d(x, Tx) + d(y, Ty)) \text{ for all } x, y \in X.$$

Then T has a unique fixed point in X.

1.1.3 Commutative and compatible mappings

The aim of this paragraph is to recall the concept of commutative mappings, weakly commutative mappings, compatible mappings, partially ordered set, fixed point, coincidence point, common fixed point and weakly compatibility of mappings.

Definition 1.3 *[26] Let A and S two self-mappings of a metric space (X, d). A and S are said to be commutative if $ASx = SAx$ for all $x \in X$.*

Definition 1.4 *[59] Let A and S two self-mappings of a metric space (X, d). A and S are said to be weakly commutative, if for all $x \in X$*

$$d(ASx, SAx) \leq d(Ax, Sx).$$

Obviously, commutative implies weakly commutative, but the converse is not true in general as it is shown in the example below.

Example 1.1 *Let $(X, d) = ([0,1], |.|)$ and d is the usual metric. Define two mappings $A, S : X \to X$ as follows*

$$A(x) = \frac{x}{2} \text{ and } S(x) = \frac{x}{2+z} \text{ for all } x \in X.$$

We get

$$\begin{aligned} |ASx - SAx| &= \left| \frac{x}{4+2x} - \frac{x}{4+x} \right| \\ &= \frac{x}{(4+x)(4+2x)} \\ &\leq \frac{x^2}{4+2x} \\ &= \frac{x}{2} - \frac{x}{2+x} \\ &= |Ax - Sx| . \end{aligned}$$

Therefore, A and S are weakly commutative, but

$$ASx = \frac{x}{4+x} > \frac{x}{4+2x} = SAx,$$

So, A and S are not commutative.

Definition 1.5 *[27] A and S are said to be compatible if*

$$\lim_{n \to \infty} d(ASx_n, SAx_n) = 0,$$

for all sequence $\{x_n\}$ in X satisfying

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \textit{ for some } t \in X.$$

Remark 1.2 *If A and S are weakly commutative, then they are weakly commutative, therefore compatible, but the converse does not hold in general.*

Example 1.2 *Let $X = \mathbb{R}$ and d be the Euclidean metric. Define two mappings $A, S : X \to X$ as follows*

$$A(x) = 2x - 1 \textit{ and } S(x) = \frac{x+1}{2} \textit{ for all } x \in X.$$

Let the sequence $\{x_n\}$ in X for all $n \geq 0$ define by $x_n = 1 + \frac{1}{n}$. We have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1$$

and

$$\begin{aligned} \lim_{n\to\infty} ASx_n &= \lim_{n\to\infty} A(1 + \frac{1}{2n}) = 1, \\ \lim_{n\to\infty} SAx_n &= \lim_{n\to\infty} S(1 + \frac{1}{n}) = 1. \end{aligned}$$

Thus, A and S are compatible.

Definition 1.6 *[14] Let $(E, \leq)$ be a partially ordered set and F a subset of E. F is said to be well ordered if every two elements of F are comparable.*

Definition 1.7 *Let $A, S : X \to X$ be two mappings. A point $u \in X$ is said to be*

i) a fixed point of A if $Au = u$,

ii) a coincidence point of A and S if $Au = Su$. The point $z = Au = Su$ is called a point of coincidence of A and S.

iii) a common fixed point of A and S if $Au = Su = u$.

Definition 1.8 *A and S are weakly compatible if they commute at their coincidence point.*

We denote by $C(A,S)$ the set of coincidence points of A and S.

Proposition 1.1 *[1] Let $A,S : X \to X$ be two mappings. If A and S have a unique point of coincidence $z = Au = Su$ and A and S are weakly compatible, then z is the unique common fixed point of A and S.*

1.2 Picard-Jungck operators

1.2.1 Jungck iterative scheme

Definition 1.9 *[60] Let X be a Banach space, Y a subset of X and $A,S : Y \to Y$ such that $A(Y) \subset S(Y)$. For $x_0 \in Y$, consider the following iterative scheme:*

$$Sx_{n+1} = Ax_n,\ n \in \mathbb{N}.$$

For $Y = X$, this scheme is called Jungck iterative scheme. It was introduced by Jungck [26] in 1976 and it reduces to the Picard iterative scheme when $S = I_X$, where I_X is the identity mapping in X.

1.2.2 Picard-Jungck operators types's

Inspired by Chandok et al. [15], we state the following definitions. In the sequel, (X,d) is a metric space and $A,S : X \to X$ are two mappings such that $A(X) \subset S(X)$.

Definition 1.10 *A and S are called Picard-Jungck operators (briefly PJO) if:*

i) A and S have a unique common fixed point z.

ii) The sequence $\{Sx_n\}$ converges to z for each $x \in X$.

If $S = I_X$, where I_X is the identity mapping in X, we obtain the definition of PO. Most of the operators such as Abbas and Khan's [2], Berinde's (theorems 3.3, 3.4) [9], Jungck's [26] and Sessa's [59] are PJO. Further, If the operators T and f in theorem 2.1 of Chandok and Karapinar [14] are weakly compatible and the set of common fixed points of T and f is well ordered, therefore T and f are PJO, see theorem 2.2 of Chandok and Karapinar [14].

Definition 1.11 *A and S are said to be weakly Picard-Jungck operators (briefly WPJO) if:*

i) A and S have at least one common fixed point.

ii) The sequence $\{Sx_n\}$ converges to a common fixed point for any $x \in X$.

If $S = I_X$, where I_X is the identity mapping in X, we get the definition of WPO.

In addition, if the operators T and f in theorem 2.1 of Chandok and Karapinar [14] are weakly compatible, so T and f are WPJO.

Definition 1.12 *A and S are called Quasi Picard-Jungck operators (briefly QPJO) if:*

i) A and S have a unique point of coincidence z or coincidence point.

ii) The sequence $\{Sx_n\}$ converges to z for each $x \in X$.

If $S = I_X$, where I_X is the identity mapping in X, we get the definition of PO.

Remark 1.3 *i) If A and S commute at z, by proposition 1.1, A and S have a unique common fixed point z and so A and S become PJO.*

ii) If we remove, the condition of weak compatibility of A and S, in theorems of Abbas and Khan [2], Berinde [10], Jungck [26] and Sessa [59], then A and S become QPJO.

Definition 1.13 *A and S are said to be Quasi weakly Picard-Jungck operators (briefly QWPJO) if:*

i) A and S have at least one point of coincidence or coincidence point.

ii) The sequence $\{Sx_n\}$ converges to a point of coincidence for any $x \in X$.

If $S = I_X$, where I_X is the identity mapping in X, we obtain the definition of WPO. The operators in theorem 3.1 of Berinde [9], theorems 2.4, 2.6 of Nashine and Samet [44] and theorem 2.1 of Chandok and Karapinar [14] are QWPJO.

Definition 1.14 *[51] Let (X, d) be a metric space and $T : X \to X$ a mapping. The fixed point problem of T is said to be well-posed if*

i) T has a unique fixed point z in X,

ii) for any sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} d(Ty_n, y_n) = 0$, we have $\lim_{n\to\infty} d(y_n, z) = 0$.

1.3 Hamilton-Jacobi equations

In this section, we will present the concepts of viscosity solutions, uniqueness, control theory and dynamic programming.

1.3.1 Viscosity solutions

This paragraph investigates the existence and the uniqueness of appropriately defined weak solutions of the initial-value problem for the Hamilton-Jacobi equation:

$$\begin{cases} u_t + H(Du, x) = 0 \ \text{ in } \ \mathbb{R}^n \times (0, \infty), \\ u = g \ \text{ on } \ \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{1.1}$$

Here the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is given, as is the initial function $g : \mathbb{R}^n \to \mathbb{R}$. The unknown is

$u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$, $u = u(x, t)$ and $Du = D_x u = (u_{x_1}, , ..., u_{x_n})$. We will write $H = H(p, x)$, so that "p" is the name of the variable for which we substitute the gradient Du in the PDE. In this section, we consider the general case that H depends also on x and more importantly is no longer necessarily convex in the variable p.

Definition 1.15 *A bounded, uniformly continuous function u is called a viscosity solution of the initial-value problem* (1.1) *for the Hamilton-Jacobi equation provided:*

$$u = g \quad \textit{on} \ \ \mathbb{R}^n \times \{t = 0\} \tag{i}$$

and

$$\textit{for each } v \in C^\infty(\mathbb{R}^n \times (0, \infty)), \tag{ii}$$

$$\begin{cases} \textit{if } u - v \textit{ has a local maximum at a point } (x_0, t_0) \in \mathbb{R}^n \times (0, \infty), \\ \textit{then } v_t(x_0, t_0) + H(Dv(x_0, t_0), x_0) \leq 0 \end{cases} \tag{1.2}$$

and

$$\begin{cases} \textit{if } u-v \textit{ has a local maximum at a point } (x_0,t_0) \in \mathbb{R}^n \times (0,\infty), \\ \qquad \textit{then } v_t(x_0,t_0) + H(Dv(x_0,t_0),x_0) \geq 0. \end{cases} \tag{1.3}$$

For the uniqueness of the viscosity solution of (1.1), assuming that the Hamiltonian H satisfies the subsequent conditions

$$\begin{aligned} |H(p,x) - H(q,x)| &\leq C\,|p-q|\,, \\ |H(p,x) - H(p,y)| &\leq C\,|x-y|\,(1+|p|)\,, \end{aligned} \tag{1.4}$$

for all $x, y, p, q \in \mathbb{R}^n$ and some constant $C \geq 0$.

To be slightly more general, let us fix a time $T > 0$ and consider the problem

$$\begin{cases} u_t + H(Du,x) = 0 \ \text{ in } \ \mathbb{R}^n \times (0,T], \\ \qquad u = g \ \text{ on } \ \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{1.5}$$

Theorem 1.8 *Under the assumption* (1.4)*, there exists at most one viscosity solution of* (1.5).

Proof. See [21]. ∎

1.3.2 Control theory and dynamic programming

Introduction to control theory

We will now study the possibility of optimally controlling the solution x of the ordinary differential equation

$$\begin{cases} \dot{x}(s) = f(x(s),\alpha(s)), \quad t < s < T, \\ \qquad x(t) = x. \end{cases} \tag{1.6}$$

Here $\cdot = \dfrac{d}{ds}$, $T > 0$ is a fixed terminal time and $x \in \mathbb{R}^n$ is a given initial point taken on by our solution x at the starting time $t \geq 0$. At later times $t < s < T$, x evolves according to the ODE, where $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ is a given bounded, Lipschitz continuous function and A is some given compact subset of $\mathbb{R}^m$. The function α appearing in (1.6) is a control, that is, some appropriate scheme for adjusting parameters from the set A as time evolves, there by affecting the dynamics of the system modelled by (1.6). Let us write

$$Ad = \{\alpha : [0,T] \to A : \alpha \text{ is measurable}\}, \tag{1.7}$$

to denote the set of admissible controls. Then since

$$|f(x,a)| \leq C,\ \ |f(x,a) - f(y,a)| \leq C|x-y|\ (x, y \in \mathbb{R}^n, a \in A), \tag{1.8}$$

for some constant C, we see that for each control $\alpha \in Ad$, the ODE (1.6) has a unique, Lipschitz continuous solution $x = x^{\alpha}$, existing on the time interval $[t,T]$ and solving the ODE for a.e. time $t < s < T$. We call x the response of the system to the control α and x the state of the system at time s.

Our goal is to find control α^* which optimally steers the system. In order to define what "optimal" means however, we must first introduce a cost criterion. Given $x \in \mathbb{R}^n$ and $0 \leq t \leq T$, let us define for each admissible control $\alpha \in Ad$ the corresponding cost

$$C_{x,t}(\alpha) := \int_t^T h(x(s), \alpha(s))ds + g(x(T)), \tag{1.9}$$

where $x = x^{\alpha}$ solves the ODE (1.6) and $h : \mathbb{R}^n \times\ A \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ are given functions. We call h the running cost per unit time and g the terminal cost, and will henceforth assume

$$\begin{cases} |h(x,a)|,\ \ |g(x)| \leq C, \\ |h(x,a) - h(y,a)|,\ \ |g(x) - g(y)| \leq C|x-y|, \\ x, y \in \mathbb{R}^n, a \in A, \end{cases} \tag{1.10}$$

for some constant C. Given now $x \in \mathbb{R}^n$ and $0 \leq t \leq T$, we would like to find if possible a control α^* which minimizes the cost functional (1.9) among all admissible controls.

Dynamic programming

The method of dynamic programming investigates the above problem by turning attention to the value function

$$u(x,t) := \inf_{\alpha \in Ad} C_{x,t}(\alpha), \quad x \in \mathbb{R}^n, 0 \le t \le T. \tag{1.11}$$

The plan is this: having defined $u(x,t)$ as the least cost given we start at the position x at time t, we want to study u as a function of x and t. We are therefore embedding our given control problem (1.6) and (1.9) into the larger class of all such problems, as x and t vary. This idea then can be used to show that u solves a certain Hamilton–Jacobi type PDE and finally to show conversely that a solution of this PDE helps us to synthesize an optimal feedback control. Hereafter, we fix $x \in \mathbb{R}^n$, $0 \le t < T$.

Theorem 1.9 *(Optimality conditions). For each $h > 0$ so small that $t + h \le T$,*

$$u(x,t) := \inf_{\alpha \in Ad} \left\{ \int_t^{t+h} h(x(s), \alpha(s))ds + u(x(t+h), t+h) \right\}, \tag{1.12}$$

where $x(\cdot) = x^{\alpha(\cdot)}(.)$, solves the ODE (1.6) *for the control $\alpha(\cdot)$.*

Proof. Choose any control $\alpha_1 \in Ad$ and solve the ODE

$$\begin{cases} \dot{x}_1(s) = f(x_1(s), \alpha_1(s)), \quad (t < s < t+h), \\ x_1(t) = x. \end{cases} \tag{1.13}$$

Fix $\epsilon > 0$ and choose then $\alpha_2 \in Ad$ so that

$$u(x_1(t+h), t+h) + \epsilon > \int_{t+h}^T h(x_2(s), \alpha_2(s))ds + g(x_2(T)), \tag{1.14}$$

where

$$\begin{cases} \dot{x}_2(s) = f(x_2(s), \alpha_2(s)), \quad (t+h < s < T), \\ x_2(t+h) = x_1(t+h). \end{cases} \tag{1.15}$$

Now define the control

$$\alpha_3(s) := \begin{cases} \alpha_1(s) \text{ if } (t \le s < t+h), \\ \alpha_2(s) \text{ if } (t+h \le s \le T) \end{cases} \tag{1.16}$$

and let

$$\begin{cases} \dot{x}_3(s) = f(x_3(s), \alpha_3(s)), \quad (t < s < T), \\ x_3(t) = x. \end{cases} \tag{1.17}$$

By the uniqueness of solutions to the differential equation (1.6), we have

$$x_3(s) := \begin{cases} x_1(s) \text{ if } (t \le s \le t+h), \\ x_2(s) \text{ if } (t+h \le s \le T). \end{cases} \tag{1.18}$$

Thus the definition(1.11) implies

$$\begin{aligned} u(x,t) &\le C_{x,t}(\alpha_3) \\ &= \int_t^T h(x_3(s), \alpha_3(s))ds + g(x_3(T)) \\ &= \int_t^{t+h} h(x_1(s), \alpha_1(s))ds + \int_{t+h}^T h(x_2(s), \alpha_2(s))ds + g(x_2(T)) \\ &\le \int_t^{t+h} h(x_1(s), \alpha_1(s))ds + u(x_1(t+h), (t+h)) + \epsilon, \end{aligned}$$

the last inequality resulting from (1.14). As $\alpha_1 \in Ad$ was arbitrary, we conclude

$$u(x,t) \le \inf_{\alpha \in Ad} \int_t^{t+h} h(x(s), \alpha(s))ds + u(x(t+h), (t+h)) + \epsilon, \tag{1.19}$$

$x = x^\alpha$ solves (1.6).

2. Fixing again $\epsilon > 0$, select now $\alpha_4 \in Ad$ so that

$$u(x,t) + \epsilon \ge \int_t^T h(x_4(s), \alpha_4(s))ds + g(x_4(T)), \tag{1.20}$$

where

$$\begin{cases} \dot{x}_4(s) = f(x_4(s), \alpha_4(s)), \quad (t < s < T), \\ x_4(t) = x. \end{cases} \tag{1.21}$$

Observe then from (1.11) that

$$u(x_4(t+h), t+h) \leq \int_{t+h}^{T} h(x_4(s), \alpha_4(s))ds + g(x_4(T)). \tag{1.22}$$

Therefore

$$u(x,t) + \epsilon \geq \inf_{\alpha \in Ad} \{ \int_{t}^{t+h} h(x(s), \alpha(s))ds + u(x\,(t+h)\,, (t+h))\}, \tag{1.23}$$

$x = x^{\alpha}$ solves (1.6). This inequality and (1.19) complete the proof of (1.12). ■

Hamilton-Jacobi-Bellman equation

Our eventual goal is writing down a PDE an "infinitesimal version"of the optimality conditions (1.12). But first we must check that the value function u is bounded and Lipschitz continuous.

Lemma 1.10 *(Estimates for a value function). There exists a constant $C \geq 0$ such that*

$$\begin{aligned} |u(x,t)| &\leq C, \\ \left|u(x,t) - u(\hat{x}, \hat{t})\right| &\leq C(|x - \hat{x}| + \left|t - \hat{t}\right|) \end{aligned}$$

for all $x, \hat{x} \in \mathbb{R}^n, 0 < t, \hat{t} < T$.

Proof. 1. Clearly hypothesis (1.10) implies u is bounded on $\mathbb{R}^n \times [0, T]$.

2. Fix $x, \hat{x} \in \mathbb{R}^n$, $0 \leq t < T$. Let $\epsilon > 0$ and then choose $\hat{\alpha} \in Ad$ so that

$$u(\hat{x}, t) + \epsilon \geq \int_{t}^{T} h(\hat{x}(s), \hat{\alpha}(s))ds + g(\hat{x}(T)), \tag{1.24}$$

where $\hat{x}$ solves the ODE

$$\begin{cases} \dot{\hat{x}}(s) = f(\hat{x}(s), \hat{\alpha}(s)), \quad t < s < T, \\ \hat{x}(t) = \hat{x}. \end{cases} \tag{1.25}$$

Then

$$\begin{aligned} u(x,t) - u(\hat{x}, t) \;\; &\leq \int_{t}^{T} h(x(s), \hat{\alpha}(s))ds + g(x(T)) \\ &\quad - \int_{t}^{T} h(\hat{x}(s), \hat{\alpha}(s))ds - g(\hat{x}(T)) + \epsilon, \end{aligned} \tag{1.26}$$

where x solves

$$\begin{cases} \dot{x}(s) = f(x(s), \hat{\alpha}(s)), \quad (t < s < T), \\ x(t) = x. \end{cases} \tag{1.27}$$

Since f is Lipschitz continuous, (1.25), (1.27) and Gronwall's inequality (pages 634-635 in [21]) imply

$$|x(s) - \hat{x}(s)| < C\,|x - \hat{x}|\,, \;\; t \leq s \leq T.$$

Hence we deduce from (1.10) and (1.25) that

$$u(x,t) - u(\hat{x},\hat{t}) \leq C\,|x - \hat{x}| + \epsilon.$$

The same argument with the roles of x and $\hat{x}$ reversed implies

$$\left|u(x,t) - u(\hat{x},\hat{t})\right| \leq C\,|x - \hat{x}|\,, (x, \hat{x} \in \mathbb{R}^n, 0 \leq t \leq T).$$

3. Now let $x \in \mathbb{R}^n$, $0 \leq t < \hat{t} \leq T$. Take $\epsilon > 0$ and choose $\alpha \in Ad$ so that

$$u(x,t) + \epsilon \geq \int_t^T h(x(s), \alpha(s))ds + g(x(T)),$$

x solving the ODE (1.6). Define

$$\hat{\alpha}(s) := \alpha(s + t - \hat{t}) \;\text{ for }\; \hat{t} \leq s \leq T$$

and let $\hat{x}$ solve

$$\begin{cases} \dot{\hat{x}}(s) = f(\hat{x}(s), \hat{\alpha}(s)), \quad \hat{t} < s < T, \\ \hat{x}(\hat{t}) = x. \end{cases}$$

Then $\hat{x}(s) = x(s + t - \hat{t})$. Hence

$$\begin{aligned} u(x,\hat{t}) - u(x,t) &\leq \int_t^T h(\hat{x}(s),\hat{\alpha}(s))ds + g(\hat{x}(T)) \\ &\quad - \int_t^T h(x(s),\alpha(s))ds - g(x(T)) + \epsilon \\ &= -\int_{T+t-\hat{t}}^T h(x(s),\alpha(s))ds + g(x(T+t-\hat{t}) - g(x(T)) + \epsilon \\ &\leq C\left|t-\hat{t}\right| + \epsilon. \end{aligned} \tag{1.28}$$

Next pick $\hat{\alpha}$ so that

$$u(x,\hat{t}) + \epsilon \geq \int_{\hat{t}}^T h(\hat{x}(s),\hat{\alpha}(s))ds + g(\hat{x}(T)),$$

where

$$\begin{cases} \dot{\hat{x}}(s) = f(\hat{x}(s),\hat{\alpha}(s)), \quad (\hat{t} < s < T), \\ \hat{x}(\hat{t}) = x. \end{cases}$$

Define

$$\alpha(s) := \begin{cases} \hat{\alpha}(s+\hat{t}-t) \text{ if } t \leq s \leq T+t-\hat{t}, \\ \hat{\alpha}(T) \text{ if } T+t-\hat{t} \leq s \leq T \end{cases}$$

and let x solve (1.6). Then $\alpha(s) = \hat{\alpha}(s+\hat{t}-t)$, $x(s) = \hat{x}(s+\hat{t}-t)$ for $t \leq s \leq T+t-\hat{t}$. Consequently

$$\begin{aligned} u(x,t) - u(x,\hat{t}) &\leq \int_t^T h(x(s),\alpha(s))ds + g(x(T)) \\ &\quad - \int_{\hat{t}}^T h(\hat{x}(s),\alpha(s))ds - g(\hat{x}(T)) + \epsilon \\ &= \int_{T+t-\hat{t}}^T h(x(s),\alpha(s))ds + g(x(T)) - g(x(T+t-\hat{t})) + \epsilon \\ &\leq C\left|t-\hat{t}\right| + \epsilon. \end{aligned}$$

This inequality and (1.28) prove

$$\left|u(x,t) - u(x,\hat{t})\right| \leq C\left|t-\hat{t}\right|, \, 0 \leq t \leq \hat{t} \leq T, x \in \mathbb{R}^n.$$

We prove next that the value function solves a Hamilton-Jacobi type partial differential equation.

■

Theorem 1.11 *(A PDE for the value function). The value function u is the unique viscosity solution of this terminal-value problem for the Hamilton-Jacobi-Bellman equation:*

$$\begin{cases} u_t + \min_{a \in A}\{f(x,a).Du + h(x,a)\} = 0 \quad in \quad \mathbb{R}^n \times (0,T), \\ u = g \quad on \quad \mathbb{R}^n \times \{t = T\}. \end{cases} \tag{1.29}$$

Remark 1.4 ***(i)*** *The Hamilton-Jacobi Bellman PDE has the form*

$$u_t + H(Du, x) = 0 \quad in \quad \mathbb{R}^n \times (0,T),$$

for the Hamiltonian

$$H(p,x) := \min_{a \in A}\{f(x,a).Du + h(x,a)\}, \quad p, x \in \mathbb{R}^n. \tag{1.30}$$

From the inequalities (1.10), *we deduce that H satisfies* (1.4).

(ii) *Since* (1.29) *is a terminal-value problem, we must specify what we mean by a solution. Let us say that a bounded and uniformly continuous function u is a viscosity solution of* (1.29) *provided:*

$$i) \; u = g \quad on \quad \mathbb{R}^n \times \{t = T\},$$

$$ii) \; for\ each\ v \in C^\infty(\mathbb{R}^n \times (0,T)),$$

$$\begin{cases} if\ u - v\ has\ a\ local\ maximum\ at\ a\ point\ (x_0, t_0) \in \mathbb{R}^n \times (0,T), \\ then\ v_t(x_0,t_0) + H(Dv(x_0,t_0), x_0) \geq 0 \end{cases} \tag{1.31}$$

and

$$\begin{cases} if\ u - v\ has\ a\ local\ maximum\ at\ a\ point\ (x_0, t_0) \in \mathbb{R}^n \times (0,\infty), \\ then\ v_t(x_0,t_0) + H(Dv(x_0,t_0), x_0) \leq 0. \end{cases} \tag{1.32}$$

We observe that for our terminal-value problem (1.29), *we reverse the sense of the inequalities from those for the initial-value problem.*

(iii) *The reader should check that if u is the viscosity solution of (1.29), then*

$$w(x,t) := u(x,T-t), \;\; x \in \mathbb{R}^n, 0 \le t \le T,$$

is the viscosity solution of the initial-value problem

$$\begin{cases} w_t + H(Dw,x) = 0 \;\; in \;\; \mathbb{R}^n \times (0,T), \\ \qquad w = g \;\; on \;\; \mathbb{R}^n \times \{t=0\}. \end{cases}$$

Proof. 1. In view of Lemma 1.3, u is bounded and Lipschitz continuous. In addition, we see directly from (1.9) and (1.11) that

$$u(x,T) = \inf_{\alpha \in Ad} C_{x,T}(\alpha) = g(x), \;\; x \in \mathbb{R}^n.$$

2. Now let $v \in C^\infty(\mathbb{R}^n \times (0,T))$ and assume that $u - v$ has a local maximum at a point $(x_0, t_0) \in \mathbb{R}^n \times (0,T)$. We must show that

$$v_t(x_0,t_0) + \min_{a \in A}\{f(x_0,a).Dv(x_0,t_0) + h(x_0,a)\} \ge 0. \tag{1.33}$$

Suppose not. So, there exists $a \in A$ and $\theta > 0$ such that

$$v_t(x,t) + f(x,a).Dv(x,t) + h(x,a) \le -\theta < 0, \tag{1.34}$$

for all points (x,t) sufficiently close to (x_0,t_0), say

$$|x - x_0| + |t - t_0| < \delta. \tag{1.35}$$

Since $u - v$ has a local maximum at a point (x_0,t_0), also we may suppose that

$$\begin{cases} (u-v)(x,t) \le (u-v)(x_0,t_0), \\ \text{for all } \; (x,t) \; \text{ satisfying (1.35).} \end{cases} \tag{1.36}$$

Consider now the constant control $\alpha(s) \equiv a, (t_0 \leq s \leq T)$ and the corresponding dynamics

$$\begin{cases} \dot{x}(s) = f(x(s), a), \quad (t_0 < s < T), \\ x(t_0) = x. \end{cases} \tag{1.37}$$

Choose $0 < h < \delta$ so small that $|x(s) - x_0| < \delta$ for $t_0 \leq s \leq t_0 + h$. Therefore

$$v_t(x(s), s) + f(x(s), a).Dv(x(s), s) + h(x(s), a) \leq -\theta, \ (t_0 \leq s \leq t_0 + h), \tag{1.38}$$

according to (1.34), (1.35). But utilizing (1.36) we find

$$\begin{aligned} u(x(t_0+h), t_0+h) - u(x_0, t_0) &\leq v(x(t_0+h), t_0+h) - v(x_0, t_0) \\ &= \int_{t_0}^{t_0+h} \frac{d}{ds} v(x(s), s) ds = \int_{t_0}^{t_0+h} v_t(x(s), s) + Dv(x(s), s).\dot{x}(s) ds \\ &= \int_{t_0}^{t_0+h} v_t(x(s), s) + f(x(s), a).Dv(x(s), s) ds. \end{aligned} \tag{1.39}$$

In addition, the optimality condition (1.12) provides us with the inequality

$$u(x_0, t_0) \leq \int_{t_0}^{t_0+h} h(x(s), a) ds + u(x(t_0+h), t_0+h). \tag{1.40}$$

Combining (1.39) and (1.40), we have

$$0 \leq \int_{t_0}^{t_0+h} v_t(x(s), s) + f(x(s), a).Dv(x(s), s) + h(x(s), a) ds \leq -\theta h,$$

according to (1.38). This contradiction realizes (1.33).

3. Now assume that $u - v$ has a local maximum at a point $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$. We must prove that

$$v_t(x_0, t_0) + \min_{a \in A} \{ f(x_0, a).Dv(x_0, t_0) + h(x_0, a) \} \leq 0. \tag{1.41}$$

Suppose not. Then there exists $\theta > 0$ such that

$$v_t(x, t) + f(x, a).Dv(x, t) + h(x, a) \geq \theta > 0, \tag{1.42}$$

for all $a \in A$ and all (x,t) sufficiently close to (x_0,t_0), say

$$|x - x_0| + |t - t_0| < \delta. \tag{1.43}$$

As $u - v$ has a local maximum at (x_0,t_0), also we may suppose that

$$\begin{cases} (u-v)(x,t) \geq (u-v)(x_0,t_0), \\ \text{for all } (x,t) \text{ verifiying (1.43).} \end{cases} \tag{1.44}$$

Choose as $0 < h < \delta$ so small that $|x(s) - x_0| < \delta$ for $t_0 \leq s \leq t_0 + h$, where x solves

$$\begin{cases} \dot{x}(s) = f(x(s), \alpha(s)), \quad (t_0 < s < T), \\ x(t_0) = x_0, \end{cases} \tag{1.45}$$

for some control $\alpha \in Ad$. This is possible owing to hypothesis (1.8). Using (1.44), we get for any control α that

$$\begin{aligned} u(x(t_0+h), t_0+h) - u(x_0,t_0) &\geq v(x(t_0+h), t_0+h) - v(x_0,t_0) \\ &= \int_{t_0}^{t_0+h} \frac{d}{ds} v(x(s),s) ds \\ &= \int_{t_0}^{t_0+h} v_t(x(s),s) + f(x(s),\alpha(s)).Dv(x(s),s)ds, \end{aligned} \tag{1.46}$$

by (1.45). Furthermore, according to the optimality condition (1.12) we can select a control $\alpha \in Ad$ so that

$$u(x_0,t_0) \geq \int_{t_0}^{t_0+h} h(x(s),\alpha(s))ds + u(x(t_0+h), t_0+h) - \frac{\theta h}{2}. \tag{1.47}$$

Combining (1.46) and (1.47), we obtain

$$\begin{aligned} \frac{\theta h}{2} &\geq \int_{t_0}^{t_0+h} v_t(x(s),s) + f(x(s),\alpha(s)).Dv(x(s),s)ds + h(x(s),\alpha(s))ds \\ &\geq \theta h, \end{aligned}$$

according to (1.42). This contradiction establishes (1.41). ■

Remark 1.5 *(Design of optimal controls). We have now shown that the value function u, defined by* (1.11)*, is the unique viscosity solution of the terminal value problem* (1.29) *for the Hamilton-Jacobi-Bellman equation. How does this PDE help us solve the problem of synthesizing an optimal control? In informal terms, the method is this. Given an initial time $0 < t \leq T$ and an initial state $x \in \mathbb{R}^n$, we consider the optimal ODE*

$$\begin{cases} \dot{x}^*(s) = f(x^*(s), \alpha^*(s)), \quad t < s < T), \\ \qquad\qquad x^*(t) = x, \end{cases} \tag{1.48}$$

where at each time s, $\alpha^* \in A$ is selected so that

$$\begin{aligned} & f(x^*(s), \alpha^*(s)).Du(x^*(s), s)ds + h(x^*(s), \alpha^*(s)) \\ = \quad & H(Du(x^*(s), s), x^*(s)). \end{aligned} \tag{1.49}$$

In other words, given the system is at the point $x^*(s)$ at time s, we adjust the optimal control value $\alpha^*(s)$ so as to attain the minimum in the definition (1.30) *of the Hamiltonian H. We call α^* so defined a feedback control.*

It is fairly easy to check that this prescription does in fact generate a minimum cost trajectory, at least in regions where u and α^* are smooth (so that (1.49) *makes sense). There are however problems in interpreting* (1.49) *at points where the gradient Du does not exist.*

Chapter 2

Two new types fixed point theorems of single-valued mappings

In this chapter, we will present two new types of fixed point theorems of single-valued mappings which were showed by Khojasteh et al. [31] and Rhoades [52], which belong to (T_3).

The chapter is organized as follows:

1. Introduction
2. Main results

2.1 Introduction

Most of the theorems such as Banach's [7], Ćirić's [16], Kannan's [29], Kirk's [33], Meir and Keeler's [42] and Suzuki's [63] belong to (T_1). Subrahmanyam's theorem [61], Ćirić's theorem [18], theorem 1 of Berinde [10] and theorems 2.1, 2.2 of Samet et al. [58] appertain to (T_3). Caristi's theorems [13, 12] appertain to (T_4). Furthermore, there are no theorems belonging to (T_2), see Kirk's survey [32].

In this section, motivated by the above, in the first paragraph, we have given a new type of fixed point theorem in complete metric spaces, was showed by Khojasteh et al. [31], an example involving this theorem and remark have been given. In the second paragraph we have given the main results published in Rhoades [52] in metric space using relation inequality contraction for two mappings, which generalizes the corresponding results of Khojasteh et al. [31].

2.2 Main results

In the first paragraph, we have given the main results published in Khojasteh et al. [31].

Theorem 2.1 *Let (X, d) be a complete metric space and let T be a self-mapping of X satisfying*

$$d(Tx, Ty) \leq \frac{d(y, Tx) + d(x, Ty)}{d(x, Tx) + d(y, Ty) + 1} d(x, y), \tag{2.1}$$

for all $x, y \in X$. Then

(i) T has at least one fixed point $\dot{x} \in X$.

(ii) $\{T^n x\}$ converges to a fixed point for all $x \in X$.

(iii) If $\dot{x}$,$\dot{y}$ are distinct fixed points of T, therefore $d(\dot{x}, \dot{y}) \geq \dfrac{1}{2}$.

Proof. Let $x_0 \in X$ be arbitrary and choose a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$. We have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \left(\frac{d(x_n, x_n) + d(x_{n-1}, x_{n+1})}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + 1} \right) d(x_{n-1}, x_n) \\ &= \left(\frac{d(x_{n-1}, x_{n+1})}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + 1} \right) d(x_{n-1}, x_n) \\ &\leq \left(\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + 1} \right) d(x_{n-1}, x_n) \\ &\leq \beta_n d(x_{n-1}, x_n), \end{aligned} \tag{2.2}$$

where

$$0 < \beta_n = \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + 1} < 1. \tag{2.3}$$

As in the proof of Lemma 2.3 of Rhoades [52], we conclude that $\{x_n\}$ is a Cauchy sequence and so converges to $\dot{x} \in X$. We claim that $\dot{x}$ is a fixed point. Note that

$$d\left(x_{n+1}, T\dot{x}\right) \leq \left(\frac{d\left(x_n, T\dot{x}\right) + d\left(\dot{x}, Tx_n\right)}{d(\dot{x}, T\dot{x}) + d(x_n, Tx_n) + 1} \right) d(x_n, \dot{x}). \tag{2.4}$$

On taking limit on both sides of (2.4), we have $d(\dot{x}, T\dot{x}) = 0$. Thus, $T\dot{x} = \dot{x}$. If there exist two distinct fixed points $\dot{x}, \dot{y} \in X$ then,

$$\begin{aligned} d(\dot{x}, \dot{y}) &= d(T\dot{x}, T\dot{y}) \\ &\leq [d(\dot{x}, T\dot{y}) + d(T\dot{x}, \dot{y})]\, d(\dot{x}, \dot{y}) \\ &= 2\,[d(\dot{x}, \dot{y})]^2 . \end{aligned} \tag{2.5}$$

Therefore, $d(\dot{x}, \dot{y}) \geq 1/2$ and we find the desired results. ■

Remark 2.1 *Note that in* (2.1) *the ratio*

$$\frac{d(y, Tx) + d(x, Ty)}{d(x, Tx) + d(y, Ty) + 1}, \tag{2.6}$$

might be greater or less than 1 *and has not introduced an upper bound. Note that if for every* $x, y \in X$ *such* $d(x, y) < 1/2$*, then we have*

$$\begin{aligned} d(y, Tx) + d(x, Ty) &\leq 2d(x, y) + d(x, Tx) + d(y, Ty) \\ &< d(x, Tx) + d(y, Ty) + 1. \end{aligned} \tag{2.7}$$

It means that

$$\frac{d(y, Tx) + d(x, Ty)}{d(x, Tx) + d(y, Ty) + 1} < 1. \tag{2.8}$$

Thus the theorem 2.1 is a special case of Banach contraction principle. Therefore, when (X, d) *is a complete metric space such that, for all* $x, y \in X, d(x, y) \geq 1/2$*, the theorem 2.1 is valuable because* (2.8) *might be greater than* 1*. The next examples show this note precisely.*

Example 2.1 *Let* $X = \{0, 1/2, 1\}$ *and* $d : X \times X \to [0, \infty)$ *be defined by:*

$$\begin{aligned} d(0, 1/2) &= 2, \qquad d(1, 1/2) = 5/2, \qquad d(0, 1) = 3, \\ d\,(0, 0) &= d(1/2, 1/2) = d(1, 1) = 0, \\ d\,(a, b) &= d(b, a), \qquad \textit{for all } \ a, b \in X. \end{aligned}$$

Let $T : X \to X$ be defined by:

$$T(0) = 0, \quad T(1/2) = 1/2, \quad T(1) = 0.$$

We have

$$\begin{aligned} d(T(0), T(1/2)) &= d(0, 1/2) \\ &= 2 \\ &< 4 \times 2 = 8. \end{aligned}$$

$$\begin{aligned} d\left(T(1), T\left(\frac{1}{2}\right)\right) &= d\left(0, \frac{1}{2}\right) \\ &= 2 < \frac{9}{8} \times \frac{5}{2} \\ &= \frac{45}{16}. \end{aligned}$$

Therefore, T satisfies all the conditions of the theorem 2.1. Also, T has two distinct fixed points $\{0, 1/2\}$ and $d(0, 1/2) = 2 > 1/2$.

The next example verifies theorem 2.1.

Example 2.2 *Let $X = \{0\} \cup [\frac{3}{4}, 1]$ be endowed with Euclidean metric and $T : X \to X$ defined by*

$$Tx = \begin{cases} 0 & \text{if} \quad x = 0, \\ 1 & \text{if } x \in [\frac{3}{4}, 1]. \end{cases}$$

For $x = 0$ and $y \in [\frac{3}{4}, 1]$, we find

$$\begin{aligned} d(T(0), Ty) &= 1, \\ \frac{d(y, T(0)) + d(0, Ty)}{d(y, Ty) + 1} d(0, y) &= \frac{y(y+1)}{2 - y}. \end{aligned}$$

Since for all $y \in [\frac{3}{4}, 1]$, $1 < \frac{21}{20} \leq \frac{y(y+1)}{2-y} \leq 2$, *the inequality* (2.1) *holds. Note that the ratio*

$$1 < \frac{7}{5} \leq r = \frac{d(y, T(0)) + d(0, Ty)}{d(y, Ty) + 1} \leq 2.$$

The other cases are obvious. Hence, T *satisfies all the conditions of theorem* 2.1 *and* T *has two fixed points* 0 *and* 1. *Furthermore,* $d(0, 1) = 1 > \frac{1}{2}$.

In the following paragraph, we have given the main results published in Rhoades [52].

Theorem 2.2 *Let* (X, d) *be a complete metric space,* S *and* T *self-mappings of* X *satisfying*

$$d(Sx, Ty) \leq N(x, y)m(x, y), \text{ for all } x, y \in X, \tag{2.9}$$

where

$$N(x, y) = \frac{\max\{d(x, y), d(x, Sx) + d(y, Ty), d(x, Ty) + d(y, Sx)\}}{d(x, Sx) + d(y, Ty) + 1} \tag{2.10}$$

and

$$m(x, y) = \max\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty) + d(y, Sx)}{2}\}. \tag{2.11}$$

Then

a) S *and* T *have at least one common fixed point* $p \in X$.

(b) For n *even,* $\{(ST)^{n/2}x\}$ *and* $\{T(ST)^{n/2}x\}$ *converge to a common fixed point for each* $x \in X$.

(c) If p *and* q *are distinct common fixed points of* S *and* T, *then* $d(p, q) \geq 1/2$.

The following Lemma will shorten the proof of the theorem 2.2.

Lemma 2.3 *Suppose that* S *and* T *satisfy the hypotheses of the theorem 2.2. Then each fixed point of* S *is a fixed point of* T *and conversely.*

Proof. Let $u \in F(S)$ and suppose that $u \notin F(T)$. From (2.10),

$$N(u, u) = \frac{d(u, Tu)}{d(u, Tu) + 1} < 1, \; m(u, u) = d(u, Tu).$$

Substituting into (2.9) gives

$$\begin{aligned} d(u,Tu) &< \frac{d(u,Tu)}{d(u,Tu)+1}d(u,Tu) \\ &< d(u,Tu), \end{aligned}$$

which is a impossible. Therefore $u \in F(T)$. Similarly, it can be shown that, if $v \in F(T)$, so $v \in F(S)$. ■

Proof. of theorem 2.2. Let $x_0 \in X$ and define a sequence $\{x_n\}$ by

$$x_{2n+1} = Sx_{2n}, \;\; x_{2n+2} = Tx_{2n+1} \text{ for all } n \in \mathbb{N}. \tag{2.12}$$

Suppose that there exists a value of n for which $x_{2n+1} = x_{2n+2}$. Then, Applying (2.12), we get

$$x_{2n+1} = Tx_{2n+1} \text{ and } x_{2n+1} \in F(T).$$

By Lemma 2.3, $x_{2n+1} \in F(S)$ and (a) is satisfied. Similarly, if there exists a value of n for which $x_{2n} = x_{2n+1}$, then $x_{2n} \in F(S) \cap F(T)$, and again (a) is satisfied. Therefore we shall assume that

$$x_n \neq x_{n+1} \;\text{ for all }\; n \in \mathbb{N}. \tag{2.13}$$

Employing (2.9), we obtain

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq N(x_{2n,2n+1})m(x_{2n}, x_{2n+1}). \end{aligned} \tag{2.14}$$

Set $d_n := d(x_n, x_{n+1})$. Using (2.10), we have

$$\begin{aligned} N(x_{2n,2n+1}) &= \frac{\max\{d_{2n}, d_{2n}+d_{2n+1}, d(x_{2n}, x_{2n+2})\}}{d_{2n}+d_{2n+1}+1} \\ &\leq \frac{\max\{d_{2n}, d_{2n}+d_{2n+1}, d_{2n}+d_{2n+1}\}}{d_{2n}+d_{2n+1}+1} \\ &= \frac{d_{2n}+d_{2n+1}}{d_{2n}+d_{2n+1}+1}. \end{aligned}$$

Set

$$0 < \beta_{2n} = \frac{d_{2n} + d_{2n+1}}{d_{2n} + d_{2n+1} + 1} < 1. \tag{2.15}$$

From (2.11) we find

$$\begin{aligned} m(x_{2n}, x_{2n+1}) &= \max\left\{d_{2n}, d_{2n+1}, \frac{d(x_{2n}, x_{2n+2})}{2}\right\} \\ &= \max\left\{d_{2n} + d_{2n+1}\right\}. \end{aligned} \tag{2.16}$$

Substituting (2.15) and (2.16) into (2.14) gives

$$d_{2n+1} \leq \beta_{2n} \max\left\{d_{2n} + d_{2n+1}\right\} = \beta_{2n} d_{2n}. \tag{2.17}$$

Similarly, it can be shown that

$$d_{2n} \leq \beta_{2n-1} d_{2n-1}. \tag{2.18}$$

Therefore, Using (2.17) and (2.18) it follows that

$$d_n \leq \beta_{n-1} \max\left\{d_{n-1} + d_n\right\} < d_{n-1}. \tag{2.19}$$

■

Lemma 2.1 *For each $n > 0, \beta_n < \beta_{n-1}$.*

Proof. of Lemma 2.4: From (2.15), $\beta_n < \beta_{n-1}$ is equivalent to

$$\frac{d_n + d_{n+1}}{d_n + d_{n+1} + 1} < \frac{d_{n-1} + d_n}{d_{n-1} + d_n + 1},\ n \geq 1,$$

which implies that $d_{n+1} < d_{n-1}$. This inequality follows from(2.19). Hence

$$d_n \leq \beta_1 d_{n-1}. \tag{2.20}$$

Therefore $\{x_n\}$is a Cauchy sequence. As X is complete, there exists a point $p \in X$ such that $\lim\limits_{n\to\infty} x_n = p$.

Using (2.9) - (2.11), (2.15) we have

$$\begin{aligned} d(p, Tp) &\leq d(p, x_{2n+1}) + d(Sx_{2n}, Tp) \\ &\leq d(p, x_{2n+1}) + N(x_{2n}, p)m(x_{2n}, p), \end{aligned} \tag{2.21}$$

where

$$\begin{aligned} N(x_{2n}, p) &= \frac{\max\{d(x_{2n}, p), d(x_{2n}, x_{2n+1}) + d(p, Tp), d(x_{2n}, Tp) + d(p, x_{2n+1})\}}{d(x_{2n}, x_{2n+1}) + d(p, Tp) + 1}, \\ m(x_{2n}, p) &= \max\{d(x_{2n}, p), d(x_{2n}, x_{2n+1}), d(p, Tp), \frac{d(x_{2n}, Tp) + d(p, x_{2n+1})}{2}\}. \end{aligned}$$

Taking the limit of both sides of (2.21) as $n \to \infty$ we obtain

$$\begin{aligned} d(p, Tp) &\leq \frac{d(p, Tp)}{d(p, Tp) + 1} d(p, Tp) \\ &< d(p, Tp), \end{aligned}$$

which implies that $p = Tp$. By Lemma 2.3, $p \in F(S)$ and (a) is satisfied. To prove (b), observe that from (2.12) and the fact that x_0 is arbitrary, we may write

$$x_{2n+1} = (ST)^{n/2}x \text{ and } x_{2n+2} = T(ST)^{n/2}x.$$

To prove (c), assume that

$$p, q \in F(S) \cap F(T) \text{ with } p \neq q.$$

By virtue of (2.10) and (2.11), we find

$$N(p, q) = 2d(p, q) \text{ and } m(p, q) = d(p, q).$$

Thus (2.9) becomes

$$d(p, q) \leq 2d^2(p, q),$$

which gives (c). ■

Corollary 2.1 *Let (X, d) be a complete metric space, T a self-mapping of X satisfying* (2.9) - (2.11) *with $S = T$. Then*

(a) T has at least one fixed point.

(b) $\{T^n x\}$ converges to a fixed point of T.

Chapter 3

Fixed point theorems of Kannan type with an application to control theory

Inspired by theorem 2.1, it is our purpose in this chapter to prove unique fixed point theorems for a self-mapping in complete metric spaces and that the fixed point problem is well-posed. Examples are furnished to illustrate the validity of our results and we give some remarks about the papers, [3], [4] and [31]. Also, to illustrate and apply our results, an application to control theory is given.

The chapter is organized as follows:

1. Introduction
2. Main results
3. Application to control theory

3.1 Introduction

In the year 1968, Kannan [29] proved that if (X, d) is complete, then T has a unique fixed point in X. Kannan [30] provided examples which show that Kannan's fixed point theorem is independent of the Banach contraction principle and Kannan mapping does not need be continuous. Kannan's theorem is also very interesting because Subrahmanyam [61] demonstrated that Kannan's theorem characterizes the metric completeness, that is a metric space (X, d) is complete if and only if every Kannan mapping on X has a fixed point. Several authors generalized Kannan's fixed point theorem, see [3], [17], [20], [22], [23], [24], [25], [34], [43], [47] and [50].

Suzuki [62] categorized the theorems which ensure the existence of a fixed point of a mapping T into the four types (T_1-T_4) mentioned in the introduction.

For more details, we refer the reader to [25]. In light of the above, Khojasteh et al. [31] established two new types of fixed point theorems of single-valued and multivalued mappings which belong to (T_3). Inspired by theorem 2.1, it is our purpose in this section to prove unique fixed point theorems for a self-mapping in complete metric spaces and that the fixed point problem is well-posed and inspired by the papers of Pathak and Shahzad [45] and Rhoades et al. [53], we investigate the possibility of optimally controlling the solution of the ordinary differential equation (3.13) by dynamic programming.

3.2 Main results

The next lemma plays a crucial role in the proof of our main theorems.

Lemma 3.1 *Let (X, d) be a metric space and $\{x_n\}$ a sequence in X such that*

$$d(x_n, x_{n+1}) \leq \beta_n d(x_{n-1}, x_n), \tag{3.1}$$

for all $n \in \mathbb{N}^*$, where

$$\beta_n = \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1}.$$

Then $\{x_n\}$ is a Cauchy sequence.

Proof. As in the proof of Lemma 2.3 of Rhoades [52], assume that

$$x_{n-1} \neq x_n \text{ for each } n \geq 1$$

and set $t_n = d(x_{n-1}, x_n)$. Therefore

$$\beta_n = \frac{t_n + t_{n+1}}{t_n + t_{n+1} + 1}. \tag{3.2}$$

Since $0 < \beta_n < 1$, we deduce from (3.1) that

$$t_{n+1} \leq \beta_n t_n < t_n \text{ for any } n \in \mathbb{N}. \tag{3.3}$$

We will prove that for all $n \geq 1$, $\beta_n < \beta_{n-1}$. Using (3.2), we obtain that

$$\beta_n < \beta_{n-1},$$

is equivalent to

$$\frac{t_n + t_{n+1}}{t_n + t_{n+1} + 1} < \frac{t_{n-1} + t_n}{t_{n-1} + t_n + 1}.$$

The above inequality yields $t_{n+1} < t_{n-1}$ which is fulfilled by (3.3) Consequently

$$t_{n+1} < \beta_1 t_n \text{ for every } n \in \mathbb{N}.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. ■

Theorem 3.2 *Let (X, d) be a complete metric space and T a mapping from X into itself satisfying the following condition*

$$d(Tx, Ty) \leq \frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} \max\{d(x, Tx), d(y, Ty)\}, \tag{3.4}$$

for all $x, y \in X$. Then

a) T has a unique fixed point $z \in X$.

b) The fixed point problem of T is well-posed.

c) T is continuous at z.

Proof. Let x_0 be an arbitrary point in X. We define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. Employing the inequality (3.4) we have

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leq \frac{d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\
&\leq \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\
&= \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} d(x_{n-1}, x_n) \\
&= \beta_n d(x_{n-1}, x_n),
\end{aligned}$$

where

$$\beta_n = \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1}.$$

Applying Lemma 3.1, we deduce that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists a point $z \in X$ such that $\lim_{n\to\infty} x_n = z$. We assert that z is a fixed point of T. If $Tz \neq z$ using the inequality (3.4) we get

$$d(Tx_n, Tz) \leq \frac{d(x_n, Tz) + d(z, x_{n+1})}{d(x_n, x_{n+1}) + d(z, Tz) + 1} \max\{d(x_n, x_{n+1}), d(z, Tz)\}. \tag{3.5}$$

Taking the limit as $n \to \infty$ in (3.5), we obtain

$$\begin{aligned}
d(Tz, z) &\leq \frac{d^2(Tz, z)}{d(Tz, z) + 1} \\
&< d(Tz, z).
\end{aligned}$$

Therefore, z is a fixed point of T. For the uniqueness, we assume that $w \neq z$ is another fixed point of T. From the inequality (3.4) we find

$$d(z, w) \leq d(Tz, Tw) \leq 0.$$

Hence, z is unique.

b) Let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} d(Ty_n, y_n) = 0$. We have

$$d(y_n, z) \leq d(y_n, Ty_n) + d(Ty_n, Tz).$$

Using the inequality (3.4) we get

$$d(Ty_n, Tz) \leq \frac{d(y_n, Tz) + d(z, Ty_n)}{d(y_n, Ty_n) + d(z, Tz) + 1} \max\{d(y_n, Ty_n), d(z, Tz)\}. \tag{3.6}$$

Therefore

$$d(Ty_n, Tz) \leq \frac{d(y_n, z) + d(z, Tz) + d(z, y_n) + d(y_n, Ty_n)}{d(y_n, Ty_n) + 1} d(y_n, Ty_n).$$

Thus,

$$\lim_{n\to\infty} d(Ty_n, Tz) = 0 \text{ and, so } \lim_{n\to\infty} d(y_n, z) = 0,$$

that is the fixed point problem of T is well-posed.

c) Let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} y_n = z$. Suppose that

$$\lim_{n\to\infty} \sup d(Ty_n, Tz) = \lim_{n\to\infty} \sup d(Ty_n, z) = l > 0.$$

Letting $n \to \infty$ in (3.6) , we obtain $l \leq \frac{l}{l+1} l < l$. Hence, $\lim_{n\to\infty} d(Ty_n, Tz) = 0$ and so T is continuous at z. The following example supports our theorem 3.2. ■

Example 3.1 *Let $X = \{0, 1, 2\}$ and $d : X \times X \to \mathbb{R}_+$ defined by:*

$$\begin{aligned} d(0,1) &= d(1,0) = 1,\ d(1,2) = d(2,1) = 2, \\ d(0,2) &= d(2,0) = 3, \\ d(0,0) &= d(1,1) = d(2,2) = 0. \end{aligned}$$

(X, d) is a complete metric space. Define $T : X \to X$ by:

$$T(0) = 0,\ T(1) = 0,\ T(2) = 1.$$

1) The cases $x = y$ and $(x, y) = (0, 1)$ are obvious.

2) For the case $(x, y) = (0, 2)$, we have

$$\begin{aligned} d(T(0), T(2)) &= d(0,1) = 1 \\ &< \frac{4}{3} \times 2 = \frac{8}{3}. \end{aligned}$$

3) For the case $(x, y) = (1, 2)$, we get

$$\begin{aligned} d(T(1), T(2)) &= d(0,1) = 1 \\ &< \frac{3}{4} \times 2 = \frac{3}{2}. \end{aligned}$$

Hence, T satisfies all the conditions of theorem 3.2 *and T has a unique fixed point* 0*. Kannan's fixed point theorem is not applicable because*

$$\begin{aligned} d(T(0), T(2)) &= 1 \\ &> \alpha(d(0, T(0)) + d(2, T(2))) \\ &= 2\alpha \end{aligned}$$

for any $\alpha \in [0, 1/2)$.

In a similar manner, we can prove the subsequent theorem. We omit the proof.

Theorem 3.3 *Let (X, d) be a complete metric space and T a mapping from X into itself such that the inequality*

$$d(Tx, Ty) \le N(x, y) \max\{d(x, Tx), d(y, Ty)\}$$

is fulfilled for all $x, y \in X$, where

$$N(x, y) = \frac{\max\{d(x, y), d(x, Tx) + d(y, Ty), d(x, Ty) + d(y, Tx)\}}{d(x, Tx) + d(y, Ty) + 1}.$$

Then

a) T has a unique fixed point $z \in X$.

b) The fixed point problem of T is well-posed.

c) T is continuous at z.

The following example illustrates our theorem 3.3.

Example 3.2 *Let $X = \{0, 1, 2, 3\}$ and $d : X \times X \to \mathbb{R}_+$ defined by:*

$$\begin{aligned} d(0,1) &= 1,\ d(1,2) = d(0,2) = 2, \\ d(1,3) &= d(0,3) = 3,\ d(2,3) = 5. \\ d(x,x) &= 0,\ \text{for all } x \in X \text{ and } d(x,y) = d(y,x),\ \text{for all } x, y \in X. \end{aligned}$$

(X, d) is a complete metric space. Define $T : X \to X$ by:

$$T(0) = 0,\ T(1) = 0,\ T(2) = 1,\ T(3) = 2.$$

1) The cases $x = y$ and $(x, y) = (0, 1)$ are obvious.

2) For the case $(x, y) = (0, 2)$, we have

$$\begin{aligned} d(T(0), T(2)) &= d(0,1) = 1 \\ &< 1 \times 2 = 2. \end{aligned}$$

3) For the case $(x, y) = (0, 3)$, we get

$$\begin{aligned} d(T(0), T(3)) &= d(0,2) \\ &< \frac{5}{6} \times 5 = \frac{25}{6}. \end{aligned}$$

4) For the case $(x, y) = (1, 2)$, we obtain

$$\begin{aligned} d(T(1), T(2)) &= d(0,1) = 1 \\ &< \frac{3}{4} \times 2 = \frac{3}{2}. \end{aligned}$$

5) For the case $(x, y) = (1, 3)$*, we find*

$$\begin{aligned} d(T(1), T(3)) &= d(0, 2) = 2 \\ &< \frac{6}{7} \times 5 = \frac{30}{7}. \end{aligned}$$

6) For the case $(x, y) = (2, 3)$*, we have*

$$\begin{aligned} d(T(2), T(3)) &= d(1, 2) = 2 \\ &< \frac{7}{8} \times 5 = \frac{35}{8}. \end{aligned}$$

Hence, T *satisfies all the conditions of theorem* 3.3 *and* T *has a unique fixed point* 0.

Remark 3.1 *theorem* 3.2 *is not applicable because*

$$\begin{aligned} d(T(2), T(3)) &= d(1, 2) = 2 \\ &> \frac{d(2, 2) + d(3, 1)\}}{d(2, 1) + d(3, 2) + 1} \max\{d(2, 1), d(3, 2)\} \\ &= \frac{3}{8} \times 5 = \frac{15}{8}. \end{aligned}$$

This shows that theorem 3.3 *is a genuine generalization of theorem* 3.2*. Also, Kannan's fixed point theorem is not applicable because* $d(T(0), T(2)) = 1 > 2\alpha$ *for every* $\alpha \in [0, 1/2)$*.*

Remark 3.2 *Khojasteh et al. [31] gave the following example.*

Example 3.3 *Let* $X = [0, 2 - \sqrt{3}]$ *be endowed with Euclidean metric and* $T : X \to X$ *defined by*

$$Tx = \begin{cases} 0 & \text{if } x \in [0, 2 - \sqrt{3}), \\ 2 - \sqrt{3} & \text{if } x = 2 - \sqrt{3}. \end{cases}$$

The authors claimed that T *satisfies all the conditions of theorem* 2.1*, but this example is false because* $d(0, 2 - \sqrt{3}) = 2 - \sqrt{3} = 0.27 < \frac{1}{2}$*. If we replace* $2 - \sqrt{3}$ *in the above example by* 1*, we get* $d(0, 1) = 1 > \frac{1}{2}$*, however, the inequality* (2.1) *does not hold for all* $x, y \in [0, 1]$*. Indeed, for*

$x = \frac{1}{2}$ *and* $y = 1$, *we obtain*

$$\begin{aligned} d(T(\frac{1}{2}), T(1)) &= d(0,1) = 1 \\ &> \frac{d(1,0) + d(\frac{1}{2}, 1)}{d(\frac{1}{2}, 0) + 1} d(\frac{1}{2}, 1) = \frac{1}{2}. \end{aligned}$$

We end this section by giving some remarks about the papers [4] and [3]. The subsequent theorem was proved by [3].

Theorem 3.4 *Let* $\{A_i\}_{i=1}^p$, $p > 1$, $p \in \mathbb{N}$, *be nonempty closed subsets of a complete metric space* (X, d) *and* $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ *fulfils one the following inequalities, then* T *has a unique fixed point* $\cap_{i=1}^p A_i$.

$$\phi(d(Tx, Ty)) \leq \phi(\alpha d(x, Tx) + \beta d(y, Ty)) - \psi(d(x, Tx), d(y, Ty)), \tag{3.7}$$

where, $x \in A_i, y \in A_{i+1}$, $i = 1, 2..., p$, $\alpha, \beta \in (0, 1)$ *such that* $\alpha + \beta \leq 1$.

$$\phi(d(Tx, Ty)) \leq \phi(\alpha d(x, Ty) + \beta d(y, Tx)) - \psi(d(x, Ty), d(y, Tx)), \tag{3.8}$$

where, $x \in A_i, y \in A_{i+1}$, $i = 1, 2..., p$, $\beta \in (0, 1)$, $\alpha \leq \frac{1}{2}$ *such that* $\alpha + \beta \leq 1$. $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is an altering distance function and* $\psi : \mathbb{R}_+^2 \to \mathbb{R}_+$ *is a continuous function with* $\psi(t, s) = 0$ *if and only if* $t = s = 0$.

Remark 3.3 *The inequalities* (3.7) *and* (3.8) *imply that*

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty), \tag{3.9}$$

$$d(Tx, Ty) \leq \alpha d(x, Ty) + \beta d(y, Tx). \tag{3.10}$$

I) If $\alpha + \beta < 1$, *employing the inequality* (3.9) *we get a special case of theorem 7 of [47].*

II) If $\alpha + \beta < 1$, *using the inequality* (3.10) *we get a special case of theorem 8 of [47].*

III) If $\alpha+\beta=1$, in the light of (3.9) *or* (3.10)*, theorem* 3.4 *becomes false in general except additional conditions are added: the continuity of T and the compactness of the metric space (X,d), see [22]. Besides, the two inequalities of Example 3.1 in [3] implying* (3.9) *and* (3.10) *and this example is not true because we can not take $\psi(t,s)=0$. A Kannan type mapping $T:X\to X$ such that for all $x,y\in X$in a complete metric space (X,d) may not have a fixed point, see [49] and [24].*

Remark 3.4 *Example 1.4. Therefore, we can not take $\psi(t,s)=0$ and $\alpha=\beta=\frac{1}{2}$ in Remark 2.1 of [3] and so Corollary 1 of [4] is also incorrect. Even a continuous Kannan type mapping such that for all $x,y\in X$, $x\neq y$*

$$d(Tx,Ty)<\frac{1}{2}(d(x,Tx)+d(y,Ty)),$$

in complete but noncompact metric space (X,d) may not have a fixed point, see [24] Example 1.5.

3.3 Application to control theory

In this section, inspired by the papers of Pathak and Shahzad [45] and Rhoades et al. [53], we investigate the possibility of optimally controlling the solution of the ordinary differential equation (3.13) via dynamic programming.

Let A be a compact subset of $\mathbb{R}^m$ and for each given $a\in A$, $F_a:\mathbb{R}^n\to\mathbb{R}^n$ is a mapping such that $F_a(x)=f(x,a)$ for all $x\in\mathbb{R}^n$, where $f:\mathbb{R}^n\times\ A\to\mathbb{R}^n$ is a given bounded continuous function which satisfies the following contractive condition

$$|f(x,a)|\leq C \quad \text{for some} \quad C>0 \tag{3.11}$$

$$|f(x,a)-f(y,a)|\leq\frac{|x-f(y,a)|+|y-f(x,a)|}{|x-f(x,a)|+|y-f(y,a)|+1}\max\{|x-f(x,a)|,|y-f(y,a)|\}. \tag{3.12}$$

Where

$$0\leq\frac{|x-f(y,a)|+|y-f(x,a)|}{|x-f(x,a)|+|y-f(y,a)|+1}<1 \text{ for all } x,y\in X.$$

Now, we will study the possibility of controlling optimally the solution $x(.)$ of the ordinary differential equation

$$\begin{cases} x'(s) = f(x(s), u(s)), \quad t < s < T, \\ x(t) = x. \end{cases} \tag{3.13}$$

Here $x'(s) = \frac{dx(s)}{ds}, T > 0$, is a fixed terminal time and $x \in \mathbb{R}^n$ is a given initial point, taken on by our solution $\mathbf{x}(.)$ at the starting time $t \geq 0$. At later times $t < s < T, \mathbf{x}(\cdot)$ evolves according to the ODE (3.13).The function $\alpha(.)$ appearing in (3.13) is a control, that is some appropriate for adjusting parameters from the set A as time evolves there by affecting the dynamics of the system modelled by (3.13). Let us write

$$Ad = \{\alpha : [0, T] \to A \text{ such that } \alpha(.) \text{ is measurable}\}$$

denote the set of admissible controls. $F_a(x) = f(x, a)$ for all $x \in \mathbb{R}^n$, employing (3.11) and (3.12) we obtain

$$\begin{aligned} |F_a(x) - F_a(y)| &\leq \frac{|x - F_a(y)| + |y - F_a(x))|}{|x - F_a(x)| + |y - F_a(y))| + 1} \max\{|x - F_a(x)|, |y - F_a(y)|\} \\ \text{for all } x, y &\in \mathbb{R}^n, \, a \in A. \end{aligned}$$

Applying theorem 3.2, we deduce that for each control $\alpha(.) \in Ad$, the ODE (3.13) has a unique continuous solution $\mathbf{x} = \mathbf{x}^{\alpha(.)}(.)$, existing on the time interval $[t, T]$ and solving the ODE for almost everywhere time $t < s < T$. We call $\mathbf{x}(.)$ the response of the system to the control $\alpha(.)$ and $\mathbf{x}(s)$ the state of the system at time s.

Our goal is to find a control $\alpha^*(.)$ which optimally steers the system. We must first introduce a cost criterion. Given $x \in \mathbb{R}^n$ and $0 \leq t \leq T$, let us define for each admissible control $\alpha(.) \in Ad$ the corresponding cost

$$P_{x,t}(\alpha(.)) := \int_t^T h(\mathbf{x}(s), \alpha(s))ds + g(\mathbf{x}(T)), \tag{3.14}$$

where $\mathbf{x} = \mathbf{x}^{\alpha(.)}(.)$ solves the ODE (3.13) and $h : \mathbb{R}^n \times A \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ are given functions.We call h the running cost per unit time and g the terminal cost and will hereafter suppose

$$\begin{cases} |H_a(x)|, \ |g(x)| \leq C \ \text{ for some } \ C > 0, \\ |H_a(x) - H_a(y)| \leq \dfrac{|x - H_a(y)| + |y - H_a(x))|}{|x - H_a(x)| + |y - H_a(y))| + 1} \max\{|x - H_a(x)|, |y - H_a(y)|\}, \\ |g(x) - g(y)| \leq \dfrac{|x - g(y)| + |y - g(x))|}{|x - g(x)| + |y - g(y))| + 1} \max\{|x - g(x)|, |y - g(y)|\}, \\ \text{for all } \ x, y \in \mathbb{R}^n, \, a \in A, \end{cases}$$

where $H_a : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping such that $H_a(x) = h(x, a)$ for all $x \in \mathbb{R}^n$. Given $x \in \mathbb{R}^n$ and $0 \leq t \leq T$, we would like to find if possible a control $\alpha^*(.)$ which minimizes the cost functional (3.14) among all other admissible controls.

To investigate the above problem we shall apply the method of dynamic programming. We now turn our attention to the value function $u(x, t)$ defined by

$$u(x, t) := \inf_{\alpha(.) \in Ad} P_{x,t}\left(\alpha(.)\right), \, x \in \mathbb{R}^n, \, 0 \leq t \leq T.$$

The idea is: having defined $u(x, t)$ as the least cost given we start at the position x at time t, we want to study u as a function of x and t. We are therefore embedding our given control problem (3.13) and (3.14) into the larger class of all such problems, as x and t vary. This idea then can be used to show that u solves a certain Hamilton–Jacobi type PDE, and finally to show conversely that a solution of this PDE helps us to synthesize an optimal feedback control.

Let us fix $x \in \mathbb{R}^n$ and $0 \leq t \leq T$. Following the technique of Evan [21, p. 553–554], the subsequent theorem gives the optimality conditions in the form (3.15).

Theorem 3.5 *For each $\xi > 0$ so small that $t + \xi \leq T$, we have*

$$u(x, t) := \inf_{u \in A} \left\{ \int_t^{t+\xi} h(x(s), u(s))ds + v(x(t + \xi), t + \xi) \right\}, \tag{3.15}$$

where $\mathbf{x} = \mathbf{x}^{\alpha(.)}$ solves the ODE (3.13) *for the control $\alpha(.)$.*

Proof. It follows as in theorem 1 of Evan [21]. ■

Chapter 4

Coincidence and common fixed points theorem with an application in dynamic programming

Our aim in this chapter is to prove coincidence and common fixed points theorem for two self-mappings in complete metric spaces. Our theorem generalizes theorem 1 of [31]. Examples are furnished to illustrate the validity of our results. We apply our theorem to realize the existence of common solutions of a system of two functional equations arising in dynamic programming.

The chapter is organized as follows:

1. Introduction
2. Main results
3. Application to dynamic programming

4.1 Introduction

Suzuki [62] categorized the theorems which ensure the existence of a fixed point of a mapping T into the four types (T_1-T_4) mentioned in the introduction. Most of the theorems such as Banach's [7], Ćirić's [16], Kannan's [29], Kirk's [33], Meir and Keeler's [42] and Suzuki's [63] belong to (T_1). Subrahmanyam's theorem [61], Ćirić's theorem [18], theorem 1 of Berinde [10] and theorems 2.1, 2.2 of Samet et al. [58] appertain to (T_3). Caristi's theorems [12, 13] appertain to (T_4). Furthermore, there are no theorems belonging to (T_2), see Kirk's survey [32].

Our purpose in this section is to prove coincidence and common fixed points theorem for two self-mappings in complete metric spaces. Our theorem generalizes theorem 1 of [31]. Examples are furnished to illustrate the validity of our results and we have applied theorem 4.1 to establish the existence of common solutions of a system of two functional equations arising in dynamic programming.

4.2 Main results

Theorem 4.1 *Let A and S be two mappings of a complete metric space (X, d) into itself verifying*

$$A(X) \subset S(X), \tag{4.1}$$

$$d(Ax, Ay) \leq N(x, y)M(x, y) \tag{4.2}$$

for all $x, y \in X$, where

$$N(x, y) = \frac{\max\{d(Sx, Sy), d(Sx, Ax) + d(Sy, Ay), d(Sx, Ay) + d(Sy, Ax)\}}{d(Sx, Ax) + d(Sy, Ay) + 1}, \tag{4.3}$$

$$M(x, y) = \max\{d(Sx, Sy), d(Sx, Ax), d(Sy, Ay), \frac{d(Sx, Ay) + d(Sy, Ax)}{2}\}. \tag{4.4}$$

Suppose that $S(X)$ is a closed subspace of X. So

i) A and S have at least one coincidence point $u \in X$ and the Jungck sequence $\{y_n\} = \{Sx_n\}$ converges to $z = Au$ for each $x \in X$. In this case, A and S are QWPJO.

ii) If there exists $u \in C(A, S)$ such that $ASu = SAu$, then $z = Au$ is an other coincidence point of A and S. In this case, A and S are QWPJO.

iii) If v is a distinct coincidence point of A and S, therefore $d(Au, Av) \geq \frac{1}{2}$.

iv) If $Au = A^2u$ for some $u \in C(A, S)$ and $ASu = SAu$, then A and S possess at least one common fixed point $z = Au$ and the Jungck sequence $\{y_n\} = \{Sx_n\}$ converges to z for any $x \in X$. In this case, A and S are WPJO.

(v) If z and w are distinct common fixed points of A and S, so $d(z,w) \geq \frac{1}{2}$.

Proof. Let x_0 be an arbitrary point in X. From (4.1), we can define inductively a sequence $\{y_n\}$ in X such that

$$y_n = Ax_n = Sx_{n+1},\ n \in \mathbb{N}. \tag{4.5}$$

Let us show that $\{y_n\}$ is a Cauchy sequence in X. If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, so $Ax_n = Sx_{n+1} = Ax_{n+1} = Sx_{n+2}$. Thus, A and S a coincidence point. Therefore, *we* assume that $y_n \neq y_{n+1}$ for each $n \in \mathbb{N}$. Using (4.2), (4.3) and (4.5) , we have

$$\begin{aligned} d(y_n, y_{n+1}) &= d(Ax_n, Ax_{n+1}) \\ &\leq N(x_n, x_{n+1})M(x_n, x_{n+1}), \end{aligned} \tag{4.6}$$

$$\begin{aligned} N(x_n, x_{n+1}) &= \frac{\max\{\ d(Sx_n, Sx_{n+1}), d(Sx_n, Ax_n) + d(Sx_{n+1}, Ax_{n+1}), d(Sx_n, Ax_{n+1})\}}{d(Sx_n, Ax_n) + d(Sx_{n+1}, Ax_{n+1}) + 1} \\ &= \frac{\max\{d(y_{n-1}, y_n), d(y_{n-1}, y_n) + d(y_n, y_{n+1}), d(y_{n-1}, y_{n+1})\}}{d(y_{n-1}, y_n) + d(y_n, y_{n+1}) + 1} \\ &\leq \frac{d(y_{n-1}, y_n) + d(y_n, y_{n+1})}{d(y_{n-1}, y_n) + d(y_n, y_{n+1}) + 1}. \end{aligned}$$

Set $t_n = d(y_{n-1}, y_n)$, $n \geq 1$. So

$$N(x_n, x_{n+1}) \leq \frac{t_n + t_{n+1}}{t_n + t_{n+1} + 1} = \beta_n. \tag{4.7}$$

By (4.4) we obtain

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), \\ &\quad d(y_n, y_{n+1}), \frac{d(y_{n-1}, y_{n+1})}{2}\} \\ &= \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}. \end{aligned} \tag{4.8}$$

Substituting (4.7) and (4.8) into (4.6), we find

$$t_{n+1} \leq \beta_n t_n \text{ for all } n \in \mathbb{N}.$$

According to Lemma 3.1, we conclude that $\{y_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, $\{y_n\}$ converges to $z \in X$. Suppose that $S(X)$ is a closed subspace of X. Therefore, $z = Su$ for some $u \in X$. We claim that $Au = z$. If $Au \neq z$, applying (4.2) we get

$$d(Au, Ax_{n+1}) \leq N(u, x_{n+1}) M(u, x_{n+1}), \tag{4.9}$$

where

$$N(u, x_{n+1}) = \frac{\max \left\{ \begin{array}{c} d(Su, Sx_{n+1}), d(Su, Au) + d(Sx_{n+1}, Ax_{n+1}), \\ d(Su, Ax_{n+1}) + d(Sx_{n+1}, Au) \end{array} \right\}}{d(Su, Au) + d(Sx_{n+1}, Ax_{n+1}) + 1},$$

$$M(u, x_{n+1}) = \max \left\{ \begin{array}{c} d(Su, Sx_{n+1}), d(Su, Au), d(Sx_{n+1}, Ax_{n+1}), \\ \dfrac{d(Su, Ax_{n+1}) + d(Sx_{n+1}, Au)}{2} \end{array} \right\}.$$

Letting $n \to \infty$ in (4.9) we obtain

$$\begin{aligned} d(Au, z) &\leq \frac{d(Au, z)}{d(Au, z) + 1} d(Au, z) \\ &< d(Au, z). \end{aligned}$$

Hence, $Au = Su = z$. Thus u is a coincidence point of A and S and the Jungck sequence $\{y_n\} = \{Sx_n\}$ converges to $z = Au$ for each $x \in X$.

ii) If there exists $u \in C\,(A, S)$ such that $ASu = SAu$, then $Az = Sz$, i.e., z is a coincidence point of A and S.

iii) If $Au \neq Av$, applying (4.2) we find

$$d(Au, Av) \leq N(u, v) M(u, v),$$

where

$$
\begin{aligned}
N(u,v) &= \frac{\max\left\{\begin{array}{c} d(Su,Sv), d(Su,Au)+d(Sv,Av), \\ d(Su,Av)+d(Sv,Au) \end{array}\right\}}{d(Su,Au)+d(Sv,Av)+1} \\
&= 2d(Au,Av),
\end{aligned}
$$

$$
\begin{aligned}
M(u,v) &= \max\left\{\begin{array}{c} d(Su,Sv), d(Su,Au), d(Sv,Av), \\ \dfrac{d(Su,Av)+d(Sv,Au)}{2} \end{array}\right\} \\
&= d(Au,Av).
\end{aligned}
$$

So, $d(Au,Av) \leq 2d^2(Au,Av)$, that is $d(Au,Av) \geq \dfrac{1}{2}$.

iv) If $Au = A^2u$ for each $u \in C(A,S)$ and $ASu = SAu$, therefore $z = Az = Sz$, i.e., A and S possess at least one common fixed point $z \in X$ and the Jungck sequence $\{y_n\} = \{Sx_n\}$ converges to $z = Au$ for any $x \in X$.

(v) If z and w are distinct common fixed points of A and S, employing (4.2) we have

$$d(z,w) = d(Az,Aw) \leq N(z,w)M(z,w),$$

where $N(z,w) = 2d(z,w)$ and $M(z,w) = d(z,w)$. Thus, $d(z,w) \geq \dfrac{1}{2}$. ■

The following examples support our theorem 4.1.

Example 4.1 *Let $X = \{0,1,2,3\}$ be endowed with the usual metric. Define $A,S : X \to X$ by:*

$$
\begin{aligned}
A(0) &= 1, A(1) = 2, A(2) = 1, A(3) = 2, \\
S(0) &= 1, S(1) = 3, S(2) = 3, S(3) = 2.
\end{aligned}
$$

We have $A(X) = \{1,2\} \subset S(X) = \{1,2,3\}$. $C(A,S) = \{0,3\}$. It is easy to see that $A^2(0) = 2 \neq 0$, $A^2(3) = 1 \neq 3$ and A and S do not commute at their coincidence points. The cases $x = y$ and $(x,y) \in \{(0,2),(0,3),(1,3)\}$ are clear.

1) For the case $(x, y) = (0, 1)$ *we get*

$$\begin{aligned} d(A(0), A(1)) &= d(1, 2) = 1 \\ &< N(0, 1)M(0, 1) = 3, \end{aligned}$$

where

$$N(0, 1) = \frac{3}{2} \quad \text{and} \quad M(0, 1) = 2.$$

2) For the case $(x, y) = (1, 2)$ *we obtain*

$$\begin{aligned} d(A(1), A(2)) &= d(2, 1) = 1 \\ &< N(1, 2)M(1, 2) = \frac{3}{2}, \end{aligned}$$

where

$$N(1, 2) = \frac{3}{4} \quad \text{and} \quad M(1, 2) = 2.$$

3) For the case $(x, y) = (2, 3)$ *we find*

$$\begin{aligned} d(A(2), A(3)) &= d(1, 2) = 1 \\ &< N(2, 3)M(2, 3) = \frac{4}{3}, \end{aligned}$$

where

$$N(2, 3) = \frac{2}{3} \quad \text{and} \quad M(2, 3) = 2.$$

Hence, all the hypotheses of theorem 4.1 *hold. Accordingly,* A *and* S *have two coincidence points* 0 *and* 3*. Moreover,* $d(A(0), A(3)) = d(1, 2) > \frac{1}{2}$.

Example 4.2 *Let* $X = \{0, 1, 2, 3\}$ *be equipped with the usual metric. Define* $A, S : X \to X$ *by:*

$$\begin{aligned} A(0) &= 1, A(1) = 2, A(2) = 1, A(3) = 0, \\ S(0) &= 1, S(1) = 2, S(2) = 3, S(3) = 0. \end{aligned}$$

We have $A(X) = \{0, 1, 2\} \subset S(X) = \{0, 1, 2, 3\}$. $C(A, S) = \{0, 1, 3\}$. *It is obvious that* $A^2(0) = 2 \neq 0$, $A^2(3) = 1 \neq 3$ *and* A *and* S *commute at their coincidence points* 0 *and* 3. *The cases* $x = y$ *and* $(x, y) \in \{(0,1), (0,2), (0,3), (1,3)\}$ *are clear.*

1) For the case $(x, y) = (1, 2)$ *we get*

$$\begin{aligned} d(A(1), A(2)) &= d(2,1) = 1 \\ &< N(1,2)M(1,2) = \frac{4}{3}, \end{aligned}$$

where

$$N(1,2) = \frac{2}{3} \quad \textit{and} \quad M(1,2) = 2.$$

2) For the case $(x, y) = (2, 3)$ *we find*

$$\begin{aligned} d(A(2), A(3)) &= d(1,0) = 1 \\ &< N(2,3)M(2,3) = 4, \end{aligned}$$

where

$$N(2,3) = \frac{4}{3} \quad \textit{and} \quad M(2,3) = 3.$$

Hence all the conditions of theorem 4.1 *hold. Consequently,* A *and* S *have three coincidence points* $0, 1$ *and* 3. *Furthermore,*

$$\begin{aligned} d(A(0), A(1)) &= d(1,2) > \frac{1}{2}, \; d(A(0), A(3)) = d(1,0) > \frac{1}{2}, \\ d(A(1), A(3)) &= d(2,0) > \frac{1}{2}. \end{aligned}$$

Example 4.3 *Let* $X = \{0, 1, 2, 3\}$ *be endowed with the usual metric. Define* $A, S : X \to X$ *by:*

$$\begin{aligned} A(0) &= 0, A(1) = 1, A(2) = 1, A(3) = 2, \\ S(0) &= 0, S(1) = 1, S(2) = 3, S(3) = 2. \end{aligned}$$

We have $A(X) = \{0, 1, 2\} \subset S(X) = \{0, 1, 2, 3\}$. $C(A, S) = \{0, 1, 3\}$. *It is evident that* $A^2(0) =$

$A(0) = 0$, $A^2(1) = A(1) = 1$, $A^2(3) = A(2) = 1 \neq 3$, A and S commute at their coincidence points 0 *and* 1 *and do not commute at their coincidence points* 3. *The cases $x = y$ and $(x, y) \in \{(0,1), (0,3), (1,2), (1,3)\}$ are obvious.*

1) For the case $(x, y) = (0, 2)$ we get

$$\begin{aligned} d(A(0), A(2)) &= d(0,1) = 1 \\ &< N(0,2)M(0,2) = 4, \end{aligned}$$

where

$$N(0,2) = \frac{4}{3} \quad \text{and} \quad M(0,2) = 3.$$

2) For the case $(x, y) = (2, 3)$ we obtain

$$\begin{aligned} d(A(2), A(3)) &= d(1,2) = 1 \\ &< N(2,3)M(2,3) = \frac{4}{3}, \end{aligned}$$

where

$$N(2,3) = \frac{2}{3} \quad \text{and} \quad M(2,3) = 2.$$

Hence, all the assumptions of theorem 4.1 *hold. Thus, A and S have two common points* 0 *and* 1 *and a coincidence point* 3. *Besides,*

$$\begin{aligned} d(A(0), A(1)) &= d(0,1) > \frac{1}{2}, \; d(A(0), A(3)) = d(0,2) > \frac{1}{2}, \\ d(A(1), A(3)) &= d(1,2) > \frac{1}{2}. \end{aligned}$$

If $S = I_X$ in theorem 4.1, *where I_X is the identity mapping in X, we have the following corollary*

Corollary 4.1 *Let A be a mapping of a complete metric space (X, d) into itself satisfying*

$$d(Ax, Ay) \leq N(x,y)M(x,y)$$

for all $x, y \in X$, *where*

$$N(x,y) = \frac{\max\{d(x,y), d(x,Ax)+d(y,Ay), d(x,Ay)+d(y,Ax)\}}{d(x,Ax)+d(y,Ay)+1}$$

and

$$M(x,y) = \max\{d(x,y), d(x,Ax), d(y,Ay), \frac{d(x,Ay)+d(y,Ax)}{2}\}.$$

Then

i) A is a WPO.

ii) If z and w are distinct fixed points of A, therefore $d(z,w) \geq \frac{1}{2}$.

Example 4.4 *Let $X = \{0,1,2,3\}$ be equipped with the usual metric. Define $A : X \to X$ by:*

$$A(0) = 0, A(1) = 1, A(2) = 1, A(3) = 0.$$

The cases $x = y$ and $(x,y) \in \{(0,1), (0,3), (1,2)\}$ are clear.

1) For the case $(x,y) = (0,2)$ we get

$$\begin{aligned} d(A(0), A(2)) &= d(0,1) = 1 \\ &< N(0,2)M(0,2) = 3, \end{aligned}$$

where

$$N(0,2) = \frac{3}{2} \text{ and } M(0,2) = 2.$$

2) For the case $(x,y) = (1,3)$ we obtain

$$\begin{aligned} d(A(1), A(3)) &= d(1,0) = 1 \\ &< N(1,3)M(1,3) = \frac{9}{4}, \end{aligned}$$

where

$$N(1,3) = \frac{3}{4} \text{ and } M(1,3) = 3.$$

3) For the case $(x, y) = (2, 3)$ *we find*

$$\begin{aligned} d(A(2), A(3)) &= d(1, 0) = 1 \\ &< N(2, 3)M(2, 3) = \frac{12}{5}, \end{aligned}$$

where

$$N(2, 3) = \frac{4}{5} \;\; \text{and} \;\; M(2, 3) = 3.$$

Hence, A satisfies all the assumptions of Corollary 4.1 *and A has two distinct fixed points* 0 *and* 1*. Besides,* $d(0, 1) = 1 > \frac{1}{2}$*. Since* $d(A(2), A(3)) = 1$ *and*

$$\frac{d(2, A(3)) + d(3, A(2))}{d(2, A(2)) + d(3, A(3)) + 1} d(2, 3) = \frac{4}{5},$$

we get $1 > \frac{4}{5}$*. Therefore, theorem 1 of [31] can not be applicable.*

Remark 4.1 *It is worth mentioning that Corollary* 4.1 *can not be applicable for the mappings A and S in Example* 4.3*, but our theorem* 4.1 *is applicable because for the case* $(x, y) = (2, 3)$ *we get*

$$\begin{aligned} d(A(2), A(3)) &= d(1, 2) = 1 \\ &> N(2, 3)M(2, 3) = \frac{2}{3}, \end{aligned}$$

where

$$N(2, 3) = \frac{2}{3}, \; M(2, 3) = 1.$$

$$\begin{aligned} d(S(2), S(3)) &= d(3, 2) = 1 \\ &> N(2, 3)M(2, 3) = \frac{2}{3}, \end{aligned}$$

where

$$N(2, 3) = \frac{2}{3}, \; M(2, 3) = 1.$$

Also, theorem 1 of [31] can not be applicable for the mappings A and S. This shows that our theorem 4.1 *is a genuine generalization of Corollary* 4.1 *and theorem 1 of [31].*

4.3 Application in dynamic programming

We apply our theorem to realize the existence of common solutions of a system of two functional equations arising in dynamic programming.

Let X and Y be Banach spaces, $S \subset X$ be the state space, $D \subset Y$ be the decision space and I_X be the identity mapping on X. $B(S)$ denotes the set of all bounded real valued functions on S and

$$d(f, g) = \sup_{x \in S} |f(x) - g(x)|.$$

It is clear that $(B(S), d)$ is a complete metric space.

As proposed in Bellman and Lee [8], the basic form of the functional equation in dynamic programming is

$$f(x) = \underset{y \in D}{opt} H(x, y, f(T(x, y))),\ x \in S,$$

where x and y denote the state and decision vectors, respectively. T denotes the transformation of the process, $f(x)$ denotes the optimal return function with the initial state x and *opt* represents sup or inf.

Many authors proved the existence and the uniqueness of solutions or common solutions for several classes of functional equations or systems of functional equations arising in dynamic programming by employing various fixed and common fixed point theorems, see Bhakta and Mitra [11], Kalinde et al. [28], Li et al. [36], Liu [37], Liu et al. [38, 39, 40, 41] and Pathak et al. [46].

In this section, applying theorem 4.1, we establish the existence of common solutions of the following system of two functional equations arising in dynamic programming.

$$f_i(x) = \underset{y \in D}{opt}\{u(x, y) + H_i(x, y, f_i(T(x, y)))\},\ x \in S,\ i = 1, 2, \tag{4.10}$$

where $u : S \times D \to S$, $T : S \times D \to S$ and $H_i : S \times D \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$.

Theorem 4.2 *Suppose that the following conditions are verified*

(c_1) u *and* H_i *are bounded for* $i = 1, 2$,

(c_2) *For all* $(x, y) \in S \times D$, $g, h \in B(S)$ *and* $t \in S$

$$|H_1(x, y, g(t)) - H_1(x, y, h(t))| \leq N(g(t), h(t))M(g(t), h(t)), \tag{4.11}$$

where

$$\begin{aligned} N(g(t), h(t)) &= \frac{\max\left\{\begin{array}{c} |A_2g(t) - A_2h(t)|, |A_2g(t) - A_1g(t)| + |A_2h(t) - A_1h(t)|, \\ |A_2g(t) - A_1h(t)| + |A_2h(t) - A_1g(t)| \end{array}\right\}}{d(A_2g, A_1g) + d(A_2h, A_1h) + 1}, \\ M(g(t), h(t)) &= \max\left\{\begin{array}{c} |A_2g(t) - A_2h(t)|, |A_2g(t) - A_1g(t)|, |A_2h(t) - A_1h(t)|, \\ \dfrac{|A_2g(t) - A_1h(t)| + |A_2h(t) - A_1g(t)|}{2} \end{array}\right\} \end{aligned}$$

and

$$A_ig_i(x) = \underset{y \in D}{opt}\{u(x, y) + H_i(x, y, g_i(T(x, y)))\},\ x \in S,\ i = 1, 2.$$

(c_3) : $A_1(B(S)) \subset A_2(B(S))$,

(c_4) : $A_1u = A_1^2u$ *for some* $u \in C(A_1, A_2)$ *and* $A_1A_2u = A_2A_1u$, $u \in B(S)$.

Then, the system of functional equations (4.10) *possesses at least one common solution in* $B(S)$. *In addition, if* z *and* w *are two distinct solutions of* (4.10), *therefore* $d(z, w) \geq \dfrac{1}{2}$.

Proof. It follows from (c_1) and (c_2) that A_1 and A_2 are self-mappings in $B(S)$. Assuming that $\underset{y \in D}{opt} = \sup_{y \in D}$. For each $g, h \in B(S)$, $x \in S$ and $\epsilon > 0$, there exists $y, z \in D$ such that

$$A_1g(x) < u(x, y) + H_1(x, y, g(T(x, y))) + \epsilon, \tag{4.12}$$

$$A_1h(x) < u(x, z) + H_1(x, z, h(T(x, z))) + \epsilon. \tag{4.13}$$

It is easy to see that

$$A_1g(x) \geq u(x, z) + H_1(x, z, g(T(x, z))), \tag{4.14}$$

$$A_1h(x) \geq u(x, y) + H_1(x, y, h(T(x, y))). \tag{4.15}$$

By virtue of (4.12) and (4.15), we infer that

$$\begin{aligned} A_1g(x) - A_1h(x) &< H_1(x, y, g(T(x, y))) - H_1(x, y, h(T(x, y))) + \epsilon \\ &\leq |H_1(x, y, g(T(x, y))) - H_1(x, y, h(T(x, y)))| + \epsilon \\ &\leq N(g(t), h(t))M(g(t), h(t)) + \epsilon. \end{aligned} \tag{4.16}$$

From (4.13) and (4.14) we conclude that

$$\begin{aligned} A_1g(x) - A_1h(x) &> H_1(x, z, g(T(x, z))) - H_1(x, z, h(T(x, z))) - \epsilon \\ &\geq -|H_1(x, z, g(T(x, z))) - H_1(x, z, h(T(x, z)))| - \epsilon \\ &\geq -N(g(t), h(t))M(g(t), h(t)) - \epsilon. \end{aligned} \tag{4.17}$$

It follows from (4.16) and (4.17) that

$$|A_1g(x) - A_1h(x)| \leq N(g(t), h(t))M(g(t), h(t)) + \epsilon.$$

Using (4.11). and the above inequality we obtain

$$|A_1g(x) - A_2h(x)| \leq N(g, h)M(g, h) + \epsilon.$$

Hence

$$d(A_1g, A_1h) \leq N(g, h)M(g, h) + \epsilon. \tag{4.18}$$

Similarly, the inequality (4.18) also holds for $opt_{y\in D} = \inf_{y\in D}$. Letting $\epsilon \to 0$ in (4.18) we deduce that

$$d(A_1g, A_1h) \leq N(g, h)M(g, h).$$

Due to theorem 4.1, A_1 and A_2 have at least one common fixed point $z \in B(S)$, i.e., z is a common solution of the system of functional equations (4.10). In addition, if z and w are two distinct solutions of (4.10), therefore $d(z, w) \geq \frac{1}{2}$. ■

Conclusion

Fixed point theory is a fascinating subject with an enormous number of applications in various fields of mathematics. We were interested in this thesis to find some applications involving new generalizations theorems in metric spaces. Our work included two principal results:

1. We established unique fixed point theorems for a self-mapping in complete metric spaces and that the fixed point problem is well-posed. Examples were provided to illustrate the validity of our results and we gave some remarks about the papers, [3], [4] and [31]. Afterwards, we applied our theorem 3.2 to study the possibility of optimally controlling the solution of an ordinary differential equation via dynamic programming.

2. We established coincidence and common fixed points theorem for two self-mappings in complete metric spaces. Our theorem generalizes theorem 1 of [31]. Suitable examples are provided to illustrate the validity of our results. We applied our theorem to establish the existence of common solutions of a system of two functional equations arising in dynamic programming.

As perspectives, we may be able to produce some results in the field of fixed point theory, by making changes in the used spaces, in the properties of the applications, or in the conditions of contraction, with applications in various fields of mathematics and in mathematical physics, etc.

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