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GLOBAL SOLUTIONS TO A SYSTEM OF STRONGLY COUPLED REACTION–DIFFUSION EQUATIONS

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1. INTRODUCTION

This article is concerned with the existence of globally bounded in time classical solutions to the following system of strongly coupled reaction–diffusion equations

$$\begin{pmatrix} u \\ v \end{pmatrix}_t - \nabla \cdot \left(\begin{pmatrix} \theta & -\alpha\beta u \\ 0 & (\varphi + \alpha\beta u) \end{pmatrix} \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix} \right) = \begin{pmatrix} \alpha\beta \nabla u \cdot \nabla v - \beta u v \\ -\alpha\beta \nabla u \cdot \nabla v + \beta u v - \lambda v \end{pmatrix} \tag{1.1}$$

in $\Omega \times (0, +\infty)$, under homogeneous boundary conditions

$$\nabla u \cdot \eta = \nabla v \cdot \eta = 0 \qquad \text{on } \partial\Omega \times (0, \infty) \tag{1.2}$$

and initial conditions

$$u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x) \qquad \text{in } \Omega. \tag{1.3}$$

Here, Ω is an open bounded regular domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ and outer normal $\eta(x)$.

The system (1.1) was proposed by Bailey in [1, p. 172] as a diffusive version of the classical Kermack-McKendrick model which represents a basic model for the description of epidemics obeying the susceptibles-infectives-removed (S-I-R) scheme [2].

The functions $u(x, t)$ and $v(x, t)$ (t time and x position) represent spatial densities of susceptibles and infectives. The factor $\alpha \geq 0$ refers to the spatial influence of infectives, $\theta > 0$ and $\varphi > 0$ to the migration-rates of susceptibles and infectives, respectively, β to the infection-rate and λ to the removal rate. Here, it is assumed that the influences are both local and isotropic: this accounts for the operator $\nabla^2 (= \Delta)$.

The model (1.1) can also be used to predict the spread of a rumour through a population. For this interpretation of the model, the susceptibles are identified with those not having heard the rumour and the infectives correspond to those who are actively spreading the rumour. In the case where $\alpha = 0$ (the right-hand side of (1.1) is then reduced to $(-\beta uv, \beta uv - \lambda v)^T$, system (1.1) was treated by Capasso [3] in $X = L^1(\Omega) \times L^1(\Omega)$ and Webb [4] in $X = C(\Omega) \times C(\Omega)$, but with the severe restrictions $\theta = \varphi$ and $n = 1$. In [5], Haraux and Kirane extended the work of Webb to n-dimensional regions ($n \geq 1$) and different diffusivities θ and φ, see also the recent paper of Fitzgibbon and Morgan [6]. These works were recently extended by Kirane [7] to the case of biatic diffusions, i.e. to the case where $\theta\Delta u$ and $\varphi\Delta v$ are replaced by $\nabla(a(u)\nabla u)$ and $\nabla(b(v)\nabla v)$, respectively, with $a(\cdot)$ and $b(\cdot)$ positive and bounded. One can also mention the papers by Capasso and Fortunato [8], and de Mottoni *et al.* [9] where the nonlinear interaction between u and v is taken of nonlocal type, i.e. equal to $u\int_\Omega k(x, y)v(y)\,dy$, where k is a nonnegative kernel with support in $\Omega \times \Omega$. At this stage, we want to mention the papers by Kim [10], Deuring [11], Posio and Tesei [12], Redlinger [13], Amann [19] and Wiegner [15] who dealt with nondiagonal systems. In [12, 13 and 19], the authors take advantage of the fact that one component is "trivially" uniformly bounded and use it to bound the other component(s). The analysis of [10] is one-dimensional while the one in [11] is multi-dimensional but restricted to small off-diagonal coefficients. The work [15] requires the strong Legendre-condition for the diffusion matrix. The diffusion matrix in (1.1) does not satisfy the strong Legendre-condition unless some restriction is imposed on u. On the other hand, a "trivial" uniform bound for one component is not available.

2. PRELIMINARY RESULTS AND NOTATIONS

The following notations will be used

$$Q := \Omega \times (0, T).$$

If $u \in L^p(D)$, the (usual) norm of u in L^p is denoted by $\|u\|_{p,D}$ for $p \in [1, +\infty]$ ($D = \Omega$ or Q).

If Z is a Banach space and $a \in \mathbb{R}^+$, $C_B([a, +\infty), Z)$ denotes the space of continuous functions $u\colon [a, +\infty) \to Z$ such that $u(t)$ remains bounded in Z for $t \geq a$. The spaces $W_p^{2,1}(Q)$ and $C^{\alpha/2,\alpha}(Q)$ are described in [17].

(IC) The initial conditions $u_0 \geq 0$, $v_0 \geq 0$ are taken in $W_p^1(\Omega)$ with $p > n + 2$.

For physical reasons, only nonnegative solutions are of interest, so from now on u and v are taken to be nonnegative.

In this range of (u, v) system (1.1) is parabolic in the sense of Petrovskii so the elliptic operator associated to it is normally elliptic [14]. According to [14], this guarantees the existence of a unique maximal classical solution.

PROPOSITION 2.1. There is a unique maximal solution $(u(\cdot, (u_0, v_0)), v(\cdot, (u_0, v_0)))$ which is smooth on $\bar{\Omega} \times (0, T_{\max}(u_0, v_0))$. The map $(t, u(u_0, v_0)) \mapsto (u(t, (u_0, v_0)), v(t, (u_0, v_0)))$ is a semi-flow on $(W_p^1)^+$ such that bounded orbits are relatively compact. If there exist $\sigma > 0$ and a continuous $C\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|u(t), v(t)\|_{C^{\sigma/2,\sigma}(Q)} \leq C(T) \qquad \text{for } 0 \leq t \leq T < \infty, \qquad T < T_{\max}, \tag{2.1}$$

then $T_{\max}(u_0, v_0) = +\infty$.

For the proof of this result, we refer to [14].

Results

Our main objective is to establish that the solution is in fact globally bounded in time. This requires the estimate (2.1). We succeeded to obtain estimate (2.1) only in the case where $\varphi \leq \theta$. In the case where $\varphi > \theta$, only some estimates of some norms of Orlicz are presented.

These results are presented in Sections 3 and 4.

3. THE CASE $\theta \geq \varphi$

Concerning Global solutions we have the following theorem.

THEOREM 3.1. Let us assume that $(u_0, v_0) \in W^{2(1-1/p)}(\Omega)$ with $\nabla u_0 \cdot \eta = 0$, $\nabla v_0 \cdot \eta = 0$ on Γ if $p > 3$. Then the solution to (1.1)–(1.3) exists globally in time whenever $\varphi < \theta$. Moreover, $v \to 0$ uniformly in x, and there exists a constant $\bar{u} \geq 0$ such that $u \to \bar{u}$ uniformly in x, and if $u_0 \neq 0$ then $\bar{u} > 0$.

The proof of theorem 3.1 is based on some a priori estimates which are collected in the following lemmata.

LEMMA 3.2. For any $0 < t < T_{\max}$, we have

$$\begin{aligned} \|u(t)\|_{2,\Omega} \leq C, &\qquad \|v(t)\|_{2,\Omega} \leq C, \\ \|\nabla u\|_{2,Q} \leq C, &\qquad \|\nabla v\|_{2,Q} \leq C. \end{aligned} \tag{3.1}$$

The following result which makes use of the maximum principle for an auxiliary nonlinear function of the unknowns is crucial for our analysis.

LEMMA 3.3. For any $0 < t < T_{\max}$, we have

$$u, v \in C_B(\mathbb{R}^+; C(\bar{\Omega})) \tag{3.2}$$

whenever $\varphi < \theta$.

LEMMA 3.4. For any $0 < t < T_{\max}$, we have

$$\|v\|_{p,Q} \leq C \tag{3.3}$$

whenever $\varphi < \theta$.

Proof of the lemmata

Let us start with the proof of lemma 3.2.

Proof of lemma 3.2. We have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left\{ (u + v)^2 + 2\frac{\theta + \varphi}{\alpha\beta} u \right\} \mathrm{d}x \\ &\quad = -2 \int_\Omega \left(\theta|\nabla u|^2 + \varphi|\nabla v|^2 + \left(\lambda + \frac{\theta + \varphi}{\alpha} \right) uv + \lambda v^2 \right) \mathrm{d}x \leq 0. \end{aligned} \tag{3.4}$$

Integrating over $(0, t)$ $(t < T_{\max})$ yields

$$\int_\Omega \left((u + v)^2 + 2\frac{\theta + \varphi}{\alpha\beta}u\right) + 2\int_Q \left((\theta|\nabla u|^2 + \varphi|\nabla v|^2) + \left(\lambda + \frac{\theta + \varphi}{\alpha}\right)uv + \lambda v^2\right)$$

$$= \int_\Omega \left((u_0 + v_0)^2 + 2\frac{\theta + \varphi}{\alpha\beta}u_0\right) \leq C,$$

where the constant C depends only on $\|u_0\|_2$ and $\|v_0\|_2$. Hence the estimates (3.1).

Remark 3.5. By the inequality (3.4), the system (1.1)-(1.3) does admit neither t-periodic nor (x, t)-periodic solutions.

Proof of lemma 3.3. As for $t < T_{\max}$ (u, v) is a classical solution, the system (1.1) can be written in the following form

$$\begin{aligned} u_t - \theta\,\nabla u &= -\alpha\beta u\,\Delta v - \beta uv, \\ v_t - \varphi\,\Delta v &= \alpha\beta u\,\Delta v + \beta uv - \lambda v. \end{aligned} \tag{1.1$'$}$$

Now, define $S(u, v)$ by

$$S(u, v) := v + u + H + H\log(-u/H),$$

where $H := (\varphi - \theta)/\alpha\beta < 0$.

Noteworthy properties of S include

$$S(-H, 0) = 0,$$

$$S(u, v) > 0 \qquad \text{for } u, v \in \mathbb{R}_+, \qquad v \neq 0,\ u \neq -H,$$

(note that $u + H + H\log(-u/H) > 0$),

$$S(u, v) \to +\infty \qquad \text{as } |(u, v)| \to +\infty \text{ or } u \to 0.$$

Now define $\Sigma(x, t) := S(u(x, t), v(x, t))$, one has

$$\begin{aligned} \Sigma_t &= (u + v)_t + Hu_t/u \\ &= \theta\,\Delta u + \varphi\,\Delta v - \lambda v + H\theta\,\Delta u/u - \alpha\beta H\,\Delta v - \beta Hv \\ &= \theta(1 + H/u)\,\Delta u + (\varphi - \alpha\beta H)\,\Delta v - (\lambda + \beta H)v, \end{aligned}$$

and

$$\Delta\Sigma = (1 + H/u)\,\Delta u + \Delta v - H|\nabla \log u|^2,$$

so that

$$\Sigma_t - \theta\,\Delta\Sigma = \theta H|\nabla \log u|^2 - (\lambda + \beta H)v. \tag{3.5}$$

The boundary and initial conditions for Σ are

$$\nabla\Sigma \cdot \eta = 0, \tag{3.6}$$

and

$$\Sigma_\delta(x) = u_\delta(x) + v_\delta(x) + H + H\log(-u_\delta(x)/H) \tag{3.7}$$

$\Sigma_\delta(x)$ is bounded, where $0 < \delta < T_{\max}$.

Now, as $v \leq \Sigma$, it is a consequence of the maximum principle that Σ exists for all $t > 0$. On the other hand, as $v \in L^\infty(0, +\infty; L^2(\Omega))$ we have

$$\Sigma \in C_B(\mathbb{R}^+, C(\bar{\Omega}))$$

by simple use of proposition (3.3) of [5].

As $u + H + H\log(-u/H) > 0$, we can deduce the following estimates

$$0 < v(x, t) < M,$$

and

$$0 < C_0(M) \leq u(x, t) \leq C_1(M) < +\infty,$$

where M depends only on $\|u_\delta\|_\infty$ and $\|v_\delta\|_\infty$, and $C_0(M)$ and $C_1(M)$ are the solutions of

$$M = \mu + H + H\log(-\mu/H).$$

Proof of lemma 3.4. We have

$$(u + v)_t - \Delta(\theta u + \varphi v) = -\lambda v. \tag{3.8}$$

Multiplying (3.8) through by $1/p(u + v)^{p-1}$, integrating by parts, using the inequality $ab \leq a^2/2 + b^2/2$, and lemma 3.3, we infer

$$\|u + v\|_{p,\Omega}^p + \frac{\lambda}{p}\int_0^T\int_\Omega v(u + v)^{p-1} \leq C\{\|\nabla u\|_{2,Q}^2 + \|\nabla v\|_{2,Q}^2\} + \|u_0 + v_0\|_{p,\Omega}^p$$

which, using lemma 3.2, provides estimate 3.3.

We are now in position to prove theorem 3.1.

Proof of theorem 3.1. The second equation of (1.1)′ can be written in the following form

$$v_t - (\varphi + \alpha\beta u)\,\Delta v = \beta uv - \lambda v.$$

Now, invoking the estimate (81) of Solonnikov [18] (see also the recent paper of Weidemaier [16]), we have

$$\|v\|_{W_p^{2,1}(Q)} \leq C(\|v\|_{p,Q} + \|\beta uv - \lambda v\|_{p,Q} + \|v_0\|_{W_p^{2(1-1/p)}(\Omega)}).$$

It is clear that using lemma 3.3 and corollary 3.6 we can have the estimates

$$\|v_t\|_{p,Q} \leq C, \qquad \|\Delta v\|_{p,Q} \leq C, \tag{3.9}$$

where C is a constant independent of t. Using the first equation of (1.1)′, lemma 3.3, corollary 3.6 and (3.9) yields

$$\|u_t\|_{p,Q} \leq C, \qquad \|\Delta u\|_{p,Q} \leq C, \tag{3.10}$$

where C is also independent of t. So, $u, v \in W_p^{2,1}(\Omega \times (0, T)) \hookrightarrow C^{\alpha/2,\alpha}(\bar{\Omega} \times [0, T])$ for $p > n + 2$ [17]. Hence (2.1) is satisfied in the case $\varphi < \theta$ and we have a global classical solution.

The case where $\theta = \varphi$ is simple to handle. In fact, the function $u + v$ satisfies a simple heat equation

$$(u + v)_t - \theta\Delta(u + v) = -\lambda v.$$

An application of the maximum principle yields

$$\|u + v\|_{\infty,\Omega} \leq C.$$

So we can conclude as before.

4. THE CASE $\theta < \varphi$

Estimates of some norms of Orlicz of u and v

In the previous, estimates were obtained for the maximums of the moduli of u and v under the condition that $\theta \geq \varphi$. In the case where this condition is noy satisfied, i.e. $\theta < \varphi$, our method of proof presented below does not give L^∞-bounds for u and v. We rather obtain some estimates in some Orlicz space. These estimates are insufficient to obtain the sought estimate (2.1) but, in the case where an L^∞-bound of u is obtained, they can be used to get the estimate (2.1) along the method used previously to prove theorem 3.1. At any rate, we now formally state the following result.

THEOREM 4.1. Suppose $\theta < \varphi$ and condition (IC) satisfied. Then

$$\int_\Omega e^{A(u+v)}\,dx \leq C,$$

where C is a constant independent of t, and $A = \alpha\beta/(\varphi + \theta) > 0$.

The proof of theorem 4.1 makes use of the following lemma.

LEMMA 4.2. For any $0 < t < T_{\max}$,

$$\mathcal{L}(t) = \int_\Omega \left(u + \frac{2\theta}{\alpha\beta}\right)^p \exp(A(u + v))\,dx, \tag{4.1}$$

where $p := (\varphi - \theta)/(\varphi + \theta)$, $A := \alpha\beta/(\varphi + \theta)$, is a Lyapunov functional whenever $\varphi > \theta$.

COROLLARY 4.3. We have

$$\|v\|_{p,Q} \leq C(p), \qquad \|u\|_{p,Q} \leq C(p) \tag{4.2}$$

and $\|uv\|_{p,Q} \leq C(p)$.

Proof of lemma 4.2. Differentiating $\mathcal{L}$ with respect to t yields

$$\mathcal{L}_t(u, v)(t) = p\int_\Omega \left(u + \frac{2\theta}{\alpha\beta}\right)^{p-1} e^{A(u+v)}u_t\,dx + A\int_\Omega \left(u + \frac{2\theta}{\alpha\beta}\right)^p e^{A(u+v)}(u + v)_t\,dx =: I + J.$$

Replacing u_t by $\theta\,\Delta u - \alpha\beta u\,\Delta v - \beta uv$ in I, and integrating by parts yields

$$I = I_1 + I_2,$$

where

$$I_1 := \int_\Omega \{\mathscr{A}^I(u)|\nabla u|^2 + \mathscr{B}^I(u)\,\nabla u\cdot\nabla v + \mathscr{C}^I(u)|\nabla u|^2\}\left(u + \frac{2\theta}{\alpha\beta}\right)^{p-2} \mathrm{e}^{A(u+v)}\,\mathrm{d}x,$$

where

$$\mathscr{A}^I(u) := -\theta p(p-1) - Ap\theta\left(u + \frac{2\theta}{\alpha\beta}\right),$$

$$\mathscr{B}^I(u) := pA\theta\left(u + \frac{2\theta}{\alpha\beta}\right) + \alpha\beta p(p-1)u + \alpha\beta Apu\left(u + \frac{2\theta}{\alpha\beta}\right) + \alpha\beta p\left(u + \frac{2\theta}{\alpha\beta}\right),$$

$$\mathscr{C}^I(u) := pA\alpha\beta u\left(u + \frac{2\theta}{\alpha\beta}\right),$$

and

$$I_2 := -p\beta \int_\Omega uv\left(u + \frac{2\theta}{\alpha\beta}\right)^{p-1} \mathrm{e}^{A(u+v)}\,\mathrm{d}x.$$

Also, replacing $(u+v)_t$ by $\Delta(\theta u + \varphi v) - \lambda v$ in J, and integrating by parts yields

$$J = J_1 + J_2,$$

where

$$J_1 := \int_\Omega \{\mathscr{A}^J(u)|\nabla u|^2 + \mathscr{B}^J(u)\nabla u\cdot\nabla v + \mathscr{C}^J(u)|\nabla v|^2\}\left(u + \frac{2\theta}{\alpha\beta}\right)^{p-2} \mathrm{e}^{A(u+v)}\,\mathrm{d}x,$$

where

$$\mathscr{A}^J(u) := -pA\theta\left(u + \frac{2\theta}{\alpha\beta}\right) - A^2\theta\left(u + \frac{2\theta}{\alpha\beta}\right)^2,$$

$$\mathscr{B}^J(u) := -\varphi pA\left(u + \frac{2\theta}{\alpha\beta}\right) - (\varphi + \theta)A^2\left(u + \frac{2\theta}{\alpha\beta}\right),$$

$$\mathscr{C}^J(u) := -A^2\varphi\left(u + \frac{2\theta}{\alpha\beta}\right)^2,$$

and

$$J_2 := -\lambda A \int_\Omega v\left(u + \frac{2\theta}{\alpha\beta}\right)^p \mathrm{e}^{A(u+v)}\,\mathrm{d}x.$$

As $P/A = (\varphi - \theta)/\alpha\beta$ (>0) we have

$$I_1 + J_1 = \int_\Omega \{\mathscr{A}^{IJ}(u)|\nabla u|^2 + \mathscr{B}^{IJ}(u)\nabla u\cdot\nabla v + \mathscr{C}^{IJ}(u)|\nabla v|^2\}\left(u + \frac{2\theta}{\alpha\beta}\right)^{p-2} \mathrm{e}^{A(u+v)}\,\mathrm{d}x,$$

where

$$\mathcal{A}^{IJ}(u) := -\theta\left(\left(A\left(u + \frac{2\theta}{\alpha\beta}\right) + p\right)^2 - p\right),$$

$$\mathcal{B}^{IJ}(u) := 2\left(\left(A\left(u + \frac{2\theta}{\alpha\beta}\right) + p\right)^2 - p\right),$$

and

$$\mathcal{C}^{IJ}(u) := \left[\left(A\left(u + \frac{2\theta}{\alpha\beta}\right) + p\right)^2 - p\right] + p - p^2.$$

The bilinear form $\mathcal{A}^{IJ}(u)\zeta^2 + \mathcal{B}^{IJ}(u)\zeta\eta + \mathcal{C}^{IJ}\eta^2$ is nonpositive because

$$0 < \left(\frac{\varphi - \theta}{\varphi + \theta}\right)^2 < p < 1.$$

This ends the proof of lemma 3.5.

Summing up, we have

$$\mathcal{L}_t(u, v) + \int_\Omega v\left(\lambda u + \left(\frac{2\lambda\theta}{\alpha\beta} + \beta p\right)\right)\left(u + \frac{2\theta}{\alpha\beta}\right)^{p-1} \mathrm{e}^{A(u+v)}\,\mathrm{d}x \le 0.$$

Integrating with respect to t yields

$$\mathcal{L}(u, v)(t) + \int_0^T \int_\Omega v\left(\lambda u + \left(\frac{2\lambda\theta}{\alpha\beta} + \beta p\right)\right)\left(u + \frac{2\theta}{\alpha\beta}\right)^{p-1} \mathrm{e}^{A(u+v)}\,\mathrm{d}x \le \mathcal{L}(u, v)(0) \le C.$$

In particular we have

$$\int_0^T \int_\Omega v\,\mathrm{e}^{A(u+v)}\,\mathrm{d}x\,\mathrm{d}t \le C, \qquad \text{and} \qquad \mathcal{L}(u, v)(t) \le C. \tag{4.3}$$

Estimates (4.2) follow from estimate (4.3).

The proof of theorem 4.1 is a consequence of lemma 4.2.

Remarks. (1) The results of theorem 4.1 remain valid for homogeneous boundary conditions of Dirichlet type for u and v, and for homogeneous boundary conditions of Dirichlet–Neumann type for (u, v) or (v, u).

(2) As is well known there are two general approaches to the theory of nonlinear parabolic problems. The first approach is based on the consideration of the initial boundary value problem for a nonlinear parabolic equation (or systems) as a Cauchy problem for the corresponding nonlinear differential-operator equation (or system) in a Banach space.

The second approach is based on the proper study of the boundary problem as an object of the theory of nonlinear partial differential equations. Here for the local existence we followed the first approach along the theory of Amann [14]. We could use the second approach using from the beginning the system in form (1.1)′ which is the form proposed in [1], and then use a continuation argument.

(3) The interaction terms—βuv and βuv—λv could be replaced either by $-uvh(u, v)$ and $uvh(u, v) - \lambda v$, respectively, with $h \in C^1(\mathbb{R}_+^2, \mathbb{R}_+)$ or by $\gamma u + \mu - \mu v$ and $-\gamma v - \mu v$, respectively, which correspond to the S-I-S model.

(4) The terms $-\alpha\beta u\,\Delta v$ and $\alpha\beta u\,\Delta v$ could be replaced by $-\rho u\,\Delta v$ and $\gamma u\,\Delta v$, respectively, where $\gamma \le \rho$. In this case Σ has to be changed into

$$u + \rho v/\gamma + H + H\log(-u/H),$$

where $H := (\varphi - \theta)/\gamma$.

5. LARGE TIME BEHAVIOUR

For the large time behaviour of solutions we have the following result.

THEOREM 5.1. Let (u, v) be the solution to (1.1)-(1.3) in the case $\varphi < \theta$. Then

$$(u, v) \to (\bar{u}, 0) \qquad \text{uniformly in } x.$$

Moreover, if $u_0 \not\equiv 0$, then $\bar{u} > 0$.

Proof. We define the Lyapunov functional

$$\mathcal{L}(u, v)(t) := \int_\Omega \left\{(u + v)^2 + 2\frac{\theta + \varphi}{\alpha\beta}u\right\} \mathrm{d}x.$$

Differentiating $\mathcal{L}$ with respect to t and using Green's identity yields

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}t} = -\int_\Omega \left(\theta|\nabla u|^2 + \varphi|\nabla v|^2 + \left(\lambda + \frac{\theta + \varphi}{\alpha}\right)uv + \lambda v^2\right) \le 0.$$

Thus $\mathcal{L}(u, v)(t) \ge 0$ is monotone nonincreasing, hence it tends to a finite limit as t goes to infinity, and there exists a sequence $\{t_k\}$ tending to infinity such that $\mathrm{d}\mathcal{L}(u, v)(t_k)/\mathrm{d}t \to 0$ as t_k goes to infinity (otherwise $\mathcal{L}(u, v)(t_k) \to -\infty$ which is absurd). Therefore,

$$\lim_{T\uparrow+\infty} \int_T^{+\infty}\int_\Omega \{\theta|\nabla u|^2 + \varphi|\nabla v|^2 + (\lambda + (\theta + \varphi)/\alpha)uv + \lambda v^2\}\,\mathrm{d}x\,\mathrm{d}t = 0.$$

It is then clear that

$$\lim_{k\uparrow+\infty} \int_{-\delta}^{+\delta}\int_\Omega \{\theta|\nabla u|^2 + \varphi|\nabla v|^2 + (\lambda + (\theta + \varphi)/\alpha)uv + \lambda v^2\}(x, t_k + \tau)\,\mathrm{d}x\,\mathrm{d}\tau = 0.$$

This implies that $|\nabla\tilde{u}| = \tilde{v} = 0$, where $(\tilde{u}, \tilde{v})$ is an element of the ω-limit set. Hence $\tilde{u}$ is a positive constant.

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