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Kernel-type estimator of the reinsurance premium for heavy-tailed loss distributions

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HIGHLIGHTS

- We generalize the classical estimator of the reinsurance premium for heavy-tailed losses.
- We propose a bias-reduced estimator of the reinsurance premium.
- The asymptotic normality of the given estimators is established.
- A small simulation study illustrates the performance of our approach.

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ABSTRACT

In this paper, we generalize the classical estimator of the reinsurance premium for heavy-tailed loss distributions with a kernel-type estimator. Since this estimator exhibits a bias, we propose its bias-reduced version by using a least-squares method. The asymptotic normality of the proposed estimators is established under suitable assumptions. A small simulation study is carried out to prove the performance of our approach.

1. Introduction

The major worry for the insurance and reinsurance companies is to determine the adequate premium. In the insurance literature, there exist several premium principles such as: expected value, variance and value-at-risk. For more details on premium principles and their properties, we refer to Goovaerts et al. (1984). Wang (1996) proposed a premium principle named proportional hazard premium (PHP) of an insured risk X, a non-negative random variable defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with continuous distribution function F, depends on the hazard function $S = 1-F$ and a parameter $r \geq 1$ called the risk aversion index or the distortion parameter. The PHP is defined as follows

$$\Pi_r = \int_0^\infty (S(x))^{1/r}\, dx.$$

In some actuarial problems, as in the reinsurance treaty, one is interested in the estimation of a premium for a given retention level $R > 0$ notation $\Pi_{r,R}$, that is, a reinsurance premium of the high layer $[R, \infty)$. This type of problem can be found whenever the insured represents a dangerous level of risk for the insurance company, and decides to give a part of this loss to another reinsurance company, because it may not have sufficient capital to cover the total risk. The reinsurance premium of the high layer is defined as follows

$$\Pi_{r,R} = \int_R^\infty (S(x))^{1/r}\, dx.$$

For heavy-tailed distributions, Beirlant et al. (2001), Necir and Boukhetala (2004), Vandewalle and Beirlant (2006) and Necir et al. (2007) have introduced and studied different estimators for $\Pi_{r,R}$, in the case of high-excess loss layers ($R \to \infty$).

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A distribution function F is said to be heavy-tailed whenever the tail function $1-F$ is a regularly varying function with index $(-1/\gamma) < 0$, i.e., for any $x > 0$,

$$1-F(x) = x^{-1/\gamma}\mathbb{L}(x),$$

where $\mathbb{L}$ is a slowly varying function at infinity, that is, $\mathbb{L}(tx)/\mathbb{L}(t) \to 1$ as $t \to \infty$. The class of regularly varying functions includes popular distributions such those Pareto's, Burr's, Student's, Fréchet's, α-stable $(0 < \alpha < 2)$, and log-gamma, which are known to be appropriate models of fitting large insurance claims, large fluctuations of prices, log-returns, and so on (see Beirlant et al., 2001).

Let $X_{1,n} \le \cdots \le X_{n,n}$, $n \ge 1$, be the order statistics pertaining to a sample $X_1, \ldots, X_n$ from X and let $k = k_n$ be an integer sequence satisfying

$$1 < k < n, k \to \infty, \quad \text{and} \quad k/n \to 0 \quad \text{as } n \to \infty.$$

Let, for $0 < s < 1$, $Q(s) = \inf\{x : F(x) \ge s\}$ be the quantile function pertaining to F. At an optimal retention level $R = R_{opt} = Q(1-k/n)$, the semi-parametric estimator for $\Pi_{r,R}$ that proposed by Necir et al. (2007) is

$$\widetilde{\Pi}_{r,\widehat{R}_{opt}} = (k/n)^{1/r}\frac{r}{1/\widehat{\gamma}^H_{n,k} - r}X_{n-k,n}, \quad \text{for } \widehat{\gamma}^H_{n,k} < 1/r,$$

where $\widehat{R}_{opt} = X_{n-k,n}$ and $\widehat{\gamma}^H_{n,k}$ is the classical Hill estimator (Hill, 1975) of the tail index γ, defined by

$$\widehat{\gamma}^H_{n,k} = \frac{1}{k}\sum_{i=1}^{k} i\left(\log X_{n-i+1,n} - \log X_{n-i,n}\right).$$

A major drawback of the Hill estimator is the discrete character of its behavior in the sense that increasing k by 1, can change the actual value of the estimate considerably. Using a kernel function K, Csörgő et al. (1985) proposed a smoother version of Hill's estimator defined by

$$\widehat{\gamma}^K_{n,k} = \frac{1}{k}\sum_{i=1}^{k} K\left(\frac{i}{k+1}\right)Z_i,$$

where $Z_i = i\left(\log X_{n-i+1,n} - \log X_{n-i,n}\right)$. The class of kernel estimators $\widehat{\gamma}^K_{n,k}$ generalizes the Hill estimator. Note that, using the uniform kernel $K = \underline{K} = \mathbf{1}_{(0,1)}$ yields Hill's estimator $\widehat{\gamma}^H_{n,k}$ as a special case.

In this paper, we propose a kernel-type estimator for the reinsurance premium $\Pi_{r,R_{opt}}$ of a heavy-tailed distribution. Thus, $\Pi_{r,R_{opt}}$ can be estimated by

$$\widehat{\Pi}^K_{r,\widehat{R}_{opt}} = (k/n)^{1/r}\frac{r}{1/\widehat{\gamma}^K_{n,k} - r}X_{n-k,n}, \quad \text{for } \widehat{\gamma}^K_{n,k} < 1/r. \tag{1.1}$$

The rest of this paper is organized as follows. In Section 2, we study the asymptotic properties of $\widehat{\Pi}^K_{r,\widehat{R}_{opt}}$ and propose its bias-reduced version whose asymptotic normality is also obtained. In Section 3, we perform a small simulation study, by sampling from Fréchet distribution, to compare these estimators. All proofs are given in Section 4.

2. Main results

Firstly, in this section, we study the asymptotic normality of $\widehat{\Pi}^K_{r,\widehat{R}_{opt}}$.

2.1. Asymptotic normality of $\widehat{\Pi}^K_{r,\widehat{R}_{opt}}$

From (1.1), it is clear that the asymptotic normality of $\widehat{\Pi}^K_{r,\widehat{R}_{opt}}$ is related to $\widehat{\gamma}^K_{n,k}$. To establish such a type of result, as usual in the extreme value theory, we need a second-order condition on the tail quantile function $\mathbb{U}$ defined, for $1 < t < \infty$, as

$$\mathbb{U}(t) = (1/(1-F))^{-1}(t) = Q(1-1/t).$$

We say that the function $\mathbb{U}$ fulfills the second-order regular variation condition with second-order parameter $\rho < 0$ if there exists a function $A(t)$ tending to 0 and not changing sign near infinity, such that for all $x > 0$

$$\lim_{t\to\infty}\frac{\log\mathbb{U}(tx) - \log\mathbb{U}(t) - \gamma\log x}{A(t)} = \frac{x^\rho - 1}{\rho}. \tag{2.2}$$

We also need the following classical conditions about the kernel K.

Condition $(\mathbb{K})$. Let K be a function defined on $(0, 1]$.

(i) $K(s) \ge 0$, whenever, $0 < s \le 1$ and $K(1) = K'(1) = 0$.
(ii) $K(\cdot)$ is differentiable, non-increasing and right continuous on $(0, 1]$.
(iii) K and K' are bounded.
(iv) $\int_0^1 K(u)du = 1$.
(v) $\int_0^1 u^{-1/2}K(u)du < \infty$.

Theorem 2.1. *Let F be a distribution function satisfying (2.2) with $\gamma \in (1/2, 1)$ and suppose that $(\mathbb{K})$ holds. Let $k = k_n$ be an integer sequence satisfying $k \to \infty$, $k/n \to 0$ and $\sqrt{k}A(n/k) = O(1)$ as $n \to \infty$. For any $1 \le r < 1/\gamma$, on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, there exists a sequence of Brownian bridges $\{\mathbb{B}_n(s); 0 \le s \le 1\}$ such that, as $n \to \infty$,*

$$\frac{(k/n)^{-1/r}\sqrt{k}}{\mathbb{U}(n/k)}\left(\widehat{\Pi}^K_{r,\widehat{R}_{opt}} - \Pi_{r,R_{opt}}\right)$$
$$= \sqrt{k}A(n/k)\mathcal{AB}_K(\gamma, r, \rho) + \mathcal{W}_{1,n} + \mathcal{W}_{2,n}(K) + o_{\mathbb{P}}(1),$$

where

$$\mathcal{AB}_K(\gamma, r, \rho)$$
$$= \frac{r}{1-r\gamma}\left(\frac{1}{r\gamma + r\rho - 1} + \frac{1}{1-r\gamma}\int_0^1 s^{-\rho}K(s)\,ds\right),$$

and

$$\begin{cases} \mathcal{W}_{1,n} = -\dfrac{r\gamma^2}{1-r\gamma}\sqrt{\dfrac{n}{k}}\mathbb{B}_n\left(1-\dfrac{k}{n}\right), \\ \mathcal{W}_{2,n}(K) = \dfrac{r\gamma}{(1-r\gamma)^2}\sqrt{\dfrac{n}{k}}\displaystyle\int_0^1 s^{-1}\mathbb{B}_n\left(1-\dfrac{sk}{n}\right)d(sK(s)). \end{cases}$$

Corollary 2.1. *Under the assumptions of Theorem 2.1, if $\sqrt{k}A(n/k) \to \lambda \in \mathbb{R}$, we have*

$$\frac{(k/n)^{-1/r}\sqrt{k}}{\mathbb{U}(n/k)}\left(\widehat{\Pi}^K_{r,\widehat{R}_{opt}} - \Pi_{r,R_{opt}}\right)$$
$$\xrightarrow{\mathcal{D}} \mathcal{N}\left(\lambda\mathcal{AB}_K(\gamma, r, \rho), \mathcal{AV}_K(\gamma, r)\right), \quad \text{as } n \to \infty,$$

where

$$\mathcal{AV}_K(\gamma, r) = \frac{r^2\gamma^4}{(1-r\gamma)^2} + \frac{r^2\gamma^2}{(1-r\gamma)^4}\int_0^1 K^2(s)ds.$$

Corollary 2.1 generalizes Theorem 2 in Necir et al. (2007, 2010) when $\lambda \neq 0$ and when we use a general kernel instead of $\underline{K}$. In view of these results, $\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}}$ is an estimator of $\Pi_{r,R_{opt}}$ with an asymptotic bias given by

$$(k/n)^{1/r}\mathbb{U}(n/k)\,A(n/k)\mathcal{AB}_K(\gamma, r, \rho).$$

For any kernel K, we can compute the asymptotic bias and variance. If $K = \underline{K}$, we have the following corollary.

Corollary 2.2. *Under the assumptions of Corollary 2.1, and in the special case where $K = \underline{K}$, we have as $n \to \infty$*

$$\frac{(k/n)^{-1/r}\sqrt{k}}{\mathbb{U}(n/k)}\left(\widehat{\Pi}^{\underline{K}}_{r,\widehat{R}_{opt}} - \Pi_{r,R_{opt}}\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(\lambda\frac{r\rho(r\gamma+r-1)}{(1-\rho)(r\gamma+r\rho-1)(1-r\gamma)^2}, \frac{r^2\gamma^2(\gamma^2+r^2\gamma^4-2r\gamma^3+1)}{(1-r\gamma)^4}\right).$$

Next, in the following subsection, we propose a bias-reduced estimator for $\Pi_{r,R_{opt}}$.

2.2. Bias-reduced estimator for $\Pi_{r,R_{opt}}$

From Theorem 2.1, we have

$$\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}} - (k/n)^{1/r}\mathbb{U}(n/k)\,A(n/k)\mathcal{AB}_K(\gamma, r, \rho) \tag{2.3}$$

is an asymptotically unbiased estimator for $\Pi_{r,R_{opt}}$. Note that γ, ρ, $\mathbb{U}(n/k)$ and $A(n/k)$ are unknown quantities that we have to estimate.

Feuerverger and Hall (1999) and Beirlant et al. (1999, 2002), using (2.2), proposed the following exponential regression model for the log-spacings of order statistics

$$Z_i \sim \left(\gamma + A\left(\frac{n}{k}\right)\left(\frac{i}{k+1}\right)^{-\rho}\right) + \varepsilon_i, \quad 1 \le i \le k, \tag{2.4}$$

where the ε_i are zero-centered error terms. We get the Hill estimator $\widehat{\gamma}^{H}_{n,k}$ when we ignore the term $A(n/k)$ in (2.4) and by taking the mean of the left-hand side of (2.4). We can exploit (2.4), using a least-squares approach, to propose a bias-reduced estimator for γ in which ρ is substituted by a consistent estimator $\widehat{\rho} = \widehat{\rho}(n, k)$ (see for instance Beirlant et al., 2002 and Fraga Alves et al., 2003) or by a canonical choice, such as $\rho = -1$ (see e.g., Feuerverger and Hall, 1999). Then, the least-squares estimators for γ and $A(n/k)$ are given by

$$\widehat{\gamma}^{LS}_{n,k}(\widehat{\rho}) = \frac{1}{k}\sum_{i=1}^{k} Z_i - \frac{\widehat{A}^{LS}_{n,k}(\widehat{\rho})}{1-\widehat{\rho}} = \widehat{\gamma}^{H}_{n,k} - \frac{\widehat{A}^{LS}_{n,k}(\widehat{\rho})}{1-\widehat{\rho}},$$

$$\widehat{A}^{LS}_{n,k}(\widehat{\rho}) = \frac{(1-2\widehat{\rho})(1-\widehat{\rho})^2}{\widehat{\rho}^2}\frac{1}{k}\sum_{i=1}^{k}\left(\left(\frac{i}{k+1}\right)^{-\widehat{\rho}} - \frac{1}{1-\widehat{\rho}}\right)Z_i.$$

We can view $\widehat{\gamma}^{LS}_{n,k}(\rho)$ as the kernel estimator

$$\widehat{\gamma}^{K_\rho}_{n,k} = \frac{1}{k}\sum_{i=1}^{k} K_\rho\left(\frac{i}{k+1}\right) Z_i,$$

where for $0 < u \le 1$

$$K_\rho(u) = \frac{1-\rho}{\rho}\underline{K}(u) + \left(1 - \frac{1-\rho}{\rho}\right)\underline{K}_\rho(u), \tag{2.5}$$

with $\underline{K}(u) = \mathbf{1}_{(0,1)}$ and $\underline{K}_\rho(u) = \left(\frac{1-\rho}{\rho}\right)\left(u^{-\rho}-1\right)\mathbf{1}_{(0,1)}$, both kernels satisfying condition $(\mathbb{K})$. On the contrary K_ρ does not satisfy statement (i) in $(\mathbb{K})$. We refer to Gomes and Martins (2004) and Gomes et al. (2007) for other techniques of bias reduction based on the estimation of the second-order parameter. Then, from (2.3) and using the above estimators for the different unknown quantities, we obtain the following bias-reduced estimator for $\Pi_{r,R_{opt}}$

$$\widetilde{\Pi}^{K}_{r,\widehat{R}_{opt}} = \widehat{\Pi}^{K}_{r,\widehat{R}_{opt}} - \left(\frac{k}{n}\right)^{1/r} \times X_{n-k,n}\widehat{A}^{LS}_{n,k}(\widehat{\rho})\,\mathcal{AB}_K\left(\widehat{\gamma}^{LS}_{n,k}(\widehat{\rho}), r, \widehat{\rho}\right). \tag{2.6}$$

The asymptotic normality of $\widetilde{\Pi}^{K}_{r,\widehat{R}_{opt}}$ is established in the following theorem.

Theorem 2.2. *Under the assumptions of Theorem 2.1, if $\widehat{\rho}$ is a consistent estimator for ρ, then we have*

$$\frac{(k/n)^{-1/r}\sqrt{k}}{\mathbb{U}(n/k)}\left(\widetilde{\Pi}^{K}_{r,\widehat{R}_{opt}} - \Pi_{r,R_{opt}}\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \widetilde{\mathcal{AV}}_K(\gamma, r, \rho)\right), \quad \text{as } n \to \infty,$$

where

$$\widetilde{\mathcal{AV}}_K(\gamma, r, \rho) = \mathcal{AV}_K(\gamma, r) + \frac{\gamma^2(1-2\rho)(1-\rho)^2}{\rho^2}\mathcal{AB}^2_K(\gamma, r, \rho) + \frac{2r\gamma^2(1-2\rho)(1-\rho)}{\rho^2(1-r\gamma)^2} \times \left(1-(1-\rho)\int_0^1 s^{-\rho}K(s)ds\right)\mathcal{AB}_K(\gamma, r, \rho).$$

We observe that $\widetilde{\Pi}^{K}_{r,\widehat{R}_{opt}}$ has a null asymptotic bias, which was not the case for $\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}}$ (see Corollary 2.1).

Corollary 2.3. *Under the same assumptions as in Theorem 2.2, and in the special case where $K = \underline{K}$, we have*

$$\frac{(k/n)^{-1/r}\sqrt{k}}{\mathbb{U}(n/k)}\left(\widetilde{\Pi}^{\underline{K}}_{r,\widehat{R}_{opt}} - \Pi_{r,R_{opt}}\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \widetilde{\mathcal{AV}}_{\underline{K}}(\gamma, r, \rho)\right), \quad \text{as } n \to \infty,$$

where

$$\widetilde{\mathcal{AV}}_{\underline{K}}(\gamma, r, \rho) = \frac{r^2\gamma^2(\gamma^2+r^2\gamma^4-2r\gamma^3+1)}{(1-r\gamma)^4} + \frac{r^2\gamma^2(1-2\rho)(r\gamma+r-1)^2}{(1-r\gamma)^4(r\gamma+r\rho-1)^2}.$$

In the special case where $K = K_\rho$, we have the estimator $\widehat{\gamma}^{LS}_{n,k}(\rho)$ coincides with $\widehat{\gamma}^{K_\rho}_{n,k}$. The goal of the next corollary is to establish the asymptotic normality of the resulting reinsurance premium estimator $\widetilde{\Pi}^{K_\rho}_{r,\widehat{R}_{opt}}$, denoted by $\widetilde{\Pi}^{LS}_{r,\widehat{R}_{opt}}$, when the least-squares method is adopted.

Corollary 2.4. *Under the same assumptions as in Theorem 2.2, and in the special case where $K = K_\rho$, we have*

$$\frac{(k/n)^{-1/r}\sqrt{k}}{\mathbb{U}(n/k)}\left(\widetilde{\Pi}^{LS}_{r,\widehat{R}_{opt}} - \Pi_{r,R_{opt}}\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \widetilde{\mathcal{AV}}_{K_\rho}(\gamma, r, \rho)\right), \quad \text{as } n \to \infty,$$

Table 1
Comparison of $\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}}$ and $\widetilde{\Pi}^{LS}_{r,\widehat{R}_{opt}}$ for 1000 samples of size $n \in \{1000, 2000, 5000\}$ of a Fréchet distribution with $\gamma = 3/4$.

n	1000		2000		5000	
r	1.10	1.20	1.10	1.20	1.10	1.20
$\Pi_{r,R_{opt}}$	3.803	8.153	3.635	8.044	3.478	7.772
$\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}}$	4.071	8.364	3.867	8.208	3.521	7.807
Bias	0.268	0.211	0.232	0.164	0.043	0.035
RMSE	0.569	0.699	0.464	0.587	0.229	0.282
$\widetilde{\Pi}^{LS}_{r,\widehat{R}_{opt}}$	3.862	8.206	3.683	8.084	3.507	7.781
Bias	0.059	0.053	0.048	0.040	0.029	0.009
RMSE	0.421	0.589	0.376	0.437	0.187	0.248

where

$$\widetilde{\mathcal{AV}}_{K_\rho}(\gamma, r, \rho) = \frac{r^2\gamma^4}{(1-r\gamma)^2} + \frac{r^2\gamma^2(1-\rho)(1-2\rho)(r\gamma\rho + r\gamma + 2r\rho - \rho - 1)}{\rho^2(1-r\gamma)^3(r\gamma + r\rho - 1)^2}.$$

3. A small simulation study

We use the statistical software **R**, see Ihaka and Gentleman (1996), to compare, in terms of bias and root of the mean squared error (RMSE), the performances of the kernel-type estimator $\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}}$ and least-squares estimator $\widetilde{\Pi}^{LS}_{r,\widehat{R}_{opt}}$. We generate 1000 samples of different sizes $n = 1000, 2000$ and 5000 from a Fréchet distribution with hazard function $S(x) = 1 - \exp\left(-x^{-1/\gamma}\right)$, $x > 0$, $\gamma = 3/4$ and the second-order parameter $\rho = -1$. For the kernel function K, we choose the uniform kernel $K = \underline{K} = \mathbf{1}_{(0,1)}$. Note that $R_{opt} = \mathbb{U}(n/k^*)$, where k^* is the optimal value of k. Several methods are available for the choice of k^*, see e.g. Danielsson et al. (2001), Cheng and Peng (2001), Neves and Fraga Alves (2004) and the references therein. In our simulation study, we use the method of Neves and Fraga Alves (2004). The simulation results are summarized in Table 1. We conclude that $\widetilde{\Pi}^{LS}_{r,\widehat{R}_{opt}}$ has smaller bias and RMSE and consequently it performs better than $\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}}$.

4. Proofs

For each integer n, let $Y_{1,n} \le \cdots \le Y_{n,n}$ be the order statistics pertaining to a sample $Y_1, \ldots, Y_n$ of independent identically distributed random variables, defined on the same probability space as the $X_i's$, with distribution function $G(y) = 1 - y^{-1}$, $y > 1$. Note that

$$\left\{X_{j,n}\right\}_{j=1}^{n} \overset{\mathcal{D}}{=} \left\{\mathbb{U}\left(Y_{j,n}\right)\right\}_{j=1}^{n}. \tag{4.7}$$

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables, defined on probability space $(\Omega, \mathcal{A}, \mathbb{P})$, uniformly distributed on $(0, 1)$ in such a way that $Y_i = G^{-1}(\xi_i)$, for all $1 \le i \le n$. Consequently, we have $Y_{i,n} = \left(1 - \xi_{i,n}\right)^{-1}$ for all $1 \le i \le n$ and $n \ge 1$, where $\xi_{1,n} \le \cdots \le \xi_{n,n}$ denote the order statistics of $\xi_1, \ldots, \xi_n$ and G^{-1} is the quantile function pertaining to G.

We will use in this section the Csörgő et al. (1986) weak approximations. On the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, there exists a sequence of Brownian bridges $\{\mathbb{B}_n(s); 0 \le s \le 1\}_{n\ge1}$, such that for every $0 \le v < 1/2$ and for all n

$$\sup_{1/n\le s\le 1-1/n} \frac{|\beta_n(s) - \mathbb{B}_n(s)|}{(s(1-s))^{1/2-v}} = O_{\mathbb{P}}\left(n^{-v}\right), \tag{4.8}$$

where the resulting uniform empirical quantile process, is denoted by

$$\beta_n(t) = \sqrt{n}(t - \mathbb{V}_n(t)), \quad 0 \le t \le 1, \tag{4.9}$$

with $\mathbb{V}_n$ is the empirical quantile function pertaining to the sample $\xi_1, \ldots, \xi_n$ which is defined by

$$\mathbb{V}_n(s) = \xi_{j,n}, \qquad \frac{j-1}{n} < s \le \frac{j}{n}, \quad j = 1, \ldots, n \quad \text{and} \quad \mathbb{V}_n(0) = \xi_{1,n}.$$

Proof of Theorem 2.1. From (4.7), we may rewrite $\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}}$ as follows

$$\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}} = \frac{r\,(k/n)^{1/r}}{1/\widehat{\gamma}^{K}_{n,k} - r}\mathbb{U}\left(Y_{n-k,n}\right), \quad \text{for } \widehat{\gamma}^{K}_{n,k} < 1/r.$$

It is easy to verify that

$$\frac{(k/n)^{-1/r}\sqrt{k}}{\mathbb{U}(n/k)}\left(\widehat{\Pi}^{K}_{r,\widehat{R}_{opt}} - \Pi_{r,R_{opt}}\right) = \sum_{i=1}^{4} T_{i,n},$$

where

$$T_{1,n} = \frac{r\sqrt{k}}{1/\widehat{\gamma}^{K}_{n,k} - r}\left(\frac{\mathbb{U}\left(Y_{n-k,n}\right)}{\mathbb{U}(n/k)} - \left(\frac{k}{n}Y_{n-k,n}\right)^{\gamma}\right),$$

$$T_{2,n} = \frac{r\sqrt{k}}{1/\widehat{\gamma}^{K}_{n,k} - r}\left(\left(\frac{k}{n}Y_{n-k,n}\right)^{\gamma} - 1\right),$$

$$T_{3,n} = \frac{r}{\left(1 - r\widehat{\gamma}^{K}_{n,k}\right)(1 - r\gamma)}\sqrt{k}\left(\widehat{\gamma}^{K}_{n,k} - \gamma\right),$$

and

$$T_{4,n} = \frac{(k/n)^{-1/r}\sqrt{k}}{\mathbb{U}(n/k)}\left(\frac{r\,(k/n)^{1/r}}{1/\gamma - r}\mathbb{U}(n/k) - \Pi_{r,R_{opt}}\right).$$

We start with the term $T_{1,n}$, according to de Haan and Ferreira (2006, page 60 and Theorem 2.3.9, page 48), for any $\delta > 0$, we have

$$\frac{\mathbb{U}\left(Y_{n-k,n}\right)}{\mathbb{U}(n/k)} - \left(\frac{k}{n}Y_{n-k,n}\right)^{\gamma} = A_0\left(\frac{n}{k}\right)\left\{\left(\frac{k}{n}Y_{n-k,n}\right)^{\gamma}\frac{\left(\frac{k}{n}Y_{n-k,n}\right)^{\rho} - 1}{\rho} + o_{\mathbb{P}}(1)\left(\frac{k}{n}Y_{n-k,n}\right)^{\gamma+\rho\pm\delta}\right\},$$

where $A_0(t) \sim A(t)$ as $t \to \infty$. Since $\frac{k}{n}Y_{n-k,n} = 1 + o_{\mathbb{P}}(1)$ and $\sqrt{k}A(n/k) = O(1)$, as $n \to \infty$, we have

$$\sqrt{k}\left(\frac{\mathbb{U}\left(Y_{n-k,n}\right)}{\mathbb{U}(n/k)} - \left(\frac{k}{n}Y_{n-k,n}\right)^{\gamma}\right) = o_{\mathbb{P}}(1),$$

and since $\widehat{\gamma}^{K}_{n,k} \overset{\mathbb{P}}{\to} \gamma$ (see Csörgő et al., 1985), then we obtain as $n \to \infty$

$$T_{1,n} = o_{\mathbb{P}}(1). \tag{4.10}$$

For the term $T_{2,n}$, the equality $Y_{n-k,n} = \left(1 - \xi_{n-k,n}\right)^{-1}$ yields

$$\sqrt{k}\left(\left(\frac{k}{n}Y_{n-k,n}\right)^{\gamma} - 1\right) = \sqrt{k}\left(\left(\frac{n}{k}\left(1 - \xi_{n-k,n}\right)\right)^{-\gamma} - 1\right).$$

Using a Taylor expansion, we get

$$\sqrt{k}\left(\left(\frac{n}{k}\left(1-\xi_{n-k,n}\right)\right)^{-\gamma}-1\right) = -\gamma\left(\lambda_n(k)\right)^{-\gamma-1}\sqrt{k}\left(\frac{n}{k}\left(1-\xi_{n-k,n}\right)-1\right),$$

where $\lambda_n(k)$ is a sequence of random variables with values in the open interval of endpoints 1 and $\frac{n}{k}\left(1-\xi_{n-k,n}\right)$. From Balkema and de Haan (1975), we have

$$\frac{n}{k}\left(1-\xi_{n-k,n}\right)\xrightarrow{\mathbb{P}}1,\quad \text{as } n\to\infty.$$

It follows that, $\lambda_n(k)\xrightarrow{\mathbb{P}}1$ as $n\to\infty$. Therefore, as $n\to\infty$,

$$\sqrt{k}\left(\left(\frac{n}{k}\left(1-\xi_{n-k,n}\right)\right)^{-\gamma}-1\right) = -\gamma\sqrt{k}\left(\frac{n}{k}\left(1-\xi_{n-k,n}\right)-1\right)\left(1+o_{\mathbb{P}}(1)\right).$$

On the other hand we have

$$\sqrt{k}\left(\frac{n}{k}\left(1-\xi_{n-k,n}\right)-1\right) = \sqrt{\frac{n}{k}}\left\{\sqrt{n}\left(\left(1-\frac{k}{n}\right)-\mathbb{V}_n\left(1-\frac{k}{n}\right)\right)\right\}.$$

Using the uniform empirical quantile process, defined in (4.9), we obtain

$$\sqrt{k}\left(\left(\frac{n}{k}\left(1-\xi_{n-k,n}\right)\right)^{-\gamma}-1\right) = -\gamma\sqrt{\frac{n}{k}}\beta_n\left(1-\frac{k}{n}\right)\left(1+o_{\mathbb{P}}(1)\right),\quad \text{as } n\to\infty.$$

Using the asymptotic approximation (4.8), we get for all large n

$$\begin{aligned}\sqrt{k}\left(\left(\frac{n}{k}\left(1-\xi_{n-k,n}\right)\right)^{-\gamma}-1\right) &= -\gamma\sqrt{\frac{n}{k}}\left\{\mathbb{B}_n\left(1-\frac{k}{n}\right)\right.\\ &\quad\left.+O_{\mathbb{P}}\left(n^{-\upsilon}\right)\left(\frac{k}{n}\right)^{1/2-\upsilon}\right\}\left(1+o_{\mathbb{P}}(1)\right),\\ &= -\gamma\sqrt{\frac{n}{k}}\mathbb{B}_n\left(1-\frac{k}{n}\right)\left(1+o_{\mathbb{P}}(1)\right).\end{aligned}$$

Consequently, since $\widehat{\gamma}^K_{n,k}\xrightarrow{\mathbb{P}}\gamma$, we obtain for all large n

$$\begin{aligned}T_{2,n} &= -\frac{r\gamma^2}{1-r\gamma}\sqrt{\frac{n}{k}}\mathbb{B}_n\left(1-\frac{k}{n}\right)\left(1+o_{\mathbb{P}}(1)\right),\\ &= \mathcal{W}_{1,n}+o_{\mathbb{P}}(1).\end{aligned}\tag{4.11}$$

For the term $T_{3,n}$, from Theorem 1 of Deme et al. (2013), we have for all large n

$$\begin{aligned}\sqrt{k}\left(\widehat{\gamma}^K_{n,k}-\gamma\right) &= \sqrt{k}A(n/k)\int_0^1 s^{-\rho}K(s)\,ds\\ &\quad+\gamma\sqrt{\frac{n}{k}}\int_0^1 s^{-1}\mathbb{B}_n\left(1-\frac{sk}{n}\right)d\left(sK(s)\right)+o_{\mathbb{P}}(1).\end{aligned}$$

Then, since $\widehat{\gamma}^K_{n,k}\xrightarrow{\mathbb{P}}\gamma$, we get as $n\to\infty$

$$\begin{aligned}T_{3,n} &= \frac{r}{(1-r\gamma)^2}\left\{\sqrt{k}A\left(\frac{n}{k}\right)\int_0^1 s^{-\rho}K(s)\,ds\right.\\ &\quad\left.+\gamma\sqrt{\frac{n}{k}}\int_0^1 s^{-1}\mathbb{B}_n\left(1-\frac{sk}{n}\right)d\left(sK(s)\right)\right\}+o_{\mathbb{P}}(1)\\ &= \frac{r}{(1-r\gamma)^2}\sqrt{k}A\left(\frac{n}{k}\right)\int_0^1 s^{-\rho}K(s)\,ds\\ &\quad+\mathcal{W}_{2,n}(K)+o_{\mathbb{P}}(1).\end{aligned}\tag{4.12}$$

For the term $T_{4,n}$, we have

$$T_{4,n} = (k/n)^{-1/r}\sqrt{k}\left(\frac{r\,(k/n)^{1/r}}{1/\gamma-r}-\frac{\Pi_{r,R_{opt}}}{\mathbb{U}(n/k)}\right),$$

where

$$\Pi_{r,R_{opt}} = \int_{\mathbb{U}(n/k)}^{\infty}\left(S(x)\right)^{1/r}dx.$$

Since $x^{-1/r}\mathbb{U}(x)\to 0$ as $x\to\infty$, then an integration by parts with a change of variables yields

$$\Pi_{r,R_{opt}} = (k/n)^{1/r}\left\{\frac{1}{r}\int_1^{\infty}x^{-1-1/r}\mathbb{U}(nx/k)\,dx-\mathbb{U}(n/k)\right\}.$$

Therefore

$$\begin{aligned}T_{4,n} &= \sqrt{k}\left\{\frac{1}{1-r\gamma}-\frac{1}{r}\int_1^{\infty}x^{-1-1/r}\frac{\mathbb{U}(nx/k)}{\mathbb{U}(n/k)}dx\right\}\\ &= -\frac{1}{r}\sqrt{k}\int_1^{\infty}x^{-1-1/r}\left(\frac{\mathbb{U}(nx/k)}{\mathbb{U}(n/k)}-x^{\gamma}\right)dx.\end{aligned}$$

From Theorem 2.3.9 of de Haan and Ferreira (2006), for $\gamma\in(1/2,1)$ and $r\in[1,1/\gamma)$, we obtain as $n\to\infty$

$$\begin{aligned}T_{4,n} &= -\frac{1}{r}\sqrt{k}A\left(\frac{n}{k}\right)\int_1^{\infty}x^{\gamma-1-1/r}\frac{x^{\rho}-1}{\rho}dx\left(1+o(1)\right)\\ &= \sqrt{k}A\left(\frac{n}{k}\right)\frac{r}{(1-r\gamma)(r\gamma+r\rho-1)}\left(1+o(1)\right).\end{aligned}\tag{4.13}$$

Combining (4.10)–(4.13), we get as $n\to\infty$

$$\begin{aligned}&\frac{(k/n)^{-1/r}\sqrt{k}}{\mathbb{U}(n/k)}\left(\widehat{\Pi}^K_{r,\widehat{R}_{opt}}-\Pi_{r,R_{opt}}\right)\\ &\quad= \sqrt{k}A(n/k)\mathcal{AB}_K(\gamma,r,\rho)+\mathcal{W}_{1,n}+\mathcal{W}_{2,n}(K)+o_{\mathbb{P}}(1).\end{aligned}$$

This finishes the proof of Theorem 2.1. □

Proof of Corollary 2.1. Since $\{\mathbb{B}_n(s);\,0\le s\le 1\}_{n\ge1}$, is a sequence of Brownian bridges, then

$$\frac{(k/n)^{-1/r}\sqrt{k}}{\mathbb{U}(n/k)}\left(\widehat{\Pi}^K_{r,\widehat{R}_{opt}}-\Pi_{r,R_{opt}}\right)\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,\mathcal{AV}_K(\gamma,r)\right),\quad \text{as } n\to\infty,$$

with

$$\begin{aligned}\mathcal{AV}_K(\gamma,r) &= \lim_{n\to\infty}E\left(\left(\mathcal{W}_{1,n}+\mathcal{W}_{2,n}(K)\right)^2\right)\\ &= \lim_{n\to\infty}\left(E\left(\mathcal{W}_{1,n}^2\right)+E\left(\mathcal{W}_{2,n}^2(K)\right)\right.\\ &\quad\left.+2E\left(\mathcal{W}_{1,n}\mathcal{W}_{2,n}(K)\right)\right).\end{aligned}$$

Elementary computation gives, as $n \to \infty$

$$E\left(\mathcal{W}_{1,n}^2\right) = \frac{r^2\gamma^4}{(1-r\gamma)^2} + o(1),$$

$$E\left(\mathcal{W}_{2,n}^2(K)\right) = \frac{r^2\gamma^2}{(1-r\gamma)^4}\int_0^1 K^2(s)ds + o(1),$$

and

$$E\left(\mathcal{W}_{1,n}\mathcal{W}_{2,n}(K)\right) = o(1).$$

Then, we get

$$\mathcal{AV}_K(\gamma, r) = \frac{r^2\gamma^4}{(1-r\gamma)^2} + \frac{r^2\gamma^2}{(1-r\gamma)^4}\int_0^1 K^2(s)ds.$$

We complete the proof of Corollary 2.1. □

Proof of Corollary 2.2. The proof is a direct result of Corollary 2.1 with the kernel $K = \underline{K} = 1_{(0,1)}$. □

Proof of Theorem 2.2. According to Theorem 2.1 and (2.6), we have

$$\frac{(k/n)^{-1/r}\sqrt{k}}{\mathbb{U}(n/k)}\left(\widetilde{\Pi}^K_{r,\widehat{R}_{opt}} - \Pi_{r,R_{opt}}\right)$$
$$= \mathcal{W}_{1,n} + \mathcal{W}_{2,n}(K) + \mathcal{W}_{3,n}(K) + o_{\mathbb{P}}(1),$$

where

$$\mathcal{W}_{3,n}(K)$$
$$= \sqrt{k}\left(A(n/k)\mathcal{AB}_K(\gamma, r, \rho) - \widehat{A}^{LS}_{n,k}(\widehat{\rho}) \times \mathcal{AB}_K\left(\widehat{\gamma}^{LS}_{n,k}(\widehat{\rho}), r, \widehat{\rho}\right)\frac{X_{n-k,n}}{\mathbb{U}(n/k)}\right)$$
$$= -\mathcal{AB}_K(\gamma, r, \rho)\sqrt{k}\left(\widehat{A}^{LS}_{n,k}(\widehat{\rho}) - A(n/k)\right)$$
$$- \sqrt{k}\widehat{A}^{LS}_{n,k}(\widehat{\rho})\left(\mathcal{AB}_K\left(\widehat{\gamma}^{LS}_{n,k}(\widehat{\rho}), r, \widehat{\rho}\right) - \mathcal{AB}_K(\gamma, r, \rho)\right)$$
$$- \sqrt{k}\widehat{A}^{LS}_{n,k}(\widehat{\rho})\mathcal{AB}_K\left(\widehat{\gamma}^{LS}_{n,k}(\widehat{\rho}), r, \widehat{\rho}\right)\left(\frac{X_{n-k,n}}{\mathbb{U}(n/k)} - 1\right).$$

From Lemma 5 of Deme et al. (2013), we have

$$\sqrt{k}\left(\widehat{A}^{LS}_{n,k}(\widehat{\rho}) - A(n/k)\right) = \gamma(1-\rho) \times \sqrt{\frac{n}{k}}\int_0^1 s^{-1}\mathbb{B}_n\left(1 - \frac{sk}{n}\right)d\left(s\left(\underline{K}(s) - K_\rho(s)\right)\right) + o_{\mathbb{P}}(1).$$

Since $\widehat{\rho}$ is a consistent estimator for ρ, then we get as $n \to \infty$

$$\sqrt{k}\widehat{A}^{LS}_{n,k}(\widehat{\rho})\left(\mathcal{AB}_K\left(\widehat{\gamma}^{LS}_{n,k}(\widehat{\rho}), r, \widehat{\rho}\right) - \mathcal{AB}_K(\gamma, r, \rho)\right) = o_{\mathbb{P}}(1).$$

Making use of Potter's inequalities (see 5th assertion of Proposition B.1.9 in de Haan and Ferreira (2006, page 367)), we obtain as $n \to \infty$

$$\sqrt{k}\widehat{A}^{LS}_{n,k}(\widehat{\rho})\mathcal{AB}_K\left(\widehat{\gamma}^{LS}_{n,k}(\widehat{\rho}), r, \widehat{\rho}\right)\left(\frac{X_{n-k,n}}{\mathbb{U}(n/k)} - 1\right) = o_{\mathbb{P}}(1).$$

Therefore

$$\mathcal{W}_{3,n}(K) = -\gamma(1-\rho)\mathcal{AB}_K(\gamma, \beta, \rho) \times \sqrt{\frac{n}{k}}\int_0^1 s^{-1}\mathbb{B}_n\left(1 - \frac{sk}{n}\right)d\left(s\left(\underline{K}(s) - K_\rho(s)\right)\right) + o_{\mathbb{P}}(1),$$

as $n \to \infty$.

It is clear that $\mathcal{W}_{1,n} + \mathcal{W}_{2,n}(K) + \mathcal{W}_{3,n}(K)$ is a Gaussian random variable with mean zero and variance

$$\widetilde{\mathcal{AV}}_K(\gamma, r, \rho) = \lim_{n\to\infty} E\left(\left(\mathcal{W}_{1,n} + \mathcal{W}_{2,n}(K) + \mathcal{W}_{3,n}(K)\right)^2\right).$$

After elementary computations, we get

$$\widetilde{\mathcal{AV}}_K(\gamma, r, \rho)$$
$$= \mathcal{AV}_K(\gamma, r) + \frac{\gamma^2(1-2\rho)(1-\rho)^2}{\rho^2}\mathcal{AB}_K^2(\gamma, r, \rho)$$
$$+ \frac{2r\gamma^2(1-2\rho)(1-\rho)}{\rho^2(1-r\gamma)^2}\left(1 - (1-\rho)\int_0^1 s^{-\rho}K(s)ds\right) \times \mathcal{AB}_K(\gamma, r, \rho).$$

This achieves the proof of Theorem 2.2. □

Proof of Corollary 2.3. The proof is a direct result of Theorem 2.2 with the kernel $K = \underline{K} = \mathbf{1}_{(0,1)}$. □

Proof of Corollary 2.4. The proof is a direct result of Theorem 2.2 with the kernel $K = K_\rho$ defined in (2.5). □

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