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Review

Path integral treatment for a Coulomb system constrained on *D*-dimensional sphere and hyperboloid

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Abstract

The propagator relating to the evolution of a particle on the *D*-sphere and the *D*-pseudosphere, subjected to the Coulomb potential, was reconsidered in the Faddeev–Senjanovic formalism. The mid-point is privileged. The space–time transformations used make it possible to regularize the singularity and to bring back the problem to its dynamical symmetry $SU(1,1)$.

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1. Introduction

Up to now, the problem of path integral formulation in curved space has not been definitively solved. This is related to the operator-ordering problem in quantum mechanics. In fact, to deduce the good effective potential due to the curvature, one has to refer to the Hamiltonian formulation. As it appears in the Lagrangian, the metric tensor depending on the position, one cannot write the kinetic part at the quantum level clearly, we

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recourse to the Hamiltonian using the Laplace–Beltrami operator. Then we introduce the momentum operator in the Hamiltonian using the Weyl order from which we deduce the Lagrangian formulation [1]. During the previous decade, a partial Lagrangian solution analogous to this procedure was proposed including a quantum equivalence principle, where all the discretization prescriptions are equivalents [2]. In our opinion, this solution is not complete because during the evolution in curved space the constraints indicating that this evolution alone is not sufficient for a complete description are essential, so we have to supply it with some constraints on the state space to ensure the good interpretability of the theory. According to this program, the quantum corrections are the product of the reduction of the phase space to an effective one using the Dirac brackets method [3,4] and up to now a concrete bond between these approaches has not been established yet. On the other hand we have an opposition. In fact, in path integral the Dirac brackets method is taken into account by using a delta functional which allows a reduction of phase space. This technique is known as Faddeev–Senjanovic formulation [5]. Furthermore, according to this technique the use of mid-point prescription is privileged [4,6,7] contrary to this quantum equivalence principle.

As the problem is still raised, let us poke the discussion by studying the case of simple curved spaces known as homogeneous spaces [8], we particularly take the D-sphere and the D-pseudosphere noted, respectively, as $SO(D+1)/SO(D)$ and $SO(D,1)/SO(D)$. These two cases had been treated before using the usual canonical method. We propose to re-examine them within the most natural framework of the constraints, i.e., the Faddeev–Senjanovic formalism. Concretely, we choose the Coulomb potential already treated by [9] with $D=3$ and by [10] with D unspecified. In the same way, we will convert the problem to the path integral proper to the dynamic symmetry $SU(1,1)$ using space–time transformations. However, in our approach the choice of these transformations is carried out so as to avoid the singularity responsible for the instability of the integrals by rejecting them to infinity. Consequently, in a stage of calculations one obtains clearly a stable path integral [11].

In Section 2, we expose the review of general Faddeev–Senjanovic formulation in the case of unspecified variety and interaction. In Section 3, we consider the case of the Coulomb interaction on the D-sphere. We consider the same problem on the D-pseudosphere in Section 4. Section 5 is devoted to concluding remarks.

2. Review of Faddeev–Senjanovic method

Let us study a particle subjected to the action of scalar and vectorial potential moving on the D-surface immersed in the space of $D+1$ dimensions. The Hamiltonian governing the dynamics of this physical system is given by

$$\mathcal{H}_T = \frac{\pi^2}{2m} - \lambda f(\mathbf{x}) + v p_\lambda + V(\mathbf{x}), \tag{1}$$

where $\pi = (\mathbf{p} - e\mathbf{A}(\mathbf{x}))$ and, $\mathbf{x}$, $\mathbf{p}$ and $\mathbf{A}$ are vectors of D-dimensions. λ is the Lagrange multiplier.

Applying the habitual Dirac procedure, the involved constraints are

$$\phi_1 = p_\lambda = 0, \tag{2}$$

$$\phi_2(\mathbf{x}) = f(\mathbf{x}) \simeq 0, \tag{3}$$

$$\phi_3(\mathbf{p},\mathbf{x}) = \{\phi_2,\mathcal{H}_T\} = \frac{1}{m}\pi_\mu\partial^\mu f(\mathbf{x}) \simeq 0, \tag{4}$$

$$\begin{aligned}\phi_4(\mathbf{p},\mathbf{x},\lambda) &= \{\phi_3,\mathcal{H}_T\}\\ &= \frac{1}{m^2}\pi_\mu\pi_\nu\partial^\mu\partial^\nu f(\mathbf{x}) + \frac{\lambda}{m}\partial^\nu f(\mathbf{x})\partial_\nu f(\mathbf{x}) + \frac{e}{m^2}\pi^\nu\partial^\mu f(\mathbf{x})F_{\mu\nu}(\mathbf{x}) - \frac{1}{m}\partial^\nu f(\mathbf{x})\partial_\nu V(\mathbf{x}),\end{aligned} \tag{5}$$

where $F_{\mu\nu}(\mathbf{x}) = \partial_\mu A_\nu(\mathbf{x}) - \partial_\nu A_\mu(\mathbf{x})$.

As the determinant $\{\phi^a,\phi^b\}$ does not vanish, the constraints are of the second class type. According to Faddeev–Senjanovic technique, the propagator is written as

$$\begin{aligned}K(f,i;T) = \int \prod_{j=1}^{N} \mathrm{d}\mathbf{x}_j \prod_{j=1}^{N+1} \frac{\mathrm{d}\mathbf{p}_j}{(2\pi)^{(D-1)}}\, \mathrm{d}\lambda_j\, \mathrm{d}p_{\lambda j}\delta(p_{\lambda j})\delta(f(\mathbf{x}_j))\delta(\phi_3(\mathbf{p}_j,\overline{\mathbf{x}}_j))\\ \times \prod_{j=1}^{N+1} \delta(\phi_4(\mathbf{p}_j,\overline{\mathbf{x}}_j,\lambda_j))\sqrt{\det\{\phi^a,\phi^b\}}\exp[\mathrm{i}(\mathbf{p}_j\Delta\mathbf{x}_j + p_{\lambda j}\Delta\lambda_j - \varepsilon\mathcal{H}_T)],\end{aligned} \tag{6}$$

where the constraints ϕ_3 and ϕ_4 are evaluated at the mid-point with $\overline{\mathbf{x}} = \frac{\mathbf{x}(t_j)+\mathbf{x}(t_{j+1})}{2}$, this choice is privileged [6,7]. The integration over the λ_j and p_{λ_j} variables is immediate and gives an infinite constant which is absorbed in a redefinition of measure. The result is then reduced to

$$\begin{aligned}K(f,i;T) = \int \prod_{j=1}^{N} \mathrm{d}\mathbf{x}_j \prod_{j=1}^{N+1} \frac{\mathrm{d}\mathbf{p}_j}{(2\pi)^{(D-1)}}\,\delta(f(\mathbf{x}_j))\delta(\phi_3(\mathbf{p}_j,\overline{\mathbf{x}}_j))|\{\phi_2(\mathbf{x}_j),\phi_3(\mathbf{p}_j,\overline{\mathbf{x}}_j)\}|\\ \times \exp\left[\mathrm{i}\left(\mathbf{p}_j\Delta\mathbf{x}_j - \varepsilon\frac{\pi^2}{2m} - \varepsilon V(\mathbf{x}_j)\right)\right],\end{aligned} \tag{7}$$

where

$$\phi_3(\mathbf{p}_j,\overline{\mathbf{x}}_j) = \frac{1}{m}\left(p_{\mu_j} - eA_\mu(\overline{\mathbf{x}}_j)\right)\partial^\mu f(\overline{\mathbf{x}}_j), \tag{8}$$

$$\left\{\phi_2(\mathbf{x}_j),\phi_3(\mathbf{p}_j,\overline{\mathbf{x}}_j)\right\} = \partial_\mu f(\mathbf{x}_j)\partial^\mu f(\overline{\mathbf{x}}_j). \tag{9}$$

We introduce the integral representation of delta function to integrate on the p^μ variables,

$$\delta(\phi_3(\mathbf{p}_j,\overline{\mathbf{x}}_j)) = \frac{1}{2\pi}\int \mathrm{d}\chi_j \exp\left[\mathrm{i}\chi_j\frac{1}{m}\left(p_{\mu j} - eA_\mu(\overline{x}_j)\right)\partial^\mu f(\overline{x}_j)\right], \tag{10}$$

one obtains

$$\begin{aligned}K(f,i;T) = \left(\frac{m}{2\pi\mathrm{i}\varepsilon}\right)^{\frac{(D-1)(N+1)}{2}} \int \prod_{j=1}^{N} \mathrm{d}\mathbf{x}_j \prod_{j=1}^{N+1} \delta\big(f(\mathbf{x}_j)\big)\sqrt{\partial_\mu f(\mathbf{x}_j)\partial^\mu f(\overline{\mathbf{x}}_j)}\\ \times \exp\left[\mathrm{i}\left(\frac{m}{2\varepsilon}\Delta\mathbf{x}_j\mathbf{\Omega}\Delta\mathbf{x}_j + e\Delta\mathbf{x}_j\mathbf{A}(\overline{\mathbf{x}}_j) - \varepsilon V(\mathbf{x}_j)\right)\right],\end{aligned} \tag{11}$$

where $\mathbf{\Omega}$ is a matrix defined by the following elements $\mathbf{\Omega}_{\mu\nu} = \delta_{\mu\nu} - \eta_\mu\eta_\nu$, with η a vector defined by

$$\eta_\mu = \partial_\mu f(\overline{\mathbf{x}}) \Bigg/ \sqrt{\left(\frac{\partial f}{\partial \overline{\mathbf{x}}}\right)^2}. \tag{12}$$

We can conclude that all the corrections brought by the constraint are gathered in the $\sqrt{\partial_\mu f(\mathbf{x}_j)\partial^\mu f(\overline{\mathbf{x}}_j)}$ term. To this level, let us turn over to the preceding remark and put the following question: if we replace in the term $\sqrt{\partial_\mu f(\mathbf{x}_j)\partial^\mu f(\overline{\mathbf{x}}_j)}$ the mid-point by an arbitrary point following the quantum equivalence principle, can we find a good quantum correction? The answer is obviously no.

In what follows we will apply this formalism to the case of a particle on D-sphere and D-pseudosphere subjected to the Coulomb potential.

3. The Coulomb problem on S^D sphere

For the case of the sphere S^D, the function $f(\mathbf{x})$, ($\mathbf{x} = x_i$, $i = 1, \ldots, D+1$), is given as

$$f(\mathbf{x}) = \mathbf{x}^2 - R^2 = \sum_{i=1}^{D+1} (x_i)^2 - R^2 = 0, \tag{13}$$

R being the radius of sphere. The Poisson bracket $\{\phi_2(\mathbf{x}_j), \phi_3(\mathbf{p}_j, \overline{\mathbf{x}}_j)\}$ is then easily evaluated and the propagator (11) is written as

$$K(f,i;T) = \lim_{N\to\infty} \int \prod_{j=1}^{N} \mathrm{d}\mathbf{x}_j \prod_{j=1}^{N+1} \left(\frac{m}{2\pi\mathrm{i}\varepsilon}\right)^{D/2} \frac{2(\mathbf{x}_j\overline{\mathbf{x}_j})}{\sqrt{\overline{\mathbf{x}_j}^2}} \delta\left(\mathbf{x}_j^2 - R^2\right) \times \prod_{j=1}^{N+1} \exp\left[\mathrm{i}\left(\frac{m}{2\varepsilon}(\Delta\mathbf{x}_j)^2 - \varepsilon V(\mathbf{x}_j)\right)\right] \tag{14}$$

with

$$\mathbf{x} = r\boldsymbol{\Omega}, \quad \boldsymbol{\Omega} = (\cos\chi \quad \sin\chi\cos\theta_1 \quad \cdots \quad \sin\chi\sin\theta_1\cdots\sin\theta_{D-2}\sin\varphi) \tag{15}$$

the variables $\chi \in [0, \pi/2]$, $\theta_1, \ldots, \theta_{D-2} \in [0, \pi]$ and $\varphi \in [0, 2\pi]$.

Thus, the expression of the propagator (14) becomes

$$K(f,i;T) = \lim_{N\to\infty} \left(\frac{m}{2\pi\mathrm{i}\varepsilon}\right)^{(N+1)D/2} \int \prod_{j=1}^{N} R^D \,\mathrm{d}\boldsymbol{\Omega}_j \prod_{j=1}^{N+1} \sqrt{\frac{1}{2}\left(1+\cos\boldsymbol{\Omega}_{j,j-1}\right)} \times \prod_{j=1}^{N+1} \exp\left[\mathrm{i}\left(\frac{mR^2}{\varepsilon}\left(1-\cos\boldsymbol{\Omega}_{j,j-1}\right) - \varepsilon V(\boldsymbol{\Omega}_j, R)\right)\right] \tag{16}$$

with

$$\mathrm{d}\boldsymbol{\Omega} = \sin^{D-1}\chi\,\mathrm{d}\chi\,\sin^{D-2}\theta_1\,\mathrm{d}\theta_1\cdots\sin\theta_{D-2}\,\mathrm{d}\theta_{D-2}\,\mathrm{d}\varphi \tag{17}$$

and

$$\cos\boldsymbol{\Omega}_{j,j-1} = \cos\Delta\chi_j - \sin\chi_j\sin\chi_{j-1}(1-\cos\Theta_{j,j-1}). \tag{18}$$

To evaluate the quantum correction, we expand

$$\langle\sqrt{\tfrac{1}{2}(1+\cos\mathbf{\Omega}_{j,j-1})}\rangle \simeq 1-\tfrac{1}{8}\langle(\Delta\mathbf{\Omega}_{j,j-1})^2\rangle \tag{19}$$

with

$$\begin{aligned}(\Delta\mathbf{\Omega}_{j,j-1})^2 \\ &= \Delta\chi_j^2+\sin\chi_j\sin\chi_{j-1}\left(\Delta\theta_{1j}^2+\sin\theta_{1j}\sin\theta_{1j-1}\left(\Delta\theta_{2j}^2+\cdots\sin\theta_{D-2j}\sin\theta_{D-2j-1}\left(\Delta\varphi_j^2\right)\right)\right)\end{aligned} \tag{20}$$

and follow the usual procedure. The correction (19) is then

$$1-\frac{1}{8}\langle(\Delta\mathbf{\Omega}_{j,j-1})^2\rangle \simeq \exp\left(\frac{\mathrm{i}\varepsilon D(D-2)}{8mR^2}\right). \tag{21}$$

Consequently, the shape of the propagator (16) becomes

$$\begin{aligned}K(f,i;T) &= \lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\varepsilon}\right)^{(N+1)D/2}\int\prod_{j=1}^{N}R^D\mathrm{d}\mathbf{\Omega}_j \\ &\quad\times\prod_{j=1}^{N+1}\exp\left[\mathrm{i}\left(\frac{mR^2}{\varepsilon}\left(1-\cos\mathbf{\Omega}_{j,j-1}\right)+\frac{\varepsilon D(D-2)}{8mR^2}-\varepsilon V(\mathbf{\Omega}_j,R)\right)\right].\end{aligned} \tag{22}$$

This last formula represents a general result where the potential is unspecified. We will consider the Coulomb potential which has a spherical symmetry.

In the spheric space $SO(D+1)/SO(D)$, this is given by

$$V(\mathbf{\Omega},R)=V(\chi,R)=-\frac{\alpha}{R}\cot\chi, \tag{23}$$

where α is the coupling constant. Knowing that the potential depends only on χ, let us proceed as usual [12] by separation of the purely angular variables ($\theta_1,\ldots,\theta_{D-2}$, φ) applying the following decomposition formula

$$\exp(z\cos\Theta)=\left(\frac{2}{z}\right)^{v}\Gamma(v)\sum_{l=0}^{\infty}(l+v)I_{l+v}(z)C_l^v(\cos\Theta),\quad v\neq 0,-1,-2,\ldots \tag{24}$$

where $I_{l+v}(z)$ are the modified Bessel functions and $C_l^v(\cos\Theta)$ are the Gegenbauer polynomials and with a notation

$$z=\frac{mR^2\sin\chi_j\sin\chi_{j-1}}{i\varepsilon}. \tag{25}$$

With this separation formula, it is possible to integrate the angular part $\Theta_{j,\,j-1}$ by using the development of the Gegenbauer polynomials and their orthogonality relations.

In fact, let us note

$$\overline{C}_n^v(\cos\Theta_{i,j-1})=\Gamma(v)(n+v)C_n^v(\cos\Theta_{i,j-1}) \tag{26}$$

and

$$\widetilde{C}_n^v(\cos\alpha)=\left(\frac{n!(n+v)2^{2v-1}}{\pi\Gamma(2v+n)}\right)^{\frac{1}{2}}\Gamma(v)C_n^v(\cos\alpha) \tag{27}$$

by applying the addition theorem to the relation (24), one obtain

$$\begin{aligned}\bar{C}_n^{\frac{P}{2}}(\cos\Theta_{j,j-1}) = 2\pi\pi^{\frac{P}{2}}\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{P-1}=0}^{k_{P-2}}\sum_{m=-k_{P-1}}^{k_{p-1}}&\left(\sin\theta_{(1)j}\sin\theta_{(1)(j-1)}\right)^{k_1}\\ &\times\left(\sin\theta_{(2)j}\sin\theta_{(2)(j-1)}\right)^{k_2}\cdots\left(\sin\theta_{(P-1)j}\sin\theta_{(P-1)(j-1)}\right)^{k_{p-1}}\\ &\times\tilde{C}_{n-k_1}^{\frac{P}{2}}(\cos\theta_{(1)j})\tilde{C}_{n-k_1}^{\frac{P}{2}+k_1}(\cos\theta_{(1)(j-1)})\\ &\times\tilde{C}_{k_1-k_2}^{\frac{(P-1)}{2}+k_2}(\cos\theta_{(2)j})\tilde{C}_{k_1-k_2}^{\frac{(P-1)}{2}+k_2}(\cos\theta_{(2)(j-1)})\\ &\times\cdots\tilde{C}_{k_{P-2}-k_{P-1}}^{1+k_{P-1}}(\cos\theta_{(p-1)j})\tilde{C}_{k_{P-2}-k_{P-1}}^{1+k_{P-1}}(\cos\theta_{(p-1)(j-1)})\\ &\times Y_{k_{P-1}}^{m*}(\theta_{pj},\varphi_j)Y_{k_{P-1}}^{m}(\theta_{(p)(j-1)},\varphi_{(j-1)}).\end{aligned}\tag{28}$$

Then, thanks to the orthogonality relations of the $\tilde{C}_m^{v}$ and $Y_l^m(\Omega)$

$$\int_0^{\pi}\mathrm{d}\alpha\sin^{2v}\alpha\tilde{C}_n^{v}(\cos\alpha)\tilde{C}_m^{v}(\cos\alpha)=\delta_{n,m}\tag{29}$$

and

$$\int Y_l^{m^*}(\Omega)Y_{\grave{l}}^{\grave{m}}(\Omega)\,\mathrm{d}\Omega=\delta_{l\grave{l}}\delta_{m\grave{m}},\tag{30}$$

the propagator (22) takes the following form

$$K(f,i;T)=\sum_{l=0}^{\infty}K_l(\chi_f,\chi_i;T)\frac{(2l+D-2)}{4(\pi)^{\frac{D}{2}}}\Gamma\left(\frac{D-2}{2}\right)C_l^{\frac{D-2}{2}}(\cos\Theta_{i,f}),\tag{31}$$

with

$$\begin{aligned}K_l(\chi_f,\chi_i;T)=\lim_{N\to\infty}\left(\frac{M}{2\pi\mathrm{i}\varepsilon}\right)^{\frac{ND}{2}}(2^{D-1}\pi)^N\int\prod_{j=1}^{N-1}R^D(\sin\chi_j)^{D-1}\mathrm{d}\chi_j\\ \times\left[\prod_{j=1}^{N}\left(\frac{\mathrm{i}\varepsilon\pi}{2MR^2\sin\chi_j\sin\chi_{j-1}}\right)^{\frac{D-2}{2}}I_{l+\frac{D-2}{2}}\left(\frac{MR^2\sin\chi_j\sin\chi_{j-1}}{\mathrm{i}\varepsilon}\right)\right]\prod_{j=1}^{N}\exp\{\mathrm{i}S_j\}\end{aligned}\tag{32}$$

and

$$S_j=\frac{mR^2}{\varepsilon}(1-\cos\Delta\chi_j)+\frac{\varepsilon D(D-2)}{8mR^2}+\varepsilon\frac{\alpha}{R}\cot\chi_j+\frac{mR^2\sin\chi_j\sin\chi_{j-1}}{\varepsilon}.\tag{33}$$

Let us simplify calculation using the asymptotic expression of the modified Bessel functions

$$I_\gamma(z)\to\left(\frac{1}{2\pi z}\right)^{\frac{1}{2}}\exp\left\{z-\frac{1}{2z}\left(\gamma^2-\frac{1}{4}\right)\right\},\quad |z|\to\infty,\ \ |\arg z|\leqslant\frac{\pi}{2}.\tag{34}$$

The radial part of the propagator (32) becomes

$$K_l(\chi_f,\chi_i;T)=\frac{1}{R^D(\sin\chi_i\sin\chi_f)^{\frac{D-1}{2}}}\lim_{N\to\infty}\int\prod_{j=1}^{N}\left(\frac{mR^2}{2\pi\mathrm{i}\varepsilon}\right)^{\frac{1}{2}}\prod_{j=1}^{N-1}\mathrm{d}\chi_j\prod_{j=1}^{N}\exp\{\mathrm{i}S_j\}\tag{35}$$

with the action

$$S_j = \frac{mR^2}{\varepsilon}\left(1-\cos\Delta\chi_j\right) + \frac{\varepsilon D(D-2)}{8mR^2} + \varepsilon\frac{\alpha}{R}\cot\chi_j - \varepsilon\frac{(l+\frac{D-1}{2})(l+\frac{D-3}{2})}{2mR^2\sin\chi_j\sin\chi_{j-1}}. \tag{36}$$

To convert this problem to that of Poschl–Teller we refer to the space–time transformation technique. Then let us introduce the Green function defined by

$$G_l(\chi_f,\chi_i;E) = \int_0^\infty \mathrm{d}T\, K_l(\chi_f,\chi_i;T)\exp(\mathrm{i}ET) \tag{37}$$

and the space–time transformation avoiding the path collapse [11]

$$\begin{aligned} &\chi \to x; \quad \exp(\mathrm{i}\chi) = -\coth\left(\frac{e^x}{2}\right), \\ &T \to S; \quad \mathrm{d}t = \frac{e^{2x}}{\sinh^2(e^x)}\,\mathrm{d}s. \end{aligned} \tag{38}$$

After some calculations, the result becomes

$$G_l(\chi_f,\chi_i;E) = \frac{1}{R^D(\sin\chi_i\sin\chi_f)^{\frac{D-1}{2}}}\sqrt{\frac{e^{x_f+x_i}}{\sinh(e^{x_f})\sinh(e^{x_i})}}\int_0^\infty \mathrm{d}S\, P_E(x_f,x_i;S) \tag{39}$$

with

$$P_E(x_f,x_i;S) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\left(\frac{mR^2}{2\pi\mathrm{i}\sigma}\right)^{\frac{1}{2}}\prod_{j=1}^{N-1}\mathrm{i}\,\mathrm{d}x_j\prod_{j=1}^{N}\exp\left\{\mathrm{i}\overline{S_j}\right\} \tag{40}$$

and

$$\begin{aligned} \overline{S_j} = &-\frac{mR^2}{2\sigma}\Delta x_j^2 + \sigma\frac{e^{2x_j}}{8mR^2}\left[e^{-2x_j} + (2l+D-2)^2\right. \\ &\left.+\frac{2mER^2+\frac{D(D-2)}{4}+2m\mathrm{i}\alpha R}{\sinh^2(e^{x_j}/2)} - \frac{2mER^2+\frac{D(D-2)}{4}-2m\mathrm{i}\alpha R}{\cosh^2(e^{x_j}/2)}\right]. \end{aligned} \tag{41}$$

At this level, we have the Poschl–Teller propagator where the singularity of the potential is rejected to the infinity thanks to the exponential. It is well known that this potential admits $SU(1,1)$ symmetry.

To pass to variety $SU(1,1)$, it is necessary to add at each step of discretization two angles (γ,β) using the following asymptotic formulas

$$\exp\left[-\frac{1}{2z}\left(p^2-\frac{1}{4}\right)\right] = \left(\frac{z}{2\pi}\right)^{\frac{1}{2}}\int_0^{2\pi}\exp[\mathrm{i}p\gamma - z(1-\cos\gamma)]\,\mathrm{d}\gamma \tag{42}$$

with

$$z = \frac{4mR^2\cosh(e^{x_j}/2)\cosh(e^{x_{j-1}}/2)}{\mathrm{i}\sigma e^{2x_j}}, \quad p^2 = \left[2mR^2\left(E-\frac{\mathrm{i}\alpha}{R}\right)+\left(\frac{D-1}{2}\right)^2\right] \tag{43}$$

and

$$\exp\left[-\frac{1}{2z'}\left(q^2-\frac{1}{4}\right)\right]=\left(\frac{z'}{2\pi}\right)^{\frac{1}{2}}\int_0^{2\pi}\exp[\mathrm{i}q\beta-z'(1-\cos\beta)]\,\mathrm{d}\beta \tag{44}$$

with

$$z'=-\frac{4mR^2\sinh(e^{x_j}/2)\sinh(e^{x_{j-1}}/2)}{\mathrm{i}\sigma e^{2x_j}},\quad q^2=\left[2mR^2\left(E+\frac{\mathrm{i}\alpha}{R}\right)+\left(\frac{D-1}{2}\right)^2\right]. \tag{45}$$

In addition, the change of variables γ_j, β_j in Euler angles $\varphi_j\in[0,2\pi]$ and $\psi_j\in[0,4\pi]$ given by the relations

$$\begin{cases}\gamma_j=\frac{1}{2}(\Delta\psi_j+\Delta\varphi_j)\\ \beta_j=\frac{1}{2}(\Delta\psi_j-\Delta\varphi_j)\end{cases},\quad \int_0^{2\pi}\mathrm{d}\gamma_j\int_0^{2\pi}\mathrm{d}\beta_j=\frac{1}{2}\int_0^{2\pi}\mathrm{d}\varphi_j\int_0^{4\pi}\mathrm{d}\psi_j, \tag{46}$$

with the condition $\varphi_0=\psi_0=0$, gives then the stable path integral

$$\begin{aligned}G_l(\chi_f,\chi_i;E)=&\frac{e^{(x_f+x_i)/2}}{R^D(\sin\chi_i\sin\chi_f)^{\frac{D-1}{2}}}\int_0^\infty\mathrm{d}S\int\mathrm{d}\varphi_f\,\mathrm{d}\psi_f\exp\left(\mathrm{i}\frac{p+q}{2}\psi_f+\mathrm{i}\frac{p-q}{2}\varphi_f\right)\\&\times\lim_{N\to\infty}\int\prod_{j=1}^{N-1}\sinh(e^{x_j})\,\mathrm{i}\,\mathrm{d}x_j\,\mathrm{d}\varphi_j\,\mathrm{d}\psi_j\\&\times\prod_{j=1}^{N}\left(\frac{mR^2}{2\pi\mathrm{i}\sigma}\right)^{\frac{1}{2}}\left(\frac{mR^2\mathrm{i}}{2\pi\sigma e^{2x_j}}\right)^{\frac{1}{2}}\left(\frac{mR^2}{2\pi\mathrm{i}\sigma e^{2x_j}}\right)^{\frac{1}{2}}\prod_{j=1}^{N}\exp\left\{\mathrm{i}\overline{S_j}\right\}\end{aligned} \tag{47}$$

with

$$\begin{aligned}\overline{S_j}=&-\frac{mR^2}{2\sigma}\Delta x_j^2+\sigma\frac{e^{2x_j}}{8mR^2}\left[e^{-2x_j}+(2l+D-2)^2\right]\\&+\frac{4mR^2\cosh(e^{x_j}/2)\cosh(e^{x_{j-1}}/2)}{\sigma e^{2x_j}}\left(1-\cos\left(\frac{1}{2}(\Delta\psi_j+\Delta\varphi_j)\right)\right)\\&-\frac{4mR^2\sinh(e^{x_j}/2)\sinh(e^{x_{j-1}}/2)}{\sigma e^{2x_j}}\left(1-\cos\left(\frac{1}{2}(\Delta\psi_j-\Delta\varphi_j)\right)\right).\end{aligned} \tag{48}$$

We introduce the following change

$$\begin{aligned}&x\to\xi\quad x=\ln\xi\\&\sigma\to\tau\quad\sigma=\tau/\xi^2,\end{aligned} \tag{49}$$

a direct calculation gives

$$\begin{aligned}G_l(\chi_f,\chi_i;E)=&\frac{1/8}{R^D(\sin\chi_i\sin\chi_f)^{\frac{D-1}{2}}}\int_0^\infty\mathrm{d}S\exp\left(\frac{\mathrm{i}S}{8mR^2}((2l+D-2)^2-1/4)\right)\\&\times\int\mathrm{d}\varphi_f\,\mathrm{d}\psi_f\exp\left(\mathrm{i}\frac{p+q}{2}\psi_f+\mathrm{i}\frac{p-q}{2}\varphi_f\right)Q(g_f,g_i;S)\end{aligned} \tag{50}$$

with $Q(g_f, g_i; S)$ is a path integral relating to the variety $SU(1,1)$ defined by

$$Q(g_f, g_i; S) = \lim_{N\to\infty} \int \left(\frac{2mR^2}{\pi i\tau}\right)^N \left(\frac{2mR^2 i}{\pi\tau}\right)^{\frac{N}{2}} \prod_{j=1}^{N-1} 2\pi^2 \, dg_j \times \exp\left\{\sum_{j=1}^{N} \left(\frac{4imR^2}{\tau}\right)\left(1 - \frac{1}{2}\mathrm{Tr}(g_j g_{j-1})\right)\right\}, \tag{51}$$

where g is an element of the pseudounitary matrix group $SU(1,1)$ parametrized as

$$g = \begin{pmatrix} e^{-\frac{i\varphi}{2}} & 0 \\ 0 & e^{\frac{i\varphi}{2}} \end{pmatrix} \begin{pmatrix} \cosh\frac{\xi}{2} & \sinh\frac{\xi}{2} \\ \sinh\frac{\xi}{2} & \cosh\frac{\xi}{2} \end{pmatrix} \begin{pmatrix} e^{-\frac{i\psi}{2}} & 0 \\ 0 & e^{\frac{i\psi}{2}} \end{pmatrix}. \tag{52}$$

The path integral $Q(g_f, g_i; S)$ can be evaluated by means of the group representation properties [13]. The result is given by

$$Q(g_f, g_i; S) = \frac{1}{2\pi^2}\left[\sum_{\sigma=\pm}\sum_{2J=0}^{\infty}(2J+1)\exp\left(-i\frac{((2J+1)^2 - 1/4)S}{8mR^2}\right)\chi^{J,\sigma}\left(g_f g_i^{-1}\right) + \sum_{\sigma=0,1/2}\int_0^\infty d\rho 2\rho \tanh\pi(\rho + i\sigma)\exp\left(-i\frac{(\rho^2 - 1/4)S}{2mR^2}\right)\chi^{-\frac{1}{2}+i\rho,\sigma}\left(g_f g_i^{-1}\right)\right], \tag{53}$$

where $\chi^{J,\sigma}$ are the character functions of the $SU(1,1)$ group given by

$$\chi^{J,\sigma}(g_f g_i^{-1}) = \sum_{M,N} d^{J,\sigma}_{M,N}(g_f) d^{J,\sigma *}_{M,N}(g_i), \tag{54}$$

$d^{J,\sigma}_{M,N}(g)$ are its unitary representations according to the Bargmann function $d^{J,\sigma}_{M,N}(\xi)$

$$d^{J,\sigma}_{M,N}(g) = e^{-iM\varphi} d^{J,\sigma}_{M,N}(\xi)\, e^{-iN\psi}. \tag{55}$$

The integration on (φ_f, ψ_f) in the formula imposes

$$\frac{p-q}{2} = M \quad \text{and} \quad \frac{p+q}{2} = N, \tag{56}$$

what allows simplification

$$G_l(\chi_f, \chi_i; E) = \frac{1/2}{R^D(\sin\chi_a \sin\chi_b)^{\frac{D-1}{2}}}\int_0^\infty dS \exp\left(\frac{iS}{8mR^2}(2l + D - 1)^2\right) \times \left[\sum_{\sigma=\pm}\sum_{2J=0}^{\infty}(2J+1)\exp\left(-i\frac{(2J+1)^2 S}{8mR^2}\right) d^{J,\sigma}_{\frac{p-q}{2},\frac{p+q}{2}}(\xi_f) d^{J,\sigma *}_{\frac{p-q}{2},\frac{p+q}{2}}(\xi_i) + \sum_{\sigma=0,1/2}\int_0^\infty d\rho 2\rho \tanh\pi(\rho + i\sigma)\exp\left(-i\frac{\rho^2 S}{2mR^2}\right) d^{-\frac{1}{2}+i\rho,\sigma}_{\frac{p-q}{2},\frac{p+q}{2}}(\xi_f) d^{-\frac{1}{2}+i\rho,\sigma *}_{\frac{p-q}{2},\frac{p+q}{2}}(\xi_i)\right]. \tag{57}$$

Following the argument presented by [9], we choose $\sigma = 0$ and consequently the functions $d^{J,+}_{M,N}$ is the only which contributes in calculation. The integral over S gives then

$$G_l(\chi_f,\chi_i;E) = \frac{mR^2}{R^D(\sin\chi_i \sin\chi_f)^{\frac{D-1}{2}}} \times \sum_{n_r=0}^{J_0} \delta\left(J_0 - n_r - l - \frac{D-3}{2}\right) d^{l-\frac{D-3}{2},+}_{\frac{p-q}{2},\frac{p+q}{2}}(\xi_f) d^{l-\frac{D-3}{2},+*}_{\frac{p-q}{2},\frac{p+q}{2}}(\xi_i),$$

$$J_0 + 1 = \frac{p-q}{2}, \quad n_r = J_0 - J. \tag{58}$$

The spectrum results from

$$\frac{p-q}{2} = n = n_r + l + \frac{D-3}{2} + 1. \tag{59}$$

The energy spectrum is then

$$E_n = \frac{1}{2mR^2}\left[n^2 - \left(\frac{D-1}{2}\right)^2\right] - \frac{Z^2e^4m}{2n^2}. \tag{60}$$

Let us replace these results in the radial propagator (35) we have

$$K_l(\chi_f,\chi_i;E) = \sum_{n=l+1}^{\infty} R_{nl}(\chi_f)R^*_{nl}(\chi_i)\exp(-\mathrm{i}E_nT), \tag{61}$$

where $R_{nl}(\chi)$ is the radial wave function

$$R_{nl}(\chi) = \left(\frac{n^2+\varepsilon_n^2}{nR^D}\right)^{\frac{1}{2}} (\sin\chi)^{\frac{D-1}{2}} d^{l-\frac{D-3}{2},+}_{n,\mathrm{i}\varepsilon_n}(\xi) \tag{62}$$

with

$$\exp(\mathrm{i}\chi) = -\coth\left(\frac{\xi}{2}\right), \quad \varepsilon_n^2 = \left(\frac{mRZe^2}{n}\right)^2. \tag{63}$$

In this part we have calculated the propagator relating to the Coulomb potential on a D-sphere and thanks to a path reparametrization which enabled us to lead to a Poschl–Teller potential. Then via integration on the compact group $SU(1,1)$, we have built the energy spectrum and the wave functions in D dimensions system.

4. The Coulomb problem on H^D pseudosphere

Let the pseudosphere H^D immersed in a $D+1$ pseudoEuclidian space defined by the equation

$$f(\mathbf{x}) = \mathbf{x}^2 - R^2 = 0, \tag{64}$$

where R being the radius and the scalar product of two vectors is defined by

$$\mathbf{ab} = (a_1b_1)^2 - \sum_{i=2}^{D+1}(a_ib_i)^2. \tag{65}$$

As previously, the propagator (11) is written as

$$K(f,i;T)=\lim_{N\to\infty}\int\prod_{j=1}^{N}\mathrm{d}\mathbf{x}_j\prod_{j=1}^{N+1}\left(\frac{m}{2\pi\mathrm{i}\varepsilon}\right)^{D/2}\frac{2(\mathbf{x}_j\overline{\mathbf{x}_j})}{\sqrt{\overline{\mathbf{x}_j}^2}}\delta(\mathbf{x}^2-R^2)$$
$$\times\prod_{j=1}^{N+1}\exp\left[\mathrm{i}\left(\frac{m}{2\varepsilon}(\Delta\mathbf{x}_j)^2-\varepsilon V(\mathbf{x}_j)\right)\right].\tag{66}$$

We introduce the adequate coordinates

$$\begin{aligned}&\mathbf{x}=r\mathbf{\Omega},\\&\mathbf{\Omega}=(\cosh\chi\quad\sinh\chi\cos\theta_1\quad\cdots\quad\sinh\chi\sin\theta_1\cdots\sin\theta_{D-2}\sin\varphi)\end{aligned}\tag{67}$$

with the variables $\chi\in[0,\infty[$, $\theta_1,\ldots,\theta_{D-2}\in[0,\pi]$ and $\varphi\in[0,2\pi]$.

The quantum fluctuations and the correction are determined following the same method and the propagator (59) is written as

$$K(f,i;T)=\lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\varepsilon}\right)^{(N+1)D/2}\int\prod_{j=1}^{N}R^D\,\mathrm{d}\mathbf{\Omega}_j$$
$$\times\prod_{j=1}^{N+1}\exp\left[\mathrm{i}\left(-\frac{mR^2}{\varepsilon}(1-\cosh\mathbf{\Omega}_{j,j-1})-\frac{\varepsilon D(D-2)}{8mR^2}-\varepsilon V(\mathbf{\Omega}_j,R)\right)\right]\tag{68}$$

with

$$\mathrm{d}\mathbf{\Omega}=\sinh^{D-1}\chi\,\mathrm{d}\chi\sin^{D-2}\theta_1\,\mathrm{d}\theta_1\cdots\sin\theta_{D-2}\,\mathrm{d}\theta_{D-2}\,\mathrm{d}\varphi\tag{69}$$

and

$$\cosh\mathbf{\Omega}_{j,j-1}=\cosh\Delta\chi_j+\sin\chi_j\sin\chi_{j-1}(1-\cos\Theta_{j,j-1}).\tag{70}$$

Here, the form of the potential is unspecified and we are interested by the Coulomb problem which has the pseudospherical symmetry.

In the hyperbolic space this is given by

$$V(\mathbf{\Omega},R)=V(\chi,R)=-\frac{\alpha}{R}(\coth\chi-1),\tag{71}$$

where α is the coupling constant. Knowing that the potential depends only on χ let us proceed as usual to the separation of the purely angular variables $(\theta_1,\ldots,\theta_{D-2},\varphi)$ using the angular decomposition, the propagator takes the form

$$K(f,i;T)=\sum_{l=0}^{\infty}K_l(\chi_f,\chi_i;T)\frac{(2l+D-2)}{4(\pi)^{\frac{D}{2}}}\Gamma\left(\frac{D-2}{2}\right)C_l^{\frac{D-2}{2}}(\cos\Theta_{i,f}),\tag{72}$$

where the radial propagator is

$$K_l(\chi_f,\chi_i;T)=\lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\varepsilon}\right)^{\frac{ND}{2}}(2^{D-1}\pi)^N\int\prod_{j=1}^{N-1}R^D(\sinh\chi_j)^{D-1}\mathrm{d}\chi_j$$
$$\times\left[\prod_{j=1}^{N}\left(\frac{\mathrm{i}\varepsilon\pi}{2MR^2\sinh\chi_j\sinh\chi_{j-1}}\right)^{\frac{D-2}{2}}I_{l+\frac{D-2}{2}}\left(\frac{MR^2\sinh\chi_j\sinh\chi_{j-1}}{\mathrm{i}\varepsilon}\right)\right]\prod_{j=1}^{N}\exp\{\mathrm{i}S_j\},\tag{73}$$

with the action

$$S_j = -\frac{mR^2}{\varepsilon}(1-\cosh\Delta\chi_j) - \frac{\varepsilon D(D-2)}{8mR^2} + \varepsilon\frac{\alpha}{R}(\coth\chi_j - 1) + \frac{mR^2\sinh\chi_j\sinh\chi_{j-1}}{\varepsilon}. \tag{74}$$

Let us simplify calculation using the asymptotic expression of the modified Bessel functions

$$I_\gamma(z) \to \left(\frac{1}{2\pi z}\right)^{\frac{1}{2}}\exp\left\{z - \frac{1}{2z}\left(\gamma^2 - \frac{1}{4}\right)\right\}, \quad |z|\to\infty,\ |\arg z| \leqslant \frac{\pi}{2}. \tag{75}$$

The radial propagator (66) becomes then

$$K_l(\chi_f,\chi_i;T) = \frac{1}{R^D(\sinh\chi_i\sinh\chi_f)^{\frac{D-1}{2}}}\lim_{N\to\infty}\int\prod_{j=1}^{N}\left(\frac{mR^2}{2\pi i\varepsilon}\right)^{\frac{1}{2}}\prod_{j=1}^{N-1}d\chi_j\prod_{j=1}^{N}\exp\{iS_j\}, \tag{76}$$

where

$$S_j = -\frac{mR^2}{\varepsilon}(1-\cosh\Delta\chi_j) - \frac{\varepsilon D(D-2)}{8mR^2} + \varepsilon\frac{\alpha}{R}(\coth\chi_j - 1) - \varepsilon\frac{\left(l+\frac{D-1}{2}\right)\left(l+\frac{D-3}{2}\right)}{2mR^2\sinh\chi_j\sinh\chi_{j-1}}. \tag{77}$$

To convert this problem to that of Poschl–Teller, we refer to the space–time transformation technique.

Let us introduce the Green function with the adequate space–time transformation

$$\begin{aligned} &\chi\to x; \quad \exp(\chi) = \coth(e^x),\\ &T\to S; \quad dt = \frac{e^{2x}}{\sinh^2 e^x}\,ds. \end{aligned} \tag{78}$$

The result of the previous changes is

$$G_l(\chi_f,\chi_i;E) = \frac{1}{R^D(\sinh\chi_a\sinh\chi_b)^{\frac{D-1}{2}}}\sqrt{\frac{e^{x_f+x_i}}{sh(e^{x_f})sh(e^{x_i})}}\int_0^\infty dS\,P_E(x_f,x_i;S) \tag{79}$$

with

$$P_E(x_f,x_i;S) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\left(\frac{mR^2}{2\pi i\sigma}\right)^{\frac{1}{2}}\prod_{j=1}^{N-1}dx_j\prod_{j=1}^{N}\exp\left\{i\overline{S_j}\right\} \tag{80}$$

and

$$\overline{S_j} = \frac{mR^2}{2\sigma}\Delta x_j^2 - \sigma\frac{e^{2x_j}}{8mR^2}\left[e^{-2x_j} + (2l+D-2)^2 + \frac{-2mER^2 + \frac{D(D-2)}{4}}{\sinh^2(e^{x_j}/2)} - \frac{-2mER^2 + \frac{D(D-2)}{4} + 4m\alpha R}{\cosh^2(e^{x_j}/2)}\right]. \tag{81}$$

Now we change $\sigma \rightarrow -\sigma$

$$\overline{S_j} = -\frac{mR^2}{2\sigma}\Delta x_j^2 + \sigma\frac{e^{2x_j}}{8mR^2}\left[e^{-2x_j} + (2l+D-2)^2 + \frac{-2mER^2 + \frac{D(D-2)}{4}}{\sinh^2(e^{x_j}/2)} - \frac{-2mER^2 + \frac{D(D-2)}{4} + 4m\alpha R}{\cosh^2(e^{x_j}/2)}\right]. \tag{82}$$

Following the same steps as previously and by making the correspondence

$$p'^2 = \left[-2mER^2 + \left(\frac{D-1}{2}\right)^2 + 4m\alpha R\right], \quad q'^2 = \left[-2mER^2 + \left(\frac{D-1}{2}\right)^2\right], \tag{83}$$

the Green function (72) becomes

$$G_l(\chi_f,\chi_i;E) = \frac{mR^2}{R^D(\sinh\chi_i \sinh\chi_f)^{\frac{D-1}{2}}}\sum_{n_r=0}^{J_0}\delta\left(J_0 - n_r - l - \frac{D-3}{2}\right)d^{J,+}_{\frac{p'-q'}{2},\frac{p'+q'}{2}}(\xi_f)d^{J,+*}_{\frac{p'-q'}{2},\frac{p'+q'}{2}}(\xi_i),$$
$$J_0 + 1 = \frac{p'-q'}{2},\ n_r = J_0 - J. \tag{84}$$

The spectrum will be given by the condition

$$\frac{p'-q'}{2} = n, \tag{85}$$

which determines the energy spectrum

$$E_n = \frac{-1}{2mR^2}\left[n^2 - \left(\frac{D-1}{2}\right)^2\right] + \frac{Ze^2}{R} - \frac{Z^2e^4m}{2n^2}. \tag{86}$$

Let us replace these results in the radial propagator, we have

$$K_l(\chi_f,\chi_i;E) = \sum_{n=l+1}^{\infty} R_{nl}(\chi_f)R^*_{nl}(\chi_i)\exp(-\mathrm{i}E_nT) \tag{87}$$

with the following radial wave functions

$$R_{nl}(\chi) = \left(\frac{n^2-\varepsilon_n^2}{nR^D}\right)^{\frac{1}{2}}(\sinh\chi)^{\frac{D-1}{2}}d^{l-\frac{D-3}{2},+}_{n,\mathrm{i}\varepsilon_n}(\xi) \tag{88}$$

and

$$\exp(\chi) = \coth(e^x), \quad \varepsilon_n^2 = \left(\frac{mRZe^2}{n}\right)^2. \tag{89}$$

5. Conclusion

In this paper we achieved a fundamental work concerning the quantification in curved spaces. This problem gives way to an open debate because its final solution has not been established yet. We have tried to deal with this by considering the sphere and the hyperboloid with D dimensions using the constraints method where one is obliged to choose the mid-point prescription contrary to what is stipulated by the quantum principle equivalence. In addition to this, we have treated the case of the Coulomb potential where we have used the space–time transformations. The latter has enabled us to avoid the singularity by projecting it to infinity and to bring the problem to its $SU(1,1)$ dynamic symmetry. We have calculated the spectrum and the wave functions, although the normalization factor is still discussed, the results agree with those of the literature.

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