



A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type

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Abstract

In this paper, we proved a common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type and a property (E.A) introduced in [M. Aamri, D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002) 181–188]. Our theorem generalizes Theorem 2.2 of [M. Aamri, D. El Moutawakil, Common fixed points under contractive conditions in symmetric spaces, Appl. Math. E-Notes 3 (2003) 156–162] and Theorem 2 of [M. Aamri, D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002) 181–188].

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1. Introduction

It is well known that the Banach contraction principle is a fundamental result in fixed point theory, which has been used and extended in many different directions. However, it has been observed in [7] that some of the defining properties of the metric are not needed in the proofs of certain metric theorems. Motivated by this fact, Hicks and Rhoades [7] established some common fixed point theorems in symmetric spaces and proved that very general probabilistic structures admit a compatible symmetric or semi-metric.

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Recall that a symmetric on a set X is a nonnegative real valued function d on $X \times X$ such that

- (i) $d(x, y) = 0$ if and only if $x = y$,
- (ii) $d(x, y) = d(y, x)$.

Let d be a symmetric on a set X and for $r > 0$ and any $x \in X$, let $B(x, r) = \{y \in X: d(x, y) < r\}$. A topology $t(d)$ on X is given by $U \in t(d)$ if and only if for each $x \in U$, $B(x, r) \subset U$ for some $r > 0$. A symmetric d is a semi-metric if for each $x \in X$ and each $r > 0$, $B(x, r)$ is a neighborhood of x in the topology $t(d)$. Note that $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ if and only if $x_n \rightarrow x$ in the topology $t(d)$.

The following two axioms were given by Wilson [10]. Let (X, d) be a symmetric space.

- (W.3) Given $\{x_n\}$, x and y in X , $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, y) = 0$ imply $x = y$.
- (W.4) Given $\{x_n\}$, $\{y_n\}$ and x in X , $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ and $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ imply that $\lim_{n \rightarrow \infty} d(y_n, x) = 0$.

It is easy to see that for a semi-metric d , if $t(d)$ is Hausdorff, then (W.3) holds.

On the other hand, the notion of the weak commutativity is introduced by Sessa [9] as follows: Two selfmappings S and T of a metric space (X, d) are said to be weakly commuting if

$$d(STx, TSx) \leq d(Sx, Tx), \quad \text{for all } x \in X.$$

Jungck [3] extended this concept in the following way: Let S and T be two selfmappings of a metric space (X, d) . S and T are said to be compatible if

$$\lim_{n \rightarrow +\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some $t \in X$.

Obviously, two weakly commuting mappings are compatible but the converse is not true as is shown in [3]. Recently, Jungck and Rhoades [4] introduced the concept of weakly compatible maps as follows: Two selfmappings S and T of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points; i.e., if $Su = Tu$ for some $u \in X$, then $STu = T Su$.

It is easy to see that two compatible maps are weakly compatible but the converse is not true. All these concepts have been frequently used to prove existence theorems in common fixed point theory.

However, the study of common fixed points of noncompatible maps is also very interesting [5,6].

On the other hand, Aamri and El Moutawakil [2] have established some new common fixed point theorems under strict contractive conditions on a metric space for mappings satisfying property (E.A) defined as follows: Let S and T be two selfmappings of a metric space (X, d) . We say that S and T satisfy property (E.A) if there exists a sequence $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t \quad \text{for some } t \in X.$$

The main purpose of this paper is to give a common fixed point theorem for selfmappings of a symmetric space under a contractive condition of integral type. These selfmappings are assumed to satisfy a new property introduced recently in [2] on a metric space, which generalizes the notion of noncompatible maps in the setting of a symmetric space.

2. Preliminaries

In the sequel, we need a function $\phi : IR_+ \rightarrow IR_+$ satisfying the condition $0 < \phi(t) < t$ for each $t > 0$. For example, we could let $\phi(t) = \alpha t$ for some $\alpha \in (0, 1)$, or $\phi(t) = t/(t + 1)$.

Definition 1. Let S and T be two selfmappings of a symmetric space (X, d) . S and T are said to be compatible if

$$\lim_{n \rightarrow +\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} d(Sx_n, t) = \lim_{n \rightarrow \infty} d(Tx_n, t) = 0$ for some $t \in X$.

Definition 2. Two selfmappings S and T of a symmetric space (X, d) are said to be weakly compatible if they commute at their coincidence points.

Definition 3. Let S and T be two selfmappings of a symmetric space (X, d) . We say that S and T satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} d(Sx_n, t) = \lim_{n \rightarrow \infty} d(Tx_n, t) = 0$$

for some $t \in X$.

Example 1. Let $X = [0, +\infty[$. Let d be a symmetric on X defined by $d(x, y) = e^{|x-y|} - 1$ for all x, y in X .

Define $S, T : X \rightarrow X$ as follows:

$$Sx = 2x + 1 \quad \text{and} \quad Tx = x + 2, \quad \text{for all } x \in X.$$

Note that the function d is not a metric. Consider the sequence $x_n = 1 + \frac{1}{n}, n = 1, 2, \dots$

Clearly

$$\lim_{n \rightarrow \infty} d(Sx_n, 3) = \lim_{n \rightarrow \infty} d(Tx_n, 3) = 0.$$

Then S and T satisfy property (E.A), but S and T are not weakly compatible.

Example 2. Let $X = IR$ with the above symmetric function d . It is easy to see that the condition (W.3) holds. Define $S, T : X \rightarrow X$ by

$$Sx = x + 1 \quad \text{and} \quad Tx = x + 2, \quad \text{for all } x \in X.$$

Suppose that property (E.A) holds. Then there exists in X a sequence $\{x_n\}$ satisfying $\lim_{n \rightarrow \infty} d(Sx_n, t) = \lim_{n \rightarrow \infty} d(Tx_n, t) = 0$ for some $t \in X$. Therefore

$$\lim_{n \rightarrow \infty} d(x_n, t - 1) = \lim_{n \rightarrow \infty} d(x_n, t - 2) = 0.$$

In view of (W.3), we conclude that $t - 1 = t - 2$, which is a contradiction. Hence S and T do not satisfy property (E.A).

It is clear from Definition 1, that two selfmappings S and T of a symmetric space (X, d) will be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} d(Sx_n, t) = \lim_{n \rightarrow \infty} d(Tx_n, t) = 0 \quad \text{for some } t \in X$$

but $\lim_{n \rightarrow +\infty} d(STx_n, TSx_n)$ is either nonzero or does not exist.

Therefore, two noncompatible selfmappings of a symmetric space (X, d) satisfy property (E.A).

Definition 4. Let (X, d) be a symmetric space. We say that (X, d) satisfies property (H.E) if given $\{x_n\}, \{y_n\}$ and x in X ,

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d(y_n, x) = 0 \quad \text{imply} \quad \lim_{n \rightarrow \infty} d(x_n, y_n) = 0.$$

Example 3.

- (i) Every metric space (X, d) satisfies property (H.E).
- (ii) Let $X = [0, +\infty)$ with the symmetric function d defined in Example 1. It is easy to see that the symmetric space (X, d) satisfies property (H.E).

3. Main results

Theorem 1. Let d be a symmetric for X that satisfies (W.3), (W.4) and (H.E). Let A, B, S and T be selfmappings of (X, d) such that

$$(1) \quad \int_0^{d(Ax, By)} \varphi(t) dt \leq \phi \left(\int_0^{\max\{d(Sx, Ty), d(Sx, By), d(By, Ty)\}} \varphi(t) dt \right)$$

for all $x, y \in X$ where $\varphi: R_+ \rightarrow R_+$ is a Lebesgue-integral mapping which is summable, nonnegative and such that

$$(2) \quad \int_0^\epsilon \varphi(t) dt > 0 \quad \text{for all } \epsilon > 0.$$

Suppose that $A(X) \subset T(X)$ and $B(X) \subset S(X)$, $\{A, S\}$ and $\{B, T\}$ are weakly compatible and $\{A, S\}$ or $\{B, T\}$ satisfies property (E.A). If the range of one of the mappings A, B, S and T is a complete subspace of X , then A, B, S and T have a unique common fixed point in X .

Proof. Suppose that B and T satisfy property (E.A). Then, there exists a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow \infty} d(Bx_n, z) = \lim_{n \rightarrow \infty} d(Tx_n, z) = 0$ for some $z \in X$. Therefore, by (H.E) we have $\lim_{n \rightarrow \infty} d(Bx_n, Tx_n) = 0$. Since $B(X) \subset S(X)$, there exists in X a sequence $\{y_n\}$ such that $Bx_n = Sy_n$. Hence, $\lim_{n \rightarrow \infty} d(Sy_n, z) = 0$. Let us show that $\lim_{n \rightarrow \infty} d(Ay_n, z) = 0$.

Suppose that $\limsup_{n \rightarrow \infty} d(Ay_n, Bx_n) > 0$. Then, using (1), we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_0^{d(Ay_n, Bx_n)} \varphi(t) dt &\leq \limsup_{n \rightarrow \infty} \phi \left(\int_0^{\max\{d(Sy_n, Tx_n), d(Sy_n, Bx_n), d(Bx_n, Tx_n)\}} \varphi(t) dt \right) \\ &= \limsup_{n \rightarrow \infty} \phi \left(\int_0^{d(Bx_n, Tx_n)} \varphi(t) dt \right) \leq \limsup_{n \rightarrow \infty} \int_0^{d(Bx_n, Tx_n)} \varphi(t) dt. \end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} \int_0^{d(Bx_n, Tx_n)} \varphi(t) dt > 0$ which is a contradiction. Then, we have that $\lim_{n \rightarrow \infty} \int_0^{d(Ay_n, Bx_n)} \varphi(t) dt$ and (2) implies that $\lim_{n \rightarrow \infty} d(Ay_n, Bx_n) = 0$. By (W.4), we deduce that $\lim_{n \rightarrow \infty} d(Ay_n, z) = 0$. Suppose that $S(X)$ is a complete subspace of X . Then, $z = Su$ for some $u \in X$. Consequently, we have

$$\lim_{n \rightarrow \infty} d(Ay_n, Su) = \lim_{n \rightarrow \infty} d(Bx_n, Su) = \lim_{n \rightarrow \infty} d(Tx_n, Su) = \lim_{n \rightarrow \infty} d(Sy_n, Su) = 0.$$

We claim that $Au = Su$. Using (1),

$$\begin{aligned} \int_0^{d(Au, Bx_n)} \varphi(t) dt &\leq \phi \left(\int_0^{\max\{d(Su, Tx_n), d(Su, Bx_n), d(Bx_n, Tx_n)\}} \varphi(t) dt \right) \\ &< \int_0^{\max\{d(Su, Tx_n), d(Su, Bx_n), d(Bx_n, Tx_n)\}} \varphi(t) dt. \end{aligned}$$

Letting $n \rightarrow \infty$, we obtain $\lim_{n \rightarrow \infty} \int_0^{d(Au, Bx_n)} \varphi(t) dt = 0$ and (2) implies that $\lim_{n \rightarrow \infty} d(Au, Bx_n) = 0$. By (W.3) we have $z = Au = Su$. The weak compatibility of A and S implies that $ASu = SAu$; i.e., $Az = Sz$. On the other hand, since $A(X) \subset T(X)$, there exists $v \in X$ such that $Au = Tv$. We claim that $Bv = Tv$. If not, condition (1) gives

$$\begin{aligned} \int_0^{d(Tv, Bv)} \varphi(t) dt &= \int_0^{d(Au, Bv)} \varphi(t) dt \leq \phi \left(\int_0^{\max\{d(Su, Tv), d(Su, Bv), d(Bv, Tv)\}} \varphi(t) dt \right) \\ &= \phi \left(\int_0^{d(Bv, Tv)} \varphi(t) dt \right) < \int_0^{d(Bv, Tv)} \varphi(t) dt, \end{aligned}$$

which is a contradiction. Hence, $\int_0^{d(Bv, Tv)} \varphi(t) dt = 0$ and (2) implies that $d(Bv, Tv) = 0$. Then, $z = Au = Su = Bv = Tv$. The weak compatibility of B and T implies that $BTv = TBv$; i.e., $Bz = Tz$. Let us show that z is a common fixed point of A, B, S and T .

If $z \neq Az$, using (1), we get

$$\begin{aligned} \int_0^{d(z, Az)} \varphi(t) dt &= \int_0^{d(Az, Bv)} \varphi(t) dt \leq \phi \left(\int_0^{\max\{d(Sz, Tv), d(Sz, Bv), d(Bv, Tv)\}} \varphi(t) dt \right) \\ &= \phi \left(\int_0^{d(z, Az)} \varphi(t) dt \right) < \int_0^{d(z, Az)} \varphi(t) dt, \end{aligned}$$

which is a contradiction. Therefore, $\int_0^{d(z, Az)} \varphi(t) dt = 0$ and (2) implies that $z = Az = Sz$.

If $z \neq Bz$, using (1), we get

$$\int_0^{d(z, Bz)} \varphi(t) dt = \int_0^{d(Az, Bz)} \varphi(t) dt \leq \phi \left(\int_0^{\max\{d(Sz, Tz), d(Sz, Bz), d(Bz, Tz)\}} \varphi(t) dt \right)$$

$$= \phi \left(\int_0^{d(z, Bz)} \varphi(t) dt \right) < \int_0^{d(z, Bz)} \varphi(t) dt,$$

which is a contradiction. Then, $\int_0^{d(z, Bz)} \varphi(t) dt = 0$ and (2) implies that $z = Bz = Tz = Az = Sz$. The proof is similar when $T(X)$ is assumed to be a complete subspace of X . The cases in which $A(X)$ or $B(X)$ is a complete subspace of X are similar to the cases in which $T(X)$ or $S(X)$ respectively is complete since $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

For the uniqueness of z , suppose that $\omega \neq z$ is another common fixed point of A, B, S and T . Using (1), we obtain

$$\begin{aligned} \int_0^{d(z, \omega)} \varphi(t) dt &= \int_0^{d(Az, B\omega)} \varphi(t) dt \leq \phi \left(\int_0^{\max\{d(Sz, T\omega), d(Sz, B\omega), d(B\omega, T\omega)\}} \varphi(t) dt \right) \\ &= \phi \left(\int_0^{d(z, \omega)} \varphi(t) dt \right) < \int_0^{d(z, \omega)} \varphi(t) dt, \end{aligned}$$

which is a contradiction. Therefore, $\int_0^{d(z, \omega)} \varphi(t) dt = 0$ and (2) implies that $z = \omega$. This completes the proof of the theorem. \square

Corollary 1. *Let d be a symmetric for X that satisfies (W.3) and (H.E). Let A and B be two weakly compatible selfmappings of (X, d) such that*

$$\int_0^{d(Ax, Ay)} \varphi(t) dt \leq \phi \left(\int_0^{\max\{d(Bx, By), d(Bx, Ay), d(Ay, By)\}} \varphi(t) dt \right)$$

for all $x, y \in X$ where $\varphi: R_+ \rightarrow R_+$ is a Lebesgue-integral mapping which is summable, non-negative and such that $\int_0^\epsilon \varphi(t) dt > 0$ for all $\epsilon > 0$.

Suppose that A and B satisfy property (E.A) and $A(X) \subset B(X)$. If the range of A or B is a complete subspace of X , then A and B have a unique common fixed point in X .

If $\varphi(t) = 1$ in Corollary 1, we obtain Theorem 2.1 of [1].

Since two noncompatible selfmappings of a symmetric space (X, d) satisfy property (E.A), we get the following result.

Corollary 2. *Let d be a symmetric for X that satisfies (W.3) and (H.E). Let A and B be two noncompatible weakly compatible selfmappings of (X, d) such that*

$$\int_0^{d(Ax, Ay)} \varphi(t) dt \leq \phi \left(\int_0^{\max\{d(Bx, By), d(Bx, Ay), d(Ay, By)\}} \varphi(t) dt \right)$$

for all $x, y \in X$ where $\varphi: R_+ \rightarrow R_+$ is a Lebesgue-integral mapping which is summable, non-negative and such that $\int_0^\epsilon \varphi(t) dt > 0$ for all $\epsilon > 0$ and $A(X) \subset B(X)$.

If the range of A or B is a complete subspace of X , then A and B have a unique common fixed point in X .

If $\varphi(t) = 1$ in Corollary 2, we obtain Corollary 2.1 of [1].

If $\varphi(t) = 1$ in Theorem 1, we obtain Theorem 2.2 of [1].

Corollary 3. *Let A, B, S and T be selfmappings of a metric space (X, d) such that*

$$\int_0^{d(Ax, By)} \varphi(t) dt \leq \phi \left(\int_0^{\max\{d(Sx, Ty), d(Sx, By), d(By, Ty)\}} \varphi(t) dt \right)$$

for all $x, y \in X$ where $\varphi: R_+ \rightarrow R_+$ is a Lebesgue-integral mapping which is summable, non-negative and such that $\int_0^\epsilon \varphi(t) dt > 0$ for all $\epsilon > 0$.

Suppose that $A(X) \subset T(X)$ and $B(X) \subset S(X)$ and A and S or B and T satisfy property (E.A). If the range of one of the mappings A, B, S and T is a complete subspace of X , then A, B, S and T have a unique common fixed point in X .

If $\varphi(t) = 1$ in Corollary 3, we obtain Theorem 2 of [2].

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References

- [1] M. Aamri, D. El Moutawakil, Common fixed points under contractive conditions in symmetric spaces, Appl. Math. E-Notes 3 (2003) 156–162.
- [2] M. Aamri, D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002) 181–188.
- [3] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9 (1986) 771–779.
- [4] G. Jungck, B.E. Rhoades, Fixed points for set valued functions without continuity, Indian J. Pure. Appl. Math. 29 (3) (1998) 227–238.
- [5] R.P. Pant, Common fixed points of contractive maps, J. Math. Anal. Appl. 226 (1998) 251–258.
- [6] R.P. Pant, Common fixed points of sequences of mappings, Ganita 47 (1996) 43–49.
- [7] T.L. Hicks, B.E. Rhoades, Fixed point theory in symmetric spaces with applications to probabilistic spaces, Non-linear Anal. 36 (1999) 331–344.
- [8] B.E. Rhoades, Two fixed point theorems for mappings satisfying a general contractive condition of integral type, Internat. J. Math. Math. Sci. 63 (2003) 4007–4013.
- [9] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math. (Beograd) 32 (46) (1982) 149–153.
- [10] W.A. Wilson, On semi-metric spaces, Amer. J. Math. 53 (1931) 361–373.