Larbi Ben M'hidi University Oum el Bouaghi

Faculty of Exact Sciences and Natural and Life Sciences

Department of Mathematics and Computer Science

Mathematical Logic

Course and Tutorial Materials

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Syllabus

Course Unit: Methodological Module 312

Field/Program: 2nd Year Bachelor's Degree in Computer Science

Semester: 3

Weekly Hours: 3 hours

Lectures (1.5 hours)

Tutorials (1.5 hours)

Credits: 4

Weighting: 2

Assessment Method: Exam (60%), Continuous Assessment (40%)

Course Objectives:

Through this course, students must understand and master the concepts of mathematical logic, and more specifically, the objects of logic, syntax, semantics, propositional logic, and predicate logic. The objective of this course is the formalization of human reasoning. The prerequisites for studying and understanding this module are: basic knowledge of mathematics and Boolean algebra.

This course covers the fundamental techniques of zero-order propositional logic and first-order predicate logic. This document offers a series of solved exercises that lead the student to a thorough understanding of the basic concepts of logic.

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Chapter 1: Introduction

Basic Concepts of Mathematical Logic

1. Introduction

Logic is the fundamental basis of all mathematical reasoning. It is crucial for formulating propositions and studying their truth value.

In this first chapter, we will introduce the basics of logic. In particular, we will present the definitions of assertions, as well as the various logical connectives.

This chapter lays the groundwork for the concepts that will appear in the following two chapters: propositional logic and first-order descriptive logic.

2. Assertion (formula or proposition)

An assertion is a mathematical statement to which one of two logical values is assigned: true (T) or false (F), but not both simultaneously.

Example 1

1. $3 + 3 = 6$ is a true statement.
2. Paris is the capital of France is a true statement.
3. The statement: It is raining is false.
4. The statement: $2 + 2 = 7$ is false.

All the previous declarative sentences are propositions: The first two sentences are propositions whose truth value is true, while the last two are mathematical propositions whose truth value is false.

Example 2

1. It will rain tomorrow
2. $4 + 5$
3. What time is it now?
4. Read it carefully please.

The previous sentences are not propositions.

3. Connectors

In any mathematical reasoning, there are five (5) logical connectives. Let P and Q be two assertions.

a. The negation (not) or (¬)

We will call the negation of P the assertion (¬P) or (not P) and which will be noted in formalized form ¬P or $\overline{P}$.

We present these definitions in the form of truth tables, where T=true, and F=false.

P	**¬p**
T	F
F	T

Table 1.1: The Negation Truth Table

b. The negation of the negation

In mathematics, a double negative is considered an affirmation.

Example 3

If p is the proposition (assertion) $X = 0$, then ¬p is the proposition $X \neq 0$.

05 is not even, therefore 05 is odd.

c. Conjunction (and) or (∧)

We will call the assertion (P and Q) a conjunction, and it will be denoted $(P \wedge Q)$.

P	Q	(P ∧ Q)
T	F	F
T	T	T
F	F	F
F	T	F

Table 1.2: The Conjunction Truth Table

We will say that the assertion P ∧ Q is false when at least one of the two assertions is false.

Example 4

P: The Earth is round (true) and Q: The sky is blue (true).

(P and Q) or (P ∧ Q) can therefore be read as: The Earth is round AND the sky is blue.

(P ∧ Q) is true

Remark

The conjunction is commutative:

(P ∧ Q) ≡ (Q ∧ P)

d. The disjunction (or) or (∨)

We will call the assertion (P ∨ Q) (P or Q) a disjunction of P or Q, and which will be denoted (P ∨ Q).

P	Q	(P ∨ Q)
T	F	T
T	T	T
F	F	F
F	T	T

Table 1.3: The Disjunction Truth Table

We will say that the assertion (P V Q) is false when both assertions p and q are false

Example 5

P: Mouhamed eats the baguette (true)

Q: Mouhamed eats the apple (false)

(P or Q) or (P V Q) can therefore be read as Mouhamed eats the baguette or Mouhamed eats the apple. (P V Q) is true.

Remark

1. Disjunction is commutative.

(P V Q) ≡ (Q V P)

2. In mathematics, "or" is non-exclusive, meaning it includes the possibility that both propositions are true. Thus, the proposition $x \cdot y = 0$ is equivalent to the proposition $x = 0$ or $y = 0$; it is true when one of the two numbers is null, and it is also true when both are null.

e. The Implication ⇒

$((\neg P) \vee Q)$, denoted "P ⇒ Q" or "P implies Q," which is false only if proposition P is true and proposition Q is false. Otherwise, the implication is true. P is then called the hypothesis and Q the conclusion. We can read it in different ways:

- If P then Q,
- For P to require Q,
- For Q to be sufficient, P is sufficient,
- P is a sufficient condition for Q,
- Q is a necessary condition for P.

P	**¬p**	**Q**	**(P ⇒ Q)**	**¬pVQ**
T	F	F	F	F
T	F	T	T	T
F	T	F	T	T
F	T	T	T	T

Table 1.4: The implication Truth Table

Example 6

Consider the propositions "P: quadrilateral ABCD is a square" and "Q: quadrilateral ABCD is a rectangle".

We then have the logical implication "P ⇒ Q", which can be read as follows: "if quadrilateral ABCD is a square, then quadrilateral ABCD is a rectangle".

Remark

The implication Q ⇒ P is called the inverse of the implication P ⇒ Q.

f. Equivalence ⇔

Given propositions P and Q, the logical equivalence of P and Q is the new proposition, denoted P ⇔ Q, which is true if and only if the biconditional P ↔ Q is a tautology.

The proposition P ⇔ Q corresponds to the propositions (P ⇒ Q) and (Q ⇒ P). We can express it as follows:

- P is equivalent to Q,
- For P, Q is necessary and sufficient,
- P is a necessary and sufficient condition for Q,
- P if and only if Q.

The Equivalence Truth Table

P	Q	(P ⇔ Q)
T	F	F
T	T	T
F	F	T
F	T	F

Table 1.5: The Equivalence Truth Table

Practice activity 1

Given that x and t are true and z is false, find the truth values of the propositions:

1. $(x \vee (t \wedge z)) \wedge (t \vee z)$.
2. $(t \Rightarrow x) \vee \neg (x \Leftrightarrow t) \wedge (z \wedge \neg x)$.

Solution

1. Let 1 = True and 0 = False.
2. $(x \vee (t \wedge z)) \wedge (t \vee z) \equiv (1 \vee (1 \wedge 0)) \wedge (1 \vee 0) \equiv (1 \vee 0) \wedge 1 \equiv 1 \wedge 1 \equiv 1$
3. $(t \Rightarrow x) \vee \neg (x \Leftrightarrow t) \wedge (z \wedge \neg x) \equiv (1 \Rightarrow 1) \vee \neg (1 \Leftrightarrow 1) \wedge (0 \wedge \neg 1) \equiv 1 \vee \neg (1) \wedge (0 \wedge 0) \equiv 1 \vee 0 \wedge 0 \equiv 0$

4. Equivalence theorems (laws)

Definition

The well-formed formulae F, G are said equivalent if and only if the truth values of F and G are the same in every interpretation.

Let A, B, and C be well-formed formulae we have the set of following true statements called equivalence laws:

- A ⇒ B ≡ ¬AVB
- A ⇔ B ≡ (A ⇒ B) ∧ (B ⇒ A)
- A ∧ B ≡ B ∧A
- A V B ≡ B V A
- (A V B) V C ≡ A V (B V C)
- (A ∧ B) ∧ C ≡ A ∧ (B ∧ C)
- A V (B ∧ C) ≡ (A V B) ∧ (A V C)
- A ∧ (B V C) ≡ (A ∧ B) V (A ∧ C)
- A V V ≡ V
- A ∧ V ≡ A
- A V F ≡ A
- A ∧ F ≡ F
- A V¬A ≡ V
- A ∧ ¬A≡ F
- (¬ (¬A)) ≡ A
- A V A ≡ A
- A ∧ A ≡ A

5. De Morgan's law

De Morgan's laws allow a conjunction to be transformed into a disjunction (and vice versa) via negation.

- $\overline{(A \wedge B)} \leftrightarrow (\overline{A}) \vee (\overline{B})$
- $\overline{(A \vee B)} \leftrightarrow (\overline{A}) \wedge (\overline{B})$.

These equivalence theorems can be used to transform a well-formed formula into another well-formed formula that is equivalent to it. This will simplify the writing of well-formed formulae.

6. Priority of logical connectors

A. In some references the order of priority of connectors is: (Highest priority)

- ¬,
- ∧,
- ∨,
- →,
- ↔.

Example 7

A ∧¬B∨C→D∧E must be read (((A∧(¬B) ∨C) →(D∧E))

Example 8

- ¬p ∧ q is read (¬p) ∧ q.
- 2. p ∧ q ⇒ r is read (p ∧ q) ⇒ r.
- 3. p ∨ q ∧ r is read p ∨ (q ∧ r).
- 4. p ∨ q ∨ r is read (p ∨ q) ∨ r

B. In some references the order of priority of connectors is: (Highest priority)

- ¬,
- →,
- ↔ ,
- ∧,
- ∨.

To remove this ambiguity: Parentheses always take precedence, therefore, a formula without parentheses is not a well-formed formula (except for Cand D)

C. The outermost parentheses are omitted by mistake.

(A∨B) becomes A ∨ B

D. When there is only one connective, the association is made from left to right.

A → B→C corresponds to ((A → B) → C)

Practice activity 2

A) Explain the implicit parentheses in:

1. a → b → c ;
2. a ∨ b ∧ c ;
3. a ∨ b ∧ c ↔ d → ¬e ∨ f ∧ g

B) Simplify the parentheses in the following formulae as much as possible (see table):

(a)	$((a \vee b))$	$((a) \wedge (b))$
$\neg(((a) \wedge b))$	$a \vee (b \wedge c)$	$(a \vee b) \wedge c$
$(a \wedge (b \rightarrow c))$	$((a \vee b) \wedge c) \leftrightarrow e$	$(((a \vee b) \wedge c) \leftrightarrow e) \rightarrow f$
$(((a \rightarrow b) \rightarrow c) \rightarrow d)$	$(a \wedge (b \wedge c))$	$(a \rightarrow (b \rightarrow c))$
$(\neg(a \vee b))$	$((a \wedge b) \rightarrow c)$	$((a \wedge b) \vee c) \leftrightarrow (e \rightarrow f)$

Table 1.6: Table of Formulae to Simplify

Solution

1. a → b → c ≡ ((a → b) → c)

2. a ∨ b ∧ c ≡ (a ∨ (b ∧ c))

3. a ∨ b ∧ c ↔ d → ¬e ∨ f ∧ g ≡ ((a ∨ (b ∧ c)) ↔ (d → ((¬e) ∨ (f ∧ g))))

$(a) \equiv a$	$((a \vee b)) \equiv a \vee b$	$((a) \wedge (b)) \equiv a \wedge b$
$\neg(((a) \wedge b)) \equiv \neg(a \vee b)$	$a \vee (b \wedge c) \equiv \neg(a \wedge b)$	$(a \vee b) \wedge c \equiv a \vee b \wedge c$
$(a \wedge (b \rightarrow c)) \equiv a \wedge (b \rightarrow c)$	$((a \vee b) \wedge c) \leftrightarrow e \equiv (a \vee b) \wedge c \leftrightarrow e$	$(((a \vee b) \wedge c) \leftrightarrow e) \rightarrow f \equiv ((a \vee b) \wedge c \leftrightarrow e) \rightarrow f$
$(((a \rightarrow b) \rightarrow c) \rightarrow d) \equiv a \rightarrow b \rightarrow c \rightarrow d$	$(a \wedge (b \wedge c)) \equiv a \wedge b \wedge c$	$(a \rightarrow (b \rightarrow c)) \equiv a \rightarrow (b \rightarrow c)$
$(\neg(a \vee b)) \equiv \neg(a \vee b)$	$((a \wedge b) \rightarrow c) \equiv a \wedge b \rightarrow c$	$((a \wedge b) \vee c) \leftrightarrow (e \rightarrow f) \equiv a \wedge b \vee c \leftrightarrow e \rightarrow f$

Table .7: Simplified Formulae

Practice activity 3

Provide the truth tables for the following formulae. Then indicate the equivalencies between the formulae.

1. ¬(p ∧ q),
2. ¬p ∨ ¬q,
3. ¬(p ∨ q),
4. ¬p ∧ ¬q,
5. p ∨ (p ∧ q),
6. p ∧ (p ∨ q),
7. p

Solution

p	q	$p \wedge q$	$\neg(p \wedge q)$
V	V	V	F
V	F	F	V
F	V	F	V
F	F	F	V

1.

p	q	$\neg p$	$\neg q$	$\neg p \vee \neg q$
V	V	F	F	F
V	F	F	V	V
F	V	V	F	V
F	F	V	V	V

2.

p	q	$p \vee q$	$\neg(p \vee q)$
V	V	V	F
V	F	V	F
F	V	V	F
F	F	F	V

3.

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
V	V	F	F	F
V	F	F	V	F
F	V	V	F	F
F	F	V	V	V

4

p	q	$p \wedge q$	$p \vee (p \wedge q)$
V	V	V	V
V	F	F	V
F	V	F	F
F	F	F	F

5.

p	q	$p \vee q$	$p \wedge (p \vee q)$
V	V	V	V
V	F	V	V
F	V	V	F
F	F	F	F

6

p
V
F

7

The equivalent formulae are:

1 and 2

3 and 4

5 and 6 and7

7. Conclusion

In this chapter, we presented the basic concepts of mathematical logic that will be used in propositional logic and predicate logic. We will address zero-order logic in the next chapter.

Chapter 2

Propositional Logic

Order 0 (Part I, Part II, Part III)

1. Introduction

In order 0 logic, we study the relationships between statements, which we will call propositions, formulae, or assertions. These relationships can be expressed through logical connectives. The main connectives are: conjunction, (inclusive) disjunction, implication, equivalence, and negation.

In other words, propositional logic, also called Boolean logic, is the simplest form of logic; it only uses variables and logical connectives. Like all logic, it is characterized by:

- A syntax: the way formulae are constructed;
- A semantics: the meaning of the formulae;
- A proof system: how to demonstrate the formulae.

Propositional Logic Part I: Syntax

To study the syntax of propositional logic is to consider formulae that are "well-written". To do this, we define an alphabet (a language we will denote L), i.e., a set of symbols, with:

- **A countable set V** = {p,q,r,...} of letters called propositional variables. These are atomic propositions such as, for example, "6 is divisible by 2";
- **Atoms:** We will call atoms, propositional variables, or elementary propositions statements whose internal structure is unknown to us, and which retain their identity throughout the propositional calculus we are dealing with. The set of propositional variables is denoted v(L). They are written in lowercase (p, q...);
- **Logical connectors**: $\neg$ (unary operator), $\Rightarrow$, $\Leftrightarrow$, $\wedge$, $\vee$ (binary operators)
- **The constants:** {true and false}. We recall that $\bot$ denotes a propositional constant that is always false, and $\top$ a propositional constant that is always true.

- **Formulae:** We will denote the formulae by capital letters of the Latin or Greek alphabet (A, B, . . . or φ...). The set of formulae, denoted F(L), is defined by: atoms are formulae (v(L) ⊆ F(L)).

Remark

- If A and B are formulae, then (A ↔B), (A→ B), (A ∧ B), (A ∨B) and (¬ B) are formulae.

- **Literal:** A literal is an atom (positive literal) or the negation of an atom (negative literal).

- **Clause:** A clause is a disjunction of literals (p1∨p2∨. . .∨ pn), the literals being able to be positive or negative.

2. Well-formed formulae

The language consists of the set of Well-Formed Formulae (also called: fbfs) or well-formed expressions defined as follows:

- **Base:** every atom is a fbf, likewise propositional constants are fbfs.
- **Induction:** if F et G are fbfs then (¬G), (F ∨ G), (F ∧ G), (F ⇒ G) et (F⇔G) are fbfs.

- **Conclusion:** all fbfs are obtained by applying the 2 rules above.

Example 1

Are the following expressions well-formed formulae?

1. p ∧ ¬q,
2. p ∨ ∨r,
3. (p ∨ ∧ (¬p)),
4. (p ∨ ¬p).

Solution

Response	T/F
1	T
2	F
3	F
4	T

3. The Propositional Language

Propositional language is composed of formulae or propositions. Propositional language is characterized by its syntax and its semantics.

Practice Activity 1

Consider the following context: "She likes apple and whipped cream." State whether the following statements are true or false.

1. "She likes apple and she likes whipped cream."
2. "She likes apple and she doesn't like whipped cream."
3. "She either likes apple or she likes whipped cream."
4. "She either likes apple or she doesn't like whipped cream."
5. "She likes apple therefore she likes whipped cream."
6. "She likes apple; therefore, she doesn't like whipped cream."
7. "She doesn't like apple; therefore, she likes whipped cream."
8. "She doesn't like apple; therefore, she doesn't like whipped cream."

Practice Activity 2

Try to determine whether the following formulas belong to propositional logic:

Let A, B, C, D be fbfs:

a) ((A ∨ (¬B)) ∧ (C∨D)): yes

b) (A ∨ B) (∧ ∨ C): no

Propositional Logic Part II: Semantics

The semantics of propositional logic is concerned with determining the truth value of a statement, that is, a formula. This involves the interpretation of a formula, which more concretely means assigning a true or false value to each of the propositional variables that compose it. For a formula with n variables, there are 2^n possible interpretations. To achieve this, we use what are called truth tables. (See Chapter 01.)

1. An Interpretation

An interpretation is a function that assigns a truth value to each propositional variable.

2. A Truth Table

is a table with several columns. The values in the cells of this table are called "truth values" (1 or T for true, 0 or F for false) in logic. The left columns define the truth values of different propositions in mathematics (propositional logic). The right column indicates the truth value of the logical expression in mathematics or logic.

3. Categories of Formulae

- **Model**: A model is an interpretation for which a formula is true.
- **Consistency**: A formula A is said to be consistent, or satisfiable, or verifiable, if there exists an interpretation of its propositional variables that makes it true. In other words, the formula A is consistent if and only if the formula A has a model.
- **Inconsistency**: A formula for which there is no interpretation that makes it true is said to be inconsistent, or unsatisfiable, or unverifiable, or more simply false. The formula $(A \wedge \neg A)$ is inconsistent.
- **Tautology**: or a valid formula, is a formula that is true in every interpretation. It is denoted as $\models A$. The formula $(A \vee \neg A)$ is a tautology.
- **Antilogy**: An unsatisfiable, falsifiable, or semantically inconsistent formula, also known as an antilogy, is a formula that is false in every interpretation.
- **Invalid Formulas**: An invalid formula is false in at least one interpretation.
- **Contingent Formulas**: A contingent formula is true in some interpretations and false in others.
- **Logical Consequence**: Let A and B be two formulae. We say that B is a valid consequence of A, denoted $(A \models B)$, if every model of A is a model of B.

Practice Activity 3:

1. Construct the truth table of the following formula: T⇒ (B ∧ C)
2. Identify the categories of the formula T⇒ (B ∧ C)
3. What we can conclude about this formula B ∧ C ⊨ T⇒ (B ∧ C)

4. CNF (Conjunctive Normal Form) and DNF (Disjunctive Normal Form)

CNF

A formula in conjunctive normal form (CNF) is a conjunction of clauses, where a clause is a disjunction of literals. CNF formulae are used in the context of automated theorem proving and in solving the SAT problem (particularly in the DPLL algorithm).

Example 2

All of the following expressions are in CNF:

- A ∧ B
- A
- (A ∨B) ∧C
- (A ∨ ¬B ∨¬C ∨ ¬D) ∧ (¬D∨ E∨F)

We obtained the CNF of a formula by applying one of the following methods

4.1 CNF Conversion Algorithm

Start

1. *Eliminate all occurrences of the connectors → and ↔ by replacing:*
   F→G with ¬F∨G
   and F↔G with (¬F∨G)∧(¬G∨F)
2. *Apply De Morgan's laws to push negations inward by replacing:*
   ¬(F∨G)\ with ¬F∧¬G
   ¬(F∧G) with ¬F∨¬G
3. *Eliminate double negations by replacing ¬¬F with F.*

4. *Apply the distributive laws by replacing:*
   F ∨(G∧H) F with (F ∨G) ∧(F ∨H)
   (F∧G) ∨H with (F ∨H) ∧(G ∨H)

4.2 Using Truth Table

To obtain the Conjunctive Normal Form (CNF) of a logical formula using a truth table, follow these steps:

1. **Construct the Truth Table** :
   - List all possible combinations of truth values for the variables in the formula.
   - Determine the output of the formula for each combination.
2. **Identify Rows for Output False:**
   - Focus on the rows where the output of the formula is **false** (i.e., where the formula evaluates to 0). These rows will help us form the clauses for CNF.
3. **Create Clauses** :
   - For each row where the output is false, create a clause that represents that row:
     - If a variable is true (1) in that row, include its negation (¬) in the clause.
     - If a variable is false (0) in that row, include the variable itself in the clause.
   - For example, if the row is P=T, Q=F, R=T (which gives an output of false), the corresponding clause would be ¬P∨Q∨¬R.
4. **Combine Clauses** :
   - Combine all the clauses from the false output rows using the logical AND operator (∧). This results in the CNF.

Example 3

Consider a formula F (P, Q) =(P∧Q) ∨¬Q.

1. **Truth Table** :

P	Q	F(P,Q)
T	T	T
T	F	T
F	T	F
F	F	F

2. **Identify False Outputs** :
   - The outputs are false for the following rows:
     - Row 3: P=F, Q=T
     - Row 4: P=F, Q=F
3. **Create Clauses** :
   - For Row 3: P=F, Q=T → Clause: P∨ ¬Q
   - For Row 4: P=F, Q=F → Clause: P∨Q
4. **Combine Clauses** :
   - CNF : (P∨ ¬Q) ∧(P∨Q)

DNF

A disjunctive normal form (DNF) is a normalization of a logical expression that is a disjunction of conjunctive clauses. It is used in automated theorem proving. A logical expression is in DNF if and only if it is a disjunction of one or more conjunctions of one or more literals.

Example 4

All of the following expressions are in DNF:

- A ∨ B
- A
- (A ∧ B) ∨C
- (A ∧ ¬B ∧¬C ∧ ¬D) ∨ (¬D∧ E∧F)

4.3. DNF Conversion Algorithm

The same as the CNF conversion method, but in steps 2 and 4, we replace ∧ by ∨ and the ∨ by ∧.

4.4. Using the Truth Table

1. **Construct the Truth Table** :
   - List all possible combinations of truth values for the variables in the formula.
   - Determine the output of the formula for each combination.
2. **Identify Rows for Output True**:
   - Focus on the rows where the output of the formula is **true** (i.e., where the formula evaluates to 1). These rows will help us form the terms for DNF.
3. **Create Minterms**:
   - For each row where the output is true, create a minterm that represents that row:
     - If a variable is true (1) in that row, include the variable itself in the minterm.
     - If a variable is false (0) in that row, include its negation (¬) in the minterm.
   - For example, if the row is P=T, Q=F, R=TP = T, Q = F, R = TP=T, Q=F, R=T (which gives an output of true), the corresponding minterm would be P∧¬Q∧RP \land \neg Q \land RP∧¬Q∧R.
4. **Combine Minterms**:
   - Combine all the minterms from the true output rows using the logical OR operator (∨). This results in the DNF.

Example 5

Consider a formula F (P,Q)=(P∧Q)∨¬Q.

1. **Truth Table** :

P	**Q**	**F(P,Q)**
T	T	T
T	F	T
F	T	F
F	F	F

2. **Identify True Outputs** :
   - The outputs are true for the following rows:
     - Row 1 : P=T, Q=T
     - Row 2 : P=T, Q=F
3. **Create Minterms**:
   - For Row 1 : P=T, Q=T → Minterm: $P \land Q$
   - For Row 2: P=T, Q=F → Minterm: $P \land \neg Q$
4. **Combine Minterms**:

- DNF: $(P \land Q) \lor (P \land \neg Q)$

Propositional Logic Part III: Proof System

1． Resolution

1.1. Clause formatting

A clause is a formula that uses only literals (positive or negative) and disjunction.

Remark

A literal on its own is a clause in which there is no disjunction. If we want to introduce a disjunction, we can always rewrite the literal l as $l \lor \bot$.

Example 6

Let p and q be propositional variables.

• ⊥, p, ⊤, ¬p, p∨q, p∨q∨¬p are clauses.

• p∧q, p∧⊤, p⇒q are not clauses

.1.2. The Refutation

A truth table is a simple and reliable way to check the validity of a logical deduction. However, this is not always true, especially if the number of propositional variables exceeds a certain number, such as 8, 16, or more. In these cases, checking the validity of a logical deduction using a truth table will not be practical.

To find an alternative way to verify the validity of a logical deduction, we must refer to mathematical theorems. More generally, if F = {F1, ..., Fn} is a set of formulae and C is a formula, we have the following deduction theorem (TD):

F |= C if only if |= (F → C)

This theorem can be expressed as follows: "C can be deduced from F if and only if (F → C) is a tautology."

Among the interesting variations of this theorem, the following theorem is particularly noteworthy:

F |= C if only if F ∪ {¬C} is inconsistant

This theorem, also known as the refutation theorem (RT), is the basis of the resolution method that we will see at the end of this chapter.

1.3. Resolution schemes

We have three main solution schemes:

A. Modus ponens: This is the simplest scheme: {Control, Control → Note} |= Note

B. Modus tollens We could have reasoned differently from the previously mentioned scheme, starting from the negation of the conclusion, that is, ¬Note, associated with our knowledge Control → Note. Using this rule, we have {Control →Note, ¬Note} |= ¬Control.

This being contradictory to the fact that we know Control, we necessarily deduce Note.

C. ***General inference rule:*** Among the various existing reasoning schemes, we will focus on the resolution rule, of which modus ponens and modus tollens are in fact special cases. This inference rule is expressed as follows:

D. $\{X \vee A, \neg X \vee B\} \models A \vee B$.

The principal of resolution

Propositional calculus transitivity rule: $p \rightarrow q, q \rightarrow r \models p \rightarrow r$

Clausal form: $\{ \neg p \vee q,\ \neg q \vee r \} \models \{ \neg p \vee r \}$

If A and B are two Complementary clauses (which respectively contain the literals Φ and $\neg\Phi$), then we can deduce the new clause C, called ***Resolvent***, obtained by joining all the literals of A and B except Φ and $\neg\Phi$.

Remark

Since the resolution principle is complete, if the set of clauses considered is inconsistent, we can always generate the ***empty clause.***

2. Proof by Refutation in Propositional Logic

• ***Method***

We must follow four steps:

1. Negation of the conclusion
2. Clause form
3. Application of the resolution principle until the clause is empty
4. Conclusion

Practice Activity 4

There is a club in Algiers that follows these rules:

a) Every player from Algiers wears shorts.

b) Every player who wears shorts is from Algiers and married.

c) Every player who is not from Algiers wears red socks.

d) Every player either wears shorts or does not wear red socks.

e) Married players do not play on Fridays.

f) A player plays on Fridays only if he is from Algiers.

Questions:

1. Translate the above statement into order 0 logic.

2. Prove, using the resolution method, that the set is unsatisfiable.

Solution

1. Translate the above statement into propositional logic

- Every player from Algiers wears shorts $A \rightarrow S$.
- Every player who wears shorts is from Algiers and married. $S \rightarrow A \wedge M$
- Every player who is not from Algiers wears red socks. $\neg A \rightarrow R$.
- Every player either wears shorts or does not wear red socks. $S \vee \neg R$.
- Married players do not play on Fridays. $M \rightarrow \neg V$.
- A player plays on Fridays only if he is from Algiers. $V \leftrightarrow A$.

2. Clausal Form

$\{\neg A \vee S, \neg S \vee A, \neg S \vee M, A \vee R, S \vee \neg R, \neg M \vee \neg V, \neg V \vee A, \neg A \vee V\}$.

Applying the resolution principle to this set produces the empty clause: this means that the set is inconsistent. (proof by refutation)

i. $A \vee R, S \vee \neg R$ ---- $A \vee S$.
ii. $A \vee S, \neg A \vee S$----S.
iii. $S, \neg S \vee M$-----M.
iv. $M, \neg M \vee \neg V$---$\neg V$.
v. $\neg V, \neg A \vee V$ ----$\neg A$.
vi. $\neg A, \neg S \vee A$----$\neg S$.
vii. $\neg S, S$----$\perp$.

Therefore, the set is unsatisfactory.

3. Conclusion

In this chapter, we presented propositional logic, defined as logic without quantifiers, which focuses solely on logical operations: negation, conjunction, disjunction, implication, and equivalence. It allows us to construct arguments based on these connectives. In the next chapter, we will discuss predicate logic, often called "first-order logic."

Chapter 3:
Predicate Logic (Order 1) (Part I, Part II,)

1. Introduction

Before studying first-order logic, we must state that propositional logic cannot solve our problems. As we know, propositional logic only allows us to describe simple language constructions, essentially consisting of statements that can be either true or false. It allows us to study the truth value of relatively inexpressive formulae within a formal framework.

Predicate calculus is considered an extension of propositional calculus that allows the introduction, alongside propositional variables, of other variables belonging to an arbitrary domain (a set of integers, real numbers, or any objects). This extension is achieved through the introduction of the two quantifiers: universal and existential, ($\forall$ and $\exists$).

2. Limitation of Order 0 Logic

Consider the following problem:

All men are mortal.

Mouhamed is a man.

So, Mouhamed is mortal.

We have already translated statements into propositional logic. Let us consider the following translation:

The statement "All men are mortal" is translated by proposition A.

"Mouhamed is a man" is translated by proposition B.

So, "Mouhamed is mortal" is translated by proposition C.

The problem can then be written, in propositional logic, as A ∧ B → C. This translation is correct. However, it is of poor quality. Indeed, it will not escape anyone's notice that the initial reasoning was valid, but its translation is not (although it is satisfiable).

3. Alphabet of Predicate Logic

To write a statement in predicate logic, we will use a richer set of symbols than in propositional logic. Therefore, predicate logic can be defined using the following alphabet:

- A set of variables {X, Y, Z, . . .}
- The universal and the existential quantifiers ∀, ∃.
- A set of constants {a, b, c, . . .}
- A set of functions {f, g, h, . . .}
- A set of predicates, or relations {P, Q, . . .}
- Logical connectives, {¬, ∧, ∨, →, . . .}, as well as parentheses "(" and ")";

Predicate Logic Part I: Syntax

1. Definitions

1.1. Predicates

A predicate is a property or relation that applies to one or more elements of a domain D. It is a function from D to {T, F}.

The previous problem can be adequately translated into predicate logic:

- Hypothesis 1: For any x belonging to domain D, if x is a man, then x is mortal. (∀x (H(x) → M(x)));
- Hypothesis 2: Socrates has the property of being a man (H(Socrate));
- Conclusion: Socrates has the characteristic of being mortal (M(Socrate)).

In this example, we used two predicates (H and M), a constant (Socrates), a variable (x), and the universal quantifier (∀). As we will see later, this reasoning, thus formalized in predicate logic, will be valid.

1.2. Terms

Any logical expression that refers to an object can be called a term. Therefore:

– Every constant is a term.

– Every variable is a term.

– And also, if f is a function and T1, ..., Tn are terms then f(T1, ..., Tn) is also a term.

1.3.Quantifiers

Two kinds of quantifiers are existed:

- The universal quantifier (∀), which stresses that the fact of all elements of a set of objects about which a predicate is expressed satisfy that predicate.
- The existential quantifier, which expresses the fact that at least one element of a set of objects about which a predicate is expressed satisfies that predicate.

1.4.Formula

A formula in predicate logic is constructed similarly to a formula in propositional logic. We only need to take into account the quantifiers, universal and existential:

1. P(x1, ..., xn) is an atomic formula ;
2. if H is a formula, then ¬H is a formula;
3. if T and H are formulae, then (T ∧H), (T ∨H), (T→ H), etc. are formulae
4. if T is a formula and X a variable, then ∀x T(X) and ∃x T(X) are formulae.

1.5. literal

It is an atomic formula or the negation of an atomic formula.

Example 1

Let F be the predicate and xi and xj be the terms. Then F (xi, xj) is a literal and ¬F (xi, xj) is also a literal.

2. Bound Variables and Free Variables

In the formula "∀x A", formula A is called the ***field*** of the quantifier ∀. The variable x is called the ***variable quantified*** by the universal quantifier ∀. The positions occupied by the variable x in formula A are called ***occurrences*** of x. In the preceding formula F, the variable x has two occurrences.

An occurrence of a variable x in a formula is said to be ***bound*** if it has an occurrence in the field of a quantifier ∀ (or ∃) in that formula. If an occurrence of a variable is not bound, it is ***free.***

In other words, free variables are variables that ***are not quantified***. Conversely, variables are said to be bound when they ***are quantified***.

Exemple2

In $(P(x) \vee \forall x Q(x))$, x is free in P(x), but it is bound in Q(x).

Exemple3

1. The statement "Some students attend all classes" can be represented by: ∃ X (Student(X) ∧ (∀Y Attends (X, Y))).

The variables X and Y are bound.

2. The statement "No student attends an uninteresting class" can be represented by: ¬∃ X (Student(X) → (Attends (X, Y) ∧ ¬Interesting(Y))).

The variable Y is free.

Practice Activity 1

Translate the following statements into Predicate Logic:

1. All professors are intelligent.
2. There is one intelligent teacher.
3. If a professor teaches ML, he/she is intelligent.
4. Everyone likes everyone else.
5. Everyone likes ice cream.
6. Everyone likes someone.
7. Someone likes everyone.
8. Some students attend all the classes.
9. All lions are ferocious.

10. Some lions do not eat meat.

11. All monkeys are mischievous.

Solution

1. ∀ x Pr(x) →Int(x)
2. ∃x Pr(x)∧Int(x)
3. ∀ x Teach(x.Ml)→ Int(x)
4. ∀ x, y likke (x, y)
5. ∀ x likke (x, ice-cream)
6. ∀ x ∃y likke(x,y)
7. ∃x ∀ y likke(x,y)
8. ∃x Studd(x) ∀ y attendd(x,y)
9. ∀ x Lion(x) →ferocious(x)
10. ∃x Lion(x) ∧¬ meat(x)

Predicate Logic (Order 1): Part II: semantics

The goal of first-order logic semantics is to:

- Define the meaning of predicate symbols
- Define the meaning of function symbols
- Define the domain in which variables take their values

To Estimate the truth value (true or false) of formulas.

1. Concept of Structure

A structure is any quadruple S = (D, C, F, R) where:

D: is a non-empty set called the domain of discourse

C: a set (empty or not) of constants

F: a set (empty or not) of functions on D

R: a set (empty or not) of relations on D

Example 4

(D, C, F, R) such that

D=N

C= {0}

F= {+, *, succ}

R= {≤} is a structure

Example 5

(D, C, F, R) such that

D = Real numbers

C = {0, 1}

F = {+, -, *}

R = ∅ is a structure

Example 6

(D, C, F, R) such that

D=N

C= {0}

F={+, -, succ}

R= {≤}

Is not a structure because subtraction is not internal in N

Example 7

(D, C, F, R) such that

D=∅

C= {0}

F= {+, *}

R= {≤}

Is not a structure because the domain is empty

2. Model Theory

As in propositional calculus, we will attempt to construct a model that allows us to derive a semantic interpretation of our formulas. However, in predicate calculus, it is not possible to apply a truth table method directly derived from propositional calculus, due to the value domains of the variables in each predicate.

In fact, for an atomic formula p(x1….,xn), we will need a function (called an interpretation function) responsible for giving meaning to the symbol p, and therefore for calculating its truth value according to the values of x1…… xn.

2.1.interpretation

We assign a meaning (a truth value) to each of the formulas by interpreting the different symbols (functions, predicates), constants and variables.

Example 8

Let: x,y € N, ∀ x ∃y (x =<y): In this case, we can see if this formula is a true proposition or a false proposition.

2.1.1. Interpretation of terms

Let F be a formula and I an interpretation of this formula. We can extend the interpretation I to the terms of F:

- to each constant symbol, we associate its value according to I;
- to each variable, we associate the variable itself;
- to each term f(t1, ..., tn), we associate the term f'(t1', ..., t'n) where t'1, ..., t'n are the interpretations of t1, ..., tn and f' is the interpretation of f.

3. Validity and Consistency (satisfaction of a formula)

3.1. Definitions

- ***Valid formula:*** A formula F is said to be valid if and only if, for any interpretation I, we have I(F) = T.
- ***Invalid formula:*** An invalid formula is false in at least one interpretation.
- ***Satisfiable formula:*** A formula A will be said to be satisfiable, or semantically consistent, if there exists an interpretation I such that I(A) = T. The interpretation is then a model of A.

- ***Unsatisfactory formula:*** An unsatisfactory formula, or semantically inconsistent formula, or antilogy, is a formula that is false in any interpretation.

- ***Contingent formula:*** A contingent formula is true in some interpretations and false in others.

- ***Logical consequence:*** Let B be a formula and Ai a family of n formulas; We say that B is a consequence of Ai if for any interpretation I such that ∀Ai; I(Ai) = T, we also have I(B) = T. We then write A1; A2; ....An |= B.

4. Equivalence of Well-formed Formulae

4.1. Equivalent formulae

Two formulae are equivalent when they have the same value in all interpretations (notation: A $\equiv$ B). Let A(x) and B(x) be two well-formed atomic formulae. Equivalent formulae in propositional logic remain equivalent in predicate logic.

- $\forall x A(x) \wedge \forall x B(x) \equiv \forall x (A(x) \wedge B(x))$.
- $\exists x A(x) \vee \exists x B(x) \equiv \exists (A(x) \vee B(x))$.
- $\neg(\forall x A(x)) \equiv \exists \neg A(x)$.
- $\neg(\exists x A(x)) \equiv \forall x \neg A(x)$.

Remark

- $\forall x A(x) \vee \forall x B(x) \not\equiv \forall x (A(x) \vee B(x))$.
- $\exists x A(x) \wedge \exists x B(x) \not\equiv \exists x (A(x) \wedge B(x))$.

4.2.Prenex form

It consists of accumulating quantifiers at the beginning of the formula. The usefulness of this form is that it highlights certain logical relationships that are not readily apparent in the usual forms of formulae.

- A matrix is a formula in predicate calculus that contains no quantifiers.

- Every formula has an equivalent prenex form.

A prenex form is a formula for predicate calculus of the form

Q1x1.........Qn xn M, where Q denotes ,∀ or ∃ and M is a matrix.

Theorem

Any formula for predicate calculus can be transformed into an equivalent formula in conjunctive prenex normal form.

A formula is said to be in prenex conjunctive normal form if it is of the form:

$$(Qv_1)(Qv_2)...(Qv_n)[A_{11} \vee A_{12}... \vee A_{1n}] \vee ... \vee [A_{m1} \wedge A_{m2} \vee ... \vee A_{mq}]$$

Where Q is either quantifier, ∀ or quantifier ∃, the vi are distinct variables that have a free occurrence in the Aij. Each Aij is an atomic formula or a negation of an atomic formula.

Construction algorithm

To construct the prenex form, five steps are required:

1. Remove equivalence and implication connectives.

2. Rename certain bound variables so that no variable is quantified twice.

3. Remove unnecessary quantifiers (those whose quantified variable does not appear in their scope), if any.

4. Move all occurrences of negation before the atoms using the rewriting rules seen for propositional logic and the following additional rules:

- $\neg\forall x A \equiv \exists x \neg A$;
- $\neg\exists x A \equiv \forall x \neg A$.

5. Move the quantifiers to the front using the following rewriting rules (and using associativity, commutativity, or variable renaming if necessary):

$$(\forall x A \wedge B) = \forall x(A \wedge B)$$
$$(\exists x A \wedge B) = \exists x(A \wedge B)$$
$$(\forall x A \vee B) = \forall x(A \vee B)$$
$$(\exists x A \vee B) = \exists x(A \vee B)$$

if B does not contain x

Example 8

Transform the following formula into prefix form

$$\forall x p(x) \wedge \exists y q(y) \rightarrow \exists y(p(y) \wedge q(y))$$

Solution

1. $\neg(\forall x p(x) \wedge \exists y q(y)) \vee \exists y(p(y) \wedge q(y))$
2. $\neg(\forall x p(x) \wedge \exists y q(y)) \vee \exists z(p(z) \wedge q(z))$
3. $(\exists x \neg p(x) \vee \forall y \neg q(y)) \vee \exists z(p(z) \wedge q(z))$
4. $\exists x \forall y \exists z(\neg p(x) \vee \neg q(y) \vee (p(z) \wedge q(z)))$

Example 9

Transform the following formula into a conjunctive prenexe form

$$\forall x[(\forall y p(x) \vee \forall z q(z; y)) \rightarrow \neg \forall y r(x; y)]$$

Solution

1. $(\forall x)[(p(x) \wedge (\forall z)q(z;y)) \rightarrow \neg(\forall y)r(x;y)]$;
2. $\forall x[(p(x) \wedge (\forall z)q(z;y)) \rightarrow \neg\forall y_1 r(x;y_1)]$;
3. $\forall x[\neg(p(x) \wedge \exists z \neg q(z;y)) \vee \exists y_1 \neg r(x;y_1)]$;
4. $\forall x \exists z \exists y_1[\neg(p(x) \wedge \neg q(z;y)) \vee \neg r(x;y_1)]$;
5. $\forall x \exists z \exists y_1[\neg(p(x) \vee \neg r(x;y_1)) \wedge (\neg q(z;y) \vee \neg r(x;y_1)]$.

4.3. Clausal Form

A literal is defined as an atomic formula or the negation of an atomic formula. A clause is a disjunction of several literals. A predicate calculus formula is said to be in clausal form if it is in the form:

$\forall x_1 \ldots \forall x_n (C_1 \vee C_2 \vee \ldots . C_n)$

Construction Algorithm

To construct the clausal form, you must:

1. Take the existential closure of D (i.e., quantify the free variables of D using the quantifier ∃),
2. Eliminate all redundant quantifiers in D,
3. Rename each quantified variable in D that appears more than once,
4. Eliminate all occurrences of connectives other than: ¬, ∧ and ∨
5. Right-shift all connectives by replacing:
   - ¬(∀xA) with ∃x(¬A)
   - ¬(∃xA) with ∀x(¬A)
   - ¬(A∨B) with ¬A∧¬B
   - ¬(A∧B) with ¬A∨¬B
   - ¬(¬A) with A

5. Conclusion

In this chapter, we presented predicate logic, which we consider richer than propositional logic due to its inclusion of universal quantifiers alongside propositional variables. In the following appendices, we offer a series of solved exercises, final exams, and re-sit exams proposed in academic years 2024-2025,2023-2024,2022-2023 in Annex A, Annex B, and Annex C respectively.

Annex A

Tutorial n° = 01(chapter 01)

Exercise 1

Let (X), (Y), and (Z) be three propositions. Give the negation of:

a) $(X) \wedge (\neg (Y) \vee (Z))$

b) $((X) \wedge (Y)) \Rightarrow (Z)$

Exercise 2

Let P P: "It is raining."

Negate the statements:

a) P

b) $(P \wedge Q)$ where Q: "I have an umbrella."

Exercise 3

Among the following assertions, which are true, which are false, and why?

1. $(2 < 3) \wedge (2 \text{ divides } 4)$
2. $(2 < 3) \wedge (2 \text{ divides } 5)$
3. $(2 < 3) \vee (2 \text{ divides } 5)$
4. $(2 < 3) \wedge \neg(2 \text{ divides } 5)$
5. $\neg(2 < 3) \vee (2 \text{ divides } 5)$

Exercise 4

Evaluate the truth value of the expression $(P \Rightarrow Q) \wedge (Q \Rightarrow R)$ for the following values:

P=T, Q=T, R=F

Exercise 5

Let the propositions (P) "I have my driver's license" and (Q) "I am over 18 years old."

Are the propositions $(P) \Rightarrow (Q)$ and $(Q) \Rightarrow (P)$ true?

What can we conclude?

Exercise 6

1. Show that the formula $(\alpha \wedge \beta) \Rightarrow \gamma$ is logically equivalent to the formula $\alpha \Rightarrow (\beta \Rightarrow \gamma)$, where α, β, and γ are any propositional variables.
2. Consider the formula $E = ((A \wedge B \wedge C) \Rightarrow (A \leftrightarrow (\neg B \vee C)))$, in which A, B, and C are propositional variables.
   Determine a formula logically equivalent to E, written without any other connector symbols than $\Rightarrow$ and $\leftrightarrow$.

Exercise 7

Construct the truth tables for the following formulae:

1. $\neg p \Rightarrow p \vee q$
2. $\neg p \vee \neg q$
3. $(P \vee Q) \wedge \neg R$

Exercise 8

If P: "I will study," Q: "I will pass," and R: "I will celebrate," write the compound statement and determine its truth value for P=T, Q=F, R=T

Statement: "If I study, then I will pass, and if I pass, then I will celebrate."

Solutions

Exercise 1

a) $(X) \wedge (\neg (Y) \vee (Z))$

$\neg ((X) \wedge(\neg(Y) \vee (Z))) \equiv (\neg(X) \vee \neg(\neg(Y)\vee (Z))$

$\equiv (\neg(X) \vee ((Y) \wedge \neg(Z))$

$\equiv (\neg(X) \vee (Y)) \wedge (\neg(X) \vee \neg(Z))$

$\equiv (\neg(X) \vee (Y)) \wedge \neg((X) \wedge (Z))$

b)

$\neg (((X) \wedge (Y)) \Rightarrow (Z)) \equiv ((X) \wedge (Y)) \wedge \neg(Z) \equiv (X) \wedge (Y) \wedge \neg(Z)$

Exercise 2

1. Negation of P :
    - $\neg P$: "It is not raining."
2. Negation of $(P\wedge Q)$:
    - $\neg(P\wedge Q)$: "It is not the case that it is raining and I have an umbrella."
    - By De Morgan's Laws :
    - $\neg P\vee\neg Q$: "It is not raining or I do not have an umbrella."

Exercise 3

1. $(2 < 3) \wedge$ (2 divides 4) : $T\wedge T=T$
2. $(2 < 3) \wedge$ (2 divides 5): $T\wedge F=F$
3. $(2 < 3) \vee$ (2 divides 5) : $T\vee F=T$
4. $(2 < 3) \wedge \neg$ (2 divides 5) : $T\wedge T=T$
5. $\neg (2 < 3) \vee$ (2 divides 5): $F\vee F=F$

Exercise 4

1. $P\Rightarrow Q$:
    - $T\Rightarrow T=T$
2. $Q\Rightarrow R$:
    - $T\Rightarrow F=F$
3. Combine :

- (T)∧(F)=F

Exercise 5

Implications

1. **P⇒Q** (If I have my driver's license, then I am over 18 years old)
    - This implication is generally **true** because, in many places, having a driver's license typically requires the individual to be at least 18 years old.
2. **Q⇒P** (If I am over 18 years old, then I have my driver's license)
    - This implication is **not necessarily true**. Just because someone is over 18 does not mean they have a driver's license; they may choose not to obtain one or may not have passed the driving test.

We conclude that the **(⇒) operator is not commutative.**

Exercise 6

Ex. **1** :

1) $(\alpha \wedge \beta) \rightarrow \gamma \equiv \neg(\alpha \wedge \beta) \vee \gamma \equiv \neg\alpha \vee \neg\beta \vee \gamma \equiv \neg\alpha \vee (\beta \rightarrow \gamma) \equiv \alpha \rightarrow (\beta \rightarrow \gamma)$

2) $E = ((A \wedge B \wedge C) \rightarrow (A \leftrightarrow (\neg B \vee C))$

$E \equiv ((A \wedge B \wedge C) \rightarrow (A \leftrightarrow (B \rightarrow C))$

$E \equiv ((A \wedge B) \rightarrow (C \rightarrow (A \leftrightarrow (B \rightarrow C))) \equiv (A \rightarrow (B \rightarrow (C \rightarrow (A \leftrightarrow (B \rightarrow C))))$ d'après 1).

Exercise 8

The compound statement is:

(P⇒Q) ∧(Q⇒R)

1. Evaluate:
    - P⇒Q
        - T⇒F=F
    - Q⇒R
        - F⇒T=T
2. Combine :
    - (F)∧(T)=F

Tutorial No = 02 part 01 (propositional logic: syntax)

Exercise 01

Transform the following sentences in propositional logic

1. If it rains, then the picnic will be canceled.
2. The cake is delicious and the coffee is hot.
3. Either the market is open or the market is closed
4. If I study hard, I will pass the exam
5. The weather is nice, but I have to work
6. If it is my birthday, then I will have a party unless it rains
7. I will go for a walk if and only if the weather is nice
8. Not all birds can fly
9. It is either raining or snowing

Exercise 02

The same question

Let p be the proposition “X estimates Y” and q be the proposition “Y estimates X”

1. **Sentence**: "X estimates Y but Y does not return his esteem."
2. **Sentence**: "X and Y esteem each other."
3. **Sentence**: "X and Y hate each other."
4. **Sentence**: "Y is esteemed by X but X is hated by Y."
5. **Sentence**: "X and Y do not hate each other."

Exercise 3

By associating the elementary statements: 'Paul is a student,' 'Quentin is a student,' and 'Rene is a student' with the propositions p, q, and r, respectively; associate each of the following sentences with the propositional formula that seems to correspond semantically:

1. Paul and Quentin are students.
2. Paul or Quentin is a student.
3. Exactly one of Paul and Quentin is a student.
4. Neither Paul nor Rene are students.
5. At least one of the three is not a student.
6. Only one among the three is not a student.
7. Exactly two among the three are students.
8. If Paul is a student, then Quentin is a student.
9. If Paul is a student, then Quentin is a student; otherwise, Quentin is not a student.
10. Paul is a student if and only if Rene is.
11. That Rene is a student is a necessary condition for Paul to be one.
12. That Rene is a student is a sufficient condition for Paul to be one.
13. That Rene is a student is a necessary and sufficient condition for Paul to be one.
14. Paul is a student only if exactly one of the other two is.
15. If Paul is a student, then at least one of the other two is not."

Exercise 4

Let R and S be two propositional variables meaning respectively "it is snowy" and "it is raining". Write a simple sentence corresponding to each of the following formulae:
1. ¬R

2. R ∧ S

3. R ∨ S

4. R ∨ ¬S

5. ¬R ∧ ¬S

6. ¬¬S

Exercise 5

Let p and q be two propositional variables such that p represents the proposition "the child knows reading" and q represents the proposition "the child knows writing". Translate the following formulae into natural language sentences:

1. $(p \land q)$
2. $(p \land \neg q)$
3. $(q \rightarrow p)$
4. $(\neg p \lor \neg q)$
5. $(p \rightarrow \neg q)$
6. $\neg \neg (p \land q)$

Exercise 6

Three tourists each make a statement:

- **1st tourist**: "We visited the Bardo Museum and the Essay Garden but not the Museum of Fine Arts."
- **2nd tourist**: "We visited the Museum of Fine Arts and the Essay Garden but not the Bardo."
- **3rd tourist**: "We visited the Bardo and the Museum of Fine Arts but not the Essay Garden "

Knowing that each tourist lies once and only once in their statement, what did they actually visit?

Exercise 7 (Home work)

After baking a cake for her four children, the mother left it to cool on the kitchen table and then went out to run an errand. When she returned, she noticed that a quarter of the cake had been eaten. Since no one else was home that day except the four children, the mother asked each of them who had eaten the cake. The four "suspects" said the following:

Chabane: Katia ate a quarter of the cake;

Saliha: I didn't eat a quarter of the cake;

Katia: Djamal ate a quarter of the cake;

Djamal: Katia lied when she said I ate a quarter of the cake.

If only one of these four statements is true and only one of the four children is guilty,

Question: Which of the four actually ate a quarter of the cake?

Exercise 08

On an island live three types of inhabitants: the Pure (who always tell the truth), the Worse (who always lie), and the Versatile (who sometimes tell the truth and other times lie, according to their whim).

Every inhabitant of the island is either a Pure, a Worse, or a Versatile. A Worse can only marry a Pure, a Pure can only marry a Worse, and a Versatile can only marry a Versatile.

A couple, Mr. and Mrs. A, make the following statements:

Mr. A: "My wife is not Versatile";

Mrs. A: "My husband is not Versatile."

What are Mr. and Mrs. A?

Exercise 09

Consider the following basic information:

p: He needs a doctor, q: He needs a paramedic, r: He had an accident, s: He is vomiting, and u: He is injured.

Translate the following formulas into natural language sentences:

(1) $((s \longrightarrow p) \wedge (r \longrightarrow q))$

(2) $(p \longrightarrow (s \vee u))$

(3) $((p \wedge q) \longrightarrow r)$

(4) $((p \wedge q) \leftrightarrow (s \wedge u))$

(5) $(\neg (s \vee u) \longrightarrow \neg p)$

Exercise 10

Among the following words, determine which are well-formed, fully parenthesized formulas of propositional calculus (strictly applying the definition seen in the course).

1- $()$
2- (p)
3- $\neg p$
4- $\neg(p)$
5- $(\neg p)$
6- $\neg pq\wedge r$
7- $p\wedge q$
8- $\vee pq$
9- $(p\wedge q)$

10- $\neg p \neg q$,

11- $\neg(\neg p)$

12- $\neg \neg \neg p$

13- $(\neg p \leftrightarrow q)$

14- $\neg (p \wedge q)$

15- $(p \vee q) \rightarrow (p \leftrightarrow q)$

16- $(p \rightarrow p)$

17- $p \vee q \wedge r$

18- $(p \rightarrow (q \wedge r))$

19- $(\neg (\neg \neg p \vee q) \rightarrow \neg r)$

20- $(((p \wedge (q \rightarrow r)) \vee (\neg p \rightarrow q)) \wedge (q \vee \neg r))$

Solutions

Exercise 01

1. R→¬ C
2. D∧H
3. O∨C
4. S→P
5. N∧W
6. B→(P∨R)
7. W↔N
8. ¬F
9. R∨S

Exercise 02

1. P∧ ¬ q
2. P∧ q
3. ¬ p ∧ ¬ q
4. P ∧ ¬ q
5. ¬ ¬p∧¬ ¬ q

Exercise 03

1. p ∧ q
2. p ∨ q
3. (p ∧ ¬q) ∨ (¬p ∧ q)
4. ¬p ∧ ¬r
5. (¬p ∨ ¬q ∨ ¬r)
6. (¬p ∧ q ∧ r) ∨ (p ∧ ¬q ∧ r) ∨ (p ∧ q ∧ ¬r)
7. ≡ 6
8. p → q
9. (p → q) ∧ (¬p → ¬q)
10. p → r

11. p → r
12. r → p
13. p ↔ r
14. p → ((q ∧ ¬r) ∨ (¬q ∧ r))
15. p → (¬q ∨ ¬r)

Exercise 04

R: “it is snowy”; S: “it is raining”

1. ¬R: It is not snowy.
2. R ∧ S: it is snowy and raining
3. R ∨ S: it is snowy or raining
4. R ∨ ¬S: it is snowy or it is not raining
5. ¬R ∧ ¬S: it is not snowy and it is not raining
6. ¬¬S: It is not true that it is not raining (or alternatively, you can say 'it is raining')."

Exercise 05

7. (p∧q): The child knows reading and the child knows writing.
8. (p∧¬q): The child knows reading but does not know writing
9. (q⟶p): If the child knows writing, then the child knows reading
10. (¬p∨¬q): The child does not know reading or the child does not know writing.
11. (p⟶¬q): If the child knows reading, then the child does not know writing
12. ¬ ¬(p∧q): "It is not the case that the child does not know reading and writing." (Alternatively: "The child knows reading and writing .")

Exercise 06

"Let B, J, and A be three propositional variables symbolizing:

- B: visit to the Bardo Museum
- J: visit to the Essay Garden
- A: visit to the Museum of Fine Arts

1st tourist: B ⋀ J ⋀ ¬ A

2nd tourist: A ⋀J ⋀ ¬ B

3rd tourist: B ⋀ A ⋀ ¬ J

1:

B ⋀ A ⋀ J, B ⋀ ¬ J ⋀ ¬ A, ¬ B ⋀ J ⋀¬ A

2:

B ⋀A ⋀ J, A ⋀¬ J ⋀ ¬ B, ¬ A ⋀ J ⋀¬ B

3:

B ⋀A ⋀J, B ⋀ ¬ A ⋀ ¬ J, ¬ B ⋀ A ⋀ ¬ J

The common proposition among the three tourists is B ⋀ A and⋀ J, so they visited the Bardo Museum, the Essay Garden, and the Museum of Fine Arts.

Thus, for verification:

- Tourist 1 lied about A
- Tourist 2 lied about B
- Tourist 3 lied about J

Exercise 7

Let K, S, and D be the propositional variables denoting:

K: Katia ate a quarter of the cake

S: Saliha ate a quarter of the cake

D: Djamel ate a quarter of the cake

The statements of the four children are:

Chaabane: K

Saliha: ¬S

Katia: D

Djamel:¬ D

Given that only one of these four propositions is true, the different truth cases are as follows:

Chabane:	K	¬K	¬K	¬K
Saliha:	S	¬S	S	S
Katia:	¬D	¬D	D	¬D
Djamel:	D	D	D	¬D

The first three combinations lead to contradictions, the fourth does not, so it will be chosen: Saliha gets to eat a quarter of the cake

Exercise 8

Let's examine the different possible scenarios: Mr. A can be either Pure, Worse, or Versatile.

- If Mr. A is Pure: then his wife is Worse, so what she says is false; consequently, her husband is Versatile, which creates a contradiction.
- If Mr. A is Worse: then his wife is Pure; what Mr. A says is false, therefore Mrs. A is Versatile: this also creates a contradiction.
- If Mr. A is Versatile: then his wife is also Versatile; here, both statements by Mr. and Mrs. A are false, which is possible because of their Versatile nature: no contradiction.

In conclusion, Mr. and Mrs. A are both Versatile.

Exercise 09

(1) $((s \longrightarrow p) \land (r \longrightarrow q))$

Translation: If he is vomiting, then he needs a doctor, and if he has had an accident, then he needs a paramedic.

(2) $(p \longrightarrow (s \lor u))$

Translation: If he needs a doctor, then he is either vomiting or injured.

(3) $((p \land q) \longrightarrow r)$

Translation: If he needs a doctor and a paramedic, then he had an accident.

(4) $((p \land q) \leftrightarrow (s \land u))$

Translation: He needs a doctor and a paramedic if and only if he is vomiting and injured.

(5) $(\neg (s \lor u) \longrightarrow \neg p)$

Translation: If he is neither vomiting nor injured, then he does not need a doctor.

Exercise 10

1. () - Not well-formed (missing content).
2. (p) - Well-formed.

3. ¬ p - Not well-formed (missing parentheses).
4. ¬(p) - Well-formed.
5. (¬p) - Well-formed.
6. ¬pq∧ r - Not well-formed (missing parentheses).
7. p∧ q - Not well-formed (missing parentheses).
8. ∨pq - Not well-formed (missing parentheses and operator).
9. (p∧ q) - Well-formed.
10. ¬p¬q - Not well-formed (missing parentheses).
11. ¬(¬p) - Well-formed.
12. ¬ ¬ ¬ p - Not well-formed (missing parentheses and has too many operators).
13. (¬p↔q) - Well-formed.
14. ¬ (p∧ q) - Well-formed.
15. (p∨q) →(p↔q) - Well-formed.
16. (p→p) - Well-formed.
17. p∨q∧ r - Not well-formed (missing parentheses).
18. (p→ (q∧ r)) - Well-formed.
19. **(¬ (¬ ¬p∨q) → ¬r)** - Well-formed.
20. **((((p∧ (q→r)) ∨(¬p→q)) ∧(q∨¬r)))** - Well-formed.

Tutorial No = 02 part 02 (propositional logic : semantics)

Exercise 01

Let the formula P defined as: (p → (q → r)) → (r ∨ ¬p)

1. Give the truth table for the formula P.
2. Indicate whether the formula is valid, satisfiable, or unsatisfiable.
3. Does the formula P have a model? If so, which one?
4. Give the CNF (Conjunctive Normal Form) and DNF (Disjunctive Normal Form) of the formula P (from the truth table)

Exercise 02

Give the truth tables for the following formulas and say whether they are valid, satisfiable, or unsatisfiable.

a. (¬P ∧ ¬Q) → (¬P ∨ R)

b. P ∧ (Q → P) → P

c. (P ∨ Q) ∧ (¬P ∧ ¬Q)

d. (P → Q) ∧ (Q ∨ R) ∧ P

e. ((P ∨ Q) → R) ↔ P

f. (P ∧ R) ∨ (Q ∧ ¬ R)

g. (¬P∧ R) ∨ (¬ Q∧ ¬ R)

Exercise 03

1. Demonstrate that the following formulae (called paradoxes of material implication) are tautologies:

a) P ⇒ (Q ⇒ P)

b) ¬P ⇒ (P ⇒ Q).

2. Do you give a translation in natural language?

Exercise 04

1) Complete the following truth table.

a	b	c	(¬ a ∧ ¬ b) ∨c	(a →b) ∧ (b→c)	(a∨b→c) ⇔(a→b)∧(b→c)

Specify whether there is a tautology in the previous table. Justify your answer.

2) Specify if there is an unsatisfiable formula. Justify your answer.

3) Provide a model for (a→b)∧(b→c). Justify your answer.

4) Give all formulas equivalent to a∨b→c.

Exercise 05

Verify using the truth tables, if

a) $p \Leftrightarrow q \models p \rightarrow q$

b) $p \Leftrightarrow \neg q \models p \rightarrow q$

c) $vrai \models r \rightarrow (s \rightarrow (t \wedge s \rightarrow r))$

d) $\{q \rightarrow (r \wedge s), \neg r \vee \neg s\} \models \neg q$

Exercise 06

Consider the following reasoning:

When it is sunny, I wear my glasses or I do not go out.

I only stay at home when I do not wear glasses and it is gray.

So if I do not wear my glasses, it is gray.

1. Formalize this reasoning using the following variables:

   S: it is sunny, L: wear my glasses, M: I stay at home.

2. Show by two methods that the reasoning above is correct (valid).

Exercise 07

Let f be a logical function with 4 logical variables, such that f=1 if and only if the number of variables of f that are '1' is not less than 2.

1. Establish the truth table for f.
2. Give the (CNF) and (DNF) of f.

Exercise 08

F: $\neg ((\neg \neg (\neg p \wedge q) \longrightarrow (p \leftrightarrow \neg r)) \vee \neg \neg s)$

Give the truth value of formula F in the following two cases:

V1: v1(p)=1, v1(q)=1, v1(r)= 0, v1(s)=1

V2: v2(p)=0, v2(q)= 0, v2(r)=0, v2(s)=1

Exercise 09

Brown, Smith, and Jones are accused of a crime. Their testimonies are as follows:

Brown: "Jones is guilty and Smith is innocent."

Jones: "If Brown is guilty, then so is Smith."

Smith: "I am innocent, but at least one of the other two is guilty."

1. After clearly explaining the choice of propositional variables, express each of their testimonies in propositional form.

2. Construct a truth table for these formulas.

3. Are the testimonies of the three suspects consistent?

4. The testimony of one of the suspects is inferred from that of another suspect. Which one?

5. The testimony of one of the suspects is inferred from that of the other two suspects. Which ones?

6. Assuming that all are innocent, who gave false testimony?

7. If all are telling the truth, which one is guilty, and which one is innocent?

Solutions

Exercise 01

p	Q	r	q→r	p→(q→r)	r∨¬p	(p→(q→r)) →(r∨¬p)
F	F	F	V	V	V	V
F	F	V	V	V	V	V
F	V	F	F	V	V	V
F	V	V	V	V	V	V
V	F	F	V	V	F	F
V	F	V	V	V	V	V
V	V	F	F	F	F	V
V	V	V	V	V	V	V

2. Not valid, satisfiable, not unsatisfiable.
3. Yes, the formula has a lot of models: P=F, Q=F, R=F
4. CNF and DNF from the truth table

Exercise 03:

p	¬ p	q	q⇒p	P ⇒q	P ⇒ (q ⇒p)	¬ p⇒ (p⇒q)
F	V	F	V	V	V	V
F	V	V	F	v	V	V
V	F	F	V	F	V	V
V	F	V	V	V	V	V

a) If we have P, any proposition implies P.
b) If P is false, P implies any proposition.

Exercise 02:

a. Valid
b. Valid
c. Unsatisfiable
d. Satisfiable
e. Satisfiable

f. –

g. -

Exercise 07

f=1 if and only if the number of variables of f that are '1' is ≥ 2. Hence, the truth table for f:

x	y	z	t	f
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	1
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	1
1	0	1	1	1
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

DNF :

$(\neg x \wedge \neg y \wedge z \wedge t) \vee (\neg x \wedge y \wedge \neg z \wedge t) \vee (\neg x \wedge y \wedge z \wedge \neg t) \vee (\neg x \wedge y \wedge z \wedge t) \vee (x \wedge \neg y \wedge \neg z \wedge t) \vee (x \wedge \neg y \wedge z \wedge \neg t) \vee (x \wedge \neg y \wedge z \wedge t) \vee (x \wedge y \wedge \neg z \wedge \neg t) \vee (x \wedge y \wedge \neg z \wedge t) \vee (x \wedge y \wedge z \wedge \neg t) \vee (x \wedge y \wedge z \wedge t)$

CNF :

$(x \vee y \vee z \vee t) \wedge (x \vee y \vee z \vee \neg t) \wedge (x \vee y \vee \neg z \vee t) \wedge (x \vee \neg y \vee z \vee t) \wedge (\neg x \vee y \vee z \vee t)$

Exercise 05

a. a logic consequence
b. not logic consequence
c. a logic consequence

Exercise 06

$((s \to (l \lor m)) \land (m \to (\neg l \land \neg s))) \to (\neg l \to \neg s)$

1. the reasoning above is correct (valid) by using the truth table you find it tautology
2. second method
3. the formula can be written like

   I= s→(l∨m)

   II=m→(¬ l ∧ ¬ s)

   III= (¬l→¬s) so we have:

   (I∧II) →III

<u>Exercise 08</u>

Valuation V1:

- v1(p)=1
- v1(q)=1
- v1(r)=0
- v1(s)=1

We have :

¬p=¬1=0

¬p∧q=0∧1=0

¬¬(¬p∧q)=¬ ¬0=0

¬r=¬0=1

p↔¬r=1↔1=1

¬¬(¬p∧q) →(p↔¬r) =0→1=1

¬¬s=¬¬1=1

(¬¬(¬p∧q) →(p↔¬r)) ∨¬¬s=1∨1=1

Calculate F

F=¬1=0

Valuation V2:

- v2(p)=0
- v2(q)=0
- v2(r)=0
- v2(s)=1

We have:

$\neg p = \neg 0 = 1$

$\neg p \wedge q = 1 \wedge 0 = 0$

$\neg\neg(\neg p \wedge q) = \neg\neg 0 = 0$

$\neg r = \neg 0 = 1$

$p \leftrightarrow \neg r = 0 \leftrightarrow 1 = 0$

$\neg\neg(\neg p \wedge q) \rightarrow (p \leftrightarrow \neg r) = 0 \rightarrow 0 = 1$

$\neg\neg s = \neg\neg 1 = 1$

$(\neg\neg(\neg p \wedge q) \rightarrow (p \leftrightarrow \neg r)) \vee \neg\neg s = 1 \vee 1 = 1$

Calculate F

$F = \neg 1 = 0$

Summary of Truth Values

- For V1 : F=0
- For V2 : F=0

Thus, the truth value of the formula F is **0** for both valuations V1 and V2.

Exercise 09

1. Propositional Variables

Let :

- B : Brown is guilty.
- S : Smith is guilty.
- J : Jones is guilty.

Testimonies in Propositional Form

- **Brown's Testimony**: "Jones is guilty and Smith is innocent."
  This can be expressed as:
  $J \land \neg S$
- **Jones' Testimony**: "If Brown is guilty, then so is Smith."
  This can be expressed as:
  $B \rightarrow S$
- **Smith's Testimony**: "I am innocent, but at least one of the other two is guilty."
  This can be expressed as:
  $\neg S \land (B \lor J)$

2. Truth Table Construction

Now we construct a truth table for B, S, and J and their combinations:

B	S	J	TB=(J∧¬S)	TJ=(B ⟹ S)	TS=(¬S∧(B∨J))
T	T	T	F	T	F
T	T	F	F	T	F
T	F	T	T	F	T
T	F	F	F	F	F
F	T	T	F	T	F
F	T	F	F	T	F
F	F	T	T	T	T
F	F	F	F	T	F

3. Are the Testimonies Consistent?

To check for consistency, we look for rows in the truth table where all testimonies are true together.

- For B=F, S=F, J=T (Row 7):
  - TB=T
  - TJ=T
  - TS=T

Thus, the testimonies are consistent.

4. Inference from One Suspect to Another

From Jones' testimony ($B \Longrightarrow S$) and Brown's ($J \land \neg S$), we see that Brown's guilt can be inferred if we assume that he is telling the truth, which means:

- If Brown is guilty, then Smith must also be guilty.

Therefore, **Jones' testimony can be inferred from Brown's testimony**.

5. Inference from Two Suspects to One

Smith's testimony ($\neg S \land (B \lor J)$) shows that he asserts he is innocent but one of the other two must be guilty. This implicates:

- Smith's testimony is inferred from the conditions posed by Brown and Jones.

Thus, **Smith's testimony can be inferred from Brown's and Jones' testimonies**.

6. Assuming All Are Innocent, Who Gave False Testimony?

If B=F, S=F, J=F, we analyze the testimonies:

- **Brown**: $TB=(J \land \neg S)$ =F (False)
- **Jones**: $TJ= (B \Longrightarrow S)$ =T (True)
- **Smith**: $TS=(\neg S \land (B \lor J))$ =F (False)

Thus, if all are innocent, **Brown and Smith gave false testimonies**.

7. If All Are Telling the Truth, Who Is Guilty or Innocent?

Assuming all are telling the truth leads to contradictions in their statements since:

- Brown's statement implies at least Jones is guilty (as Smith must be innocent).
- Jones's conditional statement forces Smith to be guilty.
- Smith asserts that at least one of the other two must be guilty.

Hence, under the assumption that all are telling the truth, the only conclusion is that it leads to inconsistencies, suggesting no single configuration allows for everyone to be truthful without contradiction.

In summary:

- There are inconsistencies if everyone is honest, leading to the conclusion that not all suspects can be innocent if the testimonies contradict each other.

Tutorial No = 02 part 02 (propositional logic : semantics)

Conjunctive and disjunctive normal forms with conversion algorithms

Exercise 01

Given the following logical expressions, determine if they are in CNF, DNF

1. (A∨B) ∧(C∨D)
2. A∧(B∨C) ∧(D)
3. (A∧B) ∨(C∧D)
4. (A∨B) ∧(C)
5. A∨(B∧C)

Exercice 02

Using the conversion algorithm,find the DNF

a. (A ∨ B ∨ C) ∧ (C ∨ ¬A)

b. (A ∨ B) ∧ (C ∨ D)

c. ¬ ((A ∨ B) → C)

d. A∧(B∨C)

e. (A∨B) ∧C

f. ¬A∨(B∧C)

Exercise 03

Using the conversion algorithm, find the CNF

a. (A ∨ B) → (C ∧ D)
b. (A ∨ (¬B ∧ (C ∨ (¬D ∧ E))))
c. A ↔ (B ∧ ¬C)
d. A∨(B∧C)

e. $(A \wedge B) \vee C$
f. $A \rightarrow (B \vee C)$
g. $\neg (A \vee B)$

Exercice 04

1. prove that the following formula is valid:

$F = (a \wedge \neg b) \vee (\neg a \wedge \neg (b \vee c)) \vee (\neg c \wedge b) \vee (b \wedge c \wedge a) \vee (c \wedge \neg a)$.

2. Deduce (without proof) what can be said about the validity of the following formula A:

$A = (a \rightarrow b) \wedge (\neg a \rightarrow (b \vee c)) \wedge (\neg c \rightarrow \neg b) \wedge ((b \wedge c) \rightarrow \neg a) \wedge (c \rightarrow a)$.

Solutions

Exercise 01

CNF, DNF

1. (A∨B) ∧(C∨D) CNF
2. A∧(B∨C) ∧(D) CNF
3. (A∧B) ∨(C∧D) DNF
4. (A∨B) ∧(C) CNF
5. A∨(B∧C) DNF

Exercise 02

a) (A ∨ B ∨ C) ∧ (C ∨ ¬A)

≡ (A ∧ (C ∨ ¬A)) ∨ (B ∧ (C ∨ ¬A)) ∨ (C ∧ (C ∨ ¬A) distributivity.

≡ ((A ∧ C)∨(A ∧ ¬A))∨((B ∧ C)∨(B ∧ ¬A))∨((C ∧ C)∨(C ∧ ¬A)) distributivity.

≡ (A ∧ C) ∨ ((B ∧ C) ∨ (B ∧ ¬A)) ∨ (C ∨ (C ∧ ¬A)) we delete (A ∧ ¬A) = False

≡ (A ∧ C) ∨ ((B ∧ C) ∨ (B ∧ ¬A)) ∨ C we a remplace (C ∨ (C ∧ ¬A)) with C

Absorption law.

≡ (A ∧ C) ∨ (B ∧ C) ∨ (B ∧ ¬A) ∨ C

b)

b) (A ∨ B) ∧ (C ∨ D).

≡ (A ∧ (C ∨ D)) ∨ (B ∧ (C ∨ D)) distributivity.

≡ ((A ∧ C) ∨ (A ∧ D)) ∨ ((B ∧ C) ∨ (B ∧ D)) distributivity.

≡ (A ∧ C) ∨ (A ∧ D) ∨ (B ∧ C) ∨ (B ∧ D)

c) ¬ ((A ∨ B) → C)

≡ ¬ (¬ (A ∨ B) ∨ C) transformation of implication

≡ (A ∨ B) ∧ ¬C

≡ ¬C ∧ (A ∨ B)

≡ (¬C ∧ A) ∨ (¬C ∧ B) distributivity.

Exercise 03

a) $(A \vee B) \rightarrow (C \wedge D)$.

$\equiv \neg (A \vee B) \vee (C \wedge D)$ *transformation of implication in disjunction.*

$\equiv (\neg A \wedge \neg B) \vee (C \wedge D)$ *Morgan law.*

$\equiv (\neg A \vee (C \wedge D)) \wedge (\neg B \vee (C \wedge D))$ *distributivity.*

$\equiv ((\neg A \vee C) \wedge (\neg A \vee D)) \wedge ((\neg B \vee C) \wedge (\neg B \vee D))$ *distributivity.*

$\equiv (\neg A \vee C) \wedge (\neg A \vee D) \wedge (\neg B \vee C) \wedge (\neg B \vee D)$

b) $(A \vee (\neg B \wedge (C \vee (\neg D \wedge E))))$

$(A \vee (\neg B \wedge (C \vee (\neg D \wedge E))))$

$\equiv (A \vee (\neg B \wedge ((C \vee \neg D) \wedge (C \vee E))))$ *distributiviy.*

$\equiv ((A \vee \neg B) \wedge (A \vee ((C \vee \neg D) \wedge (C \vee E))))$ *distributivity.*

$\equiv ((A \vee \neg B) \wedge ((A \vee C \vee \neg D) \wedge (A \vee C \vee E)))$ *distributivity.*

$\equiv (A \vee \neg B) \wedge (A \vee C \vee \neg D) \wedge (A \vee C \vee E)$

c) $A \Leftrightarrow (B \wedge \neg C)$.

$\equiv (A \rightarrow (B \wedge \neg C)) \wedge ((B \wedge \neg C) \rightarrow A)$ *transformation of* $\Leftrightarrow$ *in double* $\rightarrow$.

$\equiv (\neg A \vee (B \wedge \neg C)) \wedge (\neg (B \wedge \neg C) \vee A)$ *transformation of implication in disjunction.*

$\equiv (\neg A \vee (B \wedge \neg C)) \wedge ((\neg B \vee C) \vee A)$ *Morgan law.*

$\equiv ((\neg A \vee B) \wedge (\neg A \vee \neg C)) \wedge ((\neg B \vee C) \vee A)$ *distributivity.*

$\equiv (\neg A \vee B) \wedge (\neg A \vee \neg C) \wedge (\neg B \vee C \vee A)$.

d) A∨(B∧C)

(A∨B) ∧ (A∨C)

e) (A∧B) ∨C

(A∨C) ∧ (B∨C)

Exercise 04

1. With the truth table we find F est valid
2. $A \equiv \neg F$; then A is (inconsistency, antilogy,)

<u>Tutorial 02 part 03 (propositional logic : Proof system)</u>

<u>Exercise 01</u>

Translate this reasoning into propositional logic.

B: The weather is nice.

M: I am going to sea.

R: The tide is low.

L: The lock is closed.

(1) If the weather is nice, I am going to sea.

(2) If the tide is low, the lock is closed.

(3) If the lock is closed, I cannot go to sea.

(4) The tide is low and the weather is sunny.

Therefore,

(5) I am not going to sea.

Using truth tables, state whether this reasoning is valid.

Use propositional logic resolution method to show that C is valid.

<u>Exercise 02</u>

Use propositional logic resolution method to show whether formula C is a logical consequence of formulas A and B;

Where: $A = (p \rightarrow (q \vee r))$; $B = \neg r$ and $C = (\neg q \rightarrow \neg p)$.

<u>Exercise 03:</u>

Use the solution method to answer the following questions:

2. Prove that S is unsatisfiable:
    a. $S = \{P, \neg P \vee Q, \neg Q \vee R, \neg Q \vee \neg R\}$
    b. $S = \{R, Q \vee \neg R, S \vee \neg R, P \vee \neg Q \vee \neg S, \neg P \vee \neg Q \vee \neg S\}$
3. Is S satisfiable or unsatisfiable?

    $c = \{P \vee Q, P \vee \neg Q, R \vee Q, R \vee \neg Q\}$

Solutions

Exercise 01

1. B→M
2. R→L
3. L →non M
4. R∧B

Therefore,

5. Not M

B	M	R	L	Not M	B→M	R→L	L→ not M	R∧B	ens	Ens→ not M
V	V	V	V	F	V	V	F	V	F	V
V	V	V	F	F	V	F	V	V	F	V
V	V	F	V	F	V	V	F	F	F	V
V	V	F	F	F	V	V	V	F	F	V
V	F	V	V	V	F	V	V	V	F	V
V	F	V	F	V	F	F	V	V	F	V
V	F	F	V	V	F	V	V	F	F	V
V	F	F	F	V	F	V	V	F	F	V
F	V	V	V	F	V	V	F	F	F	V
F	V	V	F	F	V	F	V	F	F	V
F	V	F	V	F	V	V	F	F	F	V
F	V	F	F	F	V	V	V	F	F	V
F	F	V	V	V	V	V	V	F	F	V
F	F	V	F	V	V	F	V	F	F	V
F	F	F	V	V	V	V	V	F	F	V
F	F	F	F	V	V	F	V	F	F	V

Ens→ Not M: is a tautology. Therefore, there is a logical consequence between ens and not M, so this reasoning is valid.

With the resolution method

We have Ens |= ¬M   if only if   Ens, M |=. (empty clause)

1. **The negation of the conclusion is inconsistante**
2. **Clause formatting:** ¬B∨M, ¬R∨L, ¬L∨ ¬M, R,B,M

1. ¬B∨M,
2. ¬R∨L
3. ¬L∨ ¬M
4. R
5. B
6. M
7. ¬B∨¬L (1.3)
8. ¬L (7.5)
9. ¬R (2.8)
10. R¬R clause vide (9.4)

So, arriving at the empty clause, therefore Ens,M |= empty clause, therefore Ens,M is inconsistent, therefore Ens |= ¬M is and the reasoning is valid.

Exercise 02

p	**q**	**r**	¬p	¬q	¬r **B**	**q ∨ r**	**A**	**C**
V	**V**	**V**	**F**	**F**	**F**	**V**	**V**	**V**
V	**V**	**F**	**F**	**F**	**V**	**F**	**F**	**V**
V	**F**	**V**	**F**	**V**	**F**	**F**	**F**	**F**
V	**F**	**F**	**F**	**V**	**V**	**F**	**F**	**F**
F	**V**	**V**	**V**	**F**	**F**	**V**	**V**	**V**
F	**V**	**F**	**V**	**F**	**V**	**F**	**V**	**V**
F	**F**	**V**	**V**	**V**	**F**	**F**	**V**	**V**
F	**F**	**F**	**V**	**V**	**V**	**F**	**V**	**V**

With the resolution method

We have A, B |= C if only if A, B, ¬C |=. (empty clause)

we write A, B et ¬C under clausal form:

$$A = (p \rightarrow (q \vee r)) ; B = \neg r \text{ et } C = (\neg q \rightarrow \neg p).$$

A ≡ ¬P ∨ q ∨ r (is a clause),

B ≡ ¬r (is a clause),

¬C ≡ ¬q ∧ p (two clauses).

Therefore, we have:

C1 $= \neg p \lor q \lor r$

C2 $= \neg r$

C3 $= \neg q$

C4 $= p$

C5 $= q \lor r$ Res(C1, C4)

C6 $= r$ Res(C3, C5)

C7 = clause vide Res(C2, C6)

A, B, ¬C |= (empty clause) is inconsistent or unsatisfiable

Therefore, C is a logical consequence of A and B.

Exercise 03

a)

1.	P	
2.	$\neg P \lor Q$	
3.	$\neg Q \lor R$	
4.	$\neg Q \lor \neg R$	
5.	Q	$(1,1) - (2,1)$
6.	R	$(5,1) - (3,1)$
7.	$\neg R$	$(5,1) - (4,1)$
8.	$\square$	$(6,1) - (7,1)$

b)

1. R
2. $Q \vee \neg R$
3. $S \vee \neg R$
4. $P \vee \neg Q \vee \neg S$
5. $\neg P \vee \neg Q \vee \neg S$
6. Q $(1,1) - (2,2)$
7. S $(1,1) - (3,2)$
8. $P \vee \neg S$ $(6,1) - (4,2)$
9. P $(7,1) - (8,2)$
10. $\neg P \vee \neg S$ $(6,1) - (5,2)$
11. $\neg P$ $(7,1) - (10,2)$
12. $\square$ $(9,1) - (11,1)$

c)

1. $P \vee Q$
2. $P \vee \neg Q$
3. $R \vee Q$
4. $R \vee \neg Q$
5. P $(1,2) - (2,2)$
6. $P \vee R$ $(1,2) - (4,2)$
7. R $(3,2) - (4,2)$

There is no possibility of obtaining different clauses, therefore S is satisfiable.

Tutorial No = 03 part 1 (predicate logic : syntax)

Exercise 1

Formalize the following sentence in predicate logic using only one predicate: A (x, y): x loves y.

Each person loves someone and nobody loves everyone, or someone loves everyone and someone loves nobody.

Exercise 2

Formalize the following sentences using predicate logic, specifying the vocabulary used.

1. All students like logic.
2. Not all students like a subject.
3. Students who get a good grade in logic are the best.

Exercise 3

Consider the following sentences:

a- All men are mortal.

b- Socrates is mortal.

c- Socrates is a man.

Questions:

1. Identify the predicates.
2. Express a, b, c in predicate logic.
3. Can statement b be deduced from a and c? Justify your answer.

Exercise 4

Translate the following sentences into the language of first-order predicates:

a. All cats are black

b. Some cats are black

c. No cat is black

d. In the domain of natural numbers, for every natural number, there exists a prime number greater than it

Exercise 5

Formalize the following sentences using Predicate logic:

1. Whales are mammals.
2. Integers are either even or odd.
3. There exists an even integer

Exercise 6

Let:

M(x): x is a Man

¬M(x): x is a woman

P(x): x wears trousers.

C(x): x has long hair.

1. Translate the following sentences into English
   a) $\neg(\exists x(M(x) \land C(x)))$
   b) $\forall x(P(x) \to M(x))$
   c) $\forall x(M(x) \to P(x))$
2. Find a model, for example, with 2 men and 2 women, that satisfies b but not c.
3. Find a model that satisfies a, b, and c.
4. Find a model that satisfies a and c but not b.
5. Translate the following sentences into formulas:
   a) It's not only men who wear trousers.
   b) Nobody wears trousers and has long hair at the same time.

Exercice 7

Translate in predicate logic the following statements

1. Everything is relative.
2. Nothing is relative.
3. A door is either open or closed.
4. All roads lead to Rome.

5. For every integer, there is a larger integer.
6. There is an even integer.
7. There is a smaller integer.
8. All that glitters is not gold.

Solutions

Exercise1

using only one predicate: A (x, y): x loves y the formula is:

$(\forall \boldsymbol{x} \forall \boldsymbol{y} \boldsymbol{A}(\boldsymbol{x}, \boldsymbol{y}) \wedge \neg(\exists x \forall y A(x, y)) \vee (\exists x \forall y A(x, y) \wedge \exists \boldsymbol{x} \forall \boldsymbol{y} \neg \boldsymbol{A}(\boldsymbol{x}, \boldsymbol{y})$

Exercise 2

First, we need to simplify the vocabulary used the universe of discourse). In our case, we will use the following vocabulary:

Constants: log: is the constant that represents the logic.

Variables: x and y.

Unary predicates (of arity 1):

Student(x): means that "x" is a student.

Subject(y): means that "y" is a subject.

Binary predicates:

likes(x,y): means that "x" likes "y".

goodgrade(x,y): means that "x" got a good grade in "y".

Better(x,y): means that "x" is better than "y".

Now we will formalize the sentences:

1- $\forall$x(Student(x) $-\!\!\rightarrow$likes (x, log))

2- $\forall$x$\exists$y((Student(x) $\wedge$Subject(y)) $-\!\!\rightarrow\neg$likes (x, y))

3- $\forall$x$\forall$y((Student(x) $\wedge$goodgrade(x, log) $\wedge$student(y)) $\rightarrow$Better (x, y))

Exercise 3

5. The predicates are:

   Man (x) and mortal(s).

6. The sentences in predicate logic are :

a- $\forall$x(Man(x) $-\!\!\rightarrow$mortal(x)).

b- mortal (Socrate).

c - Man(socrate).

- Yes, we can deduce statement "b" from "a" and "c".

Proof:

Intuitively, "Socrates" is an instance of "x" and since "Socrates" is a man:

(Man (Socrate)= true) et ∀x(Man(x)→ mortal(x)) = true therefore "Socrate" is mortal means mortal(Socrate) = true.

We can also demonstrate a using proof theory, that is, we must prove that:

∀x(Man(x) → mortal(x)), Man (Socrate) ⊢ morael(Socrate).

The proof is as follows:

1- ⊢ ∀x(Man(x) → mortal(x))
hyp1

2- ⊢ Man (Socrate) hyp2

3- ⊢ ∀x(Man(x) → mortal(x)) → (Man (Socrate) → mortal (Socrate))

4-⊢ Man (Socrate) −→ mortal (Socrate) mp 1,3

5- ⊢ mortal(Socrate) mp 2,4

Exercice 4

a.∀x(Chat(x) → Noir(x))

b.∃x(Chat(x) ∧ Noir(x))

c. ∀x¬(Chat(x) ∧ Noir(x))

d. ∀x∃y(Pre(y) ∧ Sup (y, x))

Exercice 5

1. ∀x(Whales(x) → Mamm(x))
2. ∀x(Integer(x) → (even(x) ∨ odd(x)))
3. ∃x(Integer(x) ∧ even(x))

Exercice 6

1)

a. No man has long hair.

b. Everyone who wears trousers is a man.

Or: Only men wear trousers.

c.. All men wear trousers

2)

Domain: Sophie, Adele, Marius and Gaston. Sophie, Adele and Marius are wearing skirts, Gaston is wearing trousers. They all have long hair.

3)

Domain: Sophie, Adele, Marius and Gaston. Sophie and Adele are wearing skirts; Gaston and Marius are wearing trousers. They all have short hair.

4)

Domain: Sophie, Adele, Marius and Gaston. Everyone in trousers. The girls can wear their hair however they want and the boys have short hair.

5)

a. $\neg\forall x(P(x) \rightarrow M(x))$ or

$\exists x(P(x) \land \neg M(x))$

b. $\neg(\exists x(P(x) \land C(x)))$

Exercice 7

1. $\forall x \mathbf{r}(x)$
2. $\forall x \neg r(x)$
3. $\forall x$ (porte $(x) \rightarrow$ (ouvert $(x) \lor$ fermé (x)))
4. $\forall x$ (road (x) → lead-toRome (x))
5. $\forall x(e(x) \rightarrow \exists y(e(y) \land (y > x)))$
6. $\exists x\ (e(x) \land \text{even}(x))$
7. $\exists x\ (e(x) \land \forall y(e(y) \rightarrow x < y))$
8. $\exists x$ (glitter (x) $\land \neg$ gold (x)), $\neg\forall x$ (glitter (x) $\rightarrow$ gold (x))

Tutorial No = 03 part II (predicate logic : Semantics)

Exercise 01

1. Let P(x) be a predicate meaning "x is a cat."
2. Define a domain D consisting of three animals: {Fluffy, Max, Bella}.
3. Specify the truth values of P for each element in the domain.

Exercise 02

Consider the predicates:

- P(x):x is a cat.
- Q (x):x is a dog.

1. Use the same domain D= {Fluffy (cat), Max (dog), Bella (cat)}
2. Evaluate the statement $\forall x(P(x) \vee Q(x))$.

Exercise 03

Consider the following statements:

- $\forall x(P(x) \Longrightarrow Q(x))$
- $\forall x(Q(x))$

1. Prove or disprove $\forall x(P(x))$.

Exercise 04

Given the predicates:

- P(x):x is a student.
- Q(x):x is enrolled in the course.

1. Define a domain D= {Alice, Bob, Charlie}
2. Specify P and Q in such a way that:

- $\exists x(P(x) \wedge Q(x))$ is true.
- $\forall x(P(x) \Longrightarrow Q(x))$ is true.

Solutions

Exercise1

- P(Fluffy)=True
- P(Max)=False
- P(Bella)=True

Exercise2

Analyze each element in the domain.

- $P(Fluffy) \lor Q(Fluffy)$=True
- $P(Max) \lor Q(Max)$=True
- $P(Bella) \lor Q(Bella)$=True

Thus, $\forall x(P(x) \lor Q(x))$ is **True**.

Exercise3

Construct a counterexample where P(x) is true but Q(x) is false.
The argument does not hold, indicating that the conclusion does not necessarily follow from the premises.

Exercise 04

Possible assignements :

- P(Alice)=True
- P(Bob)=True
- P(Charlie)=False
- Q(Alice)=True
- Q(Bob)=False
- Q(Charlie)=False

This model satisfies the conditions.

<u>Annex B</u>

Final Exam Academic Year: 2024/2025

<u>Exercise 01</u>

Put a circle on the correct answer(s)?

<u>1.</u> the formula A is

a. a literal
b. a clause
c. a logic variable

<u>2.</u> the two following causes: ¬AVB ***and*** AVC are

a. complementary clauses

b. resolvent clauses

c. empty clauses

<u>3.</u> The formula ¬A VA is a

a.. tautology

b. valid

c . antilogy

d. consistency

e. contingent

<u>4.</u> The formula ¬B⇒¬A is

a. the converse of A ⇒B

b. the inverse of A ⇒B

c. the contrapositive of A ⇒B

<u>5 .</u> the general inference rule applied on

a. Two complementary clauses
b. b. two conjunctive clauses
c. c. literal and its negation

6. a consistent formula can be

a. tautology formula

b. valid formula

c. contingent formula

7. B ∧ C ⊨ A⇒ (B ∧ C) it is read

a. B ∧ C is a logic consequence of A⇒ (B ∧ C);

b. A⇒ (B ∧ C) is a logic consequence of B ∧ C

8 ***16*** interpretations is the true number of a formula with

a. ***3*** logic variables
b. ***4*** logic variables
c. ***2*** logic variables
d. ***5*** logic variables

9 The formula A ⇔ (B ∧ C) is logically equivalent to

a. ((A⇒ (B ∧ C)) ∧ ((B ∧ C) ⇒ A)))
b. ((A⇒ (B ∧ C)) V ((B ∧ C) ⇒ A)))
c. ((¬AV (B ∧ C)) ∧ ((¬BV ¬C) V A)))

10. B ∧ C ⊨ A⇒ (B ∧ C) if

a. Each model of B ∧ C is a model of A⇒ (B ∧ C);

b. Each model of A⇒ (B ∧ C) is a model of B ∧ C

Exercise 02

F is a logic function with eight interpretations; F is true when the disjunction between two logic variables is true.

1. Construct the truth table of F?
2. Give The CNF and the DNF of F?
3. Determine whether F is a tautology, satisfiable, unsatisfiable?

Exercise 03

We define the Sheffer connector, denoted by " | " (Sheffer bar), which is the NAND (not and), by: $p|q \equiv \neg (p \wedge q)$

1. Construct the truth table for the formula $(p|q)$.
2. Construct the truth table for the formula $((p|q)|(p|q))$.
3. Express the connectors ¬, ∨, and → using the Sheffer bar.

Model Answer

Exercise 01

a	B	c	F
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

1. CNF

   (a V b V c)

   DNF

   (a∧ b ∧ c) V (a ∧ b ∧ ¬ c) V (a ∧ ¬ b ∧ c) V(a∧ ¬ b∧ ¬c)V(¬a∧b∧c)V(¬a∧b∧ ¬c)V(¬a∧ ¬b∧c)

2.

   F is not a tautology

   F is not unsatisfiable

   F is satisfiable

Exercise n 03

1. p|q

p	q	p∧q	¬ (p∧q)= p|q
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

2. (p|q) | (p|q) = ¬ (p∧q) |¬ (p∧q) = ¬ ((¬ (p∧q)) ∧¬ (p∧q)) =(p∧q) ∨ (p∧q) = (p∧q)

p	q	p∧q
T	T	T
T	F	F

F	T	F
F	F	F

3.

The connector $\neg$: $\neg p \equiv \neg\,(p \wedge p) \equiv (p|p)$

The connector $\vee$: $p \vee q \equiv \neg\,(\neg p \wedge \neg q) \equiv (\neg p|\neg q) \equiv (p|p)|(q|q)$.

The connector $\rightarrow$: $p \rightarrow q \equiv \neg p \vee q \equiv \neg\,(p \wedge \neg q) \equiv p|\neg q \equiv p|(q|q)$.

Final Exam Academic Year: 2023/2024

Course questions

1. Explain why propositional logic is limited.

2. Explain the semantic relationship between a verifiable formula and a tautology.

Exercise01

Let p be the proposition "the child can laugh", q the proposition "the child can cry", and r the proposition "the child can read". Give the translation into everyday language of the following propositions:

1 : $p \land q$	6 : $p \Rightarrow (\neg q \lor \neg r)$
2 : $p \land (\neg q)$	7 : $p \Leftrightarrow r$
3 : $(q \rightarrow p)$	8 : $(\neg p \lor \neg q \lor \neg r)$
4 : $(\neg p) \lor (\neg q)$	9 : $(p \land \neg q \land \neg r) \lor (\neg p \land q \land \neg r) \lor (\neg p \land \neg q \land r)$
5 : $(\neg p) \land (\neg q)$	10. $\neg (\neg p \land \neg q \land \neg r)$

Exercise02

The Sheffer connective, denoted by "|" (Sheffer bar), which is the NAND (Not-AND) operator, is defined as:

$p \mid q \equiv \neg (p \land q)$

1. Give the truth table for the formula $(p \mid q)$.
2. Give the truth table for the formula $((p \mid q) \mid (p \mid q))$.
3. Express the connectives $\neg$, $\lor$, and $\rightarrow$ using the Sheffer bar.

Exercise 03

Puzzle:

Three people of different nationalities (Moroccan, Algerian, and Tunisian) and who play different sports (football, swimming, and tennis) live in three houses of distinct colors (white, green, and red). These three houses are located on the same street; one house is at the beginning of the street, another in the middle, and the third at the end. Each of the three houses is therefore characterized by a set of four elements (E, C, N, S), where E is the house's location on the street, C is the color of the house, N and S are the nationality and sport played by its occupant, respectively. We have the following five clues:

- Swimming is practiced in the green house.
- The green house is located before the Algerian's house.
- The Moroccan lives in the red house.
- The red house is located before the house where football is played.
- The tennis player lives at the beginning of the street.

Question: Determine the characteristics of each of the three houses.

Model Answer

Course questions

Limit 1

Propositional logic treats propositions as a whole. It cannot discuss objects, properties of objects, or establish relationships between objects.
Example: Ahmed lives in Mila.

Limit 2
Example: Consider the proposition "x is prime." The truth value of this proposition depends on the value of x (which is the object of the proposition). In propositional logic, it is not possible to formulate such a proposition because the truth value of a proposition is clear: it is either true or false. However, in predicate logic, it is possible to formulate propositions whose truth value depends on the object.
2)

A verifiable formula can be a tautology if all interpretations of the verifiable formula are true.

Exercise 01

1: $p \wedge q$: The child can laugh and cry.

2: $p \wedge (\neg q)$: The child can laugh but cannot cry.

3: $(q \rightarrow p)$: If the child can cry, then they can laugh.

4: $(\neg p) \vee (\neg q)$: The child cannot laugh or cannot cry.

5: $(\neg p) \wedge (\neg q)$: The child cannot laugh or cry.

6: $p \Rightarrow (\neg q \vee \neg r)$: If the child can laugh, then at least one of the other two actions is not performed.

7: $p \Leftrightarrow r$: That the child knows how to cry is a necessary and sufficient condition for the child to know how to laugh.

8: $(\neg p \vee \neg q \vee \neg r)$: At least one of the three actions is not fulfilled (laugh, cry, read).

9: $(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$: Only one of the three actions (laugh, cry, read) is fulfilled.

10: $\neg(\neg p \wedge \neg q \wedge \neg r)$: It is not true that the child cannot laugh, cannot cry, and cannot read.

Exercise 02

1. $p|q$

p	q	$p \wedge q$	$\neg(p \wedge q) = p|q$
V	V	V	F
V	F	F	V
F	V	F	V
F	F	F	V

2. $(p|q) | (p|q) = \neg(p \wedge q) | \neg(p \wedge q) = \neg((\neg(p \wedge q)) \wedge \neg(p \wedge q)) = (p \wedge q) \vee (p \wedge q) = (p \wedge q)$

p	q	$p \wedge q$
V	V	V
V	F	F
F	V	F
F	F	F

3.

The connector $\neg$: $\neg p \equiv \neg(p \wedge p) \equiv (p|p)$

The connector $\vee$: $p \vee q \equiv \neg(\neg p \wedge \neg q) \equiv (\neg p|\neg q) \equiv (p|p)|(q|q)$.

The connector $\rightarrow$: $p \rightarrow q \equiv \neg p \vee q \equiv \neg(p \wedge \neg q) \equiv p|\neg q \equiv p|(q|q)$.

Final Exam Academic Year: 2022/2023

Course Question

Explain two limitations of propositional logic with two illustrative examples?

Exercise 01

Let p: Ali reads books, q: Ali reads stories, and r: Ali reads newspapers.

Give a logical formula for each of the following sentences:

1. Ali reads books or stories, but does not read newspapers.

2. Ali reads both books and stories, or he reads neither books nor newspapers.

3. It is not true that Ali reads books but not newspapers.

4. It is not true that Ali reads newspapers or stories but not books.

Exercise 02

Three colleagues, Ahmed, Ali, and Mostafa, have lunch together every weekday. The following statements are true:

1. If Ahmed orders a dessert, Ali also orders one.

2. Every day, either Mostafa or Ali, but not both, orders a dessert.

3. Ahmed or Mostafa, or both, order a dessert every day.

4. If Mostafa orders a dessert, Ahmed does the same.

Questions:

Let

A: Ahmed orders a dessert,

L: Ali orders a dessert,

M: Mostafa orders a dessert.

1. Express the given information in the problem as propositional formulae.

2. What can be deduced about who orders a dessert?

Exercise 03

Using the propositional method, prove that the following formula is valid:

$F = (a \land \neg b) \lor (\neg a \land \neg (b \lor c)) \lor (\neg c \land b) \lor (b \land c \land a) \lor (c \land \neg a)$.

Deduce the validity of the following formula A:

$A = (a \rightarrow b) \land (\neg a \rightarrow (b \lor c)) \land (\neg c \rightarrow \neg b) \land ((b \land c) \rightarrow \neg a) \land (c \rightarrow a)$.

Exercise 04

Translate the following statements into predicate logic:

1. Whales are mammals.
2. All teachers who teach math are intelligent.

Model Answer

Course questions

Limit 1

Propositional logic treats propositions as a whole. It cannot discuss objects, properties of objects, or establish relationships between objects.
Example: Ahmed lives in Mila.

Limit 2
Example: Consider the proposition "x is prime." The truth value of this proposition depends on the value of x (which is the object of the proposition). In propositional logic, it is not possible to formulate such a proposition because the truth value of a proposition is clear: it is either true or false. However, in predicate logic, it is possible to formulate propositions whose truth value depends on the object.
2)

A verifiable formula can be a tautology if all interpretations of the verifiable formula are true.

Exercise 01

1. $p \lor q \land \neg r$
2. $(p \land q) \lor (\neg p \land \neg r)$
3. $\neg(p \lor \neg r)$
4. $\neg((r \lor q) \lor \neg p)$

Exercise 02

1.
    - $A \rightarrow L$
    - $(M \land \neg L) \lor (\neg M \land L)$
    - $A \lor M$
    - $M \rightarrow A$
2.

To see really who ordered a dessert, we make the truth table of the overall formula "F1" composed from the conjunction of all the previous statements in order to see all the possible models.

The overall formula:

"F1" = (A → L) ∧ ((L ∧ ¬M) ∨ (¬L ∧ M)) ∧ (A ∨ M) ∧ (M → A).

The truth table is then as follows:

Interpretations	A	L	M	A → L	(L ∧ ¬M) ∨ (¬L ∧ M)	A ∨ M	M → A	F_1
I_1	V	V	V	V	F	V	V	F
I_2	V	V	F	V	V	V	V	V
I_3	V	F	V	F	V	V	V	F
I_4	V	F	F	F	F	V	V	F
I_5	F	V	V	V	F	V	F	F
I_6	F	V	F	V	V	F	V	F
I_7	F	F	V	V	V	V	F	F
I8	F	F	F	V	F	F	V	F

The only interpretation that makes formula "F1" true is interpretation I2, in which Ahmed and Ali order dessert but Mostafa does not. This is because the truth values of propositions A and L are true, but proposition M is false (in row two of the truth table).

Exercise 03

1. To show that F is a tautology, we demonstrate that ¬F is a (inconsistent) contradiction.

F = (a ∧ ¬b) ∨ (¬a ∧ ¬ (b ∨ c)) ∨ (¬c ∧ b) ∨ (b ∧ c ∧ a) ∨ (c ∧ ¬a).

So, we have:

C1 = ¬a ∨ b

C2 = a ∨ b ∨ c

C3 = c ∨ ¬b

C4 = ¬b ∨ ¬c ∨ ¬a

C5 = ¬c ∨ a

From this, we obtain ¬F.

C6 = b ∨ c Res(C1, C2)

C7 = c Res(C3, C6)

C8 = ¬c ∨ ¬a Res(C1, C4)

C9 = ¬c Res(C5, C8)

C10 = ——————————————— Res(C7, C9)

Therefore, ¬F |=, hence: |= F is valid

2. By performing the calculation, we obtain A ≡ ¬F; therefore, A is a contradiction (inconsistent).

<u>Exercice 04</u>

1. ∀ x (Whale (x) → mammal (x))

2. ∀ x (teacher (x) ∧ teach(x.math) →intelligent(x)).

Annex C

Re-sit Exam Academic Year: 2024/2025

Exercise 01

Put true or false and correct the false expression

1. An antology formula is a formula that has at least one true interpretation
2. A model is an interpretation that makes a logic formula false
3. The formula "A" is a logic variable
4. the two following causes: ¬AVB ***and*** AVC are resolvent clauses
5. The formula ¬A ∧ A is a tautology
6. The formula ¬B⇒¬A is the contrapositive of A ⇒B
7. The general inference rule applied on two conjunctive clauses
8. a consistent formula can be tautology formula
9. B ∧ C ⊨ A⇒ (B ∧ C) it is read. B ∧ C is a logic consequence of A⇒ (B ∧ C)

10 . two equivalent logic formula have not the same interpretations in their truth table

Exercise 02

The following is the truth table of a formula F

a	**b**	**c**	**F**
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

4. Give The CNF and the DNF of F?
5. Determine whether F is a tautology, satisfiable, unsatisfiable?

Exercise 03

There are three friends: Alice, Bob, and Charlie. Each of them has a different favorite color: Red, Blue, and Green.

The following statements are true:

1. Alice does not like Green.
2. Bob likes Red or Green.
3. Charlie does not like Blue.

Use the following propositions

Let Ar be "Alice likes Red".

Let Ab be "Alice likes Blue".

Let Ag be "Alice likes Green".

Let Br be "Bob likes Red".

Let Bb be "Bob likes Blue".

Let Bg be "Bob likes Green".

Let Cr be "Charlie likes Red".

Let Cb be "Charlie likes Blue".

Let Cg be "Charlie likes Green".

Questions

1. Transform the three statements (1,2,3) in propositional logic
2. What is the favorite color of each person?

Model Answer

Exercise 01

	a	b	c	d	e						
1	X	X	X								
2	X										
3	X	X									
4			X								
5	X										
6	X	X	X								
7		X									
8		X									
9	X		X								
10	X										

Exercise 02

a	b	c	F
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

3. **CNF**

   (a V b V c)

   DNF

   (a∧ b ∧ c) V (a ∧ b ∧ ¬ c) V (a ∧ ¬ b ∧ c) V(a∧ ¬ b∧ ¬c)V(¬a∧b∧c)V(¬a∧b∧ ¬c)V(¬a∧ ¬b∧c)

4.

   F is not a tautology

   F is not unsatisfiable

   F is satisfiable

Exercise 03

Let's solve the enigma step by step using propositional logic.

Given Statements:

1. Alice does not like Green.
   - $\neg Ag$
2. Bob likes Red or Green.
   - $Br \vee Bg$
3. Charlie does not like Blue.
   - $\neg Cb$

Deductions:

1. From statement 1, since Alice does not like Green:
   - Alice must like either Red or Blue: $Ar \vee Ab$
2. From statement 3, since Charlie does not like Blue:
   - Charlie must like either Red or Green: $Cr \vee Cg$
3. From statement 2, since Bob likes Red or Green:
   - If Br is true, then Bob likes Red. If Bg is true, then Bob likes Green.

Analyzing Cases

Case 1: Assume Bob likes Red (Br)

- If Bob likes Red, then Charlie cannot like Red (since colors are unique), so Charlie must like Green:
  - Cg (Charlie likes Green)
- Since Alice cannot like Green, she must like Blue:
  - Ab (Alice likes Blue)

Conclusion from Case 1 :

- Alice : Blue

- Bob : Red
- Charlie : Green

Case 2: Assume Bob likes Green (Bg)

- If Bob likes Green, then Charlie cannot like Green (since colors are unique), so Charlie must like Red:
    - Cr(Charlie likes Red)
- Since Alice cannot like Green, she must like Blue:
    - Ab (Alice likes Blue)

Conclusion from Case 2 :

- Alice : Blue
- Bob : Green
- Charlie : Red

Summary of Conclusions

- **From Case 1** :
    - Alice : Blue
    - Bob : Red
    - Charlie : Green
- **From Case 2** :
    - Alice : Blue
    - Bob : Green
    - Charlie : Red

Final Analysis

Since both cases lead to different conclusions, we can check:

- **If Bob likes Red** :
    - Charlie must be Green, and Alice is Blue.
- **If Bob likes Green** :

    - Charlie must be Red, and Alice is still Blue.

Final Conclusion

The only consistent conclusion based on the contradictions is:

- **Alice** : Blue
- **Bob**: Red (if Charlie is Green) or Green (if Charlie is Red)
- **Charlie**: Green (if Bob is Red) or Red (if Bob is Green)

However, the valid combinations must be:

- **Alice** : Blue
- **Bob** : Red
- **Charlie** : Green

Thus, the correct assignment is:

- **Alice** : Blue
- **Bob** : Red
- **Charlie** : Green

Re-sit Exam Academic Year: 2023/2024

Exercise 01

1) State, with justification, whether the following formulas are tautologies? State whether they are satisfiable.

1-a) $(\neg A \lor \neg B) \rightarrow (A \land B)$
1-b) $((A \rightarrow B) \land (B \rightarrow \neg B)) \rightarrow \neg A$
1-c) $(((A \land B) \rightarrow C) \land \neg C) \rightarrow (A \rightarrow \neg B)$

2) Let `valid` be an algorithm that returns `true` for a formula P if the formula P is a tautology and `false` otherwise. Can the `valid` algorithm be used to test whether a formula in propositional calculus is satisfiable or not? If yes, explain how; if not, explain why.

Exercise 02

Consider the following reasoning:
- When it's sunny, I wear my sunglasses or I don't go out.
- I only stay home without sunglasses and on cloudy days.
Therefore,
If I don't wear my sunglasses, it's because it's cloudy.
1) Formalize this reasoning using propositional logic with the following variables:
S: it's sunny, L: I wear my sunglasses, M: I stay home.
2) Show that the reasoning above is correct (valid):
a. using a truth table;
b. using a resolution method.

Exercise 03

Explain the semantic relationship between the two formulas:
a. Satisfiable ----- valid
b. Contingent ----- invalid

Exercise 04

Consider the following statements:
1. If Brahim fails his exam, then he will be depressed.

2. If the weather is nice, then Brahim will go to the swimming pool.

3. At the swimming pool, Brahim does not work.

4. If Brahim does not go to the swimming pool, then he will be depressed.

5. Brahim will fail his exam if he does not work.

Question 1: Formalize the problem using propositional logic, with: R: "Brahim fails his exam", B: "The weather is nice", P: "Brahim will go to the swimming pool", T: "Brahim works", D: "Brahim is depressed".

Model Answer

Exercise 01

1. a satisfiable

A	B	¬A	¬ B	¬A∨¬B=I	A ∧ B=II	I⇒II
1	1	0	0	0	1	1
1	0	0	1	1	0	0
0	1	1	0	1	0	0
0	0	1	1	1	0	0

1.b tautology

A	B	¬A	¬ B	A⇒B=I	B ⇒¬ B=II	I∧II=III	III⇒¬A
1	1	0	0	1	0	0	1
1	0	0	1	0	1	0	1
0	1	1	0	1	0	0	1
0	0	1	1	1	1	1	1

1.c tautology

A	B	C	¬B	¬ C	A∧ B=I	I⇒C=II	II∧¬ C=III	A⇒¬B=V	III⇒V
1	1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	0	1
1	0	1	1	0	0	1	0	1	1
1	0	0	1	1	0	1	1	1	1
0	1	1	0	0	0	1	0	1	1
0	1	0	0	1	0	1	1	1	1
0	0	1	1	0	0	1	0	1	1
0	0	0	1	1	0	1	1	1	1

Exercise 02

1°) With the variables s, l and m denoting: s: it is sunny; l: I put on my glasses; m: I stay home; we can formalize the given reasoning with a formula from propositional calculus:

$$((s \rightarrow (l \vee m)) \wedge (m \rightarrow (\neg l \wedge \neg s))) \rightarrow (\neg l \rightarrow \neg s) \qquad \textbf{(A)}$$

1°) The reasoning considered is valid because formula (A) is a tautology; let's see this in the truth table, i.e., the sub-formulas:

$I = s \rightarrow (l \vee m)$, $II = m \rightarrow (\neg l \wedge \neg s)$, $III = (\neg l \rightarrow \neg s)$

s	l	m	$l \vee m$	I	$\neg l \wedge \neg s$	II	III	$I \wedge II$	(A)
0	0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1	1
0	1	0	1	1	0	1	1	1	1
0	1	1	1	1	0	0	1	0	1
1	0	0	0	0	0	1	0	0	1
1	0	1	1	1	0	0	0	0	1
1	1	0	1	1	0	1	1	1	1
1	1	1	1	1	0	0	1	0	1

Exercise 03

1. Satisfiable can become valid if all its interpretations are true.

2. Contingent can become invalid if all its interpretations are false.

Exercise 04

1. $R \rightarrow D$
2. $B \rightarrow P$
3. $P \rightarrow \neg T$
4. $\neg P \rightarrow D$
5. $\neg T \rightarrow R$.

Re-sit Exam Academic Year: 2022/2023

Exercise 01

Provide the translation into everyday language of the following propositions, where p represents the proposition "man is mortal" and q represents the proposition "man is eternal":

(1): $(p \lor q)$;

(2): $(\neg p) \lor (\neg q)$;

(3): $\neg (p \land q)$;

(4): $p \land (\neg q)$;

(5): $(p \rightarrow (\neg q))$

Exercise 02

Consider the following reasoning:

1. B→M

2. R→L

3. L →not M

4. R∧B

Therefore

5. Not M

Question: Use propositional proof to show that 5 is valid.

Exercise 03

Translate the following natural language statements into predicate logic. Use appropriate predicates and variables, and clearly define your choices.

Statement 1: "All students in the class study diligently."
Statement 2: "Some cats are not friendly."

Statement 3: "If it rains, then the ground is wet."

Statement 4: "Every dog has a owner."

Statement 5: "There exists a person who loves a cat."

Model Answer

Exercise 01

(1) Man is either mortal or eternal.

(2) Man is not mortal, or he is not eternal.

(3) It is false that "man is mortal and eternal."

(4) Man is mortal but not eternal.

(5) If man is mortal, then he is not eternal.

Exercise 02

We have Ens |= ¬M if and only if Ens, M |= (empty clause)

The negation of the conclusion is inconsistent:

1. Clause formatting: ¬B∨M, ¬R∨L, ¬L∨¬M,R,B,M

1. ¬B∨M,
2. ¬R∨L
3. ¬L∨ ¬M
4. R
5. B
6. M
7. ¬B∨¬L (1.3)
8. ¬L (7.5)
9. ¬R (2.8)
10. R¬R empty clause (9.4)

So, arriving at the empty clause, therefore Ens, M |= empty clause, therefore Ens, M is inconsistent, therefore Ens |= ¬M is and the reasoning is valid.

Exercise 03

Predicates:

- For example, you might define:

- S(x): x is a student.
- D(x): x studies diligently.
- C(x): x is a cat.
- F(x): x is friendly.
- R: It rains.
- G: The ground is wet.
- Dg(x): x is a dog.
- O(x,y): x is the owner of y.
- P(x): x is a person.
- L(x,y): x loves y.

Write the logical expressions :

- Use quantifiers (∀, ∃) appropriately.

Statement 1:

$\forall x(S(x) \Longrightarrow D(x))$

Statement 2:

$\exists x(C(x) \land \neg F(x))$

Statement 3 :

$R \Longrightarrow G$

Statement 4:

$\forall x(Dg(x) \Longrightarrow \exists y O(x,y))$

Statement 5 :

$\exists x(P(x) \land \exists y(C(y) \land L(x,y)))$

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