PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

Ministry of Higher Education and Scientific Research

LARBI BEN M'HIDI UNIVERSITY OUM EL BOUAGHI

FACULTY OF EXACT SCIENCES AND SCIENCES OF NATURE AND LIFE

DEPARTEMENT OF MATHEMATICS AND COMPUTER SCIENCE

Ph.D. Thesis Degree

Specialty : Mathematics

Option : Applied mathematics

Presented by :

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Entitled :

Average sentinel for a heat equation with incomplete data

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2023/2024

THANKS

At the beginning and above all, I give thanks to Almighty God who helped me to finish this work.

I would like to express my deep gratitude to my thesis director, Professor AYADI Abdelhamid, for his valuable help and encouragement which contributed to the successful completion of this thesis and for following me in my research.

I would like to thank Professor A. OUENNAS Adel for the honor he gives me to chair the jury for this thesis, may he be assured of our deep respect. I also thank Professor RADJAL Nadjeh, Professor BERHAIL Amel, Professor OUSSIF Taky Eddine, Professor DJEFFAL Amir and Professor Rezzoug Imad.

I sincerely thank everyone who directly or indirectly contributed to the development of this work. My thanks also go to all my colleagues in the mathematics department of the University of Oum el Bouaghi and the team of the dynamic systems and control laboratory.

Finally, I would like to thank my family for their support and encouragement.

الملخص:

تحديد الوسائل الرياضية يرتكز على إيجاد أسباب الظاهرة انطلاقا من مراقبة هذه الأخيرة وحل هذه المشكلة ويعتمد على استخدام القياس التجريبي. نقدم في هذا البحث دراسة فئة من المعادلات التي تحكمها معادلة القطع المكافئ مع معطيات غير مكتملة. لهذا الغرض نستخدم مفهوم الحارس الذي وضعه "جاك لويس ليونس" والذي هو أفضل استراتيجية للحصول على معلومات حول أسباب الظاهرة انطلاقا من المتوسط المرجح للظاهرة. وعليه نثبت وجود دالة الحارس المتوسط من خلال حل مشكلة المراقبة المتوسطة الصفرية مع القيود المفروضة على العنصر المراقب. أخيرا نركز اهتمامنا حول تحديد ومعرفة معامل التلوث الذي يظهر على معادلات القطع المكافئ ذو البيانات الناقصة.

Abstract

Identification problems consist of finding the causes of a phenomenon from an observation of it, the resolution of this type of problem is done using an experimental measurement. We propose - in this work - to study a class of problem governed by parabolic equations with incomplete data. In this case, we use the notion of sentinels introduced by J. L. Lions which leads us to the most answered strategy and which consists of obtaining information on the causes from a weighted average of the observation. Then, we prove the existence of the average sentinel function by solving a zero-average controllability problem with constraints on control. Then we identify pollution terms present on the heat equation with missing data..

Keywords: Average sentinel, Identification method, Averaged observability, Pollution term, Decomposition method, Gradient method**.**

Resumé

Les problèmes d'identification consistent à retrouver les causes d'un phénomène à partir d'une observation de celui-ci, la résolution de ce type de problème se fait à l'aide d'une mesure expérimentale. Nous proposons - dans ce travail- d'étudier une classe de problème gouvernée par des équations paraboliques avec des données incomplètes. Dans ce cas, nous utilisons la notion de sentinelles introduits par J. L. Lions qui nous amenne a la stratégie la plus répondus et qui consiste à obtenir des informations sur les causes à partir d'une moyenne pondérée de l'observation. Alors, nous prouvons l'existence de la fonction de sentinelle moyennne en résolvant un problème de la contrôlabilité moyenne à zéro avec des contraintes sur le contrôle. Puis nous identifions des termes de pollution présents sur léquation de la chaleur à données manquantes.
Mots clés: Sentinelle moyenne, Méthode d'identification, Observabilité moyenne, Terme de pollution, Méthode de décomposition, Méthode du gradient.

CONTENTS

Introduction

Questions relating to ecology, the environment and climate are today at the center of the concerns of many scientists, citizens, political parties, businesses and states. The climate plays a vital role at all levels, especially its profound modification "global warming". It is well known that air and water constitute true sources of life for flora, fauna and man. Thus, as soon as their natures are corrupted by environmental attacks, they become dangers for living beings. These may include vegetative disorders for the flora and intoxication or even cases of disease for humans. Scientists are working to determine the best palliatives for the protection and sanitation of these natural resources. They cannot, therefore, succeed in their challenge without interdisciplinary cooperation. From data observed by naturalists to equation models designed by mathematicians, including the expertise of computer engineers, digital simulation plays a very important role in mediation between scientific disciplines. Modeling these problems leads to mathematical data systems incomplete. By data we mean the initial conditions, the right hand side and possibly the boundary conditions. In almost all problems of meteorology or oceanography, we never know the initial data, we have a wide variety of possibilities when choosing the initial moment. Same thing for the problems of pollution in a lake, a river, an estuary, etc. Questions relating to ecology, the environment and climate are today at the center of the concerns of many scientists, citizens, political parties, businesses and states. The climate plays a vital role at all levels, especially its profound modification "global warming". It is well known that air and water constitute true sources of life for flora, fauna and man. Thus, as soon as their natures are corrupted by environmental attacks, they become dangers for living beings. These may include vegetative disorders for the flora and intoxication or even cases of disease for humans. Scientists are working to determine the best palliatives for the protection and sanitation of said resources natural. They cannot, therefore, succeed in their challenge without interdisciplinary cooperation. From data observed by naturalists to equation models designed by mathematicians, including the expertise of computer engineers, digital simulation plays a very important role in mediation between scientific disciplines

Modeling these problems leads to mathematical systems with incomplete data. By data we mean the initial conditions, the right hand side and possibly the boundary conditions. In almost all meteorological or oceanographic problems, we never know the initial data; we have a wide variety of possibilities when choosing the initial instant. Same thing for pollution problems in a lake, a river, an estuary, etc...

The boundary conditions may also be unknown or only partially known on a part of the boundary which may for example be inaccessible to measurements whether in biomedical situations or situations corresponding to accidents. The same goes for source terms which may be difficult to same thing for the structure of the domain which can also be imperfectly known, as for example in the management of oil wells where part of the boundary of the domain is unknown.access.

Naturally these problems are classic and the most usual idea is that of "least square", this method amounts to considering the unknowns as control variables where we seek to minimize the cost function which is the difference between the state measured on part of the domain and the state calculated by the resolution of the system considered. This leads to optimal control problems for distributed systems. In this type of method the unknowns play the same role in seeking to determine one or the other, however, there is the possibility of not being able to separate these roles.

Without neglecting this fundamental method, which remains by far the most important for this type of problem, it may be useful to try the so-called "The sentinel method".

A sentinel is a linear form acting on observations which must verify conditions of sensitivity to certain parameters of the system and insensitivity to others. So the idea of sentinels seems a little different. We then imagine that with a suitable set of sentinels we will be able to identify the interesting unknowns and eliminate the others. Suppose for example that the equation of the system describes the kinetics of a pollutant in a river or a lake and that the source is possible polluters, what is interesting in this case is obviously to know what the polluters have dumped into the river and not the state of the lake at the initial moment.

The sentinel method will therefore allow us to reconstruct a parameter or an approximation of it independently of other data that we do not want to identify, sentinels are therefore "a method of parameter identification". Identification problems have many motivations linked to important physical problems, the field of application of parameter identification methods is therefore extremely vast and the literature on the subject is abundant.

The sentinels were introduced by J. L. Lions in notes to the CRAS [46]. He subsequently published a book on this subject [43] in 1992. Many types of systems are discussed and the author studies the existence of sentinels insensitive to disturbances without constraints of sensitivity to interesting data. The study of their existence leads to the resolution of the problem of controllability of distributed systems.

Numerous theoretical and numerical results exist as well as numerous applications to real physical problems motivated by researchers and industrialists, we can cite as an example the work of G. Chavent [24]. He is also the author of work on sentinels, dealing in particular with the relationship between sentinels and least squares.

Also the work of O. Nakoulima [63, 64, 65]. We can also refer to the work of the team of J. P. Kernevez [4, 5, 6, 13] for the digital treatment of pollution identification problems in distributed systems, the detection of pollution in an aquifer [13], the determination of missing parameters in a lake and the search for pollution in a river [4]. Since then, several authors have been interested in the numerical aspect of this method. We can refer to the work of A. Traore [72, 73, 74], in which he presents sentinels adapted to the determination of environmental pollution with a numerical study on the diffusion of pollutants in a fluid medium. B. E. Ainseba's thesis [3] includes a chapter on the identification of sources of pollution in a river based on an observation. The sources are decomposed in the form of an unknown amplitude multiplied by a known function of $L^2(0, T)$.

A sentinel must still be insensitive to disturbances, but also insensitive to all the parameters that must be identified except one. For the latter, on the contrary, we impose a sensitivity constraint, we are then led to resolve controllability problems.

The objective of our work is the study of Parabolic systems with incomplete data where we seek to identify the term of pollution. So, this work is organized as follows:

In the first chapter, we give some definitions and properties of the controllable system which is the basis of the sentinel theory. For this we expose the notions of exact, weak, regional and averaged controllability with some necessary theorems. Chapter 2 introduces the problem of distributed systems with missing data and its formulation as a zero controllability problem.

We recall the principle of the sentinel method with an example on the detection of pollution in a wave equation then we study the existence and construction of the sentinel and we talk about the different types of regional sentinel, discreet and weak. In the third chapter, we analyse the problem of identification of the pollution term in a Parabolic system when the dynamics of the state is governed by a parameterized operator. In this way, we introduce a notion of average sentinel. We prove the existence of such sentinels introduced by Lions by solving a problem of null average controllability given by Zuazua. We identify the information for pollution terms by using the average sentinel. Lastly, in the fourth chapter, we analyse the identification of the amount of pollutant discharged problem by each source in a parabolic system when the dynamics of the state is governed by a parameterized unknown operator. In this way, we introduce a notion of average sentinel. The decomposition method is used to solve the equation of this problem, the gradient method is used to calculate the averaged control, the combination of two methods is used to estimate the pollution terms. Numerical example is given to confirm this result.

CHAPTER 1

Controlability of distributed systems

The controllability problem consists of the possibility of transferring the state of a system in a finite time, from an initial state to a desired state. In this section, we aim to introduce the main ideas of the controllability problem and the general resolution method.

1.1 Description of the system

Let Ω be a domain of $\mathcal{R}^N$ with boundary Γ sufficiently regular. For $T > 0$ fixed, we define $Q = \Omega \times [0, T]$, $\sum = \Gamma \times]0, T[$. We consider the system described by the state equation:

$$\begin{cases} \dfrac{\partial y(t,x)}{\partial t} = Ay(t,x) + Bu(t) & \text{in } Q, \\ y(t,\xi) = 0 & \text{on } \Sigma\ , \\ y(0,x) = y_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where B is an operator of $\mathcal{L}(R^n, X = H^1(\Omega))$ and the function u called "control" belongs to the space $U = L^2(0, T; \mathbb{R}^n)$.We assume that operator A generates a semi strongly continuous group $S(t)$, and admits an orthonormal system of eigenfunctions (ω_{ij}) associated with eigenvalues (λ_{ij}). Then the solution of this system is given by

$$y(t) = S(t)y_0 + \int_0^t S(t-s) Bu(s) ds$$

We then assume that the system (1.1) is augmented by the output:

$$z(t) = Cy(t), \tag{1.2}$$

where C is an operator of $\mathcal{L}(H^1(\Omega), O)$ such that O is the observation space.

1.2 Exact and weak controllability

In order to explain the dependence of the solution $y = y(t,x)$ of the problem with respect to the control u, we note $y_u(t) = y(t,x;u)$.

The formulation of the system controllability problem (1.1) is as follows:

Given a time $T > 0$ and a suitable initial condition y_0, does there exist a control u such that the solution $y = y_u(t)$ satisfies the condition

$$y_u(T) = y_d \text{ in } \Omega,$$

where y_d is a desired state chosen a priori.

In other words: Study the existence of a control u which returns the system to the state y_d at time $T > 0$.

Let's now introduce some concepts of exact and weak controllability.

Definition 1.2.1 *The system* (1.1) *is said to be exactly controllable in* $H^1(\Omega)$ *on* $[0,T]$ *if*

$$\forall y_d \in H^1(\Omega), \exists u \in U \textit{ such that}: y_u(T) = y_d. \tag{1.3}$$

From the definition results the following characterization:

Proposition 1.2.1 *The system* (1.1) *is said to be exactly controllable on* $[0,T]$ *iff*

$$\exists c > 0, \|y^*\|_{X^*} \leq c \|B^* S^*(.) y^*\|_{L^2(0,T;U^*)} \qquad \forall y^* \in X^*. \tag{1.4}$$

The adjoint A^* of A generates the semigroup $(S^*(t))_{t\geq 0}$ adjoint of $(S(t))$ which is also strongly continuous on the dual X^* of X, operator B^* is the adjoint of B.
The notion of exact controllability is not adapted and remains very impractical even for a source exerted on the whole domain Ω, this is why we are led to define the notion of weak controllability.

Definition 1.2.2 *The system* (1.1) *is said to be weakly controllable in* $H^1(\Omega)$ *on* $[0,T]$ *if*

$$\forall y_d \in H^1(\Omega), \forall \epsilon > 0, \exists u \in U \text{ such that}: \left\| y_{u(}T) - y_d \right\|_{H^1(\Omega)} \leq \epsilon. \tag{1.5}$$

Remark 1.2.1 *In applications, dynamic systems that are controllable over the entir domain are rare, hence the need to study this concept only over part of the domain. For this, we define the notion of regional controllability.*

1.3 Regional controllability

For regional controllability, we want the state of the system at time T to verify a desired property on a part of the domain.
Let $y_d \in H^1(\omega)$ be a given desired state where ω is a part of Ω. We define the operator

$$\chi_\omega : H^1(\Omega) \to H^1(\omega),$$

and his adjoint given by

$$(\chi_\omega^* y) = \begin{cases} y\,(x) & x \in \omega, \\ 0 & x \in \Omega/\omega, \end{cases}$$

Regional controllability is defined as follows:

Definition 1.3.1 *The system* (1.1) *is said to be exactly regionally controllable on* ω *if*

$$\forall y_d \in H^1(\omega), \exists u \in U \text{ such that}: \chi_\omega y_{u(}\,(T) = y_d. \tag{1.6}$$

Definition 1.3.2 *The system* (1.1) *is said to be weakly regionally controllable on* ω *if*

$$\forall y_d \in H^1(\omega), \forall \epsilon > 0, \exists u \in U \ \ such\ that: \left\| y_u(T) - y_d \right\|_{H^1(\omega)} \leq \epsilon. \qquad 1.7$$

Remark 1.3.1 *1- The definitions above mean that we are only interested in the state reached in the region ω.*
2- The system will also be said to be ω-exactly (resp. ω-weakly) controllable.

We consider

$$H_t : L^2(0, T, \mathcal{R}^n) \to H^1(\omega),$$

the operator definits by:

$$H_t(u) = \int_0^t S\,(t-s)\,ds, \tag{1.8}$$

H denotes the H_T operator. Regional controllability can be characterized by:

Proposition 1.3.1 *1- The system (1.1) is ω- exactly regionally controllable if and only*

$$Im\chi_\omega H = H^1(\omega):$$

2- The system (1.1) is ω- weakly regionally controllable if and only

$$\overline{Im\chi_\omega H} = H^1(\omega) \Longleftrightarrow \ker H^* \chi_\omega^* = \{0\}\,.$$

Remark 1.3.2 *-A system that is exactly (resp. weakly) controllable is exactly (resp. weakly) regionally controllable.*
- A system which is exactly (resp. weakly) regionally controllable on $\omega_1$is exactly (resp. weakly) regionally controllable on ω_2 for all $\omega_2 \subset \omega_1$.

1.4 Observability

Determining the state of a distributed parameter system from measurements is of great importance when one seeks to apply a closed-loop control to such a system. The measurements obtained are expressed by the function of the output

$$z(t) = CS(t)y_0 + CH_t u.$$

This output is the sum of a free regime with y_0 to be determined and of the controlled regime with zero initial state. The system being linear, then, we can study the observation of y_0 by supposing $u = 0$. It is therefore a question of determining y_0, solution of the equation

$$z(t) = CS(t)y_0 = Ky_0 \qquad t \in [0, T], \tag{1.9}$$

where K is a bounded linear operator, then, the adjoint operator is given by

$$K^* z = \int_0^T S^*(t)C^* z(t)dt.$$

Definition 1.4.1 *The system* (1.1) *augmented by the output* (1.2) *is said to be exactly observable on* $[0, T]$ *if*

$$X^* \subset ImK^*$$

Definition 1.4.2 *The system* (1.1) *augmented by the output* (1.2) *is said to be weakly observable on* $[0, T]$ *if*

$$kerK = \{0\}.$$

Definition 1.4.3 *The system* (1.1) *augmented by the output* (1.2) *is said to be* $\omega-$*weakly observable on* $[0, T]$ *if*

$$\ker K\chi_\omega^* = \{0\}.$$

1.5 Optimal control

In this part, we will determine the optimal control to achieve a given target. In the case where the system (1.1) is controllable, there will generally be an infinity of controls that answer the question.

- Among these controls is there one, which is of minimum standard?

- Can we explicitly determine this control according to the various parameters of the problem?

Optimization is used to find the control that gives controllability with a minimum cost given by a function

$$J(u) = \int_0^T \|u(t)\|^2 \, dt,$$

define on the space of controls U.

Let $y_d \in H^1(\Omega)$ be a desired state. We pose the problem of transferring, at a lower cost, the system (1.1) from y_0 to y_d at time T. Thus the question becomes:

Is there a energy control $u \in U$ such that $y(T) = y_d$?
The optimal control problem can be formulated as follows:

$$\begin{cases} \underset{u\in U_{ad}}{min} J(u) = \underset{u\in U_{ad}}{min} \int_0^T \|u(t)\|^2 \, dt, \\ \\ U_{ad} = \{u \in U \ / \ y(T) = y_d\}. \end{cases} \tag{1.10}$$

The objectives of this theory are:

1) Study the existence of $u \in U_{ad}$ which realizes the minimum in (1.10), we then say that u is the optimal control.

2) Give the necessary and sufficient conditions for u to be an optimal control.

3) Obtain properties of the optimal control(s) from (2).
Let's pose

$$G = \{g \in H^1(\Omega), \text{such that } g = 0 \text{ on } \omega\}.$$

$$\overline{G} = \{g \in H^1(\Omega), \text{such that } g = 0 \text{ on } \Omega/\omega\} \tag{1.11}$$

We consider the system

$$\begin{cases} \frac{\partial y}{\partial t}(t,x) & = Ay(t,x) + Bu(t) \quad Q, \\ y(t,\xi) & = 0 \quad \Sigma, \\ y(0,x) & = y_0(x) \quad \Omega, \end{cases} \tag{1.12}$$

The construction method is based on the following three steps:

Step 1 :

For $\varphi_0 \in \overline{G}$, we consider the system:

$$\begin{cases} \frac{\partial \varphi}{\partial t}(t,x) & = -A^*\varphi(t,x) \quad Q, \\ \varphi(t,\xi) & = 0 \quad \Sigma, \\ \varphi(T,x) & = \varphi_0(x) \quad \Omega, \end{cases} \tag{1.13}$$

which admits a unique solution $\varphi \in L^2(0,T;H^1(\Omega)) \cap C^0(0,T;L^2(\Omega))$

Step 2 **:**

Consider the system

$$\begin{cases} \frac{\partial \psi}{\partial t}(t,x) & = A\psi(t,x) + BB^*\varphi(t,x) \quad Q, \\ \psi(t,\xi) & = 0 \quad \Sigma, \\ \psi(0,x) & = y_0(x) \quad \Omega, \end{cases} \tag{1.14}$$

For $\varphi_0 \in \overline{G}$, equation (1.13) gives φ then equation (1.14) gives $\psi(T)$:

Then, we define the operator M by:

$$M\varphi_0 = P(\psi(T)) \text{ where } P = \chi_\omega^*\chi_\omega,$$

M is an affine operator which decomposes as

$$M\varphi_0 = P(\psi_1(T) + \psi_2(T)),$$

where ψ_1 and ψ_2 are solutions of the systems.

$$\begin{cases} \frac{\partial \psi_1}{\partial t}(t,x) & = A\psi_1(t,x) & Q, \\ \psi_1(t,\xi) & = 0 & \Sigma, \\ \psi_1(0,x) & = y_0(x) & \Omega, \end{cases} \tag{1.15}$$

$$\begin{cases} \frac{\partial \psi_2}{\partial t}(t,x) & = A\psi_2(t,x) + BB^*\varphi(t,x) & Q, \\ \psi_2(t,\xi) & = 0 & \Sigma, \\ \psi_2(0,x) & = 0 & \Omega, \end{cases} \tag{1.16}$$

Step 3 **:**

We define the linear, bounded and symmetric operator $\Lambda : \overline{G} \to \overline{G}^*$ by

$$\forall \varphi_0 \in \overline{G} : \Lambda\varphi_0 = P\psi_2(T).$$

With these notations, the problem of regional controllability leads to the resolution of the equation:

$$\Lambda\varphi_0 = P\left(y_d - \psi_1(T)\right). \tag{1.17}$$

Multiplying equation (1.17) by φ_0, we obtain

$$\langle \Lambda\varphi_0, \varphi_0 \rangle = \int_0^T \|B^*\varphi(t)\|^2 \, dt. \tag{1.18}$$

In order to guarantee the existence of the solution to the equation (1.17), we introduce the application

$$\varphi_0 \in \overline{G} \to \int_0^T \|B^*\varphi(t)\|^2 \, dt,$$

which defines a seminorm on $\overline{G}$. Then We have the result:

Proposition 1.5.1 *If the system* (1.12) *is* ω*-weakly controllable, the equation* (1.17) *admits a unique solution* $\varphi_0 \in \overline{G}$*, the control that transfers* (1.12) *into* G *at time* T *is given by*

$$u^*(t) = B^*\varphi(t,x). \tag{1.19}$$

Proof. 1) If the system (1.12) is weakly ω-controllable, then the application

$$\varphi_0 \to \|\varphi_0\|^2 = \int_0^T \|B^*\varphi(t)\|^2 \, dt,$$

defines a norm on $\overline{G}$. Indeed

$$\|\varphi_0\|_{\overline{G}}^2 = 0 \Rightarrow B^*S^*(T-t)\varphi_0 = 0 \qquad \forall t \in [0,T].$$

The system (1.12) is ω-weakly controllable, so

$$kerH^*\chi^* = \{0\}.$$

As a result

$$B^*S^*(T-t)\varphi_0 = 0 \Rightarrow \varphi_0 = 0,$$

it results that we have a norm on $\overline{G}$.

The operator Λ is an isomorphism from $\overline{G}$ to $\overline{G^*}$ with

$$\langle \Lambda\varphi_0, \varphi_0 \rangle = \int_0^T \|B^*\varphi(t,x)\|^2 \, dt = \|\varphi_0\|_{\overline{G}}^2,$$

,

hence the existence of the solution of equation (1.17).

2) Let us show that the control u^* given by (1.17) minimizes the cost function

$$J(u) = \frac{1}{2}\int_0^T \|u(t)\|^2 \, dt.$$

As the function J is quadratic and therefore strictly convex, it is sufficient to verify that

$$J'(u^*)(v-u) \geq 0.$$

We have

$$J'(u^*)(v-u^*) = \int_0^T u^*(v-u^*)\,dt = \int_0^T B^*\varphi(t,x)(v-u^*)\,dt \qquad \forall v \in U,$$

with

$$\begin{aligned}\langle B^*\varphi(t,x), \upsilon - u^*\rangle &= \langle \varphi(T), y_\upsilon(T) - y_{u^*}(T)\rangle - \langle \varphi(0), y_\upsilon(0) - y_{u^*}(0)\rangle \\ &\quad - \int_\Sigma (y_\upsilon - y_{u^*}) \frac{\partial \varphi}{\partial \nu_{A^*}} - \varphi\left(\frac{\partial y_\upsilon}{\partial \nu_A} - \frac{\partial y_{u^*}}{\partial \nu_A}\right) d\Sigma \\ &= \langle \varphi_0, y_\upsilon(T) - y_{u^*}(T)\rangle = J'(u^*)(\upsilon - u^*),\end{aligned}$$

and as we have

$$\begin{aligned}\chi_\omega y_\upsilon(T) - \chi_\omega y_{u^*}(T) &= \chi_\omega y_\upsilon(T) - y_d + y_d - \chi_\omega y_{u^*}(T) = 0 \\ &\Rightarrow \langle \varphi_0, y_\upsilon(T) - y_{u^*}(T)\rangle = 0,\end{aligned}$$

therefore

$$J'(u^*)(\upsilon - u^*) = 0.$$

which establishes the optimality of the control u^*. ■

1.6 Regional control and penalization

We assume that the non-empty set U_{ad} therefore the system (1.12) is exactly regionally controllable on U_{ad}. We want to solve the following optimization problem:

$$\begin{cases} \min\limits_{u\in U_{ad}} J(u) = \min\limits_{u\in U_{ad}} \int_0^T \|u(t)\|^2\, dt \\ U_{ad} = \{u \in U,\ y(T) - y_d \in G\}. \end{cases} \tag{1.20}$$

For all $\epsilon > 0$, consider the probleof Penalization

$$\begin{cases} \min\limits_{(u,y)\in C} J_\epsilon(u,y) \\ J_\epsilon(u,y) = \left(\int_0^T \|u(t)\|^2\, dt + \frac{1}{2\epsilon}\int_0^T \|y'\prime(t) - Ay(t) - Bu(t)\|^2\, dt\right), \end{cases} \tag{1.21}$$

where C is the set of pairs (u, y) satisfying

$$\left\{\begin{array}{lll} y'(t) - Ay(t) - Bu(t) & \in L^2(0,T;X) & \\ y(0) & = y_0 & u \in U, \\ y(T) - y_d & \in G & \end{array}\right\} \tag{1.22}$$

So we have the following result:

Proposition 1.6.1 *For all $\epsilon > 0$, the problem (1.21) admits a unique solution which we denote (u, y_ϵ). The sequence $((u_\epsilon, y_\epsilon))$ converges weakly to $(u^*; y^*)$ when ϵ tends to zero. What's more u^* is the solution of problem (1.20) given by*

$$u^*(t) = B^*p(t),$$

where $p(t)$ and $y^*(t)$ are solutions of the optimality system

$$\left\{\begin{array}{lll} y'(t) & = Ay(t) + Bu(t) & \text{on } [0,T] \\ y(0) & = y_0 & \\ p'(t) + A^*p(t) & = 0 & \text{on } [0,T] \\ p(T) & \in G^* & \end{array}\right. \tag{1.23}$$

Proof. For the demonstration see [76]. ∎

1.7 Unique extension theorems

The H.U.M method or Hilbert Uniqueness Method, was introduced by J. L. Lions [41], for the study of the controllability of the wave equation. It was then applied to a large possible problem where we construct the space of achievable states, this construction is based on a single extension (or continuation) theorem: Holmgren for waves, Mizohata or Saut-Scheurer for heat. These theorems are the basis of most methods for solving control and identification problems. We will first present Cauchy's uniqueness theorem for the linear heat equation.

Consider the solution of the parabolic equation

$$\frac{\partial y}{\partial t} + Ay = 0 \text{ in } Q \tag{1.24}$$

where A is an elliptic operator of order 2 on which the conditions will be specified for each theorem. In any case, the open Ω must be connected and $Q = \Omega \times]0,T[$.

Theorem 1.7.1 *(Cauchy uniqueness)*

Let $\Gamma_0 \subset \Gamma$ be a non-empty part of the boundary, we denote $\Sigma_0 = \Gamma_0 \times \,]0, T[$, let $y(x,t)$ verify

$$\begin{cases} \frac{\partial y}{\partial t} + Ay & = 0 \quad in\ Q, \\ y & = 0 \quad on\ \Sigma_0, \\ \frac{\partial y}{\partial v} & = 0 \quad on\ \Sigma_0, \end{cases} \tag{1.25}$$

then y is identically zero in Q. [See [69]].

The two following theorems intervene the notion of the horizontal component in an open space-time.

Definition 1.7.1 *(Horizontal component)*

Let O be an open set included in Q, we say that a point $p \in Q$ belongs to the horizontal component of O if there is a horizontal curve joining p to O, that is to say to a line whose points all have the same coordinate in time.

Now, we can state the following theorem of S. Mizohata

Theorem 1.7.2 *S. Mizohata*

Let Ω a connected open set of $\mathbb{R}^p$ and A be a second-order elliptic operator whose coefficients belong to $C^\infty(Q)$. Let y be the solution of (1.25) *for operator A and O an open set included in Q. Any solution of* (1.25) *which vanishes in O vanishes in the horizontal component of O.*

The proof of S. Mizohata (see [61]) does not lead in an obvious way to a weakening of the regularity of the coefficients of the operator. However, it is important from a practical point of view to work with irregular coefficients. The authors give the example where the parabolic equation is obtained by linearization of a non-linear operator around a solution which is not necessarily regular. The main result is the following:

Theorem 1.7.3 *(J. C. Saut et B. Scheurer)*

Let Ω be a connected open set of $\mathcal{R}^p$, A as a second-order elliptic operator defined by

$$Au = \sum_{i,j=1}^{p} a_{ij}(t,x)\frac{\partial}{\partial x_i}.\frac{\partial}{\partial x_j}u + \sum_{i=1}^{p} b_i(t,x)\frac{\partial}{\partial x_i}u + c(t,x)\,u, \tag{1.26}$$

where the coefficients of A satisfy

$$\begin{array}{lcll} a_{ij} & \in & C^1(Q) & 1 \leq i,j \leq p, \\ b_i & \in & L^{\infty}_{Loc}(Q) & 1 \leq i \leq p, \\ c & \in & L^{\infty}\left(0,T; L^1_{Loc}(Q)\ \right) & \end{array} \tag{1.27}$$

Suppose the solution to (1.25) satisfies $y \in L^2\left(0,T; H^2_{Loc}(\Omega)\ \right)$ and that it vanishes in an open set $O \subset Q$. Then y vanishes in the horizontal component of O.

1.8 Averaged control

In certain distributed systems, their parameters are not fully known and to control them, In that cas, we seek robust control strategies that are unaffected by the presence of unknown parameters, in order to effectively manage these types of systems. Zuazua on 2014 solved this problem with a new notion called "average control" which presents the average of the relative state to the unknown parameter instead of having control of the state itself.

1.8.1 Averaged controllability of distributed systems

We take Ω be a bounded domain in $\mathbb{R}^n$, with smooth boundary Γ, denote by $Q = \Omega \times (0,T)$ the space time cylinder where the equation holds, $\Sigma = \Gamma \times (0,T)$ and ω an open non-empty subset of Ω. We will assume that the parameter $\sigma \in (0,1)$, consider the following controlled heat equation depending on a parameter:

$$\left\{\begin{array}{lll} \frac{\partial y}{\partial t} - \operatorname{div}\left(a\left(x,\sigma\right)\nabla y\right) & = v\chi_{\omega} & \text{in } Q, \\ y & = 0 & \text{on } \Sigma\ , \\ y\left(x,0\right) & = y_0\left(x\right) & \text{in } \Omega, \end{array}\right. \tag{1.28}$$

The diffusion coefficient, denoted by $a(x,\sigma)$ supposed to be measurable in x, and depends continuously on the uncertainty parameter σ. We make the assumption that $y_0 \in L^2\ (\Omega)$ and $v = v(x,t) \in L^2\ (Q)$, then (1.28) has a unique solution (Lions, 1971).

$$y = y\left(x,t;\sigma\right) \in C\left(\left[0,T\right], L^2(\Omega)\right) \cap L^2(0,T;H_0^1(\Omega)), \text{for every } \sigma \in (0,1).$$

We consider the problem of average zero controllability below:

Find a control $u \in L^2(Q)$ s.t. y solution of (1.28) and verifies:

$$\int_0^1 y\left(x,t;\sigma\right) d\sigma = 0. \tag{1.29}$$

We can show that the average zero controllability (1.28) (1.29) is equivalent to an averaged observability inequality for the following system (see Zuazua, 2014)

$$\begin{cases} -\frac{\partial\varphi}{\partial t} - \operatorname{div}\left(a\left(x,\sigma\right)\nabla\varphi\right) & = 0 & \text{in } Q, \\ \varphi & = 0 & \text{on } \Sigma\ , \\ \varphi\left(x,T\right) & = \varphi_0\left(x\right) & \text{in } \Omega, \end{cases} \tag{1.30}$$

the required observability inequality is:

$$\left\| \int_0^1 \varphi\left(x,0;\sigma\right) d\sigma \right\|_{L^2(\Omega)}^2 \leq C \int_0^T \int_\omega \left| \int_0^1 \varphi\left(x,0;\sigma\right) d\sigma \right|^2 dxdt, \tag{1.31}$$

the constant C is independent of φ. To get an inequality of the form (1.31) we need the so-called Carleman inequalities (Fursikov and Imanuvilov, 1996), it's a very challenging issue.

1.8.2 Optimal averaged control for distributed systems depending upon a unknown parameter

Here, we are talking about the optimal average control of several types of distributed systems as a function of an uncertainty parameter which gives an optimality system characterizing the average optimal control as in classic distributed systems, for more examples and details to which we refer (Hafdallah & Ayadi, 2016).

Optimal Averaged control for elliptic distributed systems depending upon a unknown parameter

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ and ω an open non-empty subset of Ω. We consider the following elliptic equation controlled as a function of a parameter:

$$\begin{cases} -\operatorname{div}\left(a\left(x,\sigma\right)\nabla y\right) & =\upsilon\chi_{\omega} \quad \text{in } Q, \\ y & =0 \quad \text{on } \Gamma\ , \end{cases} \tag{1.32}$$

the coefficient of diffusion $a\left(x,\sigma\right)$ is measurable in x and depend on in a measurable manner, $\upsilon\in L^{2}\left(\Omega\right)$. For every $\sigma\in(0,1)$ the equation (1.32) has in $H_{0}^{1}\left(\Omega\right)$ a unique solution $y=y(x,\sigma)$ in $H_{0}^{1}\left(\Omega\right)$.

To check the average of the state in relation to σ.

$$z\left(x\right)=\int_{0}^{1}y\left(x,\sigma\right)d\sigma.$$

Associate to (1.32) the following quadratic cost function:

$$J(\upsilon)=\left\Vert z-z_{d}\right\Vert_{L^{2}(\Omega)}^{2}+N\left\Vert \upsilon\right\Vert_{L^{2}(\Omega)}^{2},\qquad \upsilon\in L^{2}\left(\Omega\right), \tag{1.33}$$

$z_{d}\in L^{2}\left(\Omega\right)$ and $N>0$. Then, we want to solve

$$\underset{\upsilon\in L^{2}(\Omega)}{inf}J(\upsilon)\ \ s.t.\ (1.32). \tag{1.34}$$

Theorem 1.8.1 *The unique averaged optimal control u solution to for* $(1.32)-(1.34)$ *is characterized by:*

$$\begin{cases} -\operatorname{div}\left(a\left(x,\sigma\right)\nabla y\left(u\right)\right) & =\int_{0}^{1}\varphi\left(u,\sigma\right)d\sigma\chi_{\omega} \\ -\operatorname{div}\left(a\left(x,\sigma\right)\nabla\varphi\right) & =\int_{0}^{1}y\left(u,\sigma\right)d\sigma-z_{d} \quad in\ \Omega, \\ y\left(u\right)=0 & \varphi=0 \quad on\ \Gamma, \end{cases}$$

with the variational inequality:

$$\int_{0}^{1}\varphi\left(u,\sigma\right)d\sigma+Nu=0\ \ in\ \Omega.$$

Proof. The cost fonction $J:U_{ad}\rightarrow\mathcal{R}$ is a lower semi-continuous function, strictly convex, and coercive. Hence there is a unique admissible control u solution to $(1.32)-(1.34)$. A First order Euler condition for gives

$$\int_{\Omega}\left(z\left(u\right)-z_{d}\right)\left(z\left(\upsilon\right)-z\left(u\right)\right)dx+N\int_{\Omega}u\left(\upsilon-u\right)dx=0,\ \forall\upsilon\in L^{2}\left(\Omega\right). \tag{1.35}$$

where $z(\upsilon) = \int_0^1 y(\nu,\sigma)\, d\sigma$, such that $y(\nu,\sigma)$ is the unique solution of:

$$\begin{cases} -\operatorname{div}(a(x,\sigma)\nabla y) & = \upsilon\chi_\omega & \text{in } \Omega, \\ y & = 0 & \text{on } \Gamma. \end{cases}$$

Let φ be the σ-dependent adjoint state given by

$$\begin{cases} -\operatorname{div}(a(x,\sigma)\nabla\varphi) & = z(u) - z_d & \text{in } \Omega, \\ \varphi & = 0 & \text{on } \Gamma. \end{cases}$$

Now, let is rewrite first order Euler condition (1.35) using Green formula

$$\begin{aligned} \int_\Omega (z(u) - z_d)(z(\upsilon) - z(u))\, dx &= \int_\Omega \int_0^1 (z(u) - z_d)(y(\nu,\sigma) - y(u,\sigma))\, d\sigma dx \\ &= \int_\Omega \int_0^1 \operatorname{div}(a(x,\sigma)\nabla\varphi)(y(\nu,\sigma) - y(u,\sigma))\, d\sigma dx \\ &= \int_\Omega \int_0^1 \varphi \operatorname{div}(a(x,\sigma))\nabla(y(\nu,\sigma) - y(u,\sigma))\, d\sigma dx \\ &= \int_\Omega \int_0^1 \varphi(\upsilon - u)\chi_\omega d\sigma dx. \end{aligned}$$

Then (1.35) is equivalent to

$$\int_\Omega \int_0^1 (\varphi(\upsilon - u)\chi_\omega + Nu)(\upsilon - u)\, d\sigma dx = 0, \forall \upsilon \in L^2(\Omega).$$

■

Optimal Averaged control for parabolic distributed systems depending upon an unknown parameter

Consider the following abstract second order parabolic equation :

$$\begin{cases} \frac{\partial y}{\partial t} + A(x,\sigma)\, y & = f + B(\sigma)\, \upsilon & \text{in } Q, \\ y(x,0) & = y_0(x) & \text{in } \Omega. \end{cases} \tag{1.36}$$

Let $V \subset H$ be Hilbert spaces, $f \in L^2(0,T;V')$, $\upsilon \in U_{ad} \subset L^2(0,T;V)$, where U_{ad} is a non-empty closed convex set of admissible controls, $B(\sigma) \in \mathcal{L}(U_{ad}; L^2(0,T;V'))$ is the control operator supposed also depending on σ and $y_0 \in V$ is initial state. Then, for every $\sigma \in (0,1)$ the equation (1.36) has a unique solution $y \in L^2(0,T;V)$, (Lions, 1971).

Let $z(x,t) = \int_0^1 y(x,t;\sigma)\, d\sigma \in L^2(0,T;V)$ be the averaged state with respect to σ and $z_d \in L^2(0,T;V)$. We want to solve the following quadratic optimal control problem

$$\inf_{\upsilon \in U_{ad}} J(\upsilon) \text{ with } J(\upsilon) = \|z - z_d\|^2_{L^2(0,T;V)} + N\|\upsilon\|^2_{L^2(0,T;V)} \tag{1.37}$$

where $N > 0$. Then,we have the folowing theorem:

Theorem 1.8.2 *The averaged optimal control u for* (1.36)-(1.37) *is unique and it's characterized by*

$$\begin{cases} \frac{\partial y}{\partial t}(u) + A(x,\nu)\, y & = f + B(\sigma)\, u, \\ -\frac{\partial \varphi}{\partial t}(u) + A^*(x,\nu)\,\varphi & = \int_0^1 y(x,t;u;\sigma)\,\chi_\omega d\sigma - z_d \quad in\ Q, \\ y(u)(x,0) & = y_0(x)\,, \varphi(x,T) = \quad in\ \Omega. \end{cases} \tag{1.38}$$

with the variational inequality

$$\int_0^T \int_0^1 (B^*(\sigma) + Nu, v - u)_V \, d\sigma dt \geq 0, \forall v \in U_{ad}. \tag{1.39}$$

Proof. Existence and uniqueness task follows by lower semi-continuity, strict convexity, and coercitivity of objective function J .

A first order Euler condition for (1.37) gives

$$\int_\Omega (z(u) - z_d), (z(v) - z(u))_V \, dx \quad + N \int_0^T (u, v-u)_V \, dt \geq 0 \forall v \in U_{ad} \ . \tag{1.40}$$

where $z(v) = \int_0^1 y(v,\sigma)\, d\sigma$, such that $y(v,\sigma)$ is the unique solution of

$$\begin{cases} \frac{\partial y}{\partial t} + A(x,\sigma)\, y & = B(\sigma)\, v \quad \text{in } Q, \\ y(x,0) & = 0 \quad \text{in } \Omega. \end{cases}$$

Let φ be the σ-dependent adjoint state given by

$$\begin{cases} -\frac{\partial \varphi}{\partial t} + A^*(x,\sigma)\,\varphi & = z(u) - z_d \quad \text{in } Q, \\ \varphi(x,T) & = 0 \quad \text{in } \Omega. \end{cases}$$

where $\varphi \in L^2(0,T;V)$ (Lions, 1971).

Now, let's rewrite first order Euler condition (1.40) as

$$\begin{aligned} \int_0^T (z(u) - z_d, z(v) - z(u))_V \, dx &= \int_0^T \int_0^1 (z(u) - z_d, y(\nu,\sigma) - y(u,\sigma))_V \, d\sigma dt \\ &= \int_0^T \int_0^1 \left(-\frac{\partial \varphi}{\partial t} + A^*(x,\sigma)\,\varphi, y(\nu,\sigma) - y(u,\sigma) \right)_V d\sigma dt \\ &= \int_0^T \int_0^1 \left(\varphi, \left(\frac{d}{dt} + A(x,\sigma) \right) (y(\nu,\sigma) - y(u,\sigma)) \right)_V d\sigma dt \\ &= \int_0^T \int_0^1 (\varphi, B(\sigma)(v-u))_V \, d\sigma dt \ , \end{aligned}$$

and then, (1.40) will be written as

$$\int_0^T \int_0^1 (B^*(\sigma)\varphi + Nu)(\upsilon - u)_V \, d\sigma dt, \qquad \forall u \in U_{ad} \quad .$$

■

Optimal averaged control for hyperbolic distributed systems depending upon an unknown parameter

Consider the following abstract hyperbolic problem:

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} + A(x,\sigma)\, y(u) & = f + B(\sigma)\upsilon & \text{in } Q, \\ y(x,0) & = y_0(x), \frac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega. \end{cases} \tag{1.41}$$

With V, H be Hilbert spaces and V is separable and dense in H, $f \in L^2(0,T;H)$, $y_0 \in V$, $y_1 \in H$, $B(\sigma) \in \mathcal{L}(U_{ad}, L^2(0,T;V'))$ and $U_{ad} \subset L^2(0,T;H)$. Then, for all $\sigma \in (0,1)$ the equation (1.41) has a unique solution in $L^2(0,T;V)$ (Lions & Magenes, 1972). We are interested to the optimal control problem (1.41) (1.37), we have the following theorem:

Theorem 1.8.3 *The averaged optimal control u solution for* (1.41) (1.37) *is unique and it's characterized by:*

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} + A(x,\sigma)\, y(u) & = f + B(\sigma)\, u & \\ \frac{\partial^2 \varphi}{\partial t^2} + A^*(x,\sigma)\,\varphi & = \int_0^1 y(x,t;u;\sigma)\, d\sigma - z_d & \textit{in } Q, \\ y(u)(x,0) = y_0(x), & \frac{\partial y(u)}{\partial t}(x,0) = y_1(x) & \\ \varphi(T) = 0, & \frac{\partial \varphi}{\partial t}(T) = 0 & \textit{in } \Omega, \end{cases} \tag{1.42}$$

with the variational inequality

$$\int_0^T \int_0^1 (B^*(\sigma)\varphi + Nu, \upsilon - u)_H \, d\upsilon dt \geq 0, \forall \upsilon \in U_{ad}. \tag{1.43}$$

Proof. An optimality condition writes:

$$\int_0^T (z(u) - z_d, z(\upsilon) - z(u))_V \, dt + N \int_0^T (u, \upsilon - u)_V \, dt \quad \geq 0 \qquad \forall \upsilon \in U_{ad} \tag{1.44}$$

with $z(\upsilon) = \int_0^1 y(\nu,\sigma)\, d\sigma$ which is a solution of the system:

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} + A(x,\sigma)\, y & = B(\sigma)\,\upsilon & \text{in } Q, \\ y(x,0) = 0 & \frac{\partial y}{\partial t}(x,0) = 0 & \text{in } \Omega. \end{cases}$$

We introduce the σ-dependent adjoint state by:

$$\begin{cases} \frac{\partial^2\varphi}{\partial t^2} + A^*(x,\upsilon)\,\varphi & = z(u) - z_d \quad \text{in } Q, \\ \varphi(x,T) = 0 & \frac{\partial\varphi}{\partial t}(x,T) = 0 \quad \text{in } \Omega. \end{cases}$$

Now, let's rewrite first order Euler(1.44) condition as

$$\begin{aligned} \int_0^T (z(u) - z_d, z(\upsilon) - z(u))_V\, dt & = \int_0^T \int_0^1 (z(u) - z_d, y(\nu,\sigma) - y(u,\sigma))_V\, d\sigma dt \\ & = \int_0^T \int_0^1 \left(\frac{\partial^2\varphi}{\partial t^2} + A^*(x,\sigma)\,\varphi, y(\nu,\sigma) - y(u,\sigma)\right)_V d\sigma dt \\ & = \int_0^T \int_0^1 (\varphi, B(\sigma)(\upsilon - u))_H\, d\sigma dt, \end{aligned}$$

We obtain the value of (1.43). ■

CHAPTER 2

Distributed systems with missing data

The distributed systems considered in this chapter are systems described by partial differential equations of evolution defined in a domain of $\mathcal{R}^n$ ($n = 1, 2, 3$ in applications) and for time t in an interval $(0, T)$, we have to add initial conditions and boundary conditions. We are interested in phenomena which are governed by evolution equations with missing data. The model we have is incomplete, in the sense that we do not know the initial data or certain boundary data.

2.1 Pollution detection problem

We are interested in this section in the detection of pollution in a fluid environment (Lake, River). We will assume that the pollution is due to the presence of chemical compounds (Nitrate, Lead,...) from external or sedimentary discharges. Sources of pollution disperse toxic excreta into the water over time and the study area includes obstacles (islands of land, trees,...). The mathematical transport model of a chemical substance of concentration $y(t, x)$ dissolved in a fluid leads to a convection-diffusion-reaction type equation [74]. Taking into account the main physico-chemical properties related to fluids, the modeling of the transport of a chemical compound leads to the consideration of the following terms:

- A diffusion term

$$k div(a(x)\nabla y(x, t)),$$

where k is the diffusion constant. It is a fundamental property of fluids that consists in dispersing molecules randomly throughout the domain. It is this ability that allows water to uniform color in a basin and air to maintain an odor in a closed room. The term $a(x)$ designates the transmissivity in the medium (in a homogeneous medium (water, air,...) $a(x)$ is a constant). In the case of a lake, the diffusion is given by:

$$k.\Delta y(x,t)$$

and in that of a river it is equivalent to

$$D_1 \frac{\partial^2 y}{\partial x^2}(x,t) + D_2 \frac{\partial^2 y}{\partial y^2}(x,t), \quad k, D_1, D_2 \text{ are constants.}$$

- A transport or convection term $u.\nabla y(x,t)$.

  $\overrightarrow{u}.\overrightarrow{\nabla} y(x,t)$ where $\overrightarrow{u}$ denotes the velocity field of the ‡uid.

- A reaction term R that reflects the chemical and biochemical interactions in the liquid.

- The source of pollution is described by a function $f(x,t)$, it gives rise to the polluting substances discharged into the fluid. At this level, two considerations on the sources of pollution deserve to be made for a better consideration of the polluting species: these are the distributed source terms that we will denote by $\zeta(x,t)$ and the punctual source terms at point i with coordinate x_i that we will designate by ib $\lambda_i \widehat{\zeta}_i(t)$. $\delta(x-x_i)$ where $\delta(x-x_i)$ is the Dirac function associated with point x_i. The general formulation of the source is given by:

  $$f(x,t) = \zeta(x,t) + \sum_i \lambda_i \widehat{\zeta}_i(t).\delta(x-x_i).$$

  It is assumed that polluted waters contain a wide variety of pathogenic bacteria or viruses. After a discharge is dumped in the water, the concentration of bacteria or viruses can decrease very quickly due to certain conditions (lack of nutrients, drop in temperature, sunlight,...). We then denote by y_i the concentration of a species i: The reaction term R_i is given by

  $$R_i = -k_i y_i(x,t),$$

where k_i is a kinetic constant. On the other hand, the source term can be expressed in the form

$$f_i(x,t) = \zeta(x,t) + \sum_j \lambda_j \widehat{\zeta}_i(t).\delta(x - P_i),$$

where j is the number of internal pollution sources (detritus, metals and others deposited at the bottom of rivers and lakes),λ_j is the pollution rate of the jth source, $\zeta_i(t)$ is the partial density of species i for the j th source, $\delta(x - P_i)$ designates the Dirac measure at the point P_i. We are then reduced to solving a system of equations of the form:

$$\begin{cases} \frac{\partial y_i(x,t)}{\partial t} + u.\nabla y_i(x,t) - K.\Delta y_i(x,t) = \\ = -k_i y_i(x,t) + \sum_{j=1}^{n} \lambda_j \widehat{\zeta}_i\,(t)\,\delta\,(x - P_i)\,, \quad i = 1, 2, 3, ... \end{cases}$$

In the analysis of the concentration y, we are confronted with two types of boundary conditions. To illustrate these conditions, let us subdivide the boundary of the study domain into two disjoint parts, $\partial\Omega = \Gamma_1 \cup \Gamma_1$ such that

$$y|_{\Gamma_1} = g\,(x,t)\,,$$

$$\left.\frac{\partial y}{\partial n}\right|_{\Gamma_2} = h\,(x,t)\,,$$

the edge Γ_1 is supposed to continuously diffuse a pollution discharge into the fluid, this is the case of a factory which discharges its residues into a nearby lake through an opening channel Γ_1. The edge Γ_1 can in turn be subdivided into several parts depending on the number of sources of external pollution.

The Neumann condition characterizes the exchange of concentration between the fluid and the exterior. This condition is related to the porosity of the earth. In the study of surface water (Lake, river) we can assume as an approximation, in this case, for $h = 0$, this implies that no concentration crosses the edge Γ_2, the earth is at this level impermeable. We need an initial condition

$$y(x,0) = y_0(x).$$

This condition evaluates the amount of concentration present at the start of the experiment. It is important in solving evolutionary type problems.

We summarize the dispersion of a polluting substance in a fluid by a parabolic equation of the form:

$$\begin{cases} \frac{\partial y(x,t)}{\partial t} + k\,\mathrm{div}\left(a\left(x\right)\nabla y\left(x,t\right)\right) + F\left(y,\nabla y\right) & = f\left(x,t\right) & (x,t)\in\Omega\times\left]0,T\right[\\ y|_{\Gamma_1} & = g\left(x,t\right) & (x,t)\in\Gamma_1\times\left]0,T\right[\\ \frac{\partial y}{\partial n}\Big|_{\Gamma_2} & = h\left(x,t\right) & (x,t)\in\Gamma_2\times\left]0,T\right[\\ y(x,0) & = y_0(x) & x\in\Omega, \end{cases}$$

- $\Omega \in \mathcal{R}^n$, $n = 2$ or 3 represents the field of study,
- $]0, T[$ is the study time interval $T > 0$,
- F is a nonlinear form given by

$$F\left(y,\nabla y\right) = \overrightarrow{u}.\nabla y\left(x,t\right) - \lambda y\left(x,t\right) + \mu\left|y\right|^p\left(x,t\right),$$

where λ, μ and p are real constants and $\overrightarrow{u}$ the water flow velocity.

We place ourselves here in the case of systems with incomplete data, that is to say that one of the following information:

- the coefficient k of diffusion,
- the source function $f(x,t)$,
- the initial condition $y_0(x)$,
- the boundary conditions $g(t)$ and $h(t)$,

is unknown or contains unknown parameters in its structure. We will assume that the structures of the functions f and y_0 are unknown and given in the form:

$$f\left(x,t\right) = \sum_j \lambda_j \hat{\zeta}_i\left(t\right),$$

$$y_0\left(x,t\right)=\sum_j \tau_j g_i\left(t\right).$$

The parametersλ_j and τ_j are unknown and represent pollution levels. The functions $\hat{\zeta}_j$ i and $\widehat{g}_i$ are known and respectively designate the production density functions at the source and at the initial time.

The search for this information leads us naturally to a problem of the opposite type. In solving these problems, it is necessary to have the measured data of the state y. We will note in the following y_{obs} these experimental data and then evaluate the state y according to the parameters sought.

2.2 Observation spaces

We will assume that the measurements taken y_{obs} are made in a time interval $[0,T]$, just like the calculations of the state $y(\lambda,\tau)$ is in an observation domain $O\subset\Omega$.

A usual formulation is to consider known the action on the state $y(x,t)$ of an operator linear with values in a suitable space H, i.e. defines an observation operator B allowing the searched parameters to be associated with the observed measurements $B:U\rightarrow H$. So we observe the state on a supposed domain O called observatory, and in a time interval $(0;T)$.The observatory O can be internal distributed ($O\subset\Omega$), or boundary ($O\subset\Gamma$).

Once the observed data have been obtained, the problem we are interested in this work is the following:

(q): **From the experimental measurements of the state y of the previous system, is it possible to identify the source function and / or the initial function taking into account the errors on the easurements?**

We will begin to answer this question in the general case.

2.3 Position of the problem

To illustrate our approach, we assume that the state of the system is described by y. The general structure of the partial differential equation which gosverns the state y of the studied problem is assumed to be known in the form:

$$(S)\begin{cases} \frac{\partial y}{\partial t}+F(y) &= \text{source term} \quad \text{in } \Omega\times(0,T)\,, \\ y(0) &= y_0 \quad \text{in } \Omega, \end{cases} \tag{2,1}$$

where F is a non-linear function and y_0 the initial state.
In order for the state y of the considered system to be entirely defined, it is necessary to know:

- the coefficients of the operator F, and the possible nonlinearity structure,
- the source terms,
- the initial condition,
- the boundary conditions, and
- the field of study

Which is usually not the case.
If at least one of the above information is unknown or partially known, we say that the system (P) has incomplete data. Such problems are encountered in many situations, in biomedical sciences, in meteorology, in oceanography, ..., where the initial conditions are not completely known.

2.4 Missing terms and pollution terms

Let F be a second-order elliptic operator. We assume that the first equation of the system (S) is written in the form:

$$\frac{\partial y}{\partial t}+F(y) \quad =\zeta+\lambda\widehat{\zeta} \quad \Omega\times(0,T)\,,$$

with ζ is given in a suitable space Y and $\widehat{\zeta}$ remains in the unit ball of Y and λ is a small real parameter with $\lambda\widehat{\zeta}$ is not known. We suppose that the coefficients of F and the open set are known but the initial data are incomplete. If we denote by $y(0)$ the initial condition is expressed in the form

$$y(0) = y_0 + \tau\widehat{y_0},$$

where y_0 is given and $\widehat{y_0}$ remains in the unit ball of a Hilbert or Banach space with τ small real, and we assume that the boundary conditions are known.

Our objective is to give a method allowing to obtain information on $\lambda\widehat{\zeta}$ which is not affected by the variations of the initial data around y_0. We establish thus a distinction between the term $\lambda\widehat{\zeta}$ which is said to be "of pollution" and the term $\tau\widehat{y_0}$ which is said to be "missing" which we do not seek to identify. To hope to be able to obtain some information, it is necessary to observe y: So, the problem consists in observing the state y on an accessible part of the domain and to have experimental measurements to estimate the missing data.

2.5 System observation

In a system with partially known data such as the one considered in (P), it is natural to want to reconstruct all or part of the unknown data, this is obviously impossible if nothing is observed from the system studied. Let H be the space of observed data, we cite two types of observations:

1/ The open set O can consist of several components therefore the observations are made at the points O_i, $i = 1, 2, 3$,..., and pollution sources are generated at points s_i, such a case has been studied in [32].
2/ We can consider an observatory $O \subset \Omega$. The observed data is continuous with respect to time and space. Such observations can be made by means of a ship, for example the case of an observatory boat and being an ocean or a lake,...

Moreover, one can have discontinuous observations in time.

We observe the state of the system "y" on O during the time interval $[0,T]$, so theoretically, we will have

$$y(x,t) = y_{obs} \text{ on } O \times (0,T), \tag{2.2}$$

where y_{obs} is a known measure.

Of course, experimental measurements can be in‡uenced by disturbances called noise. The noises can be due to errors on the measuring instruments or again to errors on the approximation of the equations. To account for these errors, the observation operator is defined as follows:

$$y_{obs} = m_0 + \sum_{i=1}^{n} \beta_i m_i \tag{2.3}$$

where the functions $(m_0, m_1, ...m_n)$ are given and the β_i are unknown parameters, represent the noise terms.

Remark 2.5.1 *1/ The results presented here will be in the case of an internal observatory. 2/ For dissipative systems, the open set O can be, at least theoretically, be arbitrarily small.*

The problem of identifying the reconstitution of certain unknown parameters of our system naturally leads to the notion of "idenfiability" which we are going to define in the most general way possible.

Definition 2.5.1 *We consider a system whose state denoted y depends on a vector of parameters υ, Let C be an observation operator acting on y. We define the operator $B : E \rightarrow F$ such that E is the data space and F is the measurement space. We say that B is identifiable from the observation $z = Cy$ if the application B is injective.*

To solve an identification problem, a very popular technique is the "least squares" method. In addition, at the end of the eigties, a new method was introduced: "the sentinel method". We start with a presentation of the first method.

2.6 Least squares method

It was introduced in 1795 by Gauss and Legendre for solving inverse problems. As early as 1805, Legendre presented his article "new methods for determining the orbits of comets" based on the method of least squares. Since then, this method has remained the most popular parameter identification technique for both ordinary differential equations (ODE) and partial differential equations (PDE).

Suppose that υ represents the vector of the parameters sought. The least squares technique consists in minimizing the squared distance between the observed values y_{obs} and the calculated values $y(\upsilon)$ for the v traversing the parameter space U: Thus, the identification problem comes down to solving the problem of optimization.

$$\underset{\upsilon\in U}{min} \left\| y\left(\upsilon\right) - y_{obs} \right\|^{2}, \tag{2.4}$$

where $y\left(\upsilon\right)$ is the solution of the system (P).

In the least squares method, all unknown parameters play the same role. No difference is made between the parameters (in the source terms and in the initial terms).

There are therefore risks of not being able to clearly separate the roles of each.

Moreover, the available data y_{obs} may be insufficient compared to the number of parameters sought, which leads to an infinite number of possible solutions. We have, in this case, a problem of uniqueness of the solution, also for a set of data taken from the same domain, the resolution can lead to a strong disturbance of the solution, it is a problem of stability . Faced with all these eventualities, it is generally said that the problem of least squares is badly posed, it is necessary, in this case, to introduce regularizing or stabilizing terms which reduce additional approximation errors.

2.7 Sentinels method

It responded on the one hand to the concerns cited in (Q) and on the other hand to the development of a fast algorithm in the calculation of unknown parameters. This theory has also been developed in environmental applications by its author for four years through articles and conferences to sum it all up in his book published in 1992, "Sentinels for distributed systems with incomplete data" [43]. From the numerical point of view, we agree that the sentinel method is almost equivalent to the method of classical least squares.

In 2004, O. Nakoulima (see [64]) aimed to overcome the concerns (Q) and to use Carleman's inequality to demonstrate the existence of the sentinel.
We distinguish two types of sentinels, continuous and discrete.

2.8 Continuous Sentinel

We consider an open set of $\mathcal{R}^n$, ($n = 1, 2, 3$ in the applications), bounded with boundary $\partial\Omega = \Gamma$ quite regular (class C^2 so as not to encounter any regularity problem).

Let A be an elliptic operator of second order. For $T > 0$ fixed, we define $Q = \Omega \times [0, T]$ and $\sum = \Gamma \times (0, T)$, we consider the solution $y(x, t)$ of the system:

$$\begin{cases} \frac{\partial y}{\partial t} + A(y) + f & = \zeta + \lambda\widehat{\zeta} \quad \text{in } Q, \\ y(0) & = y_0 + \tau\widehat{y_0}, \quad \text{in } \Omega, \\ y & = 0, \quad \text{on } \Gamma. \end{cases} \tag{2 \cdot 5}$$

This system is data incomplete where

- The functions $\zeta$and y_0 are given respectively in $L^2(Q)$ and $L^2(\Omega)$.
- The pollution term $\lambda\widehat{\zeta}$ and the missing term $\tau\widehat{y_0}$ are unknown respectively in $L^2(Q)$ and $L^2(\Gamma)$.
- The reals $\lambda$and τ are arbitrarily small.
- The coefficients of the operator A verifying the conditions of Saut and Sheurer's theorem.
- The operator $y \to f(y)$ is a class C^1 nonlinear function, (we can suppose that f is a function of y and ∇y).

- Equation (2.5) admits a unique weak solution in $L^2(Q)$ which we denote

$$y(x, t; \lambda, \tau) = y(\lambda, \tau)$$

The question that arises is:

(q): How can we calculate the pollution term $\lambda\widehat{\zeta}$ (or to obtain information) which is independent of the variations of the initial data around $\mathbf{y}_0$?

So we are interested in estimating the pollution term without showing any interest in the missing term.

Consider the case where the observed data y_{obs} are not noisy. The fundamental idea to answer the previous question is to take an average value, to know if something is happening. Let h_0 be a function given on $O \times (0, T)$. We then consider the averaged :

$$m(\lambda, \tau) = \int\int_{O\times(0,T)} h_0.y(x, t; \lambda, \tau)dxdt,$$

and we seek to determine the pollution term independently of the term in τ, to first order for example. But, there is in general no reason why, at first order, $m(\lambda, \tau)$ is independent of τ. In other words, there is no reason why

$$\frac{\partial m}{\partial \tau}(0,0) = \int\int_{O\times(0,T)} h_0.\frac{\partial}{\partial \tau}y(0,0)\,dxdt = 0.$$

We introduce a function w, and we give the definition of a so-called "sentinel" functional which is the average of the state y over a small domain, given by the following expression:

$$S(\lambda, \tau) = \int\int_{O\times(0,T)} (h_0 + w)\,.y(0,0)\,dxdt, \tag{2.6}$$

for functions h_0 and $w \in L^2(O \times (0, T))$.

Definition 2.8.1 *We say that S is a sentinel of Lions defined by h_0, if there exists a control ωsuch that the pair (ω, S) verifies*

$$\left.\frac{\partial S}{\partial \tau}(\lambda, \tau)\right|_{\lambda,\tau=0} = 0, \qquad \forall \widehat{y_0} \in L^2(\Omega) \tag{2.7}$$

$$\|w\|_{L^2(O\times(0,T))} = \min \tag{2.8}$$

Remark 2.8.1 *Condition* (2.7) *expresses the insensitivity of the sentinel with respect to the missing first order term and condition* (2.8) *expresses that we deviate as little as possible from the mean, it selects a single sentinel.*

Remark 2.8.2 *The choice The choice* $w = -h_0$ *gives rise to* (2.7)*. Therefore, under very general assumptions, the problem* $(2.7) - (2.8)$ *admits a unique solution. But it will be necessary to make sure that under suitable conditions,* $w \neq -h$*, the functional* $S(\lambda, \tau) = 0$ *not being likely to provide us with much information.not being likely to provide us with much information.*

2.8.1 Information provided by the sentinels

The existence and uniqueness of the function w are shown in [45]. It has been shown that the problem $(2.7) - (2.8)$ is equivalent to an exact zero controllability problem from which the HUM method, abbreviation of "Hilbert Uniquenss Method" [41], was used to establish the existence and uniqueness of the control function w. The sentinel function, once constructed, the determination of the pollution term is deduced by the Taylor expansion to order 1 of S and consider $(2.6) - (2.7)$ we have:

$$\begin{aligned} S(\lambda,\tau) &= S(0,0) + \lambda \tfrac{\partial S}{\partial \lambda}(0,0) + \tau \tfrac{\partial S}{\partial \tau}(0,0) + o\left(\|(\lambda,\tau)\|\right) \\ &= S(0,0) + \lambda \tfrac{\partial S}{\partial \lambda}(0,0) + o\left(\|(\lambda,\tau)\|\right), \end{aligned} \tag{2.9}$$

since by definition $\frac{\partial S}{\partial \tau}(\lambda,\tau)\big|_{\lambda=\tau=0} = 0$ we can deduce

$$\lambda \times \beta \approx \left(S(\lambda,\tau) - S(0,0)\right) \text{ with } \beta = \frac{\partial S}{\partial \lambda}(0,0), \tag{2.10}$$

by replacing $S(\lambda, \tau)$ by S_{obs} the observed solution

$$S_{obs} = \int\int_{O\times(0,T)} (h_0 + w)\, y_{obs}(x,t) dxdt; \tag{2.11}$$

We have the estimate

$$\lambda \times \beta \approx S_{obs} - S(0,0). \tag{2.12}$$

We will now show how, given h_0, we can construct the unique function w such that we have $(2.7) - (2.8)$.

2.8.2 Variational method

The condition of "insensitivity" of the sentinel with respect to the missing terms is equivalent to

$$S(\lambda, \tau) = \int\int_{O\times(0,T)} (h_0 + w)\, y_\tau dxdt, \tag{2.13}$$

where

$$y_\tau = \frac{\partial y}{\partial \tau}(0,0) = \lim_{\tau\to 0}\left(\frac{y(0,\tau) - y\,(0,0)}{\tau}\right),$$

then y_τ the solution of the system of equation

$$\begin{cases} \frac{\partial y_\tau}{\partial t} + Ay_\tau + f'\,(y_0)\, y_\tau & = 0 \quad \text{in } Q, \\ y_\tau\,(0) & = \widehat{y_0} \quad \text{in } \Omega, \\ y_\tau & = 0 \quad \text{on } \Sigma, \end{cases} \qquad 2.14$$

with $f\prime\,(y_0)$ designating the derivative of f at the point $y_0 = y(0,0)$. Indeed, $y(0,\tau)$ is the solution of the following system:

$$\begin{cases} \frac{\partial y(0,\tau)}{\partial t} + Ay(0,\tau) + f\,(y_0)\, y_\tau(0,\tau) & = \zeta & \text{in } Q, \\ y_\tau(0,\tau) & = y_0 + \tau\widehat{y_0} & \text{in } \Omega, \\ y_\tau(0,\tau) & = 0 & \text{on } \Sigma, \end{cases} \tag{2.15}$$

and y_0 is the solution of the system

$$\begin{cases} \frac{\partial y_0}{\partial t} + Ay_0 + f\,(y_0) & = 0 \quad \text{in } Q, \\ y_0\,(0) & = y_0 \quad \text{in } \Omega, \\ y_0 & = 0 \quad \text{on } \Sigma. \end{cases} \tag{2.16}$$

By subtracting (2.16) from (2.15) and multiplying by $\frac{1}{\tau}$, we find :

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{y(0,\tau)-y_0}{\tau}\right) + A\left(\frac{y(0,\tau)-y_0}{\tau}\right) + \frac{f(y(0,\tau))-f(y_0)}{\tau} & = 0 \quad \text{in } Q, \\ \frac{y(0,\tau)(0)-y_0(0)}{\tau} & = \widehat{y_0} \quad \text{in } \Omega, \\ \frac{y(0,\tau)-y_0}{\tau} & = 0 \quad \text{on } \Sigma. \end{cases} \tag{2.17}$$

By passing to the limit when $\tau \to 0$ in (2.17), we obtain (2.14).

2.8.3 Equivalence to a controllability problem

Let $q = q(x,t)$ be the adjoint state which is the solution of the following backward problem:

$$\begin{cases} -\frac{\partial q}{\partial t} + A^{*}q + f'(y_0)\, q & = (h+w)\,\chi_0 \quad \text{in } Q \\ q(T) & = 0 \quad \text{in } \Omega \\ q & = 0 \quad \text{on } \Sigma \end{cases} \tag{2.18}$$

The problem (2.18) admits a unique solution under very general assumptions on $f\prime(y_0)$. This solution depends on ω that we want to determine.

Lemma 2.8.1 *We assume that q is the solution of problem* (2.18)*. Then the existence problem of a sentinel insensitive to missing terms is equivalent to a zero-controllability problem, i.e.* $q(0) = 0$*.*

Proof. If we multiply the first equation of the system (2.18) by y_τ and then integrate by parts, we obtain:

$$\int_0^T \int_O (h+w)\, y_\tau dx dt = \int_O q(0)\, \widehat{y_0} dx.$$

Therefore, condition (2.13) is equivalent to

$$q(0) = 0 \tag{2.19}$$

This is a zero controllability problem. So the problem of finding a sentinel S such that (2.18) takes place, is equivalent to the following zero-controllability problem:

$\{\text{Find } \omega \in L^2(]0,T[\times O) \text{ as we have } (2.18) \text{ and } (2.19)\}$. ■

In summary, the problem of the existence of a unique sentinel amounts to solving the following optimization problem:

$$(P) : \left\{ \min_{\omega \in A} \|w\|_{L^2((0,T)\times O)}, \right\}$$

where

$$A = \left\{ w \text{ such as } \begin{cases} -\frac{\partial q}{\partial t} + A^{*}q + f'(y_0)\, q & = (h+w)\,\chi_0 \quad \text{in } Q, \\ q(T) = q(0) & = 0 \quad \text{in } \Omega, \\ q & = 0 \quad \text{on } \Sigma, \end{cases} \right\}$$

The constraint domain of problem (P) is non-empty because $\omega = -h$ gives $q \equiv 0$, consequently the problem (P) always admits a solution and only one that we denote $\widehat{\omega}$. There thus remain two problems to solve

1- Calculate $\widehat{\omega}$,

2- Make sure that bw $\widehat{\omega} \neq -h$.

A classical method to obtain the optimality system for the problem (P) is the penalization method.

2.8.4 Penalty

Let $\epsilon > 0$, we introduce the following function:

$$J_\epsilon(w,z) = \frac{1}{2}\|w\|^2_{L^2((0,T)\times O)} + \frac{1}{2\epsilon}\left\| -\frac{\partial z}{\partial t} + A^* z + f'(y_0)\,z \quad -(h+w)\,\chi_0 \right\|^2_{L^2(Q)},$$

and we consider the following problem (P_ϵ) :

$$(P_\epsilon)\begin{cases} \min J_\epsilon(w,z) \\ (w,z) \in A^\epsilon, \end{cases}$$

with

$$A^\epsilon = \left\{ (w,z) \text{ such as } \begin{cases} -\frac{\partial Z}{\partial t} + A^* z + f'(y_0)\,z & -(h+w)\,\chi_0 & \in L^2\ (Q), \\ z(T) = z(0) & = 0 & \text{in } \Omega, \\ z & = 0 & \text{on } \Sigma, \end{cases} \right\}$$

the problem (P_ϵ) admits a unique solution that we will denote (w_ϵ, z_ϵ).

The optimality system gives the existence of a function ρ_ϵ solution of the system:

$$\begin{cases} L\rho_\epsilon & = 0 \quad \text{in } Q, \\ \rho_\epsilon & = 0 \quad \text{on } \Sigma, \end{cases} \tag{2.20}$$

and characterizes the optimal control as follows:

$$\omega_\epsilon = \rho_\epsilon \chi_O, \tag{2.21}$$

where

$$L = \frac{\partial}{\partial t} + A + f'(y_0)\,I_d \quad ,$$

and by passing to the limit we obtain a function ρ solution of an optimality system of the initial problem (P). Finally the control is

$$\overline{w} = \rho\chi_O \in F,$$

where F is the Hilbert space supplemented by $L^2(Q)$ for the norm

$$\|\rho\|_F = \|\rho\|_{L^2((0,T)\times O)}.$$

In this case, we must assume that $h \notin F$ to obtain a non-identical null sentinel.
Here the notion of Lions sentinel where control and observation are in the same domain, becomes a special case. We propose for the previous definition a generalization of the notion of sentinel to the case of an observation and a control having their supports in two different open spaces.

2.9 Regional Sentinel

Modeling environmental problems leads to mathematical systems with missing data. This is the case, for example, with meteorological problems where we never know the initial data. In this work, we are concerned with the identification of the pollution terms present in the state equation of a dissipative system with incomplete initial data.
To achieve this objective, the Lions sentinel method (see [43]) is used. Here, the problem of determining a sentinel is equivalent to a zero controllability problem for which Carleman type estimates are given.
We consider here $h \in L^2(0 \times (0,T))$and ω a non-empty set of Ω, such as $\omega \neq O$. More precisely for a control function

$$w \in L^2((0,T) \times \omega).$$

We pose

$$\begin{aligned} S(\lambda,\tau) &= \int\int_{O\times(0,T)} h.y(x,t;\lambda,\tau)\,dxdt + \int\int_{O\times(0,T)} w.y(x,t;\lambda,\tau)\,dxdt, \\ &= \int\int_{O\times(0,T)} \left(h\chi_{_O} + w\chi_{_\omega}\right).y(x,t;\lambda,\tau)\,dxdt, \end{aligned} \tag{2.22}$$

where $\chi_{_O}$ and $\chi_{_w}$ are the characteristic functions of O and w respectively.

We consider the problem (2.5) with $A = \Delta$ (to simplify the calculations) and we divide the edges into two Γ_1 and Γ_2 such that $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y + f(y) & = \zeta + \lambda \widehat{\zeta} & \text{in } Q, \\ y(0) & = y_0 + \tau \widehat{y_0}, & \text{in } \Omega, \\ y & = 0 & \text{on } \Sigma_1, \\ \frac{\partial y}{\partial \nu} & = 0 & \text{on } \Sigma_2, \end{cases} \tag{2.23}$$

Definition 2.9.1 *We say that S is a sentinel defined by h, if there exists a control ω such that the pair (ω, S) holds:*

$$\frac{\partial S}{\partial \tau}(\lambda, \tau)\Big|_{\lambda=\tau=0} = 0,$$

and

$$\|w\|_{L^2(\omega \times (0,T))} = \min .$$

The existence problem of a sentinel is equivalent to a zero controllability problem:

find the optimal control $w \in L^2(]0;T[\times\omega)$ such that $q(0) = 0$ where q is the solution of the system

$$\begin{cases} -\frac{\partial q}{\partial t} - \Delta q + f'(y_0)\, q & = h\chi_0 + \omega \chi_w & \text{in } Q \\ q(T) = q(0) & = y_0 + \tau \widehat{y_0}, & \text{in } \Omega, \\ q & = 0 & \text{on } \Sigma_1 \\ \frac{\partial q}{\partial \nu} & = 0 & \text{on } \Sigma_2 \end{cases}$$

To do this, we need to solve the following optimization problem:

$$(P') \begin{cases} \min \|w\|_{L^2((\omega \times (0,T)))}, \\ w \in B, \end{cases}$$

where

$$B = \left\{ \omega \text{ such as } \begin{cases} -\frac{\partial q}{\partial t} - \Delta q + f'(y_0)\, q & = h\chi_0 + w\chi_\omega & \text{in } Q \\ q(T) = q(0) & = 0 & \text{in } \Omega, \\ q & = 0 & \text{on } \Sigma_1, \\ \frac{\partial q}{\partial \nu} & = 0 & \text{on } \Sigma_2. \end{cases} \right\}$$

2.9.1 Information provided by the sentinel

The information provided by Sentinel S, according to (2.12), is given by:

$$\int\int_{\Omega\times(0,T)} (h\chi_0 + w\chi_\omega)\, y_\lambda dxdt = \int\int_{\Omega\times(0,T)} (h\chi_0 + w\chi_\omega)\,(y_{obs} - y_0)dxdt,$$

where

$y_\lambda = \frac{\partial y}{\partial \lambda}(0,0)$ is the solution to the problem

$$\begin{cases} \frac{\partial y_\lambda}{\partial t} - \Delta y_\lambda + f\prime(y_0)\, y_\lambda & = \widehat{\zeta} \quad \text{in } Q, \\ y_\lambda(0) & = 0, \quad \text{in } \Omega, \\ y_\lambda & = 0 \quad \text{on } \Sigma_1, \\ \frac{\partial y_\lambda}{\partial \nu} & = 0 \quad \text{on } \Sigma_2, \end{cases} \qquad 2.24$$

Multiplying the first equation by q and integrating by parts, we find:

$$\begin{aligned} \int_0^T \int_{\Omega\times} qLy_\lambda dxdt \;=\; & \int_0^T \int_\Omega y_\lambda L^* q dxdt + \int_\Omega q(T)\, y_\tau(T)\, dx - \int_\Omega q(0)\, y_\tau(0)\, dx \\ & - \int_0^T \int_\Gamma q.\frac{\partial y_\lambda}{\partial \nu} d\Gamma dt + \int_0^T \int_\Gamma y_\lambda.\frac{\partial q}{\partial \nu} d\Gamma dt. \end{aligned} \qquad (2.25)$$

Since q and y_λ are solutions of (2.18)and (2.24) respectively, (2.25) becomes

$$\int_0^T \int_\Omega (h\chi_0 + w\chi_\omega)\,.y_\lambda dxdt = \int_0^T \int_\Omega q.\widehat{\zeta} dxdt.$$

Finally, we obtain

$$\int_0^T \int_\Omega (h\chi_0 + w\chi_\omega)\,.\,(m_0 - y_0)\, dxdt = \int_0^T \int_\Omega q.\lambda\widehat{\zeta} dxdt. \qquad (2.26)$$

Therefore, knowledge of the optimal control w provides information on the pollution term $\lambda\widehat{\zeta}$.

2.9.2 Construction of a sentinel

We have shown that the existence of a sentinel is equivalent to a zero controllability problem, the essential tool to solve the existence problem is a Carleman-type observability inequality. To do this, we introduce the space

$$V=\left\{ \upsilon\in C^{\infty}\left(Q\right) ,\upsilon=\frac{\partial\upsilon}{\partial t}=0\text{ on }\Sigma_{1}\text{ and }\frac{\partial\upsilon}{\partial\nu}=0\text{ on }\Sigma\ _{2}\right\} .$$

Then the following theorem gives a Carleman inequality. (For the proof see [59]).

Theorem 2.9.1 *There is a positive constant $C=C\left(\Omega,\omega,0,T,f^{\prime}\left(y_{0}\right) \right)$ such that:*

$$\int_{0}^{T}\int_{\Omega}\frac{1}{\theta^{2}}\left\vert u\right\vert ^{2}dxdt\leq C\left[\int_{0}^{T}\int_{\Omega}\left\vert Lu\right\vert ^{2}dxdt+\int_{0}^{T}\int_{\omega}\left\vert u\right\vert ^{2}dxdt\right] \text{ for all }v\in V, \quad . \tag{2.27}$$

where $\theta\in C^{2}\left(Q\right)$ positive with $\frac{1}{\theta}$ bounded and "L" is a differentiable operator defined by:

$$L \;=\; \frac{\partial}{\partial t}+A+f^{\prime}(y_{0})I_{d}.$$

Remark 2.9.1 *if $Lu=0$, the inequality obtained is called "Inequality of observability" because if $u=0$ on $\omega\times\left(0,T\right)$ implies that $u=0$ on $\Omega\times\left(0,T\right)$ whole.*

We define a symmetric bilinear form of $V\times V$ in $\mathcal{R}$ by

$$a\left(u,v\right) =\int_{0}^{T}\int_{\Omega}Lu.Lvdxdt+\int_{0}^{T}\int_{\omega}u.vdxdt, \tag{2.28}$$

we deduce the following lemma:

Lemma 2.9.1 *The bilinear form $a(.,.)$ is a scalar product.*

Proof. *We have $a\left(.,.\right)$ a bilinear, symmetric and positive form*

$$a\left(u,u\right) =\int_{0}^{T}\int_{\Omega}\left\vert Lu\right\vert ^{2}dxdt+\int_{0}^{T}\int_{w}\left\vert u\right\vert ^{2}dxdt\geq0\text{ for all }u\in V$$

It remains to show that

$$a\left(u,u\right) =0\Longrightarrow u=0,$$

we have

$$a\left(u,u\right) =0\Longrightarrow\int_{0}^{T}\int_{\Omega}\left\vert Lu\right\vert ^{2}dxdt=0\text{ and }\int_{0}^{T}\int_{w}\left\vert u\right\vert ^{2}dxdt=0\ ,$$

from (2.27) *we deduce*

$$\int_0^T \int_\Omega \frac{1}{\theta^2} |u|^2 \, dxdt = 0 \Longrightarrow u = 0 \text{ in } Q;$$

so, $a(,)$ *is a scalar product.* ■

Let W be the space completed by V for the norm $\|.\|_V$ defined by:

$$\|u\|_W = \sqrt{a(u,u)}, \tag{2.29}$$

Then W is a Hilbert space and moreover V is dense in W We can specify the structure of the elements of W. Indeed, let $H_\theta(Q)$ be the Hilbert space with weight defined by:

$$H_\theta(Q) = \left\{ \upsilon \in L^2(Q) \text{ such as } \int_0^T \int_\Omega \frac{1}{\theta^2} |\upsilon|^2 \, dxdt < \infty \right\}, \tag{2.30}$$

provided with the norm

$$\|\upsilon\|_\theta = \left(\int_0^T \int_\Omega \frac{1}{\theta^2} |\upsilon|^2 \, dxdt \right)^{\frac{1}{2}}.$$

Then, by applying Carleman's inequality (2.27) we find

$$\|\upsilon\|_\theta \le C \|\upsilon\|_W .$$

Which shows that W continuously injects into $H_\theta(Q)$.

Now, We consider the following linear application:

$$\begin{aligned} l: \quad & V \to R \\ & \upsilon \mapsto l(\upsilon) = \textstyle\int_0^T \int_\Omega h\chi_O \upsilon dxdt, \end{aligned}$$

where the value of h is given in $L^2(Q)$.

Lemma 2.9.2 *We suppose that*

$$h \in L^2(Q) \text{ and } (\theta h) \in L^2(Q). \tag{2.31}$$

So l *is continuous.*

$$|l(\upsilon)| \le \left(\int_0^T \int_\Omega |\theta h \chi_O|^2 \, dxdt \right)^{\frac{1}{2}} \left(\int_0^T \int_\Omega \frac{1}{\theta^2} |\upsilon|^2 \, dxdt \right)^{\frac{1}{2}}. \tag{2.32}$$

Proof. According to the Cauchy Schwartz inequality. As $\theta h \in L^2(Q)$, we obtain:

$$\left(\int_0^T \int_\Omega |\theta h \chi_O|^2 \, dxdt\right)^{\frac{1}{2}} \leq C_1 \text{ constant}, \tag{2.33}$$

moreover

$$\int_0^T \int_\Omega \frac{1}{\theta^2} |v|^2 \, dxdt \leq C \left(\int_0^T \int_\Omega |Lv|^2 \, dxdt + \int_0^T \int_\omega |v|^2 \, dxdt\right) = C \|v\|_W^2 .$$

Thus

$$|Lv| \leq C_1 \sqrt{C} \, \|v\|_W^2 .$$

so l is continuous on W. ■

As a result of the above, we have the following proposition:

Proposition 2.9.1 *We assume that assumption* (2.31) *is satisfied. Then there exists a unique element* $\overline{u} \in W$ *solution of the problem*

$$a(\overline{u}, v) = \int_0^T \int_\Omega h \chi_o v dxdt \quad \forall v \in W. \tag{2.34}$$

Proof. *The application* l *is linear and continuous on* W *and as the bilinear, symmetric form* $a(,)$ *is continuous, coercive, then according to the Lax-Milgram theorem, there exists a unique element* $\overline{u} \in W$ *solution of* (2.34). ■

We will now show that the set of solutions which satisfies $(2.18) - (2.19)$ is non-empty.

Proposition 2.9.2 *Under the hypothesis* (2.31), *let* $\overline{u} \in W$ *be the unique solution of* (2.34), *we pose*

$$\begin{cases} w &= -\overline{u}\chi_\omega, \\ q &= L\overline{u}, \end{cases} \tag{2.35}$$

So

1) (ω, q) *is a solution of the system* $(2.18) - (2.19)$ *i.e. there exists a unique sentinel insensitive to missing terms.*

2) We have

$$\begin{array}{ll} \|\overline{u}\|_W & \leq C\,\|\theta h\chi_O\|_{L^2(Q)}\,, \quad C \text{ is aconstant,} \\ \|w\|_{L^2(\omega\times(0,T))} & \leq C\,\|\theta h\chi_O\|_{L^2(Q)}\,, \\ \|q\|_{L^2(Q)} & \leq C\,\|\theta h\chi_O\|_{L^2(Q)}\,. \end{array}$$

Proof. 1) As u is solution of (2.34) then

$$\int_0^T\int_\Omega L\overline{u}L\upsilon dxdt + \int_0^T\int_w \overline{u}\upsilon dxdt = \int_0^T\int_\Omega h\chi_O\upsilon dxdt,\ \forall\upsilon\in W, \tag{2.36}$$

but (w,q) satisfies (2.35), then (2.36) is written

$$\int_0^T\int_\Omega qL\upsilon dxdt - \int_0^T\int_\omega w\upsilon dxdt = \int_0^T\int_\Omega h\chi_O\upsilon dxdt,\ \forall\upsilon\in W, \tag{2.37}$$

then

$$\int_0^T\int_\Omega qL\upsilon dxdt = \int_0^T\int_\Omega (h\chi_O + w\chi_\omega)\,\upsilon dxdt,\ \forall\upsilon\in W, \tag{2.38}$$

We have to show that q is a solution of the system $(2.18)-(2.19)$. Equation (2.38) is true in particular for $\upsilon\in D(Q)\subset V\subset W$, that is to say:

$$\langle q, L\upsilon\rangle_{L^2(Q)} = \langle (h\chi_O + w\chi_\omega)\,,\upsilon\rangle_{L^2(Q)}\,\forall\upsilon\in D(Q),$$

hence in the sense of $D(Q)$, we have:

$$\begin{array}{ll} L^*q & = h\chi_O + w\chi_\omega \\ & \Rightarrow L^*q\in L^2\left(Q\right). \end{array} \tag{2.39}$$

Moreover, we have $q\in L^2(Q)$ and by application of the Lions-Magnenes theorem [51], the trace functions q at $t=0$, $t=T$ and q on Σ exist.

We multiply (2.39) by $\rho\in C^\infty(\overline{Q})$ and we integrate by parts over Q, we obtain:

$$\begin{array}{ll} \int_0^T\int_\Omega \rho L^*qdxdt & = \int_\Omega q.L\rho dxdt - \int_\Omega q\left(T\right)\rho\left(T\right)dx + \int_\Omega q\left(0\right)\rho\left(0\right)dx \\ & - \int_0^T\int_\Gamma \frac{\partial q}{\partial\nu}.\rho d\Gamma dt + \int_0^T\int_\Gamma \frac{\partial\rho}{\partial\nu}.qd\Gamma dt \\ & = \int_0^T\int_\Omega \left(h\chi_O + w\chi_\omega\ \right)\rho dxdt. \end{array} \tag{2.40}$$

Now using equation (2.39)

$$-\int_{\Omega} q\left(T\right)\rho\left(T\right)dx+\int_{\Omega} q\left(0\right)\rho\left(0\right)dx-\int_0^T\int_{\Gamma}\frac{\partial q}{\partial \nu}.\rho d\Gamma dt+\int_0^T\int_{\Gamma}\frac{\partial \rho}{\partial \nu}.q d\Gamma dt=0\ \forall \rho\in C^{\infty}(\overline{Q}),$$

We take $\rho \in W$, we obtain:

$$-\int_{\Omega} q\left(T\right)\rho\left(T\right)dx+\int_{\Omega} q\left(0\right)\rho\left(0\right)dx-\int_0^T\int_{\Gamma_2}\frac{\partial q}{\partial \nu}.\rho d\Gamma_2 dt+\int_0^T\int_{\Gamma_1}\frac{\partial \rho}{\partial \nu}.q d\Gamma_1 dt=0\ \forall \rho\in C^{\infty}(\overline{Q}), \tag{2.41}$$

for $\rho \in V$, such that$\rho(0)=\rho(T)=0$ and $\rho|_{\Sigma_2}=0$. Then it comes:

$$\int_0^T\int_{\Gamma_1}\frac{\partial \rho}{\partial \nu}.q d\Gamma_1 dt=0\Rightarrow q=0 \text{ on } \Sigma_1.$$

We take(2.41) with in particular for $\rho \in V$, such that $\rho(0)=\rho(T)=0$ so:

$$\int_0^T\int_{\Gamma_2}\frac{\partial q}{\partial \nu}.\rho d\Gamma_2 dt=0\Rightarrow \frac{\partial q}{\partial \nu}=0 \text{ on } \Sigma_2.$$

The same is true for $\rho \in$ W, such that $\rho(0)=0$, so

$$\int_{\Omega} q\left(T\right)\rho\left(T\right)dx=0\Rightarrow q\left(T\right)=0 \text{ in } \Omega.$$

Finally, we have

$$\int_{\Omega} q\left(0\right)\rho\left(0\right)dx=0\Rightarrow q\left(0\right)=0 \text{ in } \Omega.$$

Thus the couple (w,q) is a solution of the problem $(2.18)-(2.19)$.

2) Now, we prove the estimates.

Let $\upsilon=\overline{u}$ in equation (2.34). By the Cauchy-Schwartz inequality, it comes that:

$$\begin{aligned}\int_0^T\int_{\Omega}\left|L\overline{u}\right|^2 dxdt+\int_0^T\int_{\omega}\left|\overline{u}\right|^2 dxdt &= \int_0^T\int_{\Omega} h\chi_O\overline{u}dxdt,\\ &\leq \left\|\theta h\chi_O\right\|_{L^2(Q)}\left\|\tfrac{1}{\theta}\overline{u}\right\|_{L^2(Q)},\end{aligned}$$

moreover, the Carleman inequality gives:

$$\left\|\frac{1}{\theta}\overline{u}\right\|_{L^2(Q)}\leq C\left\|\overline{u}\right\|_W. \tag{2.42}$$

Using inequality (2.42), we find:

$$\|\overline{u}\|_W^2 = \int_0^T \int_\Omega |L\overline{u}|^2 \, dxdt + \int_0^T \int_\omega |\overline{u}|^2 \, dxdt \leq \sqrt{C} \, \|\theta h \chi_O\|_{L^2(Q)} \, \|\overline{u}\|_W \, , \tag{2.43}$$

thus

$$\|\overline{u}\|_W \leq \sqrt{C} \, \|\theta h \chi_O\|_{L^2(Q)} \, . \tag{2.44}$$

So the first estimate of the proposition is realised.

Moreover, since $(\omega; q)$ satisfies $(2, 35)$, then the equation $(2, 43)$ is written:

$$\|q\|_{L^2(Q)}^2 + \|w\|_{L^2(\omega \times (0,T))}^2 \leq \sqrt{C} \, \|\theta h \chi_O\|_{L^2(Q)} \, \|\overline{u}\|_W \, , \tag{2.45}$$

From (2.44) and (2.45) he comes

$$\|q\|_{L^2(Q)}^2 + \|w\|_{L^2(\omega \times (0,T))}^2 \leq C \, \|\theta h \chi_O\|_{L^2(Q)}^2 \, , \tag{2.46}$$

We then deduce the second and the third estimates of the proposition. ■

The following theorem gives the existence of solution for the problem (P').

Theorem 2.9.2 *Under the assumptions of the previous proposition, there exists a unique pair* $(\widehat{w}, \widehat{q})$ *solution of the problem* (P')*, such as*

$$\widehat{w} = -\widehat{\rho} \chi_\omega,$$

where $\widehat{\rho}$ *is the solution of the system*

$$\begin{cases} L\widehat{\rho} & = 0 \quad in \ Q \\ \widehat{\rho} & = 0 \quad on \ \Sigma_1, \\ \frac{\partial \widehat{\rho}}{\partial \nu} & = 0 \quad on \ \Sigma_2. \end{cases}$$

Proof. If the assumptions of proposition (2.5) are satisfied, the domain B is then non-empty. Also, it's closed. The application $\omega \to \|\omega\|_{L^2(w \times (0,T))}$ is continuous, coercive and strictly convex. Then, we deduce that there is a unique solution for the problem $(P\prime)$ that we note $(\widehat{\omega}, \widehat{q}) \in B$ which satisfies:

$$\|\widehat{w}\|_{L^2(\omega \times (0,T))}^2 \leq \|w\|_{L^2(\omega \times (0,T))} \qquad \forall w \in B.$$

We use the penalization method to obtain the optimality system for $(\widehat{w}, \widehat{q})$:

Let $\varepsilon > 0$, we introduce the following function:

$$J_{\varepsilon}(w, z) = \frac{1}{2}\|w\|^2_{L^2(\omega\times(0,T))} + \frac{1}{2\varepsilon}\|L^* z - h\chi_O - w\chi_\omega\|^2_{L^2(Q)},$$

and we consider the following problem (P_ε)

$$(P_\varepsilon)\begin{cases} \min J_\varepsilon(w, z) \\ (w, z) \in B_\varepsilon \end{cases},$$

with

$$B_\varepsilon = \left\{ (w, z) \text{ such as } \begin{cases} L^* z - h\chi_O - w\chi_\omega \in L^2(Q) & \\ z(T) = z(0) & = 0 \quad \text{in } \Omega, \\ z & = 0 \quad \text{on } \Sigma, \end{cases} \right\}$$

the problem (P_ε) admits a unique solution that we will denote $(w_\varepsilon, z_\varepsilon)$ such that:

$$\begin{cases} \begin{matrix} w_\epsilon \rightharpoonup \widehat{w} \\ \epsilon \to 0 \end{matrix} & \text{weakly in } L^2(\omega \times (0,T)), \\ \begin{matrix} z_\varepsilon \rightharpoonup \widehat{q} \\ \epsilon \to 0 \end{matrix} & \text{weakly in } W. \end{cases}$$

Then, $(\widehat{\omega}, \widehat{q})$ is the unique solution of the problem (P) if and only if there exists a function $\widehat{\rho}$ such that $(\widehat{\omega}, \widehat{q}, \widehat{\rho})$ is solution of the following optimality system

$$\begin{cases} L^*\widehat{q} & = h\chi_O + \widehat{w}\chi_\omega & \text{in } Q \\ \widehat{q}(T) = \widehat{q}(0) & = 0 & \text{in } \Omega \\ \widehat{q} & = 0 & \text{on } \Sigma_1 \\ \frac{\partial \widehat{\rho}}{\partial \nu} & = 0 & \text{on } \Sigma_2. \end{cases} \text{ and } \begin{cases} L\widehat{\rho} & = 0 & \text{in } Q \\ \widehat{\rho} & = 0 & \text{on } \Sigma_1, \\ \frac{\partial \widehat{\rho}}{\partial \nu} & = 0 & \text{on } \Sigma_2. \end{cases}$$

with

$$\widehat{w} = -\widehat{\rho}\chi_\omega, \quad \widehat{\rho} \in W.$$

■

2.10 Pointwise sentinel

Jean Pierre Kernevez and his team were the first to have proposed numerical results on sentinels in applications related to environmental pollution in the 90s [see $4, 5, 6, 13$] . These authors agree to define the source of pollution as being a continuous function with respect to time at a specific position in space. We place ourselves in the problem case (2.5), which can be written in the following form

$$\begin{cases} \frac{\partial y}{\partial t} + Ay + f(y) & = \zeta + \Sigma_{i=1}^{N} \lambda_i \widehat{\zeta_i} \delta(x - a_i) & \text{in } Q, \\ y(0) & = y_0 + \Sigma_{j=1}^{M} \tau_j \widehat{y_{0_j}} & \text{in } \Omega, \\ y & = 0 & \text{on } \Sigma, \end{cases} \tag{2.47}$$

where

- The a_i are the observation points,
- $\delta(x - a_i)$ is the Dirac function at point a_i.
- Source functions $\lambda_i \widehat{\zeta_i}$ that we assume in $L^2(0, T)$.

Now the problem can be formulated as follows:
Find (λ_i , $i = 1, ..., N$) best representing the flow which produced the measurement y_{obs}.
We define a function $\omega_i \in L^2(O)$ specific to the i^{th} component of λ must be constructed as:

$$(w_i, y_{obs})_{L^2(O)} = \lambda_i, \tag{2.48}$$

where y_{obs} is the measured state function. If such a function ω_i exists, its uniqueness is ensured by choosing the minimum norm function.
The function w_i designates the sentinel allowing the determination of the parameter λ_i, so to estimate all the λ_i, we have to calculate the whole family of functions $(\omega_i)_{i=1,N}$.
We have:

$$y(\lambda, \tau) = \Sigma_{i=1}^{N} \lambda_i y_i + \Sigma_{j=1}^{M} \tau_j y_{\tau_j}, \tag{2.49}$$

where $(y_i)_{i=1,N}$ and $(y_j)_{j=1,M}$ are respectively the solutions of the following equations:

$$\begin{cases} \frac{\partial y_i}{\partial t} + Ay_i + f'(y_0) y_i & = \widehat{\zeta_i} \delta(x - a_i) & \text{in } Q, \\ y_i(0) & = 0 & \text{in } \Omega, \\ y_i & = 0 & \text{on } \Sigma, \end{cases}$$

$$\begin{cases} \frac{\partial y_{\tau_j}}{\partial t} + Ay_{\tau_j} + f'(y_0) y_{\tau_j} & = 0 & \text{in } Q, \\ y_{\tau_j}(0) & = \widehat{y_0}_j & \text{in } \Omega, \\ y_{\tau_j} & = 0 & \text{on } \Sigma. \end{cases}$$

Then the scalar product of w_i with $y(\lambda, \tau)$ becomes:

$$(w_i, y(\lambda, \tau))_{L^2(O)} = \Sigma_{i=1}^{N} \lambda_i (w_i, y_i)_{L^2(O)} + \Sigma_{j=1}^{M} \tau_j \left(w_i, y_{\tau_j}\right)_{L^2(O)},$$

Therefore, formula (2.48) is equivalent to:

$$\begin{cases} (w_i, y_k)_{L^2(O)} & = 1 \quad \text{if } i = k, \\ (w_i, y_k)_{L^2(O)} & = 0 \quad \forall k = \overline{1, N}, \text{ if } i \neq k, \\ \left(w_i, y_{\tau_j}\right)_{L^2(O)} & = 0 \end{cases}$$

Then, we use the conjoint state noted q_i solution of the adjoint system, by multiplying q_i by y_i (resp.y_{τ_j}) and integrating by part in space and time, the previous equalities become:

$$\begin{cases} \int_0^T \widehat{\zeta}_i (t) q_i (a_i, t) dt & = 1 \quad 1 \leq i \leq N, \\ \int_\Omega y_0 (x) q_i (x, 0) dx & = 0. \end{cases}$$

This result is fundamental for the calculation of the sentinels. In effect, it summarizes the fact that the sentinel is sensitive to λ_i and insensitive to all the other parameters of equation (2.47). It also allows the previous equalities to be interpreted as an optimal control problem.
This discrete method has been used to determine pollution in an aquifer [25], in a lake [45] and in a river [4].

2.11 Discriminating Sentinel

We consider the problem (2.23), the observed data can be affected by measurement errors or "noise"effects, so

$$y_{obs} = m_0 + \Sigma_{i=1}^{n} \beta_i m_i,$$

where the functions $m_0, m_1, ..., m_n$ are known in $L^2(O \times (0,T))$ but the $\beta_i \neq 0$ are not known, we say that the β_i are the noise terms. The problem now is:
Can we obtain information about $\lambda\widehat{\zeta}$ which is not affected by the variations of $y(0)$ around y_0, and which are not affected by noise $\beta_i m_i$, $i = 1, ..., n$?

In such a situation, in addition to the hypotheses (2.7 − (2.8) it would be necessary to associate a condition of insensitivity of the sentinel to the effects of noises

$$\int_0^T \int_O (h+w) .m_i dxdt = 0 \qquad 1 \leq i \leq n. \tag{2.50}$$

Such a sentinel is called "discriminant" [43, 65, 74].

Remark 2.11.1 *1) The discriminating sentinel is not sensitive to missing terms (in τ), it is also not sensitive to noise (in β), it can therefore differentiate what comes from terms in λ from what comes from terms in β (discriminating properties).*
2) If $\omega \neq O \subset \Omega$, the condition (2.50) *becomes*

$$\int_0^T \int_O hm_i dxdt + \int_0^T \int_\omega wm_i dxdt = 0 \qquad 1 \leq i \leq n. \tag{2.51}$$

3) We assume that $\omega \subset O \subset \Omega$, because the m_i are supported in $O \times (0,T)$.

Definition 2.11.1 *We say that S is a discriminating sentinel (or sentinel for an observation with noise) defined by h if there exists a control ω such that:*

$$\begin{cases} 1/ \ \frac{\partial S}{\partial \tau}(0,0) & = 0 \qquad \forall \widehat{y_0} \in L^2(\Omega), \\ 2/ \ \int_0^T \int_O h.m_i dxdt + \int_0^T \int_\omega w.m_i dxdt & = 0 \qquad 1 \leq i \leq n, \\ 3/ \ \|w\|_{L^2(\omega \times (0,T))} & = \min \end{cases}$$

Now, we will show that the existence of a discriminant sentinel insensitive to noise terms and missing terms is equivalent to a zero controllability problem with constraints on the control. Let K be the vector subspace of $L^2(\omega \times (0,T))$ generated by the $(m_i \chi_\omega, 1 \leq i \leq n)$, which are assumed to be linearly independent. So we have the following lemma:

Lemma 2.11.1 *The application*

$$\begin{aligned} f : K & \rightarrow \mathcal{R}^n \\ k & \rightarrow f(k) = \left(\int_0^T \int_\omega k.m_1 dxdt, \ \int_0^T \int_\omega k.m_2 dxdt, ..., \ \int_0^T \int_\omega k.m_n dxdt \right), \end{aligned}$$

is an isomorphism. Moreover, the condition of insensitivity to noise terms is equivalent to

$$\exists k_0 \in K \ \textit{such as} \ W = k_0 + k \ \textit{with} \ k \in K^\perp, \tag{2.52}$$

Proof. The application f is linear and bijective, because

$$\begin{aligned}\ker f &= \{k \in K,\ f(k) = 0\},\\ &= \left\{k \in K,\ \int_0^T \int_\omega k.m_i dxdt \ = 0 \qquad \forall i = 1, n, \right\},\\ &= \left\{k = \Sigma_{i=1}^n \alpha_i m_i, \quad \int_0^T \int_\omega k.\Sigma_{i=1}^n \alpha_i m_i dxdt \ = 0, \right\},\\ &= \left\{k \in K, \|k\|^2_{L^2(\omega\times(0,T))} = 0\right\} = \{0\},\end{aligned}$$

hence f is injective and therefore is isomorphism. Therefore, we have

$$\forall \alpha \in \mathbb{R}^n, \exists! k_0 \in K \text{ such as } f(k_0) = \alpha,$$

which is equivalent to

$$\forall \alpha_i \in \mathcal{R}, \exists! k_0 \in K \text{ such as } \int_0^T \int_\omega k_0.m_i dxdt \ = \alpha_i \quad i = 1, ..., n.$$

From the condition of insensitivity there comes

$$-\int_0^T \int_O h.m_i dxdt = \int_0^T \int_\omega w.m_i dxdt \qquad \forall i = 1, ..., n$$

for

$$\alpha_i = -\int_0^T \int_O h.m_i dxdt,$$

we deduce that

$$\int_0^T \int_w k_0.m_i dxdt \ = \int_0^T \int_\omega \omega.m_i dxdt,$$

as a result, we conclude

$$w - k_0 \in K^\perp.$$

Thus

$$w - k_0 = k \text{ with } k \in K^\perp \text{ and } k_0 \in K.$$

■

The existence problem of a discriminant sentinel is then equivalent to the following problem:

$$(P_2) \left\{ \begin{array}{c} \min \|k\|_{L^2(\omega\times(0,T))} \\ k \in A_2 \end{array} \right. ,$$

where

$$A_2 = \left\{ k \text{ such as } \left\{ \begin{array}{lll} -\frac{\partial q}{\partial t} - \Delta q + f'(y_0)\, q & = h^* \chi_0 + k\chi_w & \text{in } Q, \\ q(T) = q(0) & = 0, & \text{in } \Omega, , \\ q & = 0 & \text{on } \Sigma_1, \\ \frac{\partial q}{\partial \nu} & = 0 & \text{on } \Sigma_2, \end{array} \right. \right\} \qquad ((2.53))$$

where $h^* = h\chi_0 + k_0\chi_\omega$. We define a symmetric bilinear form of $V \times V$ in $\mathcal{R}$ by:

$$a(u, \upsilon) = \int_0^T \int_\Omega Lu.L\upsilon dxdt + \int_0^T \int_\omega (u - Pu) . (\upsilon - P\upsilon)\, dxdt,$$

P is the orthogonal projection of $L^2(w \times (0, T))$ on K, and V is a space given by (2.27):

For the domain of the constraints to be non-empty, we use a Carleman inequality adapted to our problem. The following proposition shows that the domain A_2 is non-empty.

Proposition 2.11.1 *Let $\overline{u} \in W$ be the unique solution of*

$$a(\overline{u}, \upsilon) = \int_0^T \int_\Omega h\chi_0 \upsilon dxdt \qquad \forall \upsilon \in V. \tag{2.54}$$

We suppose that/

i) $\{\nexists k \in K$ such as : $\frac{\partial k}{\partial t} - \Delta k + f'(y_0)\, k \;\; = 0$ in $\omega \times (0, T)$,

ii) $h \in L^2(Q)$ and $\theta h \in L^2(Q)$, with

$$\left\{ \begin{array}{ll} \overline{k} & = -(\overline{u} - P\overline{u})\, \chi_\omega \\ \overline{q} & = L\overline{u}. \end{array} \right. \tag{2.55}$$

Then $(\overline{k}, \overline{q})$ is solution of the system

$$\left\{ \begin{array}{lll} -\frac{\partial q}{\partial t} - \Delta q + f'(y_0)\, q & = h\chi_0 + k_0\chi_\omega + w\chi_\omega & \text{in } Q, \\ q(T) = q(0) & = 0 & \text{in } \Omega, , \\ q & = 0 & \text{on } \Sigma_1, \\ \frac{\partial q}{\partial \nu} & = 0 & \text{on } \Sigma_2, \end{array} \right. \tag{2.56}$$

i.e. there is a discriminating sentinel. Moreover, we have:

$$\begin{array}{ll} \|u\|_W & \leq C \left\|\theta\left(h\chi_0 + k_0\chi_\omega\right)\right\|_{L^2(Q)}, \\ \left\|\overline{k}\right\|_{L^2(\omega\times(0,T))} & \leq C \left\|\theta\left(h\chi_0 + k_0\chi_\omega\right)\right\|_{L^2(Q)}, \\ \|\overline{q}\| & \leq C \left\|\theta\left(h\chi_0 + k_0\chi_\omega\right)\right\|_{L^2(Q)}, \end{array} \tag{2.57}$$

where C is a positive constant, which is not the same each time.

This problem is equivalent to a zero-controllability problem with constraints on the non-trivial control ($k \in K^{\perp}$).

Theorem 2.11.1 *Under the assumptions of the previous proposition, there exists a unique pair $(\widehat{k}, \widehat{q})$solution of the problem (P_2):*

Proof. The assumptions of the previous proposition are fulfilled, hence the domain A_2 is non-empty, moreover it is closed. On the other hand, the application $k \rightarrow \left\|\overline{k}\right\|_{L^2(\omega\times(0,T))}$ is continuous, coercive and strictly convex, so there exists one and only one solution for (P_2) which we note $(\widehat{k}, \widehat{q}) \in A_2$ and which satisfies:

$$\left\|\widehat{k}\right\|_{L^2(\omega\times(0,T))} \leq \|k\|_{L^2(\omega\times(0,T))}, \forall (k, q) \in A_2.$$

■

2.12 Weak Sentinel

In this section, we use a notion called "weak sentinel" to study the estimation of the pollution term of weakly controllable distributed systems independently of the missing term. Consider the system (2.5) with

$$\|\widehat{y_0}\|_{L^2(\Omega)} \leq 1, \qquad \left\|\widehat{\zeta}\right\|_{L^2(Q)} \leq 1,$$

and $\omega = 0$ we have the following definition:

Definition 2.12.1 *We say that the functional S defined by*

$$S(\lambda, \tau) = \int_0 \int_0^T (h + w) . y(x; t; \lambda, \tau)\, dxdt,$$

is a weak sentinel (or approximate sentinel) if it exists, for all "$\epsilon > 0$,a control $w_\epsilon$such as:

$$\left.\frac{\partial S}{\partial \tau}(\lambda, \tau)\right|_{\tau=0,\lambda=0} \leq \epsilon, \qquad \forall \widehat{y_0} \in L^2(\Omega) \tag{2.58}$$

$$\|w_\epsilon\|_{L^2(O\times(0,T))} = \min . \tag{2.59}$$

To construct the weak sentinel, we must determine ω_ϵ which satisfies the conditions (2.58) and (2.59). The existence problem of a weak sentinel is then equivalent to the following problem :

$$(P_3)\left\{\begin{array}{c} \min \|w_\epsilon\|_{L^2(O\times(0,T))} \\ w_\epsilon \in A_3 \end{array}\right. ,$$

$$A_3 = \left\{ w_\epsilon \text{ such as } \left\{\begin{array}{lll} -\frac{\partial q}{\partial t} - \Delta q + f'(y_0)\, q & = (h + w_\epsilon)\, \chi_O & \text{in } Q, \\ q\,(T) & = 0, & \text{in } \Omega, , \\ q & = 0 & \text{on } \Sigma. \end{array}\right. \right\} \tag{2.60}$$

Remark 2.12.1 *The weak sentinel method approach is equivalent to a weak controllability problem.*

Theorem 2.12.1 *If the system in* (2.60)*is weakly controllable, then for any " positive, there exists a function $w_\epsilon \in L^{2(}O \times (0,T))$ which satisfies the conditions* (2.58) *and* (2.59)*.*

Proof. If the system in (2.60) is weakly controllable, then for $q(0) \in L^2(\Omega)$ and for all positive ϵ, there exists a function $w_\epsilon \in L^2\,(O \times (0,T))$ such that

$$\begin{array}{ll} \int_\Omega q(0).\widehat{y_0}dx & \leq \epsilon, \\ \|q(0)\|_{L^2(\Omega)} & \leq \epsilon, \end{array} \tag{2.61}$$

and so

$$\frac{\partial S}{\partial \tau}(\lambda,\tau) = \int_0^T \int_O (h + w_\epsilon)\,.y_\tau(x,t,\lambda,\tau)dxdt,$$

where y_τ is the solution of (2.14), so

$$\frac{\partial S}{\partial \tau}(\lambda,\tau) \quad \leq \int_\Omega q(0).\widehat{y_0}dx \quad \leq \epsilon,$$

which proves (2.58) and (2.59). ■

The constraint (2.60) can therefore theoretically be approximated with arbitrary precision. We can therefore use approximate controllability algorithm.

CHAPTER 3

Average Sentinel for a parabolic equation with incomplete data

In the modeling of the evolution system type, the source terms as well as the initial or boundary conditions may be unknow. In this chapter, we analyze the problem of identification of pollution term (the source) in a systems gouvred by a parabolic equation depending on unknown parameters in a deterministic manner [39]. We look for a control, independent of the values of these parameters, that need to introduce the sentinel in an averaged sense to be made precise [50]. The notion of averaged control for parameter-dependent family of parabolic system is introduced by Zuazua [78] and the sentinel method introduced by J.L.Lions [43, 46] is adapted to the estimation of this incomplet or unknow data in the problems gouverned by parabolic system in general, for exemple, pollution in river or a lake. So since the introduction of the sentinel method many authors developed several applications, such as in environment, in ecology [39].

The notion of average sentinel is very interested in the idenfication of the missing data when the system is depending on unknown parameters.

3.1 Setting of the problem

Let $\Omega \subset \mathcal{R}^d$, $d \succeq 1$, be a bounded domain with smooth boundary Γ and ω be an open non-empty subset of Ω [61] . Denote by $Q = \Omega \times [0, T]$ the space time cylinder where the equation holds and $\sum = \Gamma \times]0, T[$ for the lateral boundary.Let $y_\theta\,(x, t) = y(x, t, \theta)$ be the solution of the following system

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) - \operatorname{div}(a(x,\theta)\nabla y_\theta(x,t) & = f(x,t) \quad \text{in } Q, \\ y_\theta & = 0 \qquad\quad \text{on } \Sigma, \\ y_\theta(x,0) & = \; y_0\,(x) \quad \text{in } \Omega, \end{cases} \tag{3.1}$$

Where the diffusivity coefficients $a(x, \theta)$, taken to be scalar to simplify the study, are assumed to be bounded above and below by positive constants, and to depend on the uncertainty parameter θ where $\theta \in (0, 1)$ in a continuous manner. However, the dynamics of the state is governed by a parametrized operator $A(\theta)$ with $A(\theta) = \operatorname{div}(a(\theta)\nabla y)$.

We take that $y_0 \in L^2(\Omega)$, $f \in L^2(Q)$ and so that the system (3.1) admits a unique solution

$$y_\theta = y(x, t; \theta) \in C(0, T, L^2(\Omega)) \cap L^2(0, T, H_0^1(\Omega)) \qquad \text{for all } \theta \in (0, 1)$$

The motivation of the problem we consider is the following:

We address to the system (3.1) whose initial datum and the source term are unknown and the effective value of the parameter being unknown [65],

$$\begin{aligned} f &= \zeta + \lambda\widehat{\zeta}, \\ y_0 &= g + \tau\widehat{g}, \end{aligned} \tag{3.2}$$

where ζ and g are given. However, the terms $\lambda\widehat{\zeta}$ and $\tau\widehat{g}$ is unknow function with λ, τ are a small reals parameters [64].

We aim at choosing a control that would perform optimally in an averaged sense, i. e. so that, rather than controlling specific realisations of the adjoint state, the average with respect to is controlled. This allows building a control independent of the parameter and define the average sentinel to obtain a good estamation of the source term which called pollution term independly of the initial condition called missing data.

The notion of sentinel permits to distinguish and to analyse two types of incomplete data, the pollution term at which we look for information independently of the missing term that we do not want to identify. Typically, the average sentinel is a linear functional sensitive to certain parameters we are trying to evaluate, and insensitive to others which do not interest us. So we show that this functional can be associated to our system and allows to characterize the pollution term.

In this part, we study this system with incomplete initial data, we use the average sentinel concept, which relies on the following three objects: some state equation, some observation function and some control function to be determined.

Let $y(x,t,\lambda,\tau,\theta) = y(\lambda,\tau,\theta)$ be the unique solution of the problem in eqn.(3.1). We denote by an observation which is a measure of the concentration of the pollution taken on a non-empty open subset ω at the interval time $(0,T)$.

$$\int_0^1 y(x,t,\lambda,\tau,\theta)d\theta = y_{obs}(x,t), \text{ for all } (x,t) \in \overline{\omega} \times (0,T). \tag{3.3}$$

an observation which is a measure of the concentration of the pollution taken at the interval time $(0,T)$ and on a non empty open subset $O \subset \Omega$ called observatory.

Let h denote a function belonging to the space $L^2(\omega \times (0,T))$, for any control function $u \in L^2(\omega)$, We define the functional $S_m(\lambda,\tau)$ in the following manner:

$$S_m(\lambda,\tau) = \int_0^1 \int_0^T \int_O hy(x,t,\lambda,\tau)dxdtd\theta + \int_0^1 \int_0^T \int_\omega uy(x,t,\lambda,\tau)dxdtd\theta.$$

$$S_m(\lambda,\tau) = \int_0^1 \int_0^T \int_O (h\chi_O y(x,t,\lambda,\tau) + u\chi_\omega y(x,t,\lambda,\tau))dxdtd\theta. \tag{3.4}$$

where χ_O and χ_ω are the characteristic functions for the open sets O and ω respectively, such that

$$\chi_O \ : \ L^2(\Omega) \to L^2(O)$$
$$\chi_\omega \ : \ L^2(\Omega) \to L^2(\omega)$$

Definition 3.1.1 *Let S_m be a real function in equation (3.4) depending only on the parameters λ and τ. S_m is said a average sentinel defined by h if the following conditions are satisfied:*

i)

$$\frac{\partial S_m}{\partial \tau}(\lambda,\tau)\Big|_{\lambda=0,\tau=0}=0. \tag{3.5}$$

ii) there exists a average control $u \in L^2(\omega)$ such that:

$$\|u\|_{L^2(\omega)}=\min_{v\in U}\|v\| \tag{3.6}$$

where $U=\left\{u\in L^2(\omega),\ \textit{such that}\ \frac{\partial S_m}{\partial \tau}(\lambda,\tau)\Big|_{\lambda=0,\tau=0}=0\right\}.$

3.2 Average controllability problem

We take the function y_θ^0 which that solves the problem (3.1) for $\lambda=\tau=0$:

$$\begin{cases} \frac{\partial y_\theta^0}{\partial t}(x,t)+\operatorname{div}(a(x,\theta)\nabla y_\theta^0(x,t) & =\zeta \quad \text{in } Q,\\ y_\theta^0 & =0 \quad \text{on}\Sigma,\\ y_\theta^0(x,0) & =g\ (x) \quad \text{in } \Omega, \end{cases} \tag{3.7}$$

noting:

$$y_m\left(x,t,\lambda,\tau\right)=\int_0^1 y\left(x,t,\lambda,\tau\right)d\theta.$$

We take the function y_θ^τ and y_m^τ defined as:

$$y_\theta^\tau=\frac{\partial y_\theta}{\partial \tau}(\lambda,\tau)\Big|_{\lambda=0,\tau=0}$$

and

$$y_m^\tau=\frac{\partial y_m}{\partial \tau}(\lambda,\tau)\Big|_{\lambda=0,\tau=0}$$

with y_θ^τ the unique solution of the system:

$$\begin{cases} \frac{\partial y_\theta^\tau}{\partial t}(x,t)+\operatorname{div}(a(x,\theta)\nabla y_\theta^\tau(x,t) & =\zeta \quad \text{in } Q,\\ y_\theta^\tau & =0 \quad \text{on}\Sigma,\\ y_\theta^\tau(x,0) & =\widehat{g}\,(x) \quad \text{in } \Omega, \end{cases} \tag{3.8}$$

Remark 3.2.1 *.The condition in equation*(3.5) *holds if and only if:*

$$\int_0^1 \int_0^T \int_O (h\chi_{_O} + u\chi_{_\omega}) y_\theta^\tau(x,t) dx dt d\theta = 0. \tag{3.9}$$

In order to realize this equation (3.9), we introduce the classical adjoint system of (3.8):

$$\begin{cases} -\dfrac{\partial q_\theta}{\partial t} + \operatorname{div}(a(x,\theta)\nabla q_\theta) & = h\chi_{_O} + u\chi_{_\omega} & \text{in } Q, \\ q_\theta & = 0 & \text{on } \Sigma, \\ q_\theta(x,T) & = 0 & \text{in } \Omega, \end{cases} \tag{3.10}$$

with $q_\theta = q(x,t,\theta)$.

Theorem 3.2.1 *Let q_θ be the solution to the backward problem* (3.10)*, then the existence of an average sentinel insensitive to the missing data is equivalent to the average null-controllability problem*

$$q_m\,(x,0) = \int_0^1 q_\theta(x,0) d\theta = 0, \tag{3.11}$$

Proof. Multiplying the first equation in (3.10) by y_θ^τ, and integrating by parts over $(0,1)\times Q$, then we find:

$$\int_0^1 \int_\Omega q_\theta(x,T) y_\theta^\tau(T) dx d\theta - \int_0^1 \int_\Omega q_\theta(x,0) y_\theta^\tau(0) dx$$
$$= \int_\Omega \int_0^1 (h\chi_{_O} + u\chi_{_\omega}) y_\theta^\tau(x,t) d\theta dx,$$

then

$$\int_0^1 \int_\Omega q_\theta(x,0) \widehat{g}\,(x)\, dx d\theta + \int_\Omega \int_0^1 (h\chi_{_O} + u\chi_{_\omega}) y_\theta^\tau(x,t) d\theta dx = 0$$

Since $\widehat{g}\,(x)$ is independent of θ, then if (3.9) is verified we will have

$$\int_0^1 q(x,0,\theta) d\theta = 0.$$

■

3.3 Characterization of optimal control

The problem of the average null-controllability can be formulated as follows:

find a control u so that the solution of (3.10) satisfies (3.11) :

$$\int_0^1 q_\theta(x,0)d\theta = 0,$$

To proof this problem we use the idea of zuazua [78] and see[70] .

Theorem 3.3.1 *Solving a problem of the average null-controllability is equivalent to finding a control u so that the solution of system* (3.10) *satisfies equation* (3.11) *such that*

$$u\,(x,t) = -\int_0^1 \rho_\theta(x,t)\chi_\omega d\theta \tag{3.12}$$

where ρ_θ is the solution of system (3.16)*.*

Proof. To satisfy equation (3.11), we separate q in (3.10) into two components:

$$q(x,t,\theta) = q_1(x,t,\theta) + q_2(x,t,\theta)$$

and we take

$$\begin{cases} q_{1,\theta}(x,t) & = q_1(x,t,\theta) \\ q_{2,\theta}(x,t) & = q_2(x,t,\theta) \end{cases}$$

which are the solutions of the following systems:

$$\begin{cases} -\dfrac{\partial q_{1,\theta}}{\partial t}(x,t) + \operatorname{div}(a(x,\theta)\nabla q_{1,\theta}(x,t) & = h_{\chi_o} & \text{in } Q, \\ q_{1,\theta}\ (x,t) & = 0 & \text{on } \Sigma, \\ q_{1,\theta}(x,T) & = 0 & \text{in } \Omega, \end{cases} \tag{3.13}$$

and

$$\begin{cases} -\dfrac{\partial q_{2,\theta}}{\partial t}(x,t) + \operatorname{div}(a(x,\theta)\nabla q_{1,\theta}(x,t) & = u_{\chi_\omega} & \text{in } Q, \\ q_{2,\theta}\ (x,t) & = 0 & \text{on } \Sigma, \\ q_{2,\theta}(x,T) & = 0 & \text{in } \Omega, \end{cases} \tag{3.14}$$

thus, there exists a control u such that:

$$\int_0^1 q_{1,\theta}(x,t)d\theta + \int_0^1 q_{2,\theta}(x,t)d\theta = 0. \tag{3.15}$$

In that case, the solution of the adjoint system (3.16) of equation (3.14) depends also on the parameter θ :

$$\begin{cases} \dfrac{\partial \rho_\theta}{\partial t}(x,t) + \operatorname{div}(a(x,\theta)\nabla \rho_\theta(x,t) & = 0 & \text{in } Q, \\ \rho_\theta & = 0 & \text{on}\Sigma, \\ \rho_\theta(x,0) & = \rho_0\,(x) & \text{in } \Omega, \end{cases} \tag{3.16}$$

with $\rho_0(x)$ is a unknown term independent of θ.

We aim to determine the value of ρ_0 such that the averaged control u is defined as:

$$u|_{\omega}(x,t) = -\int_0^1 \rho_\theta(x,t)\chi_\omega d\theta. \tag{3.17}$$

Making the assumption that $q_{1,\theta}$ is independant of θ at time zero.

We then take equation (3.16) in (3.13), we find:

$$\begin{cases} -\dfrac{\partial q_{2,\theta}}{\theta\partial t}(x,t) + \operatorname{div}(a(x,\theta)\nabla q_{2,\theta}(x,t) & = -\int_0^1 \rho_\theta(x,t)\chi_\omega d\theta \quad \text{in } Q, \\ q_{2,\theta} & = 0 \quad \text{on } \Sigma, \\ q_{2,\theta}(x,T) & = 0 \quad \text{in } \Omega, \end{cases} \tag{3.18}$$

Then, for given $\rho_0(x)$, the first equation in ((3.16)) have unique solution. To find ρ_0, such that the solution in (3.16) satisfied equation (3.19)

$$\int_0^1 q_{2,\theta}(x,t)d\theta = -\int_0^1 q_{1,\theta}(x,t)d\theta. \tag{3.19}$$

We define an linear operator Λ by:

$$\Lambda(\rho_0(x)) = -\int_0^1 q_{1,\theta}(x,0)d\theta. \tag{3.20}$$

To achieve this , we multiply in equation (3.18) by $\widehat{\rho}$ where $\widehat{\rho}_0$ is the solution in equation (3.16) correspondent to $\widehat{\rho}_0$ who is independent of θ and if we integrated by part over $(0,T)$, we obtain:

$$-\langle \widehat{\rho}_\theta(T), q_{2,\theta}(x,T)\rangle_{L^2(\Omega)} + \langle \widehat{\rho}_\theta(0), q_{2,\theta}(x,0)\rangle_{L^2(\Omega)} = \left\langle \widehat{\rho}_0, -\int_0^1 \rho_\theta\chi_\omega d\theta \right\rangle_{L^2(\Omega)}$$

if we integrate for θ over $(0,1)$, we will have

$$\int_0^1 \langle \widehat{\rho}_0, q_{2,\theta}(0)\rangle \, d\theta = \int_0^T \int_\omega \int_0^1 \left\langle \widehat{\rho}_\theta, \int_0^1 \rho_\theta\chi_\omega d\theta \right\rangle d\theta dx dt,$$

we take

$$\Lambda\widehat{\rho}_0 = -\int_0^1 q_{2,\theta}(x,0)d\theta.$$

then, we find:

$$\langle \widehat{\rho}_0, \Lambda\widehat{\rho}_0\rangle = \int_0^T \int_\omega \int_0^1 \left\langle \widehat{\rho}_0, \int_0^1 \rho_\theta\chi_\omega d\theta \right\rangle d\theta dx dt,$$

if we take $\widehat{\rho}_0 = \rho_0$ then, we find:

$$\langle \rho_0, \Lambda \rho_0 \rangle = \int_0^T \int_\omega \left| \int_0^1 \rho_\theta d\theta \right|^2 d\theta dx dt,$$

We then introduce, the norm defined by

$$\|\rho_0\|_F^2 = \int_0^T \int_\omega \left(\int_0^1 \rho(x,t,\theta) \right)^2 d\theta dx dt, \tag{3.21}$$

by the result of Mizohata [61], we find:

$$\|\rho_0\|_F = 0 \text{ wich give } \rho = 0 \text{ on } \omega \times (0,T).$$

The space $L^2(\Omega)$ is not complete for the norm in equation (3.21), then we introduce F as a Hilbert space completed of $L^2(\Omega)$, we note F' the dual space of F, then the linear operator $\wedge$ such as:

$$\Lambda : F \to F'$$

is an isomorphism, which gives the result. ■

3.4 Identification of the pollution term

To show how the sentinel defined above permits to estimate the pollutin term, we consider m_0 be the measured state of the system on the observatory O during the interval $[0,T]$, then the measured sentinel associate to m_0 is given by :

$$S_{obs}(\lambda,\tau) = \int_0^1 \int\int_{\Omega\times]0,T[} (h\chi_{O} + u\chi_{\omega}) m_0(x,t,\lambda,\tau) dx dt d\theta.$$

Theorem 3.4.1 *The pollution term is identified as follows:*

$$\int_0^1 \int_\Omega q(h)\widehat{f} d\Omega = S_{obs}(\lambda,\tau) - S(0,0)$$

Proof. We know that

$$S_{obs}(\lambda,\tau) = S(0,0) + \lambda \frac{\partial S}{\partial \lambda}(\lambda,\tau)\Big|_{\lambda=0,\tau=0} + O(\lambda,\tau)$$

with

$$\frac{\partial S}{\partial \lambda}(\lambda, \tau) = \int_0^1 \int_\Omega (h\chi_{_O} + w\chi_{_\omega}) y_\lambda dx dt$$

and

$$\lambda \frac{\partial S}{\partial \lambda}(\lambda, \tau)\bigg|_{\lambda=0,\tau=0} = S_{obs}(\lambda, \tau) - S(0,0)$$

Hence

$$\int_0^1 \int_\Omega q(h)\widehat{f} d\Omega = S_{obs}(\lambda, \tau) - S(0,0)$$

where y_θ^λ is the solution of the following system:

$$\begin{cases} \frac{\partial y_\theta^\lambda}{\partial t}(x,t) - \operatorname{div}(a(x,\theta)\nabla y_\theta^\lambda(x,t) & = \widehat{f} \quad \text{in } Q, \\ y_\theta^\lambda & = 0 \quad \text{in} \Sigma, \\ y_\theta^\lambda(x,0) & = 0 \quad \text{in } \Omega, \end{cases}$$

■

CHAPTER 4

Average sentinel of a parabolic equation with an unknown reaction

In this chapter, we analyse the identification of the amount of pollutant discharged problem by each source in a heat system when the dynamics of the state is governed by a parameterized unknown operator. In that case, we introduce a notion of average sentinel.

We consider in this work a water lake polluted by a chemical species. The phenomena we have take into account are the dispersion and the consumption of the pollutant [39].
One may think of a lake polluted by biological oxygen demand (BOD) and of unknown consumption proportional to the concentration of BOD. The physical problem is to identify the amount of pollutant discharged by each source [39]. Measurements are available to achieve this goal. These are the averaged pollutant concentrations measured at a few points, which we call the observatory. The notion of averaged control for a parameter-dependent family of parabolic systems is introduced by Zuazua [78], and the sentinel method introduced by Lions [51] is adapted to the estimation of this incomplete or unknown data in the problems governed by parabolic system in general, for example, pollution in lakes or in a river. So, since the introduction of the sentinel method many authors developed several applications, such as in the environment, in ecology [29].
The notion of sentinel is very interested in the identification of the missing data when the

system depends on unknown parameters, for instance we refer to [1, 28, 29, 70]. With σ null, the problem becomes a classical control problem [70], [78], if σ is different from zero and at the same time is given, the problem has been studied by Kernevez [39], now, if σ is different from zero and at the same time unknown, then the problem has become delicate, because of the nonlinearity in σ, for this we have introduced the decomposition method to obtain an independent problem system in σ, from which we can make an efficient calculation algorithm, which gives the novelty of our work.

The difference between our problem setting in this section and the problem setting in Kernevez [39] is the parameter σ, that is to say, the difficulty of the problem resists in the change of the parameter, which becomes an unknown parameter, which gives a nonlinear problem with σ.

To resolve this problem, we use in the first part, the decomposition method to isolate the parameter σ see Lions [42], in the second, we use the gradient method to identify the averaged control parameter, this method is more efficient, it is confirmed by the example given in the numerical part, in the third, we calculate the average solution and the averaged parameter control, and in the last, we give the numerical example to confirm our result.

4.1 Setting of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, denote the water field, with smooth boundary Γ and design with ω be an open non empty subset of Ω. Denote by $Q = \Omega \times (0, T)$ the space-time cylinder where the equation holds and $\Sigma = \Gamma \times (0, T)$ for the lateral boundary, we will assume that the parameter $\sigma \in (0, 1)$, and $z(t, x)$ is the solution of the following system :

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,x) + Az(t,x) & = \sum_{i=1}^{N_1} \lambda_i s_i(t)\delta(x - a_i) \quad \text{in } Q, \\ \frac{\partial z}{\partial \eta} & = 0 \quad \text{on } \Sigma, \\ z(0,x) & = \sum_{j=1}^{N_2} \tau_J \chi_J(x) \quad \text{in } \Omega, \end{cases} \tag{4.1}$$

where

$Az = -\Delta z + \sigma z$,

$\frac{\partial z}{\partial \eta} = \nabla z . \eta$, η is the unit co-normal vector,

N_1 is the discharge number,

$\lambda_i s_i(t)$ is the flow rate of the i-th source,

$\sum_{i=1}^{N_1} \lambda_i s_i(t)$ is the total flow rate,

$\delta(x - a_i)$ is the Dirac mass at the discharge point a_i,

N_2 is the number of missing terms.

The positive parameter σ characterizes a first-order chemical reaction of disappearance supposed in $(0, 1)$. That is to say that the consumption of pollutants is of the form σz.
The points $a_i = (a_{i1}, a_{i2})$, $1 \leq i \leq N_1$ are located in Ω , and are the sources of pollution.
λ_i is the i-th source intensity of the pollutant discharge.

s_i is defined from $(0, T)$ to $\mathbb{R}$, it's the shape of the discharge of the i-th source of pollution on

a period of T hours.
The indicator function of the element Ω_J is $\chi_j(x) = \begin{cases} 1 \text{ if } x \in \Omega_J \\ 0 \text{ if } x \notin \Omega_J \end{cases}$.
For all s_i and χ_j are given, but the terms $\lambda_i s_i$ and $\tau_j \chi_j$ are unknown functions.
The term $\tau_j \chi_j$ describe the missing data and $\lambda_i s_i$ the pollution term.
This work aims to identify the average pollution term of the system not affected by the missing term.
There are two possible approaches to this problem, one is more classical and uses the least square method (see G.Chavent [24]), but the problem in this method that the pollution and the missing terms play the same role, so we can not separate them.
The other is the sentinel method introduced by J.L.Lions [49], which is used to study systems of incomplete data.
The notion permits to distinguish and to analyze two types of incomplete data, the pollution term and the missing terms.
So, we show that this functional can be associated to our system and allows to characterize the pollution terms.[39].

Let us denote:

τ_J is the initial condition on element Ω, $1 \leq J \leq N_2$.

$\lambda = \lambda_1, ..., \lambda_i, ..., \lambda_{N_1}$ and $\tau = \tau_1, ..., \tau_J, ..., \tau_{N_2}$,

$\nu = \lambda_1, ..., \lambda_i, ..., \lambda_{N_1}; \tau_1, ..., \tau_J, ..., \tau_{N_2} = (\lambda, \tau)$ of length $N = N_1 + N_2$.

To overcome the non-linearity of the solution with the parameter σ.

4.2 Decomposition method

We assume that the system (4.1) admits a solution and we write it in the series form:

$$z(t,x;\sigma) = \sum_{i=0}^{\infty} z_i(t,x)\sigma^i, \tag{4.2}$$

we replace it in the first equation of system (4.1), and by identification, we get

$$\sum_{i=0}^{\infty} \frac{\partial z_i}{\partial t}(t,x) - \Delta z_i(t,x)\sigma^i + \sum_{i=0}^{\infty} z_i(t,x)\sigma^{i+1} = \sum_{i=1}^{\infty} \lambda_i s_i(t)\delta(x-a_i),$$

which is equivalent to say

$$\left(\frac{\partial z_0}{\partial t} - \Delta z_0\right)\sigma^0 + \sum_{i=1}^{N_1}\left(\frac{\partial z_i}{\partial t} - \Delta z_i\right)\sigma^i + \sum_{i=1}^{N_1} z_{i-1}\sigma^i = \sum_{i=1}^{N_1} \lambda_i s_i(t)\delta(x-a_i),$$

then

$$\left(\frac{\partial z_0}{\partial t} - \Delta z_0 - \sum_{i=1}^{N_1} \lambda_i s_i(t)\delta(x-a_i)\right)\sigma^0 + \sum_{i=1}^{N_1}\left(\frac{\partial z_i}{\partial t} - \Delta z_i + z_{i-1}\right)\sigma^i = 0,$$

this is equivalent to

$$\begin{cases} \frac{\partial z_0}{\partial t} - \Delta z_0 - \sum_{i=1}^{N_1} \lambda_i s_i(t)\delta(x-a_i) & = 0, \\ \frac{\partial z_i}{\partial t} - \Delta z_i + z_{i-1} & = 0, \quad i = \overline{1,\infty}, \end{cases} \tag{4.3}$$

adding the initial condition and the boundary conditions, the precedent system becomes :

$$\begin{cases} \frac{\partial z_0}{\partial t} - \Delta z_0 & = \sum\limits_{i=1}^{N_1} \lambda_i s_i(t) \delta(x - a_i) & \text{in } Q, \\ \frac{\partial z_0}{\partial \eta} & = 0 & \text{on } \Sigma, \\ z_0\,(0, x) & = \sum\limits_{i=1}^{N_2} \tau_j \chi_{\Omega_j} & \text{in } \Omega, \end{cases} \tag{4.4}$$

and

$$\begin{cases} \frac{\partial z_i}{\partial t} - \Delta z_i & = -z_{i-1} & \text{in } Q, \\ \frac{\partial z_i}{\partial \eta} & = 0 & \text{on } \Sigma, \quad 1 \\ z_i\,(0, x) & = 0 & \text{in } \Omega, \end{cases} \tag{4.5}$$

for all $i = \overline{1, \infty}$.

Then, the averaged solution denoted $\overline{z}(t,x)$ is:

$$\overline{z}(t,x) = \int_0^1 z(t,x;\sigma) d\sigma = \sum_{i=1}^{\infty} \frac{1}{i+1} z_i\,(t,x)\,. \tag{4.6}$$

Theorem 4.2.1 *The average solution given by* ((4.6)) *is well-defined. (see in Figure* (4.1)*)*

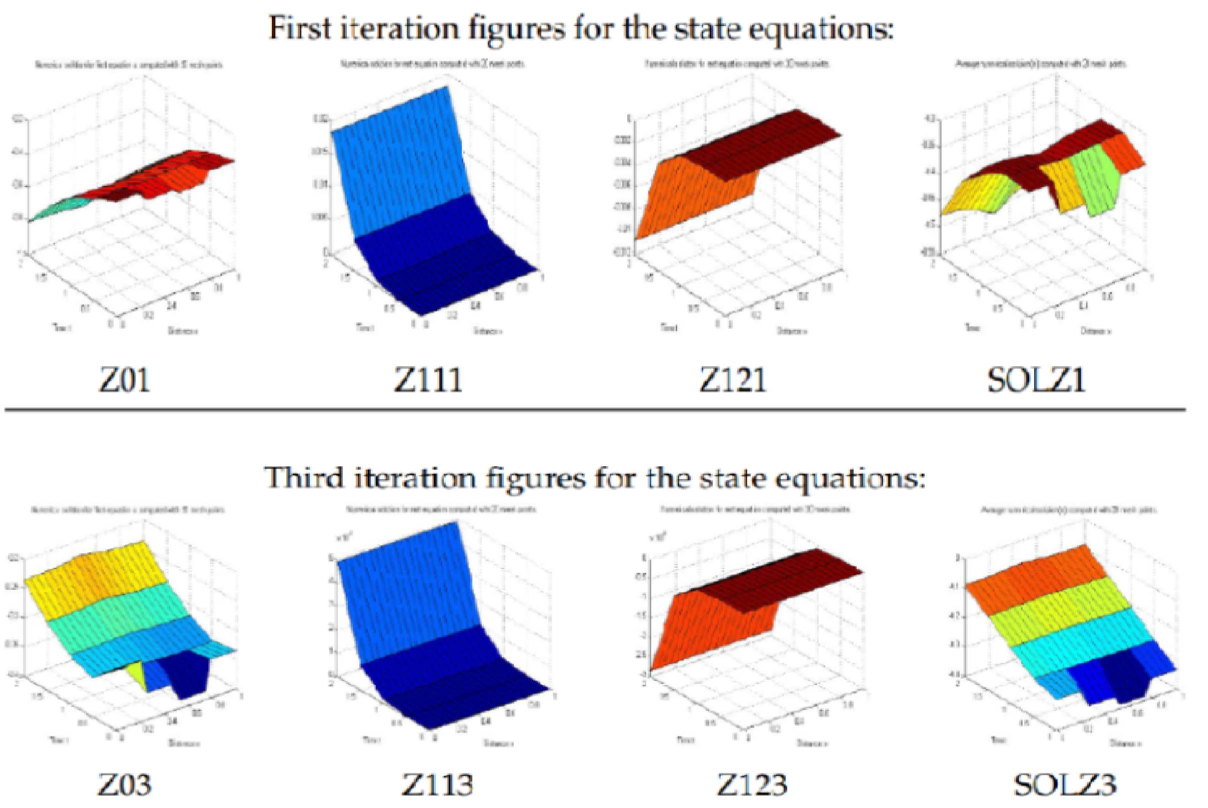

Figure 4.1: The approximation of the average solution of state equation.

The general term of the average solution given by (4.6) *is alternated and decreases in absolute value towards zero. So this series is convergent(d'Alembert's theorem).*

Moreover, suppose that the sensors provide some punctual averaged observation in the points $\{x_k\}$ given by

$$\{z(t, x_k)\}, k = \overline{1, M}, \tag{4.7}$$

where M is the number of observation sensors.

We suppose that the available data are continuous-time average observations of the pollutant concentration at each of these M observation points.

Suppose we do not know the parameters ν. In the counterpart, we have at our disposal $z(x_k, t; \nu)$ at M points x_k, the time history, as time t varies in the time interval $(0, T)$, of the average pollutant concentration

$$\overline{z} : t \to \overline{z}(t, x_k; \nu), \quad 1 \leq k \leq M.$$

Let's define the operator B between $\mathcal{R}^N$ and $H = L^2\left(0, T; \mathbb{R}^M\right)$,

where

$$\overline{z} = B\nu, \tag{4.8}$$

where $\overline{z}$ is the average calculated observation corresponding to the parameter ν.

Let's z_{dm} the given average observation vector defined on the interval $[0, T]$, then we go to find $\overline{\nu}$ such that

$$z_{dm} = B\overline{\nu}, \tag{4.9}$$

with $\overline{\nu} \in \mathcal{R}^N$.

Then, we define for that the cost function

$$J(\nu) = \frac{1}{2}|B\nu - z_{dm}|_H^2, \tag{4.10}$$

where $|.|_H$ denotes the norm of H.

We take

$$\Lambda = B^* \times B, \tag{4.11}$$

where Λ is the $N \times N$ matrix.

The minimum of (4.11) is characterized by $B^*B\overline{\nu} - B^*z_{dm} = 0$, so we have

$$\Lambda\overline{\nu} = B^*z_{dm}. \tag{4.12}$$

We essentially suppose, B one-to- one. Then, it is well-know that Λ is strictly positive definite, Λ^{-1} exists then

$$\overline{\nu} = \Lambda^{-1}B^*z_{dm}, \tag{4.13}$$

if e_n denotes the n-th vector of the canonical basis of $\mathcal{R}^N$, the n-th component of $\overline{\nu}$ is given by $\overline{\nu}_n = \langle \overline{\nu}, e_n \rangle$ and $\overline{\nu}_n = \langle \Lambda^{-1}B^*z_{dm}, e_n \rangle$, Λ^{-1} is symmetric, then

$$\overline{\nu}_n = \left\langle B^*z_{dm}, \Lambda^{-1}e_n \right\rangle, \tag{4.14}$$

where ω_n is defined by $\Lambda^{-1}e_n = \omega_n$, then

$$\Lambda\omega_n = e_n, \tag{4.15}$$

Remark 4.2.1 *We solve this equation* (4.15) *using the gradient method.*

4.3 Gradient methods

To solve this equation $\Lambda\omega_n = e_n$,, since the matrix Λ is symmetric, then the resolution of the system (4.15) is equivalent to the minimization of $J(w) = \frac{1}{2} \left\langle (\Lambda\omega_n, w) \right\rangle_H - \left\langle (e_n, w) \right\rangle_{\mathbb{R}^N}$, for all w on $\mathcal{R}^N$. (See in Figure 4.2).

To solve this equation $\Lambda\omega_n = e_n$,, since the matrix Λ is symmetric, then the resolution of the system (4.15) is equivalent to the minimization of $J(w) = \frac{1}{2} \left\langle (\Lambda\omega_n, w) \right\rangle_H - \left\langle (e_n, w) \right\rangle_{R^N}$, for all w on $\mathcal{R}^N$.

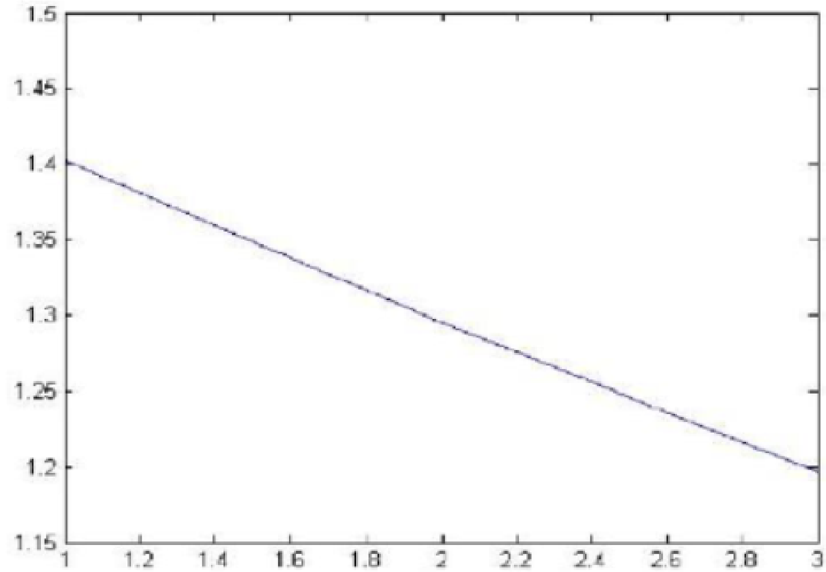

Figure 4.2: Error of the gradient relative to the number of iterations.

Proof. However the gradient methods are based upon the fact that, if we give the vector of controls $\gamma = (\lambda, \tau)$, the cost function $J(\gamma)$ and its gradient $J'(\gamma)$ are obtained by solving the following two systems of equations by the decomposition method for, respectively, the state ρ and the adjoint state q :

$$\begin{cases} \rho' + A\rho & = \sum_{i=1}^{N_1} \lambda_i s_i(t)\delta(x - a_i) & \text{in } Q, \\ \frac{\partial \rho}{\partial \eta} & = 0 & \text{on } \Sigma, \\ \rho\,(x, 0) & = \sum_{j=1}^{N_2} \tau_j \chi_{\Omega_j} & \text{in } \Omega, \end{cases} \tag{4.16}$$

and

$$\begin{cases} -q' + A^* q & = \sum_{k=1}^{M} w_k(t)\delta(x - x_k) & \text{in } Q, \\ \frac{\partial q}{\partial \eta} & = 0 & \text{on } \Sigma, \\ q\,(x, T) & = 0 & \text{in } \Omega, \end{cases} \tag{4.17}$$

with $w_k(t) = \overline{\rho}(x_k, t)$ where $\overline{\rho}$ is the average of ρ with σ. ∎

Lemma 4.3.1 *The components of the gradient of J are given by*

$$\begin{cases} \left\{ \int_0^T s_i(t)\overline{q}(t, a_i)dt - \delta_{in} \right\}_{1 \leq i \leq N_1} \\ \left\{ \int_{\Omega_j} \overline{q}\,(0, x)\, dx \right\}_{1 \leq j \leq N_2} \end{cases} \tag{4.18}$$

where $\overline{q}$ is the average of q with σ (see in Figure.4.3).

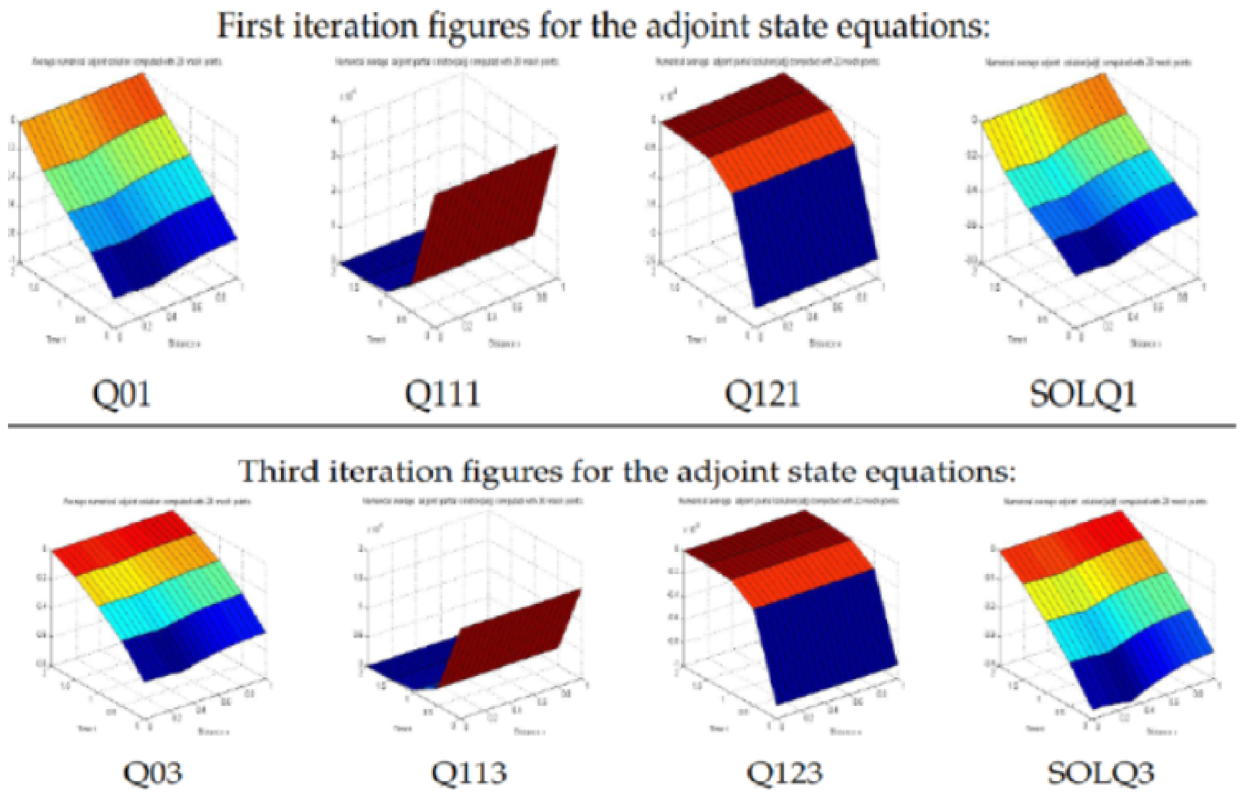

Figure 4.3: The approximation of the average solution of adjoint state equation.

Proof. We take the derivative following the direction e_i, then

$$\langle \Lambda\omega_n, e_i\rangle = \langle e_n, e_i\rangle = \delta_{ni} = \begin{cases} 1 & \text{if } n = i, \\ 0 & \text{if } n \neq i, \end{cases}$$

then, we have $\langle B^* B\omega_n, e_i\rangle_{R^N} = \langle B\omega_n, Be_i\rangle_H$, since

$$\begin{aligned} \langle B\omega_n, Be_i\rangle_H &= \int_0^T \sum_{k=1}^M \overline{\rho}(t, x_k)\overline{\Psi_i}(t, x_k)dt \\ &= \sum_{k=1}^M \int_0^T \int_0^1 \overline{\rho}(t, x_k) \int_\Omega \delta\left(x - x_k\right) \Psi_i(t, x; \sigma) dx d\sigma dt \\ &= \int_0^1 \left\langle \sum_{k=1}^M \overline{\rho}(t, x_k)\delta\left(x - x_k\right), \Psi_i(t, x; \sigma)\right\rangle_{L^2(0,T;\Omega)} d\sigma \end{aligned}$$

since $\sum_{k=1}^M \overline{\rho}(t, x_k)\delta\left(x - x_k\right) = -\frac{\partial q}{\partial t} + A^* q$, then

$$\begin{aligned} \int_0^1 \left\langle \sum_{k=1}^M \overline{\rho}(t, x_k)\delta\left(x - x_k\right), \Psi_i\right\rangle_{L^2(0,T;\Omega)} d\sigma &= \int_0^1 \left\langle -\frac{\partial q}{\partial t} + A^* q, \Psi_i\right\rangle_{L^2(0,T;\Omega)} d\sigma \\ &= \int_0^1 \left\langle q, \frac{\partial \Psi_i}{\partial t} + A\Psi_i\right\rangle_{L^2(0,T;\Omega)} d\sigma, \end{aligned}$$

since $\frac{\partial \Psi_i}{\partial t} + A\Psi_i = s_i(t)\delta(x-a_i)$, then

$$\begin{aligned} \int_0^1 \langle q, s_i(t)\delta(x-a_i)\rangle_{L^2(0,T;\Omega)} d\sigma &= \langle s_i(t)\delta(x-a_i), \overline{q}\rangle_{L^2(0,T)}, \langle,\rangle \text{ is commutative}), \\ &= \langle s_i(t), \overline{q}(t,a_i)\rangle_{L^2(0,T)} \\ &= \int_0^T s_i(t)\overline{q}(t,a_i)dt - \delta_{in}, \end{aligned}$$

then

$$gradJ = \begin{cases} \left\{\int_0^T s_i(t)\overline{q}(t,a_i)dt - \delta_{in}\right\}_{1\le i\le N_1} \\ \left\{\int_{\Omega_j} \overline{q}(x,0)dx\right\}_{1\le j\le N_2} \end{cases}$$

■

4.4 Average solution of the heat equation with an unknown parameter

We have

$$z(t,x;\sigma) = \sum_{i=0}^{\infty} \sigma^i z_i(t,x), \tag{4.19}$$

then the averaged solution denoted $\overline{z}(t,x)$ is

$$\begin{aligned} \overline{z}(t,x) &= \int_0^1 z(t,x;\sigma)d\sigma, \\ &= \sum_{i=1}^{\infty} z_i(t,x)\int_0^1 \sigma^i d\sigma, \end{aligned} \tag{4.20}$$

then,

$$\overline{z}(t,x) = \sum_{i=1}^{\infty} \frac{1}{i+1} z_i(t,x). \tag{4.21}$$

In the same way, we calculate the average adjoint state noted $\overline{q}(t,x)$

$$\overline{q}(t,x) = \sum_{j=1}^{\infty} \frac{1}{j+1} q_j(t,x). \tag{4.22}$$

Remark 4.4.1 *The average solution $\overline{q}$ is also well defined (see Theorem 3.1).*

We deduce the components of the average gradient :

$$\begin{cases} \left\{\int_0^T s_i(t)\overline{q}(t,a_i)dt - \delta_{in}\right\}_{1\le i\le N_1} \\ \left\{\int_{\Omega_j} \overline{q}(x,0)dx\right\}_{1\le j\le N_2} \end{cases} \tag{4.23}$$

Now, we give two positive parameters ϵ and θ, and we take :

$$\nu_{j+1} = \nu_j + \theta gradJ. \tag{4.24}$$

If $\nu_{j+1} - \nu_j \leq \epsilon \Rightarrow$ stop.

then,

$$u_n = (z_{dm}, Bw_n)_H = \nu_j, \tag{4.25}$$

else, we repeat the calculation.

4.5 Numerical application

To determine the average solution $z(t,x)$, we take $x \in [0,1]$, $t \in [0,2]$, we are able to composethis accordingto the established stucture, $dx = \frac{1}{19}$, dt $= \frac{1}{2}$.

We take in (4.4) : $N_1 = 1$, $\lambda_{\ 1} = -0.2115$, $a_1 = x(10)$, $z_{Obs}(t) = exp(-t)$, $S(t) = t$, $N_2 = 4$, $\tau_j = -0.3192,\ -0.3839,\ -0.4128,\ -0.3554$ for $j = \overline{1,4}$.

We calculate z_0, by solving the equation (4.4) and we calculate z_i , $i = \overline{1,12}$, by solving the equations (4.5). To calculate $z(t)$, we use the equation (4.7) and we take $M = 1$, $x_{obs} = x(7)$, to calculate the approximate average solution, then we find these results at $t = 1$

$$\overline{z}\,(t,x) = \begin{matrix} -0.2087 & -0.2087 & -0.2086 & -0.2085 & -0.2082 \\ -0.2078 & -0.2073 & -0.2067 & -0.2060 & -0.2054 \\ -0.2063 & -0.2073 & -0.2081 & -0.2089 & -0.2095 \\ -0.2099 & -0.2103 & -0.2105 & -0.2106 & -0.2106 \end{matrix}$$

$$\overline{q}\,(t,x) = \begin{matrix} -0.1530 & -0.1530 & -0.1537 & -0.1550 & -0.1570 \\ -0.1595 & -0.1615 & -0.1559 & -0.1197 & -0.1439 \\ -0.1389 & -0.1344 & -0.1305 & -0.1272 & -0.1245 \\ -0.1224 & -0.1207 & -0.1197 & -0.1191 & -0.1191 \end{matrix}$$

To make the graph of the norm of the gradient vector with $\theta = 0.1$ and $\varepsilon = 0.01$,

we find the control $u = 0.1722 - 0.2846 - 0.3427 - 0.3753 - 0.3258$.

Conclusion and perspectives

This thesis is devoted to parabolic problems including data without missing (unknown or partially known), where we dealt with the identification of the pollution term which is independent of variations of the missing term.

In the first section, we introduced the sentinel method, the most commonly used strategy to obtain information about missing terms by calculating a weighted average of observations. The search for this information naturally leads us to an opposite problem.

In the second part, we carried out a study of pararbolic systems with incomplete data on the edge and in the initial data where we used the sentinel method of J. L. Lions to answer the question.

This showed that we can estimate the pollution term independently of other data that we do not want to identify. This method consists of transposing a problem of identification or estimation of incomplete data into a problem of average exact or averaged weak controllability with constraints on averaged control. In solving these problems, it is necessary to have measured data of the state, for this, we have treated the two observation cases, with and without noise.

In the third part, our goal was the estimation of the pollution term of a parabolic system where the initial condition and the boundary condition are partially known. With the same average sentinel method we answered the question.

In the forth chapter, we have used the method of decomposing the solution of the system in series allowed us to transfer the problem to a series of simple problems. The calculation of the average sentinel requires the resolution of a very large linear system (not recommended for numerical calculation) this made us think of an indirect method (optimization method). In addition, the calculation of the adjoint state allows the components of the gradient vector to be calculated. This new idea led to surprising results, we could demonstrate the convergence of the series in a numerical way. It is believed that this new idea can be applied in several areas: physical problems, wave problems : acoustics, bilinear control

These results open up numerical perspectives of this method. The digital simulation tools available can still be improved to respond to the many problems current environmental conditions. Today, we hope that the development of a new techniques will allow a better estimation of unknown parameters in systems polluted.

However, much remains to be done theoretically, such as the search for an average observatory which brings us back to averaged optimal control and therefore to the estimation of the pollution term instead of xing it at the start of the work, and the link remains to be made with other approaches: no-regret and least-regret control and especially robust control.

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