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Stability analysis and optimal control of a time-fractional diffusion systems : a numerical approach

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Stability Analysis and Optimal Control of a time-Fractional Diffusion Systems : a Numerical Approach

ملخص : تتناول هذه الأطروحة أساسيات في نظرية التحكم، مع التركيز على كل من الأنظمة الخطية وغير الخطية. حيث نميز الفرق بين المسائل المباشرة وغير المباشرة في التحكم، فنجد أن المسالة غير المباشرة تهدف إلى تقدير البيانات المفقودة استنادًا إلى نتائج جزئية، بينما تسعى المسالة المباشرة إلى إيجاد الحلول بناءً على الظروف الأولية المعطاة. فتؤكد أيضا على أن إمكانية المراقبة والتحكم تعتبر عنصرا مهما، تتضمن أيضا طرقًا لتقدير البيانات الأولية المفقودة من خلال الملاحظات. كما تناقش الأطروحة المسائل الثنائية، واستخدام أجهزة الاستشعار، وتأثيراتها على التحكم الأمثل، مع عرض تقنيات مثل مبدأ بونترياغين الأقصى.

يبرز العمل أهمية الاستقرار في الأنظمة الديناميكية، مع توضيح تحليل الاستقرار وطرائق الاستقرار من خلال التحكم المرتد (النظام الديناميكي المالي). كما أن الأطروحة تشمل تطبيقين رئيسيين : الأول يتناول مسالة ستيفان، المتمثل انتقال الحرارة وتغيير الطور، وتطبيق طريقة الحارس غير الخطية. أما التطبيق الثاني، فيركز على الأنظمة الهجينة ذات الرتبة الكسرية، التي تدمج الديناميكيات المستمرة والمتقطعة، وتعرض تعقيدات إمكانية المراقبة والتحكم في هذا السياق.

كلمات مفتاحية : التحكم الأمثل، مبدأ بونترياجين الأقصى، مسالة ستيفان، الاستقرار، المشتقات الكسرية، الأنظمة الهجينة، المراقبة، المشاهدة، مسائل الحدود الحرة.

Stability Analysis and Optimal Control of a time-Fractional Diffusion Systems : a Numerical Approach

Abstract : This thesis addresses fundamental control theory challenges, focusing on linear and nonlinear systems. Introduces the distinction between direct and indirect problems in control, where the indirect problem aims to estimate missing data based on partial results. In contrast, the direct problem seeks solutions given initial conditions. The thesis emphasizes the importance of observability and controllability, exploring methods to estimate missing initial data through observations. It discusses dual problems, the use of sensors, and the implications for optimal control, showcasing techniques like Pontryaguine's maximum principle.

The work highlights the importance of stability in dynamic systems, detailing stability analysis and methods for stabilization through feedback control (the financial dynamical system). It also includes two key applications : the first explores the Stefan problem, a critical challenge in heat transfer and phase transition, and we apply the nonlinear sentinel method. The second focuses on hybrid systems of fractional order, which integrate continuous and discrete dynamics, presenting the complexities of controllability and observability within this context.
Keywords : Optimal control, Pontryaguine's maximum principle, Stefan problem, stability, fractional derivatives, hybrid systems, controllability, observability, free-boundary problems.

Stability Analysis and Optimal Control of a time-Fractional Diffusion Systems : a Numerical Approach

Résumé : Cette thèse aborde des défis fondamentaux dans la théorie du contrôle, en se concentrant à la fois sur les systèmes linéaires et non linéaires. Elle introduit la distinction entre les problèmes directs et indirects en contrôle, où le problème indirect vise à estimer des données manquantes à partir de résultats partiels. En revanche, le problème direct cherche des solutions à partir de conditions initiales données. La thèse met l'accent sur l'importance de l'observabilité et de la contrôlabilité, en explorant des méthodes pour estimer les données initiales manquantes à travers des observations. Elle discute des problèmes duals, de l'utilisation de capteurs, et des implications pour le contrôle optimal, en mettant en avant des techniques telles que le principe du maximum de Pontryaguine.

Ce travail souligne l'importance de la stabilité dans les systèmes dynamiques, en détaillant l'analyse de la stabilité et les méthodes de stabilisation via le contrôle par rétroaction (le système dynamique financier). La thèse présente deux applications clés : la première explore le problème de Stefan, un défi crucial dans le transfert de chaleur et la transition de phase, et on applique de la méthode sentinelle non linéaire. La deuxième se concentre sur les systèmes hybrides d'ordre fractionnaire, qui intègrent des dynamiques continues et discrètes, présentant les complexités de la contrôlabilité et de l'observabilité dans ce contexte.
Mots clés : Contrôle optimal, principe du maximum de Pontryaguine, problème de Stefan, stabilité, dérivées fractionnaires, systèmes hybrides, contrôlabilité, observabilité, problèmes de frontière libre.

Dedication

I dedicate this work to my parents (Abdelkader and Nadia), whose guidance and unwavering belief in me have always inspired and strengthened me. His values and lessons have shaped my journey in ways I will forever be grateful for.

To my sisters (Ibtisam and Rayenne) and brother (Mohamed Anis), I owe my deepest thanks for their unconditional support, love, and encouragement, which have carried me through every challenge and triumph. Their presence in my life has been constant in moments of difficulty and joy.

I also extend my heartfelt gratitude to my friend Dounia, whose constant encouragement, understanding, and companionship have given me the strength and motivation to continue on this path. Their belief in me and their invaluable friendship has been an enduring source of strength throughout this remarkable journey.

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I want to begin by expressing my heartfelt thanks to our university's esteemed Director, Professor Dibi Zohir, for his immense support and visionary leadership in serving the university and elevating its stature. His wise guidance and continued encouragement have been a great source of inspiration. I am proud and grateful to have been under his leadership.

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I also thank Dr. Dehilis Sofiane, for his valuable guidance and great interest in my success. His insightful advice and the support he provided during the examination period motivated me to give my best. I feel incredibly fortunate to have been surrounded by such an experienced and visionary committee chair.

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I also thank professor Oussaeif Taki-Eddine, my co-supervisor, for his constant support and valuable contributions. His willingness to assist at any time and his encouragement for me to try new things were key factors in the progress of my thesis. Having supervisors and professors committed to supporting their students and accompanying them with their experiences was a blessing.

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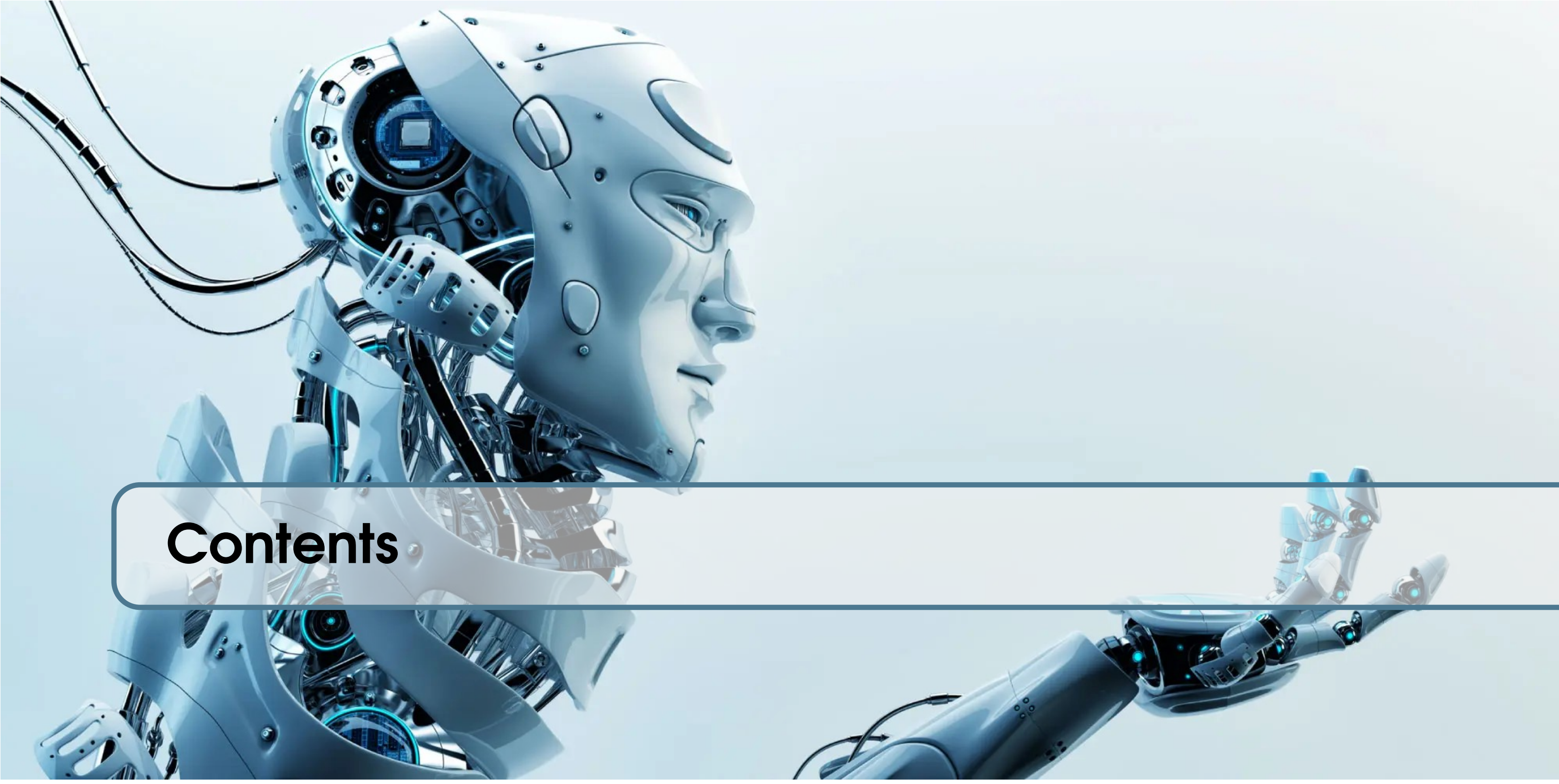

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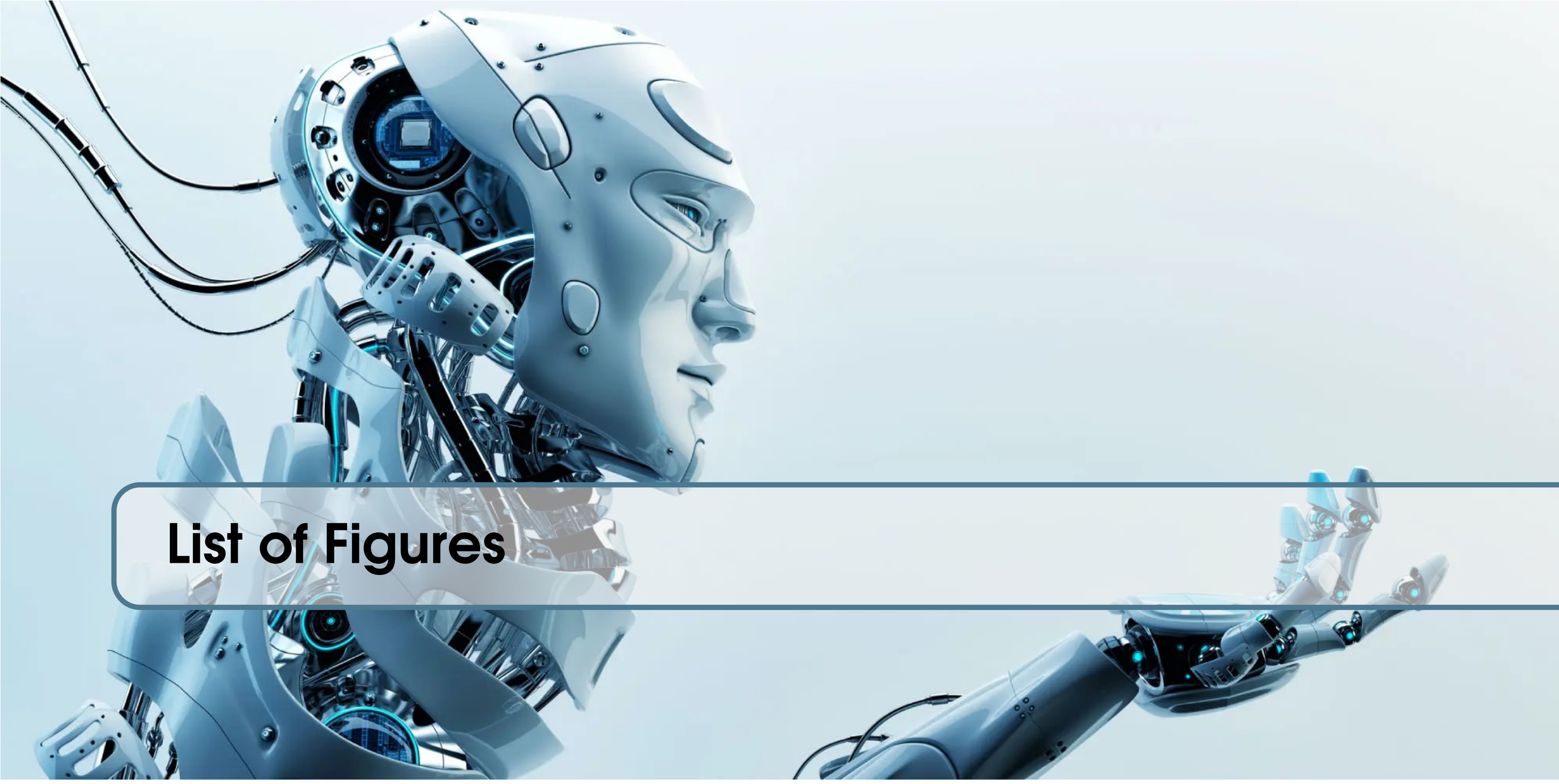

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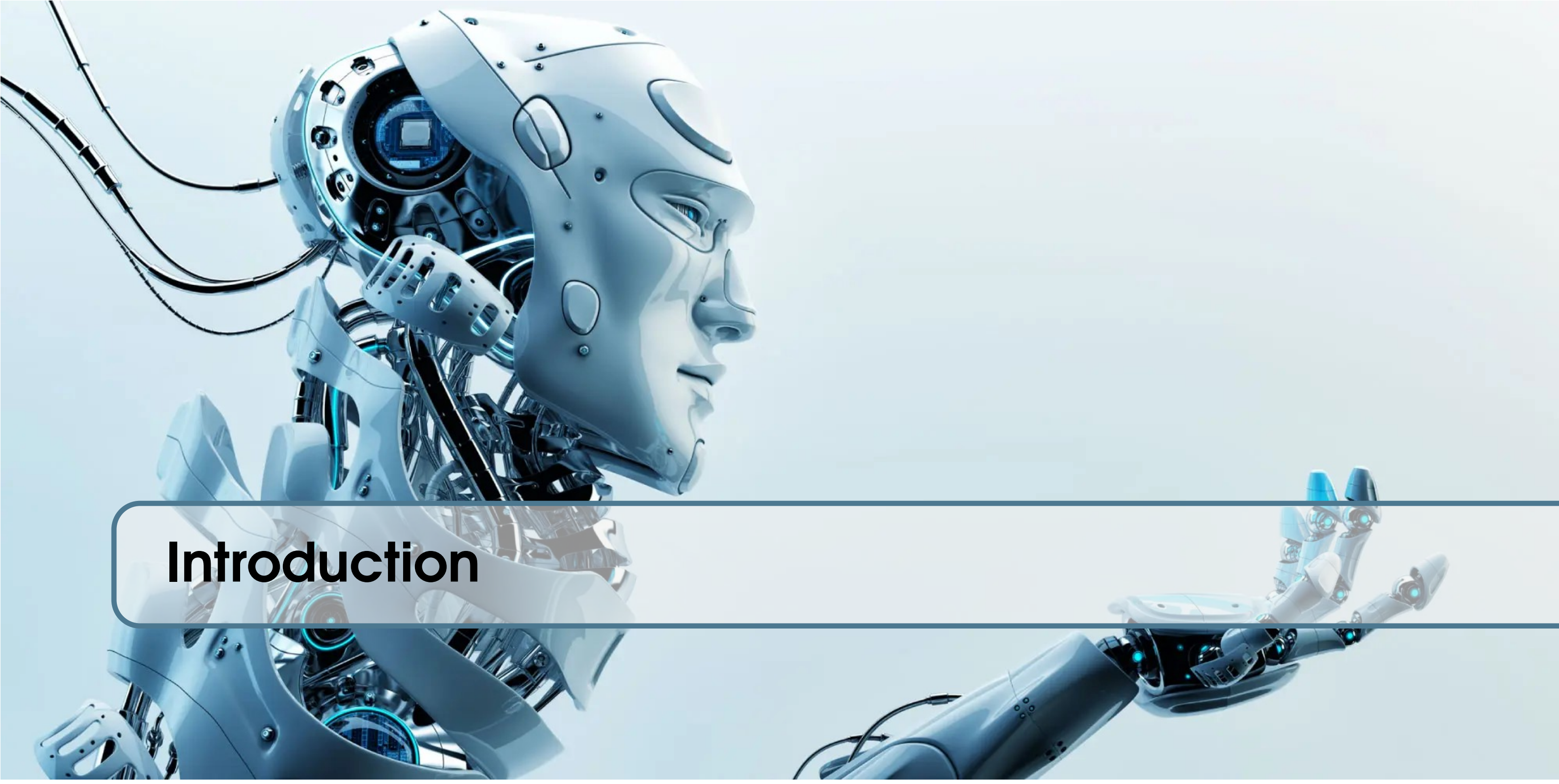


Introduction

Control theory, a mathematical discipline with roots dating back to the 19th century, has matured significantly throughout the 20th century and into the 21st century [**12**]. It addresses the regulation and manipulation of dynamic systems across various fields, from engineering and biology to economics and environmental science. This field bridges theoretical foundations and practical applications, providing solutions to complex problems through advanced mathematical frameworks and computational tools. Over the years, control theory has evolved to incorporate optimization techniques, fractional calculus, semigroup theory, and hybrid models, offering new approaches to longstanding challenges. This evolution is driven by the need to manage linear and nonlinear systems, tackle inverse problems, apply optimal control, and study partial differential equations (PDEs) [**37**].

Linear systems are easier to analyze due to their more straightforward structure and predictable behaviour, typically modelled using ordinary differential equations (ODEs). Techniques such as Laplace transforms and transfer function analysis are integral to their analysis, providing clear insights into system dynamics. However, many real-world phenomena exhibit nonlinear behaviour characterized by multiple equilibria, limit cycles, and chaotic dynamics. Linearization methods often approximate nonlinear behaviour around equilibrium points, but these methods struggle to capture the full range of global dynamics. Nonlinear control strategies are therefore necessary to handle such complexities. Techniques such as feedback linearization, sliding mode control, and Lyapunov-based stability analysis are employed to manage nonlinearities, ensuring system stability and performance even in the face of significant disturbances. The development of these techniques reflects the increasing complexity of systems encountered in practical applications, where purely linear models fall short. For instance, the advent of robust control methods, such as sliding mode control, which ensures system stability regardless of disturbances, has been transformative. These methods enable more precise control of systems like robotic arms or autonomous vehicles, where maintaining performance under various operating conditions is critical. The introduction of nonlinear control techniques also aligns with advancements in computational capabilities, allowing the real-time implementation of complex control strategies in embedded systems [**49**].

Inverse problems are a critical aspect of control theory. These problems involve determining unknown system parameters from observed outputs. Regularization techniques are essential for ensuring the stability and feasibility of solutions to these inverse problems, especially when they are ill-posed. Regularization stabilizes solutions by incorporating a penalty term into the objective function, improving solutions robustness in the presence of noise and incomplete data. These techniques are crucial for tasks such as state estimation and sensor fusion.

Optimal control theory is a cornerstone of control theory, focusing on finding control inputs that minimize a certain cost function while achieving desired system behavior [**1**], [**9**], [**27**]. Techniques like Pontryaguine's maximum principle [**21**] and dynamic programming [**20**] are crucial for solving optimization problems across linear and nonlinear systems. The principle of duality is central to control theory, linking controllability and observability [**44**]. Controllability ensures that a system can be steered to a desired state, while observability allows for determining internal states from outputs [**2**], [**19**]. This dual relationship is crucial in designing control systems, as it underpins the theoretical foundations for feedback control and state estimation [**28**]. Pontryaguine's maximum principle, for instance, provides a necessary and sufficient condition for optimal control in dynamic systems, connecting state trajectories with control inputs optimally [**20**]. The principle is widely used in applications, where optimal control solutions are necessary to design trajectories for spacecraft that minimize fuel consumption and maximize mission performance. Furthermore, the principle of duality is fundamental to real-time control systems, where it enables the design of optimal control laws that adapt to changing conditions, ensuring efficient performance under various operating scenarios. Integrating Pontryaguine's maximum principle into control theory also reflects the field's growing emphasis on interdisciplinary applications, bridging the gap between pure mathematics and engineering practice.

Functional analysis is critical in control theory, particularly in studying partial differential equations (PDEs). Key mathematical tools include Lebesgue spaces [**17**] and Sobolev spaces [**47**], which provide a rigorous framework for analyzing PDEs. These spaces are crucial for studying systems governed by partial derivatives, such as heat conduction and wave propagation. Fractional derivatives and fractional-order systems have gained importance in this context, offering a way to model systems with memory and nonlocal interactions [**5**], [**32**]. Semigroup theory, particularly the Hille-Yosida theorem, is widely applied in analyzing infinite-dimensional systems, providing a robust framework for solving time-evolution problems [**25**], [**33**]. These tools enable the rigorous treatment of dynamical systems that evolve over time, capturing phenomena such as heat conduction and wave propagation, which are common in engineering and physics.

The sentinel method provides a framework for anomaly detection within distributed systems by monitoring a state variable to identify irregularities, such as pollutant sources or temperature deviations [**7**], [**41**], [**39**], [**38**]. This method uses PDEs to model phenomena such as pollutant transport and diffusion, refining detection and localization capabilities in environmental monitoring. The sentinel method is applicable across various fields, from industrial processes like alloy cooling to complex biological systems, showcasing its broad applicability and potential for enhancing safety and efficiency [**8**]. This approach is crucial in environmental monitoring, where early detection of pollution sources can prevent environmental damage and improve public health outcomes. For example, Glowinski and Lions demonstrated their application to detect chemical discharges in rivers and lakes, enhancing our understanding of complex interactions in environmental systems. The sentinel method is effective for environmental monitoring and finds applications in critical infrastructure protection, where it can detect anomalies in systems such as power grids or communication networks. Integrating this method into real-time monitoring systems provides a proactive approach to prevent disruptions and mitigate risks, making it an invaluable tool for maintaining system reliability and security.

A significant application of control theory involves the Stefan problem [**4**], [**30**], a classical model of phase transitions, such as ice melting into water. Introduced by Josef Stefan in the 19th century, the problem is governed by PDEs with moving boundaries, where the position of the boundary evolves based on energy equilibrium conditions. The nonlinear sentinel method is applied to tackle the complexities of this free-boundary problem, such as [**29**], [**34**], [**45**], providing innovative solutions for industrial processes like alloy cooling and semiconductor manufacturing.

Hybrid systems, systems that integrate continuous and discrete dynamics, are increasingly important in modern applications such as robotics, automotive systems, and telecommunications [**14**], [**31**], [**35**]. Fractional-order dynamics play a critical role in modelling these systems. For example, viscoelastic materials and biological networks exhibit complex dynamic behaviours that fractional calculus can accurately describe. The Caputo formulation of fractional derivatives allows the practical inclusion of memory in hybrid models, allowing precise modelling and control strategies for systems with state jumps and memory-dependent dynamics. These models are particularly useful in applications where past states influence current behaviour, such as in modelling ageing processes in biological tissues or stress relaxation in materials.

The structure of this thesis unfolds as follows :

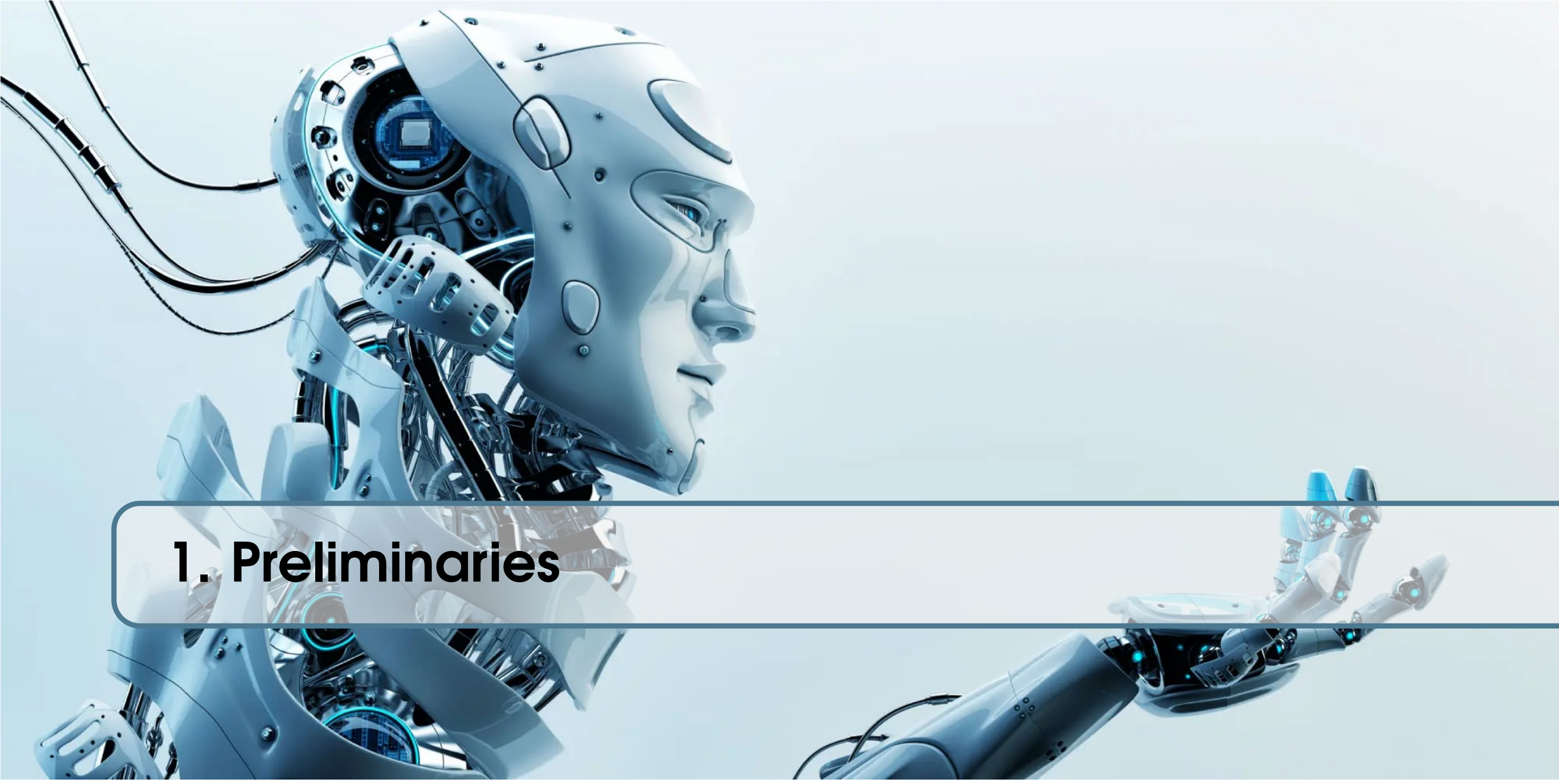
Chapter 1 : This introductory chapter establishes foundational theories necessary for subsequent discussions. It includes essential reminders regarding functional analysis encompassing Lebesgue and Sobolev spaces, pivotal results such as existence and uniqueness theorems, and semigroup theory, which is integral in addressing evolution problems. Resolution methodologies via semigroups, commonly utilized within dynamical system studies, are also explored. Finally, key optimization concepts, both constrained and unconstrained minimization techniques, are discussed alongside theoretical results affirming solutions' existence and uniqueness in optimization contexts.

Chapter 2 : This chapter presents an overview of control theory, emphasizing controllability and observability concerning localized versus distributed systems. In addition, it investigates the stability and stabilization processes applicable to localized frameworks, illustrated through examples involving dynamic financial systems. Necessary conditions and the development of relevant theorems to identify optimal controls applicable across linear and nonlinear contexts are also provided, laying essential groundwork for understanding complex control architectures.

Chapter 3 : Here, we introduce sentinel methodologies illustrated through concrete examples concerning pollution detection within fluid environments. Subsequent examination revolves around the existence and construction of these sentinels, detailing various classifications, including regional, discrete, and weak discriminating sentinels, thus enhancing understanding of their potential utilization for monitoring and detection mechanisms in complex system environments.

Chapter 4 : This chapter investigates applications of sentinel methodologies, specifically targeting nonlinear challenges exemplified through first-order Stefan problem analyses. Numerical simulations support the effectiveness of this methodology, demonstrating its applicability to resolve practical issues closely related to transient phase modelling efforts.

Chapter 5 : Conclusively, this chapter delineates hybrid system fundamentals, outlining definitions and characteristics intrinsic to them while addressing questions surrounding optimal controls applicable within fractional hybrid constructs featuring jumps. Concrete examples serve as demonstrations showcasing the relevance of the methodologies presented throughout the preceding chapters when tackling complexities inherent in diverse applications concerning hybrid frameworks.



1. Preliminaries

Let Ω be an open of $\mathbb{R}^n$. To gain a more comprehensive understanding and explore detailed proofs of the theorems and corollaries presented in this chapter, the readers are advised to refer to : [**15**] for Lebesgue space, [**47**] for Sobolev space, [**5**] for fractional derivative, [**33**] for semigroups, and [**10**], [**17**], [**24**] for the optimization.

1.1 Lebesgue space

Lebesgue space has had an essential impact on mathematics, particularly in functional analysis, measurement theory, integration, and approximation theory by providing a framework for analyzing the integrability and properties of functions. The integration and convergence principles for work in the space $L^p(\Omega)$ can be generalized. The Lebesgue space $L^p(\Omega)$, where $p \geq 1$, includes all measurable functions for which the p-Th power of the absolute value of the function is integrated on across the entire space about a specified measure. These spaces help us understand convergence, continuity, and other essential function characteristics in analysis, and they are widely used in various mathematical subjects such as harmonic analysis, partial differential equations, and probability theory.

In general, the application $g \to \|g\|_{\mathscr{L}^p(\Omega)}$ defines a semi-norm on $\mathscr{L}^p(\Omega)$ because it may assign a zero value to distinct equivalent functions almost everywhere.

However, it follows directly that if $\|g\|_{\mathscr{L}^p(\Omega)} = 0$ then $g = 0$ almost everywhere in Ω.

Consequently, given two measurable functions $g, f : \Omega \to \mathbb{R}$, we say that g is equivalent to f, and we write $g \sim f$, if $g(x) = f(x)$ almost everywhere $x \in \Omega$.

Note that $\sim$ is an equivalence relation in the class of fundamental measurable functions. The space quotient of spaces $\mathscr{L}^p(\Omega)$ by the equality equivalence relation g is denoted $L^p(\Omega)$ almost everywhere.

Definition 1.1 We design the space of integrated functions by $L^1(\Omega)$ on Ω with values in $\mathbb{R}$. We pose

$$\|g\|_{L^1(\Omega)} = \int_\Omega |g(x)|\,dx. \tag{1.1}$$

Definition 1.2 With $1 \leq p < \infty$, given a measurable function g from Ω in $\mathbb{R}$ (or in $\mathbb{C}$), we note

$$\int_\Omega |g(x)|^p\,dx < \infty$$

we note

$$\|g\|_{L^p(\Omega)} = \left(\int_\Omega |g(x)|^p\,dx\right)^{\frac{1}{p}}. \tag{1.2}$$

Definition 1.3 Given a measurable function g from Ω in $\mathbb{R}$ (or in $\mathbb{C}$), we note

$$L^\infty(\Omega) = \{g : \Omega \longrightarrow \mathbb{R}; g \text{ is measurable and } \exists M \text{ such that } |g(x)| \leq M \text{ for almost everywhere on } \Omega\}, \quad (1.3)$$

and we note

$$\|g\|_{L^\infty(\Omega)} = \inf\{M; |g(x)| \leq M \text{ almost everywhere on } \Omega\}.$$

For all $p \in [1, +\infty]$, we denote $q = \begin{cases} 1 \text{ if } p = +\infty, \\ \dfrac{p}{p-1} \text{ if } 1 < p < +\infty, \\ +\infty \text{ if } p = 1. \end{cases}$

One can prove that $L^p(\Omega)$ equipped with the norm $\|\cdot\|_{L^p}$ is a Banach space for all $1 \leq p \leq \infty$. In particular, L^2 is a Hilbert space.

Lemma 1.1 — Hölder Inequality. Let $g \in L^p(\Omega)$ and $h \in L^q(\Omega)$, than $g.h \in L^1$ and

$$\int_\Omega |g(x)h(x)|\, dx \leq \|g\|_{L^p(\Omega)} \|h\|_{L^q(\Omega)}. \quad (1.4)$$

Remark If $p = q = 2$, then the Hölder inequality becomes the Schwartz inequality.

Theorem 1.1 L^p is a vector space and $\|\cdot\|_{L^p}$ is a norm for everything $1 \leq p \leq \infty$.

Lemma 1.2 — Minkowsky Inequality. If $g, h \in L^p(\Omega)$. Then

$$\|g+h\|_p \leq \|g\|_p + \|h\|_p \quad (1.5)$$

Next, we are interested in the Riesz representation theorem. This theorem is the cornerstone of the theory of L^p spaces and plays an essential role in understanding the duality between L^p and L^q spaces.

Theorem 1.2 — Riesz Representation. Either $1 < p < \infty$ and $\psi \in (L^p)'$. So there exists $v \in L^q$ unique such that

$$\langle \psi, g\rangle = \int_\Omega vg, \forall g \in L^p, \quad (1.6)$$

and

$$\|v\|_{L^{p'}} = \|\psi\|_{(L^p)'}. \quad (1.7)$$

1.2 Sobolev spaces

Sobolev spaces is crucial in purposeful analysis, partial differential equations (PDE) theory, and the mathematical modelling of physical phenomena. Sobolev spaces generalize classical notions of derivatives and integrals, enabling the analysis of functions that may lack pointwise differentiability but are differentiable in an averaged sense.

Distribution

In this part, let Ω a non empty open of $\mathbb{R}^n$.

Definition 1.4 If K is a compact of Ω we denote by $D_K(\Omega)$ the space of all test functions $\varphi \in D(\Omega)$.

Definition 1.5 We call the space of test functions and note $D(\Omega)$ the set :

$$D(\Omega) = \left\{\varphi \in C^\infty(\Omega) / \exists\, K \text{ compact}, K \subset \Omega, \varphi \in D_K(\Omega)\right\}. \quad (1.8)$$

Definition 1.6 We call a distribution any continuous linear form on $D(\Omega)$, and we denote by $D'(\Omega)$ the space of distribution.

Definition 1.7 Let T and element of $D'(\Omega)$; for all α of $\mathbb{N}^n$, we call derivative of order α of T and we note $D^\alpha T$ the application :

$$\begin{array}{rcl} D^\alpha & : & D(\Omega) \to \mathbb{C} \\ \varphi & \to & D^\alpha T(\varphi) = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle . \end{array}$$

Definition 1.8 Let a sequence $(T_n)_n$ of the distribution. We called the $(T_n)_n$ converge into $T \in D'$ if the numerical sequence $(\langle T_n, \varphi \rangle)_n$ converge into $\langle T, \varphi \rangle$ for every function φ in D.

Proposition 1.1 For all T of $D'(\Omega)$ and for all α of $\mathbb{N}^n$, $D^\alpha T$ is a distribution.

$H^k(\Omega)$ spaces

Definition 1.9 Let $k \in \mathbb{N}$, $H^k(\Omega)$ the set of functions defined on Ω whose derivatives are partials (in the sense of distributions) of order less than or equal to k are square functions integrable on Ω.

Remark

1. $H^k(\Omega)$ is a subspace of $L^2(\Omega)$ and $H^0(\Omega) = L^2(\Omega)$.
2. $H^1(\Omega) = \{u \in L^2(\Omega), \nabla u \in (L^2(\Omega))^n\}$.

Topological Structure

We can structure the space $H^k(\Omega)$ into a pre-Hilbertian space by equipping it with the following scalar product

$$\begin{aligned} (u|v)_{H^k(\Omega)} &= \sum_{|\alpha| \le k} (D^\alpha u | D^\alpha v)_{L^2(\Omega)} \\ &= \sum_{|\alpha| \le k} \int_\Omega D^\alpha u D^\alpha v dx. \end{aligned}$$

Particular case

$$\begin{aligned} \|u\|_{H^1(\Omega)} &= \sqrt{(u|v)_{H^1(\Omega)}} \\ &= \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{(L^2(\Omega))^n} \right)^{\frac{1}{2}}. \end{aligned}$$

$H_0^k(\Omega)$ spaces

We have for all $k \in \mathbb{N}$.

Definition 1.10 We denote by $H_0^k(\Omega)$ the adhesion of $D(\Omega)$ in the Hilbertian space $H^k(\Omega)$.

Remark $H_0^k(\Omega)$ is a Hilbert space.

Definition 1.11 We denote by $H^{-k}(\Omega)$ the topological dual of $H_0^k(\Omega)$.

Green's formula

Green's formula is a fundamental theorem in vector calculus that establishes a relationship between a line integral around the boundary of a region in the plane and a double integral over the interior of that region.

First Green's formula

$$\int_\Omega \frac{\partial f}{\partial x_i} g = -\int_\Omega f \frac{\partial g}{\partial x_i} + \int_\Gamma f g (\vec{n} \cdot \vec{e}_i) d\sigma, \quad \forall f, g \in H^1(\Omega) \tag{1.9}$$

where $\vec{n}$ is the unit outward normal vector to Γ. Note that the surface integral is well-defined since $f, g \in L^2(\Gamma)$.

Similarly, one can define the normal derivative
Second Green's formula

$$-\int_{\Omega} \Delta f\, g = \int_{\Omega} \nabla f \cdot \nabla g - \int_{\Gamma} \frac{\partial f}{\partial n} g\, d\sigma, \quad \forall f, g \in H^2(\Omega). \tag{1.10}$$

1.3 Fractional derivative

The fractional derivative is a mathematical extension of classical differentiation to non-integer orders, forming a key concept in fractional calculus. Unlike integer-order derivatives, fractional derivatives incorporate a memory effect, making them essential for modelling systems with history-dependent dynamics. The most common definitions include the Riemann-Liouville derivative, which is based on fractional integration and involves an integral kernel weighted by the Gamma function, and the Caputo derivative, which rearranges the order of differentiation and integration to align with classical initial conditions, making it more suitable for physical problems.

Caputo derivative

Definition 1.12 Let $\alpha > 0, n \in \mathbb{N}, I$ is the interval $-\infty \leq a \leq b \leq +\infty$, $f, \psi \in C^n(I)$ two functions such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in I$. The left ψ-Caputo fractional derivative of f of order α is given by

$$^C D_{a+}^{\alpha,\psi} f(x) = I_{a+}^{n-\alpha,\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n f(x),$$

and the right ψ-Caputo fractional derivative of f by

$$^C D_{b-}^{\alpha,\psi} f(x) = I_{b_-}^{n-\alpha,\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n f(x),$$

where

$$n = [\alpha] + 1 \text{ for } \alpha \notin \mathbb{N}, \; n = \alpha \text{ for } \alpha \in \mathbb{N}.$$

Remark Given $\alpha = m \in \mathbb{N}$,

$$^C D_{a+}^{\alpha,\psi} f(x) = f_{\psi}^{[m]}(x) \text{ and } {}^C D_{b-}^{\alpha,\psi} f(x) = (-1)^m f_{\psi}^{[m]}(x),$$

and if $\alpha \notin \mathbb{N}$, then

$$^C D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \psi'(t) \left(\psi(x) - \psi(t)\right)^{n-\alpha-1} f_{\psi}^{[n]}(t) dt,$$

and

$$^C D_{b_-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b \psi'(t) \left(\psi(x) - \psi(t)\right)^{n-\alpha-1} (-1)^n f_{\psi}^{[n]}(t) dt,$$

where the function $\Gamma(n-\alpha)$ represents the Gamma function defined for $x > 0$ as :

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt,$$

where n is an integer and α is a parameter, it is evaluated using this integral representation.
When $\alpha \in (0,1)$, we have

$$^C D_{a^+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \left(\psi'(x) - \psi(t)\right)^{-\alpha} f'(t) dt,$$

and

$$^C D_{b^-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_x^b \left(\psi(t) - \psi'(x)\right)^{-\alpha} f'(t) dt.$$

Riemann-Liouville derivative

Definition 1.13 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f(t)$ is defined using the concept of fractional integration. For a function $f(t)$ that is sufficiently well-behaved, it is given by:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau,$$

where:

- $n = [\alpha]$ (the smallest integer greater than or equal to α),
- $\Gamma(\cdot)$ is the Gamma function,
- a is the lower limit of the integral (often taken as $a = 0$ for simplicity),
- $\dfrac{d^n}{dt^n}$ denotes the n-th ordinary derivative.

1.4 Semigroups

Semigroups provide a mathematical framework for modelling the time evolution of linear infinite-dimensional systems, such as systems governed by PDEs or delay differential equations.

Specifically, semigroups describe the solution of the abstract Cauchy problem :

$$\begin{cases} \dot{x}(t) = Ax(t), \\ x(0) = x_0, \end{cases}$$

where A is the generator of the semigroups.

Semigroups are essential for representing solutions to linear dynamical systems and analyzing stability, controllability, and observability. And approximation and numerical method for infinite-dimensional systems. The foundational result in the theory of semigroups of linear operators is the Hille-Yosida theorem, which describes the time evolution of systems governed by linear dynamics.

In all of this section, X is a Banach space.

1.4.1 Uniformly continuous semigroups

Definition 1.14 Let A be a parameter family $S(t)$, $0 \leq t < \infty$, of bounded linear operators from X into X is a semigroup of bounded linear operator on X if

1. $S(0) = I$,
2. $S(t+s) = S(t)S(s)$ for every t, $s \geq 0$.

Uniformly continuous semigroups play a vital role in solving evolution equations, where the state of a system evolves smoothly over time, i.e.

$$\lim_{t \to 0} \|S(t) - I\| = 0.$$

Definition 1.15 The linear operator

$$\begin{aligned} A &: X \to X, \\ A &= \lim_{t \to 0} \frac{S(t) - I}{t}, \end{aligned}$$

is called the infinitesimal generator of a uniformly continuous semigroup $\{S(t)\}_{t \geq 0}$.

Lemma 1.3 There exist a unique uniformly continuous semigroup $\{S(t)\}_{t \geq 0}$, with the operator A

$$S(t) = e^{tA}, \forall t \geq 0.$$

having the operator A as a generator.

Theorem 1.3 A is a bounded linear operator if and only if A is the infinitesimal generator of a uniformly continuous semigroup.

Theorem 1.4 Let $T(t)$ and $S(t)$ be uniformly continuous semigroups of bounded linear operators. If

$$\lim_{t\to 0}\frac{T(t)-I}{t} = A = \lim_{t\to 0}\frac{S(t)-I}{t},$$

then $T(t) = S(t)$ for $t \geq 0$.

Corollary 1.1 Let $S(t)$ be a uniformly continuous semigroup of bounded linear operators. Then

1. There exists a constant $\omega \geq 0$ such that $\|S(t)\| \leq e^{\omega t}$.
2. There exists a unique bounded linear operator A such that $S(t) = e^{tA}$.
3. The operator A is the infinitesimal generator of $S(t)$.
4. $t \to S(t)$ is the differentiable in norm and

$$\frac{dS(t)}{dt} = AS(t) = S(t)A.$$

1.4.2 Strongly continuous semigroups

Definition 1.16 We called C_0-semigroup (or strongly continuous semigroup, a semigroup of class C_0) of bounded linear operators, the family $\{S(t)\}_{t\geq 0}$ if

1. $S(0) = I$,
2. $S(t+s) = S(t)S(s), \forall t,s \geq 0$,
3. $\lim\limits_{t\to 0} S(t)x = x, \forall x \in X$.

Definition 1.17 We called the infinitesimal generator of C_0-semigroup $\{S(t)\}_{t\geq 0}$, the operator A defined on the set

$$D(A) = \left\{ x \in X : \lim_{t\to 0}\frac{S(t)x-x}{t} \text{ exist} \right\},$$

by

$$Ax = \lim_{t\to 0}\frac{S(t)x-x}{t}, \forall x \in D(A).$$

is the infinitesimal generator of the semigroup $S(t), D(A)$ is the domain of A.

Theorem 1.5 Let $S(t)$ be a C_0-semigroup. There exist a constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|S(t)\| \leq Me^{\omega t} \text{ for } 0 \leq t < \infty.$$

Corollary 1.2 If $S(t)$ is a C_0-semigroup then for every $x \in X$, $t \to S(t)x$ is a continuous function from $\mathbb{R}_0^+$ into X.

Theorem 1.6 Let $S(t)$ be a C_0-semigroup and let A be its infinitesimal generator. Then

1. For $x \in X$, $\int_0^t S(s)xds \in D(A)$ and

$$A\left(\int_0^t S(s)xds\right) = S(t)x - x.$$

2. For $x \in D(A)$, $S(t)x \in D(A)$ and

$$\frac{d}{dt}S(t)x = AS(t)x = S(t)Ax.$$

Proposition 1.2 Let $\{S(t)\}_{t\geq 0} \in SG(M,\omega)$ and A is the infinitesimal generator. If $x \in D(A)$, then $S(t)x \in D(A)$ and we have the inequality

$$S(t)Ax = AS(t)x, \forall t \geq 0.$$

Lemma 1.4 Let $\{S(t)\}_{t\geq 0}$ is a C_0-semigroup. Then

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} S(s)xds = S(t)x, \forall x \in X \text{ and } t \geq 0.$$

Corollary 1.3 If A is the infinitesimal generator of a C_0-semigroup $S(t)$ then $D(A)$, the domain of A, is dense in X, and A is a closed linear operator.

Theorem 1.7 Let $T(t)$ and $S(t)$ be C_0-semigroup of bounded linear operators with infinitesimal generators A and B respectively. If $A = B$ then $T(t) = S(t)$ for $t \geq 0$.

1.4.3 Hille-Yosida theorem

The Hille-Yosida theorem provides a framework for analyzing how solutions to differential equations evolve, ensuring the existence of a strongly continuous semigroup for a given generator A.

Theorem 1.8 — Hille-Yosida , General case. A linear operator A is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on X satisfying $\|S(t)\| \leq Me^{\omega t}, t \geq 0$, with $\omega \geq 0, M \geq 1$, if and only if :

1. A is closed and $\overline{D(A)} = X$,
2. For all $\lambda \in \mathbb{C}$ such that $Re\lambda > \omega$, we have $\lambda \in \rho(A)$ and

$$\|R(\lambda,A)^n\| \leq \frac{M}{(Re\lambda-\omega)^n}, \forall n \in \mathbb{N}.$$

This corollary use to proof the theorem of Hille-Yosida.

Corollary 1.4 Let A be the infinitesimal generator of C_0-semigroup of contractions $S(t)$. If A_λ is the Yosida approximation of A, then

$$S(t)x = \lim_{\lambda\to\infty} e^{tA_\lambda}x \text{ for } x \in X.$$

1.5 Optimization

Optimization involves identifying the most efficient solution to a problem while satisfying constraints. This process is used in engineering, economics, and operations research. It involves finding the maximum ergonomic design for diverse applications, such as nuclear-strength vegetation or strategic planning. Optimization aims to maximize efficiency, minimize costs, or achieve unique goals within defined constraints. This subject encompasses a variety of strategies and algorithms geared toward enhancing structures and techniques throughout memorable domains. In the real world, there are many diverse applications of optimization. For example, in economic sectors, it is critical for efficient portfolio management, legal responsibility, control, and sales control in airlines, and index fund management to decrease dangers and maximize returns, or in electric energy corporations use generator dedication choices primarily based totally on forecasted calls for and working costs to ensure efficient power generation.

1.5.1 General notions

The general form of an optimization problem is written as follows

$$\begin{cases} \min J(x) \\ g(x) \leq 0, \\ h(x) = 0, \\ x \in \mathbb{R}^n. \end{cases} \tag{1.11}$$

with $J : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the cost function, $g : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ is a function that represents unequal constraints. This function is written as follows

$$g(x) = (g_1(x), ..., g_p(x)),$$

and $h : \mathbb{R}^n \longrightarrow \mathbb{R}^q$ is a function that represents equal constraints. This function is written as follows

$$h(x) = (h_1(x), ..., h_q(x)),$$

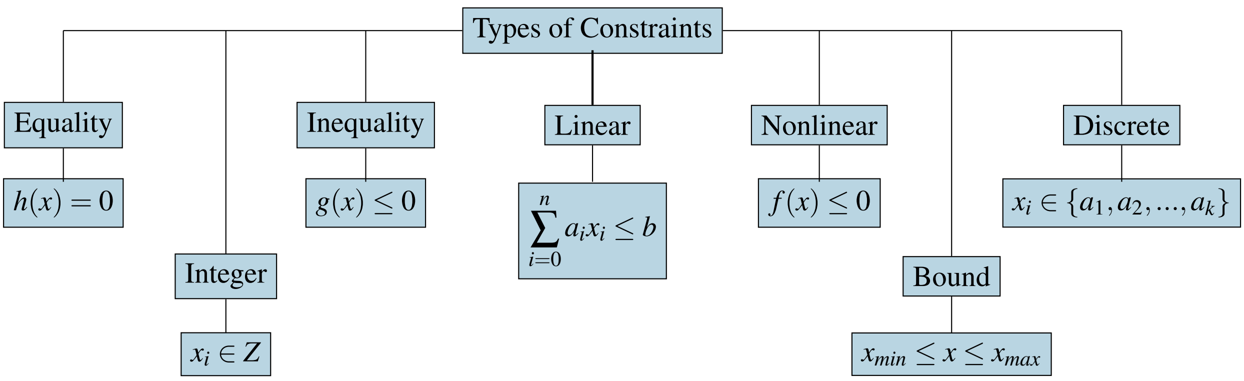


Figure 1.1: Types of constraints.

Convex functions and sets are critical in optimization because they guarantee global solutions. For example, minimizing a convex cost function over a convex set ensures that any local minimum is also a global minimum.

Local and global minimum or maximum

Let K be a set of constraints ; i.e.,

$$K = \{x \in \mathbb{R}^n / g(x) \leq 0, h(x) = 0\}. \tag{1.12}$$

We assume that K is non-empty ; an element x of K will be said to be realizable.

Definition 1.18 K is a set nonempty of a real Hilbert space H and f a function K in $\mathbb{R}$. We say that $x^* \in K$ satisfies a local minimum of f if

$$\forall x \in B(x^*) \cap K \,;\, f(x^*) \leq f(x).$$

Or a local maximum if

$$\forall x \in B(x^*) \cap K \,;\, f(x^*) \geq f(x),$$

such as

$$B(x^*, \rho) = \{x \in H / \|x - x^*\| \leq \rho\},$$

where $\|\cdot\|$ means the norm of H.

Definition 1.19 We say that $x^* \in K$ achieves a global minimum of f if

$$\forall x \in K \; ; \; f(x^*) \leq f(x).$$

Or a global maximum if

$$\forall x \in K \; ; \; f(x^*) \geq f(x).$$

Proposition 1.3 If x^* satisfies a (global or local) maximum ; then x^* satisfies a (global or local) minimum ; i.e.,

$$\max\{f(x), x \in K\} = -\min\{-f(x), x \in K\}.$$

Notions of convexity

Definition 1.20 — Convex set. If

$$\forall (x,y) \in K \times K \; ; \forall t \in [0,1] \; ; tx + (1-t)y \in K, \tag{1.13}$$

then $K \subset H$ is convex. Or if K contains all segments connecting any two of its points, then we can say K is convex.

Definition 1.21 — Convex function. If K is convex and if

$$\forall (x,y) \in K \times K \; ; \forall t \in [0,1] \; ; J(tx + (1-t)y) \leq tJ(x) + (1-t)J(y),$$

then, $J : K \subset H \to \mathbb{R} \cup \{+\infty\}$ is convex.

Definition 1.22 Let $J : K \subset H \to \mathbb{R} \cup \{+\infty\}$ is convex, the domain of J given by

$$Dom\ J = \{x \in H : J(x) < +\infty\},$$

is convex.

Definition 1.23 — Strictly convex function. If k is convex and if

$$\forall (x,y) \in K \times K \; ; x \neq y \; ; \forall t \in [0,1] \; ;$$
$$J(tx + (1-t)y) < tJ(x) + (1-t)J(y),$$

then, $J : K \subset H \to \mathbb{R} \cup \{+\infty\}$ is strictly convex.

Differentiability of convex function

Definition 1.24 Let J be a function of H into $\mathbb{R} \cup \{+\infty\}$. We say that J is Gâteaux differentiable at $w \in DomJ$ if the directional derivative

$$J'(w;v) = \lim_{h \to 0^+} \frac{J(w+hv) - J(w)}{h} = \langle \nabla J(w), h \rangle, \tag{1.14}$$

exist in the all of direction v of H if the application $v \to J'(w;v)$ is linear and continuous.

Theorem 1.9 Let $J : K \subset H \to \mathbb{R} \cup \{+\infty\}$ Gâteaux differentiable in K, with K convex.
Then J is convex if and only if

$$\forall (w,v) \in K \times K, \; J(v) \geq J(w) + (\nabla J(w), v - w). \tag{1.15}$$

Theorem 1.10 Let $J : K \subset H \to \mathbb{R} \cup \{+\infty\}$, Gâteaux differentiable in K, with K convex

J is convex if and only if ∇J is a monoton opeartor, i.e.,

$$\forall (w,v) \in K \times K,\ (\nabla J(w) - \nabla J(v), w - v) \geq 0.$$

1.5.2 Unconstrained minimization

The unconstrained problem is written as follows

$$\begin{cases} \min J(x) \\ x \in \mathbb{R}^n, \end{cases} \tag{1.16}$$

with J is a function from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$.

Definition 1.25 We say that $J : H \longrightarrow \mathbb{R}$ is coercive if

$$\lim_{\|x\| \to +\infty} J(x) = +\infty.$$

with $\|\cdot\|$ means the norm of the Hilbert space H.

Definition 1.26 A function $J : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be proper if:
1. $J(x) < +\infty$ for at least one $x \in \mathbb{R}^n$ (i.e., J is not identically $+\infty$),
2. $J(x) > -\infty$ for all $x \in \mathbb{R}^n$ (i.e., J does not take the value $-\infty$).

Theorem 1.11 — Existence. Let $J : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ continu, proper and coercive. Then (1.16) admit at least one solution.

We don't always have a unique solution, so we give below a criterion for uniqueness.

Theorem 1.12 Let $J : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ be strictly convex. Then the problem (1.16) admits at most one solution.

Theorem 1.13 Let J a function $\mathbb{C}^1$ of $\mathbb{R}^n$ in $\mathbb{R}$. We suppose that there exists $\alpha > 0$ such that

$$\forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n, (\nabla J(x) - \nabla J(y), x - y) \geq \alpha \|x - y\|^2. \tag{1.17}$$

Then J is strictly convex and coercive, and problem (1.16) admits a unique solution.

Optimality condition

The following conditions are differential conditions which relate to the derivative of the function to be minimized.

Theorem 1.14 — First-order necessary optimality conditions. Let H be the Hilbert space and $J : H \longrightarrow \mathbb{R}$ a derivative Gâteaux functional on H. If x^* achieves a minimum (global or local) of J over H, then

$$\nabla J(x^*) = 0.$$

Definition 1.27 A point x^* of H verify $\nabla J(x^*) = 0$ is called critical point or stationary point.

Theorem 1.15 — First-order necessary and sufficient conditions. Let $J : H \longrightarrow \mathbb{R}$ a derivative Gâteaux and convex over H. A point x^* achieve a global minimum of J over H if and only if $\nabla J(x^*) = 0$.

Theorem 1.16 — Second order necessary condition. Let x^* be a (local) minimum of J and J be twice differentiable on H. Then
1. $\nabla J(x^*) = 0$,
2. $\forall x \in H \ \ (D^2 J(x^*)x, x) \geq 0$.

Theorem 1.17 — Second order sufficient condition. Let J twice differentiable on H verify $\nabla J(x^*) = 0$ and $\exists \alpha > 0, \forall x \in H, (D^2 J(x^*)x, x) \geq \alpha \|x\|^2$. Then the function J admit a strict local minimum in x^*.

1.5.3 Constrained minimization

There are different types of constraints in figure (1.1). These constraints are determined based on the feasible region considering the intersection of all constraints.

The constrained minimization problem is as follows

$$\begin{cases} \min J(x) \\ x \in K, \end{cases} \tag{1.18}$$

where K is a non-empty closed subset of $\mathbb{R}^n$ is called the set of constraints.

Theorem 1.18 Suppose that J is continue, K is a non-empty closed subset of $\mathbb{R}^n$ and one of the following conditions is fulfilled :

1. Let K is bounded,
2. Let J is coercive.

Then problem (1.18) admits at least one solution.

Theorem 1.19 Under the assumption of theorem (1.18) if J is strictly convex and if K is convex, then the problem (1.18) admits a unique solution.

First-order optimality condition

Theorem 1.20 If J is a derivative function (Gâteaux) and if K is a closed convex, then any solution x^* of (1.11) satisfies a necessary first-order optimality condition :

$$\forall x \in K, (\nabla J(x^*), x - x^*) \geq 0. \tag{1.19}$$

Theorem 1.21 — Necessary and sufficient condition of the first order. Suppose J convex, Gâteaux derivative, and K closed convex. Let x^* an element of K, the condition (1.19) is a necessary and sufficient condition of the problem (1.18).

Constraints in equality

$$\begin{cases} \min J(x) \\ h(x) = 0, \\ x \in \mathbb{R}^n, \end{cases} \tag{1.20}$$

where $h(x) = (h_1(x), \ldots\ldots, h_p(x))$ and h is continue of $\mathbb{R}^n$ in $\mathbb{R}^p$.

Theorem 1.22 — First-order necessary condition-constraints. Let

1. J and h are of class C^1 on $\mathbb{R}^n$,
2. the problem (1.11) has a solution x^*,
3. the vectors p of $\mathbb{R}^n$; $(\nabla h_1(x^*), \ldots\ldots, \nabla h_p(x^*))$ are linearly independent.

Then, there exists p real $(\lambda_1^*, \ldots\ldots, \lambda_p^*)$ such as

$$\nabla J(x^*) + \sum_{j=1}^{p} \nabla h_j(x^*) = 0.$$

Definition 1.28 The reals λ_j^* obtained by the previous theorem are called the Lagrange multiplier.

Equality and inequality constraints

We use problem (1.18) with (1.12).

Definition 1.29 An element $x^* \in \mathbb{R}^n$ is said to be *regular* for the constraints h and g if :

- x^* is feasible : $h(x^*) = 0$ and $g(x^*) \leq 0$,
- the vectors $\nabla h_i(x^*)$, $i = 1, \ldots, p$, are linearly independent,
- there exists $d \neq 0 \in \mathbb{R}^n$ such that :

$$(\nabla h_i(x^*), d) = 0, \quad \forall i = 1, \ldots, p, \quad \text{and} \quad (\nabla g_j(x^*), d) < 0, \quad \forall j \in I(x^*),$$

where $I(x^*) = \{j \mid g_j(x^*) = 0\}$ is the set of active constraints at x^*.

Theorem 1.23 We assume that J, h and g are of class C^1. Let x^* a solution of the problem (1.11). Then there exists $\lambda^* = (\lambda_1^*, \ldots.., \lambda_p^*) \in \mathbb{R}^p, \mu^* = (\mu_1^*, \ldots..., \mu_q^*) \in \mathbb{R}_+^q$ and $\mu_0^* \in \mathbb{R}^+$ such as

$$\forall j \in \{0, \ldots., q\} \ , \mu_j^* \geq 0,$$
$$h(x^*) = 0, \ g(x^*) \leq 0,$$
$$\forall j \in \{0, \ldots., q\} \ , \mu_j^* g_j(x^*) = 0,$$
$$\mu_0^* \nabla J(x^*) + \sum_{i=1}^{p} \lambda_i^* \nabla h_i(x^*) + \sum_{j=1}^{q} \mu_j^* \nabla g_j(x^*) = 0.$$

Theorem 1.24 — Karush-Kuhn-Tucker conditions. Let J,h and g be of class C^1. Let x^* a solution of the problem (1.11). We think that x^* is regular for constraints h and g. Then there exist $\lambda^* = (\lambda_1^*, \ldots.., \lambda_p^*) \in \mathbb{R}^p$ and $\mu^* = (\mu_1^*, \ldots..., \mu_q^*) \in \mathbb{R}^q$ such as

$$\forall j \in \{0, \ldots., q\} \ , \mu_j^* \geq 0, \tag{1.21}$$

$$h(x^*) = 0 \ , \ g(x^*) \leq 0, \tag{1.22}$$

$$\forall j \in \{0, \ldots., q\} \ , \mu_j^* g_j(x^*) = 0, \tag{1.23}$$

$$\nabla J(x^*) + \sum_{i=1}^{p} \lambda_i^* \nabla h_i(x^*) + \sum_{j=1}^{q} \mu_j^* \nabla g_j(x^*) = 0. \tag{1.24}$$

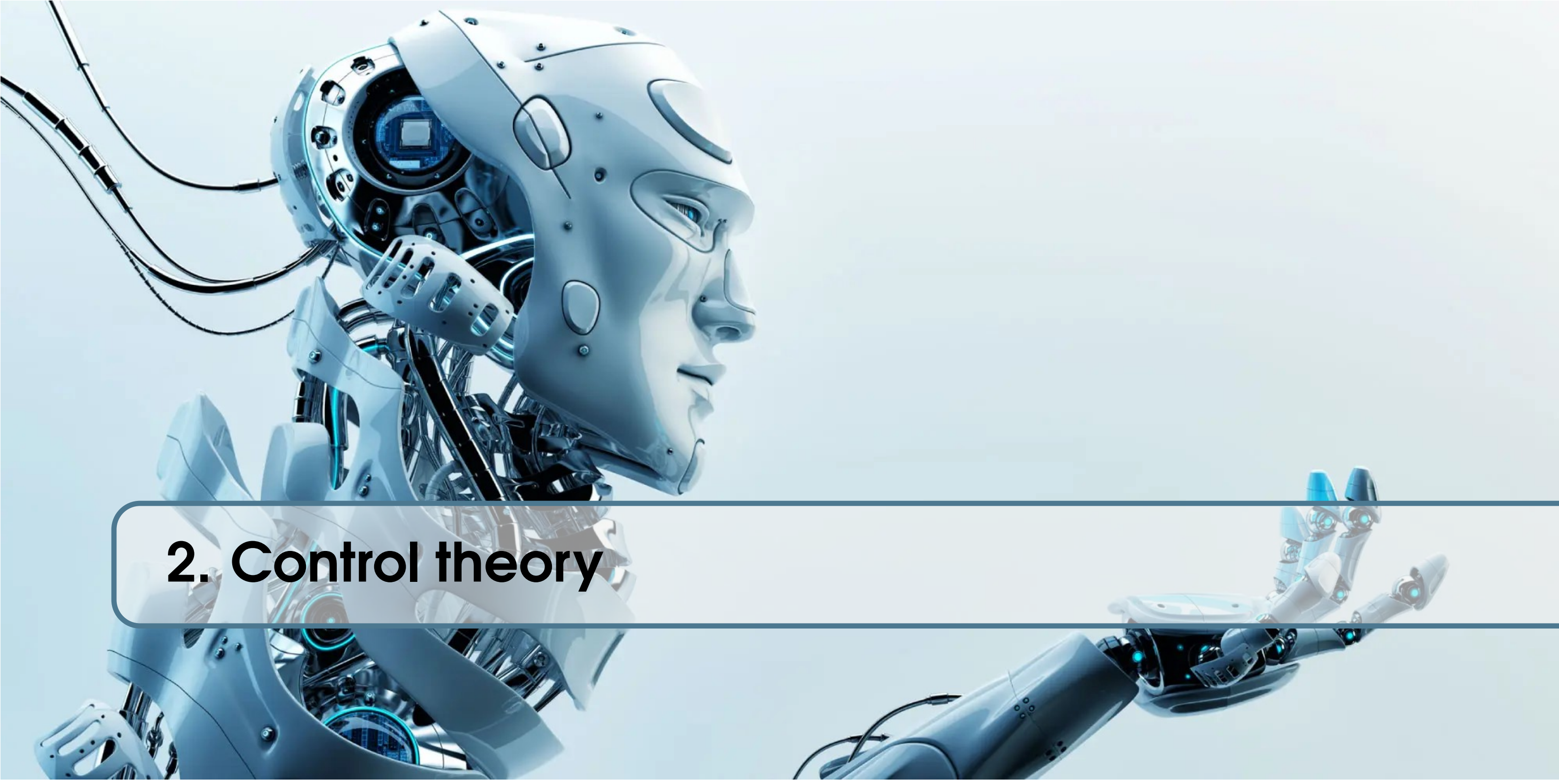
Definition 1.30 We define the Lagrangian of problem (1.11) as the function given in $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q$ by

$$L(x, \lambda, \mu) = J(x) + \sum_{i=1}^{p} \lambda_i h_i(x) + \sum_{j=1}^{q} \mu_j g_j(x).$$

The relation (1.24) is then written

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0.$$

Theorem 1.25 — Necessary and sufficient conditions. Let J, h and g be C^1, and J, g be convex, h be affine, and x^* be regular for constraints h and g. Then x^* solution of problem (1.11) if and only if the conditions (1.21)-(1.24) are satisfied.



2. Control theory

For more information on the proofs of the theorems and corollaries of this chapter, you can read [**1**], [**2**], [**3**], [**9**], [**20**], [**21**], [**40**], [**26**], [**28**], [**37**], [**46**] and [**49**].

2.1 Localized sytems

The state equation

The state of a system is the minimal set of variables required to predict its behaviour over time $(t \geq t_0)$. These are the state variables. All of these variables constitute what we call the state of the system. The solution to the equation is called the trajectory. We will designate by $\mathbb{R}^n$.

The state space $\mathbb{R}^n$ represents all possible admissible values of the system's state variables, encompassing the system's dynamic behaviour. The state equation is written in the form

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), t \geq t_0, \\ x(t_0) = x_0, \end{cases} \tag{2.1}$$

for all $t, x(t) \in X$ is a vector of n components and for t,$u(t)$ is an input (control) vector such that $u(t) \in U$ with U in $\mathbb{R}^p$.

This equation takes one of the following forms ($t_0 = 0$) :

1. In the case of continuous linear systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), t > 0, \\ x(0) = x_0, \end{cases} \tag{2.2}$$

with output :

$$y(t) = Cx(t) + Du(t), t \geq 0. \tag{2.3}$$

2. In the case of discrete linear systems

$$\begin{cases} x_{k+1} = Ax_k + Bu_k, k > 0, \\ x_0 \text{ given}, \end{cases} \tag{2.4}$$

with output :

$$y_k = Cx_k + Du_k, k \geq 0. \tag{2.5}$$

In the previous equations and the all of this part :

A, B, C and D have dimension $(n \times n), (n \times p), (q \times n)$, and $(p \times q)$ respectively.
Solution of the state equation
Let us consider the homogenuous linear system described by the equation

$$\begin{cases} \dot{x}(t) = Ax(t), t > 0, \\ x(0) = x_0, \end{cases} \tag{2.6}$$

where state $x : [0,T] \longrightarrow \mathbb{R}^n$ and $A \in M_{n,n}(\mathbb{R}^n)$ represents the dynamics of the system. By using the exponential of the matrix, the solution is in the form

$$x(t) = e^{At}x_0. \tag{2.7}$$

In the linear non-homogenuous case, the equation is

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), t > 0, \\ x(0) = x_0, \end{cases} \tag{2.8}$$

where $B \in M_{n,p}(\mathbb{R}^n)$. The solution is given by :

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds. \tag{2.9}$$

2.1.1 Controllability

The notion of controllability is one of the best-known concepts.
Consider the linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), 0 < t < T, \\ x(0) = x_0, \end{cases} \tag{2.10}$$

where $A \in \mathcal{L}(\mathbb{R}^n), B \in \mathcal{L}(\mathbb{R}^p;\mathbb{R}^n), u \in L^2(0,T;\mathbb{R}^p)$ and $x \in C[0,T;\mathbb{R}^n]$.
The purpose of this is to know if we can bring the system from an initial state x_0 to a desired state given by x_d.

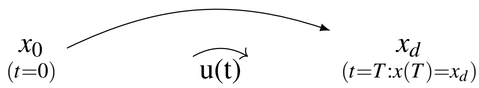


Figure 2.1: Controllability.

Figure (2.1) is an illustration of controllability : A system can move from the initial state x_0 to the desired state y_d using an appropriate control input $u(t)$.

Definition 2.1 A system is controllable if, for any initial state x_0 and desired state x_d, a control u exists such that the system reaches x_d at time T i.e.,

$$x(T) = x_d. \tag{2.11}$$

Definition 2.2 — Controllable space. We call U_{ad} controllable space the set of elements that can be reached from the state x_0. We have

$$U_{ad} = \left\{ x_d \in \mathbb{R}^n / \exists u \in L^2(0,T,\mathbb{R}^p) \text{ realizing } x(T) = x_d \right\}, \tag{2.12}$$

and since the solution to (2.10) is

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds,$$

then the system is controllable if for all x_d, there exists $u \in L^2(0,T;\mathbb{R}^p)$ achieving

$$x_d = x(T) = e^{AT}x_0 + \int_0^T e^{A(T-s)}Bu(s)ds.$$

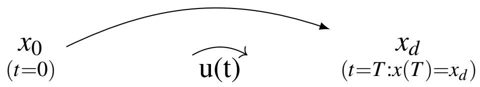


Figure 2.2: Trajectories of controllability.

Figure (2.2) shows possible paths for the state of the system to achieve the desired outcomes.

Proposition 2.1 The set U_{ad} is a vector subspace of $\mathbb{R}^n$.

Proposition 2.2 The dimension of the control space U_{ad} is equal to the rank of the order matrix $(n \times np)$

$$\left[B/AB/A^2B/..../A^{n-1}B\right].$$

Proposition 2.3 If $rg\left[B/AB/A^2B/..../A^{n-1}B\right]$, all points of $\mathbb{R}^n$ are accessible from the origin.

Theorem 2.1 — Kalmann. The system (2.10) is controllable if and only if

$$rg\left[B/AB/A^2B/..../A^{n-1}B\right] = n.$$

2.1.2 Observability

Observability refers to the ability to reconstruct the initial state of a system x_0 based solely on the output measurements over a finite time interval.

We have the following system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), 0 < t < T, \\ x(0) = x_0, \end{cases} \tag{2.13}$$

with

$$y(t) = Cx(t), \tag{2.14}$$

where $y \in L^2(0,T;\mathbb{R}^q)$ and $C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^q)$. We have

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds.$$

So, it can be written as

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-s)}Bu(s)ds.$$

Definition 2.3 The system (2.13) with the output equation (2.14) is said to be observable on $[0,T]$ if the data of an output y by (2.14) makes it possible to determine a unique x_0.

Proposition 2.4 The system (2.14) is observable if and only if

$$rg\left[C^t/A^tC^t/(A^t)^2C^t/..../(A^t)^{n-1}C^t\right] = n. \tag{2.15}$$

Duality

The duality principle states that controllability and observability are mathematically equivalent in dual systems. Specifically, a system is observable if and only if its dual system is controllable. We have the systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y = Cx(t), \end{cases} \tag{2.16}$$

and

$$\begin{cases} \dot{x}(t) = \bar{A}x(t) + \bar{B}u(t), \\ y = \bar{C}x(t), \end{cases} \tag{2.17}$$

where $\bar{A} = A^t, \bar{B} = C^t$, and $\bar{C} = B^t$.

Proposition 2.5 The system (2.16) is observable if and only if

$$rg\left[C^t/A^tC^t/(A^t)^2C^t/..../(A^t)^{n-1}C^t\right] = n. \tag{2.18}$$

which is equivalent to $rg\left[\bar{B}/\bar{A}\bar{B}/(\bar{A})^2\bar{B}/..../(\bar{A})^{n-1}\bar{B}\right] = n$ i.e., (2.17) controllable.

2.1.3 Stability and stabilization

There are less strong notions that are sufficient in applications, such as stability. Consider the following uncontrolled system :

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \forall t \geq 0, \\ x(0) = x_0. \end{cases} \tag{2.19}$$

We call an equilibrium point of the previous system (if it exists) any point $x^* \in \mathbb{R}^n$ such that

$f(t, x^*) = 0$ for all $t \geq 0$.

The system (2.19) is said to be stable if a small shift away from the equilibrium point at time $s > 0$ leads to a convergence of the solution towards the equilibrium point at any time $t \geq s$.

Definition 2.4
- The system (2.19) is said to be stable in the Lyapunov sense if for all $s > 0$ and all $\varepsilon > 0$, there exists an $\alpha > 0$ such that

$$\|x(s) - x^*\| \leq \alpha \Longrightarrow \|x(t) - x^*\| \leq \varepsilon \text{ for all } t \geq s,$$

where x^* is an equilibrium point of the system (2.19).

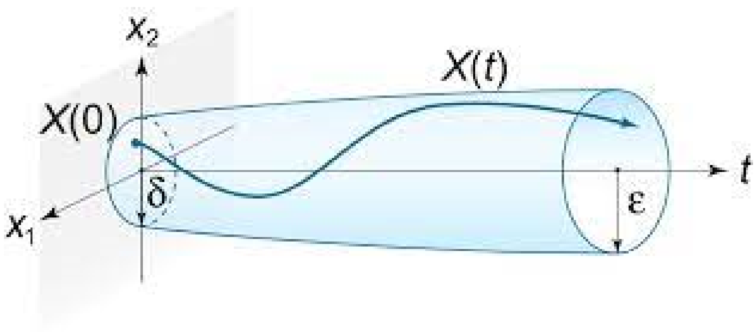


Figure 2.3: The equilibrium point is stable.

- The system (2.19) is said to be asymptotically stable in the sense of Lyapunov if it is stable in the sense of Lyapunov and for all $s > 0$, there exists a $\beta > 0$ such that

$$\|x(s) - x^*\| \leq \beta \Longrightarrow \lim_{t \to +\infty} \|x(t) - x^*\| = 0,$$

where x^* is an equilibrium point of the system (2.19).

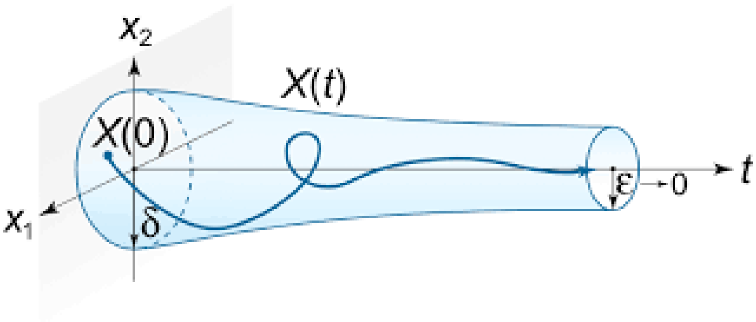


Figure 2.4: The equilibrium point is asymptotically stable.

Let us now return to the autonomous linear case of which $f(t,x(t)) = Ax(t)$ with $A \in M_{nn}$ a bounded constant matrix. In this case, $x^* = 0$ is an equilibrium point.

Theorem 2.2 The following two properties are equivalent :

- The linear system

$$\begin{cases} \dot{x}(t) = Ax(t), \forall t \geq 0, \\ x(0) = x_0. \end{cases} \quad . \tag{2.20}$$

is asymptotically stable ; that is, $\lim_{t\to\infty} x(t) = 0$ for any initial condition x_0.

- The matrix A is stable ; that is to say, the real parts of all the eigenvalues of A are strictly negative.

However, the matrix A is only sometimes stable. So, we can modify it by adding a term BF so that the new perturbed matrix $A+BF$ is stable. This is called a stabilization system (2.20).

Definition 2.5 The system (2.20) is said to be stabilized by feedback if there exists a feedback matrix F giving control over the feedback $u = Fx$ by adding it to the system (2.20) we obtain

$$\begin{cases} \dot{x}(t) = (A+BF)x(t), \forall t \geq 0, \\ x(0) = x_0, \end{cases}$$

with $A+BF$ a stable matrix.

Theorem 2.3 If the system (2.20) is controllable, it is stabilized.

2.2 Distributed systems

While localized systems deal with finite-dimensional state spaces, distributed systems extend these concepts to infinite-dimensional spaces.

Let the system be represented by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), 0 < t < T, \\ x(0) = x_0 \in D(A), \end{cases} \tag{2.21}$$

where A is an infinitesimal generator of a strongly continuous semigroup denoted $\{S(t)\}_{t\geq 0}$ in a state space X (Hilbert separable), and B is bounded by a control space U in X.

Remark When A is unbounded, $D(A) \neq X$, and therefore, any element of X can't be reached from an initial state x_0.

1. We precisely control the designed fixed state; this is exact controllability.
2. That we can only control a neighborhood of this state is low controllability.

Proposition 2.6 Let V, W, and Z are Banach spaces and let the operators $F \in \mathcal{L}(V,Z)$ and $G \in \mathcal{L}(W,Z)$. Suppose that $Im(F) \subset Im(G)$ then we have

$$\exists \gamma > 0 / \|F^* z^*\|_{V^*} \leq \gamma \|G^* z^*\|_{W^*} ; \forall z^* \in Z^*.$$

Proposition 2.7 Let $F \in \mathcal{L}(Z,V)$ and $G \in \mathcal{L}(Z,W)$ where V, W, and Z are Banach spaces. If there exists $\gamma > 0$ such that

$$\|Fz\|_V \leq \gamma \|Gz\|_W ,$$

then, $Im(F^*) \subset Im(G^*)$

Corollary 2.1 If V,W and Z are reflexive Banach spaces and $F \in \mathcal{L}(V,Z)$ and $G \in \mathcal{L}(W,Z)$, then, according to both previous properties, there is an equivalence between

1. $Im(F) \subset Im(G)$,
2. $\exists \gamma > 0 / \|F^* z^*\|_{V^*} \leq \gamma \|G^* z^*\|_{W^*}$.

Proposition 2.8 If V,W and Z are reflexive Banach spaces and $F \in \mathcal{L}(V,Z)$ and $G \in \mathcal{L}(W,Z)$, then there is an equivalence between

1. $\ker(G^*) \subset \ker(F^*)$,
2. $\overline{Im(G)} \supset \overline{Im(F)}$.

In what follows, the control and observation state spaces are Hilbert spaces.

2.2.1 Controllability

Exact controllability

We have the system given by the state equation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), 0 < t < T, \\ x(0) = x_0. \end{cases} \tag{2.22}$$

Definition 2.6 Let x_0 and $x_d \in X$. A system is exactly controllable over $[0,T]$, if for any initial state x_0 and target state x_d, there exists a control $u \in L^2(0,T,U)$ such that $x(T) = x_d$.

Proposition 2.9 If $B \in \mathcal{L}(U,X)$ is the uniform limit of a sequence of operators (B_n) which belongs a finite-dimensional subspaces of $\mathcal{L}(U,X)$, then the operator H

$$\begin{aligned} L^2(0,T,U) &\rightarrow X \\ u &\rightarrow Hu = \int_0^T S(T-s)Bu(s)ds, \end{aligned} \tag{2.23}$$

is compact.

Weak controllability

We have the following system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), 0 < t < T, \\ x(0) = x_0, \end{cases} \tag{2.24}$$

where X and U are the state and control Hilbert spaces. A generates a strongly continuous semigroup $\{S(t)\}_{t \geq 0}$ in X and $B \in \mathcal{L}(U,X)$.

The solution of (2.24) is in the form

$$x(t) = S(t)x_0 + \int_0^t S(t-s)Bu(s)ds. \tag{2.25}$$

Furthermore for all $x \in X$, we denote

$$\Omega_T = \left\{ x_d \in X : \exists u \in L^2(0,T;U) : x(T) = x_d \right\}; \tag{2.26}$$

which represents the set of all states reachable at time T from state x. We have

$$\Omega_T = Im(H_T). \tag{2.27}$$

Definition 2.7 We will say that system (2.24) is weakly controllable over $[0,T]$ if for all $x_d \in X$ and for all $\varepsilon > 0$,

$$\exists u \in L^2(0,T;U) \text{ such that } \|x(T) - x_d\|_X < \varepsilon \tag{2.28}$$

with (2.27) and the adjoint $H^* : X \to L^2(0,T;U^*)$ is defined by

$$H^*x = B^*S^*(T-\cdot)x.$$

Proposition 2.10 There is an equivalence between

1. The system (2.24) is weakly controllable.
2. $\overline{Im(H)} = X$.
3. $\overline{Im(HH^*)} = X$.
4. H^* injective $\iff Ker(H^*) = \{0\}$

Remark

- Weak controllability is for all $t, Im(H_t)$ is everywhere dense in X.
- The system (2.24) is weakly controllable if the set of desired states $\underset{t\geq 0}{\cup}\Omega_t$ is dense everywhere in X.

Proposition 2.11 System (2.24) is weakly controllable if and only if

$$\overline{\underset{t\geq 0}{\cup} Im(S(t)B)} = X. \tag{2.29}$$

Corollary 2.2 A necessary and sufficient condition for system (2.24) to be weakly controllable is that

$$\left[\int_0^t S(s)BB^*S(s)xds = 0, \forall t \geq 0\right] \Longrightarrow x = 0, \tag{2.30}$$

with (**??**), the condition (2.30) can still be written as $HH^*x = 0 \Longrightarrow x \equiv 0$.

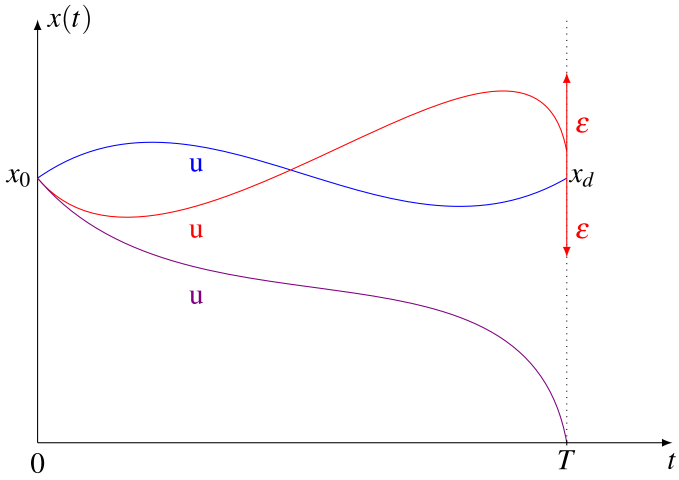


Figure 2.5: Types of controllability.

2.2.2 Observability

Let Ω be an open bounded of $\mathbb{R}^n$ represent the geometric domain in which the system (2.22) evolves and let $T > 0$. We assume that $\Gamma = \partial\Omega$ is quite regular.

The output of system (2.22) is

$$y(t) = Cx(t). \tag{2.31}$$

where $C \in \mathcal{L}(X,O)$, U, and O are the control and observation spaces, assumed by Hilbert. We have $x_u(t)$ the solution of the solution (2.22).

If A is self-adjoint then (2.22) admits a unique strongly continuous weak solution in $[0,T]$ which is given by

$$x_u(t) = S(t)x_0 + \int_0^t S(t-s)Bu(s)ds.$$

The measurements are expressed by (2.31), which gives

$$y(t) = CS(t)x_0 + CHu.$$

This output $y(t)$ is composed of two components :

- A free response associated with the initial state x_0, and
- A controlled response corresponding to the effect of the control u, assuming a zero initial state.

Since the system is linear, we can analyze the observation of x_0 by assuming $u = 0$. This reduces the problem to determining x_0, which satisfies the equation :

$$y(t) = CS(t)x_0 = Nx_0, t \in [0,T], \tag{2.32}$$

where N is a bounded linear operator $N : X \to O$.The adjoint $N^*$of N, $N^* : O^* \to X^*$ is given by

$$N^*y = \int_0^T S^*(t)C^*y(t)dt,$$

and contains all the measurements on $[0,T]$.

Definition 2.8 The system (2.22) with the output (2.31) is said to be exactly observable on $[0,T]$ if $X^* \subset ImN^*$.

Proposition 2.12 The system (2.22) with the output (2.31) is exactly observable on $[0,T]$ if and only if there exists $\gamma > 0$ such that

$$\|x_0\|_X \leq \gamma \|CS^*(.)x_0\|_{L^q(0,T;O)}.$$

Weak observability

Definition 2.9 The system (2.22) with the output (2.31) is said to be weakly observable on $[0,T]$ if $\ker(N) = \{0\}$.

Remark This definition translates the injectivity of the operator N.

Proposition 2.13 There is an equivalence between the following statements

- (2.22) and (2.31) is weakly observable.
- $\overline{Im(N^*)} = X$.
- $\overline{Im(N^*N)} = X$.
- $\overline{\bigcup_{t\geq 0} Im(S^*(t)C^*)} = X$.

One can use pseudo-inverse techniques to characterize the exact (respectively weak) observability. By introducing the operator $N^*N : X \to X^*$ defined by

$$N^*Nx = \int_0^T S^*(t)C^*CS(t)xdt.$$

We show that if N^*N is positive (positive definite), then the system (2.22) and (2.31) is weakly (respectively exactly) observable, and the initial state can be determined from (2.32) by

$$x_0 = (N^*N)^{-1}N^*y,$$

Either

$$x_0 = (\int_0^T S^*(t)C^*CS(t)dt)^{-1} \int_0^T S^*(t)C^*y(t)dt.$$

Duality observation control

The controllability of (2.22) amounts to determinate u such that $Hu = x_d$, where x_d given in X.

Let us then consider

$$\begin{aligned}\langle Hu, x\rangle_X &= \left\langle \int_0^T S(T-s)Bu(s)ds, x\right\rangle \\ &= \int_0^T \langle u(s), B^*S^*(T-s)x\rangle\, ds \\ &= \langle u, H^*x\rangle_{L^2(0,T;U)},\end{aligned}$$

The adjoint H^* of H can be interpreted as an output operator which to given x in X associates the observation

$$H^*x = B^*S^*(T-\cdot)x \in L^2(0,T;U). \tag{2.33}$$

Definition 2.10 We will say that the controlled system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), 0 < t < T, \\ x(0) = x_0, \end{cases} \tag{2.34}$$

and the system observed

$$\begin{cases} \dot{x}(t) = \tilde{A}x(t) + Bu(t), 0 < t < T, \\ y(t) = Cx(t), \end{cases} \tag{2.35}$$

are dual to each other if $\tilde{A} = A^*$ et $B^* = C$. With this definition and (2.33), the adjoint H^* of H is given by $H^*x = B^*S^*x = CSx = Nx$.

Proposition 2.14 If the two systems (2.34) and (2.35) are adjoint, a necessary and sufficient condition for (2.35) to be weakly observable is that (2.34) is weakly controllable.

Regional controllability and observability

We have Ω an open domain bounded by $\mathbb{R}^n$, with regular Γ boundary and $T > 0$. Let X (respectively U) be a separable Hilbert space that designates the state space (respectively, control). We will take $X = L^2(\Omega)$ and $U = \mathbb{R}^p$.

Consider a system defined by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), 0 < t < T, \\ x(0) = x_0 \in D(A). \end{cases} \tag{2.36}$$

We suppose that the operator A generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on $X, B \in \mathcal{L}(U,X)$ and $u \in L^2(0,T;U)$.

The system (2.22) admits a unique weak solution given by

$$x_u(t) = S(t)x_0 + \int_0^t S(t-s)Bu(s)ds, \tag{2.37}$$

where $x_u \in L^2(0,T;U)$ and $x_u(T) \in X$.

With the output

$$y(t) = Cx(t), \tag{2.38}$$

where $C \in \mathcal{L}(X,O)$ and O is the assumed Hilbert observation space. Consider a subdomain (a region) w of $\Omega, w \subset \Omega$, assumed to be nonempty and not necessarily connected. We have the restriction function

$$\begin{aligned}\chi_w \quad &: \quad L^2(\Omega) \to L^2(w) \\ x \quad &\to \quad \chi_w x = x|_w,\end{aligned}$$

whose adjoint $\chi_w : L^2(w) \to L^2(\Omega)$ is defined by

$$(\chi_w^* x)(z) = \begin{cases} x(z), z \in w, \\ 0, z \in \Omega \backslash w. \end{cases}$$

Regional controllability

Definition 2.11 1. The system (2.36) is said to be exactly regionally controllable on w (or exactly w-controllable) if for all $x_d \in X$ there exists a control $u \in U$ such that

$$x_u(T)|_w = x_d.$$

2. The system (2.36) is said to be weakly regionally controllable on w (or weakly $w-$controllable) if for all $x_d \in X$ and $\forall \varepsilon \geq 0$ there exists a control $u \in U$ such that

$$\|x_u(T)|_w - x_d\|_{L^2(w)} \leq \varepsilon,$$

where $x_u(.)$ is given by (2.37).

Remark If the system is excited by a punctual or boundary action, the operator B is no longer bounded, so the choice of spaces must be reviewed.

For the exact regional controllability (respectively low) study without loss of generation reality, we can assume that $x_0 = 0$.

Proposition 2.15 There is an equivalence between

1. Exact regional controllability and $Im\chi_w H = L^2(w)$.
2. Exact low controllability and $\overline{Im\chi_w H} = L^2(w)$.

Proposition 2.16 If $u \in L^2(0,T;U)$, then system (2.36) is exactly regionally controllable if and only if for all $x^* \in L^2(w)$ there exists $\gamma > 0$, such that

$$\gamma \|B^* S^*(\cdot)\chi_w^* x^*\|_{L^2(0,T;U)} \geq \|x^*\|_{L^2(w)}.$$

Proposition 2.17 1. The system (2.36) is exactly regional controllable if and only if

$$\ker \chi_w + ImH = L^2(\Omega).$$

2. The system (2.36) is weakly regional controllable if and only if

$$\ker \chi_w + \overline{ImH} = L^2(\Omega).$$

Proposition 2.18 The system (2.36) is weakly regional controllable on w if and only if

$$\overline{\bigcup_{n\geq 0} \chi_w A^n BU} = L^2(w), \text{ for all } t \in [0,T].$$

Remark 1. A system that is exactly (resp. weakly) controllable is exactly (resp. weakly) regional controllable.
2. A system that is exactly (resp. weakly) regional controllable on w_1 is exactly (resp. weakly) regional controllable on w_2, for all $w_2 \subset w_1$.

Regional Observability The problem consists of the reconstruction of the initial state, which is supposed to be unknown, in the subregion w. If we consider the following decomposition :

$$x_0 = \begin{cases} x_e, z \in w, \\ x_u, z \in \Omega \backslash w, \end{cases} \tag{2.39}$$

where x_e is the state to be estimated and x_u is the undesired state, then the problem consists of reconstructing x_e with the knowledge of (2.36) and (2.38).

If the system (2.36) is autonomous, (2.38) gives

$$y(t) = CS(t)x_0 = N(t)x_0, \tag{2.40}$$

where N is an operator $L^2(\Omega) \to L^2(0,T,;X)$. The adjoint N^* is given by

$$N^* x = \int_0^T S^*(s)C^* x(s)ds. \tag{2.41}$$

We recall that the system (2.36) with output (2.38) is said to be weakly observable if $\ker N = \{0\}$. Consider now the restriction map

$$\begin{aligned} \gamma \quad &: \quad L^2(\Omega) \to L^2(\omega) \\ y \quad &\to \quad \gamma x = x|_{\omega}, \end{aligned} \tag{2.42}$$

where $x|_w$ is the restriction of x to w. Then, we have the following.

Definition 2.12 The system (2.36) with (2.38) is said to be w-regionally observable if

$$Im\gamma N^* = L^2(w). \tag{2.43}$$

The system (2.36) with (2.38) is said to be w-weakly regionally observable if

$$\overline{Im\gamma N^*} = L^2(w). \tag{2.44}$$

From the above definition, we deduce the following property.

Proposition 2.19 The system (2.36) with (2.38) is w-regionally observable if there exists $\upsilon > 0$ such that

$$\forall x_0 \in L^2(w), \|\gamma x_0\|_{L^2(w)} \leq \upsilon \|N\gamma^* x_0\|_{L^2(0,T;Z)}. \tag{2.45}$$

Remark The regional observability concept is more convenient in the analysis of real systems. This is clear because :

1. The definitions (2.43) and (2.44) are general and can be applied to the case where $w = \Omega$.
2. There exist systems that are not observable but regionally observable.

2.3 Optimal control

Among all C^1 curves $x : [a,b] \to \mathbb{R}$ satisfying given boundary conditions

$$x(a) = x_0, x(b) = x_1,$$

find (local) minima of the cost functional

$$J(x) = \int_a^b L(y, x(y), x'(y))dy,$$

where the function $L : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called the Lagrangian of the problem.

The calculus of variations determines the finding of functions that minimize or maximize (optimize) a functional (given). This is the same in the finite-dimensional optimization problem but involves finding optimal functions rather than scalar values. This consists of solving equations called Euler-Lagrange equations, which arise from the principle of variation. Solutions often involve solving the Euler-Lagrange equations to find the optimal function $x(y)$.

After that, the calculus of variations will continue if optimal control theory is used. Historically, optimal control theory has been closely linked to classical mechanics, particularly to the variational principles of mechanics. The key point of this theory is the Pontryagin maximum principle, formulated by L. S. Pontryaguine in 1956, which gives a necessary condition of optimality and thus allows the calculation of optimal trajectories.

2.3.1 Problem of optimal control

The goal of optimal control is to minimize a functional cost $J(u)$ subject to dynamics. The general form of the optimal control problems is as follows :

$$\begin{cases} \min J(T,u) = g(T, x(T)) + \int_0^T f_0(t, x(t), u(t))dt, \\ \dot{x}(t) = f(t, x(t), u(t)), \\ x(0) = x_0, \\ x(T) = x_1, \\ u \in U, t \in I = [0,T], \end{cases} \tag{2.46}$$

where x_0 is the initial position, $x(T)$ is its terminal position. $u(t)$ is the control. U is the set of admissible controls. And $J(T,u)$ is the cost of the objective function. This function divided on two parts, the first $g(T,x(T))$ is the final cost and the second $\int_0^T f_0(t,x(t),u(t))dt$ is the integral cost.

the main is to minimize the cost function

$$J(T,u) = g(T,x(T)) + \int_0^T f_0(t,x(t),u(t))dt.$$

We have three problems :

1. **Lagrange problem** with the cost function

$$J(T,u) = \int_0^T f_0(t,x(t),u(t))dt,$$

   where $g = 0$ and $J(T,u) = \int_0^T f_0(t,x(t),u(t))dt$ the integral cost. This problem is useful for objectives that focus on trajectory quality.

2. **Mayer problem** with the cost function

$$J(T,u) = g(T,x(T)),$$

   where $f_0 = 0$. This problem is useful when the end state is the primary concern.

3. **Mayer-Lagrange problem** with the cost function

$$J(T,u) = g(T,x(T)) + \int_0^T f_0(t,x(t),u(t))dt. \tag{2.47}$$

This problem combines terminal and integral costs, which is useful when both types of cost are essential.

2.3.2 Optimal control for the linear system

This system is written as follows

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + r(t), t \in I, \\ x(0) = x_0, \end{cases} \tag{2.48}$$

where $I = [0,T], A, B$ and r are three applications in $M_n(\mathbb{R}), M_{n,m}(\mathbb{R})$ and $\mathbb{R}^m$.

The solution $x(t) : I \to \mathbb{R}^n$ of this system is

$$x(t) = R(t)x_0 + \int_{t_0}^T R(t)M(s)^{-1}(B(s)u(s) + r(s))ds, \forall t \in I,$$

with $R(t) : I \to M_n(\mathbb{R})$ the resolvent of the homogeneous system $\dot{x}(t) = A(t)x(t)$ defined by

$$\begin{aligned} \dot{R}(t) &= A(t)R(t), \\ R(0) &= Id. \end{aligned}$$

Before we discuss the theorem, we need to define the following definition.

Definition 2.13 A control $u^*(t), t \in [0,T]$ is said optimal control if $u^*(t)$ is extrema and $J(u^*(t)) < J(u(t))$, for any extrema control $u(t), t \in [0,T]$.

Theorem 2.4 — Maximum principle. Consider the linear control system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) + r(t), t \in I, \\ x(0) = x_0, \end{cases} \tag{2.49}$$

where the constraint domain $\Omega \subset \mathbb{R}^m$ on the control is compact. Let the control u be extremal on $[0,T]$ if and only if there exists a nontrivial solution $p(t)$ of the equation $\dot{p}(t) = -p(t)A(t)$ with the transversality condition $p(T) = p_T$ (given) such that

$$p(t)B(t)u(t) = \max_{v\in\Omega} p(t)B(t)v.$$

The linear case's maximum principle (the previous theorem) gives a necessary condition for optimality. Also, there exists control for the a linear system with a quadratic cost, i.e.

$$J(u) = x(T)^t Qx(T) + \int_0^T \left(x(t)^t W(t)x(t) + u(t)^t U(t)u(t)\right) dt, \quad (2.50)$$

where $U(t) \in M_m(\mathbb{R})$ is symmetric positive matrix, $W(t) \in M_n(\mathbb{R})$ is symmetric positive matrix, and $Q \in M_n(\mathbb{R})$ is a symmetric positive matrix.

This quadratic cost (2.50) is often very natural in a problem because of its advantageous mathematical properties, its suitability with common physical interpretations, and its ability to simplify complex problems by linearization.

2.3.3 Optimal control for the nonlinear system

Consider the general nonlinear system of the control

$$\dot{x}(t) = f(t, x(t), u(t)), \quad (2.51)$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a function of class C^1.

Definition 2.14 The Hamiltonian of the system (2.51) is the function :

$$\begin{array}{rcl} H & : & \mathbb{R} \times \mathbb{R}^n \times (\mathbb{R}^n - \{0\}) \times \mathbb{R}^m \to \mathbb{R} \\ (t,x,p,u) & \to & H(t,x,p,u) = p_0 f_0 + \langle p, f(t,x,u)\rangle, \end{array}$$

where p is the adjoint vector.

Proposition 2.20 Let u be a singular control in $[0,T]$ for the control system (2.51), and let $x(t)$ be the associated singular trajectory. Then, there exists an absolutely continuous map $p : [0,T] \to \mathbb{R}^n - \{0\}$, called adjoint vector, such that the following equations hold for almost all $t \in [0,T]$

$$\begin{array}{rcl} \dot{x}(t) & = & \dfrac{\partial H}{\partial p}(t,x(t),p(t),u(t)), \\ \dot{p}(t) & = & -\dfrac{\partial H}{\partial x}(t,x(t),p(t),u(t), \\ \dfrac{\partial H}{\partial u}(t,x(t),p(t),u(t)) & = & 0, \end{array}$$

Definition 2.15 where H is the Hamiltonian of the system.

Theorem 2.5 — Maximum principle of Pontryaguine. Consider the control system in $\mathbb{R}^n$

$$\dot{x}(t) = f(t, x(t), u(t)), \quad (2.52)$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is of class C^1 and and where the controls are measurable and bounded applications defined on an interval $[0, t_e(u)[$ of $\mathbb{R}^+$ and with values in $\Omega \subset \mathbb{R}^m$. Let M_0 and M_1 be two subsets of $\mathbb{R}^n$. We denote by U the set of admissible controls u whose associated trajectories connect an initial point of M_0 to an endpoint of M_1 in time $t(u) < t_e(u)$. Furthermore, we define the cost of a control u on $[0,T]$

$$J(t,u) = \int_0^t f_0(s,x(s),u(s))ds + g(t,x(t)),$$

where $f_0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are $C^1, x(t)$ is the trajectory solution of (2.52) associated with control u.

We consider the following optimal control problem : determine a trajectory connecting M_0 to M_1 and minimizing the cost. The final time can be fixed or not.

If the control $u \in U$ associated with the trajectory $x(t)$ is optimal on $[0,T]$, then there exists an absolutely continuous application $p(t) : [0,T] \to \mathbb{R}^n$ called adjoint vector, and a real $p_0 \leq 0$, such that the pair $(p(t), p_0)$ is non-trivial, and such that, for almost all $t \in [0,T]$,

$$\begin{aligned}\dot{x}(t) &= \frac{\partial H}{\partial p}(t,x(t),p(t),p_0,u(t)),\\ \dot{p}(t) &= -\frac{\partial H}{\partial x}(t,x(t),p(t),p_0,u(t)),\end{aligned}$$

where $H(t,x,p,p_0,u) = \langle p, f(t,x,u)\rangle + p_0 f_0(t,x,u)$ is the Hamiltonian of the system, and we have the maximization condition almost everywhere on $[0,T]$

$$H(t,x(t),p(t),p_0,u(t)) = \max_{v\in\Omega} H(t,x(t),p(t),p_0,v).$$

If in addition the final time to reach the target M_1 is not fixed, we have the condition at the final time T

$$\max_{v\in\Omega} H(T,x(T),p(T),p_0,v) = -p_0 \frac{\partial g}{\partial t}(T,x(T)).$$

If furthermore M_0 and M_1 (or just one of the two sets) are varieties of $\mathbb{R}^n$ having tangent spaces at $x(0) \in M_0$ and $x(T) \in M_1$, then the adjoint vector can be constructed to satisfy the transversality conditions at both ends.

$$p(0) \perp T_{x(0)}M_1,$$

and

$$p(T) - p_0 \frac{\partial g}{\partial x}(T,x(T)) \perp T_{x(T)}M_1.$$

More than one version of the principle maximum of Pontryaguine (constraints in the state) exists, and this problem was more complicated.

2.4 Financial dynamical system

Economic dynamic systems were developed to model intricate interactions between economic variables such as interest rates, investment levels, inflation, and other macroeconomic parameters. The first formal model of a financial dynamic system based on differential equations was introduced by Richard Goodwin in 1967. His model of business cycle dynamics was inspired by the Lotka-Volterra predator-prey equations, initially used in biology to describe interactions between animal populations. Goodwin demonstrated that business cycles, characterized by phases of growth and recession, could be modelled using nonlinear differential equations. He focused on the relationship between capital (investment) and labour (employment), showing that their interaction could generate cyclical business fluctuations.

During the 1970*s*, models emerged that incorporated inflation dynamics and interest rates. These two variables are critical in financial systems, as they directly impact investment decisions, consumption, and monetary policy.

Economist Hyman Minsky developed a theory addressing the dynamics of financial instability, showing that financial systems do not naturally stabilize but instead tend to evolve toward crises. His model posits that periods of economic stability foster riskier financial behaviour, leading to debt accumulation and financial crises. Nonlinear models are essential for analyzing complex financial systems, as interactions between economic variables are often highly nonlinear.

The study of financial dynamics is crucial for predicting crises (by modelling tipping points in market behaviour), analyzing economic policies (to assess how changes in fiscal or monetary policies affect business cycles), and understanding financial instability (by examining how risky behaviour can lead to imbalances, as described by Minsky).

2.4.1 Setting of the problem

Numerous authors have described the financial dynamic model in this section as composed of four interconnected subblocks : production, money supply, stock, and labour force. Together, these sub-blocks capture the key elements of an economic system. The behaviour of the financial system can be modelled using three first-order differential equations that govern the evolution of critical variables over time.

In this formulation, the first variable, x represents the interest rate, which plays a central role in the influence of investment and savings. The second variable, y corresponds to investment demand, reflecting the capital expenditure needed for growth and production. The third variable, z denotes the price index, an indicator of inflation and economic purchasing power.

These variables interact dynamically through a system of differential equations, capturing complex feedback loops between interest rates, investment, and price levels. The resulting dynamical system provides a framework for analyzing the economy's evolution under different conditions, helping to predict economic cycles, inflationary pressures, and investment behaviours. By studying the relationships between these variables, economists can gain insight into how changes in one aspect of the system (such as a change in interest rates) can propagate through the economy, affecting production, employment, and overall financial stability.

$$\begin{cases} \dot{x} = z + (y-a)x, \\ \dot{y} = 1 - by - x^2, \\ \dot{z} = -x - cz, \end{cases} \tag{2.53}$$

where $a \geq 0$ is the savings amount, $b \geq 0$ is the cost of pre-investment, and $c \geq 0$ is the elasticity of commercial demands.

Two factors cause significant changes in the interest rate : Contradictions in the investment market are the surplus between investment and savings and structural adjustment from the prices. This is expressed in the first equation. The rate of change is related to the cost of investment and the interest rate, as given in the second equation. Change is affected by inflation rates ; therefore, at the same time, it can be expressed by the nominal interest rate and the real interest rate, which is formulated in the third equation.

2.4.2 Stability analysis

For stability, we linearize the system (because it is nonlinear) and study its stability using the central manifold theorem.

Fixed point

To calculate the fixed points, we will solve the equations $f(x,y,z) = 0$ such as :

$$f(x,y,z) = \begin{pmatrix} z + (y-a)x \\ 1 - by - x^2 \\ -x - cz \end{pmatrix}. \tag{2.54}$$

Then the fixed point is : $q\left(0, \frac{1}{b}, 0\right)$ if $c - b - abc < 0$ and the other fixed points are : $q\left(0, \frac{1}{b}, 0\right)$ and $P_{\pm}\left(\pm\sqrt{\frac{c-b-abc}{c}}, \frac{1+ac}{c}, \pm\frac{1}{c}\sqrt{\frac{c-b-abc}{c}}\right)$ if $c - b - abc \geq 0$.

Stability

We will study the stability of the Jacobian matrix for the first fixed point.

■ **For q** $\left(0, \frac{1}{b}, 0\right)$ **if** $\mathbf{c - b - abc \leq 0}$

We suppose that $X = x, Y = y - \frac{1}{b}$ and $Z = z$. So the system becomes as follows :

$$\begin{cases} \dot{X} = (\frac{1}{b} - a)X + Z + XY, \\ \dot{Y} = -bY - X^2, \\ \dot{Z} = -X - CZ. \end{cases} \tag{2.55}$$

Through changes in variables. The point q becomes $q(0,0,0)$ is a fixed point of the system (2.55).
The Jacobian matrix :

$$J = \begin{pmatrix} \frac{1}{b} - a + Y & X & 1 \\ -2X & -b & 0 \\ -1 & 0 & -c \end{pmatrix},$$

in the point $q(0,0,0)$:

$$J = \begin{pmatrix} \frac{1}{b} - a & 0 & 1 \\ 0 & -b & 0 \\ -1 & 0 & -c \end{pmatrix}.$$

The eigenvalues of this matrix are given stability. The fixed point is stable if all eigenvalues have a negative real part.

The characteristic polynomial associated with the Jacobian matrix :

$$\det(J - \lambda I) = (-\lambda - b)(\lambda^2 + (c + a - \frac{1}{b})\lambda + 1 + ac - \frac{c}{b}).$$

We find that three eigenvalues $\lambda_1 = -b$ and λ_2, λ_3 were determined by :

$$\lambda^2 + (c + a - \frac{1}{b})\lambda + 1 + ac - \frac{c}{b} = 0, \tag{2.56}$$

$$\Delta = (c + a - \frac{1}{b})^2 - 4(1 + ac - \frac{c}{b}).$$

For $c - b - abc < 0$, then $1 + ac - \frac{c}{b} > 0$. The solutions of the polynomial (2.56) λ_2 and λ_3, can be divided into three cases :

1. **Case 01 :** $\mathbf{c - b - abc < 0}$, $\mathbf{c + a - \frac{1}{b} > 0}$, then $\Delta > 0$.
   So,

$$\begin{cases} \lambda_2 = \dfrac{-(c + a - \frac{1}{b}) - \sqrt{(c + a - \frac{1}{b})^2 - 4(1 + ac - \frac{c}{b})}}{2} < 0, \\ \lambda_3 = \dfrac{-(c + a - \frac{1}{b}) + \sqrt{(c + a - \frac{1}{b})^2 - 4(1 + ac - \frac{c}{b})}}{2} < 0. \end{cases}$$

   Then, the fixed point $q\left(0, \frac{1}{b}, 0\right)$ of the system (1.13) under the conditions of the first case is stable.

2. **Case 02 :** $\mathbf{c-b-abc<0}$, $\mathbf{c+a-}\frac{1}{b}\mathbf{<0}$, then $\Delta>0$.
So,

$$\begin{cases} \lambda_2=\dfrac{-(c+a-\frac{1}{b})-\sqrt{(c+a-\frac{1}{b})^2-4(1+ac-\frac{c}{b})}}{2}>0, \\ \lambda_3=\dfrac{-(c+a-\frac{1}{b})+\sqrt{(c+a-\frac{1}{b})^2-4(1+ac-\frac{c}{b})}}{2}>0. \end{cases}$$

Then, the fixed point $q\left(0,\frac{1}{b},0\right)$ of the system (2.53) and under the conditions of the first case, they are unstable.

3. **Case 03 :** $\mathbf{c-b-abc=0}$, then

$$\Delta=(c+a-\frac{1}{b})^2.$$

So,

$$\begin{cases} \lambda_2=0 \\ \lambda_3=-(c+a-\frac{1}{b}) \end{cases}$$

For the sign of the third eigenvalue, we have two cases :

a. If $a=0$, $b=c$ and $0<c<1$, then

$$\lambda_3=-(c+a-\frac{1}{b})=\frac{1-c^2}{c}>0.$$

Then, the fixed point q is unstable.

b. If $a=0$, $b=c$ and $c>1$, then

$$\lambda_3=-(c+a-\frac{1}{b})=\frac{1-c^2}{c}<0.$$

We use the central manifold theorem to ensure the stability of the fixed point. The associated eigenvectors are respectively :
For $\lambda_1=-b$

$$JX=\lambda_1X\ ;\ X\in\mathbb{R}^3,$$

$$\begin{pmatrix} \frac{1}{b}-a & 0 & 1 \\ 0 & -b & 0 \\ -1 & 0 & -c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -bx \\ -by \\ -bz \end{pmatrix},$$

$$\Longrightarrow\begin{cases} (\frac{1}{b}-a)x+z=-bx, \\ -by=-by, \\ -x-cz=-bz. \end{cases}$$

From the equation (2), we have : $y\in\mathbb{R}$.
From the equations (1) and (3), we have :

$$(\frac{c}{b}-ca+cb+1-ab+b^2-1)x=0,$$

$\Longrightarrow x=0$ this implies that $z=0$.
The eigenvector is $(0,1,0)^t$

For $\lambda_2 = 0$

$$JX = \lambda_2 X \; ; X \in \mathbb{R}^3,$$

$$\begin{pmatrix} \frac{1}{b} - a & 0 & 1 \\ 0 & -b & 0 \\ -1 & 0 & -c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\Longrightarrow \begin{cases} (\frac{1}{b} - a)x + z = 0, \\ -by = 0, \\ -x - cz = 0. \end{cases}$$

From the equation (2), we have : $y = 0$.
From the equation (3), we have : $z = -\frac{1}{c}x$
The eigenvector is $(1, 0, -\frac{1}{c})^t$.
For $\lambda_3 = \frac{1-c^2}{c} < 0$

$$JX = \lambda_3 X \; ; X \in \mathbb{R}^3,$$

$$\begin{pmatrix} \frac{1}{b} - a & 0 & 1 \\ 0 & -b & 0 \\ -1 & 0 & -c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1-c^2}{c} x \\ \frac{1-c^2}{c} y \\ \frac{1-c^2}{c} z \end{pmatrix},$$

$$\Longrightarrow \begin{cases} (\frac{1}{b} - a)x + z = \frac{1-c^2}{c} x, \\ -by = \frac{1-c^2}{c} y, \\ -x - cz = \frac{1-c^2}{c} z. \end{cases}$$

From the equation (2), we have : $y \in \mathbb{R}$.
From the equation (3), we have : $x = -\frac{1}{c}z$.
The eigenvector is $(-\frac{1}{c}, 0, 1)^t$.
The eigenvectors of the eigenvalues λ_1 and λ_3 are extended in the stable subspace E^s, and the eigenvalues λ_2 is extended in the center subspace E^c.
Do the linear transformation for the characteristic base $(X, Y, Z)^t$.

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = T \begin{pmatrix} y \\ v \\ w \end{pmatrix}, T = \begin{pmatrix} 1 & -\frac{1}{c} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{c} & 1 & 0 \end{pmatrix},$$

$$T^{-1} = \begin{pmatrix} \frac{c^2}{c^2-1} & 0 & \frac{c}{c^2-1} \\ \frac{c}{1-c^2} & 0 & \frac{c^2}{1-c^2} \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.57}$$

We pose that $X = u - \frac{1}{c}v, Y = w, Z = -\frac{1}{c}u + v, X' = u' - \frac{1}{c}v', Y' = w'$ and $Z' = -\frac{1}{c}u' + v'$. Putting this changes in the system (2.55), we find it :

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{c^2-1}{c} & 0 \\ 0 & 0 & -\frac{1}{b} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} \frac{c^2(u-\frac{1}{c}v)w}{c^2-1} \\ \frac{c(u-\frac{1}{c}v)w}{1-c^2} \\ -(u-\frac{1}{c}v)^2 \end{pmatrix}. \tag{2.58}$$

For the new variables $(u,v,w)^t$, the eigenvalues do not change and the corresponding eigenvectors are respectively : $\lambda_1 = -b$ the associated eigenvector is $(0,0,1)^t, \lambda_2 = 0$ the associated eigenvector is $(1,0,0)^t$, and $\lambda_3 = \frac{1-c^2}{c} < 0$ the associated eigenvector is $(0,1,0)^t$.
So, $E^c = u$, $E^s =$ the space $\{(u,v,w)^t)\}$, and $E^i = \varnothing$.
The linear part of the system (2.58) has been decoupled :

i.

$$u' = \frac{c^2}{c^2-1}(u - \frac{1}{c}v)w. \tag{2.59}$$

ii.

$$\begin{pmatrix} v' \\ w' \end{pmatrix} = \begin{pmatrix} \frac{c^2}{c^2-1} & 0 \\ 0 & -b \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} \frac{c}{1-c^2}(u-\frac{1}{c}v)w \\ -(u-\frac{1}{c}v)^2 \end{pmatrix}. \tag{2.60}$$

The central manifold W^c is the curve, which is the tangent of the central subspace E^c, and we try to get the equation of W^c.
Now just get the functions $h_1(u)$ and $h_2(u)$ that deviate from the u axis towards the v axis and the w axis after having the tangent of the fixed point $(0,0,0)$.
The central manifold :

$$W^c = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} / \begin{pmatrix} v \\ w \end{pmatrix} = h(U) = \begin{pmatrix} h_1(u) \\ h_2(u) \end{pmatrix} \right\}, h(0) = 0, Dh(u) = 0, \tag{2.61}$$

such as $h(U) : \mathbb{R} \longrightarrow \mathbb{R}^2$ is the differential in the neighborhood of the point $(0,0,0)$.To get $h(u)$, the form of power series is used. Suppose that :

$$\begin{pmatrix} v \\ w \end{pmatrix} = h(U) = \begin{pmatrix} a_1u^2 + b_1u^3 + c_1u^4 + \ldots \\ a_2u^2 + b_2u^3 + c_2u^4 + \ldots \end{pmatrix}. \tag{2.62}$$

Put (2.62) in (2.60), we obtain :

$$\begin{aligned} &\begin{pmatrix} h_1'(u) \\ h_2'(u) \end{pmatrix} \left(\frac{c^2}{c^2-1}(u - \frac{1}{c}h_1(u))h_2(u) \right) \\ &= \begin{pmatrix} \frac{c^2}{c^2-1} & 0 \\ 0 & -b \end{pmatrix} \begin{pmatrix} h_1(u) \\ h_2(u) \end{pmatrix} + \begin{pmatrix} \frac{c}{1-c^2}(u-\frac{1}{c}h_1(u))h_2(u) \\ -(u-\frac{1}{c}h_1(u))^2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{c^2}{c^2-1}h_1(u) + \frac{c}{1-c^2}(u-\frac{1}{c}h_1(u))h_2(u) \\ -bh_2(u) - (u-\frac{1}{c}h_1(u))^2 \end{pmatrix}. \end{aligned} \tag{2.63}$$

We have : $\dot{h}_i = \frac{\partial h_i(u)}{\partial u} = 2a_iu + 3b_iu^2 + 4c_iu^3$. Put h and $\dot{h}$ of (2.62) of shape (2.63) :

$$\begin{aligned} &\begin{pmatrix} 2a_1u + 3b_1u^2 + 4c_1u^3 \\ 2a_2u + 3b_2u^2 + 4c_2u^3 \end{pmatrix} \left(\frac{c^2}{c^2-1}(u - \frac{1}{c}h_1(u))h_2(u) \right) \\ &= \begin{pmatrix} \frac{c^2-1}{c}(a_1u^2 + b_1u^3 + c_1u^4) + \frac{c}{c^2-1}(u - \frac{1}{c}(a_1u^2 + b_1u^3 + c_1u^4)) \\ -b(a_2u^2 + b_2u^3 + c_2u^4) - (u - \frac{1}{c}(a_1u^2 + b_1u^3 + c_1u^4))^2 \end{pmatrix}. \end{aligned}$$

Equation v :

$$\begin{cases} o(u^2): \ \dfrac{c^2-1}{c}a_1 = 0, \\ o(u^3): \ \dfrac{c^2-1}{c}b_1 + \dfrac{c}{c^2-1}a_2 = 0, \\ o(u^4): \ \dfrac{2a_1a_2c^2}{c^2-1} = \dfrac{c^2-1}{c}c_1 + \dfrac{c}{1-c^2}(b_2 - \dfrac{a_1a_2}{c}). \end{cases} \tag{2.64}$$

Equation w :

$$\begin{cases} o(u^2): \ \dfrac{c^2-1}{c}c_1 + \dfrac{c}{1-c^2}b_2 = 0, \\ o(u^3): \ \dfrac{2a_1}{c} - b_1b_2 = 0, \\ o(u^4): \ \dfrac{2a_2^2c^2}{c^2-1} = \dfrac{2b_1}{c} - bc_2 - \dfrac{a_1^2}{c^2}. \end{cases} \tag{2.65}$$

From (2.64) and (2.65), we have :

$$\begin{cases} a_1 = 0, b_1 = -\dfrac{c^2}{b(c^2-1)^2}, c_1 = 0, \\ a_2 = -\dfrac{1}{b}, b_2 = 0, c_2 = \dfrac{-2c}{b^3(c^2-1)}(b + c(c^2-1)). \end{cases}$$

So,

$$W^c = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} / \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} h_1(u) \\ h_2(u) \end{pmatrix} = \begin{pmatrix} f_1(u) \\ f_2(u) \end{pmatrix} \right\},$$

with

$$f_1(u) = -\frac{c^2}{b(c^2-1)^2}u^3 + o(u^5),$$

$$f_2(u) = -\frac{1}{b}u^2 - \frac{2c}{b^3(c^2-1)}(b + c(c^2-1))u^4 + o(u^5).$$

By neglecting $h_1(u)$ and $h_2(u)$ of W^c to (2.59), we find that the equation that determines the central manifold is the differential equation.

$$u' = -\frac{c^2}{b(c^2-1)^2}u^3 - \frac{2c}{b^3(c^2-1)}(b + c(c^2-1))u^5 + o(u^6). \tag{2.66}$$

The reason is that the coefficient u^3 is negative. Then, the manifold on the central manifold is gradually tilted to be stable. According to the center variety theorem, the fixed point $(0,0,0)$ corresponds to the fixed point $(0, \frac{1}{b}, 0)$ in the system (2.53) is also gradually inclined to be stable. We can conclude that the bifurcation occurs at the equilibrium $q(0, \frac{1}{b}, 0)$ in the system (2.53) if $c = 1$ under the condition of the third case.

4. **Case 02 :** $\mathbf{c - b - abc < 0}$, $\mathbf{c + a -} \frac{1}{b} \mathbf{= 0}$, then $c^2 < 1$ and

$$\Delta = (c + a - \frac{1}{b})^2 - 4(1 + ac - \frac{c}{b}) < 0.$$

So,

$$\begin{cases} \lambda_1 = -b < 0, \\ \lambda_2 = -\sqrt{1 + ac - \dfrac{c}{b}}i, \\ \lambda_3 = +\sqrt{1 + ac - \dfrac{c}{b}}i. \end{cases}$$

The eigenvectors λ_2 and λ_3 are pure imaginary. Let us take a look at the following.
$\alpha = -(c+a-\frac{1}{b})$, and $\frac{\partial \alpha}{\partial a}\Big|_{a=\alpha_0} = -1 \neq 0$, then the intersection of Hopf bifurcation conditions are satisfied. This shows that when a passes a_0, the orbit line of the system exceeds the imaginary axis. So at the point $q(0,\frac{1}{b},0)$ we have a Hopf bifurcation under the conditions of the fourth case.

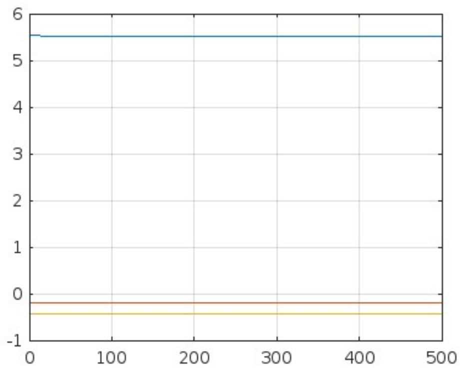


Figure 2.6: Lyapunov exponent.

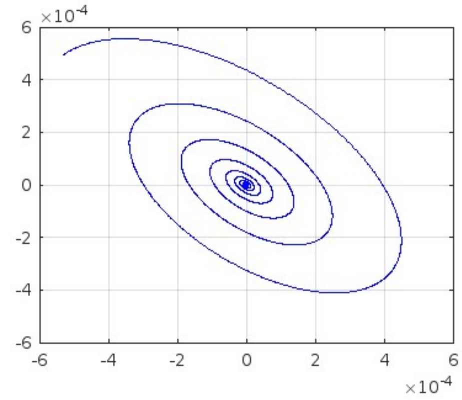


Figure 2.7: Chaotic attractor for $a = 4.5, b = 0.2$ and $c = 0.6$.

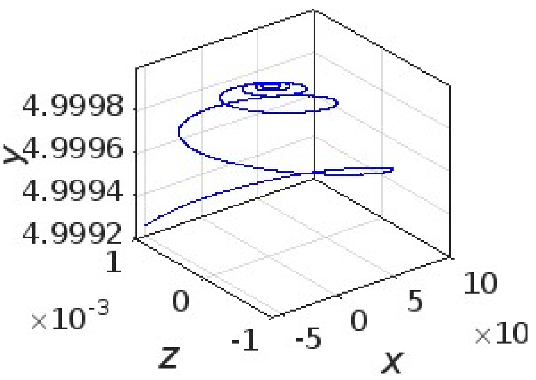


Figure 2.8: Chaotic attractor.

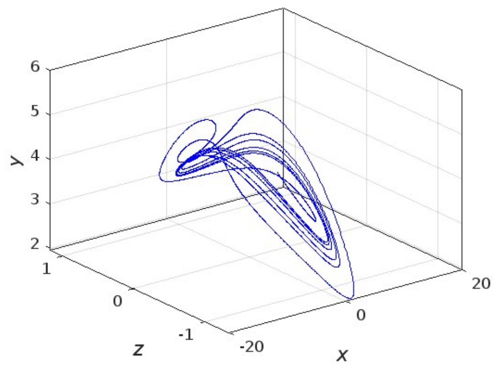


Figure 2.9: Chaotic attractor for $a = 3, b = 0.1$ and $c = 1$.

2.4.3 Stabilization

In this section, we use feedback control to stabilize the system and give some numerical results.

$$\begin{cases} \dot{x} = z + (y-a)x, \\ \dot{y} = 1 - by - x^2, \\ \dot{z} = -x - cz. \end{cases} \tag{2.67}$$

If $c > \frac{1}{2}$, then $c + a - \frac{1}{b}$ is always positive and the fixed point q is stable.
We suppose that $a = 4.5, b = 0.2, c < \frac{1}{2}$ then the system is unstable at the point $(0,5,0)$.

Stabilization via nonlinear control

The control system is

$$\begin{cases} \dot{x_1} = z_1 + (y_1 - a)x_1 + u_1, \\ \dot{y_1} = 1 - by_1 - x_1^2 + u_2, \\ \dot{z_1} = -x_1 - cz_1, \end{cases} \tag{2.68}$$

where the controller is $U = (u_1, u_2)$.

Proposition 2.21 The controller is given as follows :

$$\begin{cases} u_1(t) = x_1 y_1 - xy, \\ u_2(t) = (x - x_1)(x + x_1). \end{cases} \tag{2.69}$$

Proof. We have

$$e_1 = x_1 - x, e_2 = y_2 - y, e_3 = z_3 - z,$$

then the error dynamic system between the slave system and the master system is as follow

$$\begin{cases} \dot{e}_1 = e_3 + x_1 y_1 - xy - ae_1 + u_1, \\ \dot{e}_2 = -be_2 + (x - x_1)(x + x_1) + u_2, \\ \dot{e}_3 = -e_1 - ce_3. \end{cases} \tag{2.70}$$

The controllers u_1 and u_2, are given by :

$$\begin{cases} u_1(t) = x_1 y_1 - xy, \\ u_2(t) = (x - x_1)(x + x_1). \end{cases} \tag{2.71}$$

We obtain

$$\begin{cases} \dot{e}_1 = e_3 - ae_1, \\ \dot{e}_2 = -be_2, \\ \dot{e}_3 = -e_1 - ce_3. \end{cases} \tag{2.72}$$

■

Stabilization by linear control

Theorem 2.6 The controlled system (2.72) is stable.

Proof. Suppose the Lyapunov function is as follows

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2).$$

The differential Lyapunov is

$$\begin{aligned} \frac{dV}{dt} &= \dot{e}_1.e_1 + \dot{e}_2.e_2 + \dot{e}_3.e_3 \\ &= (e_3 - ae_1)e_1 + (-be_2)e_2 - (e_1 + ce_3)e_3 \\ &= e_1 e_3 - ae_1^2 - be_2^2 - e_1 e_3 - ce_3^2 \\ &= -(ae_1^2 + be_2^2 + ce_3^2) < 0. \end{aligned}$$

Then, the system is stable. ■

According to Lyapunov, if the exponents are negatives, then the system is stable.

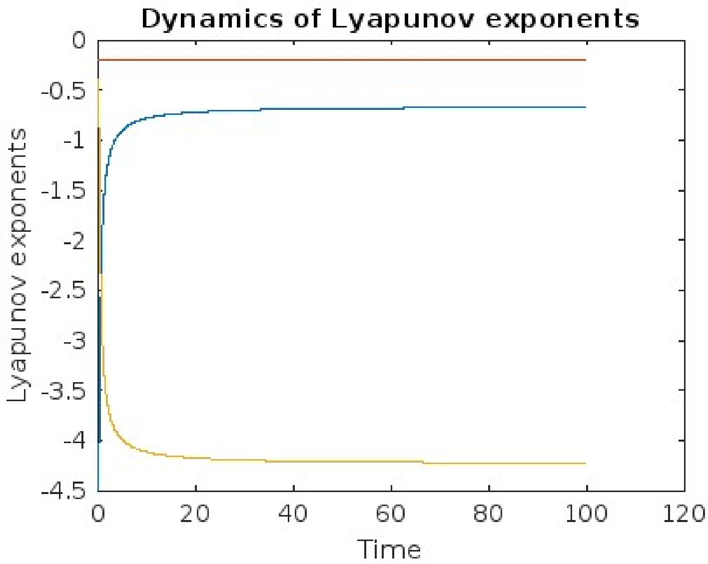


Figure 2.10: Lyapunov exponent.

2.5 Controlling of a novel Lorenz system

Hyperchaotic systems with more than one positive Lyapunov exponent represent an advanced class of nonlinear dynamical systems that exhibit highly complex and unpredictable behaviour. This dual instability makes them more intricate than standard chaotic systems, with applications spanning secure communication, cryptography, signal processing, neural networks, and laser physics. However, their complex algebraic structure often challenges the analysis of their boundedness, synchronization, and control.

The study of hyperchaos was first initiated with the Rossler system, followed by other notable examples such as the hyperchaotic Lorenz-Haken system, Chua's circuit and Matsumoto circuit. These systems have become benchmarks in the nonlinear dynamics community, providing insight into attractor topology, instability, and synchronization phenomena. Understanding the boundedness of hyperchaotic systems is particularly critical, as it ensures that the system trajectories remain finite, enabling robust control and synchronization schemes. Moreover, boundedness plays a vital role in practical applications, such as estimating fractal dimensions of attractors and identifying hidden attractors, which could lead to unexpected and undesirable system behaviour.

Synchronization of chaotic and hyperchaotic systems has emerged as a key area of research due to its wide applicability in secure communication, where chaotic signals are used to encrypt and transmit information. The achievement of synchronization between the master system (driver) and the slave system (response) requires the design of control strategies that ensure asymptotic convergence of the slave system states with those of the master system. Techniques like linear feedback, adaptive, and sliding mode control have been developed to address this synchronization problem.

This part introduces a novel 4D hyperchaotic Lorenz system characterized by additional nonlinear terms and high-dimensional behaviour. The system is analyzed for boundedness using Lyapunov function theory and optimization methods, providing explicit conditions under which the trajectories remain confined within a specific range. Furthermore, a linear feedback controller is designed to synchronize the master and slave systems, ensuring asymptotic global stability. Numerical simulations validate the theoretical results and demonstrate the effectiveness of the proposed control scheme.

The findings of this study contribute to the growing body of research on hyperchaotic systems, providing new insights into their dynamics, boundedness, and synchronization. These results have implications for practical applications in secure communication, nonlinear control, and studying high-dimensional chaotic systems.

2.5.1 Dynamics of the system

The novel 4D hyperchaotic Lorenz system is defined by :

$$\begin{cases} \dot{x} = a(y-x)+w, \\ \dot{y} = cx - xz - y, \\ \dot{z} = xy - bz, \\ \dot{w} = rw - yz, \end{cases}$$

where x, y, z, w are state variables, and a, b, c, r are system parameters.

- **Hyperchaos** : Identified by two positive Lyapunov exponents, indicating exponential divergence along two dimensions.
- **System parameters** : For $a = 10$, $b = \frac{8}{3}$, $c = 28$, $r = -1$, the system exhibits hyperchaotic behavior.

2.5.2 Boundedness analysis

To ensure that the system's states remain finite, a Lyapunov function is constructed :

$$V_1(y,z) = y^2 + (z-c)^2. \tag{2.73}$$

Time derivative of V_1 :

$$\dot{V}_1 = 2y\dot{y} + 2(z-c)\dot{z}. \tag{2.74}$$

Substituting the system equations :

$$\dot{V}_1 = -2y^2 - 2bz^2 + 2cbz. \tag{2.75}$$

Boundary analysis : The boundary set is defined as :

$$\Gamma = \left\{ (y,z) \;\middle|\; \frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} = 1 \right\}. \tag{2.76}$$

Maximum value of V_1 : Using the Lagrange multiplier method, the maximum value of V_1, denoted R^2, is :

$$R^2 = \begin{cases} \frac{b^2c^2}{4(b-1)} & \text{if } b \geq 2, \\ c^2 & \text{if } b < 2. \end{cases} \tag{2.77}$$

Bounded region : The bounded set for the system is :

$$\Omega = \left\{ (x,y,z,w) \;\middle|\; x^2 \leq \frac{R^2 + Rc - raR}{r^2a^2},\; y^2 + (z-c)^2 \leq R^2,\; w^2 \leq \frac{R^2 + Rc}{r^2} \right\}. \tag{2.78}$$

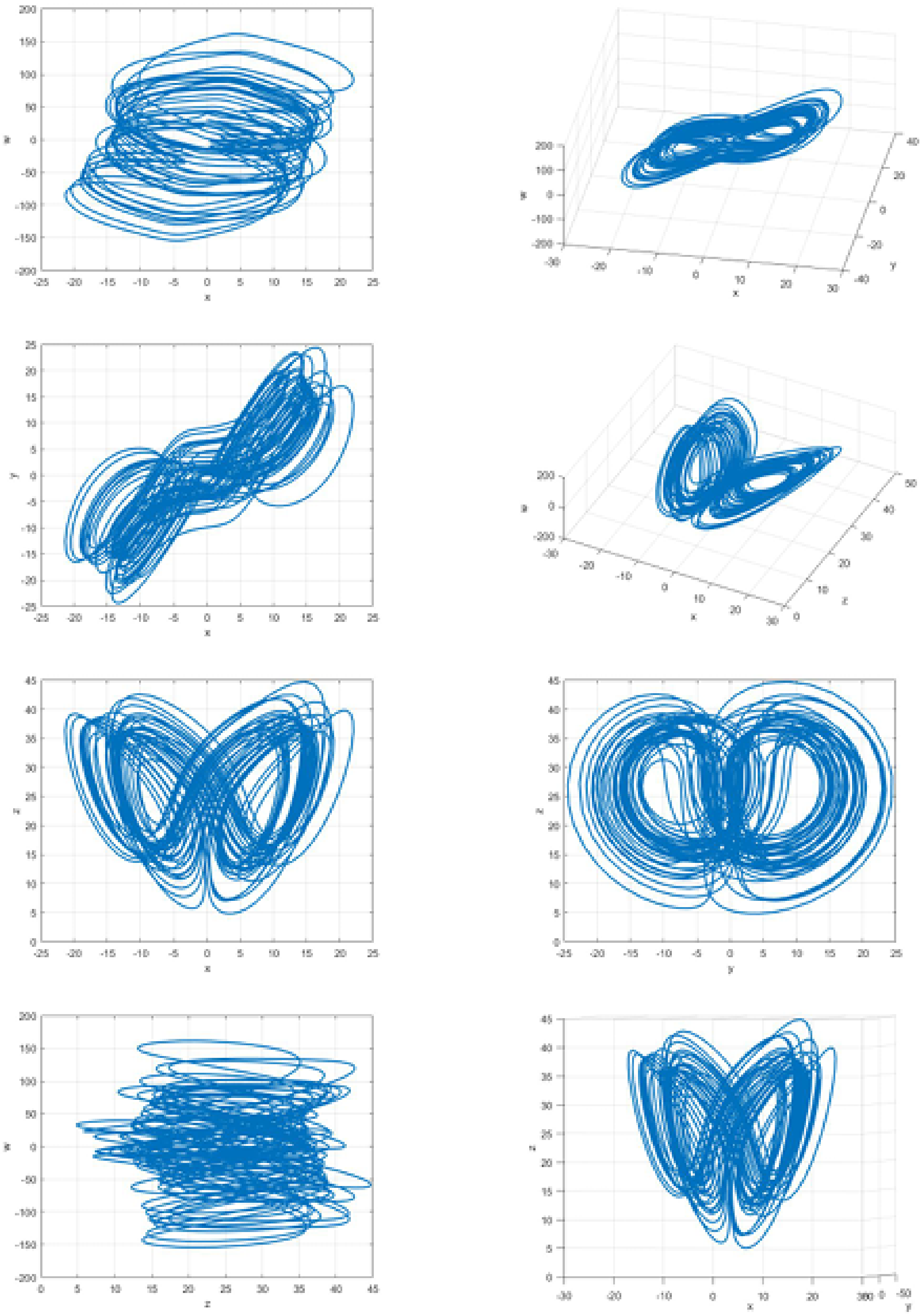


Figure 2.11: Hyperchaotic attractor of the system.

The figure (2.11) shows the boundedness of the system and its hyperchaotic behaviour. The trajectories do not escape to infinity, forming a complex, attractor-like structure indicative of hyperchaotic dynamics with more than one positive Lyapunov exponent.

2.5.3 Controlling of the system

A linear feedback controller is designed to synchronize a slave system with the master system. The slave system is given by :

$$\begin{cases} \dot{x}_1 = a(y_1 - x_1) + w_1 - k_1(w_1 - w), \\ \dot{y}_1 = cx_1 - x_1 z_1 - y_1 - k_2(y_1 - y), \\ \dot{z}_1 = x_1 y_1 - b z_1, \\ \dot{w}_1 = r w_1 - y_1 z_1, \end{cases}$$

where $k_1 > 0$, $k_2 > 0$ are control gains.

Error dynamics : Define the synchronization errors as :

$$e_1 = x_1 - x, \quad e_2 = y_1 - y, \quad e_3 = z_1 - z, \quad e_4 = w_1 - w.$$

The error dynamics are :

$$\begin{cases} \dot{e}_1 = a(e_2 - e_1), \\ \dot{e}_2 = (c - z)e_1 - x e_3 - e_1 e_3 - (k_2 + 1)e_2, \\ \dot{e}_3 = y e_1 + x e_2 + e_1 e_2 - b e_3, \\ \dot{e}_4 = r e_4 - y e_3 - z e_2 - e_2 e_3. \end{cases}$$

Lyapunov function for synchronization : A Lyapunov function is chosen as :

$$V(e_1, e_2, e_3) = \sigma e_1^2 + e_2^2 + e_3^2, \quad \sigma > \frac{R^2}{4ab}.$$

Its time derivative along the error dynamics is :

$$\dot{V} = -\sigma a e_1^2 - (k_2 + 1)e_2^2 - b e_3^2 + (\sigma a + R + 2c)|e_1||e_2| + R|e_1||e_3|.$$

For $\dot{V} < 0$, the control gains must satisfy :

$$k_2 > \frac{b(\sigma a + R + 2c)^2}{4ab\sigma - R^2} - 1.$$

2.5.4 Numerical simulations

Using MATLAB, simulations are conducted with :

- Initial conditions for the master : $(-1,1,-2,-3)$,
- Initial conditions for the slave : $(-7,2,3,2)$,
- Parameters : $a=10$, $b=\frac{8}{3}$, $c=28$, $r=-1$, $k_2=4249$.

The results show successful synchronization as the errors e_1,e_2,e_3,e_4 asymptotically approach zero.

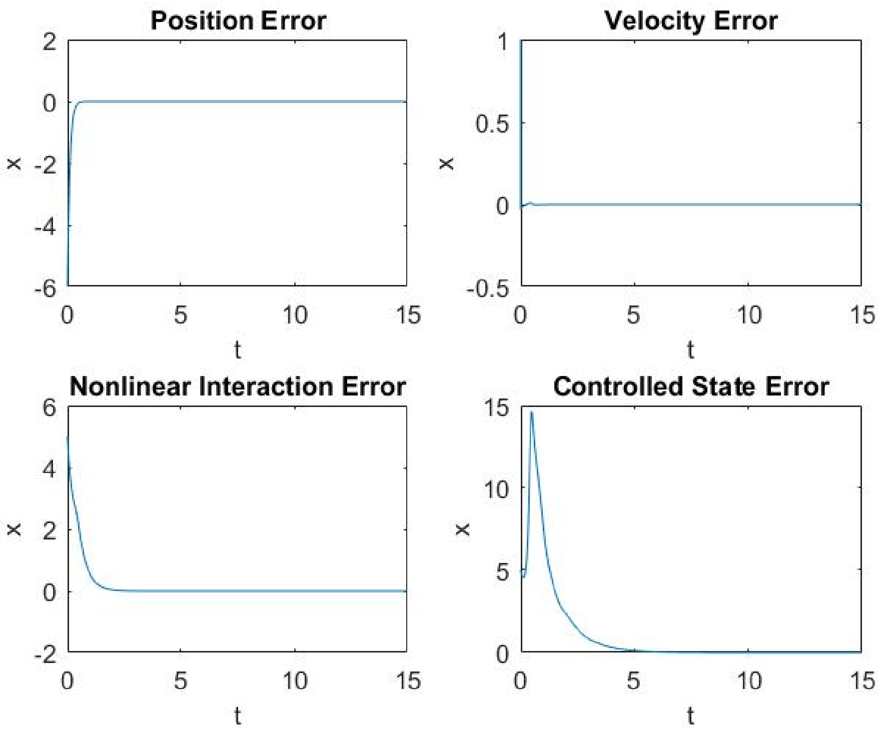
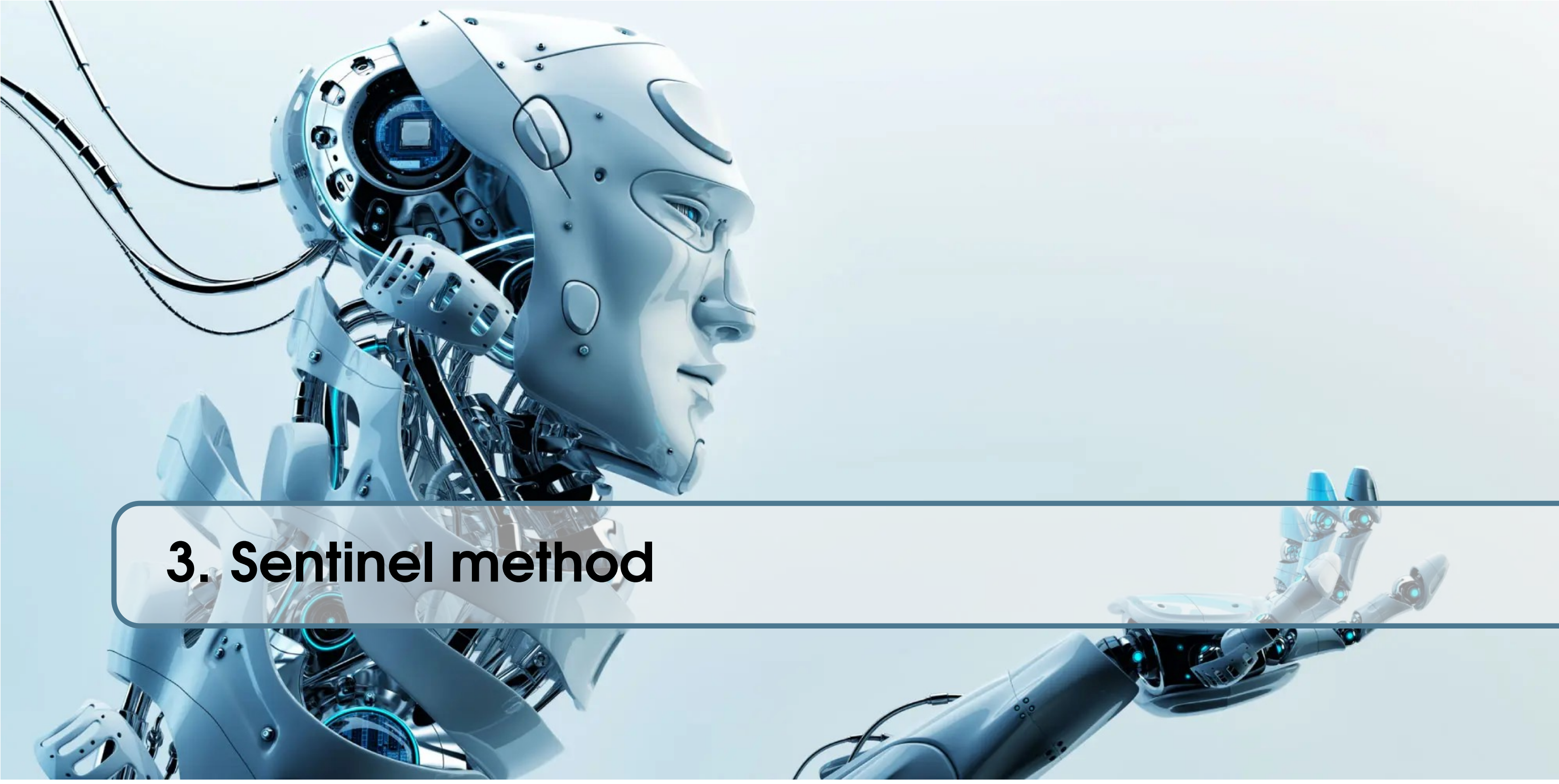


Figure 2.12: Master-Slave Synchronization in Nonlinear Chaotic Systems.

The figure (2.12) shows :

- The errors start with non-zero initial values and decay to zero as time progresses, verifying successful synchronization.
- The plots highlight the effectiveness of the linear feedback control strategy in driving the slave system to match the master system states.



3. Sentinel method

The sentinel method is a mathematical technique applied to detect anomalies in distributed systems, such as identifying pollution sources in fluid environments or monitoring temperature irregularities in heat transfer systems. Observing the variable y, representing its state, is essential to extract meaningful information about the system.

To gain a deeper understanding of the sentinel method and review the detailed proofs of the theorems and corollaries outlined in this chapter, it is recommended for readers to refer to [**8**] [**7**], [**11**], [**18**], [**41**], [**30**].

3.1 Pollution detection problem

This section models the detection of pollution caused by chemical discharges in fluid environments (e.g., rivers and lakes). The key goal is to identify and understand the transport, reaction, and diffusion mechanisms of pollutants in these environments to determine the unknown source and initial conditions.

The pollutant concentration $x(z,t)$ evolves according to a parabolic convection-diffusion-reaction equation, which integrates :

- **Diffusion term**

$$k div \cdot \big(a(z)\nabla x\big) \tag{3.1}$$

where k is a diffusion constant and $a(z)$ is the transitivity in the medium. This term models a random molecular dispersion in a medium.

  - In homogeneous medium : For example, in still water, the diffusion operator simplifies to:

$$k\Delta x(z,t). \tag{3.2}$$

  - In anisotropic environments : For example, in rivers, the term accounts for directional differences :

$$D_1\frac{\partial^2 x}{\partial z^2}+D_2\frac{\partial^2 x}{\partial t^2}, \tag{3.3}$$

where D_1 and D_2 are constants reflecting medium properties.

- **Convection term**

$$u\cdot\nabla x \tag{3.4}$$

This term describes pollutant transport due to fluid velocity u.
For example, in a flowing river, this term dominates, moving the pollutant downstream.

- **Reaction term**

$$R_i=-k_ix_i(z,t) \tag{3.5}$$

This term reflects chemical or biochemical interactions, such as the degradation of pollutants. Reaction rates are quantified by k_i, which depend on temperature, sunlight, or nutrient levels.

- **Source terms**
  - Distributed source : Represents pollutants continuously dispersed across a region :

$$\zeta(z,t). \tag{3.6}$$

  - Point source : Models localized pollution, such as a factory outlet :

$$\lambda_i \hat{\zeta}_i(t) \delta(z - z_i), \tag{3.7}$$

where δ is the Dirac function.

The general formulation is given by

$$f_i(z,t) = \sum_j \lambda_j \hat{\zeta}_i(t).\delta(z - P_i).$$

For the boundary and initial conditions, we have the following.

- **Boundary conditions**
  - Dirichlet (Γ_1) : Specifies the pollutant concentration at the boundaries (e.g., factory discharge channel)

$$x\big|_{\Gamma_1} = g(z,t). \tag{3.8}$$

  - Neumann (Γ_2) : Describes the flux exchange between the fluid and the surrounding medium (e.g., impermeable soil)

$$\frac{\partial x}{\partial n}\Big|_{\Gamma_2} = h(z,t). \tag{3.9}$$

The boundary Γ_1 is assumed to continuously release pollutants into the fluid, as in the case where a factory discharges waste into a nearby lake through an open channel Γ_1. This boundary can also be divided into segments for various external sources of pollution.

The Neumann boundary condition governs the flux of pollutant concentration exchanged between the fluid and the surrounding environment, a phenomenon influenced by the porosity of the soil. In the context of surface water studies (e.g., lakes or rivers), it can be approximated that for $h = 0$, no pollutant concentration crosses the boundary Γ_2, which implies that the soil at this boundary is impermeable.

- **Initial condition**

$$x(z,0) = x_0(z) \tag{3.10}$$

Describes the initial pollutant concentration at the start of the observation period. This is critical for solving time-dependent (evolutionary) equations.

3.2 Position of the problem

To illustrate our approach, we assume that the state of the system is described by x. The general structure of the partial differential equation which governs the state x of the problem studied is assumed to be known in the form:

$$\begin{cases} \dfrac{\partial x}{\partial t} + F(x) = \text{source term} \quad \Omega \times]0,T[, \\ \qquad x(t=0) = x_0 \quad \Omega, \end{cases} \tag{3.11}$$

where F is a nonlinear function and x_0 is the initial state.

- the coefficients of the operator F, and the possible non-linearity structure,
- Sources term,

- Initial condition,
- Boundary conditions, and
- the field of study Ω.

This is generally not the case.

If at least one of the above information is unknown or partially known, we say that the system (3.11) has incomplete data. We encounter this type of problem in many situations, in biomedical sciences, meteorology, oceanography, etc., where the initial conditions are not completely known.

3.2.1 Missing terms and pollution terms

Let F be a second-order elliptic operator. We assume that the first equation of the system is written in the form :

$$\frac{\partial x}{\partial t} + F(x) = \zeta + \lambda \hat{\zeta} \quad \Omega \times]0,T[,$$

with ζ given in a suitable space X and $\lambda \hat{\zeta}$ remains in the unit ball of X and λ is a small real parameter with $\lambda \hat{\zeta}$ is not known. We assume that the F and the open coefficients are known, but the initial data are incomplete. If we designate by $x(0)$, the initial condition is expressed in the form $x(0) = x_0 + \tau \hat{x}_0$ or x_0 where $\hat{x}_0$ is given and $\hat{x}_0$ remains in the unit ball of a Hilbert or Banach space with small real, and we assume that the boundary conditions are known.

Our objective is to provide a method to obtain information on $\lambda \hat{\zeta}$ that is not affected by variations in the initial data around x_0. We thus establish a distinction between the term $\lambda \hat{\zeta}$ which is called "pollution", and the term $\tau \hat{x}_0$, which is said to be "missing" which we do not seek to identify. To hope to obtain some information, we must observe x. So, the problem consists of observing the state x in an accessible part of the domain and having the experimental measurements to estimate the missing data.

3.2.2 Observation system

In systems with partially known data, reconstructing unknown data requires system observations. Observations can take two main forms :

1. **Discrete Observations :** Observations are made at specific points O_i (e.g., observation locations), while pollution sources are at points S_i.
2. **Continuous Observations :** Observations are made continuously over time and space, such as through a moving observatory (e.g., a ship monitoring an ocean or lake). Observations can also be discontinuous in time.

The state of the system, denoted as x, is observed in O during a time interval $[0,T]$, represented as:

$$x(z,t) = x_{obs} \quad \text{on } O \times (0,T), \tag{3.12}$$

where x_{obs} is the measured data. However, experimental measurements can be affected by noise arising from instrument errors or approximations in the equations. To account for this, the observation operator is defined as:

$$x_{obs} = m_0 + \sum_{i=1}^{n} \beta_i m_i, \tag{3.13}$$

where $m_0, m_1, \ldots, m_n$ are known functions and β_i are unknown noise parameters.

The concept of *identifiability* is introduced to address the problem of reconstructing unknown parameters. A system is identifiable if the operator mapping the data space E to the measurement space F via the observation $y = Cx$ is injective, ensuring unique parameter determination. Formally, we define:

Definition 3.1 We consider a system whose state, denoted x, depends on a vector of parameters v. Let C be an observation operator that acts on x. We define the operator $B : E \to F$, where E is the data space and F is the measurement space. We say that B is identifiable from observation $y = Cx$ if the application B is injective.

A very popular technique for solving an identification problem is the “least squares” method. Furthermore, at the end of the 1980s, the sentinel method was new.

3.3 Sentinel method

This theory addressed, on the one hand, the challenges outlined and, on the other hand, the need for a fast algorithm to compute unknown parameters. Over a span of four years, the author extended the theory to environmental applications, disseminating findings through research articles and conferences and ultimately consolidating the work in the 1992 book "Sentinels for Distributed Systems with Incomplete Data".

From a numerical perspective, it is widely recognized that the sentinel method is nearly equivalent to the classical least squares approach.

In 2004, O. Nakoulima aimed to resolve the issues by leveraging Carleman's inequality to establish the existence of sentinels.

3.3.1 Continuous sentinel

We consider an open Ω of $\mathbb{R}^n$, ($n = 1,2,3$ in applications), bounded by a fairly regular boundary $\partial\Omega = \Gamma$ (of class C^2 so as not to encounter regularity problems).

Let A be a second-order elliptic operator. For fixed $T > 0$, we define $Q = \Omega \times [0,T]$ and $\Sigma = \Gamma\times]0,T[$; we consider the solution $x(z,t)$ of the system :

$$\begin{cases} \dfrac{\partial x}{\partial t} + Ax + f(x) = \zeta + \lambda\hat{\zeta} \quad \text{in } Q \\ x(0) = x_0 + \tau\hat{x}_0 \quad \text{in } \Omega \\ x = 0 \quad \text{on } \Sigma \end{cases}$$

This system has incomplete data where

- The functions ζ and x_0 are given in $L^2(\Omega)$, respectively.
- The pollution term $\lambda\hat{\zeta}$ and the missing term $\tau\hat{x}_0$ are unknown in $L^2(\Omega)$, respectively.
- The real numbers λ and τ are arbitrarily small.
- The operator A is a second-order elliptic differential operator:

$$Ay = -\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y}{\partial x_j}\right), \tag{3.14}$$

- The operator $x \longrightarrow f(x)$ is a nonlinear function of class C^1 ; (we can assume that f is a function of x and ∇x).

The equation admits a unique weak solution in $L^2(Q)$ which we note

$$x(z,t,\lambda,\tau) = x(\lambda,\tau)$$

The question that arises is:

$$\begin{cases} \text{How can we calculate the pollution term } \lambda\hat{\zeta} \text{ (or obtain information)} \\ \text{which is independent of the variations of the initial data author of } x_0\text{?} \end{cases}$$

In this part, we consider the observatory $O \subset \Omega$ and

$$x_{obs} = m_0(z,t) \text{ on } O \times (0,T). \tag{3.15}$$

Let $h_0 \in L^2(O \times (0,T))$ such that

$$\int\int_{O\times(0,T)} h_0(z,t)dzdt = 1, \tag{3.16}$$

and

$$\iint_{O\times(0,T)} h_0\, y(x,t,\lambda,\tau)\, dx\, dt$$

is an average. Then, we consider the functional

$$S(\lambda,\tau) = \int\int_{O\times(0,T)} (h_0 + w)x(\lambda,\tau)dzdt, \tag{3.17}$$

where we determine the function w in the following manner.

$$\frac{\partial S}{\partial \tau}(0,0) = 0, \tag{3.18}$$

and

$$\|w\|_{L^2(O\times(0,T))} = \min. \tag{3.19}$$

This is the definition of the sentinel.

Remark The condition (3.18) represents the insensitivity of the sentinel function to the missing term $\tau\hat{x}_0$, allowing the pollution term to be determined independently of the unobserved data. Meanwhile, condition (3.19) ensures that the sentinel function remains as close as possible to an average value.

Information provided by sentinels

The functional of the sentinel gives information about $\lambda\hat{\varepsilon}$ i.e., we can write this equation when w is well defined

$$S(0,0) = \int\int_{O\times(0,T)} (h_0 + w)x_0 dzdt, \tag{3.20}$$

where x_0 is the solution of the problem

$$\begin{cases} \dfrac{\partial x_0}{\partial t} + Ax_0 + f(x_0) = \varepsilon \text{ in } Q\times(0,T), \\ x_0(0) = x_0 \text{ in } \Omega, \\ x_0 = 0 \text{ on } \Sigma. \end{cases} \tag{3.21}$$

Using the Taylor development of $S(\lambda,\tau)$

$$\begin{aligned} S(\lambda,\tau) &= S(0,0) + \lambda\frac{\partial S}{\partial \lambda}(0,0) + \tau\frac{\partial S}{\partial \tau}(0,0) + o(\|(\lambda,\tau)\|) \\ &= S(0,0) + \lambda\frac{\partial S}{\partial \lambda}(0,0) + o(\|(\lambda,\tau)\|). \end{aligned}$$

So,

$$\begin{aligned} S_{obs}(\lambda,\tau) &= \int\int (h_0 + w)x_{obs} dzdt \\ &= \int\int (h_0 + w)m_0 dzdt. \end{aligned}$$

Then, we get

$$\lambda\frac{\partial S}{\partial \lambda}(0,0) = (S_{obs} - S(0,0)). \tag{3.22}$$

Finding the sentinel involves determining a control u that meets specific conditions, such as (3.18) and (3.19). J.-L. Lions developed the Hilbert Uniqueness Method (HUM) to solve this problem. HUM ensures that the adjoint system has a unique solution that can be used to construct the control function u. This function satisfies the desired controllability conditions, driving the system to zero.

Variational method

Firstly, we suppose that $\lim_{\tau\to 0}\dfrac{x(0,\tau) - x(0,0)}{\tau} = x_\tau(0,0)$ i.e., $\lambda = 0$ and $\tau = 0$. We have $x(0,\tau)$ the solution of

$$\begin{cases} \dfrac{\partial x(0,\tau)}{\partial t} + Ax(0,\tau) + f(x(0,\tau)) = \varepsilon \text{ in } Q\times(0,T), \\ x(0,\tau)(0) = x_0 + \tau\hat{x}_0 \text{ in } \Omega, \\ x(0,\tau) = 0 \text{ on } \Sigma. \end{cases} \tag{3.23}$$

where x_0 is the solution of (3.21). By subtracting (3.21) and (3.23) and multiplying by $\frac{1}{\tau}$, and a passage to the limit when $\tau \to 0$ verifies that x_τ is a solution of

$$\begin{cases} \dfrac{\partial x_\tau}{\partial t} + Ax_\tau + f'(x_0)x_\tau = 0 \text{ in } Q \times (0,T), \\ x_\tau(0) = x_0 \text{ in } \Omega, \\ x_\tau = 0 \text{ on } \Sigma. \end{cases} \tag{3.24}$$

Then, condition (3.18) is equivalent to

$$\int\int_{O\times(0,T)} (h_0 + w)x_\tau dzdt = 0 \tag{3.25}$$

Equivalent to a control problem

Secondly, consider the adjoint problem

$$\begin{cases} \dfrac{\partial q}{\partial t} + A^*q + f'(x_0)q = (h_0 + w)\chi_O \text{ in } Q \times (0,T), \\ q(T) = x_0 \text{ in } \Omega, \\ q = 0 \text{ on } \Sigma. \end{cases} \tag{3.26}$$

We multiply the first equation of (3.26) by x_τ and integrate over Q to obtain

$$\int_Q \left(\frac{\partial q}{\partial t} + A^*q + f'(x_0)q \right) x_\tau dzdt = \int_Q (h_0 + w)\chi_O x_\tau dzdt,$$

with integration by parts and the two conditions of (3.24) and (3.26)

$$\int_O (h_0 + w)x_\tau dzdt = \int_\Omega q(0)\hat{x}_0 dz. \tag{3.27}$$

Then, condition (3.18) becomes

$$q(0) = 0. \tag{3.28}$$

Our problem is a problem of null controllability because we have (3.19) and (3.28), and we research $w \in L^2(O)$.

Third, we want to prove the existence of an optimal control. So we can separate the adjoint problem into two components.

$$q = q_0 + y,$$

where q_0 is the solution of the problem

$$\begin{cases} \dfrac{\partial q_0}{\partial t} + A^*q_0 + f'(y_0)q_0 = h_0\chi_O \text{ in } Q, \\ q_0(T) = 0 \text{ in } \Omega, \\ q_0 = 0 \text{ on } \Sigma. \end{cases} \tag{3.29}$$

and y is the solution of the problem

$$\begin{cases} \dfrac{\partial y}{\partial t} + A^*y + f'(x_0)y = w\chi_O \text{ in } Q, \\ y(T) = 0 \text{ in } \Omega, \\ y = 0 \text{ on } \Sigma. \end{cases} \tag{3.30}$$

We need to search w such that

$$y(0,w) = -q_0(0).$$

We have an optimization problem

$$(P)\left\{\min_{w\in U}\|w\|_{L^2(O)}\right\}, \tag{3.31}$$

with

$$U=\left\{w,\left\{\begin{array}{c}-\dfrac{\partial y}{\partial t}+A^*y+f'(x_0)y=w\chi_O \text{ in } Q,\\ y(T)=0 \text{ in } \Omega,\\ y=0 \text{ on } \Sigma,\\ y(0)=-q_0(0),\end{array}\right.\right\} \tag{3.32}$$

In this step, we need to use the penalty method of the optimization theory technique. This method transforms a constrained problem into an unconstrained problem by modifying the objective function. It does so by adding a penalty term that increases when constraints are violated, discouraging solutions that do not satisfy them.

Penalization

Proposition 3.1 The problem (3.31) admits a unique solution.

We introduce the functional for $\varepsilon>0$

$$J_\varepsilon(w,y)=\frac{1}{2}\|w\|^2_{L^2(O)}+\frac{1}{2\varepsilon}\left\|-\frac{\partial y}{\partial t}+A^*y+f'(x_0)y-w\chi y\right\|^2_{L^2(\Omega)}. \tag{3.33}$$

Let the problem

$$(P_\varepsilon)\left\{\min_{w\in U}\|w\|_{L^2(O)}\right\}, \tag{3.34}$$

with

$$U_\varepsilon=\left\{(w,y),\left\{\begin{array}{c}-\dfrac{\partial y}{\partial t}+A^*y+f'(x_0)y-w\chi_O \text{ in } L^2(Q),\\ y(T)=0 \text{ in } \Omega,\\ y=0 \text{ on } \Sigma,\\ y(0)=-q_0(0),\end{array}\right.\right\} \tag{3.35}$$

Proposition 3.2 For all ε, Problem (3.34) admits a solution $(w_\varepsilon,y_\varepsilon)$ which converges weakly to $(\hat{w},\hat{y_\varepsilon})$ when $\varepsilon\to 0$, where $\hat{w}$ is the solution of problem (3.31) and $\hat{q}_1$ the solution of problem (3.30) associated with w.

Proposition 3.3 $(w_\varepsilon,y_\varepsilon)$ is a unique solution to the problem (3.34), if and only if there exists a function $\rho_\varepsilon\in L^2(Q)$ such that $(w_\varepsilon,y_\varepsilon,\rho_\varepsilon)$ is a solution to the following optimality system

$$\left\{\begin{array}{c}-\dfrac{\partial y_\varepsilon}{\partial t}+A^*y_\varepsilon+f'(z_0)y_\varepsilon=w\chi_O-\varepsilon\rho_\varepsilon \text{ in } L^2(Q),\\ y_\varepsilon(T)=0 \text{ in } \Omega,\\ y_\varepsilon=0 \text{ on } \Sigma,\\ y_\varepsilon(0)=-q_0(0).\end{array}\right. \tag{3.36}$$

$$\left\{\begin{array}{c}\dfrac{\partial \rho_\varepsilon}{\partial t}+A^*\rho_\varepsilon+f'(x_0)\rho_\varepsilon=0\\ \rho_\varepsilon=0 \text{ on } \Sigma\end{array}\right. \tag{3.37}$$

without any information about on ρ_ε at $t=0$ or $t=T$, such that

$$w_\varepsilon\chi_O=-\rho_\varepsilon. \tag{3.38}$$

We can proof that $\rho_\varepsilon \to \rho$ when $\varepsilon \to 0$ such that ρ is a solution to the optimality system of the problem (3.31)

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + A^*\rho + f'(x_0)\rho = 0 \text{ in } Q, \\ \rho = 0 \text{ on } \Sigma, \\ \rho(0) = \rho_0 \text{ in } \Omega. \end{cases} \tag{3.39}$$

From (3.38) and proposition (3.1) we deduce that

$$\hat{w} = \rho \chi_O. \tag{3.40}$$

The problem 3.30 becomes

$$\begin{cases} \dfrac{\partial y}{\partial t} + A^* y + f'(x_0) y = \rho \chi_O \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \\ y(T) = 0 \text{ in } \Omega. \end{cases} \tag{3.41}$$

And the problem of the existence of $\hat{w}$ becomes

$$\begin{cases} \text{search } \rho_0 \text{ verified} \\ y(0,\rho) = -q_0(0), \\ \hat{w} = \rho \chi_O. \end{cases} \tag{3.42}$$

We define the operator

$$\Lambda \rho_0 = y(0,\rho). \tag{3.43}$$

This operator is well defined. We must, therefore, resolve this in a suitable functional space

$$\Lambda \rho_0 = -q_0(0), \tag{3.44}$$

we multiply (3.41) by ρ, where $\tilde{\rho}$ is the solution of (3.39) corresponding to $\tilde{\rho}_0$ and by integrating by parts we obtain

$$\langle \Lambda \rho_0, \tilde{\rho}_0 \rangle_{L^2(\Omega)} = \int\int_{O\times(0,T)} \rho \tilde{\rho}\, dxdt, \tag{3.45}$$

for $\tilde{\rho}_0 = \rho_0$

$$\langle \Lambda \rho_0, \rho_0 \rangle_{L^2(\Omega)} = \int\int_{O\times(0,T)} \rho_0^2\, dxdt. \tag{3.46}$$

Then,

$$\|\rho_0\| = \left(\int\int_{O\times(0,T)} \rho^2\, dxdt \right)^{\frac{1}{2}}. \tag{3.47}$$

The quantity (3.47) is definite as a norm in $L^2(\Omega)$. If$\|\rho_0\| = 0$, then $\rho = 0$ on O, and if the coefficients of A and $f'(x_0)$ are regular, then from Mizohata's uniqueness theorem $\rho = 0$ on Q. Let F be the complete Hilbert subspace in $L^2(\Omega)$ for the norm (3.47).

Theorem 3.1 The operator Λ is an isomorphism of F on F_0 and it is also symmetric ; $\Lambda^* = \Lambda$, then equation (3.44) has a unique solution

$$\rho_0 = -\Lambda^{-1} q_0(0),$$

Then, the existence and the uniqueness of the sentinel were proven by

$$S(\lambda,\tau) = \int\int_{O\times(0,T)} (h_0 + \rho) x(z,t,\lambda,\tau)\, dzdt.$$

Pollution term

Theorem 3.2 Since x_0, ρ and ρ_0 are calculated on O, then, the pollution term $\lambda\hat{\varepsilon}$ is estimated by the relation

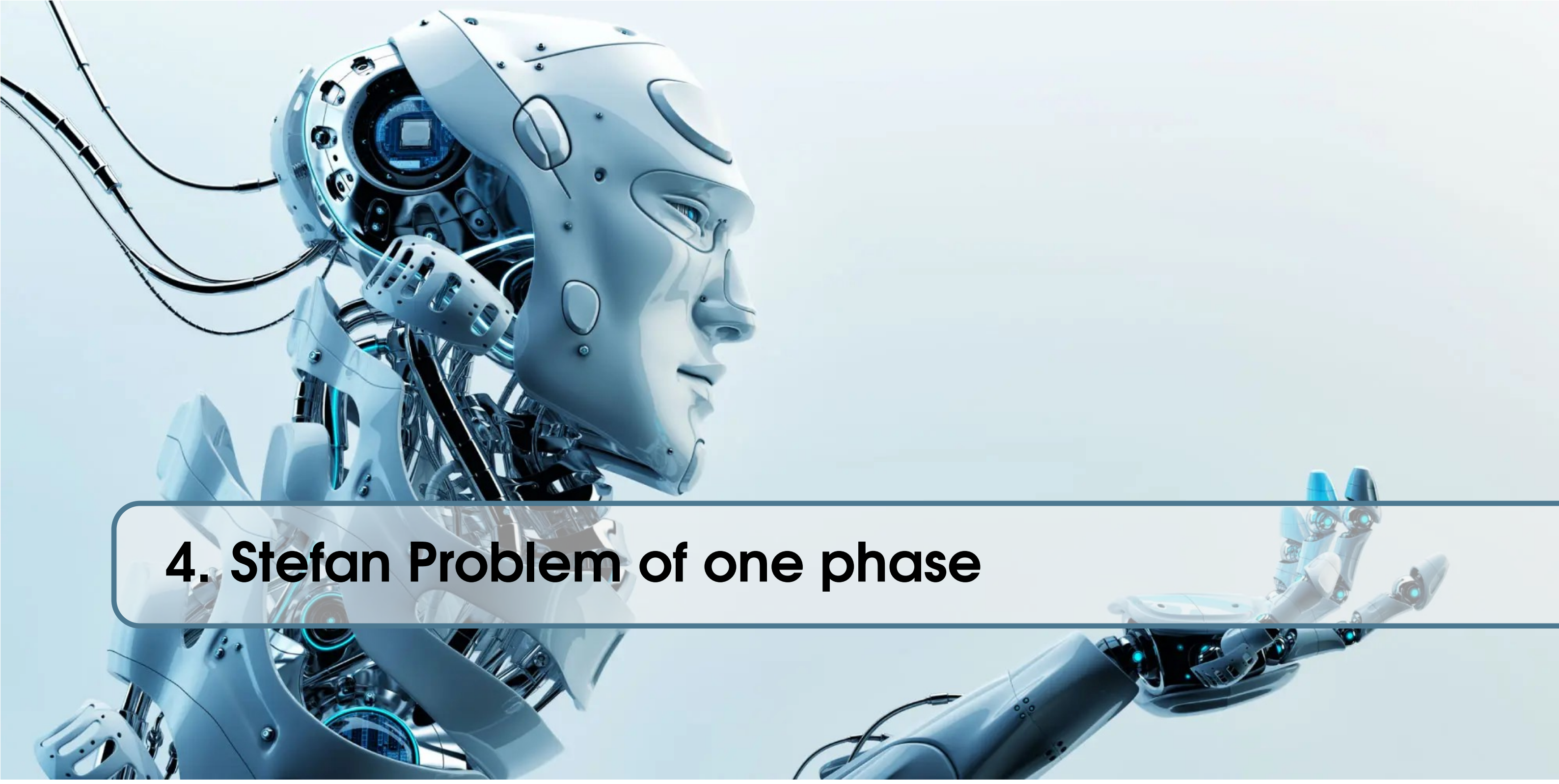
$$\int\int_{O\times(0,T)}(q_0+z)\lambda\hat{\varepsilon}dzdt = \int\int_{O\times(0,T)}(h_0+\rho)(x_{obs}-x_0)dzdt.$$

where x_0, q_0 and y are, respectively, the solutions of (3.21), (3.29) and (3.30) and x_{obs} is the state observed on O during the time interval $[0,T]$.

3.3.2 Types of the sentinel method

There are other types of sentinel methods :

- **Regional sentinel** : According to the name, the word regional, that is, in a region or part of a spatial domain, is applied only to problems where regional controllability is satisfied.
- **Discrete sentinel** : Discrete means specific domain points are observed. This type can be used to determine the pollution in a river.
- **Weak sentinel** : According to their name, we use it when the problem is weakly controllable; this method finds approximate information.
- **Discriminating sentinel** : This type of sentinel is used for noisy observations and makes it possible to differentiate the effects of noise from other parameters of interest. Designed to distinguish between multiple phenomena or states based on defined criteria, it ensures that disturbances do not affect the quality of the information obtained while also being used to identify distinct properties in a system.



4. Stefan Problem of one phase

4.1 Introduction

Stefan's problem is a fundamental mathematical model that describes phase transitions between two states of matter, such as the transformation of ice into water. It is presented as a boundary problem that involves partial differential equations (PDEs), where the position of the boundary between phases changes over time. The classical Stefan problem focuses on the evolution of this boundary, requiring the solution of the heat equations in each phase, with initial and boundary conditions. At the phase change interface, the temperature is kept constant at the transition temperature, and the Stefan condition, which expresses an energy equilibrium, allows the position of this mobile interface to be calculated. The problem was developed by the Slovenian physicist Josef Stefan, who proposed it at the end of the nineteenth century. However, similar issues had already been addressed before, notably by Lamé and Clapeyron in 1831, when they were studying the formation of the Earth's crust.

For additional information and further details regarding the works related to the Stefan problem in mathematics, please refer to cite [**4**], [**11**], [**19**], [**23**], [**29**], [**34**], [**45**], [**48**] and [**50**].

This problem is an example of the deformation of the interface between the liquid and solid phases. The deformation theory facilitates the mathematical representation of domain boundaries as functions of time and space. By introducing a deformation parameter α, we can systematically study how boundary perturbations influence the dynamics of the heat equation. This approach derivates precise conditions for existence, uniqueness, and controllability, as shown in figure 4.1.

Then, we applied the sentinel method in the article of [**11**]. He introduced the nonlinear sentinel, and we applied this method to identify the boundary, i.e., the boundary moves each time. This method aims to investigate and estimate the unknown part.

In the beginning, we gave

$$x_{\alpha_i} = \frac{\partial x(\alpha)}{\partial \alpha_i},$$

where $\alpha_i \in \mathbb{R}^n$, which satisfies this relation

$$x(\alpha) = \sum_{i=1}^{n} \alpha_i x_{\alpha_i}.$$

The sentinel item defined by

$$S(\alpha) = \sum_{i=1}^{n} \alpha_i (u, x_{\alpha_i}) = \alpha_{i_0},$$

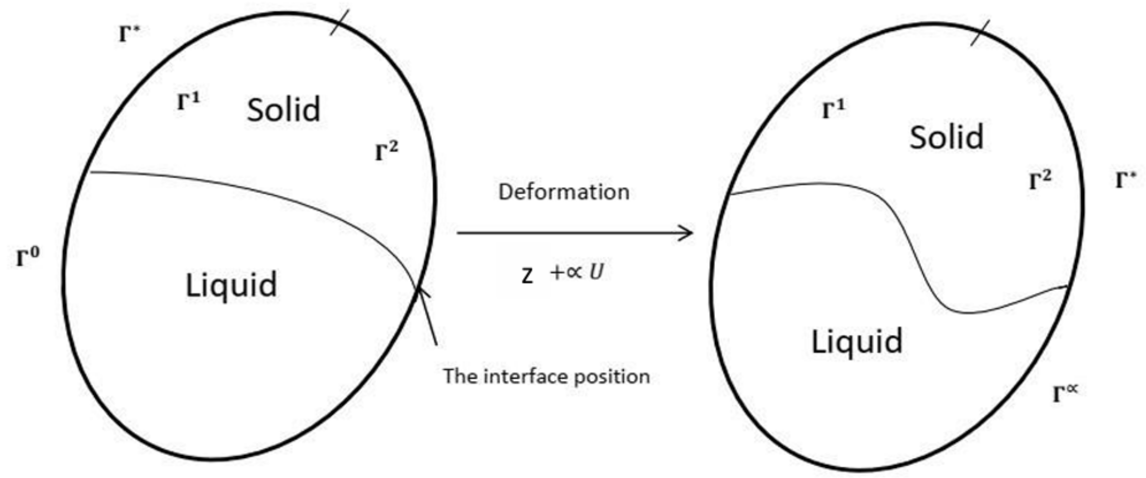


Figure 4.1: Deformation of the interface.

where the parameter i_0 used to estimate the system and

$$\begin{cases} (u, x_{\alpha_i}) = 0, & \forall i \neq i_0, \\ (u, x_{\alpha_{i_0}}) = 1, & \text{Otherwise.} \end{cases}$$

we suppose that u_j and S_j are certain functions, and the sentinel item is associated, respectively, with the parameter α_j. From this point of view, we assume

$$(u_j, x_{\alpha_i}) = \delta_{ij}, \ \forall i, j = \overline{1,n}.$$

Then, we have

$$S(\alpha) = (u_j, x(\alpha))_{j=1,2,\cdots,n}.$$

and

$$D_\alpha S = Id.$$

To compute the function $(u_j)_{j=1,2,\cdots,\infty}$, we use the linearization of the state. We have

$$S(\tilde{\alpha}, \alpha) = (u_j(\tilde{\alpha}), x(\alpha))_{j=1,2,\cdots,\infty},$$

where this is the new sentinel.

4.2 Setting of the problem

We have that $\Omega_0 \subset \mathbb{R}^2$ is an open subset with the smooth boundary $\partial\Omega_0 = \Gamma^* \cup \Gamma_0$, and $\Gamma^* \cap \Gamma_0 = \varnothing$. The deformation of Ω_0 defined by

$$\Omega_\alpha = \{z + \alpha(z)U(z), z \in \Omega_0\}, \tag{4.1}$$

where $\alpha(z)$ is a $\mathbb{C}^2$-function such that Γ^* remains invariant by the deformation αU and U is a known transverse vector field of the class $\mathbb{C}^\infty$.

$\partial\Omega_\alpha = \Gamma^* \cup \Gamma_\alpha$ is the boundary of Ω_α such that $\Gamma^* \cap \Gamma_\alpha = \varnothing$ with

$$\Gamma_\alpha = \{z + \alpha(z)U(z), \ z\Gamma_0\}, \tag{4.2}$$

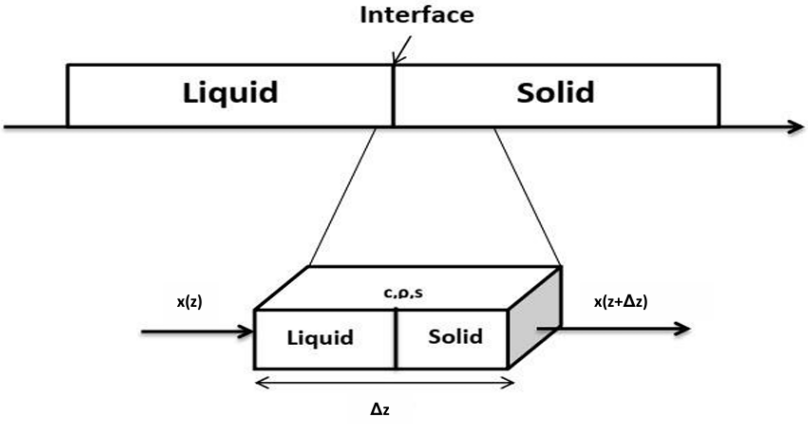


Figure 4.2: One-phase Stefan problem.

Now, let $x = x(z,t;\alpha)$ be the solution of the following problem

$$\begin{cases} \dfrac{\partial x}{\partial t} = C\dfrac{\partial^2 x}{\partial z^2} \text{ in } Q_\alpha \times]0,T[, \\ -k\dfrac{\partial x}{\partial z}(0,T) = u(t) \text{ on } \Sigma_\alpha = \Gamma_\alpha \times]0,T[, \\ x(s(t),t) = x_m \text{ on } \Sigma^* = \Gamma^* \times]0,T[, \\ x(z,0) = x_0(z) \text{ in } \Omega_\alpha, \end{cases} \tag{4.3}$$

where $u(t) \in L^2\left(]0,T[,H^{\frac{3}{2}}(\Gamma)\right)$, k is the thermal conductivity, x_m is the limit of $x(z,t)$ when $t \to \infty$ and $s(t)$ is the solution of the following nonlinear differential equation

$$\dot{s}(t) = -\beta x_z(s(t),t),$$

where $\beta = \frac{k}{\rho \Delta H^*}$ such that ΔH^* is the latent heat of fusion. Hence, we have

$$x \in L^2\left(]0,T[,H^2(\Omega_\alpha)\right) \text{ and } \left.\frac{\partial x}{\partial n}\right|_{\Gamma_\alpha} \in L^2\left(]0,T[,H^{\frac{1}{2}}(\Gamma)\right).$$

we consider $\omega \subset \Omega_\alpha$ an open subset for all α such that $O = \omega \times]0,T[$ and we observe the solution $x(z,t;\tilde{\alpha})$ in O such that

$$x_{obs} = x(z,t;\tilde{\alpha}), \text{ for all } (z,t) \in O.$$

4.3 Application of the method

In this section, we should note that the parametrization of Γ_α is given by

$$\Gamma_\alpha = \{z(s) + \alpha(s)U(s), s \in [0,1], z(s) \in \Gamma_0\}, \tag{4.4}$$

where α is $\mathbb{C}^2$-function over $[0,1]$, i.e. it belongs to $L^2(]0,1[)$. Next, with the decomposition of α over the basis of functions $(b_j)_{j=1,2,\cdots,\infty}$ in $\mathbb{C}^2(0,1)$ and $\alpha \in l^2(\mathbb{R})$, the parametrization (4.4) can be reexpressed as

$$\Gamma_\alpha = \left\{ z(s) + \sum_{j=1}^{\infty} \alpha_j(s) b_j(s) U(s) \right\}, \tag{4.5}$$

where $s \in [0,1]$ and $z(s) \in \Gamma_0$.

Proposition 4.1 Let $S(\tilde{\alpha},\alpha)$ be a sentinel item defined as follows:

$$S: \left| \begin{array}{c} l^2(\mathbb{R})\times l^2(\mathbb{R}) \longrightarrow l^2(\mathbb{R}) \\ (\tilde{\alpha},\alpha) \longrightarrow (\int_O u_i(\tilde{\alpha})x(\alpha)dzdt)_{i=1,2,\cdots,\infty} \end{array} \right. , \tag{4.6}$$

where $x(\alpha)=x(z,t;\alpha)$ is the solution of the first problem and the functions $u_i(\tilde{\alpha})_{i=1,2,\cdots,\infty}$ that need to be found in such a way that

$$D_\alpha S(\hat{\alpha},\hat{\alpha}) = Id + M, \quad \forall\hat{\alpha}\in l^2(\mathbb{R}), \tag{4.7}$$

and

$$u_i(\tilde{\alpha}) = \min \|\phi\|_{L^2(O)}, \quad i=1,2,\cdots,\infty, \tag{4.8}$$

where $M\in\mathcal{L}(l^2(\mathbb{R}))$ such that

$$\|(M_i)\|_{l^2(\mathbb{R})} = \frac{\varepsilon}{i}, \quad \text{for } i=1,2,\cdots,\infty, \tag{4.9}$$

in which M_i is the i^{th}-line of M and $D_\alpha S(\hat{\alpha},\hat{\alpha})$ is the differential of S with respect to its second parameter at the point $(\hat{\alpha},\hat{\alpha})$. Then, $S(\tilde{\alpha},\alpha)$, which is defined by (4.6-4.8), exists and unique.

To prove this proposition, we have steps.

Proof. **Firstly :** This step shows that the sentinel function $S(\alpha,\alpha)$ depends smoothly on α and can be differentiated. This ensures that the method is computationally feasible and allows for gradient-based analysis.The solution of the one-phase Stefan problem $x(z,t;\tilde{\alpha})$ is differentiable with respect to the parameter α. For each j, define

$$x_{\alpha_j} = \frac{\partial x(\tilde{\alpha})}{\partial\alpha_j}, \quad j=1,2,\cdots,\infty,$$

the function x_{α_j} satisfies the following problem

$$\left\{ \begin{array}{c} \dfrac{\partial x_{\alpha_j}}{\partial t} = \Delta x_{\alpha_j} \text{ in } Q_\alpha = \Omega_\alpha\times]0,T[, \\ -k\dfrac{\partial x_{\alpha_j}}{\partial z}(0,T) = -b_j\,(\nabla x(\tilde{\alpha}).U) \text{ on } \Sigma_\alpha = \Gamma_\alpha\times]0,T[, \\ x_{\alpha_j}(s(t),t) = 0 \text{ on } \Sigma^* = \Gamma^*\times]0,T[, \\ x_{\alpha_j}(z,0) = 0 \text{ in } \Omega_\alpha, \end{array} \right. \tag{4.10}$$

where $x(\tilde{\alpha}) = x(z,t;\tilde{\alpha})$ solves (4.3) with data $\tilde{\alpha}$ and y_m that is dependent of α_j. This leads to the general form of the matrix $D_\alpha S(\tilde{\alpha},\tilde{\alpha})$

$$(D_\alpha S(\tilde{\alpha},\tilde{\alpha}))_{ij} = \left(\int_O u_i(\tilde{\alpha})y_{\alpha_j}dzdt\right)_{i=1,2,\cdots,\infty} \tag{4.11}$$

$$= \delta_{ij} + (M)_{ij}, \; j=1,2,\cdots,\infty, \tag{4.12}$$

where $\|M\| = \dfrac{\varepsilon}{i}$. Now, (4.7) reads

$$\int_O u_i(\tilde{\alpha})y_{\alpha_j}dxdt = \delta_{ij} + (M)_{ij}, \quad j=1,2,\cdots,\infty, \tag{4.13}$$

where i is fixed and the matrix M_{ij} represents the perturbation terms.

Secondly : Here, we prove that the sentinel method satisfies the approximate controllability condition, meaning we can construct control functions $u_i(\alpha)$ that approximate any desired boundary behaviour.

The backward problem is given by

$$\begin{cases} -\dfrac{\partial q_i}{\partial t} - \Delta q_i = u_i(\tilde{\alpha})|_O \text{ in } Q_{\tilde{\alpha}} = \Omega_{\tilde{\alpha}} \times]0,T[, \\ q_i = 0 \text{ on } \Sigma_{\tilde{\alpha}} = \Gamma_{\tilde{\alpha}} \times]0,T[, \\ q_i(.,T) = 0 \text{ in } \Omega_{\tilde{\alpha}}, \end{cases} \tag{4.14}$$

where $q_i \in L^2\left(]0,T[, H_0^1(\Omega_{\tilde{\alpha}}) \cap H^2(\Omega_{\tilde{\alpha}})\right)$ is the solution of the adjoint problem.

Multiply the first equation of (4.14) by y_{α_j}, and apply Green's formula. This leads to

$$\int_O u_i(\tilde{\alpha}) \frac{\partial x_{\alpha_j}}{\partial z} dz dt = \int_O \frac{b_j}{k} (\nabla x(\tilde{\alpha}).U) \frac{\partial q_i}{\partial n} d\Sigma. \tag{4.15}$$

Define a linear continuous operator $B \in \mathcal{L}(L^2(O); l^2(\mathbb{R}))$ as follows

$$B : \left| \begin{array}{c} L^2(O) \longrightarrow l^2(\mathbb{R}) \\ u_i(\tilde{\alpha}) \longrightarrow \left(\int_{\Sigma_{\tilde{\alpha}}} \dfrac{b_j}{k} (\nabla x(\tilde{\alpha}).U) \dfrac{\partial q_i}{\partial n} d\Sigma \right)_{j=1,2,\cdots,\infty} \end{array} \right. . \tag{4.16}$$

Equation (4.15) allows rewriting (4.11) as

$$(D_\alpha S(\tilde{\alpha}, \tilde{\alpha}))_{ij} = (B u_i(\tilde{\alpha}))_j. \tag{4.17}$$

As a result, we have exact control over the problem considered. The goal is to find $u_i(\tilde{\alpha}) \in L^2(O)$ that minimizes the norm and satisfies $B u_i(\tilde{\alpha}) = y$ with $y \in l^2(\mathbb{R})$.

Thirdly : The final step establishes uniqueness using the Fenchel-Rockafellar duality method. This ensures that for each parameter α, there is a unique sentinel function $u_i \alpha$ that minimizes the norm while satisfying the controllability constraints.

Next, we demonstrate that the image of B, denoted $\overline{Im(B)}$, equals $l^2(\mathbb{R})$, ensuring that the controllability condition is satisfied. Consider the adjoint operator B^* given by

$$B^* : \left| \begin{array}{c} l^2(\mathbb{R}) \longrightarrow L^2(O) \\ (\sigma_j)_{j=1,2,\cdots,\infty} \longrightarrow \phi|_O \end{array} \right. , \tag{4.18}$$

where ϕ solves the following problem:

$$\begin{cases} \dfrac{\partial \phi}{\partial t} = \Delta \phi, \\ \phi(s(t),t) = 0, \\ -k \dfrac{\partial \phi}{\partial z} = -(\nabla y(\tilde{\alpha}).U) \sum_{j=1}^{\infty} \sigma_j b_j, \\ \phi(z,0) = 0. \end{cases} \tag{4.19}$$

With (4.19), we have

$$(u, \phi)_{L^2(\Theta)} = \sum_{j=1}^{\infty} \sigma_j \int_{\Sigma_{\tilde{\alpha}}} b_j (\nabla x(\tilde{\alpha}).U) \frac{\partial q_i}{\partial n} d\Sigma \tag{4.20}$$

$$= (\sigma, Bu)_{l^2(\mathbb{R})}, \tag{4.21}$$

it's shown that $\phi|_O = B^* \sigma$. Suppose that $B^* \sigma = \phi|_O = 0$, i.e. $\phi = 0$ in O. By the unique continuation theorem, we have:

$$-(\nabla x(\tilde{\alpha}).U) \sum_{j=1}^{\infty} \sigma_j b_j = 0. \tag{4.22}$$

Since $(b_j)_{j=1,2,\cdots,\infty}$ is a basis of $l^2(\mathbb{R})$ whenever either $\{(\nabla x(\tilde{\alpha}).U) = 0\}$ or $\left\{\sigma_j = 0, j = 1,2,\cdots,\infty\right\}$. Now, we decompose the field U on the $\nu_{\tilde{\alpha}}$ and the tangent vectors $\tau_{\tilde{\alpha}}$ on $\Gamma_{\tilde{\alpha}}$ to obtain

$$\nabla x(\tilde{\alpha}).U(z) = \nabla x(\tilde{\alpha}).(a\nu_{\tilde{\alpha}}(z)) + \nabla x(\tilde{\alpha}).(b\tau_{\tilde{\alpha}}(z)), \ \forall z \in \Gamma_{\tilde{\alpha}}.$$

In other words, since $x(\tilde{\alpha}) = 0$ on $\Gamma_{\tilde{\alpha}}$, we have

$$\nabla x(\tilde{\alpha}).U(z) = a\frac{\partial x(\tilde{\alpha})}{\partial \nu_{\tilde{\alpha}}}(z), \forall z \in \Gamma_{\tilde{\alpha}}. \tag{4.23}$$

From the Cauchy uniqueness, we can have

$$\left|\frac{\partial x(\tilde{\alpha})}{\partial \nu_{\tilde{\alpha}}}\right._{\Gamma_{\tilde{\alpha}}} \neq 0,$$

otherwise, we have

$$x(\tilde{\alpha}) = 0 \text{ in } Q_{\tilde{\alpha}}.$$

Thus $(\nabla x(\tilde{\alpha}).U) \neq 0$ and B^* is injective, implying that $\overline{Im(B)} = l^2(\mathbb{R})$, i.e., $\forall \rho > 0$, $\forall y \in l^2(\mathbb{R})$, $\exists u_i(\tilde{\alpha}) \in L^2(O)$ such that

$$\|Bu_i(\tilde{\alpha}) - y\|_{l^2(\mathbb{R})} \leq \rho. \tag{4.24}$$

Fourthly : The admissible set is given by $U_{ad} = \left\{u \in L^2(O) : \|Bu - y\|_{l^2(\mathbb{R})} \leq \rho, y \in l^2(\mathbb{R})\right\}$, where U_{ad} is a nonempty convex and closed set in $L^2(O)$.

There exists a unique solution of the minimization problem $u_i(\tilde{\alpha})$ satisfying (4.8)

$$\min_{\omega \in U_{ad}} \frac{1}{2} \|u\|^2_{L^2(O)}. \tag{4.25}$$

Let F and G be two functions defined as

$$F(u) = \frac{1}{2} \|u\|^2_{L^2(O)} \tag{4.26}$$

and

$$G(w) = \begin{cases} 0 \quad \text{if} \quad \|w - y\|_{l^2(\mathbb{R})} \leq \rho, \\ \quad +\infty \quad \text{otherwise} \end{cases}. \tag{4.27}$$

So, problem (4.25) write as follows

$$\min_{\omega \in L^2(O)} F(u) + G(w). \tag{4.28}$$

We get

$$u_i(\tilde{\alpha}) = B^* \sigma^*, \tag{4.29}$$

expresses the optimal control function by the duality theorem of Fenchel-Rockafellar (The Fenchel-Rockafellar duality framework is a powerful tool for optimization problems, particularly in systems governed by partial differential equations like the Stefan problem. Its strength lies in the ability to handle constraints systematically while ensuring both existence and uniqueness of solutions), where σ^* is the solution of the dual of (4.25)

$$\min_{\sigma \in l^2(\mathbb{R})} F^*(B^* \sigma) + G^*(-\sigma), \tag{4.30}$$

where F^* and G^* are the conjugates of F and G such that $F^* = F$ and G^* is given by

$$\begin{aligned} G^*(\sigma) &= \sup_{u \in l^2(\mathbb{R})} (u, \sigma)_{l^2(\mathbb{R})} - G(w) \\ &= (z, \sigma)_{l^2(\mathbb{R})} + \rho \|\sigma\|_{l^2(\mathbb{R})}, \end{aligned} \tag{4.31}$$

where $\overline{B(0,\rho)}$ is $l^2(\mathbb{R})$, which is a closed ball with center 0 and radius ρ. This would immediately turn (4.19) to be as

$$\min_{\sigma\in l^2(\mathbb{R})} J(\sigma) = F(\phi) + \rho\,\|\sigma\|_{l^2(\mathbb{R})} - (y,\sigma)_{l^2(\mathbb{R})}, \tag{4.32}$$

where ϕ is the solution of (4.19). ■

Lemma 4.1 $\sigma^* = 0$ is the solution of (4.32) if and only if $\|y\|_{l^2(\mathbb{R})} \le \rho$.

Proof. $\Rightarrow$) It's assumed that $\sigma^* = 0$. Then, with (4.29), we can have $u_i(\tilde{\alpha}) = 0$, then we get

$$Bu_i(\tilde{\alpha}) - y = -y.$$

This means

$$\|Bu_i(\tilde{\alpha}) - y\|_{l^2(\mathbb{R})} \le \rho,$$

i.e.

$$\|y\|_{l^2(\mathbb{R})} \le \rho.$$

$\Leftarrow$) If $\|y\|_{l^2(\mathbb{R})} \le \rho$. Then, the optimal solution to the minimization problem $u_i(\tilde{\alpha}) = 0$.
Consequently, due to B^* is injective, then (4.29) yields that $\sigma^* = 0$. ■

Based on the previous discussion and to demonstrate the existence and uniqueness of equations (4.7) and (4.8), we derive the following

$$\left(\frac{\partial J}{\partial \sigma}, \delta\sigma\right)_{l^2(\mathbb{R})} = \left(BB^*\sigma + \rho\frac{\sigma}{\|\sigma\|_{l^2(\mathbb{R})}} - y, \delta\sigma\right)_{l^2(\mathbb{R})}, \tag{4.33}$$

for any $\delta\sigma \in l^2(\mathbb{R})$ and $\sigma \neq 0$, the idea is to find the σ that minimizes this expression. The control σ^* is

$$BB^*\sigma - y = -\rho\frac{\sigma}{\|\sigma\|_{l^2(\mathbb{R})}}. \tag{4.34}$$

This is a fixed-point equation for σ^*.
With $u_i(\tilde{\alpha}) = B^*\sigma^*$, we obtain

$$\|Bu_i(\tilde{\alpha}) - y\|_{l^2(\mathbb{R})} = \rho, \tag{4.35}$$

indicates that the control is chosen to minimize the error between the target y and the current state.
Now, we choose to simplify the analysis where y_j are the components of the vector y in a canonical basis

$$y_j = \delta_{ij}, \quad j = 1,2,\cdots,\infty, \tag{4.36}$$

where y_j is the generic coordinate of y on a canonical basis.
The final result

$$D_\alpha(S(\tilde{\alpha},\tilde{\alpha})) = Id + M, \;\; \forall\tilde{\alpha} \in l^2(\mathbb{R}), \tag{4.37}$$

shows that the differential of the sentinel function is invertible, implying the existence and uniqueness of the sentinel functions. The identity operator Id ensures that the system is well posed.

After proving the existence and uniqueness of the sentinel solution through the lemma, the next step is to represent the sentinel function in a form that allows for iterative computation. Then, we transform the considered problem into a fixed-point problem.

Now, we differentiate $S(\alpha)$ and take $\tilde{\alpha} = \alpha^k$ and $\bar{\alpha} = \alpha^{k+1}$, we obtain the iteration calculate

$$\alpha^{k+1} = \alpha^k + S(\alpha^k, \bar{\alpha}) - S(\alpha^k, \alpha^k),$$

where

$$S(\alpha^k, \bar{\alpha}) = \left(\int_O w_i(\alpha^k) x_{obs} dz dt \right)_{i=1,2,\cdots,\infty},$$

and

$$S(\alpha^k, \alpha^k) = \left(\int_O w_i(\alpha^k) x(z,t,\alpha^k) dz dt \right)_{i=1,2,\cdots,\infty}.$$

Theorem 4.1 The sequence $\left(\alpha^k\right)_{k=0,1,\cdots,\infty}$ in which

$$\begin{cases} \alpha^0 \in l^2(\mathbb{R}), \\ \alpha^{k+1} = \alpha^k + S(\alpha^k, \bar{\alpha}) - S(\alpha^k, \alpha^k), \end{cases}$$

locally converges in $l^2(\mathbb{R})$.

The local convergence of this scheme is assured. This means that if one starts with an initial estimate sufficiently close to the exact solution, the iterative scheme will converge with this solution after several iterations.

4.4 Numerical scheme

The numerical scheme described in the paper is designed to solve the one-phase Stefan problem using the sentinel method. It aims to transform the problem into a fixed-point problem and demonstrate local convergence.

The fixed-point transformation is critical in breaking down the complexity of tracking the moving boundary in the Stefan problem. Using the sentinel method, which observes temperature variations and boundary changes, the scheme iterates to adjust the boundary's position, refining its estimate step by step.

Now, to carry such a transformation, we differentiate $S(\alpha) = S(\tilde{\alpha}, \alpha)$ to obtain

$$S(\tilde{\alpha}, \alpha) = S(\tilde{\alpha}, \tilde{\alpha}) + D_\alpha S(\tilde{\alpha}, \tilde{\alpha}).(\alpha - \tilde{\alpha}) + o(|\alpha - \tilde{\alpha}|).$$

Accordingly, for $\bar{\alpha} = \alpha$, we have

$$S(\tilde{\alpha}, \bar{\alpha}) = S(\tilde{\alpha}, \tilde{\alpha}) + \bar{\alpha} - \tilde{\alpha} + M(\alpha - \tilde{\alpha}) + o(|\bar{\alpha} - \tilde{\alpha}|).$$

We neglect the higher-order terms of the above equality by taking $\tilde{\alpha} = \alpha^k$ and $\bar{\alpha} = \alpha^{k+1}$, we obtain

$$\alpha^{k+1} = \alpha^k + S(\alpha^k, \bar{\alpha}) - S(\alpha^k, \alpha^k),$$

where

$$S(\alpha^k, \bar{\alpha}) = \left(\int_O w_i(\alpha^k) x_{obs} dz dt \right)_{i=1,2,\cdots,\infty},$$

and

$$S(\alpha^k, \alpha^k) = \left(\int_O w_i(\alpha^k) x(y,t,\alpha^k) dx dt \right)_{i=1,2,\cdots,\infty}.$$

Theorem 4.2 The sequence $\left(\alpha^k\right)_{k=0,1,\cdots,\infty}$ in which

$$\begin{cases} \alpha^0 \in l^2(\mathbb{R}) \\ \alpha^{k+1} = \alpha^k + S(\alpha^k, \bar{\alpha}) - S(\alpha^k, \alpha^k) \end{cases}$$

locally converges in $l^2(\mathbb{R})$.

Proof. The numerical scheme here is a method to solve the fixed point problem

$$\alpha^{k+1} = g(\alpha^k),$$

where g is an operator defined from $l^2(\mathbb{R})$ to itself. For $\mu \in l^2(\mathbb{R})$, we obtain

$$g'(\mu) = Id + D_{\tilde{\alpha}}S(\mu,\tilde{\alpha}) - D_{\tilde{\alpha}}S(\mu,\mu) - D_{\alpha}S(\mu,\mu),$$

which implies $\mu = \bar{\alpha}$ and so we have

$$g'(\bar{\alpha}) = Id - D_{\alpha}S(\bar{\alpha},\bar{\alpha}).$$

Now, in light of Proposition 4.1, we can get

$$g'(\bar{\alpha}) = -M,\ M \in L(l^2(\mathbb{R})).$$

This consequently yields

$$\begin{aligned} \left\|g'(\bar{\alpha})\right\|_{HS}^2 &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(g'(\bar{\alpha}))_{ij}^2 \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(g'(\bar{\alpha}))_{ij}^2 \\ &= \sum_{i=1}^{\infty}\|(M_i)\|_{l^2(\mathbb{R})}^2 \\ &= \varepsilon^2\sum_{i=1}^{\infty}\frac{1}{i^2}. \end{aligned}$$

This means

$$\left\|g'(\bar{\alpha})\right\|_{HS} \leq 1.$$

Hence, the local convergence of the sequence is satisfied. ■

Thus, the numerical scheme efficiently updates the boundary condition for the Stefan problem, ensuring that it gradually approaches the proper solution.

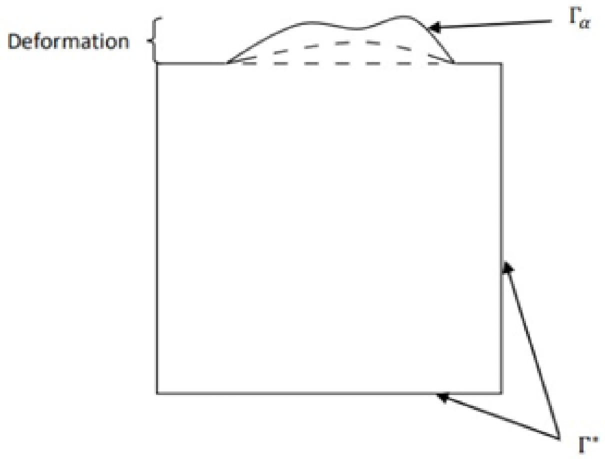


Figure 4.3: Deformation of the boundary.

The figure (4.3) shows how the moving boundary Γ_α evolves over several iterations. Initially flat, the boundary undergoes significant deformation before stabilizing to the final configuration.

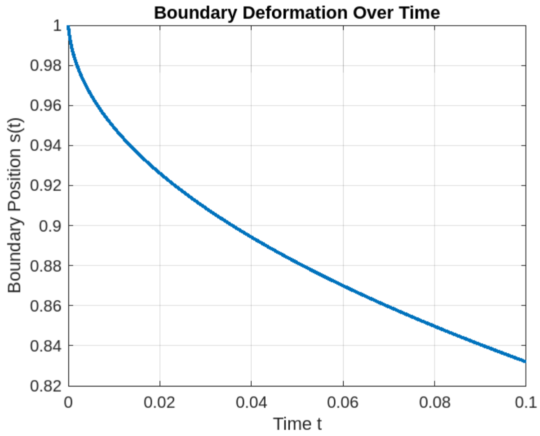


Figure 4.4: Boundary deformation over time.

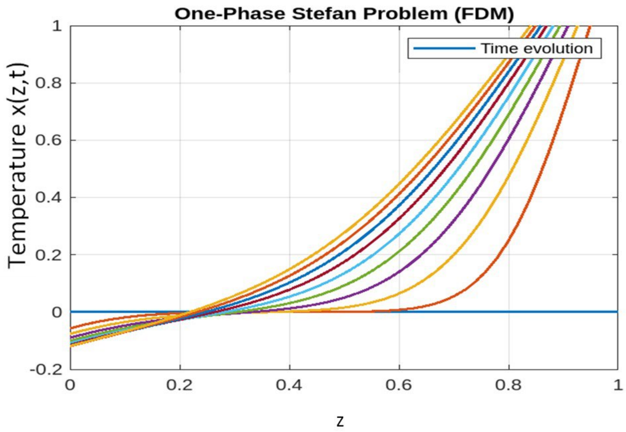


Figure 4.5: One-Phase Stefan problem (FDM).

The sentinel method reliably identifies the unknown boundary in the one-phase Stefan problem through iterative updates. The numerical results validate the theoretical guarantees of existence, uniqueness, and local convergence, demonstrating the effectiveness and robustness of the method. Future work will focus on extending this approach to two-phase Stefan problems and improving computational efficiency.

5. Fractional Hybrid Optimal System

For more information and a deeper exploration of hybrid systems studies, consult the relevant literature cited in [**14**], [**22**], [**27**], [**31**] and [**35**].

5.1 Introduction

Hybrid systems are pivotal in advanced technological fields, including robotics, automotive engineering, and telecommunications, where the integration of continuous and discrete dynamics is essential. These systems model processes with dual dynamics, enabling seamless transitions between physical behaviours (e.g. motion) and logical decisions (e.g. switching actions). However, their versatility introduces analytical challenges, particularly in safety-critical applications requiring precision and adaptability.

Including fractional-order dynamics extends the capabilities of the traditional hybrid model by capturing memory effects and nonlocal behaviour, which is essential for systems influenced by historical dependencies. Fractional derivatives, such as Caputo and Riemann-Liouville, provide a more nuanced representation of dynamics, allowing for the development of sophisticated control strategies. For instance, fractional dynamics can model viscoelastic materials or biological systems more accurately than integer-order models.

We propose a framework for fractional hybrid systems with state jumps that address the complexities introduced by abrupt state changes and memory-dependent dynamics. Extending optimal control principles to these systems enables precise modelling and control of complex processes such as robotics and biomedical systems.

Consider a robotic arm operating in a dynamic environment. Traditional hybrid models struggle to account for the system's memory effects, such as friction or inertia, while transitioning smoothly between states like grasping or releasing objects. Our approach captures these hereditary properties by incorporating fractional-order dynamics, providing enhanced precision and stability.

A diagram summarizing the hybrid system structure, including continuous dynamics, discrete events, and state jumps, can be added here to outline the problem visually.

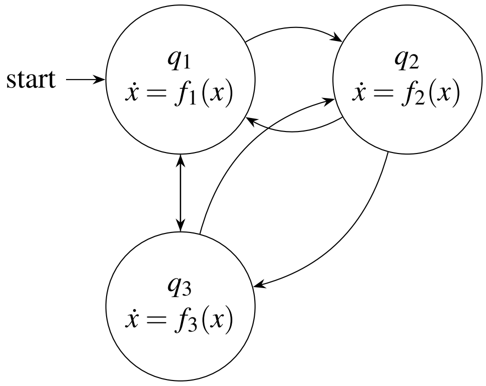


Figure 5.1: Hybrid System with Mode Transitions

This figure (5.1) represents a hybrid system with three discrete modes (q_1, q_2, q_3), where each mode follows its own continuous dynamics ($\dot{x} = f_i(x)$). The system starts in mode q_1 and transitions between modes based on discrete switching rules, represented by the arrows. These transitions can be triggered by external inputs, system conditions, or control actions. Such models are widely used in robotics, multi-agent systems, and energy management, where both continuous evolution and discrete decisions are essential.

Definition 5.1 A hybrid system is a 9-tuple

$$H = \{Q, X, U, F, M, \Lambda, \Omega, \Gamma, H\}, \tag{5.1}$$

where $Q = \{1, 2, 3, ..., N_q\}$ is a set of discrete states, X is a collection of continuous state spaces, U is a collection of continuous control space, F is a collection of vector fields for the continuous dynamics, M is the collection of switching manifolds, Λ is the collection of reset map, Ω is the collection of discrete control spaces, Γ is a discrete transition map and H is a collection of constraint functions.

5.2 Fractional hybrid optimal control problem with state jump

The fractional hybrid optimal control problem framework combines the fractional-order Caputo derivative with hybrid systems theory, extending traditional optimal control by introducing memory effects and jump conditions. The reset map depends on discrete state transitions and the continuous state trajectory, ensuring seamless integration of hybrid and fractional dynamics

We have the following problem

$$\min \ \phi(x(a), x(b)) + I_{a^+}^{\beta} [F(x, u, t)](b). \tag{5.2}$$

Such that

$$\begin{cases} {}_cD_{a^+}^{\alpha} x(t) = f_{q(t)}(x(t), u(t)), \forall t \in [a, b], \\ x(a) = x_a, \\ x(b) = x_b, \\ \varphi(x(t_j^-), x(t_j^+)) = x(t_j^+) - x(t_j^-) - \delta_{(q(x(t_j^-)), q(x(t_j^+)))}(x(t_j^-)), \forall t_j \in \Theta_t = t_1, t_2, t_3, ..., t_{swt}. \end{cases}$$

where

- **Terminal Cost Function** ($\phi(x(a), x(b))$): Evaluates the penalty associated with deviations of the initial state $x(a)$ and final state $x(b)$ from desired values.
- **Fractional Integral Term** ($I_{a^+}^{\beta}[F(x, u, t)](b)$): Represents the Riemann-Liouville fractional integral of order β over the interval $[a, b]$, acting on the function $F(x, u, t)$.
- **Caputo Fractional Derivative** (${}_cD_{a^+}^{\alpha} x(t)$): Denotes the Caputo-type fractional derivative of order α for the state $x(t)$, computed starting from time a.
- **Mode-Dependent Dynamics** ($f_{q(t)}(x(t), u(t))$): Describes the continuous-time evolution of the system, governed by the discrete state $q(t)$, which determines the active dynamics (e.g., motion, braking).
- **Reset Condition** ($\varphi(x(t_j^-), x(t_j^+))$): Specifies the relationship between the state $x(t_j^-)$ (before a switching event) and $x(t_j^+)$ (after the event) at switching times $t_j \in \Theta_t$, where Θ_t is the set of all switching instances.

- **Switching Count** (N_{swt}): Indicates the total number of discrete state transitions (switches) in $q(t)$, with $j = 1,2,\dots,N_{swt}$ enumerating each switching event.
- **Reset Map** ($\delta_{(q(x(t_j^-)),q(x(t_j^+)))}(x(t_j^-))$): Defines the state jump magnitude during a discrete transition, dependent on both the pre-switch discrete state $q(x(t_j^-))$ and post-switch state $q(x(t_j^+))$.

The transversality condition for the general boundary condition

This condition is defined by

$$I_{b^-}^{1-\alpha}(p(a)) = -p^0\partial_1\varphi(x^*(a),x^*(b)) - \partial_1 g(x^*(a),x^*(b))^T \times \psi, \tag{5.3}$$
$$I_{b^-}^{1-\alpha}(p(b)) = -p^0\partial_2\varphi(x^*(a),x^*(b)) - \partial_2 g(x^*(a),x^*(b))^T \times \psi, \tag{5.4}$$

with $[t_j,t_{j+1}] \subset [0,1]$ and $\varphi(x(t_0),x(t_f)) = 0$ because not all initial values will be defined.
Let us introduce a new time variable :

$$\tau \in [0,1]; \varepsilon = t_{j+1} - t_j; j = 1,...,N_{exe} - 1$$

with the number of switchings is then given as $N_{\mathrm{swt}} = N_{\mathrm{exe}} - 1$ and the function $\tilde{t}_j(\tau) = t_j + \varepsilon_j\tau$, we obtain a set of linear initial value problems

$$_cD_{a^+}^{\alpha}(\tilde{t}_j(\tau)) = \frac{\varepsilon_j^{-\alpha}}{\Gamma(\alpha)(\alpha)}\tau^{1-\alpha} \tag{5.5}$$

with $\tilde{t}_j(0) = t_j$, and the continuity condition $\tilde{t}_j(0) = \tilde{t}_{j-1}(1)$ with $j = 1,...,N_{swt}$.

All the variables $x_j(t)$ continuous valued states, $u_j(t)$ continuous valued controls, $q_j(t)$ the discrete and $\lambda_j(t)$ the adjoint state (or costate) are now functions depending on τ, i.e. $x_j(\tilde{t}_j(\tau)), u_j(\tilde{t}_j(\tau)), \lambda_j(\tilde{t}_j(\tau)), q_j(\tilde{t}_j(\tau))$, we write $x_j(\tau), u_j(\tau), \lambda_j(\tau)$ and $q_j(\tau)$.

Then, the Fractional Hybrid optimal control problem be transformed into the new time domain $\tau \in [0,1]$ as

$$\min\ \phi(\tilde{x}_j(a),\tilde{x}_j(b)) + I_{a^+}^{\beta}\left[F(\tilde{x}_j,\tilde{u}_j,\cdot)\right](b). \tag{5.6}$$

Such that

$$\begin{cases} {}_cD_{a^+}^{\alpha}(\tilde{x}_j(\tau)) = \varepsilon_j^{\alpha} f_{\tilde{q}_j}(\tilde{x}_j(\tau),\tilde{u}_j(\tau)),\ \text{for a.e. } \tau \in [0,1], \\ \tilde{x}_0(0) = x_0, \\ \tilde{x}_{N_{exe}-1}(1) = x_f, \\ \varphi(\tilde{x}_{j-1}(1),\tilde{x}_j(0)) = \tilde{x}_j(0) - \tilde{x}_{j-1}(1) - \delta_{(\tilde{q}_{j-1}(1),\tilde{q}_j(0))}(\tilde{x}_{j-1}(1)), j = 1,...,N_{swt}, \\ {}_cD_{a^+}^{\alpha}(\tilde{t}_j(\tau)) = \frac{\varepsilon_j}{\Gamma(\alpha)(\alpha)}\tau^{1-\alpha}, \\ \tilde{t}_j(0) - t_j = 0, \\ \tilde{t}_0(0) = t_0, \\ \tilde{t}_{N_{exe}-1}(1) = t_f. \end{cases}$$

Defining the Hamiltonian of the transformed problem as :

$$\begin{aligned} \tilde{H}(\tilde{x}(\tau),\tilde{p}(\tau),\tilde{\rho}(\tau),\tilde{u}(\tau),\varepsilon) &= \sum_{j=0}^{N_{exe}-1} \tilde{H}(\tilde{x}_j(\tau),\tilde{p}_j(\tau),\tilde{\rho}_j(\tau),\tilde{u}_j(\tau),\varepsilon_j) \\ &= \sum_{j=0}^{N_{exe}-1} \left\langle \tilde{p}_j, \varepsilon_j f_{\tilde{q}_j}(x_j,u_j,\tau)\right\rangle_{\mathbb{R}^n} + \left\langle \tilde{\rho}_j, {}_cD_{a^+}^{\alpha}(\tilde{t}_j(\tau))\right\rangle + p^0\frac{(b-t)^{\beta-1}}{\Gamma(\beta)}F(x,u,t). \end{aligned}$$

The Adjoint differential equation

$$D_{b^-}^{\alpha}(\tilde{p}(\tau)) = -\partial_1\tilde{H}(\tilde{x}(\tau),\tilde{p}(\tau),\tilde{\rho}(\tau),\tilde{u}(\tau),\varepsilon); \tag{5.7}$$
$$D_{b^-}^{\alpha}(\tilde{\rho}(\tau)) = -\frac{\partial\tilde{H}}{\partial\tilde{t}_j}(\tilde{x}(\tau),\tilde{p}(\tau),\tilde{\rho}(\tau),\tilde{u}(\tau),\varepsilon); \tag{5.8}$$

with $\tilde{\rho}$ is the costate that adjoints the differential equation of the transformed time $\tilde{t}_j(.)$ to the Hamiltonian.

The transversality condition of the adjoint vector

$$\tilde{p}_{j-1}(1) = \left(\frac{\partial\varphi}{\partial\tilde{x}_{j-1}(1)}\right)^T (\tilde{x}_{j-1}(1),\ldots).\psi_j; \tag{5.9}$$

$$\tilde{p}_j(0) = -\left(\frac{\partial\varphi}{\partial\tilde{x}_j(0)}\right)^T (\tilde{x}_j(0),\ldots).\psi_j. \tag{5.10}$$

with

$$\frac{\partial\varphi}{\partial\tilde{x}_{j-1}(1)} = \frac{\partial\varphi}{\partial\tilde{x}_{j-1}(1)}(\tilde{x}_{j-1}(1),\tilde{x}_j(0)) \tag{5.11}$$

$$= -I - \frac{\partial\delta_{(\tilde{q}_{j-1}(1),\tilde{q}_j(0))}(\tilde{x}_{j-1}(1))}{\partial\tilde{x}_{j-1}(1)}; \tag{5.12}$$

and

$$\frac{\partial\varphi}{\partial\tilde{x}_j(0)} = I.$$

Then, the transversality conditions are

$$\tilde{p}_{j-1}(1) = -\left(I + \frac{\partial\delta_{(\tilde{q}_{j-1}(1),\tilde{q}_j(0))}(\tilde{x}_{j-1}(1))}{\partial\tilde{x}_{j-1}(1)}\right)^T \psi_j; \tag{5.13}$$

$$\tilde{p}_j(0) = -\psi_j; \tag{5.14}$$

then,

$$\tilde{p}_{j-1}(1) = \tilde{p}_j(0) + \left(\frac{\partial\delta_{(\tilde{q}_{j-1}(1),\tilde{q}_j(0))}(\tilde{x}_{j-1}(1))}{\partial\tilde{x}_{j-1}(1)}\right)^T \tilde{p}_j(0);$$

with $\tilde{p}_{j-1}(1) = p(t_j^-); \tilde{p}_j(0) = p(t_j^+); \tilde{x}_{j-1}(1) = x(t_j^-)$ and $\tilde{x}_j(0) = x(t_j^+)$, so we have

$$p(t_j^-) = p(t_j^+) + \left(\frac{\partial\delta_{(q(t_j^-),q(t_j^+))}(x(t_j^-))}{\partial x(t_j^-)}\right)^T p(t_j^+).$$

5.3 Hybrid fractional-order system in robotics

Consider a robotic arm used in industrial applications, where the control of the arm's position and velocity can be modelled using fractional-order dynamics. The system can also experience discrete events, such as triggering a grasping mechanism when an object is detected.

5.3.1 Model description

- **Continuous Dynamics:** The movement of the robotic arm can be described by a fractional-order model that accounts for the effects of friction and inertia. The dynamics of the joint position $x(t)$ could be represented as :

$$D^\alpha x(t) = -bD^\beta x(t) + ku(t), \quad t \in [t_k, t_{k+1}) \tag{5.15}$$

  where :
  - D^α is the fractional derivative of order α (e.g., $0 < \alpha < 1$),
  - D^β represents a fractional-order velocity term,
  - b is the damping coefficient,
  - k is a gain factor,
  - $u(t)$ is the control input (e.g., torque applied to the joint).

- **Discrete Events:** When the robotic arm approaches an object, a sensor triggers a discrete event that activates a gripper mechanism. This can be modelled as:

$$x(t_k^+) = \Delta_k(x(t_k^-), g_k), \tag{5.16}$$

  where :
  - g_k represents the gripper position or state (e.g., open or closed),
  - Δ_k modifies the joint position based on the gripper activation.
- **Output Equation :** The output can be the end effector position or velocity of the arm, defined as:

$$y(t) = x(t) + du(t), \tag{5.17}$$

  where d is a transformation matrix that relates the joint position to the end effector position.

5.3.2 Results

The proposed fractional hybrid optimal control framework is applied to a robotic arm system with memory-dependent dynamics and state jumps. Below, we elaborate on the key results derived from the Pontryagin Maximum Principle (PMP) and their implications.

The optimization problem minimizes a hybrid cost function combining terminal state penalties and a fractional integral of control effort:

$$J = \phi(x(a), x(b)) + I_{a+}^{\beta}\left[u^2\right](b), \tag{5.18}$$

such that

$$\begin{cases} {}_cD_{a+}^{\alpha}x(t) = -b_cD_{a+}^{\beta}x(t) + ku(t), \\ x(t_j^+) = \Delta_j(x(t_j^-), g_j), \end{cases}$$

where :

- $\phi(x(a), x(b))$ penalizes deviations from desired initial/final states (e.g., joint angles).
- $I_{a+}^{\beta}\left[u^2\right](b) = \frac{1}{\Gamma(\beta)}\int_a^b (b-t)^{\beta-1}u^2(t)\,dt$ quantifies the energy expenditure with a fractional weighting kernel. This kernel emphasizes recent control actions more heavily, aligning with the memory-dependent nature of fractional dynamics

Using the PMP for fractional hybrid systems, we define the Hamiltonian:

$$H(x, p, u, t) = p(t)\left(-b_cD_{a+}^{\beta}x(t) + ku(t)\right) + \frac{(b-t)^{\beta-1}}{\Gamma(\beta)}u^2(t), \tag{5.19}$$

where $p(t)$ is the costate variable. The first term represents the system's fractional dynamics weighted by the costate, while the second term penalizes control effort.

The costate evolves according to the backward fractional differential equation:

$${}_cD_{b-}^{\alpha}p(t) = -\frac{\partial H}{\partial x} = b_cD_{b-}^{\beta}p(t). \tag{5.20}$$

Minimizing H with respect to $u(t)$

$$\frac{\partial H}{\partial u} = kp(t) + \frac{2(b-t)^{\beta-1}}{\Gamma(\beta)}u(t) = 0,$$

then, we conclude the optimal control given as follows

$$u^*(t) = -\frac{k\Gamma(\beta)}{2(b-t)^{\beta-1}}p(t).$$

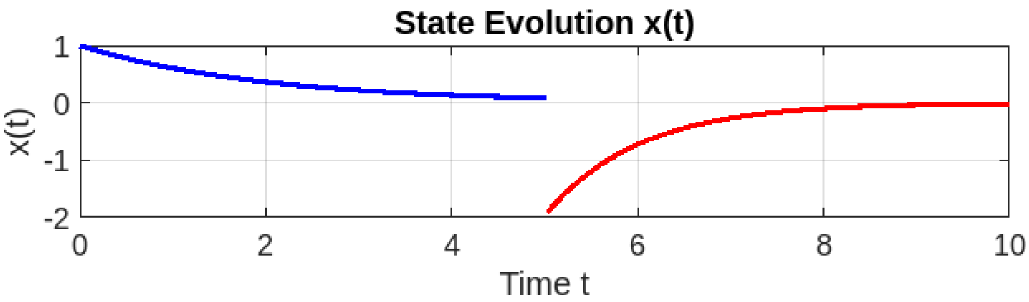
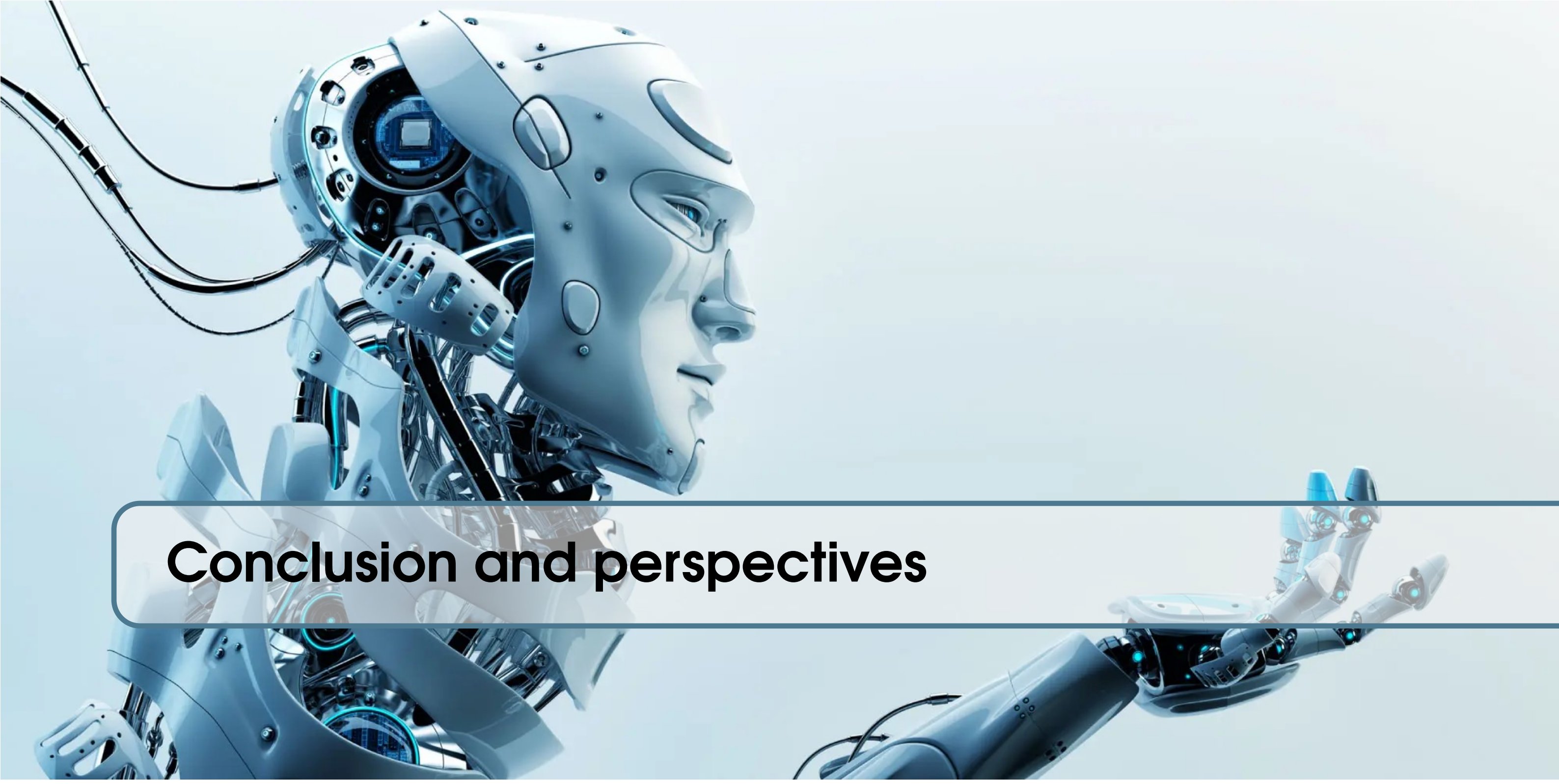


Figure 5.2: Evolution of the state.

This figure (5.2) shows the evolution of the state with state jump.



Conclusion and perspectives

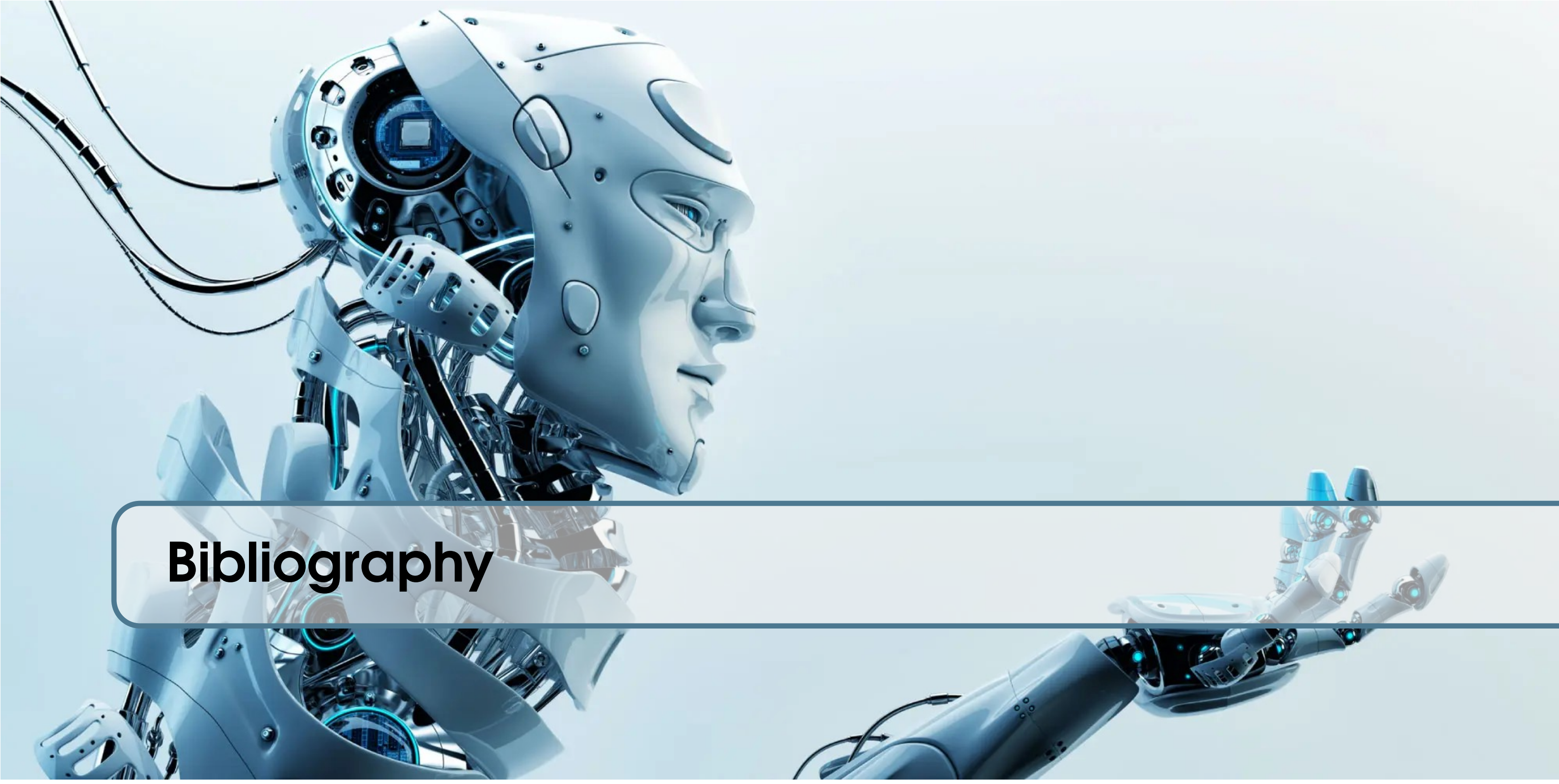
In conclusion, this thesis investigates various challenges and solutions in the field of control theory, with a particular focus on both linear and nonlinear systems. Explore essential concepts such as controllability, observability, and stability while delving into advanced topics, including optimal control and hybrid systems with fractional-order dynamics. This work provides significant insights into the mathematical foundations of control systems by addressing direct and inverse problems and utilizing methods such as duality principles and optimal control theory. It highlights their practical applications to real-world phenomena.

The study extends classical control theory by introducing hybrid systems encompassing continuous and discrete behaviours. The framework is enriched by considering fractional-order dynamics, which effectively models memory effects and nonlocal interactions in complex systems. By leveraging established techniques such as Pontryagin's maximum principle, dynamic programming, the Riccati algorithm, semigroup theory, and optimization methods, this research offers efficient solutions for managing highly intricate systems.

From our perspective, the central objective of this work is to develop an optimal control strategy tailored for fractional hybrid systems. These systems, which combine continuous and discrete dynamics, pose significant challenges for control design, particularly when fractional calculus is employed. Fractional calculus is a powerful mathematical tool for accurately modelling systems with memory effects or hereditary properties, which are prevalent in many physical and engineering contexts. Incorporating fractional derivatives and integrals enables a more precise representation of long-term system behaviour than traditional integer-order models.

The framework established in this thesis allows for formulating hybrid optimal control problems in which fractional differential equations govern the state evolution. The aim is to derive optimal control laws that ensure smooth transitions between discrete events while maintaining stability and performance. This requires solving the continuous dynamics, typically described by fractional differential equations, alongside the discrete switching behaviour at predetermined time instances. Numerical simulations use appropriate discretization methods to validate theoretical results, solve fractional differential equations, and implement the proposed control strategies. These simulations assess the performance of the control strategies in various scenarios, confirming their effectiveness and aligning with the theoretical predictions.

In the final phase of this research, the analysis is extended to hybrid systems governed by partial differential equations (PDEs) with fractional orders. Such hybrid PDE systems arise in numerous real-world applications, including fluid dynamics, population modelling, and spatially distributed systems. The study explores complex behaviours such as anomalous diffusion and spatial memory effects by incorporating fractional operators into these PDE models. Numerical solutions to these hybrid PDEs provide deeper insight into the control and stability of systems exhibiting spatial and temporal complexities. Ultimately, this research contributes to the advancement of fractional hybrid systems and proposes novel strategies for their optimal control, offering valuable applications across engineering and scientific disciplines.



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