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Common fixed point theorems of Gregus type for weakly compatible mappings satisfying generalized contractive conditions

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Abstract

We prove a common fixed point theorem of Gregus type for four mappings satisfying a generalized contractive condition in metric spaces using the concept of weak compatibility which generalizes theorems of [I. Altun, D. Turkoglu, B.E. Rhoades, Fixed points of weakly compatible mappings satisfying a general contractive condition of integral type, Fixed Point Theory Appl. 2007 (2007), article ID 17301; A. Djoudi, L. Nisse, Gregus type fixed points for weakly compatible mappings, Bull. Belg. Math. Soc. 10 (2003) 369–378; A. Djoudi, A. Aliouche, Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type, J. Math. Anal. Appl. 329 (1) (2007) 31–45; P. Vijayaraju, B.E. Rhoades, R. Mohanraj, A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci. 15 (2005) 2359–2364; X. Zhang, Common fixed point theorems for some new generalized contractive type mappings, J. Math. Anal. Appl. 333 (2) (2007) 780–786]. We prove also a common fixed point theorem which generalizes Theorem 3.5 of [H.K. Pathak, M.S. Khan, T. Rakesh, A common fixed point theorem and its application to nonlinear integral equations, Comput. Math. Appl. 53 (2007) 961–971] and common fixed point theorems of Gregus type using a strict generalized contractive condition, a property (E.A) and a common property (E.A).

Keywords: Weakly compatible mappings; Common fixed point; Metric space; Property (E.A); Common property (E.A); Generalized contractive condition

1. Introduction

Let S and T be self-mappings of a metric space (X,d). S and T are commuting if $STx = TSx$ for all $x \in X$. Sessa [21] defined S and T to be weakly commuting if for all $x \in X$

$$d(STx, TSx) \leqslant d(Tx, Sx).$$

Jungck [8] defined S and T to be compatible as a generalization of weakly commuting if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

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It is easy to show that commuting implies weakly commuting implies compatible and there are examples in the literature verifying that the inclusions are proper, see [8] and [21].

Jungck et al. [9] defined S and T to be compatible mappings of type (A) if

$$\lim_{n\to\infty} d\big(STx_n, T^2x_n\big) = 0 \quad \text{and} \quad \lim_{n\to\infty} d\big(TSx_n, S^2x_n\big) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Examples are given to show that the two concepts of compatibility are independent, see [9].

Recently, Pathak and Khan [14] defined S and T to be compatible mappings of type (B) as a generalization of compatible mappings of type (A) if

$$\lim_{n\to\infty} d\big(TSx_n, S^2x_n\big) \leqslant \frac{1}{2}\Big[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d\big(Tt, T^2x_n\big)\Big] \quad \text{and}$$
$$\lim_{n\to\infty} d\big(STx_n, T^2x_n\big) \leqslant \frac{1}{2}\Big[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d\big(St, S^2x_n\big)\Big],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Clearly, compatible mappings of type (A) are compatible mappings of type (B), but the converse is not true, see [14]. However, compatibility, compatibility of type (A) and compatibility of type (B) are equivalent if S and T are continuous, see [14].

Pathak et al. [15] defined S and T to be compatible mappings of type (P) if

$$\lim_{n\to\infty} d\big(S^2x_n, T^2x_n\big) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

However, compatibility, compatibility of type (A) and compatibility of type (P) are equivalent if S and T are continuous, see [15].

Pathak et al. [16] defined S and T to be compatible mappings of type (C) as a generalization of compatible mappings of type (A) if

$$\lim_{n\to\infty} d\big(TSx_n, S^2x_n\big) \leqslant \frac{1}{3}\Big[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d\big(Tt, S^2x_n\big) + \lim_{n\to\infty} d\big(Tt, T^2x_n\big)\Big] \quad \text{and}$$
$$\lim_{n\to\infty} d\big(STx_n, T^2x_n\big) \leqslant \frac{1}{3}\Big[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d\big(St, T^2x_n\big) + \lim_{n\to\infty} d\big(St, S^2x_n\big)\Big],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Compatibility, compatibility of type (A) and compatibility of type (C) are equivalent if S and T are continuous, see [16].

2. Preliminaries

Definition 2.1. (See [10].) S and T are said to be weakly compatible if they commute at their coincidence points; i.e., if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

Lemma 2.2. *(See [8,9,14–16].) If S and T are compatible, or compatible of type* (A)*, or compatible of type* (P)*, or compatible of type* (B)*, or compatible of type* (C)*, then they are weakly compatible.*

It was shown in [6] that the converse is not true in general.

Definition 2.3. (See [13].) S and T are said to be R-weakly commuting if there exists an $R > 0$ such that

$$d(STx, TSx) \leqslant Rd(Tx, Sx) \quad \text{for all } x \in X. \tag{2.1}$$

Definition 2.4. (See [13].) S and T are pointwise R-weakly commuting if for all $x \in X$, there exists an $R > 0$ such that (2.1) holds.

It was proved in [13] that R-weak commutativity is equivalent to commutativity at coincidence points; i.e., S and T are pointwise R-weakly commuting if and only if they are weakly compatible.

Definition 2.5. (See [1].) Let $S, T : X \to X$. The pair (S, T) satisfies property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \in X. \tag{2.2}$$

It is clear from the definition of compatibility that the pair (S, T) of a metric space (X, d) is noncompatible if there exists at least one sequence $\{x_n\}$ in X such that (2.2) holds but, $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either nonzero or does not exist. Therefore, two noncompatible mappings of a metric space (X, d) satisfy property (E.A).

Definition 2.6. (See [12].) Let $A, S, B, T : X \to X$. The pairs (A, S) and (B, T) satisfy a common property (E.A) if there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t \in X. \tag{2.3}$$

If $B = A$ and $T = S$ in (2.3), we obtain the definition of property (E.A).

Several authors have proved fixed point theorems for mappings satisfying contractive conditions of integral type. See [2–4,6,19,20,23]. Recently, Zhang [24] proved a common fixed point theorem using a new generalized contractive condition for a pair of self-mappings in a metric space. This theorem extend results in [4,19] and [20].

Let $A \in (0, \infty]$, $R_A^+ = [0, A)$ and $F : R_A^+ \to \mathbb{R}$ satisfying

(i) $F(0) = 0$ and $F(t) > 0$ for each $t \in (0, A)$,
(ii) F is nondecreasing on R_A^+,
(iii) F is continuous.

Define $\mathcal{F}[0, A) = \{F\colon\ F$ satisfies (i)–(iii)$\}$.

The following examples were given in [24].

(1) Let $F(t) = t$, then $F \in \mathcal{F}[0, A)$ for each $A \in (0, +\infty]$.
(2) Suppose that φ is nonnegative, Lebesgue integrable on $[0, A)$ and satisfies

$$\int_0^{\epsilon} \varphi(t)\,dt > 0 \quad \text{for each } \epsilon \in (0, A).$$

Let $F(t) = \int_0^t \varphi(s)\,ds$, then $F \in [0, A)$.

(3) Suppose that ψ is nonnegative, Lebesgue integrable on $[0, A)$ and satisfies

$$\int_0^{\epsilon} \psi(t)\,dt > 0 \quad \text{for each } \epsilon \in (0, A)$$

and φ is nonnegative, Lebesgue integrable on $[0, \int_0^A \psi(s)\,ds)$ and satisfies

$$\int_0^{\epsilon} \varphi(t)\,dt > 0 \quad \text{for each } \epsilon \in \left(0, \int_0^A \psi(s)\,ds\right).$$

Let $F(t) = \int_0^{\int_0^t \psi(s)\,ds} \varphi(u)\,du$, then $F \in \mathcal{F}[0, A)$.

(4) If $G \in [0, A)$ and $F \in \digamma[0, G(A-0))$, then a composition mapping $F \circ G \in \digamma[0, A)$. For instance, let $H(t) = \int_0^{F(t)} \varphi(s)\,ds$, then $H \in \digamma[0, A)$ whenever $F \in \digamma[0, A)$ and φ is nonnegative, Lebesgue integrable on $\digamma[0, F(A-0))$ and satisfies

$$\int_0^{\epsilon} \varphi(t)\,dt > 0 \quad \text{for each } \epsilon \in \big(0, F(A-0)\big).$$

Lemma 2.7. *(See [24].) Let $A \in (0, +\infty]$, $F \in \digamma[0, A)$. If $\lim_{n\to\infty} F(\epsilon_n) = 0$ for $\epsilon_n \in R_A^+$, then $\lim_{n\to\infty} \epsilon_n = 0$.*

Let $A \in (0, +\infty]$, $\psi : R_A^+ \to \mathbb{R}_+$ satisfying

(i) $\psi(t) < t$ for each $t \in (0, A)$,
(ii) ψ is nondecreasing and upper semi-continuous.

Define $\Psi[0, A) = \{\psi\colon\ \psi$ satisfies (i) and (ii) above$\}$.

Lemma 2.8. *(See [24].) If $\psi \in \Psi[0, A)$, then $\psi(0) = 0$.*

Lemma 2.9. *(See [22].) For any $t \in (0, A)$, $\psi(t) < t$ iff $\lim_{n\to+\infty} \psi^n(t) = 0$, where ψ^n denotes the n-times repeated composition of ψ with itself.*

Lemma 2.10. *(See [17].) If $\psi_i \in \Psi(0, A)$ for all $i \in I$, where I is a finite indexing set, then there exists some $\psi \in \Psi$ such that*:

$$\max\big\{\psi_i(t),\ i \in I\big\} \leqslant \psi(t) \quad \textit{for all } t > 0.$$

Theorem 2.11. *(See [7].) Let C be a nonempty closed convex subset of a Banach space X and T be a mapping from C into itself satisfying the inequality*

$$\|Tx - Ty\| \leqslant a\|x - y\| + b\|x - Tx\| + c\|y - Ty\|$$

for all x, y in, where $a > 0$, $b, c \geqslant 0$, $a + b + c = 1$. Then, T has a unique fixed point.

Several authors have generalized Theorem 2.11, see [5,6,11,14,16].

In [5], there is a problem in the proof of $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$ if n is odd. In fact, assume that $y_n \neq y_{n+1}$ for all n. Applying the following inequality

$$\begin{aligned} d^p(Ax, By) \leqslant \psi\Big[&ad^p(Sx, Ty) + (1-a)\max\Big\{d(Ax, Sx), d(By, Ty), \big(d(Ax, Sx)\big)^{\frac{1}{2}} \cdot \big(d(Ax, Ty)\big)^{\frac{1}{2}}, \\ &\big(d(Sx, By)\big)^{\frac{1}{2}} \cdot \big(d(Ax, Ty)\big)^{\frac{1}{2}}\Big\}^p\Big] \end{aligned}$$

for $x = x_{2n+2}$ and $y = x_{2n+1}$, we have

$$\begin{aligned} d^p(Ax_{2n+2}, Bx_{2n+1}) &= d^p(y_{2n+1}, y_{2n+2}) \\ &\leqslant \psi\big(ad^p(y_{2n}, y_{2n+1}) \\ &\quad + (1-a)\max\big\{d^p(y_{2n}, y_{2n+1}), d^p(y_{2n+1}, y_{2n+2}), \\ &\quad d^{\frac{p}{2}}(y_{2n+1}, y_{2n+2}) \cdot d^{\frac{p}{2}}(y_{2n}, y_{2n+2})\big\}\big). \end{aligned}$$

Therefore

$$\begin{aligned} d^p(y_{2n+1}, y_{2n+2}) \leqslant \psi\big(&ad^p(y_{2n+1}, y_{2n+2}) \\ &+ (1-a)\max\big\{d^p(y_{2n+1}, y_{2n+2}), d^p(y_{2n}, y_{2n+1}), \\ &\big[d(y_{2n+1}, y_{2n+2}) \cdot \big(d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})\big)\big]^{\frac{p}{2}}\big\}\big). \end{aligned}$$

If $d(y_{2n}, y_{2n+1}) \leqslant d(y_{2n+1}, y_{2n+2})$ in the above inequality, then

$$d^p(y_{2n+1}, y_{2n+2}) \leqslant \psi\big(ad^p(y_{2n+1}, y_{2n+2}) + (1-a)\max\big\{d^p(y_{2n+1}, y_{2n+2}), 2^{\frac{p}{2}} d^p(y_{2n+1}, y_{2n+2})\big\}\big).$$

Since $2^{\frac{p}{2}} d^p(y_{2n+1}, y_{2n+2}) > d^p(y_{2n+1}, y_{2n+2})$, it follows that

$$\begin{aligned} d^p(y_{2n+1}, y_{2n+2}) &\leqslant \psi\big(ad^p(y_{2n+1}, y_{2n+2}) + 2^{\frac{p}{2}}(1-a)d^p(y_{2n+1}, y_{2n+2})\big) \\ &= \psi\big(\big(a + 2^{\frac{p}{2}}(1-a)\big)d^p(y_{2n+1}, y_{2n+2})\big) \\ &< \big(a + 2^{\frac{p}{2}}(1-a)\big)d^p(y_{2n+1}, y_{2n+2}). \end{aligned}$$

As $a + 2^{\frac{p}{2}}(1-a) \geqslant 1$, we cannot get a contradiction. Therefore, the term $d^{\frac{p}{2}}(Ax, Sx) \cdot d^{\frac{p}{2}}(Ax, Ty)$ should be replaced by

$$\min\big\{d^{\frac{p}{2}}(Ax, Sx) \cdot d^{\frac{p}{2}}(Ax, Ty), d^{\frac{p}{2}}(By, Ty) \cdot d^{\frac{p}{2}}(Sx, By)\big\}.$$

Then, in [6], the term $(\int_0^{d(Ax,Sx)} \psi(t)\,dt)^{\frac{1}{2}} \cdot (\int_0^{d(Ax,Ty)} \psi(t)\,dt)^{\frac{1}{2}}$ should be replaced by

$$\min\left\{\left(\int_0^{d(Ax,Sx)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Ax,Ty)} \psi(t)\,dt\right)^{\frac{1}{2}}, \left(\int_0^{d(By,Ty)} \psi(t)\,dt\right)^{\frac{1}{2}} \cdot \left(\int_0^{d(Sx,By)} \psi(t)\,dt\right)^{\frac{1}{2}}\right\}$$

and so in the revised paper, the term $(F(d(Ax, Sx)))^{\frac{1}{2}} \cdot (F(d(Ax, Ty)))^{\frac{1}{2}}$ should be replaced by

$$\min\big\{\big(F\big(d(Ax, Sx)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Ax, Ty)\big)\big)^{\frac{1}{2}}, \big(F\big(d(By, Ty)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Sx, By)\big)\big)^{\frac{1}{2}}\big\}.$$

The same problem appears in [14] with the term $\frac{d^p(Ax,Ty)+d^p(Sx,By)}{2}$ if n is even and n is odd and so this term should be deleted or replaced by $\min\{d^{\frac{p}{2}}(Ax, Sx) \cdot d^{\frac{p}{2}}(Ax, Ty), d^{\frac{p}{2}}(By, Ty) \cdot d^{\frac{p}{2}}(Sx, By)\}$.

3. Main results

Let $D = \sup\{d(x, y)\colon x, y \in X\}$. Set $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$. Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X), \tag{3.1}$$

$$\begin{aligned} \big(F\big(d(Ax, By)\big)\big)^p \leqslant \psi\big[&a\big(F\big(d(Sx, Ty)\big)\big)^p \\ &+ (1-a)\max\big\{F\big(d(Ax, Sx)\big), F\big(d(By, Ty)\big), \\ &\min\big\{\big(F\big(d(Ax, Sx)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Ax, Ty)\big)\big)^{\frac{1}{2}}, \big(F\big(d(By, Ty)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Sx, By)\big)\big)^{\frac{1}{2}}\big\}, \\ &\big(F\big(d(Sx, By)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Ax, Ty)\big)\big)^{\frac{1}{2}}\big\}^p\big], \end{aligned} \tag{3.2}$$

for all x, y in X, where $0 \leqslant a \leqslant 1$, $p \geqslant 1$, $F \in \mathcal{F}[0, A)$ and $\psi \in \Psi[0, F(A-0))$.

By (3.1), we can define inductively a sequence $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \tag{3.3}$$

for all $n = 0, 1, 2, \ldots$.

Lemma 3.1. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying* (3.1) *and* (3.2). *Then, the sequence $\{y_n\}$ defined by* (3.3) *is a Cauchy sequence in X.*

Proof. First of all, assume that $y_n \neq y_{n+1}$ for all n. Applying (3.2) and (3.3) we have

$$\begin{aligned}\big(F\big(d(y_{2n}, y_{2n+1})\big)\big)^p &= \big(F\big(d(Ax_{2n}, Bx_{2n+1})\big)\big)^p \\ &\leqslant \psi\big[a\big(F\big(d(y_{2n-1}, y_{2n})\big)\big)^p + (1-a)\max\big\{F\big(d(y_{2n-1}, y_{2n})\big), F\big(d(y_{2n}, y_{2n+1})\big)\big\}^p\big].\end{aligned} \tag{3.4}$$

If $F(d(y_{2n-1}, y_{2n})) \leqslant F(d(y_{2n}, y_{2n+1}))$ in (3.4), then

$$\big(F\big(d(y_{2n}, y_{2n+1})\big)\big)^p \leqslant \psi\big(\big(F\big(d(y_{2n}, y_{2n+1})\big)\big)^p\big) < \big(F\big(d(y_{2n}, y_{2n+1})\big)\big)^p$$

which is a contradiction. Therefore

$$\big(F\big(d(y_{2n}, y_{2n+1})\big)\big)^p \leqslant \psi\big(\big(F\big(d(y_{2n-1}, y_{2n})\big)\big)^p\big).$$

Similarly, we get

$$\big(F\big(d(y_{2n+1}, y_{2n+2})\big)\big)^p \leqslant \psi\big(\big(F\big(d(y_{2n}, y_{2n+1})\big)\big)^p\big).$$

By induction, we obtain

$$\big(F\big(d(y_n, y_{n+1})\big)\big)^p \leqslant \psi\big(\big(F\big(d(y_{n-1}, y_n)\big)\big)^p\big) \leqslant \cdots \leqslant \psi^n\big(\big(F\big(d(y_0, y_1)\big)\big)^p\big).$$

Using Lemma 2.9, it follows that

$$\lim_{n\to\infty} F\big(d(y_n, y_{n+1})\big) = 0, \tag{3.5}$$

and Lemma 2.7 implies that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{3.6}$$

Now, we show that $\{y_n\}$ is a Cauchy sequence in X. By (3.6), it suffices to show that the subsequence $\{y_{2n}\}$ of $\{y_n\}$ is a Cauchy sequence in X. Suppose not. As in [5] we have

$$d(y_{2n(k)}, y_{2m(k)}) \to \varepsilon \quad \text{as } k \to \infty, \tag{3.7}$$

$$d(y_{2n(k)}, y_{2m(k)-1}) \to \varepsilon \quad \text{and} \quad d(y_{2n(k)+1}, y_{2m(k)-1}) \to \varepsilon \quad \text{as } k \to \infty. \tag{3.8}$$

Using (3.3) we get

$$d(y_{2n(k)}, y_{2m(k)}) \leqslant d(y_{2n(k)}, y_{2n(k)+1}) + d(Ax_{2m(k)}, Bx_{2n(k)+1}).$$

By (3.2) and (3.6) we obtain

$$\begin{aligned}\lim_{k\to\infty} F\big(d(y_{2n(k)}, y_{2m(k)})\big) &\leqslant \lim_{k\to\infty} F\big(d(Ax_{2m(k)}, Bx_{2n(k)+1})\big) \\ &\leqslant \lim_{k\to\infty}\big(\psi\big[a\big(F\big(d(Sx_{2m(k)}, Tx_{2n(k)+1})\big)\big)^p \\ &\quad + (1-a)\max\big\{F\big(d(Ax_{2m(k)}, Sx_{2m(k)})\big), F\big(d(Bx_{2n(k)+1}, Tx_{2n(k)+1})\big), \\ &\quad \big(F\big(d(Ax_{2m(k)}, Sx_{2m(k)})\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Ax_{2m(k)}, Tx_{2n(k)+1})\big)\big)^{\frac{1}{2}}, \\ &\quad \big(F\big(d(Sx_{2m(k)}, Bx_{2n(k)+1})\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Ax_{2m(k)}, Tx_{2n(k)+1})\big)\big)^{\frac{1}{2}}\big\}\big]^p\big)^{\frac{1}{p}} \\ &= \lim_{k\to\infty}\big(\psi\big[a\big(F\big(d(y_{2m(k)-1}, y_{2n(k)})\big)\big)^p \\ &\quad + (1-a)\max\big\{F\big(d(y_{2m(k)}, y_{2m(k)-1})\big), F\big(d(y_{2n(k)+1}, y_{2n(k)})\big), \\ &\quad \big(F\big(d(y_{2m(k)}, y_{2m(k)-1})\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(y_{2m(k)}, y_{2n(k)})\big)\big)^{\frac{1}{2}}, \\ &\quad \big(F\big(d(y_{2m(k)-1}, y_{2n(k)+1})\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(y_{2m(k)}, y_{2n(k)})\big)\big)^{\frac{1}{2}}\big\}\big]^p\big)^{\frac{1}{p}}.\end{aligned} \tag{3.9}$$

Applying (3.5), (3.7), (3.8) and (3.9) we find as $k \to \infty$

$$F(\epsilon) \leqslant \big[\psi\big(a\big(F(\epsilon)\big)^p + (1-a)\big(F(\epsilon)\big)^p\big)\big]^{\frac{1}{p}} < F(\epsilon)$$

which is a contradiction. Hence, $\{y_n\}$ is a Cauchy sequence in X. □

Theorem 3.2. *Let A, B, S and T be mappings from a metric space (X,d) into itself satisfying* (3.1) *and* (3.2). *Suppose that one of $S(X)$ or $T(X)$ or $B(X)$ or $A(X)$ is complete and the pairs (A,S) and (B,T) are weakly compatible. Then, A, B, S and T have a unique common fixed point in X.*

Proof. By Lemma 3.1, the sequence $\{y_{2n+1}\} = \{Sx_{2n+2}\} \subset S(X)$ is a Cauchy sequence in $S(X)$. Since $S(X)$ is complete, it converges to a point $z = Su$ for some $u \in X$. Therefore, the subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Tx_{2n+1}\}$ also converge to z. If $Au \neq z$, using (3.2) we get

$$\begin{aligned}\big(F\big(d(Au, Bx_{2n+1})\big)\big)^p \leqslant \psi\big[&a\big(F\big(d(Su, Tx_{2n+1})\big)\big)^p \\ &+ (1-a)\max\big\{F\big(d(Au, Su)\big), F\big(d(Bx_{2n+1}, Tx_{2n+1})\big), \\ &\big(F\big(d(Au, Su)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Au, Tx_{2n+1})\big)\big)^{\frac{1}{2}}, \\ &\big(F\big(d(Su, Bx_{2n+1})\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Au, Tx_{2n+1})\big)\big)^{\frac{1}{2}}\big\}^p\big].\end{aligned}$$

Letting $n \to \infty$ we obtain

$$\big(F\big(d(Au, z)\big)\big)^p \leqslant \psi\big((1-a)\big(F\big(d(Au, z)\big)\big)^p\big) < \big(F\big(d(Au, z)\big)\big)^p$$

which is a contradiction. Then, $z = Au = Su$. As $A(X) \subset T(X)$, there exists a $v \in X$ such that $z = Tv$. If $z \neq Bv$, applying (3.2) we have

$$\begin{aligned}\big(F\big(d(z, Bv)\big)\big)^p &= \big(F\big(d(Au, Bv)\big)\big)^p \\ &\leqslant \psi\big[a\big(F\big(d(Su, Tv)\big)\big)^p \\ &\quad + (1-a)\max\big\{F\big(d(Au, Su)\big), F\big(d(Bv, Tv)\big), \big(F\big(d(Au, Su)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Au, Tv)\big)\big)^{\frac{1}{2}}, \\ &\quad \big(F\big(d(Su, Bv)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Au, Tv)\big)\big)^{\frac{1}{2}}\big\}^p\big] \\ &= \psi\big((1-a)\big(F\big(d(z, Bv)\big)\big)^p\big) \\ &< \big(F\big(d(z, Bv)\big)\big)^p\end{aligned}$$

which is a contradiction. Therefore, $z = Bv = Tv$. As the pair (A,S) is weakly compatible, we have $SAu = ASu$; i.e., $Az = Sz$. If $Az \neq z$, using (3.2) we obtain

$$\begin{aligned}\big(F\big(d(Az, z)\big)\big)^p &= \big(F\big(d(Az, Bv)\big)\big)^p \\ &\leqslant \psi\big[a\big(F\big(d(Sz, Tv)\big)\big)^p \\ &\quad + (1-a)\max\big\{F\big(d(Az, Sz)\big), F\big(d(Bv, Tv)\big), \big(F\big(d(Az, Sz)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Az, Tv)\big)\big)^{\frac{1}{2}}, \\ &\quad \big(F\big(d(Sz, Bv)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Az, Tv)\big)\big)^{\frac{1}{2}}\big\}^p\big] \\ &= \psi\big(\big(F\big(d(Az, z)\big)\big)^p\big) \\ &< \big(F\big(d(Az, z)\big)\big)^p\end{aligned}$$

which is a contradiction. So, $z = Az = Sz$. Similarly, we can prove that $z = Bz = Tz$. The same result of Theorem 3.2 holds if we assume that $T(X)$ or $B(X)$ or $A(X)$ is complete instead of $S(X)$.

Suppose there exists an n such that $y_n = y_{n+1}$. Therefore, $y_n = y_{n+k}$ for $k \geqslant 1$ and so there exists $u, v \in X$ such that $Au = Su$ and $Bv = Tv$. As in Theorem 3.2, we can prove that $z = Az = Bz = Tz$. The uniqueness of z follows from (3.2). □

If $F(t) = \int_0^t \varphi(s)\,ds$ in Theorem 3.2, where $t \in [0, A)$, $\int_0^\epsilon \varphi(t)\,dt > 0$ for each $\epsilon \in (0, A)$, we get Theorem 3 of [6].

If $F(t) = t$ in Theorem 3.2, where $t \in [0, A)$, we obtain Theorem 7 of [5].

If $B = A$ and $T = S$ in Theorem 3.2, we get a corollary which generalizes Corollary 1 of [6].

If $p = a = 1$, $S = T = I_X$ and $F(t) = \int_0^t \varphi(s)\,ds$ in Theorem 3.2, where $t \in [0, A)$, $\int_0^\epsilon \varphi(t)\,dt > 0$ for each $\epsilon \in (0, A)$, we obtain Lemma 1 of [20].

If $p = a = 1$, $B = A$ and $S = T = I_X$ in Theorem 3.2, we obtain a corollary which generalizes Theorem 2.1 of [4].

Now, we prove a common fixed point theorem of Gregus type using a strict contraction of integral type and property (E.A) which generalizes Theorem 4 of [6].

Theorem 3.3. *Let A, B, S and T be mappings from a metric space (X,d) into itself satisfying* (3.1) *and*

$$F\big(d(Ax,By)\big) < aF\big(d(Sx,Ty)\big) + (1-a)\max\Big\{F\big(d(Ax,Sx)\big), F\big(d(By,Ty)\big), \big(F\big(d(Ax,Sx)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Ax,Ty)\big)\big)^{\frac{1}{2}}, \big(F\big(d(Sx,By)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Ax,Ty)\big)\big)^{\frac{1}{2}}\Big\}, \tag{3.10}$$

for all x, y in X for which the right-hand side of (3.10) *is positive, where $0 < a < 1$ and $F \in \mathcal{F}[0,A)$. Suppose that (A,S) or (B,T) satisfies property* (E.A)*, one of $A(X)$, $B(X)$, $S(X)$, $T(X)$ is a closed subspace of X and the pairs (A,S) and (B,T) are weakly compatible. Then, A, B, S and T have a unique common fixed point in X.*

Proof. Suppose that (B,T) satisfies property (E.A). Then, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Therefore, $\lim_{n\to\infty} d(Bx_n, Tx_n) = 0$. Since $B(X) \subset S(X)$, there exists in X a sequence $\{y_n\}$ such that $Bx_n = Sy_n$. Hence, $\lim_{n\to\infty} Sy_n = z$. Let us show that $\lim_{n\to\infty} Ay_n = z$.

Suppose that $\limsup_{n\to\infty} d(Ay_n, z) = \varepsilon > 0$. Applying (3.10) we get

$$\begin{aligned} F\big(d(Ay_n,Bx_n)\big) &< aF\big(d(Sy_n,Tx_n)\big)^p \\ &\quad + (1-a)\max\Big\{F\big(d(Ay_n,Sy_n)\big), F\big(d(Bx_n,Tx_n)\big), \\ &\quad \big(F\big(d(Ay_n,Sy_n)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Ay_n,Tx_n)\big)\big)^{\frac{1}{2}}, \big(F\big(d(Sy_n,Bx_n)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Ay_n,Tx_n)\big)\big)^{\frac{1}{2}}\Big\} \\ &= aF\big(d(Bx_n,Tx_n)\big) \\ &\quad + (1-a)\max\Big\{F\big(d(Ay_n,Bx_n)\big), F\big(d(Bx_n,Tx_n)\big), \\ &\quad \big(F\big(d(Ay_n,Bx_n)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Ay_n,Tx_n)\big)\big)^{\frac{1}{2}}\Big\}. \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$F(\epsilon) \leqslant (1-a)F(\epsilon) < F(\epsilon)$$

which is a contradiction. Hence, $\varepsilon = 0$; i.e., $\lim_{n\to\infty} Ay_n = z$. Suppose that $S(X)$ is a closed subspace of X. Then, $z = Su$ for some $u \in X$. If $Au \neq z$, using (3.10) we get

$$\begin{aligned} F\big(d(Au,Bx_{2n+1})\big) &< aF\big(d(Su,Tx_{2n+1})\big) \\ &\quad + (1-a)\max\Big\{F\big(d(Au,Su)\big), F\big(d(Bx_{2n+1},Tx_{2n+1})\big), \\ &\quad \big(F\big(d(Au,Su)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Au,Tx_{2n+1})\big)\big)^{\frac{1}{2}}, \\ &\quad \big(F\big(d(Su,Bx_{2n+1})\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Au,Tx_{2n+1})\big)\big)^{\frac{1}{2}}\Big\}. \end{aligned}$$

Letting $n \to \infty$ we obtain

$$F\big(d(Au,z)\big) \leqslant (1-a)F\big(d(Au,z)\big) < F\big(d(Au,z)\big)$$

which is a contradiction. Then, $z = Au = Su$. Since $A(X) \subset T(X)$, there exists a $v \in X$ such that $z = Tv$. If $z \neq Bv$, applying (3.10) we have

$$\begin{aligned} F\big(d(z,Bv)\big) &= F\big(d(Au,Bv)\big) \\ &< aF\big(d(Su,Tv)\big) \\ &\quad + (1-a)\max\Big\{F\big(d(Au,Su)\big), F\big(d(Bv,Tv)\big), \big(F\big(d(Au,Su)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Au,Tv)\big)\big)^{\frac{1}{2}}, \\ &\quad \big(F\big(d(Su,Bv)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Au,Tv)\big)\big)^{\frac{1}{2}}\Big\} \end{aligned}$$

$$= (1-a)F\big(d(z, Bv)\big)$$
$$< F\big(d(z, Bv)\big)$$

which is a contradiction. Therefore, $z = Bv = Tv$. As the pair (A, S) is weakly compatible, we have $SAu = ASu$; i.e., $Az = Sz$. If $Az \neq z$, using (3.10) we obtain

$$\begin{aligned} F\big(d(Az, z)\big) &= F\big(d(Az, Bv)\big) \\ &< aF\big(d(Sz, Tv)\big) \\ &\quad + (1-a)\max\big\{F\big(d(Az, Sz)\big), F\big(d(Bv, Tv)\big), \big(F\big(d(Az, Sz)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Az, Tv)\big)\big)^{\frac{1}{2}}, \\ &\quad \big(F\big(d(Sz, Bv)\big)\big)^{\frac{1}{2}} \cdot \big(F\big(d(Az, Tv)\big)\big)^{\frac{1}{2}}\big\} \\ &= F\big(d(Az, z)\big) \end{aligned}$$

which is a contradiction. Hence, $z = Az = Sz$. In the same manner, we can prove that $z = Bz = Tz$. The same result of Theorem 3.3 holds if we assume that $S(X)$ or $B(X)$ or $T(X)$ is complete instead of $A(X)$. The uniqueness of z follows from (3.10). □

Now, we prove a common fixed point theorem of Gregus type using a strict contraction of integral type and a common property (E.A) which generalizes Theorem 5 of [6].

Theorem 3.4. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying* (3.10) *for which the right-hand side of* (3.10) *is positive. Suppose that (A, S) and (B, T) satisfy a common property* (E.A)*, $S(X)$ and $T(X)$ are closed subspaces of X and the pairs (A, S) and (B, T) are weakly compatible. Then, A, B, S and T have a unique common fixed point in X.*

Proof. Suppose that (A, S) and (B, T) satisfy a common property (E.A). Then, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$ for some $z \in X$. Assume that $S(X)$ and $T(X)$ are closed subspaces of X. Therefore, $z = Su = Tv$ for some $u, v \in X$. If $Au \neq z$, Applying (3.10) we get

$$\begin{aligned} F\big(d(Au, By_n)\big) &< aF\big(d(Su, Ty_n)\big) \\ &\quad + (1-a)\max\big\{F\big(d(Au, Su)\big), F\big(d(By_n, Ty_n)\big), F\big(d(Au, Su)\big)^{\frac{1}{2}} \cdot F\big(d(Au, Ty_n)\big)^{\frac{1}{2}}, \\ &\quad F\big(d(Su, By_n)\big)^{\frac{1}{2}} \cdot F\big(d(Au, Ty_n)\big)^{\frac{1}{2}}\big\}. \end{aligned}$$

Letting $n \to \infty$ we obtain

$$F\big(d(Au, z)\big) \leqslant (1-a)F\big(d(Au, z)\big) < F\big(d(Au, z)\big)$$

which is a contradiction. So, $z = Au = Su$. The rest of the proof follows as in Theorem 3.3. □

Similarly, we can prove the following theorem which generalizes Theorem 2.1 of [3] and Theorem 1 of [24].

Theorem 3.5. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying* (3.1) *and*

$$F\big(d(Ax, By)\big) \leqslant \psi\left(F\left(\max\left\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{d(Ax, Ty) + d(Sx, By)}{2}\right\}\right)\right)$$

for all x, y in X, $F \in \digamma[0, A)$ and $\psi \in \Psi[0, F(A-0))$. Suppose that one of $A(X)$ or $B(X)$ or $S(X)$ or $T(X)$ is complete and the pairs (A, S) and (B, T) are weakly compatible. Then, A, B, S and T have a unique common fixed point in X.

Now, we are going to generalize Theorem 3.5 of [18].

Lemma 3.6. *Let A, B, S and T be self-mappings of a metric space (X,d) satisfying* (3.1) *and*

$$\begin{aligned}\big[F\big(d(Ax,By)\big)\big]^{2p} \leqslant{}& a\psi_0\big(\big[F\big(d(Sx,Ty)\big)\big]^{2p}\big)\\ &+(1-a)\max\Big\{\psi_1\big(\big[F\big(d(Sx,Ty)\big)\big]^{2p}\big), \psi_2\big(\big[F\big(d(Ax,Sx)\big)\big]^{q}\cdot\big[F\big(d(By,Ty)\big)\big]^{q'}\big),\\ &\psi_3\big(\big[F\big(d(Sx,By)\big)\big]^{r}\cdot\big[F\big(d(Ax,Ty)\big)\big]^{r'}\big),\\ &\psi_4\Big(\frac{1}{2}\big[F\big(d(Ax,Sx)\big)\big]^{s}\cdot\big[F\big(d(Ax,Ty)\big)\big]^{s'}\Big),\\ &\psi_5\Big(\frac{1}{2}\big[F\big(d(By,Ty)\big)\big]^{l}\cdot\big[F\big(d(Sx,By)\big)\big]^{l'}\Big)\Big\}\end{aligned}\tag{3.11}$$

for all $x,y\in X$, where $\psi_i\in\Psi[0,F(A-0))$, $F\in F[0,A)$ satisfying $F(2t)\leqslant 2F(t)$ for all $t>0$, $i=0,1,2,3,4,5$, $0\leqslant a\leqslant 1$ and $0<p,q,q',r,r',s,s'',l,l'\leqslant 1$, such that $2p=q+q'=r+r'=s+s'=l+l'$. Then, the sequence $\{y_n\}$ defined by (3.3) *is a Cauchy sequence in X.*

Proof. First of all, assume that $y_n\neq y_{n+1}$ for all n. Using (3.3) and (3.11) we have

$$\begin{aligned}\big(F\big(d(y_{2n},y_{2n+1})\big)\big)^{p} ={}& \big(F\big(d(Ax_{2n},Bx_{2n+1})\big)\big)^{2p}\\ \leqslant{}& a\psi_0\big(\big[F\big(d(y_{2n-1},y_{2n})\big)\big]^{2p}\big)\\ &+(1-a)\max\Big\{\psi_1\big(\big[F\big(d(y_{2n-1},y_{2n})\big)\big]^{2p}\big),\\ &\psi_2\big(\big[F\big(d(y_{2n-1},y_{2n})\big)\big]^{q}\cdot\big[F\big(d(y_{2n},y_{2n+1})\big)\big]^{q'}\big),\\ &\psi_5\Big(\frac{1}{2}\big[F\big(d(y_{2n},y_{2n+1})\big)\big]^{l}\cdot\big[F\big(d(y_{2n-1},y_{2n+1})\big)\big]^{l'}\Big)\Big\}\\ \leqslant{}& a\psi_0\big(\big[F\big(d(y_{2n-1},y_{2n})\big)\big]^{2p}\big)\\ &+(1-a)\max\Big\{\psi_1\big(\big[F\big(d(y_{2n-1},y_{2n})\big)\big]^{2p}\big),\\ &\psi_2\big(\big[F\big(d(y_{2n-1},y_{2n})\big)\big]^{q}\cdot\big[F\big(d(y_{2n},y_{2n+1})\big)\big]^{q'}\big),\\ &\psi_5\Big(\frac{1}{2}\big[F\big(d(y_{2n},y_{2n+1})\big)\big]^{l}\cdot\big(\big[F\big(d(y_{2n-1},y_{2n})\big)\big]^{l'}+\big[F\big(d(y_{2n},y_{2n+1})\big)\big]^{l'}\big)\Big)\Big\}.\end{aligned}$$

If $F(d(y_{2n-1},y_{2n}))\leqslant F(d(y_{2n},y_{2n+1}))$ in the above inequality, then

$$\begin{aligned}\big(F\big(d(y_{2n},y_{2n+1})\big)\big)^{2p} \leqslant{}& a\psi_0\big(\big[F\big(d(y_{2n},y_{2n+1})\big)\big]^{2p}\big)\\ &+(1-a)\max\big\{\psi_1\big(\big[F\big(d(y_{2n},y_{2n+1})\big)\big]^{2p}\big), \psi_2\big(\big[F\big(d(y_{2n},y_{2n+1})\big)\big]^{2p}\big),\\ &\psi_5\big(\big[F\big(d(y_{2n},y_{2n+1})\big)\big]^{2p}\big)\big\}.\end{aligned}$$

Applying Lemma 2.10, it follows that

$$\begin{aligned}\big(F\big(d(y_{2n},y_{2n+1})\big)\big)^{2p} &\leqslant a\psi\big(\big[F\big(d(y_{2n},y_{2n+1})\big)\big]^{2p}\big)+(1-a)\psi\big(\big[F\big(d(y_{2n},y_{2n+1})\big)\big]^{2p}\big)\\ &< \big(F\big(d(y_{2n},y_{2n+1})\big)\big)^{2p}\end{aligned}$$

which is a contradiction. Therefore

$$\big(F\big(d(y_{2n},y_{2n+1})\big)\big)^{p}\leqslant\psi\big(\big(F\big(d(y_{2n-1},y_{2n})\big)\big)^{p}\big).$$

In the same manner, we get

$$\big(F\big(d(y_{2n+1},y_{2n+2})\big)\big)^{p}\leqslant\psi\big(\big(F\big(d(y_{2n},y_{2n+1})\big)\big)^{p}\big).$$

The rest of the proof follows as in Lemma 3.1. Hence, $\{y_n\}$ is a Cauchy sequence in X. □

Theorem 3.7. *Let A, B, S and T be mappings from a metric space (X, d) into itself satisfying* (3.1) *and* (3.11). *Suppose that one of $S(X)$ or $T(X)$ or $A(X)$ or $B(X)$ is complete and the pairs (A, S) and (B, T) are weakly compatible. Then A, B, S and T have a unique common fixed point in X.*

Proof. By Lemma 3.6, the sequence $\{y_{2n+1}\} = \{Sx_{2n+2}\} \subset S(X)$ is a Cauchy sequence in $S(X)$. Since $S(X)$ is complete, it converges to a point $z = Su$ for some $u \in X$. Therefore, the subsequences $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Tx_{2n+1}\}$ also converge to z. If $Au \neq z$, using (3.11) we get

$$\begin{aligned}
\big[F\big(d(Au, Bx_{2n+1})\big)\big]^{2p} \leqslant{}& a\psi_0\big(\big[F\big(d(Su, Tx_{2n+1})\big)\big]^{2p}\big) \\
&+ (1-a)\max\Big\{\psi_1\big(\big[F\big(d(Su, Tx_{2n+1})\big)\big]^{2p}\big), \\
&\psi_2\big(\big[F\big(d(Au, Su)\big)\big]^{q} \cdot \big[F\big(d(Bx_{2n+1}, Tx_{2n+1})\big)\big]^{q'}\big), \\
&\psi_3\big(\big[F\big(d(Su, Bx_{2n+1})\big)\big]^{r} \cdot \big[F\big(d(Au, Tx_{2n+1})\big)\big]^{r'}\big), \\
&\psi_4\Big(\frac{1}{2}\big[F\big(d(Au, Su)\big)\big]^{s} \cdot \big[F\big(d(Au, Tx_{2n+1})\big)\big]^{s'}\Big), \\
&\psi_5\Big(\frac{1}{2}\big[F\big(d(Bx_{2n+1}, Tx_{2n+1})\big)\big]^{l} \cdot \big[F\big(d(Su, Bx_{2n+1})\big)\big]^{l'}\Big)\Big\}.
\end{aligned}$$

Letting $n \to \infty$ we obtain

$$\big[F\big(d(Au, z)\big)\big]^{2p} \leqslant (1-a)\psi\Big(\frac{1}{2}\big[F\big(d(Au, z)\big)\big]^{2p}\Big) < \big[F\big(d(Au, z)\big)\big]^{2p}$$

which is a contradiction. Then $z = Au = Su$. As $A(X) \subset T(X)$, there exists a $v \in X$ such that $z = Tv$. If $z \neq Bv$, applying (3.11) we have

$$\big[F\big(d(z, Bv)\big)\big]^{2p} = \big[F\big(d(Au, Bv)\big)\big]^{2p} \leqslant \psi_5\Big(\frac{1}{2}\big[F\big(d(z, Bv)\big)\big]^{2p}\Big)$$

which is a contradiction. Therefore, $z = Bv = Tv$. As the pair (A, S) is weakly compatible, we have $SAu = ASu$; i.e., $Az = Sz$. If $Az \neq z$, using (3.11) we obtain

$$\begin{aligned}
\big[F\big(d(Az, Bv)\big)\big]^{2p} &\leqslant a\psi_0\big(\big[F\big(d(Sz, Tv)\big)\big]^{2p}\big) \\
&\quad + (1-a)\max\big\{\psi_1\big(\big[F\big(d(Sz, Tv)\big)\big]^{2p}\big), \psi_3\big(\big[F\big(d(Sz, Bv)\big)\big]^{r} \cdot \big[F\big(d(Az, Tv)\big)\big]^{r'}\big)\big\} \\
&= a\psi\big(\big[F\big(d(Az, z)\big)\big]^{2p}\big) + (1-a)\psi\big(\big[F\big(d(Az, z)\big)\big]^{2p}\big) \\
&< \big[F\big(d(Az, z)\big)\big]^{2p}
\end{aligned}$$

which is a contradiction. So, $z = Az = Sz$. Similarly, we can prove that $z = Bz = Tz$. The same result of Theorem 3.2 holds if we assume that $T(X)$ or $B(X)$ or $A(X)$ is complete instead of $S(X)$. The uniqueness of z follows from (3.11). □

If $F(t) = \int_0^t \varphi(s)\,ds$ in Theorem 3.7, where $t \in [0, A)$, $\int_0^\epsilon \varphi(t)\,dt > 0$ for each $\epsilon \in (0, A)$, we get Theorem 3 of [6] or $F(t) = t$ in Theorem 3.7, where $t \in [0, A)$, we obtain a generalization of a Theorem 3.5 of [18].

The following example shows that Theorem 3.2 is a generalization of Theorem 7 of [5] if $\psi(t) = ht$ for all $t > 0$, $h \in [0, 1)$.

Example 3.8. Let $X = \{\frac{1}{n}\colon n \in \mathbb{N}^*\} \cup \{0\}$ with the Euclidean metric and A, B, S and T are self mappings of X defined by

$$A\left(\frac{1}{n}\right)=\begin{cases}\frac{1}{n+1}, & \text{if } n \text{ is odd},\\ \frac{1}{n+2}, & \text{if } n \text{ is even},\\ 0, & \text{if } n=\infty,\end{cases}\qquad B\left(\frac{1}{n}\right)=\begin{cases}\frac{1}{n+1}, & \text{if } n \text{ is even},\\ \frac{1}{n+2}, & \text{if } n \text{ is odd},\\ 0, & \text{if } n=\infty,\end{cases}$$

$$S\left(\frac{1}{n}\right)=T\left(\frac{1}{n}\right)=\frac{1}{n}\quad \text{for all } n\in\mathbb{N}^*\cup\{\infty\}.$$

Clearly, $D=1$, $A(X)\subset T(X)$, $B(X)\subset S(X)$; $S(X)$ is a complete subspace of X and the pairs (A,S) and (B,T) are weakly compatible.

Now suppose that the contractive condition of Theorem 7 of [5] is satisfied if $\psi(t)=ht$ for all $t>0$; i.e., there exists $h\in[0,1)$ such that $d^p(Ax,By)\leqslant hM_p(x,y)$ for all $x;y\in X$, where

$$\begin{aligned}M_p(x,y)=ad^p(Sx,Ty)+(1-a)\max\Big\{&d^p(Ax,Sx),d^p(By,Ty),\\ &\min\big\{d^{\frac{p}{2}}(Ax,Sx)\cdot d^{\frac{p}{2}}(Ax,Ty),d^{\frac{p}{2}}(By,Ty)\cdot d^{\frac{p}{2}}(Sx,By)\big\},\\ &d^{\frac{p}{2}}(Sx,By)\cdot d^{\frac{p}{2}}(Ax,Ty),\frac{d^p(Ax,Sx)+d^p(By,Ty)}{2}\Big\}.\end{aligned}$$

Therefore, for all $x\neq y$ we have $\frac{d^p(Ax,By)}{M_p(x,y)}\leqslant h<1$. Using Example 4 of [20] we obtain

$$\sup_{x\neq y}\frac{d^p(Ax,By)}{M_p(x,y)}=\sup_{m\in\mathbb{N}^*}\frac{\frac{1}{(m+1)^p(m+2)^p}}{a\frac{1}{m^p(m+1)^p}+(1-a)\frac{1}{m^p(m+1)^p}}=\sup_{m\in\mathbb{N}^*}\frac{m^p}{(m+2)^p}=1.$$

So, there is no $h\in[0,1)$ such that $d^p(Ax,By)\leqslant hM_p(x,y)$. Hence, Theorem 7 of [5] cannot be used if $\psi(t)=ht$ for all $t>0$. On the other hand, the inequality (3.2) is satisfied. To see this, let $F(s)=s^{\frac{1}{s}}$ and $\psi(t)=\frac{t}{2}$. Then $F\in\digamma[0,A)$ and $\psi\in\Psi[0,e^{\frac{1}{e}}]$, where $A=e>D$. Using Example 4 of [20] we have

$$d(Ax,By)\leqslant\left|\frac{1}{n+1}-\frac{1}{m+1}\right|.$$

Therefore

$$F\big(d(Ax,By)\big)\leqslant F\left(\left|\frac{1}{n+1}-\frac{1}{m+1}\right|\right)=\left|\frac{1}{n+1}-\frac{1}{m+1}\right|^{\frac{1}{|\frac{1}{n+1}-\frac{1}{m+1}|}}.$$

Using Example 3.6 of [4], we get

$$F\big(d(Ax,By)\big)\leqslant\frac{1}{2}\left|\frac{1}{n}-\frac{1}{m}\right|^{\frac{1}{|\frac{1}{n}-\frac{1}{m}|}}=\frac{1}{2}F\big(d(Sx,Ty)\big)$$

and so

$$\begin{aligned}\big[F\big(d(Ax,By)\big)\big]^p&\leqslant\frac{1}{2^p}\big[F\big(d(Sx,Ty)\big)\big]^p\\ &\leqslant\frac{1}{2}\big[F\big(d(Sx,Ty)\big)\big]^p\\ &=\psi\big(\big[F\big(d(Sx,Ty)\big)\big]^p\big)\\ &\leqslant\psi\big[a\big(F\big(d(Sx,Ty)\big)\big)^p\\ &\quad+(1-a)\max\big\{F\big(d(Ax,Sx)\big),F\big(d(By,Ty)\big),\\ &\quad\min\big\{\big(F\big(d(Ax,Sx)\big)\big)^{\frac{1}{2}}\cdot\big(F\big(d(Ax,Ty)\big)\big)^{\frac{1}{2}},\big(F\big(d(By,Ty)\big)\big)^{\frac{1}{2}}\cdot\big(F\big(d(Sx,By)\big)\big)^{\frac{1}{2}}\big\},\\ &\quad\big(F\big(d(Sx,By)\big)\big)^{\frac{1}{2}}\cdot\big(F\big(d(Ax,Ty)\big)\big)^{\frac{1}{2}}\big\}^p\big].\end{aligned}$$

In the same manner, we can prove that the inequality (3.11) is satisfied, but Theorem 3.5 of [18] cannot be used if $\psi_i(t)=ht$, $i=0,1,2,3,4,5$, for all $t>0$, where $h\in[0,1)$.

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