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UNDER THE SUPERVISION OF
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Entitled

Study of Some Nonlinear Evolution Boundary Value Problems with Nonlocal Conditions

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ملخص

في هذه الرسالة ، تم النظر في تطوير طريقة "المتراجحات الطاقوية" ، مطبقة على فئة معينة من المسائل الحدودية الزمنية الكسرية (TFBVPs) بشروط حدودية غير محلية.

أولاً ، أنشأنا صيغة تكامل معممة شبيهة بعبارة Leibniz للعمل على حل المسألة المرتبطة بشروط حدودية تكاملية بحتة. بعد ذلك ، نقوم بجعل المسألة TFBVP متجانسة عندما ترفق بشروط حدودية مختلطة من نوع Neumann-Integral لإثبات وجود حلها القوي ووحدانيته.

نعطي أمثلة منهجية لتوضيح أهمية النتائج التي تم الحصول عليها.

الكلمات المفتاحية: مسألة زمنية كسرية ، تقدير مسبق ، شروط غير محلية ، حل قوي

Abstract

In this dissertation, the development of the " Energy Inequality " method, applied to a certain class of time fractional boundary value problems with nonlocal boundary conditions, is considered.

First, we establish a generalized Leibniz-like formula of integration to work out the solvability of the problem asociated purely integral conditions. Then, we homogenize the problem with mixed boundary conditions of type Neumann-Integral to prove the existence and uniqueness of its strong solution. We systematically give examples to illustrate the usefulness of the obtained results.

Keywords : Time fractional problem, a priori estimate, nonlocal boundary conditions, strong solution

Résumé

Dans cette thèse, le développement de la méthode des " Inégalités énergétiques ", appliquée à une certaine classe de problèmes d'évolution fractionnaires avec des conditions aux limites non locales, est considéré.

Premièrement, nous établissons une formule d'intégration généralisée de type Leibniz pour déterminer la résolvabilité du problème quand associé des conditions purement intégrales. Ensuite, nous homogénéisons le problème avec des conditions aux limites mixtes de type Neumann-Integral pour prouver l'existence et l'unicité de sa solution forte. Nous donnons systématiquement des exemples pour illustrer l'utilité des résultats obtenus.

Mots-clés : Problème d'évolution fractionnaire, estimation à priori, conditions aux limites non locales, solution forte

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K. Aggoun

Dedication

To my parents, brother and sisters

To my wife, son and daughter

Contents

Notations and conventions

- α is a real number in $(0,1)$.
- T is a known positive real number.
- Q is the domain $(0,1)\times(0,T)$.
- $\lceil x \rceil$ is the ceiling function.
- $\delta_{k,j}$ is the Kronecker delta.
- $\|f\|_X$ is the norm of the function f in the space X.
- $X \oplus Y$ is the direct sum.
- $\overline{X}, \overline{X}^Y$ is the closure of X (in Y).
- $M^\perp$ is the orthogonal complement of M.
- $span(S)$ is the linear span (hull) of the set S.
- $\begin{pmatrix} m \\ n \end{pmatrix}$ is the binomial coefficient.
- $C^k\left(\overline{\Omega}\right)$ denotes the space of k-fold differentiable functions on $\overline{\Omega}$.
- $L^2(\Omega)$ is the space of measurable square-integrable functions on Ω.
- B is a Banach space.
- H is a Hilbert space.
- $C_0(\Omega)$ is the space of continuous functions with compact support in Ω.
- $\Im_x^k u$ is the primitivation operator of order k with respect to the space variable x.
- B_2^k is the Bouziani space.
- $L^2\left(B_2^k, \Omega\right)$ is the Hilbert space of measurable square integrable functions on Ω taking values in B_2^k.
- $D'(\Omega)$ is the space of distributions.
- $\hookrightarrow$ denotes the canonical injection.
- I^α denotes the Riemann-Liouville integral operator.
- ∂_{0t}^α denotes the **C**aputo **t**ime fractional derivative of order $\boldsymbol{\alpha}$ with lower bound **0**.
- E_α and $E_{\alpha,\beta}$ are the Mittag-Leffler functions.
- $D(\mathscr{L})$ the domain of definition of the operator $\mathscr{L}$.
- $G(\mathscr{L})$ the graph of the operator $\mathscr{L}$.
- $R(\mathscr{L})$ the range of the operator $\mathscr{L}$.

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Introduction

We study in the present thesis certain classes of fractional boundary value problems (FBVPs) with nonlocal boundary conditions using the so called "energy inequality" method, also known as the "a priori estimate" method. It contains original and interesting results that have been discussed in national/international conferences and published in renowed international journals. More precisely, we give an important development of the "energy inequality" method to new classes of non-classical fractional problems described in the rectangle $Q = (0,1) \times (0,T)$, $T > 0$, by the fractional partial differential equation (FPDE)

$$\partial_{0t}^{\alpha} u + (-1)^m \frac{\partial^m}{\partial x^m}\left(a(x,t)\frac{\partial^m u}{\partial x^m}\right) = f(x,t), \tag{1}$$

where $m \in \mathbb{N}^*$, $\partial_{0t}^{\alpha} u$ denotes the Caputo time fractional derivative of order $0 < \alpha < 1$ with lower bound 0.

There are plenty of problems involving fractional derivatives and nonlocal conditions, we can cite [2], [6], [1], [28], [5] and [27], and

many instances of problems described by equation (1) have been also investigated. For example the case $m = 1$ in [14], [3] and [17]. Other authors studied the integer order case ([7], [16], [9], [18] and [19]), but only few of them were in the fractional order case.

The fractional calculus

Computational and Fractional Analysis nowadays are more and more in the center of mathematics and of other related sciences either by themselves because of their rapid development, which is based on very old foundations, or because they cover a great variety of applications in the real world.

Fractional Calculus has a history of more than 300 years, yet its applicability in different domains has been realised only recently. In the last three decades, the subject witnessed exponential growth and a number of researchers around the globe are actively working on this topic.

But what does a fractional derivative mean ?

The n-th derivative of a function f at a point x noted $f^{(n)}(x)$ is a local property only when n is an integer, this is not the case for non-integer power derivatives. In other words, a non-integer fractional derivative of order α (real or complex) of a function f at $x = a$ depends on all values of f, even those far away from a. Therefore, it is expected that the fractional derivative operation involves some sort of boundary conditions, involving information on the function further out. For more about the fractional calculus, the reader can refer to the books [20], [23] or [22].

The approaches to fractional calculus

There are different approaches to the fractional calculus which, not being all equivalent, have lead to a certain degree of confusion and several misunderstandings in the literature. Based on the **Riemann-Liouville** integral operator, the fractional derivative in the **Caputo** sense (see page 78 of [23] and page 90 of [12]) has a main role in the problems studied in this thesis.

Nonlocal boundary conditions

We use this type of conditions when it is hard or impossible to determine the value of the unknown function or its derivatives on the boundary.

In [10], A.A.Dezin showed for the first time that some problems can not be well-posed unless associated nonlocal boundary conditions.

Well-posed in the Hadamard sense means :

- a solution exists,
- the solution is unique,
- the solution's behaviour changes continuously with the initial conditions.

This type of non-standard conditions especially the integral conditions reflects a certain reality in the mathematical modeling of some natural problems in several fields such as biotechnology [26], biology [21], thermal conduction theory [8], in semiconductors [4], in plasma physics [25]...

The basic singnification of the integral conditions is : mean eg.

$$\int_0^1 u(x,t)\,dx = 0,$$

n-th moment eg.

$$\int_0^1 x^n u(x,t)\,dx = 0,$$

total flow, total energy, total mass, ...

Energy inequality method

In our study, we establish existence theorems, uniqueness of strong solution, its continuous dependence on the data, as well as its continuity on the parameters. These results are obtained using the "Energy Inequality" method which is an efficient functional analysis method for the study of many problems in mathematical physics based on the technique of multipliers. This method results from the ideas introduced by J.Leray [15] and L. Garding [11] in their works, and those developed in the works of A.A. Dezin [10], N.J. Yurchuk [30], [31], [32] and V.I. Korzyuk [13].

The method consists of

Reducing the problem posed to an equivalent operational form :

$$\mathscr{L}u = \mathcal{F},$$

where the operator $\mathscr{L}$ is considered from a Banach space B into a Hilbert space H properly chosen. We establish the a priori estimate for the operator $\mathscr{L}$. We then prove the density of the range of this operator in the arrival space H.

More precisely we will follow in this work the following diagram:

For the operator $\mathscr{L}$, of domain $D(\mathscr{L})$, generated by the problem considered, we prove the inequality of the energy of the type

$$\|u\|_B \leq c\,\|\mathscr{L}u\|_H \tag{2}$$

where c is a positive constant independent of the function u. This demonstration is based on a precise analysis of the quantities obtained (seen as energies) by multiplying the equation given by a **multiplier** operator $\mathtt{M}u$ containing the function u and/or its derivatives and primitives, then integrating on the domain. The choice of the operator $\mathtt{M}u$ is fundamental, it is dictated by the equation and the boundary conditions; there is no constructive method or algorithm to find the multiplier for a given problem, it needs human intuition. Then in the strong topologies of the spaces B and H we construct the **closure** $\overline{\mathscr{L}}$ of the operator $\mathscr{L}$, and the solution of the equation $\overline{\mathscr{L}}u = \mathcal{F}$, is called a **strong** solution of the problem considered. With the help of a passage to the limit, we extend the inequality (2) to $D\left(\overline{\mathscr{L}}\right)$ and thus we assure the existence of the solution over $R\left(\overline{\mathscr{L}}\right)$ the range of operator $\overline{\mathscr{L}}$. Since the range of the operator $\overline{\mathscr{L}}$ is closed in H and

$$R\left(\overline{\mathscr{L}}\right) = \overline{R(\mathscr{L})},$$

to prove the existence of the strong solution for any $\mathcal{F} \in H$, it suffices to establish the density of $R(\mathscr{L})$ in H, which is obtained using regularization operators. The uniqueness derives from the energy inequality (2). The choice of the operators of regularization is related to the character of the studied problem. In our case we use the regularization operators with respect to the variable t.

Contribution

In addition to the different techniques and methods related to the fractional calculus and the "energy inequality" approach (finding the appropriate functional spaces, multipliers and regularization operators), the new Generalized formula of integration given in the proposition 1.12 and the Homogenization 3.1.1 are our most relevant contribution in this thesis.

Outline

After the introduction, where we present the history and interest of the topic studied,the thesis is composed mainly of three other chapters.

The second chapter of Preliminaries where we introduce some definitions, theorems and propositions regarding the fractional calculus and functional analysis like the Bouziani space and some useful inequalities. As we remind other identities, we prove other important propositions (see the Contribution section above) that are crucial for the sequel.

In the third chapter, we work out the solvability of a class of FBVPs with the purely integral conditions

$$\int_0^1 x^k u(x,t)\,dx = 0, \qquad k = \overline{0, 2m-1}. \tag{3}$$

We start the first section by setting the problem, after which, in the second and third sections, we apply the "energy inequality" method to prove the existence and uniqueness of the solution. Finally, in the fourth section, we present an example that illustrates the obtained results usefulness.

A fourth chapter is devoted to the study of a class of FBVPs with mixed boundary conditions of type integral-Neumann:

the integral conditions

$$\begin{cases} \int_0^1 u(x,t)\,dx = 0, \\ \\ \int_0^1 xu(x,t)\,dx = 0, \end{cases},$$

and the Neumann conditions

$$\begin{cases} \dfrac{\partial^k}{\partial x^k} u(0,t) = g_k(t) \\ \\ \dfrac{\partial^k}{\partial x^k} u(1,t) = \psi_k(t) \end{cases} \qquad k = \overline{1, m-1}.$$

it contains four sections as well. The first section is a homogenization of the problem where the non-homogeneous Neumann conditions become homogeneous (get rid of the functions $g_k(t)$ and $\psi_k(t)$). In the second section, we establish the a priori estimate and its consequences. After that, the existence is proved in the third section, where we show the range density of the operator associated with the problem. The fourth section contains an illustration to the proved results.

Chapter 1

Preliminaries

In this chapter, we wish to collect certain results, which we shall use frequently in the sequel. In particular, we collect the definitions of the fractional operators, other preliminary results, such as relevant theorems of the functional analysis and particular spaces . At the end, we give some inequalities and identities which will be of use to us in the sequel as well.

Contents

In the whole document, α is a real number in $(0,1)$.

1.1 Fractional calculus

In this section, we briefly recall some definitions and properties concerning fractional calculus.

Definition 1.1. The Gamma function, known as an extension of the factorial function, is given by

$$\Gamma(\alpha) = \int_0^\infty \mu^{\alpha-1} e^{-t} d\mu.$$

Definition 1.2. For any positive real β, The Mittag-Leffler function is the complex function defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

if $\beta = 1$, $E_{\alpha,\beta}$ is abbreviated as E_α.

Definition 1.3. For any differentiable function v, Let I^α denotes the Riemann-Liouville integral operator defined for by

$$I^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(\tau)}{(t-\tau)^{1-\alpha}} d\tau.$$

Definition 1.4. The **C**aputo **t**ime fractional derivative of order $\boldsymbol{\alpha}$ with lower bound **0** noted simply ∂_{0t}^α is defined by

$$\begin{aligned} \partial_{0t}^\alpha v(t) &= I^{1-\alpha} v'(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{v'(\tau)}{(t-\tau)^\alpha} d\tau, \qquad t > 0. \end{aligned}$$

In our case $(1-\alpha > 0)$, we have another equivalent formula of the Caputo fractional derivative.

Proposition 1.1. *∂_{0t}^α is also defined by*

$$\partial_{0t}^\alpha v(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{v(\tau) - v(0)}{(t-\tau)^\alpha} d\tau. \tag{1.1}$$

Proof. We have

$$
\begin{aligned}
I^{1-\alpha}v'(t) &= \frac{d}{dt} I^{2-\alpha}v'(t) \\
&= \frac{1}{\Gamma(2-\alpha)} \frac{d}{dt} \int_0^t v'(\tau)(t-\tau)^{1-\alpha} d\tau \\
&= \frac{1}{\Gamma(2-\alpha)} \frac{d}{dt} \left\{ \left[v(\tau)(t-\tau)^{1-\alpha} \right]_0^t + (1-\alpha) \int_0^t \frac{v(\tau)}{(t-\tau)^{\alpha}} d\tau \right\} \\
&= \frac{1-\alpha}{\Gamma(2-\alpha)} \frac{d}{dt} \left\{ \int_0^t \frac{v(\tau)}{(t-\tau)^{\alpha}} d\tau - v(0) \frac{t^{1-\alpha}}{1-\alpha} \right\} \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left\{ \int_0^t \frac{v(\tau)}{(t-\tau)^{\alpha}} d\tau - v(0) \int_0^t \frac{d\tau}{(t-\tau)^{\alpha}} \right\} \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{v(\tau)-v(0)}{(t-\tau)^{\alpha}} d\tau.
\end{aligned}
$$

□

1.2 Functional analysis

In this section, we focus on some particular material of the functional analysis that will be of great use later. The reader should be aware of the basics : Completeness, Scalar product, Banach and Hilbert spaces, bounded and unbounded operators ...

1.2.1 Linear operators

Let E and F be normed spaces.

Definition 1.5 (Closed operator)**.** The operator $\mathscr{L} : D(\mathscr{L}) \subset E \longrightarrow F$ is **closed** if and only if its graph $G(\mathscr{L})$ is closed in $E \times F$.

Remark 1.1. A closed operator may also be defined by :

For any sequence $(u_n) \subset D(\mathscr{L})$ such that

$$
\begin{cases} u_n \xrightarrow{E} u \\ \mathscr{L}u_n \xrightarrow{F} v \end{cases},
$$

we have

$$
\begin{cases} \mathscr{L}u = v \\ u \in D(\mathscr{L}) \end{cases}.
$$

Definition 1.6 (Closable operator)**.**

A linear operator $\mathscr{L} : D(\mathscr{L}) \longrightarrow H$ from its domain $D(\mathscr{L})$ into a Hilbert space H is **closable** if it has a **closed** extension.

Remark 1.2. To prove that a linear operator $\mathscr{L}$ is closable, we often prove the following :
For any sequence $(u_n) \subset D(\mathscr{L})$ such that

$$\begin{cases} u_n \xrightarrow{E} 0 \\ \\ \mathscr{L}u_n \xrightarrow{F} v \end{cases},$$

we have $v = 0$.

1.2.2 Orthogonality and density

Let M be a subspace of a real Hilbert space H.

Definition 1.7 (Orthogonal complement)**.** The **orthogonal complement** of M is defined by

$$M^{\perp} = \{x \in H : (x, y)_H = 0, \forall y \in M\}.$$

Proposition 1.2. *$M^{\perp}$ is a closed subspace in H.*

Proof. Let x be an element of the closure of $M^{\perp}$. So, there exists a sequence (x_n) in $M^{\perp}$ such that $x_n \longrightarrow x$. The linearity and the continuity of the scalar product allow us to write, for any $y \in M^{\perp}$,

$$\begin{aligned} (x, y)_H &= (\lim x_n, y)_H \\ &= \lim (x_n, y)_H \\ &= 0. \end{aligned}$$

Hence $x \in M^{\perp}$. □

Remark 1.3. These properties are obvious :

1. $M \subset N$ implies that $N^{\perp} \subset M^{\perp}$.
2. $M \subset M^{\perp^{\perp}}$.

Let M be a closed subspace of H. In this situation, we are able to express the least distance property in a geometrical manner.

Theorem 1.1 (Projection theorem)**.** *For a closed subspace M of H the point $x_0 \in M$ is the closest point to an element $x \in H$ if and only if*

$$x - x_0 \in M^{\perp}.$$

x_0 is said to be the **orthogonal projection** of x on M. Hence, we have the following corollary.

Corollary 1.1. *For a closed subspace M of H, we have*

$$M \oplus M^{\perp} = H.$$

Also, we have

Proposition 1.3. *If M is a closed subspace of H, then $M^{\perp^{\perp}} = M$.*

Proof. Using remark 1.3, it suffices to prove that $M^{\perp^{\perp}} \subset M$. Let $x \in M^{\perp^{\perp}}$, then $x = x_0 + y_0$ where $x_0 \in M$ and $y_0 \in M^{\perp}$. We have

$$0 = (x, y_0)_H = \underbrace{(x_0, y_0)_H}_{0} + (y_0, y_0)_H = \|y_0\|_H^2 .$$

Hence, $y_0 = 0$ and consequently, $x \in M$. □

Definition 1.8. Given a set of vectors S in H, the **linear span** of S is defined by

$$span\,(S) = \left\{ \sum_{i=1}^{k} \lambda_i v_i | k \in \mathbb{N}, \lambda_i \in \mathbb{R}, v_i \in S \right\}.$$

It is the smallest subspace containing S. Obviously, if S is a subspace of H, then $span\,(S) = S$.

Proposition 1.4. *If S is a set in H, then $S^{\perp^{\perp}} = \overline{span\,(S)}$.*

Proof. We proceed to inclusion in both directions.

1. $S^{\perp^{\perp}} \subset \overline{span\,(S)}$: note that $\overline{span\,(S)}$ is the smallest closed subspace containing S. Therefore, since $S \subset \overline{span\,(S)}$, we get by the remark 1.3 that $\overline{span\,(S)}^{\perp} \subset S^{\perp}$ and hence $S^{\perp^{\perp}} \subset \overline{span\,(S)}^{\perp^{\perp}}$ which is equal to $\overline{span\,(S)}$ thanks to proposition 1.3.

2. $S^{\perp^{\perp}} \supset \overline{span\,(S)}$: let x be the limit of (x_n) a sequence in $span\,(S)$ and $z \in S^{\perp}$. Then, we car write

$$x_n = \sum_{i=1}^{k_n} \lambda_{i_n} x_{i_n}, \ k_n \in \mathbb{N}, \lambda_{i_n} \in \mathbb{R}, x_{i_n} \in S,$$

and

$$(z, s)_H = 0, \ \forall s \in S.$$

Using the linearity and the continuity of the scalar product, one can get

$$\begin{aligned}(x, z)_H &= \lim \sum_{i=1}^{k_n} \lambda_{i_n} \underbrace{(x_n, z)_H}_{0} \\ &= 0,\end{aligned}$$

which means that $x \in S^{\perp^{\perp}}$. □

Corollary 1.2. *$span\,(S)$ is dense in H if and only if $S^{\perp} = 0$.*

The two upcoming spaces are very important to the main results.

1.2.3 Bouziani space $B_2^k(0, 1)$

To define properly the bouziani space, we need first

1.2.3.1 The primitivation operator

Definition 1.9. The primitivation operator of order k with respect to the space variable x noted $\Im_x^k$ is defined by $\Im_x^0 u = u$ and for $k \geq 1$

$$\Im_x^k u(x,t) = \frac{1}{(k-1)!} \int_0^x \frac{u(\xi,t)}{(x-\xi)^{1-k}} d\xi.$$

- We define on $C_0(0,1)$ the bilinear form given by

$$((u,v)) = \int_0^1 \Im_x^k u \Im_x^k v dx, \tag{1.2}$$

  $C_0(0,1)$ is not complete for this bilinear form considered as a scalar product, therefore we define

1.2.3.2 The Bouziani space

Definition 1.10. Denoted $B_2^k(0,1)$, the Bouziani space is the completion of the space $C_0(0,1)$ for the scalar product (1.2), its associated norm is

$$\begin{aligned} \|u\|_{B_2^k(0,1)} &= \sqrt{((u,u))} \\ &= \left\| \Im_x^k u \right\|_{L^2(0,1)}. \end{aligned}$$

1.2.4 The space $L^2\left(B_2^k(0,1),(0,T)\right)$

Definition 1.11. On the rectangle $Q = (0,1) \times (0,T)$, we define $L^2\left(B_2^k(0,1),(0,T)\right)$ as the space of measurable functions on $(0,T)$ taking values in $B_2^k(0,1)$.

This space also has a structure of a Hilbert space inherited from that of Bouziani space $B_2^k(0,1)$. Its scalar product is given by

$$(u,v)_{L^2\left(B_2^k(0,1),(0,T)\right)} = \int_0^T (u,v)_{B_2^k(0,1)} dt,$$

thus, its norm is

$$\|u\|^2_{L^2\left(B_2^k(0,1),(0,T)\right)} = \int_0^T \|u\|^2_{B_2^k(0,1)} dt.$$

1.3 Some useful inequalities

Proposition 1.5 (Poincaré-like inequality). *For $k \in \mathbb{N}$*

$$\|u\|^2_{B_2^k(0,1)} \leq \frac{1}{2^k} \|u\|^2_{L^2(0,1)}. \tag{1.3}$$

Proof. See corollary of lemma 1 in [7] for $b = 1$. □

Proposition 1.6. *For any function $f(\cdot, t)$ in $B_2^m(0,1)$ we have*

$$I^{\alpha+1}\int_0^1 (\Im_x^m f)^2\, dx \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|f\|^2_{L^2\left(B_2^m(0,1),(0,T)\right)}. \tag{1.4}$$

Proof. The explicit formula of left hand side is

$$\frac{1}{\Gamma(\alpha+1)}\int_0^t (t-\tau)^\alpha \int_0^1 (\Im_x^m f)^2\, dx d\tau,$$

it is sufficient to notice that $t-\tau \leq T$ and that the integrand is positive. □

Proposition 1.7. *for any absolutely continuous function $v(t)$ on the interval $(0,T)$, we have the inequality*

$$2v(t)\,\partial_{0t}^\alpha v(t) \geq \partial_{0t}^\alpha (v(t))^2, \qquad 0<\alpha<1.$$

Proof. See lemma 1 in [3]. □

Proposition 1.8. *Let a non-negative absolutely continuous function $y(t)$ satisfy the inequality*

$$\partial_{0t}^\alpha y(t) \leq c_1 y(t) + c_2(t), \qquad 0<\alpha<1,$$

for almost all t in $[0,T]$, where $c_1 > 0$ and $c_2(t)$ is an integrable non-negative function on $[0,T]$.Then

$$y(t) \leq y(0)\, E_\alpha(c_1 t^\alpha) + \Gamma(\alpha)\, E_{\alpha,\alpha}(c_1 t^\alpha)\, I^\alpha c_2(t).$$

Proof. See Lemma 2 in [3]. □

- Cauchy inequality with ε

  $$2AB \leq \varepsilon A^2 + \frac{1}{\varepsilon}B^2 \tag{1.5}$$

  where A and B are real numbers.

1.4 Important identities

1.4.1 Vandermonde-like identity

Proposition 1.9. *For any positive integers m and k we have*

$$\sum_{i=0}^{k}\binom{m-1}{2k-1-i}\binom{k}{i} = \binom{m+k-1}{2k-1}. \tag{1.6}$$

Proof. The well known Vandermonde's identity is given by

$$\binom{m+n}{k} = \sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i},$$

(see page 8 of [24]), if we replace m, n and k respectively with $m-1$, k and $2k-1$, we get

$$\binom{m+k-1}{2k-1} = \sum_{i=0}^{2k-1}\binom{m-1}{i}\binom{k}{2k-i-1}.$$

Note that the index i goes from 0 to $2k-1$, so we can substitute it by $2k-1-i$ to get

$$\sum_{i=0}^{2k-1}\binom{m-1}{2k-i-1}\binom{k}{i}=\binom{m+k-1}{2k-1},$$

but

$$i>k\Rightarrow\binom{k}{i}=0,$$

so we can take the index i goes to k, that ends the proof. □

1.4.2 the coefficient K_n^k

- Inspired by the Pascal triangle, we introduce the coefficient K_n^k defined recurrently for all $(k,n)\in\mathbb{N}^2$ by

$$\begin{cases} K_n^k=0, & k>n \\ K_0^0=2, & \\ K_{\mathrm{k}}^0=0, & k\geq 1 \\ K_k^k=(-1)^{\mathrm{k}}, & k\geq 1 \\ K_n^k=-\left(K_{n-1}^{k-1}+K_{n-2}^{k-1}\right), & 1\leq k\leq n \end{cases}. \tag{1.7}$$

- Here are some values for $n\leq 6$.

Table 1.1: Values of the sequence $K_{n,k}:=K_n^k$ for $n\leq 6$.

$n\backslash k$	***0***	***1***	***2***	***3***	***4***	***5***	***6***
0	2	0	0	0	0	0	0
1	0	-1	0	0	0	0	0
2	0	-2	1	0	0	0	0
3	0	0	3	-1	0	0	0
4	0	0	2	-4	1	0	0
5	0	0	0	-5	5	-1	0
6	0	0	0	-2	9	-6	1

It has the form of half a lower triangular matrix. The general term of the sequence K_n^k is given in the following proposition.

Proposition 1.10. *For all $(k,n)\in\mathbb{N}^2$ we have*

$$\begin{cases} K_0^0=2, & \\ K_k^0=0, & k\geq 1 \\ K_n^k=(-1)^k\dfrac{n}{k}\dbinom{k}{n-k} & k,n\geq 1 \end{cases}.$$

Remark 1.4. If $0\leq k<\left\lceil\frac{n}{2}\right\rceil$, then $k<n-k$, consequently $K_n^k=0$.

Proof. Besides the trivial cases, we have, for $k \geq 1$

$$\begin{aligned} K_k^k &= (-1)^k \frac{k}{k} \binom{k}{k-k} \\ &= (-1)^k, \end{aligned}$$

that is the fourth relation of (1.7) is fulfilled.

Now, we show that the last relation of (1.7) is also satisfied. Let k and n be integers such that $1 \leq k < n$, we have

$$K_{n-1}^{k-1} + K_{n-2}^{k-1} = \frac{(-1)^{k-1} (n-1) (k-2)!}{(n-k)! (2k-n-1)!} + \frac{(-1)^{k-1} (n-2) (k-2)!}{(n-k-1)! (2k-n)!},$$

the common denominator is $(n-k)! (2k-n)!$, therefore, we multiply the numerator and the denominator of the two fractions in the above equation by $(2k-n)$ and $(n-k)$ to get

$$\begin{aligned} K_{n-1}^{k-1} + K_{n-2}^{k-1} &= \frac{(-1)^{k-1} (2k-n) (n-1) (k-2)! + (-1)^{k-1} (n-k) (n-2) (k-2)!}{(n-k)! (2k-n)!} \\ &= \frac{(-1)^{k-1} (k-2)! [(2k-n) (n-1) + (n-k) (n-2)]}{(n-k)! (2k-n)!} \\ &= \frac{(-1)^{k-1} (k-2)! (kn-n)}{(n-k)! (2k-n)!} \\ &= \frac{(-1)^{k-1} n (k-1)!}{(n-k)! (2k-n)!}, \end{aligned}$$

hence,

$$\begin{aligned} -\left(K_{n-1}^{k-1} + K_{n-2}^{k-1}\right) &= \frac{(-1)^k n (k-1)!}{(n-k)! (2k-n)!} \\ &= (-1)^k \frac{n}{k} \binom{k}{n-k} \\ &= K_n^k. \end{aligned}$$

□

1.4.3 The coefficient A_m^k

- We also introduce a similar coefficient A_m^k defined for all $(m,k) \in \mathbb{N}^2$

$$\begin{cases} A_m^0 = 2, & m \geq 0 \\ A_m^k = (-1)^k \frac{m}{k} \binom{m+k-1}{2k-1}, & k > 0 \end{cases}. \tag{1.8}$$

- Here are some values for $m \leq 5$, it has the form of a lower triangular matrix.

Table 1.2: Values of the sequence $A_{m,k} := A_m^k$ for $m \leq 5$.

$m \backslash k$	**0**	**1**	**2**	**3**	**4**	**5**
0	2	0	0	0	0	0
1	2	-1	0	0	0	0
2	2	-4	1	0	0	0
3	2	-9	6	-1	0	0
4	2	-16	20	8	1	0
5	2	-25	50	-35	10	-1

1.4.4 Relation between the coefficients K_n^k and A_m^k

Proposition 1.11. *For all* $(m,k) \in \mathbb{N}^2$

$$A_m^k = \sum_{n=k}^{2k} \binom{m}{n} K_n^k. \tag{1.9}$$

Remark 1.5. By noticing that $n > m \Rightarrow \binom{m}{n} = 0$, one could have taken the sum from $n = k$ to $\min\{m, 2k\}$ as improvement.

Proof. Recall the Vandermonde-like formula (1.6)

$$\binom{m+k-1}{2k-1} = \sum_{i=0}^{k} \binom{m-1}{2k-1-i} \binom{k}{i},$$

in the right hand side, we substitue the index i by n such that $n = 2k - i$

$$\binom{m+k-1}{2k-1} = \sum_{n=k}^{2k} \binom{m-1}{n-1} \binom{k}{2k-n},$$

since

$$\binom{k}{2k-n} = \binom{k}{n-k},$$

one can get

$$\binom{m+k-1}{2k-1} = \sum_{n=k}^{2k} \binom{m-1}{n-1} \binom{k}{n-k}.$$

Now, multiplying both sides by $(-1)^k \frac{m}{k}$, we obtain

$$\underbrace{(-1)^k \frac{m}{k} \binom{m+k-1}{2k-1}}_{A_m^k} = \sum_{n=k}^{2k} \underbrace{\frac{m}{\boldsymbol{n}} \binom{m-1}{n-1}}_{\binom{m}{n}} \underbrace{(-1)^k \frac{\boldsymbol{n}}{k} \binom{k}{n-k}}_{K_n^k}.$$

□

The aforementioned coefficients and their relation given in proposition 1.11 are crucial for the next proposition.

1.4.5 Leibniz-like formula of integration

Before we state the generalization, we would like to have a look at some examples with concrete values of m.

Starting from the integral

$$2\left(-1\right)^{m}\int_{0}^{1}a\frac{\partial^{m}u}{\partial x^{m}}\Im_{x}^{m}udx,$$

and performing a series of integrations, our aim is to keep only the **even** powers of the function u or its primitives so it will be seen as energies.

Let us start with

1. $m=1$, using the integration by parts with respect to the space variable x, one can easily get

$$-2\int_{0}^{1}a\frac{\partial u}{\partial x}\Im_{x}udx=-2\left[au\Im_{x}u\right]_{0}^{1}+2\int_{0}^{1}au^{2}dx+2\int_{0}^{1}\frac{\partial a}{\partial x}u\Im_{x}udx,$$

and by noticing that a primtive of $2u\Im_{x}u$ is $\left(\Im_{x}u\right)^{2}$, we obtain

$$-2\int_{0}^{1}a\frac{\partial u}{\partial x}\Im_{x}udx=-2\underline{\left[au\Im_{x}u\right]_{0}^{1}}+2\int_{0}^{1}au^{2}dx+\underline{\left[\frac{\partial a}{\partial x}\left(\Im_{x}u\right)^{2}\right]_{0}^{1}}-\int_{0}^{1}\frac{\partial^{2}a}{\partial x^{2}}\left(\Im_{x}u\right)^{2}dx,$$

if for some reason, like the integral conditions (3), the underlined quantities go zero, we get to

$$-2\int_{0}^{1}a\frac{\partial u}{\partial x}\Im_{x}udx=\mathbf{2}\int_{0}^{1}au^{2}dx-\int_{0}^{1}\frac{\partial^{2}a}{\partial x^{2}}\left(\Im_{x}u\right)^{2}dx.$$

The numbers $[2,-1]$ are the row, in the table (1.2) displaying some values of A_{m}^{k}, corresponding to $m=1$.

2. $m=2$, iterating again and again the process of integration by parts we can show that

$$2\int_{0}^{1}a\frac{\partial^{2}u}{\partial x^{2}}\Im_{x}^{2}udx=\mathbf{2}\int_{0}^{1}au^{2}dx\,\mathbf{-4}\int_{0}^{1}\frac{\partial^{2}a}{\partial x^{2}}\left(\Im_{x}u\right)^{2}dx+\int_{0}^{1}\frac{\partial^{4}a}{\partial x^{4k}}\left(\Im_{x}^{2}u\right)^{2}dx.$$

$[2,-4,1]$ is the row corresponding to $m=2$.

3. $m=3$, similarly, one can get

$$-2\int_{0}^{1}a\frac{\partial^{3}u}{\partial x^{3}}\Im_{x}^{3}udx=\mathbf{2}\int_{0}^{1}au^{2}dx\,\mathbf{-9}\int_{0}^{1}\frac{\partial^{2}a}{\partial x^{2}}\left(\Im_{x}u\right)^{2}dx+\mathbf{6}\int_{0}^{1}\frac{\partial^{4}a}{\partial x^{4k}}\left(\Im_{x}^{2}u\right)^{2}dx$$
$$-\int_{0}^{1}\frac{\partial^{6}a}{\partial x^{6k}}\left(\Im_{x}^{3}u\right)^{2}dx.$$

$[2,-9,6,-1]$ corresponds to $m=3$. In general, we have

Proposition 1.12. *Let a function u satisfies the integral conditions (3) and*

$$u \in B_2^k(0,1), \frac{\partial^k u}{\partial x^k} \in L^2(0,1), \qquad k = \overline{0,m}, \tag{1.10}$$

then for all $a(\cdot,t) \in C^{2m}([0,1])$ we have

$$2(-1)^m \int_0^1 a \frac{\partial^m u}{\partial x^m} \Im_x^m u dx = \sum_{k=0}^m A_m^k \int_0^1 \frac{\partial^{2k} a}{\partial x^{2k}} \left(\Im_x^k u\right)^2 dx.$$

Proof. Let u verifies (1.10) and $a(\cdot,t) \in C^{2m}([0,1])$, we first show by induction on k that the integration by parts k times ($0 \leq k \leq m$) using integral conditions (3) gives

$$\int_0^1 a \frac{\partial^m u}{\partial x^m} \Im_x^m u dx = (-1)^k \sum_{n=0}^k \binom{k}{n} \int_0^1 \frac{\partial^n a}{\partial x^n} \frac{\partial^{m-k} u}{\partial x^{m-k}} \Im_x^{m+n-k} u dx, \tag{1.11}$$

indeed a simple integration by parts of the right-hand side using condition (3) leads to

$$\int_0^1 a \frac{\partial^m u}{\partial x^m} \Im_x^m u dx = (-1)^{k+1} \sum_{n=0}^k \binom{k}{n} \int_0^1 \frac{\partial^{n+1} a}{\partial x^{n+1}} \frac{\partial^{m-(k+1)} u}{\partial x^{m-(k+1)}} \Im_x^{m+n-k} u dx$$
$$+ (-1)^{k+1} \sum_{n=0}^k \binom{k}{n} \int_0^1 \frac{\partial^n a}{\partial x^n} \frac{\partial^{m-(k+1)} u}{\partial x^{m-(k+1)}} \Im_x^{m+n-(k+1)} u dx, \tag{1.12}$$

if we replace n by $n-1$ in the first term of the right-hand side of the above equality we get

$$\int_0^1 a \frac{\partial^m u}{\partial x^m} \Im_x^m u dx = (-1)^{k+1} \sum_{n=1}^{k+1} \binom{k}{n-1} \int_0^1 \frac{\partial^n a}{\partial x^n} \frac{\partial^{m-(k+1)} u}{\partial x^{m-(k+1)}} \Im_x^{m+n-(k+1)} u dx$$
$$+ (-1)^{k+1} \sum_{n=0}^k \binom{k}{n} \int_0^1 \frac{\partial^n a}{\partial x^n} \frac{\partial^{m-(k+1)} u}{\partial x^{m-(k+1)}} \Im_x^{m+n-(k+1)} u dx, \tag{1.13}$$

hence

$$\int_0^1 a \frac{\partial^m u}{\partial x^m} \Im_x^m u dx =$$
$$(-1)^{k+1} \sum_{n=1}^k \left\{ \binom{k}{n} + \binom{k}{n-1} \right\} \int_0^1 \frac{\partial^n a}{\partial x^n} \frac{\partial^{m-(k+1)} u}{\partial x^{m-(k+1)}} \Im_x^{m+n-(k+1)} u dx$$
$$+ (-1)^{k+1} \left(\int_0^1 a \frac{\partial^{m-(k+1)} u}{\partial x^{m-(k+1)}} \Im_x^{m-(k+1)} u dx + \int_0^1 \frac{\partial^{k+1} a}{\partial x^{k+1}} \frac{\partial^{m-(k+1)} u}{\partial x^{m-(k+1)}} \Im_x^m u dx \right), \tag{1.14}$$

The sum in equation (1.14) goes from 1 to k and, using the fact that

$$\binom{k+1}{0} = \binom{k+1}{k+1} = 1$$

and

$$\binom{k}{n} + \binom{k}{n-1} = \binom{k+1}{n},$$

the last two integrals in the above equation can be viewed as the first and last terms corresponding to the values of $n = 0$ and $n = k+1$. Consequently, the equation (1.14) becomes

$$\int_0^1 a\frac{\partial^m u}{\partial x^m}\Im_x^m u dx = (-1)^{k+1}\sum_{n=0}^{k+1}\binom{k+1}{n}\int_0^1 \frac{\partial^n a}{\partial x^n}\frac{\partial^{m-(k+1)}u}{\partial x^{m-(k+1)}}\Im_x^{m+n-(k+1)}u dx, \tag{1.15}$$

and the identity (1.11) is proved.

The particular case of identity (1.11)

$$\boxed{2(-1)^m\int_0^1 a\frac{\partial^m u}{\partial x^m}\Im_x^m u dx = 2\sum_{n=0}^{m}\binom{m}{n}\int_0^1 \frac{\partial^n a}{\partial x^n}u\Im_x^n u dx} \tag{1.16}$$

is needed.

Secondly we prove by induction on n ($n < m$) that

$$\boxed{2\int_0^1 \frac{\partial^n a}{\partial x^n}u\Im_x^n u dx = \sum_{k=\lceil \frac{n}{2}\rceil}^{n} K_n^k\int_0^1 \frac{\partial^{2k} a}{\partial x^{2k}}\left(\Im_x^k u\right)^2 dx,} \tag{1.17}$$

We used the ceiling function (instead of the floor $\lfloor . \rfloor$) which gives

$$\left\lceil \frac{n}{2}\right\rceil = \begin{cases} \frac{n}{2} & n \text{ is even} \\ \frac{n+1}{2} & n \text{ is odd} \end{cases}.$$

Assume equality (1.17) holds for n and n–1, notice that taking k starts from $\left\lceil \frac{n}{2}\right\rceil$ is just an optimization (see remark 1.4), so we can take k starting from 0, and we will not be forced to distinguish whether n is even or odd. Hence

$$2\int_0^1 \frac{\partial^n a}{\partial x^n}u\Im_x^n u dx = \sum_{k=0}^{n} K_n^k\int_0^1 \frac{\partial^{2k} a}{\partial x^{2k}}\left(\Im_x^k u\right)^2 dx \tag{1.18}$$

and

$$2\int_0^1 \frac{\partial^{n-1} a}{\partial x^{n-1}}u\Im_x^{n-1} u dx = \sum_{k=0}^{n-1} K_{n-1}^k\int_0^1 \frac{\partial^{2k} a}{\partial x^{2k}}\left(\Im_x^k u\right)^2 dx. \tag{1.19}$$

Replacing u by $\Im_x u$ and a by $\frac{\partial^2 a}{\partial x^2}$ in equations (1.18) and (1.19) yields

$$2\int_0^1 \frac{\partial^{n+2}a}{\partial x^{n+2}} \Im_x u \Im_x^{n+1} u dx = \sum_{k=0}^{n} K_n^k \int_0^1 \frac{\partial^{2(k+1)}a}{\partial x^{2(k+1)}} \left(\Im_x^{k+1} u\right)^2 dx \tag{1.20}$$

and

$$2\int_0^1 \frac{\partial^{n+1}a}{\partial x^{n+1}} \Im_x u \Im_x^{n} u dx = \sum_{k=0}^{n-1} K_{n-1}^k \int_0^1 \frac{\partial^{2(k+1)}a}{\partial x^{2(k+1)}} \left(\Im_x^{k+1} u\right)^2 dx. \tag{1.21}$$

A simple integration by parts using condition (3) gives

$$2\int_0^1 \frac{\partial^{n+1}a}{\partial x^{n+1}} u \Im_x^{n+1} u dx = -2\int_0^1 \frac{\partial^{n+2}a}{\partial x^{n+2}} \Im_x u \Im_x^{n+1} u dx - 2\int_0^1 \frac{\partial^{n+1}a}{\partial x^{n+1}} \Im_x u \Im_x^{n} u dx \tag{1.22}$$

and the substitution of equations (1.20) and (1.21) in (1.22) leads to

$$\begin{aligned} 2\int_0^1 \frac{\partial^{n+1}a}{\partial x^{n+1}} u \Im_x^{n+1} u dx = &-\sum_{k=0}^{n-1} \left(K_n^k + K_{n-1}^k\right) \int_0^1 \frac{\partial^{2(k+1)}a}{\partial x^{2(k+1)}} \left(\Im_x^{k+1} u\right)^2 dx \\ &- K_n^n \int_0^1 \frac{\partial^{2(n+1)}a}{\partial x^{2(n+1)}} \left(\Im_x^{n+1} u\right)^2 dx, \end{aligned} \tag{1.23}$$

notice that $-K_n^n = K_{n+1}^{n+1}$ and from the recurrence formula in (1.7) we have $-\left(K_n^k + K_{n-1}^k\right) = K_{n+1}^{k+1}$, then replacing k by $k-1$ in the above equation gives

$$2\int_0^1 \frac{\partial^{n+1}a}{\partial x^{n+1}} u \Im_x^{n+1} u dx = \sum_{k=0}^{n+1} K_{n+1}^k \int_0^1 \frac{\partial^{2k}a}{\partial x^{2k}} \left(\Im_x^{k} u\right)^2 dx, \tag{1.24}$$

since $K_{n+1}^0 = 0$.This proves identity (1.17).
We substitute equation (1.17) in (1.16) to get

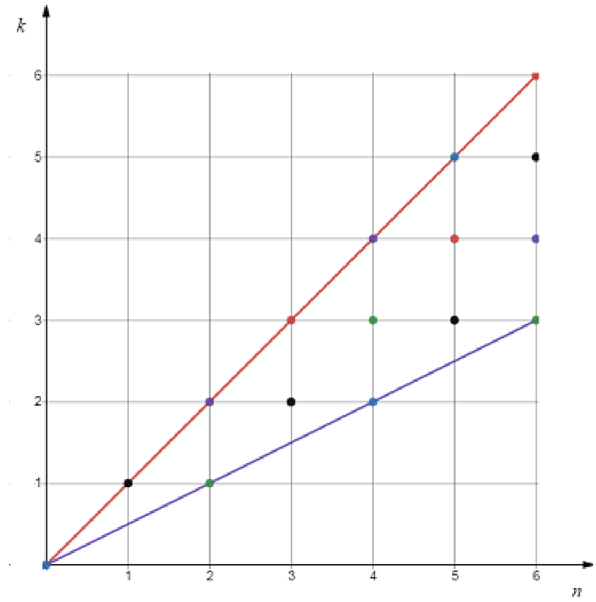
$$2(-1)^m \int_0^1 a \frac{\partial^m u}{\partial x^m} \Im_x^m u dx = \sum_{n=0}^{m} \binom{m}{n} \sum_{k=\lceil \frac{n}{2} \rceil}^{n} K_n^k \int_0^1 \frac{\partial^{2k}a}{\partial x^{2k}} \left(\Im_x^{k} u\right)^2 dx. \tag{1.25}$$

Now, observe that if we expand the sums, each term is of the form

$$\binom{m}{n} K_n^k \int_0^1 \frac{\partial^{2k}a}{\partial x^{2k}} \left(\Im_x^{k} u\right)^2 dx$$

presented in the figure below with the point of coordinates (n,k) for $m = 6$, effective points are between the lines of equations $k = n$ and $k = \frac{n}{2}$.

Figure 1.1: Display of integrals in the sums of (1.25)



To invert the sums, it is sufficient to see that $0 \leq k \leq m$ and $k \leq n \leq 2k$, so (1.25) can be written

$$2\left(-1\right)^m \int_0^1 a \frac{\partial^m u}{\partial x^m} \Im_x^m u dx = \sum_{k=0}^{m} \underbrace{\sum_{n=k}^{2k} \binom{m}{n} K_n^k}_{A_m^k} \int_0^1 \frac{\partial^{2k} a}{\partial x^{2k}} \left(\Im_x^k u\right)^2 dx,$$

and the proposition 1.11 finishes the proof. □

Chapter 2

FBVP with purely integral conditions

In this chapter, we study the existence and uniquenes of the strong solution of a class of FBVPs with Purely integral conditions. After setting the problem, we get to the section 2.2 in which we choose convenient spaces and multiplier to ascertain the a priori estimate that will be extended to conclude the uniqueness of the strong solution and its dependence on the data given in the problem. Finally, we prove the existence in section 2.3 relying on the fact that the operator generated by our problem is dense, then we present an example in which we illustrate the usefulness of the results obtained previously .

Contents

2.1 Problem setting

The problem is the following

Let Q be a rectangle defined by $Q = (0,1) \times (0,T)$ and considering the fractional partial differential equation

$$\partial_{0t}^{\alpha} u + (-1)^m \frac{\partial^m}{\partial x^m}\left(a(x,t)\frac{\partial^m u}{\partial x^m}\right) = f(x,t) \tag{2.1}$$

where $m \geq 1$ and the function $a(\cdot,t) \in C^{2m}([0,1])$, subject to the initial condition

$$u(x,0) = \varphi(x), \qquad x \in (0,1) \tag{2.2}$$

and the boundary integral conditions

$$\int_0^1 x^k u(x,t)\,dx = 0, \qquad k = \overline{0,2m-1}. \tag{2.3}$$

Dictated by the “energy inequality“ method, we start by the

2.2 A priori estimate

To establish the existence and uniqueness of the solution of the problem (2.1)-(2.3), we write it in an equivalent operator form so that it can be viewed as the solution of this operator equation

$$\mathscr{L}u = \mathcal{F}.$$

Recall that $\mathscr{L} = (\mathcal{L}, \ell)$ acts from a Banach space B to a Hilbert one H, we shall define precisely the functional spaces and the operator. To do so, we need the following lemma.

Lemma 2.1. *The space*

$$B = \left\{ u \in L^2\left(B_2^k(0,1),(0,T)\right), k = \overline{0,m} / \frac{\partial^k u}{\partial x^k} \in L^2(Q), k = \overline{0,m},\right.$$

$$\left. \partial_{0t}^{\alpha} u \in L^2\left(B_2^m(0,1),(0,T)\right)\right\}$$

endowed by the finite norm

$$\|u\|_B^2 = \sup_{0\leq t\leq T} I^{1-\alpha}\|u\|^2_{B_2^m(0,1)} + \sum_{k=0}^{m-1}\|u\|^2_{L^2\left(B_2^k(0,1),(0,T)\right)},$$

is a Banach space.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in B, that is : for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $p, q \geq N$ we have

$$\left\|u_p - u_q\right\|_B \leq \varepsilon,$$

we would like to show that $(u_n)_{n\in\mathbb{N}}$ is convergent.

The above inequality implies that

$$\begin{cases} \sup\limits_{0\leq t\leq T} I^{1-\alpha}\left\|u_p-u_q\right\|^2_{B_2^m(0,1)}\leq \varepsilon^2 \\ \left\|u_p-u_q\right\|_{L^2\left(B_2^k(0,1),(0,T)\right)}\leq \varepsilon \qquad k=\overline{0,m-1} \end{cases}. \tag{2.4}$$

The second inequality of (2.4) means that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in each complete space $L^2\left(B_2^k(0,1),(0,T)\right)$, $k=\overline{0,m-1}$, so it is convergent in each one.

And the first inequality of (2.4) can be written

$$\int_0^t \frac{\left\|u_p-u_q\right\|^2_{B_2^m(0,1)}}{(t-\tau)^\alpha}d\tau\leq \varepsilon^2\Gamma(1-\alpha), \qquad t\in(0,T),$$

for any strictly positive constant $\rho\leq t$ we have

$$\int_\rho^t \left\|u_p-u_q\right\|^2_{B_2^m(0,1)}d\tau\leq \varepsilon^2\Gamma(1-\alpha)(t-\rho)^\alpha, \qquad t\in(0,T),$$

then, by taking $\rho\to 0$, one can get

$$\left\|u_p-u_q\right\|_{L^2\left(B_2^m(0,1),(0,T)\right)}d\tau\leq T^{\frac{\alpha}{2}}\varepsilon\sqrt{\Gamma(1-\alpha)},$$

hence, $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $L^2\left(B_2^m(0,1),(0,T)\right)$ which is also complete, therefore it is convergent. □

We also introduce the Hilbert space H consisting of vector-valued functions

$$\mathcal{F}=(f,\varphi)$$

with finite norm

$$\|\mathcal{F}\|_H^2=\|f\|^2_{L^2(Q)}+\|\varphi\|^2_{L^2(0,1)},$$

and we set

$$\begin{cases} \mathcal{L}u=\partial_{0t}^\alpha u+(-1)^m\frac{\partial^m}{\partial x^m}\left(a(x,t)\frac{\partial^m u}{\partial x^m}\right) \\ \ell u=u(x,0) \end{cases}.$$

The domain of definition of the operator $\mathscr{L}$

$$D_\varphi(\mathscr{L})=\left\{u\in B/\int_0^1 x^k u(x,t)\,dx=0, k=\overline{0,2m-1}; \ell u=\varphi\right\}.$$

Before stating the main result, we also need the two following lemmas.

Lemma 2.2. *For any $u\in D_\varphi(\mathscr{L})$ we have*

$$\Im_x^m\left(\partial_{0t}^\alpha u\right)=\partial_{0t}^\alpha\left(\Im_x^m u\right).$$

Proof. Each one of the operators $\Im_x^m$ and ∂_{0t}^{α} acts with respect to an independant variable, and by noticing that $\Im_x^m = \underbrace{\Im_x \Im_x ... \Im_x}_{m \text{ times}}$ ($\Im_x := \Im_x^1$), it is sufficient to prove that

$$\Im_x \left(\partial_{0t}^{\alpha} u\right) = \partial_{0t}^{\alpha} \left(\Im_x u\right).$$

We have

$$\Im_x \left(\partial_{0t}^{\alpha} u\right) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \int_0^t \frac{\frac{\partial u(\xi,\tau)}{\partial \tau}}{(t-\tau)^{\alpha}} d\tau d\xi$$

the integrand

$$\frac{\frac{\partial u(\xi,\tau)}{\partial \tau}}{(t-\tau)^{\alpha}}$$

is a continuous mapping, so using Fubini's theorem we can switch to

$$\Im_x \left(\partial_{0t}^{\alpha} u\right) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\int_0^x \frac{\partial u(\xi,\tau)}{\partial \tau} d\xi}{(t-\tau)^{\alpha}} d\tau,$$

and from the leibnitz rule of differentiation of integrals (see page 17 in [29]), we can write

$$\begin{aligned} \Im_x \left(\partial_{0t}^{\alpha} u\right) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{\partial}{\partial \tau} \int_0^x u(\xi,\tau)\, d\xi}{(t-\tau)^{\alpha}} d\tau, \\ &= \partial_{0t}^{\alpha} \left(\Im_x u\right). \end{aligned}$$

□

Lemma 2.3. *For all u in $D_{\varphi}(\mathscr{L})$, we have*

$$\int_0^t \int_0^1 \partial_{0\tau}^{\alpha} \left(\Im_x^m u\right)^2 dx d\tau = I^{1-\alpha} \|u\|^2_{B_2^m(0,1)} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \|\varphi\|^2_{B_2^m(0,1)}.$$

Proof. We have

$$\begin{aligned}\int_0^t\int_0^1 \partial_{0\tau}^{\alpha}\left(\Im_x^m u\right)^2 dxd\tau &= \int_0^1\int_0^t \partial_{0\tau}^{\alpha}\left(\Im_x^m u\right)^2 d\tau dx \\ &= \int_0^1\int_0^t I^{1-\alpha}\frac{d}{d\tau}\left(\Im_x^m u\right)^2 d\tau dx \\ &= \int_0^1 I^{1-\alpha}\left\{\int_0^t \frac{d}{d\tau}\left(\Im_x^m u\right)^2 d\tau\right\} dx \\ &= I^{1-\alpha}\int_0^1 \left[\left(\Im_x^m u\right)^2\right]_0^t dx \\ &= I^{1-\alpha}\int_0^1 \left(\Im_x^m u\right)^2 dx - I^{1-\alpha}\int_0^1 \left(\Im_x^m \varphi\right)^2 dx \\ &= I^{1-\alpha}\left\|u\right\|^2_{B_2^m(0,1)} - I^{1-\alpha}\left\|\varphi\right\|^2_{B_2^m(0,1)}.\end{aligned}$$

The fractional Riemann-Liouville integral of the constant $\left(\left\|\varphi\right\|^2_{B_2^m(0,1)}\right)$ given by

$$I^{1-\alpha}\left\|\varphi\right\|^2_{B_2^m(0,1)} = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\left\|\varphi\right\|^2_{B_2^m(0,1)},$$

ends the proof. □

Our first main result is the next theorem.

Theorem 2.1. *Assume the function a satisfies the conditions*

$$0 < c_k \leq A_m^k \frac{\partial^{2k} a}{\partial x^{2k}}, \qquad k = \overline{0, m}, \tag{2.5}$$

where $\left\{c_k, k = \overline{0, m}\right\}$ are real constants. Then there exists a positive constant c not depending on u such that

$$\left\|u\right\|_B \leq c\left\|\mathscr{L}u\right\|_H \tag{2.6}$$

for all u in $D_\varphi(\mathscr{L})$.

Proof. We consider the scalar product in the space $B_2^m(0,1)$ of the equation (2.1) by the multiplier

$$\mathtt{M}u := 2u,$$

that is

$$\left(\partial_{0t}^{\alpha}u + (-1)^m \frac{\partial^m}{\partial x^m}\left(a\frac{\partial^m u}{\partial x^m}\right), 2u\right)_{B_2^m(0,1)} = (f, 2u)_{B_2^m(0,1)},$$

or

$$\left(\Im_x^m\left(\partial_{0t}^{\alpha}u\right) + (-1)^m\left(a\frac{\partial^m u}{\partial x^m}\right), 2\Im_x^m u\right)_{L^2(0,1)} = \left(\Im_x^m f, 2\Im_x^m u\right)_{L^2(0,1)}.$$

In light of lemma 2.2 we can write

$$2\int_0^1 \partial_{0t}^{\alpha}\left(\Im_x^m u\right)\Im_x^m u dx + 2\left(-1\right)^m \int_0^1 a\frac{\partial^m u}{\partial x^m}\Im_x^m u dx = 2\int_0^1 \Im_x^m f \Im_x^m u dx. \tag{2.7}$$

We need to estimate each inegral term of the above equation.
Using proposition 1.7 on the first term of left hand side of the equation (2.7) we obtain

$$2\int_0^1 \partial_{0t}^{\alpha}\left(\Im_x^m u\right)\Im_x^m u dx \geq \int_0^1 \partial_{0t}^{\alpha}\left(\Im_x^m u\right)^2 dx. \tag{2.8}$$

The second term in the left hand side is estimated from the proposition 1.12 and the assumption (2.5) as follows

$$\begin{aligned} 2\left(-1\right)^m \int_0^1 a\frac{\partial^m u}{\partial x^m}\Im_x^m u dx &= \sum_{k=0}^{m} A_m^k \int_0^1 \frac{\partial^{2k} a}{\partial x^{2k}}\left(\Im_x^k u\right)^2 dx \\ &\geq \sum_{k=0}^{m} c_k \int_0^1 \left(\Im_x^k u\right)^2 dx, \end{aligned} \tag{2.9}$$

for the right hand side term of the equation (2.7), Cauchy inequality (1.5) with $\varepsilon = c_m + 1$ gives

$$2\int_0^1 \Im_x^m f \Im_x^m u dx \leq \left(c_m+1\right)\int_0^1 \left(\Im_x^m u\right)^2 dx + \frac{1}{\left(c_m+1\right)}\int_0^1 \left(\Im_x^m f\right)^2 dx. \tag{2.10}$$

We substitute the estimations (2.8),(2.9), and (2.10) in (2.7), then simplifying the term $c_m \int_0^1 \left(\Im_x^m u\right)^2 dx$ yields

$$\int_0^1 \partial_{0t}^{\alpha}\left(\Im_x^m u\right)^2 dx + \sum_{k=0}^{m-1} c_k \int_0^1 \left(\Im_x^k u\right)^2 dx \leq \int_0^1 \left(\Im_x^m u\right)^2 dx + \frac{1}{\left(c_m+1\right)}\int_0^1 \left(\Im_x^m f\right)^2 dx. \tag{2.11}$$

Now, dropping the positive terms

$$\sum_{k=0}^{m-1} c_k \int_0^1 \left(\Im_x^k u\right)^2 dx$$

in the inequality (2.11), replacing t by τ and integrating with respect to τ from 0 to t, we obtain

$$\int_0^t\int_0^1 \partial_{0\tau}^{\alpha}\left(\Im_x^m u\right)^2 dx d\tau \leq \int_0^t\int_0^1 \left(\Im_x^m u\right)^2 dx d\tau + \int_0^t\int_0^1 \left(\Im_x^m f\right)^2 dx d\tau, \tag{2.12}$$

and by setting

$$y\left(t\right) = \int_0^t\int_0^1 \left(\Im_x^m u\right)^2 dx d\tau$$

inequality (2.12) through the proposition 1.8 induces

$$y(t) \leq \Gamma(\alpha) E_{\alpha,\alpha}(t^\alpha) I^{\alpha+1} \int_0^1 (\Im_x^m f)^2 \, dx. \tag{2.13}$$

Combining (2.13) and (1.4) , inequality (2.11) becomes

$$\int_0^t \int_0^1 \partial_{0t}^\alpha (\Im_x^m u)^2 \, dx d\tau + \sum_{k=0}^{m-1} c_k \int_0^t \int_0^1 \left(\Im_x^k u\right)^2 dx d\tau$$
$$\leq \Gamma(\alpha) E_{\alpha,\alpha}(t^\alpha) I^{\alpha+1} \int_0^1 (\Im_x^m f)^2 \, dx + \frac{1}{(c_m+1)} \int_0^t \int_0^1 (\Im_x^m f)^2 \, dx d\tau.$$

For the first term of the above inequality, we use lemma 2.3, and for the right hand side terms, we apply the inequality (1.4). Hence,

$$I^{1-\alpha} \|u\|^2_{B_2^m(0,1)} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \|\varphi\|^2_{B_2^m(0,1)} + \sum_{k=0}^{m-1} c_k \int_0^t \|u\|^2_{B_2^k(0,1)} d\tau$$
$$\leq \frac{\Gamma(\alpha) E_{\alpha,\alpha}(t^\alpha) T^\alpha}{\Gamma(\alpha+1)} \|f\|^2_{L^2(B_2^m(0,1),(0,T))} + \frac{1}{(c_m+1)} \int_0^t \|f\|^2_{B_2^m(0,1)} d\tau. \tag{2.14}$$

The term $\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \|\varphi\|^2_{B_2^m(0,1)}$ can be transferred to the right hand side of the inequality above and becomes $\frac{T^{1-\alpha}}{2^m \Gamma(2-\alpha)} \|\varphi\|^2_{L^2(0,1)}$ using proposition 1.5 and the fact that $t \leq T$. By noticing that

$$\frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} = \frac{1}{\alpha},$$

(2.14) gives through proposition 1.5

$$I^{1-\alpha} \|u\|^2_{B_2^m(0,1)} + \sum_{k=0}^{m-1} c_k \int_0^t \|u\|^2_{B_2^k(0,1)} d\tau$$
$$\leq \frac{1}{2^m} \left(\frac{E_{\alpha,\alpha}(t^\alpha) T^\alpha}{\alpha} + \frac{1}{(c_m+1)} \right) \|f\|^2_{L^2(Q)} + \frac{T^{1-\alpha}}{2^m \Gamma(2-\alpha)} \|\varphi\|^2_{L^2(0,1)}.$$

Since the right hand side of the above inequality does not depend ont t, we can take the upper bound with respect to t over $[0, T]$ to get

$$\sup_{0 \leq t \leq T} I^{1-\alpha} \|u\|^2_{B_2^m(0,1)} + \sum_{k=0}^{m-1} c_k \|u\|^2_{L^2(B_2^k(0,1),(0,T))} \leq \gamma \left(\|f\|^2_{L^2(Q)} + \|\varphi\|^2_{L^2(0,1)} \right),$$

where

$$\gamma = \frac{1}{2^m} \max \left\{ \frac{E_{\alpha,\alpha}(T^\alpha) T^\alpha}{\alpha} + \frac{1}{(c_m+1)}, \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \right\},$$

or

$$\min_{0\leq k<m}\{1,c_k\}\underbrace{\left(\sup_{0\leq t\leq T} I^{1-\alpha}\|u\|^2_{B_2^m(0,1)}+\sum_{k=0}^{m-1}\|u\|^2_{L^2\left(B_2^k(0,1),(0,T)\right)}\right)}_{\|u\|_B^2}\leq\gamma\underbrace{\left(\|f\|^2_{L^2(Q)}+\|\varphi\|^2_{L^2(0,1)}\right)}_{\|\mathcal{F}\|_H^2}.$$

Finally, the a priori estimate (2.6) follows taking

$$c=\left(\frac{\gamma}{\min\limits_{0\leq k<m}\{1,c_k\}}\right)^{1/2}.$$

□

Proposition 2.1. *The operator $\mathscr{L}$ from B to H has a closure $\overline{\mathscr{L}}$.*

Proof. In light of the remark 1.2, we take a sequence $(u_n)\subset D_\varphi(\mathscr{L})$ such that

$$\begin{cases} u_n\xrightarrow{B}0\\ \mathscr{L}u_n\xrightarrow{H}\mathcal{F}=(f,\varphi)\end{cases},$$

recall that

$$\mathscr{L}u_n\xrightarrow{H}\mathcal{F}\Leftrightarrow\begin{cases}\mathcal{L}u_n\xrightarrow{L^2(Q)}f\\ \ell u_n\xrightarrow{L^2(0,1)}\varphi\end{cases}.$$

We need to show that $f=0$ and $\varphi=0$.

Since $u_n\xrightarrow{B}0$, then $u_n\xrightarrow{D'(Q)}0$. By virtue of the continuity of derivation of $D'(Q)$ in itself, we get $\mathcal{L}u_n\xrightarrow{D'(Q)}0$. But we already know that $\mathcal{L}u_n\xrightarrow{L^2(Q)}f$ which implies that $\mathcal{L}u_n\xrightarrow{D'(Q)}f$. Hence, we conclude that $f=0$ by the uniqueness of the limit in $D'(Q)$.

On the other hand , $u_n\xrightarrow{B}0$ and $\|\ell u_n\|_{L^2(0,1)}\leq\|u_n\|_B$ result in $\ell u_n\xrightarrow{L^2(0,1)}0$, and the uniqueness of the limit in the space $L^2(0,1)$ allows us to conclude that $\varphi=0$. □

Remark 2.1. A solution to the operator equation $\overline{\mathscr{L}}u=\mathcal{F}$ is called a **strong solution** to our problem.

Consequently, the a priori estimate (2.6) can be extended to cover strong solutions by passing to the limit.

Corollary 2.1. *Under assumptions (2.5) there exists a positive constant c such that*

$$\|u\|_B\leq c\left\|\overline{\mathscr{L}}u\right\|_H \tag{2.15}$$

for all u in $D_\varphi(\overline{\mathscr{L}})$.

This corollary asserts that, if a strong solution exists, it is unique and depends continuously on (f,φ), with u considered in the topology of B and (f,φ) in the topology of H.

As our second main result, the existence demonstration comes is the next section.

2.3 Existence of the solution

We aim to show the range density of the noperator $\mathscr{L}$ in Hilbert space H, that is $\overline{R(\mathscr{L})} = H$. Recall that $\mathscr{L} = (\mathcal{L}, \ell)$ and $H = L^2(Q) \times L^2(0,1)$,We start by the case u belongs to $D_0(\mathscr{L})$ (i.e. $\ell u = 0$). Thus, using corollary 1.2, we are going to show that

$$R(\mathcal{L})^{\perp} = \{0\}.$$

The follwoing lemma is needed.

Lemma 2.4. *The range $R(\overline{\mathscr{L}})$ is equal to the closure of the range $R(\mathscr{L})$.*

Proof. 1. It follows from the definition of $\overline{\mathscr{L}}$ that $R(\overline{\mathscr{L}}) \subset \overline{R(\mathscr{L})}$. It remains to prove the other inclusion.

2. Let's take $v \in \overline{R(\mathscr{L})}$, if $v \in R(\mathscr{L})$ nothing to prove, $\overline{\mathscr{L}}$ is an extension of $\mathscr{L}$. Now, let $(v_n) \subset R(\mathscr{L})$ be a sequence converging to v, so there exists a corresponding sequence (u_n) in $D_{\varphi}(\mathscr{L})$ such that $\mathscr{L}u_n = v_n$. Using the a priori estimate (2.6), we can write

$$\|u_n - u_{n'}\|_B \leq c\,\|\mathscr{L}u_n - \mathscr{L}u_{n'}\|_H,$$

which means, when $n, n' \to \infty$, that (u_n) is a Cauchy sequence in the Banach space B. Consequently, $u_n \xrightarrow{B} u$ and $\overline{\mathscr{L}}u = v$ (see remark 1.1). □

Our second main result relies on the following.

Theorem 2.2. *Assume for all u in $D_0(\mathscr{L})$*

$$(\mathcal{L}u, \psi)_{L^2(Q)} = 0, \tag{2.16}$$

then under assumptions (2.5) ψ vanishes a.e in $L^2(Q)$.

Proof. Let us assume that a function $\theta(x,t)$ satisfies integral conditions (2.3) such that $\theta \in B_2^k(0,1)$ and $\frac{\partial^k \theta}{\partial x^k} \in L^2(0,1)$, $k = \overline{0,m}$. Then, we can set for all s in $[0,t]$

$$u(x,t) = \begin{cases} 0 & 0 \leq t \leq s \\ \int_s^t \theta(x,\tau)\,d\tau & s \leq t \leq T \end{cases}.$$

Clearly, $u \in D_0(\mathscr{L})$ and u equals zero in the neighborhood of $(0,s)$, it follows that the assumption (3.16) becomes

$$\int_Q \partial_{0t}^{\alpha} \left(\int_0^t \theta(x,\tau)\,d\tau \right) \psi(x,t)\,dxdt$$

$$+ (-1)^m \int_Q \frac{\partial^m}{\partial x^m} \left(a(x,t) \frac{\partial^m}{\partial x^m} \int_0^t \theta(x,\tau)\,d\tau \right) \psi(x,t)\,dxdt = 0. \tag{2.17}$$

We now express ψ in terms of θ:

$$\psi(x,t) = 2(-1)^m \int_0^t \Im_x^{2m} \theta(x,\tau)\,d\tau$$

to get from equation (3.17)

$$2\int_Q \left(\int_0^t \Im_x^m \theta(x,\tau)\,d\tau\right) \partial_{0t}^{\alpha} \left(\int_0^t \Im_x^m \theta(x,\tau)\,d\tau\right) dxdt$$

$$+2(-1)^m \int_Q a(x,t) \left(\int_0^t \frac{\partial^m}{\partial x^m}\theta(x,\tau)\,d\tau\right) \left(\int_0^t \Im_x^m \theta(x,\tau)\,d\tau\right) dxdt = 0. \quad (2.18)$$

From proposition 1.7 we have

$$2\left(\int_0^t \Im_x^m \theta(x,\tau)\,d\tau\right) \partial_{0t}^{\alpha} \left(\int_0^t \Im_x^m \theta(x,\tau)\,d\tau\right) \geq \partial_{0t}^{\alpha} \left(\int_0^t \Im_x^m \theta(x,\tau)\,d\tau\right)^2,$$

and using proposition 1.12 one can get

$$2(-1)^m \int_0^1 a(x,t) \left(\int_0^t \frac{\partial^m}{\partial x^m}\theta(x,\tau)\,d\tau\right) \left(\int_0^t \Im_x^m \theta(x,\tau)\,d\tau\right) dx =$$

$$\sum_{k=0}^m A_m^k \int_0^1 \frac{\partial^{2k} a}{\partial x^{2k}} \left(\int_0^t \Im_x^k \theta(x,\tau)\,d\tau\right)^2 dx.$$

Hence by using assumptions (2.5) and substituting the last two relations in equation (3.18) we deduce

$$\int_Q \partial_{0t}^{\alpha} \left(\int_0^t \Im_x^m \theta(x,\tau)\,d\tau\right)^2 dxdt + \sum_{k=0}^m c_k \int_Q \left(\int_0^t \Im_x^k \theta(x,\tau)\,d\tau\right)^2 dxdt \leq 0,$$

by taking the upper bound ovet $[0,T]$ of the above inequality yields

$$\sup_{0\leq t\leq T} I^{1-\alpha} \left\| \int_0^t \Im_x^m \theta(x,\tau)\,d\tau \right\|_{L^2(0,1)}^2 + \sum_{k=0}^m c_k \left\| \int_0^t \Im_x^k \theta(x,\tau)\,d\tau \right\|_{L^2(Q)}^2 \leq 0.$$

The same process can be repeated by integrating step by step along the reactangle Q, i.e. integrating over $Q_\tau = (0,1)\times(0,\tau)$ for any $\tau \leq T$. Consequently, θ is identically zero and $\psi = 0$ a.e in $L^2(Q)$. □

Similarly, we consider the general case. For any $\mathcal{F} = (\psi,\omega)$ in $H = L^2(Q)\times L^2(0,1)$ such that $(\mathscr{L}u,\mathcal{F})_H = 0$, we have

$$(\mathcal{L}u,\psi)_{L^2(Q)} + (\ell u,\omega)_{L^2(0,1)} = 0. \quad (2.19)$$

Putting u in $D_0(\mathscr{L})$ and using the previous theorem, equation 2.19 becomes

$$(\ell u,\omega)_{L^2(0,1)} = 0,$$

taking into consideration the fact that operator ℓ is everywhere dense in $L^2(0,1)$ implies that $\omega = 0$, and so follows the density in the general case $u \in D_\varphi(\mathscr{L})$.

2.4 Illustration ($m = 2$)

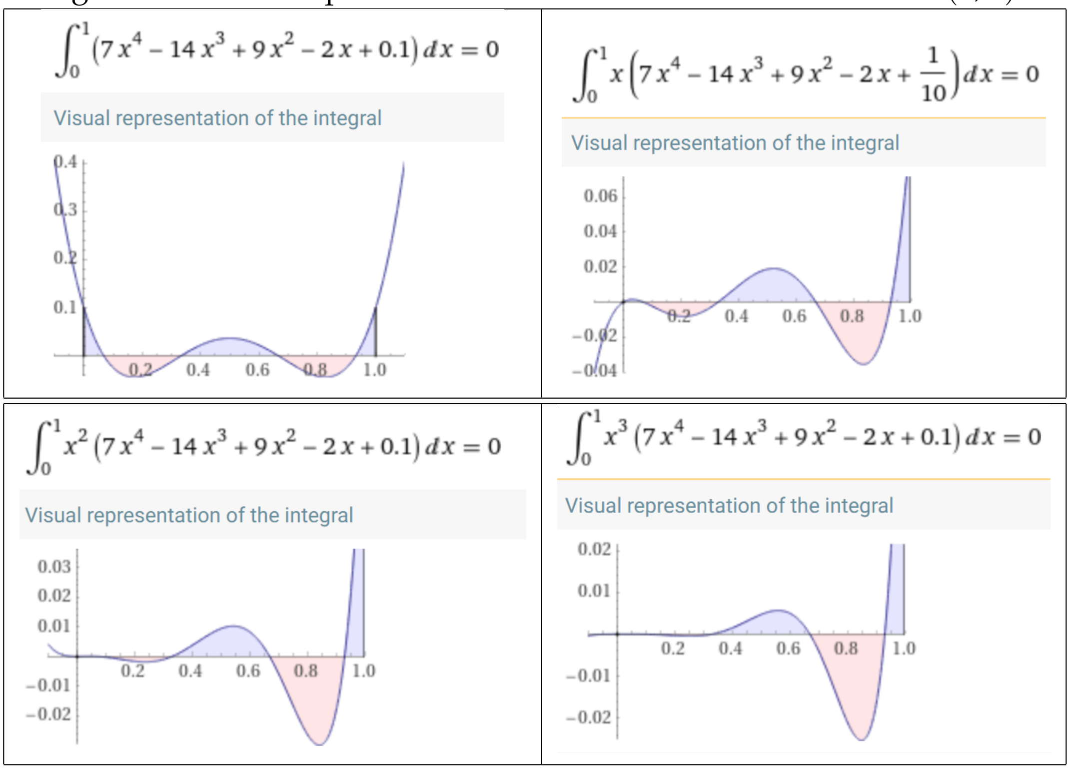
In this section, we take concrete data in the studied problem then we give the exact unique solution.

Example 2.1. Let $\beta = 7(189\ln 2 - 131)$, $\gamma = (1416 - 2043\ln 2)$, $\delta = \dfrac{15}{2}(114\ln 2 - 79)$, $\mu = \dfrac{1}{10}(535 - 774\ln 2)$ and

$$\omega(x) = 7x^4 - 14x^3 + 9x^2 - 2x + \frac{1}{10}.$$

We see in the figures below that the function ω satisfies the integral conditions (2.3), that is all the k-th moments are zero with $k = \overline{0..3}$, (Blue area=Red area).

Figure 2.1: Visual representation of the zero moments of ω over $(0, 1)$.



In the problem (2.1)-(2.3) we set

$$a(x,t) = t - \frac{x^2}{2} + 1,$$

$$\varphi(x) = \beta x^3 + \gamma x^2 + \delta x + \mu + \frac{3}{20(x+1)},$$

and

$$f(x,t) = \frac{\omega(x)}{\Gamma(2-\alpha)} t^{1-\alpha} - 504tx^2 + 18(14t - \beta)x + 2\left(84t^2 + 75t - \gamma\right) + \frac{3(12t - x^2 + 4x + 11)}{10(x+1)^5}.$$

Obviously, $f \in L^2(Q)$ and the function a satisfies the assumptions of theorem 2.1, since

$$A_2^0 = 2,\ A_2^1 = -4,\ A_2^2 = 1,$$

and

$$a \geq \frac{1}{2},\ \frac{\partial^2 a}{\partial x^2} = -1,\ \frac{\partial^4 a}{\partial x^4} = 0,$$

we can take $(c_0, c_1, c_2) = (1, 4, 0)$.

We conclude that problem (2.1)-(2.3) admits a unique solution for the given data.

The reader can check easily that the function φ also satisfies the integral conditions (2.3), so does the function u such that

$$u(x,t) = t\omega(x) + \varphi(x).$$

Moreover, we can show by an elementary calculation that u fulfills equation (2.1) (Clearly, $u(x,0) = \varphi(x)$). Therefore it is the desired **unique** solution.

Chapter 3

FBVP with Mixed conditions : Neumann-integral

In this chapter, we are going to propose an application of the "energy inequality" method to a class of fractional boundary value problems with Neumann-integral conditions. The problem is the following :

In the rectangle $Q = (0,1) \times (0,T)$, we consider the fractional equation

$$\partial_{0t}^{\alpha} v + (-1)^m \frac{\partial^m}{\partial x^m}\left(a(x,t)\frac{\partial^m v}{\partial x^m}\right) = h(x,t) \tag{3.1}$$

where a is a continuous function satisfying $0 < c_0 \leq a \leq c_1$, subject to the initial condition

$$v(x,0) = \phi(x), \qquad x \in (0,1), \tag{3.2}$$

the boundary integral conditions

$$\begin{cases} \int_0^1 v(x,t)\,dx = 0, \\ \int_0^1 xv(x,t)\,dx = 0, \end{cases} \qquad t \in (0,T) \tag{3.3}$$

and the Neumann conditions

$$\begin{cases} \frac{\partial^k}{\partial x^k} v(0,t) = g_k(t) \\ \frac{\partial^k}{\partial x^k} v(1,t) = \psi_k(t) \end{cases} \qquad t \in (0,T), k = \overline{1, m-1}. \tag{3.4}$$

Contents

The proposed problem contains the non-homogeneous Neumann conditions (3.4), so to work out the solvability of the problem

- First, we start by the homogenization of our problem.
- In section 3.2, we establish an a priori estimate of the problem's strong solution to ascertain its uniqueness and data dependance in case of existence.
- Finally, we prove the existence in section 3.3 relying on the density of the generated operator's range, and we finish by giving an example, where m=2, to illustrate the usefulness of the obtained results.

3.1 Problem setting

3.1.1 Homogenization of the Neumann boundary conditions

We show that we can get an equivalent problem to ours (3.1)-(3.4), with homogeneous Neumann conditions. Before establishing the homogenization for any m, let's see how it works in the following examples.

3.1.1.1 Example (m=2)

$k = 1$ follows from (3.4), then we build 2 polynomials of degree (at least) 4 fulfilling the following conditions

$$\begin{cases} \int_0^1 p(x)\,dx = \int_0^1 xp(x)\,dx = 0 \\ \int_0^1 q(x)\,dx = \int_0^1 xq(x)\,dx = 0 \\ \frac{\partial}{\partial x}p(0) = 4x^3 - \frac{7}{2}x^2 - \frac{3}{2}x + 1\big|_0 = 1 \\ \frac{\partial}{\partial x}p(1) = 4x^3 - \frac{7}{2}x^2 - \frac{3}{2}x + 1\big|_1 = 0 \\ \frac{\partial}{\partial x}q(1) = 4x^3 - \frac{7}{2}x^2 + \frac{1}{2}x\Big|_0 = 1 \\ \frac{\partial}{\partial x}q(0) = 4x^3 - \frac{7}{2}x^2 + \frac{1}{2}x\Big|_0 = 0 \end{cases} .$$

It is easy to check that

$$p(x) = x^4 - \frac{7}{6}x^3 - \frac{3}{4}x^2 + x - \frac{19}{120},$$
$$q(x) = x^4 - \frac{7}{6}x^3 + \frac{1}{4}x^2 + \frac{1}{120},$$

are convenient (they are found using a simple system of linear equations from the conditions above).

The first and second equations of the previous brace mean that the polynomials p and q satisfy the integral conditions (3.3), so does the function$w$ defined by

$$w(x,t) = p(x)\,g(t) + q(x)\,\psi(t).$$

Obviously, we have

$$\frac{\partial}{\partial x}w(0,t) = g(t)$$

and

$$\frac{\partial}{\partial x} w(1,t) = \psi(t)$$

Hence, we take

$$v(x,t) = u(x,t) - w(x,t),$$

and the resulting problem (with the unknown function u) is homogreneous. Let's take $m = 3$.

3.1.1.2 Example (m=3)

$k = 1, 2$ dictated by the Neumann conditions (3.4), then we build 4 $(2m-2)$ polynmials of degree at least 6 $(2m)$:

$$p_1(x) = x^6 - \frac{16}{3}x^5 + \frac{28}{3}x^4 - \frac{53}{9}x^3 + x - \frac{187}{1260},$$

$$p_2(x) = x^6 - \frac{52}{15}x^5 + \frac{53}{12}x^4 - \frac{22}{9}x^3 + \frac{1}{2}x^2 - \frac{1}{252},$$

$$q_1(x) = x^6 - \frac{16}{3}x^5 + \frac{25}{3}x^4 - \frac{35}{9}x^3 + \frac{13}{252},$$

$$q_2(x) = x^6 - \frac{38}{15}x^5 + \frac{25}{12}x^4 - \frac{5}{9}x^3 + \frac{1}{630}.$$

The reader can check that

$$p_1'(0) = 1, p_1'(1) = 0, p_1''(0) = 0, p_1''(1) = 0,$$

$$p_2'(0) = 0, p_2'(1) = 0, p_2''(0) = 1, p_2''(1) = 0,$$

$$q_1'(0) = 0, q_1'(1) = 1, q_1''(0) = 0, q_1''(1) = 0,$$

$$q_2'(0) = 0, q_2'(1) = 0, q_2''(0) = 0, q_2''(1) = 1.$$

And each polynomial satisfies the integral conditions (3.3), see figures below

Figure 3.1: Visual representation of p_1 satisfying the integral conditions

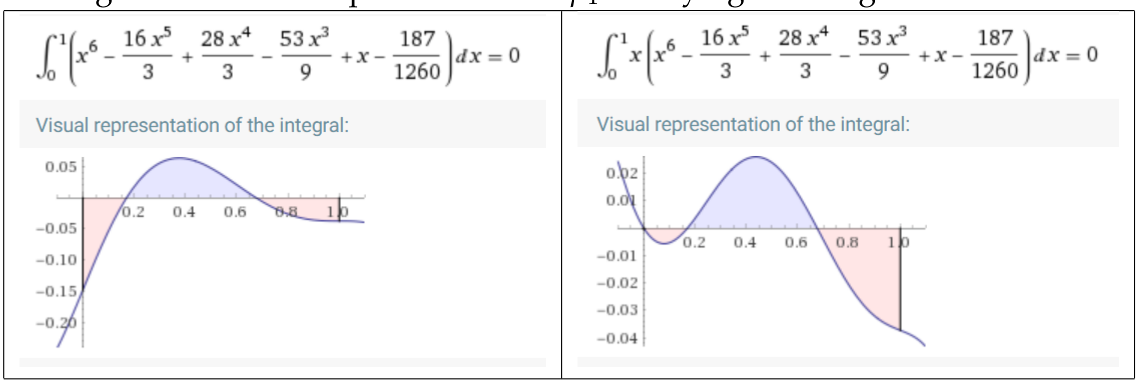


Figure 3.2: Visual representation of p_2 satisfying the integral conditions

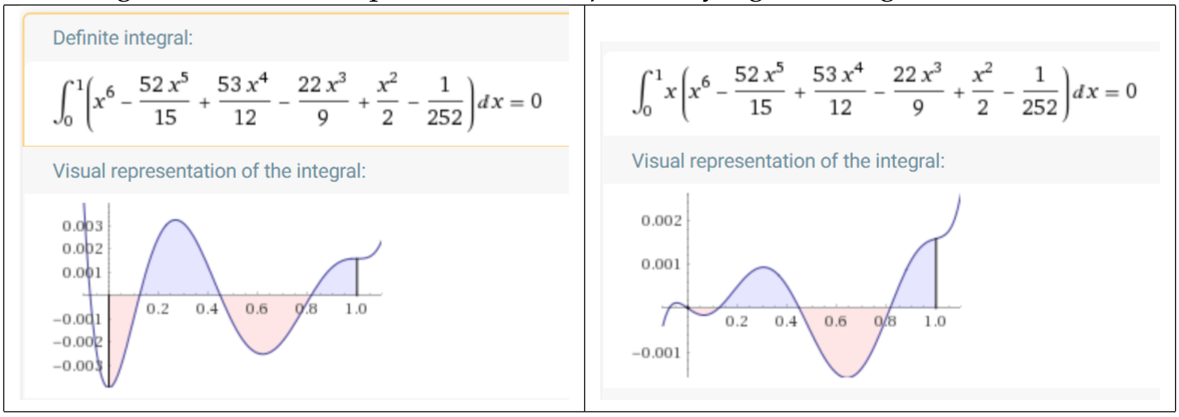


Figure 3.3: Visual representation of q_1 satisfying the integral conditions

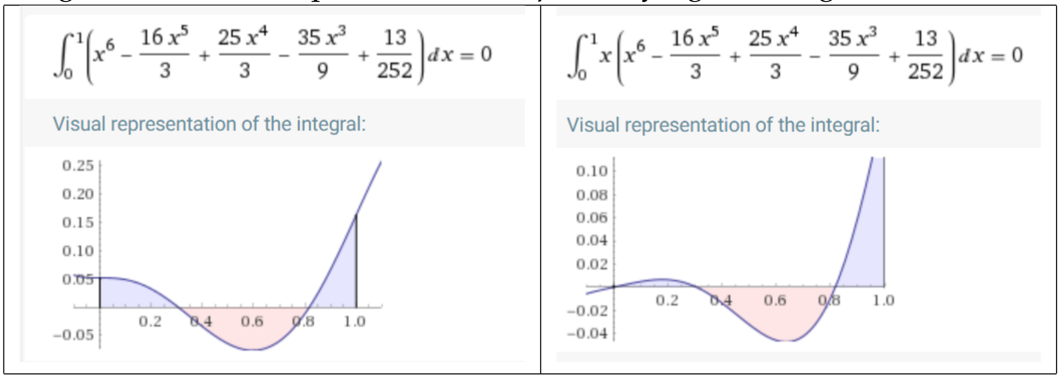
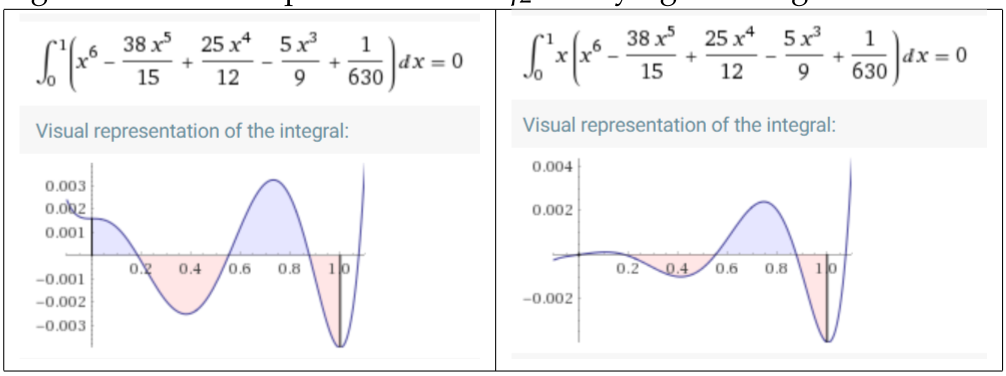


Figure 3.4: Visual representation of q_2 satisfying the integral conditions



Now, we set

$$w(x,t) = p_1(x)\,g_1(t) + p_2(x)\,g_2(t) + q_1(x)\,\psi_1(t) + q_2(x)\,\psi_2(t).$$

Clearly, w satisfies the two integral conditions, also we have

$$\begin{aligned}\frac{\partial}{\partial x} w(0,t) &= p_1'(0)\,g_1(t) + p_2'(0)\,g_2(t) + q_1'(0)\,\psi_1(t) + q_2'(0)\,\psi_2(t)\\ &= 1g_1(t) + 0g_2(t) + 0\psi_1(t) + 0\psi_2(t) = g_1(t),\\ \frac{\partial^2}{\partial x^2} w(0,t) &= p_1''(0)\,g_1(t) + p_2''(0)\,g_2(t) + q_1''(0)\,\psi_1(t) + q_2''(0)\,\psi_2(t)\end{aligned}$$

$$= 0g_1(t) + 1g_2(t) + 0\psi_1(t) + 0\psi_2(t) = g_2(t),$$

$$\frac{\partial}{\partial x} w(1,t) = p_1'(1) g_1(t) + p_2'(1) g_2(t) + q_1'(1)\psi_1(t) + q_2'(1)\psi_2(t)$$

$$= 0g_1(t) + 0g_2(t) + 1\psi_1(t) + 0\psi_2(t) = \psi_1(t)$$

and

$$\frac{\partial^2}{\partial x^2} w(1,t) = p_1''(1) g_1(t) + p_2''(1) g_2(t) + q_1''(1)\psi_1(t) + q_2''(1)\psi_2(t)$$

$$= 0g_1(t) + 0g_2(t) + 0\psi_1(t) + 1\psi_2(t) = \psi_2(t).$$

Consequently, by setting

$$v(x,t) = u(x,t) - w(x,t),$$

the resulting problem (with the unknown function u) is homogreneous.

3.1.1.3 The general case

Recall $\delta_{k,j}$ is the Kronecker delta defined by

$$\begin{cases} \delta_{k,j} = 1, & k = j \\ \delta_{k,j} = 0, & k \neq j \end{cases}.$$

.

We consider $\{p_1, ..., p_{m-1}, q_1, ..., q_{m-1}\}$ a set of $2m-2$ polynomials of degree $2m$ satisfying

$$\int_0^1 p_k(x)\,dx = \int_0^1 x p_k(x)\,dx = 0, \qquad k = \overline{1, m-1},$$

$$\int_0^1 q_k(x)\,dx = \int_0^1 x q_k(x)\,dx = 0, \qquad k = \overline{1, m-1},$$

and for $1 \leq k, j \leq m-1$

$$\begin{cases} \dfrac{\partial^k}{\partial x^k} p_j(0) = \delta_{k,j} \\ \dfrac{\partial^k}{\partial x^k} p_j(1) = 0 \\ \dfrac{\partial^k}{\partial x^k} q_j(1) = \delta_{k,j} \\ \dfrac{\partial^k}{\partial x^k} q_j(0) = 0 \end{cases}.$$

So by setting $u = v - w$ where

$$w(x,t) = \sum_{j=1}^{m-1} p_j(x) g_j(t) + \sum_{j=1}^{m-1} q_j(x)\psi_j(t)$$

taking into consideration that

$$f = h - \partial_{0t}^{\alpha} w - (-1)^m \frac{\partial^m}{\partial x^m}\left(a \frac{\partial^m w}{\partial x^m}\right)$$

and

$$\varphi = \phi - \ell w,$$

the problem (3.1)-(3.4) is equivalent to the following:

3.1.2 The homogeneous equivalent problem

$$\mathcal{L}u = \partial_{0t}^{\alpha} u + (-1)^m \frac{\partial^m}{\partial x^m}\left(a(x,t) \frac{\partial^m u}{\partial x^m}\right) = f(x,t), \tag{3.5}$$

subject to the initial condition

$$\ell u = u(x,0) = \varphi(x), x \in (0,1), \tag{3.6}$$

the integral conditions

$$\begin{cases} \int_0^1 u(x,t)\, dx = 0, \\ \\ \int_0^1 xu(x,t)\, dx = 0, \end{cases}, \tag{3.7}$$

and the Newman conditions

$$\begin{cases} \frac{\partial^k}{\partial x^k} u(0,t) = \mathbf{0} \\ \\ \frac{\partial^k}{\partial x^k} u(1,t) = \mathbf{0} \end{cases} \quad k = \overline{1, m-1}. \tag{3.8}$$

This is the problem we are going to study in the following sections.

3.2 A priori estimate and consequences

As seen in the previous chapter, to establish the existence and uniqueness of the solution of problem (3.5)-(3.8) we write it in the equivalent operator form

$$\mathscr{L}u = \mathcal{F},$$

where $\mathscr{L} = (\mathcal{L}, \ell)$ acts from B to H with domain of definition $D_{\varphi}(\mathscr{L})$ of functions $u \in L^2(0,1)$ satisfying (3.7), (3.8) and

$$\Im_x^k u, \frac{\partial^k u}{\partial x^k} \in L^2(0,1), k = \overline{1,m}.$$

B is a Banach space of functions u endowed by the finite norm

$$\|u\|_B^2 = \sup_{0 \le t \le T} I^{1-\alpha} \|u\|^2_{B_2^m(0,1)} + \|u\|^2_{L^2(Q)},$$

(see lemma 2.1) and H is the Hilbert space consisting of vector-valued functions $\mathcal{F} = (f, \varphi)$ with finite norm

$$\|\mathcal{F}\|_H^2 = \|f\|^2_{L^2\left(B_2^m(0,1),(0,T)\right)} + \|\varphi\|^2_{L^2(0,1)} .$$

As a main result, the "a priori estimate" is proved in the next theorem.

Theorem 3.1. *There exists a positive constant c not depending on u such that*

$$\|u\|_B \leq c\,\|\mathscr{L}u\|_H \tag{3.9}$$

for all u in $D_\varphi(\mathscr{L})$.

Proof. We take the scalar product in space $B_2^m(0,1)$ of the equation (3.1) by the multiplier

$$\mathsf{M}u := 2\frac{\partial^m}{\partial x^m}\left(\frac{1}{a}\Im_x^m u\right),$$

that is

$$2\left(\Im_x^m\left(\partial_{0t}^\alpha u\right) + (-1)^m a\frac{\partial^m u}{\partial x^m}, \frac{1}{a}\Im_x^m u\right)_{L^2(0,1)} = 2\left(\Im_x^m f, \frac{1}{a}\Im_x^m u\right)_{L^2(0,1)},$$

using lemma 2.2, this can be written

$$2\int_0^1 \partial_{0t}^\alpha\left(\Im_x^m u\right)\frac{1}{a}\Im_x^m u dx + 2(-1)^m\int_0^1 \frac{\partial^m u}{\partial x^m}\Im_x^m u dx = 2\int_0^1 \Im_x^m f\frac{1}{a}\Im_x^m u dx. \tag{3.10}$$

For the first term of left hand side of equation (3.10), lemma 1.7 using the positive boundness of the function a implies the existance of a positive constant c_2 such that

$$2\int_0^1 \partial_{0t}^\alpha\left(\Im_x^m u\right)\frac{1}{a}\Im_x^m u dx \geq c_2\int_0^1 \partial_{0t}^\alpha\left(\Im_x^m u\right)^2 dx, \tag{3.11}$$

for the second term of the left hand side, integration by parts $m-1$ times using Neumann conditions (3.8) gives

$$2(-1)^m\int_0^1 \frac{\partial^m u}{\partial x^m}\Im_x^m u dx = -2\int_0^1 \frac{\partial u}{\partial x}\Im_x u dx,$$

and one last integration using integral conditions (3.7) leads to

$$2(-1)^m\int_0^1 \frac{\partial^m u}{\partial x^m}\Im_x^m u dx = 2\int_0^1 u^2 dx. \tag{3.12}$$

For the right hand side of equation (3.10) we use the Cauchy inequality with $\varepsilon = c_2$, that is

$$2\int_0^1 \Im_x^m f\frac{1}{a}\Im_x^m u dx \leq c_2\int_0^1 \left(\Im_x^m u\right) dx + \frac{1}{c_2}\int_0^1 \left(\Im_x^m f\right)^2 dx. \tag{3.13}$$

In light of (3.11)-(3.13), we deduce from the inequality (3.10) that

$$\int_0^1 \partial_{0t}^{\alpha} \left(\Im_x^m u\right)^2 dx + \frac{2}{c_2} \int_0^1 u^2 dx \le \int_0^1 \left(\Im_x^m u\right)^2 dx + \frac{1}{c_2^2} \int_0^1 \left(\Im_x^m f\right)^2 dx. \tag{3.14}$$

Now, in the above inequality, we drop the positive term

$$\frac{2}{c_2} \int_0^1 u^2 dx$$

and substitute t by τ, then we integrate with respect to τ from 0 to t to obtain

$$\int_0^t \int_0^1 \partial_{0t}^{\alpha} \left(\Im_x^m u\right)^2 dx d\tau \le \int_0^t \int_0^1 \left(\Im_x^m u\right)^2 dx d\tau + \frac{1}{c_2^2} \int_0^t \int_0^1 \left(\Im_x^m f\right)^2 dx d\tau,$$

from which lemma 1.8 implies

$$\int_0^t \int_0^1 \left(\Im_x^m u\right)^2 dx d\tau \le \frac{\Gamma(\alpha) E_{\alpha,\alpha}(t^{\alpha})}{c_2^2} I^{\alpha+1} \int_0^1 \left(\Im_x^m f\right)^2 dx.$$

Taking into consideration the proposition 1.6

$$I^{\alpha+1} \int_0^1 \left(\Im_x^m f\right)^2 dx \le \frac{T^{\alpha}}{\Gamma(\alpha+1)} \|f\|^2_{L^2\left(B_2^m(0,1),(0,T)\right)}$$

and lemma 2.3

$$\int_0^t \int_0^1 \partial_{0\tau}^{\alpha} \left(\Im_x^m u\right)^2 dx d\tau = I^{1-\alpha} \|u\|^2_{B_2^m(0,1)} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \|\varphi\|^2_{B_2^m(0,1)},$$

one can get, from the inequality (3.14), using corollary 1.5

$$I^{1-\alpha} \|u\|^2_{B_2^m(0,1)} + \frac{2}{c_2} \int_0^t \|u\|^2_{L^2(0,1)} d\tau$$

$$\le \gamma \left(\|f\|^2_{L^2\left(B_2^m(0,1),(0,T)\right)} + \|\varphi\|^2_{L^2(0,1)} \right)$$

where

$$\gamma = \max\left\{ \frac{\Gamma(\alpha) E_{\alpha,\alpha}(T^{\alpha}) T^{\alpha}}{c_2^2 \Gamma(1+\alpha)} + \frac{1}{c_2^2}, \frac{T^{1-\alpha}}{2^m \Gamma(2-\alpha)} \right\},$$

since the right hand side of the above inequality does not depend on t, we can take the upper bound for both sides with respect to t over $[0, T]$ and the a priori estimate 3.9 follows for

$$c = \left(\frac{\gamma}{\min\left\{1, \frac{2}{c_2}\right\}} \right)^{\frac{1}{2}}.$$

□

Proposition 3.1. *The operator $\mathscr{L}$ from B to H has a closure $\overline{\mathscr{L}}$.*

Proof. Similar to the proposition 2.1. □

Consequently the a priori estimate (3.9) can be extended to cover **strong** solutions by passing to the limit.

Corollary 3.1. *There exists a positive constant c such that*

$$\|u\|_B \leq c \left\|\overline{\mathscr{L}}u\right\|_H \tag{3.15}$$

for all u in $D_\varphi\left(\overline{\mathscr{L}}\right)$.

The uniqueness and continuous dependence of the strong solution on the problem data is now guaranteed in case of existence.

3.3 Existence of the solution

As explained in Section 2.3, we aim to show that is $\overline{R\left(\mathscr{L}\right)} = H$. We start by the case u belongs to $D_0\left(\mathscr{L}\right)$ (i.e. $\ell u = 0$), after which follows the density in the general case $u \in D_\varphi\left(\mathscr{L}\right)$. To do so, notice that $R\left(\overline{\mathscr{L}}\right) = R\left(\mathscr{L}\right)$ (see lemma 2.4). Another main result derives from the next theorem.

Theorem 3.2. *Assume for all u in* $D_0\left(\mathscr{L}\right)$

$$\left(\mathcal{L}u, \psi\right)_{L^2\left(B_2^m(0,1),(0,T)\right)} = 0, \tag{3.16}$$

then ψ vanishes a.e in $L^2\left(B_2^m\left(0,1\right),\left(0,T\right)\right)$.

Proof. First consider the scalar product $\left(\mathcal{L}u, \psi\right)_{B_2^m(0,1)}$, assume a function $\theta\left(x,t\right) \in L^2\left(0,1\right)$ satisfies boundary conditions (3.7)-(3.8) and $\Im_x^k\theta, \dfrac{\partial^k\theta}{\partial x^k} \in L^2\left(0,1\right), k = \overline{1,m}$. Then we can set

$$u\left(x,t\right) = \int_0^t \theta\left(x,\tau\right)d\tau$$

so we have

$$\left(\mathcal{L}u, \psi\right)_{B_2^m(0,1)} = \left(\Im_x^m\left(\partial_{0t}^\alpha u\right) + \left(-1\right)^m a\frac{\partial^m u}{\partial x^m}, \Im_x^m\psi\right)_{L^2(0,1)}$$

$$= \int_0^1 \partial_{0t}^\alpha\left(\int_0^t \Im_x^m\theta\left(x,\tau\right)d\tau\right)\Im_x^m\psi dx + \left(-1\right)^m\int_0^1 a\Im_x^m\psi\int_0^t \frac{\partial^m}{\partial x^m}\theta\left(x,\tau\right)d\tau dx \tag{3.17}$$

We now express ψ in terms of θ:

$$\psi\left(x,t\right) = \frac{\partial^m}{\partial x^m}\left(\frac{1}{a}\int_0^t \Im_x^m\theta\left(x,\tau\right)d\tau\right)$$

to get from equation (3.17)

$$\left(\mathcal{L}u, \psi\right)_{B_2^m(0,1)} = \int_0^1 \frac{1}{a}\partial_{0t}^\alpha\left(\int_0^t \Im_x^m\theta\left(x,\tau\right)d\tau\right)\left(\int_0^t \Im_x^m\theta\left(x,\tau\right)d\tau\right)dx$$

$$+(-1)^m \int_0^1 \int_0^t \Im_x^m \theta(x,\tau)\, d\tau \int_0^t \frac{\partial^m}{\partial x^m}\theta(x,\tau)\, d\tau dx. \tag{3.18}$$

From lemma 1.7 we have

$$\int_0^1 \frac{1}{a}\partial_{0t}^{\alpha}\left(\int_0^t \Im_x^m \theta(x,\tau)\, d\tau\right)\left(\int_0^t \Im_x^m \theta(x,\tau)\, d\tau\right) dx \geq \frac{c_2}{2}\int_0^1 \partial_{0t}^{\alpha}\left(\int_0^t \Im_x^m \theta(x,\tau)\, d\tau\right)^2 dx,$$

and using boundary conditions (3.3), (3.8) one can get

$$(-1)^m \int_0^1 \int_0^t \Im_x^m \theta(x,\tau)\, d\tau \int_0^t \frac{\partial^m}{\partial x^m}\theta(x,\tau)\, d\tau dx = \int_0^1 \left(\int_0^t \theta(x,\tau)\, d\tau\right)^2 dx.$$

Hence, in light of the last two relations, the substitution of t by τ in equation (3.18) then integrating with respect to τ over $[0,t]$ yields

$$\frac{c_2}{2} I^{1-\alpha}\left\|\left(\int_0^t \Im_x^m \theta(x,\tau)\, d\tau\right)\right\|^2_{L^2(0,1)} + \left\|\left(\int_0^t \theta(x,\tau)\, d\tau\right)\right\|^2_{L^2(Q)} \leq 0,$$

by taking the upper bound ovet $[0,T]$ for both sides of the above inequality yields

$$\gamma\left(\sup_{0\leq t\leq T} I^{1-\alpha}\left\|\int_0^t \Im_x^m \theta(x,\tau)\, d\tau\right\|^2_{L^2(0,1)} + \left\|\left(\int_0^t \theta(x,\tau)\, d\tau\right)\right\|^2_{L^2(Q)}\right) \leq 0,$$

where $\gamma = \min\left\{1, \frac{c_2}{2}\right\}$. Consequently, $\theta = 0$ and $\psi = 0$ a.e in $L^2\left(B_2^m(0,1),(0,T)\right)$.

□

Now consider the general case. Let $\mathscr{L}_0$ denotes the operator $(\mathcal{L}_0, 0)$, If we use the fact that $\mathscr{L} - \mathscr{L}_0 = (\mathcal{L} - \mathcal{L}_0, \ell)$ maps continuously B into H, we conclude that $R(\mathscr{L})$ is dense in H.

3.4 Illustration ($m = 2$)

Starting with the homogenization of the problem (3.1)-(3.4), we take from the **example** 3.1.1.1

$$p(x) = x^4 - \frac{7}{6}x^3 - \frac{3}{4}x^2 + x - \frac{19}{120}, q(x) = x^4 - \frac{7}{6}x^3 + \frac{1}{4}x^2 + \frac{1}{120}$$

and

$$w(x,t) = p(x) g(t) + q(x)\psi(t)$$

to ensure that the problem is equivalent to the homogeneous one (3.5)-(3.8).

Now, let

$$b_1 = 2880 \ln 2 - 2020,\ b_2 = 3066 - 4320 \ln 2,\ b_3 = 816 \ln 2 - 469,$$

$$\omega(x) = b_1 x^3 + b_2 x^2 + b_3 x - 96 \left(x + \frac{1}{x+1} \right),$$

$$a(x,t) = t - \frac{x^2}{2} + 1,$$

$$\varphi(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$$

and

$$f(x,t) = \frac{\omega(x)}{\Gamma(2-\alpha)} t^{-\alpha} - \frac{2}{(x+1)^5} \begin{pmatrix} (-73 + 324t + 5tb_2 + 9tb_1)\, x \\ + (-216t - 164 + 45tb_1 + 10tb_2)\, x^2 \\ + (-120t - 110 + 90tb_1 + 10tb_2)\, x^3 \\ + (-60t + 125 + 90tb_1 + 5tb_2)\, x^4 \\ + (259 - 12t + 45tb_1 + tb_2)\, x^5 \\ + (162 + 9tb_1)\, x^6 \\ +36x^7 + 1044t + 1152t^2 + tb_2 - 11 \end{pmatrix}.$$

Obviously, the function a is positively bounded, f is in $L^2\left(B_2^m(0,1), (0,T)\right)$. Thus the function u given by

$$u(x,t) = \varphi(x) + t\omega(x)$$

satisfies the equation (3.5) and fulfills :

- Initial condition (3.6)
$$u(x,0) = \varphi(x),$$
- Integral conditions (3.7) and Neumann conditions (3.8).

The function u is the desired **unique** solution.

Conclusion

In this thesis, we have studied some fractional evolution problems with nonlocal boundary conditions. More precisely we have studied a class of problems with purely integral conditions, then with mixed conditions of the integral-Neumann type. The approach followed to do this is the "energy inequality" method also known as the "a priori estimate" method. This method consists mainly in establishing the estimate and the demonstration of the range density of the operator generated by the problem.

In the chapter of Preliminaries, we introduced several concepts and mathematical tools among which we quote:

- Banach and Hilbert spaces as well as their properties such as completeness, orthogonality and density.
- Bouziani space B_2^k and the space $L^2\left(B_2^k, (0,T)\right)$.
- Several equalities and inequalities either known and referenced or demonstrated here.

In the 3^{rd} chapter , we proved the "strong" well-posedness of a the problem associated purely integral conditions using a new generalized formula of integration. Thus, we would like to note that the solvability of the problem obtained by replacing the operator $\mathcal{L}$ with

$$\mathcal{L}u \equiv \partial_{0t}^{\alpha} u + (-1)^m \frac{\partial^m}{\partial x^m}\left(a\left(x,t\right)\frac{\partial^m u}{\partial x^m}\right) + b\left(x,t\right)u$$

which some instances have been studied by many authors, can be established in the same way assuming the boundness of function b.

In the 4^{th} chapter, we proved the existence and uniqueness of the strong solution of the problem associated non-homogeneous mixed nonlocal conditions of type : integral-Neumann.

As perspectives, we would like to improve the application of the method either by posing less demanding hypotheses in the main theorems, or by starting other classes involving other types of derivatives (Riemann-Liouville or mixed ...)

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