Applied Mathematics and Mechanics (English Edition), 2006, **27**(9):1217–1222

FIXED POINT THEOREMS FOR MAPPINGS SATISFYING IMPLICIT RELATION ON TWO COMPLETE AND COMPACT METRIC SPACES *

Abdlkrim Aliouche[1], Brian Fisher[2]

(1. Department of Mathematics, University of Lari Ben M'Hidi,
Oum EI Bouaghi, 04000, Algria;
2. Department of Mathematics, University of Leicester,
Leicester, LE1 7RH, U. K.)

(Communicated by DING Xie-ping)

Abstract: First, the implicit relations were given. A common fixed point theorem was proved for two mappings satisfying implicit relation functions. A further fixed point theorem was proved for mappings satisfying implicit relation functions on two compact metric spaces.

Key words: complete metric space; compact metric space; fixed points; implicit relation

Chinese Library Classification: O177.91
2000 Mathematics Subject Classification: 54H25
Digital Object Identifier(DOI): 10.1007/s 10483-006-0908-z

1 Preliminaries

The following theorems were proved in Ref.[1].

Theorem 1 *Let (X, d) and (Y, ρ) be complete metric spaces, T a mapping of X into Y and S a mapping of Y into X satisfying the inequalities:*

$$\rho(Tx, TSy) \le f(d(x, Sy), \rho(y, Tx), \rho(y, TSy)),$$
$$d(Sy, STx) \le g(\rho(y, Tx), d(x, Sy), d(x, STx)),$$

for all x in X and y in Y, where $f, g \in \digamma$. Then ST has a unique fixed point u in X and TS has a unique fixed point v in Y. Further, $Tu = v$ and $Sv = u$.

Theorem 2 *Let (X, d) and (Y, ρ) be compact metric spaces, T a continuous mapping of X into Y and S a continuous mapping of Y into X satisfying the inequalities:*

$$\rho(Tx, TSy) < f(d(x, Sy), \rho(y, Tx), \rho(y, TSy))$$

for all x in X and y in Y with $x \neq Sy$, where $f \in \digamma^*$, and

$$d(Sy, STx) < g(\rho(y, Tx), d(x, Sy), d(x, STx))$$

for all x in X and y in Y with $y \neq Tx$, where $g \in \digamma^*$. Then ST has a unique fixed point u in X and TS has a unique fixed point v in Y. Further, $Tu = v$ and $Sv = u$.

* Received Aug.4, 2005; Revised May 30, 2006
Corresponding author Abdlkrim Aliouche, Professor, Doctor, E-mail: abdmath@hotmail.com

2 Fixed Points on Complete Metric Spaces

Implicit relations

We denote by $\mathbb{R}_+$ the set of the non-negative reals and by $\mathcal{F}$ the set of all real functions $f : R_+^4 \to \mathbb{R}$ such that:

(i) f is upper semi-continuous in each coordinate variable;

(ii) if either $f(u, v, 0, u) \le 0$ or $f(u, v, u, 0) \le 0$ for all $u, v \ge 0$, then there exists a real constant $0 \le c < 1$ such that $u \le cv$.

Example 1 $f(t_1, t_2, t_3, t_4) = t_1 - c \max\{t_2, t_3, t_4\}, \ 0 \le c < 1$.

(i) is clear since f is continuous.

Suppose that $u, v \ge 0$ and then $f(u, v, 0, u) = u - c \max\{v, u\} \le 0$. If $v \le u$, then $u \le cu < u$, a contradiction. Therefore, $u \le cv$. Similarly, if $f(u, v, u, 0) \le 0$ then $u \le cv$. The proof of (ii) is completed.

Example 2 $f(t_1, t_2, t_3, t_4) = t_1^2 - c_1 \max\{t_2^2, t_3^2, t_4^2\} - c_2 \max\{t_1t_3, t_2t_4\} - c_3t_3t_4$, where $c_1, c_2, c_3 \in \mathbb{R}_+$, and $c_1 + c_2 < 1$.

(i) is clear since f is continuous.

Suppose that $u, v \ge 0$ and then $f(u, v, 0, u) = u^2 - c_1 \max\{v^2, u^2\} - c_2uv \le 0$. If $v \le u$, then $u^2 \le (c_1 + c_2)u^2 < u^2$, a contradiction. Therefore, $u \le av$, where $a = \sqrt{c_1 + c_2} < 1$.

Similarly, if $f(u, v, u, 0) \le 0$ then $u \le bv$, where $b = c_1/(1 - c_2) < 1$. We then take $c = \max\{a, b\} < 1$, thus the proof of (ii) is completed.

Example 3 $f(t_1, t_2, t_3, t_4) = t_1^3 - \alpha t_1^2 t_2 - \beta t_1t_3t_4 - \gamma t_2t_3^2 - \delta t_3t_4^2$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}_+$ and $\alpha + \gamma < 1$.

(i) is clear since f is continuous.

Suppose that $u, v \ge 0$ and then $f(u, v, 0, u) = u^3 - \alpha u^2 v = u^2(u - \alpha v) \le 0$. Then $u \le \alpha v$.

Similarly, if $f(u, v, u, 0) \le 0$ then $u \le (\alpha + \gamma)v$. We can then take $c = \max\{\alpha, \gamma\} < 1$, thus the proof of (ii) is completed.

Example 4 $f(t_1, t_2, t_3, t_4) = t_1^3 - c\frac{t_1^2t_2^2 + t_3^2t_4^2}{t_2+t_3+t_4+1}$, where $0 < c < 1$.

(i) is clear since f is continuous.

Suppose that $u, v \ge 0$ and then

$$f(u, v, 0, u) = u^3 - c\frac{u^2v^2}{v + u + 1} \le 0.$$

Then,

$$u \le c\frac{v^2}{v + u + 1} < cv.$$

Similarly, if $f(u, v, u, 0) \le 0$ then $u \le cv$, thus the proof of (ii) is completed.

Example 5 $f(t_1, t_2, t_3, t_4) = (1 + pt_2)t_1 - p \max\{t_1t_2, t_3t_4\} - c \max\{t_2, t_3, t_4\}$, where $0 < c < 1$ and $p \ge 0$.

(i) is clear since f is continuous.

Suppose that $u, v \ge 0$ and then

$$f(u, v, 0, u) = (1 + pv)u - puv - c \max\{v, u\} \le 0.$$

If $v \le u$, then $u \le cu < u$, a contradiction. Therefore, $u \le cv$.

Similarly, if $f(u, v, u, 0) \le 0$ then $u \le cv$, thus the proof of (ii) is completed.

Example 6 $f(t_1, t_2, t_3, t_4) = t_1 - c \max\{t_2, t_3, t_4, b\sqrt{t_3t_4}\}$, where $b \ge 0$ and $0 < c < 1$.

(i) is clear since f is continuous and (ii) follows as Example 1.

Example 7 $f(t_t, t_2, t_3, t_4) = t_1 - (\alpha t_2^p + \beta t_3^p + \gamma t_4^p)^{\frac{1}{p}}$, where $p > 0$ and $0 < \alpha, \beta, \gamma, \alpha + \beta + \gamma < 1$.

(i) is clear since f is continuous.

Suppose that $u, v \geq 0$ and then $f(u, v, 0, u) = u - (\alpha v^p + \gamma u^p)^{\frac{1}{p}} \leq 0$ and $u \leq av$, where

$$a = \left(\frac{\alpha}{1-\gamma}\right)^{\frac{1}{p}} < 1.$$

Similarly, if $f(u, v, u, 0) \leq 0$ then $u \leq bv$, where

$$b = \left(\frac{\alpha}{1-\beta}\right)^{\frac{1}{p}} < 1.$$

We can then take $c = \max\{a, b\} < 1$, thus the proof of (ii) is completed.

Now, we prove the following fixed point theorem in a complete metric space using an implicit relation.

Theorem 3 *Let (X, d) and (Y, ρ) be complete metric spaces and let S, T be mappings of Y into X satisfying the inequalities*:

$$f(\rho(Tx, TSy), d(x, Sy), \rho(y, Tx), \rho(y, TSy)) \leq 0, \tag{1}$$
$$g(d(Sy, STx), \rho(y, Tx), d(x, Sy), d(x, STx)) \leq 0 \tag{2}$$

for all x in X and y in Y, where $f, g \in \mathcal{F}$. Then ST has a unique fixed point u in X and TS has a unique fixed point v in Y. Further, $Tu = v$ and $Sv = u$.

Proof Let x_0 be an arbitrary point in X. We define the sequences $\{x_n\}$ and $\{y_n\}$ in X and Y inductively by

$$y_n = Tx_{n-1}, \quad x_n = Sy_n$$

for $n = 1, 2, \cdots$.

Applying the inequality (1), we get

$$\begin{aligned} & f(\rho(Tx_{n-1}, TSy_n), d(x_{n-1}, Sy_n), \rho(y_n, Tx_{n-1}), \rho(y_n, TSy_n)) \\ = {} & f(\rho(y_n, y_{n+1}), d(x_{n-1}, x_n), 0, \rho(y_n, y_{n+1})) \leq 0, \end{aligned}$$

and from (ii), we have

$$\rho(y_n, y_{n+1}) \leq cd(x_{n-1}, x_n), \tag{3}$$

where $c = \max\{a, b\}$, a, b are real constants satisfying (ii) for f and g, respectively.

Using the inequality (2), we have

$$\begin{aligned} & g(d(Sy_n, STx_n), \rho(y_n, Tx_n), d(x_n, Sy_n), d(x_n, STx_n)) \\ = {} & g(d(x_n, x_{n+1}), \rho(y_n, y_{n+1}), 0, d(x_n, x_{n+1})) \leq 0, \end{aligned}$$

and from (ii), we have

$$d(x_n, x_{n+1}) \leq c\rho(y_n, y_{n+1}). \tag{4}$$

From the inequalities (3) and (4), we now obtain

$$d(x_n, x_{n+1}) \leq c^2 d(x_{n-1}, x_n)$$

and it follows by induction that

$$d(x_n, x_{n+1}) \leq c^{2n} d(x_0, x_1) \tag{5}$$

for $n = 1, 2, \cdots$.

Similarly,

$$\rho(y_n, y_{n+1}) \leq c^{2n-2} d(x_0, x_1) \tag{6}$$

for $n = 1, 2, \cdots$.

Since $0 < c < 1$, it follows from the inequalities (5) and (6) that $\{x_n\}$ is a Cauchy sequence in X with a limit u in X and $\{y_n\}$ is a Cauchy sequence in Y with a limit v in Y.

If $Tu \neq v$, then using the inequality (1), we get

$$\begin{aligned} & f(\rho(Tu, TSy_{n-1}), d(u, Sy_{n-1}), \rho(y_{n-1}, Tu), \rho(y_{n-1}, TSy_{n-1})) \\ = & f(\rho(Tu, y_n), d(u, x_{n-1}), \rho(y_{n-1}, Tu), \rho(y_{n-1}, y_n)) \\ \leq & 0. \end{aligned}$$

Letting n tend to infinity and using (i) we obtain

$$\rho(Tu, v) \leq f(0, \rho(v, Tu), 0).$$

From (ii) it follows that $\rho(Tu, v) = 0$ and so $Tu = v$.

It follows similarly that $Sv = u$. It then follows that $STu = Sv = u$ and $TSv = Tu = v$.

To prove the uniqueness, suppose that TS has a second fixed point v' in Y. Using the inequality (1), we get

$$f(\rho(Tu, TSv'), d(u, Sv'), \rho(v', v), 0) = f(\rho(v, v'), d(Sv, Sv'), \rho(v', v), 0) \leq 0,$$

and from (ii) it follows that

$$\rho(v, v') \leq cd(Sv, Sv'). \tag{7}$$

If $Sv \neq Sv'$, then using the inequality (2), we obtain

$$g(d(Sv, Sv'), \rho(v, v'), d(Sv, Sv'), 0) \leq 0,$$

and from (ii) it follows that

$$d(Sv, Sv') \leq c\rho(v, v'). \tag{8}$$

Using the inequalities (7) and (8), we see that $v = v'$, then the uniqueness of v is proved.

The uniqeness of u follows similarly. This completes the proof of Theorem 3.

3 Fixed Points on Compact Metric Spaces

Implicit relations

We now denote by $\mathcal{F}^*$ the set of all real functions $f : R_+^4 \to \mathbb{R}$ which are upper semi-continuous in each coordinate variable such that

(iii) if either $f(u, v, 0, u) < 0$ or $f(u, v, u, 0) < 0$ for all $u, v \geq 0$, then $u < v$.

Example 8 $f(t_1, t_2, t_3, t_4) = t_1 - c \max\{t_2, t_3, t_4\}$, $0 < c \leq 1$.

Example 9 $f(t_1, t_2, t_3, t_4) = t_1^2 - c_1 \max\{t_2^2, t_3^2, t_4^2\} - c_2 \max\{t_1 t_3, t_2 t_4\} - c_3 t_3 t_4$, where $c_1, c_2, c_3 \in \mathbb{R}_+$, and $c_1 + c_2 \leq 1$.

Example 10 $f(t_1, t_2, t_3, t_4) = t_1^3 - \alpha t_1^2 t_2 - \beta t_1 t_3 t_4 - \gamma t_2 t_3^2 - \delta t_3 t_4^2$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}_+$, $\alpha + \gamma \leq 1$.

Example 11 $f(t_1, t_2, t_3, t_4) = t_1^3 - c\frac{t_3^2 t_4^2 + t_5^2 t_6^2}{t_2 + t_3 + t_4 + 1}$, where $0 < c \leq 1$.

Example 12 $f(t_1, t_2, t_3, t_4) = (1 + pt_2)t_1 - p \max\{t_1 t_2, t_3 t_4\} - c \max\{t_2, t_3, t_4\}$, where $p \geq 0$ and $0 < c \leq 1$.

Example 13 $f(t_1, t_2, t_3, t_4) = t_1 - c \max\{t_2, t_3, t_4, b\sqrt{t_3 t_4}\}$, where $b \geq 0$ and $0 < c \leq 1$.

Example 14 $f(t_1, t_2, t_3, t_4) = t_1 - (\alpha t_2^p + \beta t_3^p + \gamma t_4^p)^{\frac{1}{p}}$, where $p > 0$, $\alpha, \beta, \gamma \in \mathbb{R}_+$ and $\alpha + \beta + \gamma \leq 1$.

The proofs that these functions are in $\mathcal{F}^*$ follow as above.

Now, we prove the following fixed point theorem in a compact metric space using an implicit relation.

Theorem 4 *Let (X, d) and (Y, ρ) be compact metric spaces, T a continuous mapping of X into Y and S a continuous mapping of Y into X satisfying the inequalities*:

$$f(\rho(Tx, TSy), d(x, Sy), \rho(y, Tx), \rho(y, TSy)) < 0 \tag{9}$$

for all x in X and y in Y with $x \neq Sy$, where $f \in \digamma^*$, and

$$g(d(Sy, STx), \rho(y, Tx), d(x, Sy), d(x, STx)) < 0 \tag{10}$$

for all x in X and y in Y with $y \neq Tx$, where $g \in \digamma^*$. Then ST has a unique fixed point u in X and TS has a unique fixed point v in Y. Further, $Tu = v$ and $Sv = u$.

Proof The function $\varphi : X \to \mathbb{R}_+$ defined by $\varphi(x) = d(x, STx)$ is continuous on X. Since X is compact, there exists a point u in X such that

$$\varphi(u) = d(u, STu) = \min\{d(x, STx) : \ x \in X\}.$$

We will suppose that $Tu \neq TSTu$ so that $u \neq STu$. Using the inequality (10), we have

$$g(d(STu, STSTu), \rho(Tu, TSTu), 0, d(STu, STSTu)) < 0,$$

and from (iii), it follows that

$$d(STu, STSTu) < \rho(Tu, TSTu). \tag{11}$$

Using the inequality (9), we have

$$f(\rho(Tu, TSTu), d(u, STu), 0, \rho(Tu, TSTu)) < 0,$$

and from (iii), it follows that

$$\rho(Tu, TSTu) < d(u, STu). \tag{12}$$

We now deduce from the inequalities (11) and (12) that

$$\varphi(STu) = d(STu, STSTu) < \rho(Tu, TSTu) < d(u, STu) = \varphi(u),$$

a contradiction and so $TSTu = Tu$.

Putting $Tu = w$ and $Sw = z$, we get

$$ST(STu) = S(TSTu) = STu = Sw = z$$

and

$$w = Tu = TS(Tu) = T(STu) = Tz,$$

then the existence of z and w is proved.

To prove uniqueness, suppose that ST has a second distinct fixed point z'. Using the inequality (10), we obtain

$$g(d(STz, STz'), \rho(Tz, Tz'), d(z', z), 0) = g(d(z, z'), \rho(Tz, Tz'), d(z, z'), 0) < 0,$$

and it follows from (iii) that

$$d(z, z') < \rho(Tz, Tz'). \tag{13}$$

Using the inequality (9), we have

$$f(\rho(Tz, TSTz'), d(z, z'), \rho(Tz', Tz), 0) = f(\rho(Tz, Tz'), d(z, z'), \rho(Tz', Tz), 0) < 0$$

and it follows from (iii) that

$$\rho(Tz, Tz') < d(z, z'). \tag{14}$$

The inequalities (13) and (14) lead to a conradiction and so $z = z'$, then the uniqueness of z is proved.

We can prove similarly that w is unique. This completes the proof of Theorem 4.

Remark

If

$$\begin{aligned} & f(\rho(Tx, TSy), d(x, Sy), \rho(y, Tx), \rho(y, TSy)) \\ = & \rho(Tx, TSy) - c\max\{d(x, Sy), \rho(y, Tx), \rho(y, TSy)\} \end{aligned}$$

and

$$\begin{aligned} & g(d(Sy, STx), \rho(y, Tx), d(x, Sy), d(x, STx)) \\ = & d(Sy, STx) - c\max\{\rho(y, Tx), d(x, Sy), d(x, STx)\}, \end{aligned}$$

where $0 \leq c < 1$, we obtain the result of Ref.[2].

Acknowledgement The authors would like to thank Dr. Telci for his paper[1].

References

[1] Telci M. Fixed points on two complete and compact metric spaces[J]. *Applied Mathematics and Mechanics* (*English Edition*), 2001, **22**(5):564–568.

[2] Fisher B. Fixed point on two metric spaces[J]. *Glasnik Mat,* 1981, **16**(36):333–337.